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Chapter 1

Formal Power Series

1.1 Intro

Definition 1.1.1. Let R be integral domain, then we define $R[[x]] = \{\sum_{n\geq 0} a_i x^i : a_i \in R\}$ to be the ***ring of formal power series*** with the usual operation.

Proposition 1.1.2. *If R is integral domain then $R[[x]]$ is an integral domain.*

Proof. Exercise. ♡

Remark 1.1.3. We get the following operations:

1. We can define $\frac{A(x)}{B(x)} = C(x)$, if $A(x) = B(x)C(x)$.
2. We can define $[x^n]A(x)$ to be the coefficient x^n of $A(x)$, i.e. $[x^n]A(x) = a_n$ if $A(x) = \sum_{i\geq 0} a_i x^i$.
3. We can define the ***valuation*** $\mathrm{Val}_x(A)$ to be $\min\{m : [x^m]A(x) \neq 0\}$ or ∞ if $A(x) = 0$. We have:
   (a) $\mathrm{Val}_x(A + B) \geq \min\{\mathrm{Val}_x(A), \mathrm{Val}_x(B)\}$.
   (b) $\mathrm{Val}_x(AB) = \mathrm{Val}_x(A) + \mathrm{Val}_x(B)$.

Remark 1.1.4. As we can define a valuation on $R[[x]]$, we see we can define a topology on $R[[x]]$ as follows: as a matric space, let $0 < \epsilon < 1$, then we define $d_\epsilon(A(x), B(x)) = \epsilon^{\mathrm{Val}_x(A-B)}$.

In particular, we can take limits as follows: suppose we have $A_i \in R[[x]]$, then we define

$$\lim_{n\to\infty} A_n(x) = A(x)$$

if and only if

$$\lim_{n\to\infty} \mathrm{Val}_x(A_n(x) - A(x)) = \infty$$

in the standard topology in $\mathbb{R}$.

Thus, infintie sums and infinite products can be defined. Indeed, we just consider $\sum_{n=0}^{\infty} A_n(x) = \lim_{N\to\infty} \sum_{n=0}^{N} A_n(x)$. In other word, the limit exists if and only if the coefficients of A_i are eventually constant.

Remark 1.1.5. In a lot of cases, we also want to consider multivariate formal power series. In particular, we see $R[[x,y]] := R[[x]][[y]] = R[[y]][[x]]$ and the two at the right are canonically isomorphic as rings. However, we note the topology is a little bit tricky as for $R[[x]][[y]]$ uses Val_y and $R[[y]][[x]]$ uses Val_x.

On the other hand, we can also define $\mathrm{Val}_{x,y} A(x,y)$ as follows: $\mathrm{Val}_{x,y} A = \min\{m+n : [x^m y^n]A(x,y) \neq 0\}$ or equal ∞ if $A(x,y) = 0$.

Definition 1.1.6. We define $R[[x]]_+$ to be the set of formal power series with no constant, i.e. $A(x) \in R[[x]]_+$ iff $[x^0]A(x) = 0$. For $A(x) \in R[[x]]$ we also define $A_+(x) = A(x) - [x^0]A(x) \in R[[x]]_+$.

Remark 1.1.7. Given $A(x), B(x) \in R[[x]]$, then:

1. we can define $A(B(x)) := \sum_{n\geq 0} a_n B(x)^n$. This is not always defined. We have the following properties, as one can prove:
   (a) $A(B(x))$ is defined if and only if $A[x] \in R[x]$ or $B(x) \in R[[x]]_+$.
   (b) For $B(x) \in R[[x]]_+$, the "evaluation map" $ev_B : R[[x]] \to R[[x]]$ given by $A(x) \mapsto A(B(x))$ is a ring homomorphism. If $B(x) \neq 0$ then this map is injective.
2. we say $A(x), B(x) \in R[[x]]$ are ***compositional inverses*** if and only if $A(B(x)) = B(A(x)) = x$.

Proposition 1.1.8. *Let $A(x), B(x) \in R[[x]]$, then $A(B(x)) = x$ implies $B(A(x)) = x$.*

Proof. Note $A(B(x)) = x$, we see we can apply B to both side and get $B(A(B(x))) = B(x)$. On the other hand, note if $B(A(x)) = x$ then we can apply ev_B to both side and get $B(A(B(x))) = B(x)$, but ev_B is injective, which forces us to have $B(A(x)) = x$. ♡

Proposition 1.1.9. *$A(x)$ has a multiplicative inverse if and only if a_0 is invertible in R.*

Proof. ($\Rightarrow$): exercise.

($\Leftarrow$): Consider $G(x) = \sum_{n\geq 0}(-1)^n a_0^{-n-1} x^n$. Then we see $(a_0 + x)G(x) = 1$ and now apply ev_{A_+} to both side of the equation and we get $(a_0 + A_+(x))G(A_+(x)) = 1$, where $a_0 + A_+(x) = A(x)$ and hence we are done, i.e. $G(A_+(x))$ is the inverse of $A(x)$. ♡

Definition 1.1.10. Let $A(x) \in R[[x]]$, then we define:

1. $\frac{d}{dx}A(x) = A'(x) = \sum_{n\geq 0} n a_n x^{n-1}$.
2. If $\mathbb{Q} \subseteq R$, then we can define the ***formal integral*** to be $\int_x A(x) = \sum_{n\geq 0} \frac{a_n}{n+1} x^{n+1}$.

Proposition 1.1.11. *We have:*

1. $\frac{d}{dx}(A(x) + cB(x)) = A'(x) + cB'(x)$ *for* $c \in R$
2. $\frac{d}{dx}(A(x)B(x)) = A'(x)B(x) + A(x)B'(x)$.

3. *$\frac{d}{dx}A(B(x)) = A'(B(x))B'(x)$ if composition is defined.*

Example 1.1.12. We define some special formal power series:

1. $\exp(x) := \sum_{n\geq 0} \frac{1}{n!}x^n \in \mathbb{Q}[[x]]$. Properties:
   (a) $\exp(x+y) = \exp(x) + \exp(y)$
   (b) $\frac{d}{dx}\exp(x) = \exp(x)$

   We may also write $\exp(x) = e^x$.
2. $L(x) := \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}x^n \in \mathbb{Q}[[x]]$. Properties:
   (a) $L(x)$ is the compositional inverse of $\exp_+(x) = e^x - 1$.
   (b) $L(\frac{x}{1-x}) = -L(-x) = \sum_{n\geq 1}\frac{1}{n}x^n$. This is the compositional inverse of $1 - e^{-x}$.
   (c) $L(x+y+xy) = L(x) + L(y)$.
   (d) $\frac{d}{dx}L(x) = \frac{1}{1+x}$

   We may also write $\log(1+x) := L(x)$ and $\log(\frac{1}{1-x}) := L(\frac{x}{1-x})$.
3. $B(x,y) = \sum_{n\geq 0}\binom{y}{n}x^n \in \mathbb{Q}[y][[x]]$. Properties:
   (a) $B(x,y)B(x,z) = B(x,y+z)$.
   (b) $\frac{\partial}{\partial x}B(x,y) = yB(x,y-1)$.
   (c) $B(x,y) = \exp(y\log(1+x))$.

   We may also write $B(x,y) = (1+x)^y$.

Lemma 1.1.13 (Hensel's Lemma)**.** *Let $F(x,t) \in R[[t,x]]$ and $F'(t,x) := \frac{\partial}{\partial x}F(t,x)$. Suppose $F(0,0) = 0$, $F'(0,0)$ is invertible, then there exists unique $f(t) \in R[[t]]_+$ such that $F(t,f(t)) = 0$.*

Remark 1.1.14. Two key ideas for the proof here:

1. Linear approximation: for any $A(x) \in R[[x]]$, we have $A(u) = A(v) + (u-v)A'(v) \pmod{(u-v)^2}$.
2. Newton's method: we will construct $f_0(t) = 0$, then $f_{n+1}(t) = f_n(t) - \frac{F(t,f_n(t))}{F'(t,f_n(t))}$ and we take the limit to define $f(t) = \lim_{n\to\infty} f_n(t)$. Then we use linear approximation to prove the limit exists.

Now we give the detailed proof of Hensel's lemma.

Proof. Existence: let $f_0(t) = 0$, $f_{n+1}(t) = f_n(t) - \frac{F(t,f_n(t))}{F'(t,f_n(t))}$. By induction, $f_n(t) \in R[[t]]_+$ for all n, hence the right hand side is defined. Now apply the linear approximation with $A(x) = F(t,x)$, $u = f_n(t), v = f_{n-1}(t)$ we get

$$F(t,f_n(t)) = F(t,t_{n-1}(t)) + (f_n(t) - f_{n-1}(t))F'(t,f_{n-1}(t)) \pmod{(f_n(t) - f_{n-1}(t))^2}$$

Thus we see $F(t,f_{n+1}(t))$ is divisible by $(f_n(t) - f_{n-1}(t))^2$.

On the other hand, since $F'(t,f_n(t))$ is invertible, from the definition of f_n we see

$$f_{n+1}(t) - f_n(t)$$

is divisible by $F(t,f_n(t))$. Thus we see

$$\operatorname{Val}_t(f_n(t) - f_{n-1}(t)) \leq 2\operatorname{Val}_t F(t,f_n(t)) \leq 2\operatorname{Val}_t(f_{n+1}(t) - f_n(t))$$

and hence $\lim_{n\to\infty} \mathrm{Val}_t(f_{n+1}(t) - f_n(t)) = \infty$ and thus $f(t) = \lim_{n\to\infty} f_n(t) = \sum_{n\geq 0}(f_{n+1}(t) - f_n(t))$ exists. Now take limit of the defining equation of f_n, we get

$$\lim_{n\to\infty} f_{n+1}(t) = \lim_{n\to\infty} f_n(t) - \frac{F(t, f_n(t))}{F'(t, f_n(t))}$$

$$f(t) = f(t) - \frac{F(t, f(t))}{F'(t, f(t))}$$

where $F'(t, f(t)) \neq 0$ and hence $F(t, f(t)) = 0$ as desired.

Uniqueness: Suppose $F(t, f(t)) = F(t, g(t)) = 0$. Apply linear approximation with $A(x) = F(t, x)$ and $u = g(t), v = f(t)$ we get

$$F(t, g(t)) = F(t, f(t)) + (g(t) - f(t))F'(t, f(t)) \pmod{(g(t) - f(t))^2}$$

and hence $g(t) - f(t)$ is divisible by $(g(t) - f(t))^2$. Since $g(t) - f(t)$ is not a unit in $R[[t]]$ we must have $g(t) - f(t) = 0$ and the proof follows. ♡

Proposition 1.1.15. *$A(x) \in R[[x]]_+$ has a compositional inverse if and only if $[x']A(x)$ is invertible.*

Proof. $(\Rightarrow)$: Exercise.

$(\Leftarrow)$: Let $F(x, \lambda) = x - A(\lambda)$ and $F'(x, \lambda) = \frac{\partial}{\partial \lambda} F(x, \lambda)$. Then $F(0, 0) = 0 - A(0) = 0$ and $F'(0, 0) = A'(0)$ by definition is invertible. Now by Hensel's lemma we can find $B(\lambda) \in R[[x]]_+$ such that $F(x, B(x)) = 0 = x - A(B(x))$ as desired. ♡

Proposition 1.1.16. *If F is a field with $\mathrm{Char}(F) \neq 2$, and $A(x) \in \mathbb{F}[[x]]$. Then $A(x)$ has a square root in $F[[x]]$ if and only if $\mathrm{Val}_x A(x) = 2m$ and $[x^m]A(x)$ is a square in F.*

Proof. $(\Rightarrow)$: Exercise.

$(\Leftarrow)$: Write $A(x) = a^2x^{2m} +$ higher terms. Let $F(x, y) = (1 + y)^2 - \frac{A(x)}{a^2x^{2m}}$. Then we see $F'(x, y) := \frac{\partial}{\partial y} F(x, y) = 2(1 + y)$. Then $F(0, 0) = 0$ and $F'(0, 0) = 2$ and so we can use Hensel's lemma and get $f(x) \in F[[x]]_+$ such that $F(x, f(x)) = 0$ and hence $a^2x^{2m}(1 + f(x))^2 = A(x)$, i.e. $ax^m(1 + f(x))$ is a square root of $A(x)$ as desired. ♡

Remark 1.1.17. If F is not a field then the argument breaks. In particular, $\frac{A(x)}{a^2x^{2m}}$ is not necessarily defined. For example, given $A(x, y) \in \mathbb{Q}[[x, y]]$, then we cannot use this method. Thus, how can we tell if $A(x, y)$ has a square root?

1.2 Formal Laurent Series

Remark 1.2.1. In $\mathbb{Q}[[x]]$ we have $\frac{e^x}{x}$ is not defined, which is not good and we want it to equal $\sum_{n\geq -1}\frac{1}{(n+1)!}x^n$. However, on the other hand we have the following pathological example: consider

$$\sum_{n\in\mathbb{Z}} x^n = \sum_{n\geq 0} x^n + \sum_{n\geq 0} x^{-n-1} = \frac{1}{1-x} + \frac{x^{-1}}{1-x^{-1}} = 0$$

which is not what we want.

Definition 1.2.2. We define the ***ring of formal Laurent series*** $R((x))$ as

$$R((x)) = \{\sum_{n\geq N} a_n x^n : N\in\mathbb{Z}, a_n\in R\}$$

with the normal formal series operation.

Remark 1.2.3. We also have the coefficient extraction $[x^n]A(x)$ for $A(x)\in R((x))$. Also valuation make sense like usual, hence limits. For division, we note $R((x))$ is an integral domain, and $A(x)\in R((x))$ has a multiplicative inverse if and only if $[x^{\mathrm{Val}_x(A(x))}]A(x)$ is invertible in R. In particular, if F is a field then $F((x))$ is a field.

Definition 1.2.4. We define $A(B(x)) = \sum_{n\geq N} a_n B(x)^n$ to be the composition.

Remark 1.2.5. A sufficient condition for $A(B(x))$ to be defined is that $B(x)$ has a multiplicative inverse and $\mathrm{Val}_x(B(x)) > 0$.

Remark 1.2.6. We also have formal derivatives, and the formal integral only exists if $[x^{-1}]A(x) = 0$.

Definition 1.2.7. We define the ***formal residue*** of $A(x)\in R((x))$ to be the coefficient $[x^{-1}]A(x)$.

Remark 1.2.8. The reason why we want this is that it behaves like a definite integral.

Proposition 1.2.9.

1. $[x^{-1}](A(x)+cB(x)) = [x^{-1}]A(x) + c[x^{-1}]B(x)$ *with* $c\in R$.
2. $[x^{-1}]A'(x) = 0$. *We should think this as fundamental theorem of calculus.*
3. $[x^{-1}]A'(x)B(x) = -[x^{-1}]A(x)B'(x)$. *We should think this as integral by part.*
4. $[x^{-1}]A(x) = \frac{1}{\mathrm{Val}_y(B(y))}[y^{-1}]A(B(y))B'(y)$. *Think this as substitution formula.*

Remark 1.2.10. More caution: for formal power series (FPS) $R[[x]][[y]] = R[[y]][[x]]$. However, for formal Laurent series (FLS), we have $R((x))((y)) \neq R((y))((x))$ and we cannot put x and y on equal footing. Thus we have to make a choice.

In $\mathbb{Q}((x))((y))$, we have $(x+y)^{-1} = \sum_{n\geq 0} x^{-n-1}y^n$. In $\mathbb{Q}((y))((x))$ we then get $(x+y)^{-1} = \sum_{n\geq 0} y^{-n-1}x^n$ and it is completely different from $\sum_{n\geq 0} x^{-n-1}y^n$. This is because in the two rings we used different valuations,

Theorem 1.2.11 (Lagrange Implicit Function Theorem)**.** *Assume* $\mathbb{Q}\subseteq R$ *and* $\phi(\lambda)\in R[[\lambda]]$ *is invertible. Then:*

1. *There exists unique $A(x) \in R[[x]]_+$ such that $A(x) = x\phi(A(x))$. This equation is called the LIFT equation.*
2. $[x^n]A(x) = \frac{1}{n}[\lambda^{n-1}]\phi(\lambda)^n$ *for* $n \geq 1$.
3. *For any $f(\lambda) \in R((\lambda))$, $[x^n]f(A(x)) = \frac{1}{n}[\lambda^{n-1}]f'(\lambda)\phi(\lambda)^n$ for $n \neq 0$ and for $n = 0$ we have*

$$[x^0]f(A(x)) = [\lambda^0]f(\lambda) + [\lambda^{-1}]f'(\lambda)\log(\frac{\phi(\lambda)}{\phi(0)})$$

where the second term is zero when $\mathrm{Val}_\lambda f(\lambda) \geq 0$.

Remark 1.2.12 (Key Ideas).

1. Hensel's lemma: used to prove (1).
2. Compositional inverses: let $B(\lambda) = \frac{\lambda}{\phi(\lambda)} \in R[[\lambda]]_+$, then the LIFT equation holds iff $B(A(x)) = x$ iff A is the compositional inverse of B.
3. Formal residues: $[x^n]f(A(x)) = [x^{-1}]x^{-n-1}f(A(x))$. To show this, we use substitution with $B(\lambda)$ and use integration by parts.

Theorem 1.2.13 (Alternate LIFT). *Let $\phi(\lambda), f(\lambda) \in \mathbb{Q}[[\lambda]]$, $A(x) \in \mathbb{Q}[[x]]_+$ be formal power series. Assume $\phi(\lambda)$ is invertible. Suppose $A(x) = x\phi(A(x))$. Then for all $n \geq 0$,*

$$[x^n]f(A(x)) = [\lambda^n]f(\lambda)\left(1 - \lambda\frac{\phi'(\lambda)}{\phi(\lambda)}\right)\phi(\lambda)^n$$

Proof. We first prove our claim for the special case $f(x) = f_m(x) = (\frac{x}{\phi(x)})^m$ where $m \geq 0$. In this case we have $f(A(x)) = x^m$ as we note $A(x) = x\phi(A(x)) \Rightarrow \frac{A(x)}{\phi(A(x))} = x \Rightarrow \left(\frac{A(x)}{\phi(A(x))}\right)^m = x^m$. Thus $[x^n]f(A(x)) = 0$ if $n \neq m$. Now let $n = m$. In this case, we have $[x^n]f(A(x)) = 1$ and hence it suffices to show

$$[\lambda^n]f(\lambda)(1 - \lambda\frac{\phi'(\lambda)}{\phi(\lambda)})\phi(\lambda)^n = 1$$

Observe we have

$$\begin{aligned} f(\lambda)(1 - \lambda\frac{\phi'(\lambda)}{\phi(\lambda)})\phi(\lambda)^n &= \frac{\lambda^n}{\phi(\lambda)^n}(1 - \lambda\frac{\phi'(\lambda)}{\phi(\lambda)})\phi(\lambda)^n \\ &= \lambda^n - \lambda^{n+1}\frac{\phi'(\lambda)}{\phi(\lambda)} \end{aligned}$$

This implies $\deg(\lambda^{n+1}\frac{\phi'(\lambda)}{\phi(\lambda)}) \geq n + 1$ because $\phi(\lambda)$ invertible implies $\phi'(\lambda)/\phi(\lambda) \in \mathbb{Q}[[x]]$ and hence we get

$$[\lambda^n]f(\lambda)(1 - \lambda\frac{\phi'(\lambda)}{\phi(\lambda)})\phi(\lambda)^n = 1$$

as desired. This proves the special case for $f(x) = f_m(x)$.

Now we note that the operation $[x^n]F(x)$ is linear, i.e. if $F(x) = G(x) + cH(x)$ then $[x^n]F(x) = [x^n]G(x) + c[x^n]H(x)$ for $c \in \mathbb{Q}$ and $G(x), H(x) \in \mathbb{Q}[[x]]$.

In particular, this means the following. Suppose $F(x) = \sum_{i=0}^m a_i f_i(x)$ where the $f_i(x)$ are defined above, then we see $F(A(x)) = \sum_{i=0}^n a_i f_i(A(x))$ by definition of composition. Thus, we see

$$\begin{aligned}
[x^n]F(A(x)) = [x^n]\sum_{i=0}^m a_i f_i(A(x)) &= \sum_{i=0}^m a_i[x^n]f_i(A(x)) \\
&= \sum_{i=0}^m a_i[\lambda^n]f_i(\lambda)c(\lambda), \quad \text{where } c(\lambda) = (1-\lambda\frac{\phi'(\lambda)}{\phi(\lambda)})\phi(\lambda)^n \\
&= [\lambda^n]\sum_{i=0}^m a_i f_i(\lambda)c(\lambda) \\
&= [\lambda^n]F(\lambda)c(\lambda) \\
&= [\lambda^n]F(\lambda)(1-\lambda\frac{\phi'(\lambda)}{\phi(\lambda)})\phi(\lambda)^n
\end{aligned}$$

In other word, now we have proved the claim for f_i and all $\mathbb{Q}$-linear combinations of f_i.

Now we consider the infinite case, i.e. we will show f_i has the claim holds implies $F(x)$ has the claim holds, where $F(x) = \sum_{i=0}^\infty a_i f_i(x)$ and we assume the sum converges. In this case, we get, with identical argument as above, we see we get $F(x) = \sum_{i=0}^\infty a_i f_i(x)$ implies $[x^n]F(A(x)) = [\lambda^n]F(\lambda)(1-\lambda\frac{\phi'(\lambda)}{\phi(\lambda)})\phi(\lambda)^n$ where we see the summation is justified because since $\sum_{i=0}^\infty a_i f_i(x)$ converges (recall $\sum_{i\geq 0} A_i(x)$ converges means the coefficients are eventually constant) we can find $N > 0$ so that $[x^n]F(A(x)) = [x^n]\sum_{i=0}^N a_i f_i(A(x)) = [x^n]\sum_{i=0}^N a_i f_i(x)c(x)$, where $c(x) = (1 - x\frac{\phi'(x)}{\phi(x)})\phi(x)^n$. However, on the other hand, note since $\sum_{i=0}^\infty a_i f_i(x)c(x)$ also converges (it is a product of two formal power series), we see we can find $M > 0$ so that $[x^n]\sum_{i=0}^M a_i f_i(x)c(x) = [x^n]\sum_{i=0}^\infty a_i f_i(x)c(x) = [x^n]F(x)c(x)$. Then let $W = \max\{M, N\}$ we see $[x^n]F(A(x)) = [x^n]F(x)c(x)$ as desired.

Finally, to prove our claim for $F(x) \in \mathbb{Q}[[x]]$, we note $F(x) = \sum_{k\geq 0} a_k x^k$ implies $F(x) = \sum_{k\geq 0} a_i\phi(x)^k f_k(x)$ as $f_k(x) = \frac{x^k}{\phi(x)^k}$. We note this $\sum_{k\geq 0} a_i\phi(x)^k f_k(x)$ always converges because its degree is bounded from below by $f_k(x)$. ♡

Remark 1.2.14. In practice, $\phi(\lambda)$ does not need to be invertible. We just need $\phi(0) \neq 0$. This is because every integral domain is contained in some field.

Example 1.2.15. Suppose $A(x) = xe^{A(x)}$, compute $A(x)$.

We use LIFT (2) with $\phi(\lambda) = e^\lambda$. Then

$$[x^n]A(x) = \frac{1}{n}[\lambda^{n-1}](e^\lambda)^n = \frac{1}{n}[\lambda^{n-1}]e^{n\lambda} = \frac{1}{n}(\frac{n^{n-1}}{(n-1)!}) = \frac{n^{n-1}}{n!}$$

for all $n \geq 1$. This means $A(x) = \sum_{n\geq 1}\frac{n^{n-1}}{n!}x^n$.

Example 1.2.16. Find $B = B(x,y) \in \mathbb{Q}[[x,y]]$ that solves $xB^3 - B + y = 0$.

Isolate x we get $x = \frac{B-t}{B^3}$ and substitute $A = B - y$ we get $A = x(A+y)^3$. Thus we can use LIFT to solve it. Let $R = \mathbb{Q}[[y]]$ (or let $R = \mathbb{Q}((y))$) and $\phi(\lambda) =$

$(\lambda+y)^3 \in R[[\lambda]]$. Thus we see

$$[x^n]A(x) = \frac{1}{n}[\lambda^{n-1}]((\lambda+y)^3)^n = \frac{1}{n}[\lambda^{n-1}](\lambda+y)^{3n} = \frac{1}{n}\binom{3n}{n-1}y^{2n+1}$$

for all $n \geq 1$. Then $B = y + A = y + \sum_{n\geq 1}\frac{1}{n}\binom{3n}{n}x^n y^{2n+1}$

Example 1.2.17. Suppose $H(x-x^2) = (1-x)^{-1}$, compute $H(x)$.

Let $B(x) = x - x^2$ and $f(x) = (1-x)^{-1}$. Let $A(x)$ be the compositional inverse of $B(x)$. Then $H(B(x)) = f(x)$ is the same as by apply ev_A to both side and get $H(x) = f(A(x))$. To find $A(x)$, let $\phi(\lambda) = \frac{\lambda}{B(\lambda)} = (1-\lambda)^{-1}$. Then $A(x) = x\phi(A(x))$. Use LIFT (3), we have

$$\begin{aligned}[x^n]H(x) = [x^n]f(A(x)) &= \frac{1}{n}[\lambda^{n-1}](1-\lambda)^{-2}((1-\lambda)^{-1})^n \\ &= \frac{1}{n}[\lambda^{n-1}](1-\lambda)^{-n-2} = \frac{1}{n}\binom{2n}{n-1}\end{aligned}$$

Thus $[x^0]H(x) = [x^0]f(A(x)) = 1$ and so $H(x) = 1+\sum_{n\geq 1}\frac{1}{n}\binom{2n}{n-1}x^n = \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n$.

1.3 Algebraic Transformations

Remark 1.3.1. Suppose we have polynomial $p(z_0, z_1, z_2, ...) \in R[z_0, z_1, z_2, ...]$. For example, $p(z_0, z_1, z_2, ...) = z_1 z_4^2 + z_{999} \in \mathbb{Q}[z_0, z_1, z_2, ...]$. Given such a polynomial, we get a function $\phi_p : R[[x]] \to R$ given by

$$U(x) = \sum_{n\geq 0} u_n x^n \mapsto p(u_0, u_1, u_2, ...)$$

We call such ϕ_p an ***algebraic function***.

For example, given $p(z_0, z_1, z_2, ...) = z_1 z_4^2 + z_{999}$ then $p(\sum_{n\geq 0} nx^n) = 1\cdot 4^2 + 999$.

Similarly given $p(z_1, z_2, ...) \in R[z_1, z_2, ...]$, we get a function $\phi_p : R[[x]]_+ \to R$ given by $\sum_{n\geq 1} u_n x^n \mapsto p(u_1, u_2, ...)$, which is called an algebraic function as well.

For example, if $p(z_0, z_1, ...) = z_n$ then $\phi_p = [x^n]$.

Definition 1.3.2. An ***algebraic transformation*** of FPS is a function $f : A \to B$ with $A, B \in \{R[[x]], R[[x]]_+\}$ such that $[x^n] \circ f$ is an algebraic function for all $n \geq 0$.

Example 1.3.3.

1. Squaring map: $sq : R[[x]] \to R[[x]]$ is given by $U(x) \mapsto U(x)^2$. We see $[x^n] \circ sq : U(x) \mapsto [x^n]U(x)^2 = \sum_{k=0}^n u_k u_{n-k}$, which is a polynomial in $u_0, u_1, ...$ and hence it is an algebraic transformation.
2. Differentiation map: $\frac{d}{dx} : R[[x]] \to R[[x]]$ is given by $U(x) \mapsto U'(x)$. This is algebraic transformation since $[x^n]U'(x) = (n+1)u_{n+1}$, which is a polynomial in $u_0, u_1, ...$ as desired.

3. Evaluation map: Given $B(x) \in R[[x]]_+$, the evaluation map $ev_B : R[[x]] \to R[[x]]$ is given by $U \mapsto U(B(x))$. We have $[x^n] \circ ev_B(u(x))$ is a linear function of $u_0, u_1, u_2, ...$, as one can show.
4. Application map: given $A(x) \in R[[x]]$, we get $ap_A : R[[x]]_+ \to R[[x]]$ is given by $U(x) \mapsto A(U(X))$. This time $[x^n] \circ ap_A(U(x))$ is a degree n polynomial of $u_1, u_2, u_3, ...$, as one can show.

Definition 1.3.4. Let $\mathbb{T}_f = \mathbb{T}_{full} = \mathbb{T}_{full}(R[[x]])$ be the set of algebraic transformations $R[[x]] \to R[[x]]$.

We use $\mathbb{T}$ be the set of algebraic transformations $R[[x]]_+ \to R[[x]]$.

We use $\mathbb{T}_+$ be the set of algebraic transformations $R[[x]]_+ \to R[[x]]_+$.

Remark 1.3.5. In particular we have

$$\mathbb{T}_+ \hookrightarrow \mathbb{T} \twoheadleftarrow \mathbb{T}_f$$

We also note $g \in \mathbb{T}$, $h \in \mathbb{T}_+$ then $g \circ h \in \mathbb{T}$. To see why composition is again algebraic trans, just note composition of polynomials is still polynomial.

On the other hand, $f \in \mathbb{T}_f$ and $g \in \mathbb{T}$ then $f \circ g \in \mathbb{T}$.

Remark 1.3.6. Note we get a ring structure on $\mathbb{T}$. Note the codomain of $\mathbb{T}$ is a ring, hence we can define $f, g \in \mathbb{T}$ then we define $f + g : U(x) \mapsto f(U) + g(U(x))$. Define the similar point-wise subtraction and multiplications, we get a ring.

The 0 element will be the map $U(x) \mapsto 0$ and the 1 element will be $U(x) \mapsto 1$.

We note $\mathbb{T}_f$ is a ring, but $\mathbb{T}_+$ is not a ring as it is missing the multiplicative identity.

Remark 1.3.7. We can also take limits of algebraic transformations. Given $f_0, f_1, f_2, ... \in \mathbb{T}$, or in $\mathbb{T}_f, \mathbb{T}_+$. Then we say

$$\lim_{n\to\infty} f_n = f \in \mathbb{T}$$

if and only if $\lim_{n\to\infty} f_n[U(x)] = f(U(x))$ for all FPS $U(x) \in R[[x]]_+$. Hence we can also talk about infinite sums and products of algebraic transformations.

Remark 1.3.8. In particular we can define composition with FPS. If $A(x) = \sum_{n\geq 0} a_n x^n \in R[[x]]$, and $f \in \mathbb{T}_+(R[[x]])$. Then we can define $A(f) = \sum_{n\geq 0} a_n f^n \in \mathbb{T}$. As an exercise, show that $A(f) = ap_A \circ f$.

Example 1.3.9. Let $f = ev_{x^2}$ and $g = \frac{d}{dx}$. Then what does $\exp(f^2 g) \circ f$ do?

Apply this to $U(x)$, we see

$$\begin{aligned}\exp(f^2 g) \circ f[U(x)] &= \exp(f^2 g)[U(x^2)] \\ &= \exp(f(U(x^2))^2 \cdot g(U(x^2))) \\ &= \exp(U(x^4)^2 \cdot 2xU'(x^2))\end{aligned}$$

Remark 1.3.10. Caution: note algebraic transformations are not formal power series. For example, if we have $A(x), B(x) \in R[[x]], U(x) \in R[[x]]_+$. If $A(U(x)) = B(U(x))$ we know in fact $A = B$.

However, on the other hand, if we have $f, g \in \mathbb{T}$, then $f[U(x)] = g[U(x)]$ does not imply $f = g$.

1.4 Exercises 1

In this section, R will always be integral domain.

Example 1.4.1. Prove $R[[x]]$ is integral domain.

Solution. Let $A(x) := \sum a_i x^i, B(x) := \sum b_i x^i \in R[[x]]$ such that

$$A(x)B(x) = \sum_{n\geq 0}(\sum_{i=0}^{n} a_i b_{n-i})x^n = 0$$

and we proceed with induction. Note we must have $a_0 b_0 = 0$ and hence $a_0 = 0$ or $b_0 = 0$. WLOG say $a_0 = 0$ and $b_0 \neq 0$ (if they are both zero then we proceed to a_1 and b_1, until we get $a_n = 0$ and $b_n \neq 0$). Then we claim, by induction, that $a_0, ..., a_{n-1} = 0$ implies $a_n = 0$. Indeed, note we have $\sum_{i=0}^{n} a_i b_{n-i} = 0$ with $a_0 = ... = a_{n-1} = 0$ and hence we get $a_n b_0 = 0$ and since $b_0 \neq 0$ we are forced to have $a_n = 0$ as desired. ♠

Example 1.4.2. Suppose $A_0(x), A_1(x), ... \in R[[x]]$ is a sequence of formal power seires and $A(x) \in R[[x]]$. Show $\lim_{n\to\infty} A_n(x) = A(x)$ if and only if for all $n \in \mathbb{N}$ there exists $K > 0$ and such that $k > K$ implies $[x^n]A_k(x) = [x^n]A(x)$.

Solution. Suppose $\lim A_n(x) = A(x)$. Let $n \in \mathbb{N}$ be given. By definition of limit we see $\lim_{n\to\infty} \mathrm{Val}_x(A_n(x) - A(x)) = \infty$, i.e. given any $N \in \mathbb{N}$, we can find $K \in \mathbb{N}$ so that for $k > K$ we have $\mathrm{Val}_x(A_k(x) - A(x)) > N$. However this means $A_k(x) - A(x)$ agrees on the first N terms, i.e. for any $n < N$ we have $[x^n]A_k(x) = [x^n]A(x)$. Since we can let $N \to \infty$ we see for all $n \in \mathbb{N}$ we get our desired $K > 0$.

Suppose the converse holds. Then we see let N be arbitrary, we can find a common $K > 0$ so that for $n = 0, 1, 2, 3, ..., N, N + 1$, we have $k > K$ implies $[x^n]A_k(x) = [x^n]A(x)$ at the same time (e.g. let $K = \max\{K_0, ..., K_{N+1}\}$). This means $\mathrm{Val}_x(A_k(x) - A(x)) > N$ for $k > K$, i.e. $\lim_{n\to\infty} \mathrm{Val}_x(A_n(x) - A(x)) = \infty$ as desired. ♠

Example 1.4.3. Show $\sum_{n\geq 0} A_n(x)$ converges if and only if $\lim_{n\to\infty} A_n(x) = 0$.

Solution. Suppose $\sum_{n\geq 0} A_n(x)$ converges to $A(x) \in R[[x]]$. Now let $n \in \mathbb{N}$ be arbitrary, we need to find $K > 0$ so that for all $k > K$ we have $[x^n]A_k(x) = 0$.

Suppose for some particular n we have for all $k > 0$ that $[x^n]A_k(x) \neq 0$. In particular this means for all $m > 1$ we have

$$[x^n]\sum_{i=0}^{m-1} A_i(x) \neq [x^n]\sum_{i=0}^{m} A_i(x)$$

In other word, this means for n, there does not exist $K > 0$ so that for all $k > K$ we have $[x^n]\sum_{i=0}^{k} A_i(x) = a_n$ for some fixed constant $a_n \in R$. This shows the $\sum_{n\geq 0} A_n(x)$ does not converge.

Conversely, suppose $\lim_{n\to\infty} A_n(x) = 0$. This means for all $n \in \mathbb{N}$ we can find $K > 0$ so $k > K$ means $[x^n]A_k(x) = 0$, i.e. for K big enough, we have $[x^n]A_K(x) = 0$ for all n and K. Thus just define $A(x) = \sum_{n\geq 0} a_n x^n$ where

$$a_n = \sum_{i=0}^{K_n}[x^n]A_i(x)$$

where K_n is the max number that $[x^n]A_i(x)$ is not zero. One should verify this is indeed the limit.

♠

Example 1.4.4. Show that

$$\prod_{k\geq 0}(1 + x^{2^k}) = \sum_{n\geq 0} x^n$$

Solution. We use induction on $\prod_{k=0}^{n}(1 + x^{2^k})$ to show it is equal to

$$1 + x + x^2 + ... + x^{\sum_{i=0}^{n} 2^i}$$

For $n = 1$, we get $\prod_{k=0}^{1}(1 + x^{2^0}) = 1 + x$, thus the base case holds. Then we get

$$\begin{aligned}\prod_{k=0}^{n}(1 + x^{2^k}) &= (1 + x + x^2 + ... + x^{\sum_{i=0}^{n-1} 2^i})(1 + x^{2^n}) \\ &= 1 + x + x^2 + ... + x^{\sum_{i=0}^{n-1} 2^i} + x^{2^n} + ... + x^{\sum_{i=0}^{n-1} 2^i + 2^n}\end{aligned}$$

where we note by induction we can prove $\sum_{i=0}^{n-1} 2^i = 2^n - 1$, which completes our induction.

This immediately tell us

$$\lim_{n\to\infty}\prod_{k\geq 0}(1 + x^{2^k}) = \lim_{n\to\infty}\sum_{i=0}^{2^{n-1}-1} x^i = \sum_{n\geq 0} x^n$$

♠

Example 1.4.5. Given an example of a sequence $A_0(x,y), A_1(x,y), \ldots \in \mathbb{Q}[[x,y]]$ for which $\sum_{n\to\infty} A_n(x,y)$ converges using the limit defined by Val_x but does not converge using the limit Val_y.

Solution. Consider $A_n(x,y) = yx^n$, then we see $A(x,y) = \sum_{n\geq 0} A_n(x,y) = \sum_{n\geq 0} yx^n$. We see (we will use $A_n(x)$ as we assume y is a coefficient)

$$\mathrm{Val}_x(A_n(x) - A(x)) = \mathrm{Val}_x(\sum_{m\geq 0, m\neq n} yx^m) = n$$

and hence $n \to \infty$ gives us $\lim_{n\to\infty} \mathrm{Val}_x(A_n(x) - A(x)) = \infty$ as desired. On the other hand, we note $\mathrm{Val}_y(A_n(y) - A(y)) = 0$ as

$$A_n(y) - A(y) = x^n y - (\sum_{n\geq 0} x^n) y = f(x) y$$

with $f(x) \neq 0$, i.e. $\min(n : [y^n]A_n(y) - A(y) = 0)$ is 0 as it is missing a constant term. ♠

Example 1.4.6. Suppose $A(x,y) \in \mathbb{Q}[y][[x]]$ is not a polynomial. Explain why in the context of formal power series, it makes sense to substitute any rational number for y, but for $x = 0$ is the only rational we are allowed to substitute.

Solution. Just note if $A(x,y) = \sum_{n\geq 0} x^n$ then any substitution $x \neq 0$ makes no sense at all. On the other hand since the coefficient ring is $\mathbb{Q}[y]$ we see in each case it is finite and we still get a valid power series in $\mathbb{Q}[[x]]$ after substituting $y \in \mathbb{Q}$. ♠

Example 1.4.7. If $A(x) = \sum a_n x^n \in \mathbb{Q}[[x]]$, show that

$$\sum a_{2n} x^{2n} = \frac{1}{2}(A(x) + A(-x)), \quad \sum a_{2n+1} x^{2n+1} = \frac{1}{2}(A(x) - A(-x))$$

Solution. We show the first one the second one is similar. Note

$$A(x) + A(-x) = \sum a_n x^n + \sum (-1)^n a_n x^n = \sum_{\substack{n\equiv 0 \pmod 2 \\ n\geq 0}} 2a_n x^n$$

and the proof follows. ♠

Example 1.4.8. If $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{m\geq 0} b_m x^m$, show that

$$[x^n]A(B(x)) = \sum_{k=1}^{n} \sum_{\substack{j_1,\ldots,j_k\geq 1 \\ j_1+\ldots+j_k=n}} a_k b_{j_1} \ldots b_{j_k}$$

Solution. Just note by definition we have $A(B(x)) = \sum_{n\geq 0} a_n B(x)^n$ where we note

$$B(x)^n = \sum_{n\geq 0} \left(\sum_{\substack{j_1,\ldots,j_n\geq 1\\ j_1+\ldots j_n=n}} b_{j_1}\ldots b_{j_n} \right) x^n$$

Therefore it is not hard to see to get the coefficient of x^n we need to run over all possible coefficient of x^m for $B(x)^m$ with coefficient a_{n-m}. In other word we get

$$[x^n]A(B(x)) = \sum_{k=1}^{n} \left(\sum_{\substack{j_1,\ldots,j_k\geq 1\\ j_1+\ldots j_k=n}} b_{j_1}\ldots b_{j_k} \right)$$

♠

Example 1.4.9. Prove the composition map $R[[x]] \times R[[x]]_+ \to R[[x]]$ given by

$$(A(x), B(x)) \mapsto A(B(x))$$

is continuous. That is, if $\lim A_n(x) \to A(x)$ and $\lim B_n(x) \to B(x)$ then $\lim A_n(B_n(x)) = A(B(x))$.

Solution. Just note

$$[x^n]A(B(x)) = \sum_{k=1}^{n} \sum_{\substack{j_1,\ldots,j_k\geq 1\\ j_1+\ldots+j_k=n}} a_k b_{j_1}\ldots b_{j_k}$$

and hence if each A_n, B_n's coefficient are eventually constant, we see the composition's coefficient is also eventually constant. ♠

Example 1.4.10. Prove the geometric series formula: if $a(x) \in R[[x]]$, $r(x) \in R[[x]]_+$ then

$$\sum_{n\geq 0} a(x)r(x)^n = \frac{a(x)}{1-r(x)}$$

Solution. We first note $\sum_{n\geq 0} x^n = \frac{1}{1-x}$ by definition. Next note composition with $r(x)$ is well-defined, hence we get $\sum_{n\geq 0} r(x)^n = \frac{1}{1-r(x)}$ and now multiply by $a(x)$. ♠

Example 1.4.11. Rewrite the properties of $L(x)$ using $L(x) = \log(1+x)$. Why it is consistent to also write $L(\frac{x}{1-x}) = \log(\frac{1}{1-x})$?

Solution. The four properties using log notation are just:

1. $\log(1+x)$ is the compositional inverse of $e^x - 1$.
2. $\log(\frac{1}{1-x}) = -\log(1-x) = \sum_{n\geq 1} \frac{1}{n}x^n$, which is the compositional inverse of $1 - e^{-x}$.

3. $\log(1+x+y+xy) = \log((1+x)(1+y)) = \log(1+x) + \log(1+y)$.
4. $\frac{d}{dx}\log(1+x) = \frac{1}{1+x}$.

To see why it is consistent to write $L(\frac{x}{1-x}) = \log(\frac{1}{1-x})$ we just note $L(\frac{x}{1-x}) = \log(1+\frac{x}{1-x}) = \log(\frac{1}{1-x})$. ♠

Example 1.4.12. Prove $L(x) = \log(1+x)$ is the compositional inverse of $e^x - 1$.

Solution. For one thing, note $e^{\log(1+x)} - 1 = 1 + x - 1 = x$ from calculus. To prove it formally, note $\frac{d}{dx}L(\exp_+(x)) = L'(\exp_+(x)) \cdot \exp(x) = \frac{1}{\exp(x)} \cdot \exp(x) = 1$ and this happends if and only if $L(\exp_+(x)) = x$. ♠

Example 1.4.13. Verify the binomial series $(1-x)^y \in \mathbb{Q}[y][[x]]$. Define $\left(\!\binom{y}{n}\!\right)$: $\frac{1}{n!}\prod_{k=0}^{n-1}(y+k)$. Show that

$$\left(\!\binom{y}{n}\!\right) = \binom{y+n-1}{n}$$

and

$$(1-x)^{-y} = B(-x,-y) = \sum_{n\geq 0}\left(\!\binom{y}{n}\!\right)x^n$$

Solution. We note by definition $B(-x,y) = (1+x)^y = \sum_{n\geq 0}(-1)^n\binom{y}{n}x^n$ where $(-1)^n\binom{y}{n}$ is indeed a element of $\mathbb{Q}[y]$ as desired. To show $\left(\!\binom{y}{n}\!\right) = \binom{y+n-1}{n}$ we just note

$$\prod_{k=0}^{n-1}(y+k) = \prod_{k=0}^{n-1}((y+n-1)-k) = \prod_{j=0}^{n-1}(y+j)$$

On the other hand note $B(-x,-y) = \sum_{n\geq 0}(-1)^n\binom{-y}{n}x^n$. In particular note $\binom{-y}{n} = \frac{1}{n}\prod_{k=0}^{n-1}(-y-k) = (-1)^n\prod_{k=0}^{n-1}(y+k)$ and hence the proof follows. ♠

Example 1.4.14. By expanding $(1-x)^y(1+x)^y = (1-x^2)^y$ and comparing coefficients, prove

$$\sum_{k=0}^{2n}(-1)^{n-1}\binom{n}{k}\binom{y}{2n-k} = \binom{y}{2n}$$

Solution. Expand and compare coefficients. ♠

Example 1.4.15. Use Hensel's lemma to prove the equation $x^2e^{A(x)} = A(x)e^x$ has a unique solution $A(x) \in \mathbb{Q}[[x]]_+$.

Solution. We are looking for some $f(t)$ so that $t^2e^{f(t)} - f(t)e^t = 0$. Thus we let $F(x,t) = x^2e^t - te^x$ and set $F'(x,t) = \frac{\partial}{\partial t}F(x,t) = x^2e^t - e^x$. In particular then we see $F(0,0) = 0 - 0 = 0$ and $F'(0,0) = 0 - 1 = -1 \in \mathbb{Q}^*$. Hence by Hensel's lemma there exists $f(t) \in \mathbb{Q}[[t]]_+$ such that $F(x, f(x)) = 0$ as desired. ♠

Example 1.4.16. Consider the equation $e^{xB(x)} = 1 + xB(x)^2$. Show this has a unique solution $B(x) \in \mathbb{Q}[[x]]_+$ by using Hensel's lemma with $F(x,y) = e^{xy} - 1 - xy^2$ and $F'(x,y) = \frac{\partial}{\partial y} F(x,y)$.

Solution. Note $F(0,0) = 1 - 1 - 0 = 0$ and $F'(x,y) = xe^{xy} - 2xy \Rightarrow F'(0,0) = 0$. We cannot do use Hensel's lemma! However, we can fix this by get rid of the x in the derivative. ♠

Example 1.4.17. Show the equation $e^{xB(x)} = 1 + xB(x)^2$ has a unique solution $B(x) \in \mathbb{Q}[[x]]_+$ by Hensel's lemma using $F(x,y) = \frac{e^{xy}-1-xy^2}{x}$.

Solution. We note $F(x,y) = 0$ as $e^{xy} - 1 = \sum_{n\geq 1} \frac{x^n y^n}{n!} \Rightarrow \frac{e^{xy}-1}{x} = \sum_{n\geq 1} \frac{x^{n-1}y^n}{n!}$ and so when substitute $x = y = 0$ we get 0. Next we note $F'(x,y) := \frac{\partial}{\partial y} F(x,y) = e^{xy} - 2y$ and hence $F'(0,0) = 1$ which is invertible. Thus we can find unique $B(x)$ so that $F(x, B(x)) = 0$ where we see $F(x,B(x)) = \frac{e^{xB(x)}-1-xB(x)^2}{x} = 0$ which implies $e^{xB(x)} - 1 - xB(x)^2 = 0$ as desired. ♠

Example 1.4.18. Prove the linear approximation: for $A(x) \in R[[x]]$ we have

$$A(u) = A(v) + (u-v)A'(v) \pmod{(u-v)^2}$$

in $R[[u,v]]$.

Solution. We will show $A(x+y) = A(y) + xA'(y) \pmod{x^2}$, then given any u, v, let $x = u - v$ and $y = v$ we see $x + y = u$ and hence we get

$$A(u) = A(v) + (u-v)A'(v) \pmod{(u-v)^2}$$

as desired.

Let $A(t) = \sum_{n\geq 0} a_n t^n \in R[[x]]$. Then $A(x+y) = \sum_{n\geq 0} a_n (x+y)^n$. On the other hand we have

$$A(y) + xA'(y) = \sum_{n\geq 0} a_n y^n + \sum_{n\geq 1} n a_n x y^{n-1}$$

and hence

$$A(x+y) - A(y) = \sum_{n\geq 0} a_n \left(\sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i} \right) = \sum_{n\geq 0} a_n x \left(\sum_{i=1}^{n} \binom{n}{i} x^{i-1} y^{n-i} \right)$$

Thus

$$A(x+y) - A(y) - xA'(y) = \sum_{n\geq 0} a_n x \left(\sum_{i=1}^{n} \binom{n}{i} x^{i-1} y^{n-i} - n y^{n-1} \right)$$

where we note $\binom{n}{1}y^{n-1} = ny^{n-1}$ and hence

$$\sum_{i=1}^{n} \binom{n}{i} x^{i-1}y^{n-i} - ny^{n-1} = x\left(\sum_{i=2}^{n} \binom{n}{i} x^{i-2}y^{n-i}\right)$$

and so

$$A(x+y) - A(y) - xA'(y) = \sum_{n\geq 0} a_n x^2 \left(\sum_{i=2}^{n} \binom{n}{i} x^{i-2}y^{n-i}t\right) \equiv 0 \pmod{x^2}$$

♠

Example 1.4.19. Prove $R((x))$ is an integral domain.

Solution. Similar to how to prove $R[[x]]$ is integral domain. ♠

Example 1.4.20. Prove the "integration by parts" formula for formal residues.

Solution. Say $A(x) = \sum_{x\geq N} a_i x^i$ and $B(x) = \sum_{x\geq N} b_i x^i$ where

$$N = \min\{\mathrm{Val}_x A, \mathrm{Val}_x B\}$$

A particular term of $[x^{-1}]A'(x)B(x)$ looks like $i \cdot ([x^i]A(x)) \cdot ([x^{-i}]B(x))$, i.e.

$$[x^{-1}]A'(x)B(x) = \sum_{i\geq N-1} i \cdot [x^i]A(x) \cdot [x^{-i}]B(x)$$

On the other hand we see with similar argument we get

$$[x^{-1}]B'(x)A(x) = \sum_{i\geq N-1} i \cdot [x^i]B(x) \cdot [x^{-i}]A(x)$$

Now make a substitution $i = -j$ we get

$$[x^{-1}]B'(x)A(x) = \sum_{-j\geq N-1} -j[x^{-j}]B(x) \cdot [x^j]A(x)$$

where we note the sum is actually fintie and bounded by $N-1$ and $-N+1$, hence the proof follows.

♠

Example 1.4.21. For $A(x) \in \mathbb{Q}((x))$, show that

$$[x^{-1}]e^x A(x) = [x^{-1}]A(\log(1+x))$$

Solution. We use substitution formula (one could try to prove it, if they want, but I'm lazy) with $W(x) = e^x A(x)$ and $B(x) = \log(1+x)$. Thus we see $[x^{-1}]W(x) = \frac{1}{\mathrm{Val}_y(B(x))}[x^{-1}]W(B(x))B'(x)$. We see $W(B(x)) = e^{\log(1+x)} \cdot A(\log(1+x)) = (1+x)A(\log(1+x))$. On the other hand note $B'(x) = \frac{1}{1+x}$ and hence we see

$$[x^{-1}]e^x A(x) = [x^{-1}](1+x)A(\log(1+x)) \cdot \frac{1}{1+x} = [x^{-1}]A(\log(1+x))$$

♠

Example 1.4.22. Verify that

$$\frac{1}{x+y} = \sum_{n\geq 0}(-1)^n x^{-n-1}y^n \in \mathbb{Q}((x))((y))$$

and

$$\frac{1}{x+y} = \sum_{n\geq 0}(-1)^n x^n y^{-n-1} \in \mathbb{Q}((y))((x))$$

Solution. We verify the first one. Note in $\mathbb{Q}((x))((y))$ we have

$$\begin{aligned}(x+y)\sum_{n\geq 0}(-1)^n x^{-n-1}y^n &= \sum_{n\geq 0}(-1)^n x^{-n}y^n + \sum_{n\geq 0}(-1)^n x^{-n-1}y^{n+1} \\ &= (1 - x^{-1}y^1 + x^{-2}y^2 - x^{-3}y^3 + \ldots) \\ &\quad + (x^{-1}y^1 - x^{-2}y^2 + x^{-3}y^3 - \ldots) \\ &= 1\end{aligned}$$

and hence the proof follows. We remark $\sum_{n\geq 0}(-1)^n x^n y^{-n-1}$ is not an element of $\mathbb{Q}((x))((y))$ because in the series the exponent of y goes to $-\infty$, contradicts the definition. ♠

Example 1.4.23. Suppose $A = x(1-A)^{-2}$. Use LIFT to solve for $A = A(x) \in \mathbb{Q}[[x]]$.

Solution. Let $\phi(x) = (1-x)^{-2}$, then we see $A = x\phi(A(x))$. In particular, we see by LIFT.(2) we get

$$[x^n]A(x) = \frac{1}{n}[x^{n-1}]\phi(x)^n = \frac{1}{n}[x^{n-1}](1-x)^{-2n}$$

where we see

$$(1-x)^{-2n} = \sum_{j\geq 0}\left(\!\!\binom{2n}{j}\!\!\right)x^j \Rightarrow [x^{n-1}](1-x)^{-2n} = \left(\!\!\binom{2n}{n-1}\!\!\right)$$

and hence

$$[x^n]A(x) = \frac{1}{n}\binom{3n-2}{n-1}$$

♠

Example 1.4.24. Suppose $xB^2 - B + 1 = 0$. Use LIFT to solve for $B = B(x) \in \mathbb{Q}[[x]]$.

Solution. Let $x = \frac{B-1}{B^2}$ and hence let $A = B - 1$ this means $x = \frac{A}{(A+1)^2} \Rightarrow A = x(A+1)^2$. Now use LIFT with $\phi(x) = (x+1)^2$ and A, we get

$$[x^n]A(x) = \frac{1}{n}[x^{n-1}](1+x)^{2n} = \frac{1}{n}\binom{2n}{n-1}$$

and hence

$$B(x) = 1 + \sum_{n\geq 1}\frac{1}{n}\binom{2n}{n-1}x^n$$

♠

Example 1.4.25. What is the compositional inverse of xe^{-x}?

Solution. Let $A(x) = xe^{-x}$ and suppose $B(x)$ is the compositional inverse of $A(x)$. This means $A(B(x)) = B(A(x)) = x$. In particular suppose $B(x) = x\phi(B(x))$ then we see this is the same as $B(A(x)) = A(x)\phi(B(A(x))) \Rightarrow x = A(x)\phi(x)$, i.e. we let $\phi(x) = \frac{x}{A(x)} = e^x$. Then we get $B(x) = x\phi(B(x)) = xe^{B(x)}$. However by some example we did, we see $B(x) = xe^{B(x)}$ has unique solution given by $\sum_{n\geq 1}\frac{n^{n-1}}{n!}x^n$. ♠

Example 1.4.26. Compute $A(x)^{100}$ where we have the relation $A = x(1-A)^{-2} \in \mathbb{Q}[[x]]$.

Solution. We note by taking $f(x) = x^{100}$ and use LIFT.(3) with $\phi(x) = (1-x)^{-2}$, we see we have

$$[x^n]f(A(x)) = \frac{1}{n}[x^{n-1}]99x^{99}(1-x)^{-2n}$$

where we see

$$99x^{99}(1-x)^{-2n} = \sum_{j\geq 0} 99\left(\!\!\binom{2n}{j}\!\!\right)x^{j+99}$$

and hence for $n \geq 100$ we have

$$[x^{n-1}]99x^{99}(1-x)^{-2n} = 99\left(\!\!\binom{2n}{n-99}\!\!\right)$$

and thus

$$A^{100} = \sum_{n\geq 100}\frac{1}{n}99\binom{3n-100}{n-99}x^n$$

♠

Example 1.4.27. Note we can also solve $xB^2 - B + 1 = 0$ using quadratic formula, i.e. $B = \frac{1\pm\sqrt{1^2-4x}}{2x}$ and we get two solutions. However by LIFT we know the solution is unique, so what happended?

Solution. One of them is not in $\mathbb{Q}[[x]]$, as one should verify. ♠

Example 1.4.28. Show that $f : \mathbb{Z}[[x]] \to \mathbb{Z}$, $f[U(x)] = U(0)U'(0)U''(0)$ is an algebraic function.

Solution. Note $U(0) = [x^0]U(x)$, $U'(0) = [x^1]U(x)$ and $U''(0) = 2[x^2]U(x)$. Thus consider the polynomial $p(u_0, u_1, ...) = u_0 + u_1 + 2u_2$ and we see $f[U(x)]$ is given by $\sum_{n\geq 0} u_n x^n \mapsto p(u_0, u_1, ...)$ as desired. ♠

Example 1.4.29. Give an example of a function $f : \mathbb{R}[[x]] \to \mathbb{R}$ that is not an algebraic function.

Solution. Consider $\sum_{n\geq 0} u_n x^n \mapsto e^{u_0}$, which is not a polynomial function (it is a formal power series in u_0 but not polynomial in u_0) in terms of $u_0, u_1, ...$. ♠

Example 1.4.30. For $A(x) \in R[[x]]$, we define the ***multiplication map***

$$\begin{aligned} m_A = m_{A(x)} : R[[x]] &\to R[[x]] \\ U(x) &\mapsto A(x)U(x) \end{aligned}$$

Explain why m_A is an algebraic transformation.

Solution. Observe $[x^n](m_A(\sum_{i\geq 0} u_i x^i)) = \sum_{i+j=n} u_i a_j$, which is a polynomial in terms of $u_0, u_1, ...$, i.e. it is an algebraic function. Thus by definition we see m_A is algebraic transformation. ♠

Example 1.4.31. Consider the identity map $\mathrm{Id} : R[[x]] \to R[[x]]$, show Id is an algebraic transformation and express Id as multiplication map and evaluation map.

Solution. Note $[x^n](\sum_{m\geq 0} u_m x^m) = u_m$, which is algebraic function as desired. In terms of multiplication, Id is given by $m_{1(x)}$, the constant polynomial 1. In terms of evaluation map, Id is given by $x \in R[[x]]_+$, i.e. $\mathrm{Id} = ev_x$. ♠

Example 1.4.32. Define the ***self-composition map*** $sc : R[[x]]_+ \to R[[x]]_+$ given by $sc[U(x)] = U(U(x))$. Show sc is an algebraic transformation.

Solution. Note

$$u_n U(x)^n = u_n \sum_{m\geq 0} \left(\sum_{j_1+...+j_n=m} u_{j_1}...u_{j_n} \right) x^m$$

and hence we see

$$\begin{aligned} [x^n] \sum_{j\geq 0} u_j U(x)^j &= \sum_{i=0}^{n} [x^n] u_i U(x)^i \\ &= \sum_{i=0}^{n} u_i \left(\sum_{j_{i,1}+...+j_{i,i}=n} u_{j_{i,1}} u_{j_{i,2}} ... u_{j_{i,i}} \right) \end{aligned}$$

which is a polynomial in terms of $u_0, u_1, ...$ as desired. ♠

1.5 Tutorial 1(Abel's Extension)

Theorem 1.5.1 (Abel's Extension of Binomial Theorem). *We have*

$$(\alpha+\beta)(\alpha+\beta+n)^{n-1} = \alpha\beta\sum_{k=0}^{n}\binom{n}{k}(\alpha+k)^{k-1}(\beta+n-k)^{n-k-1}$$

Example 1.5.2. By expanding $e^{(\alpha+\beta)x} = e^{\alpha x}e^{\beta x}$, prove $(\alpha+\beta)^n = \sum_{k=0}^{n}\binom{n}{k}\alpha^k\beta^{n-k}$.

Proof. Just note $e^{(\alpha+\beta)x} = \sum_{n\geq 0}\frac{(\alpha+\beta)^n}{n!}x^n$. On the other hand, we see we have

$$\begin{aligned} e^{\alpha x}e^{\beta x} &= \left(\sum_{n\geq 0}\frac{\alpha^n}{n!}x^n\right)\left(\sum_{n\geq 0}\frac{\beta^n}{n!}x^n\right) = \sum_{n\geq 0}\left(\sum_{i+j=n}\frac{\alpha^i}{i!}\frac{\beta^j}{j!}\right)x^n \\ &= \sum_{n\geq 0}\left(\sum_{i=0}^{n}\frac{1}{i!(n-i)!}\alpha^i\beta^{n-i}\right)x^n \end{aligned}$$

♡

Example 1.5.3. Similarly prove $(\alpha+\beta)^{\underline{n}} = \sum_{k=0}^{n}\binom{n}{k}\alpha^{\underline{k}}\beta^{\underline{n-k}}$ where $y^{\underline{m}} = \prod_{i=0}^{m-1}(y-i)$.

Proof. Use $(1+x)^{\alpha+\beta} = (1+x)^\alpha(1+x)^\beta$ and compare the coefficients. ♡

Remark 1.5.4. What are those two examples have in common?

We started with some function $F(x,\gamma)$, then the LHS in our solution is just $F(x,\alpha+\beta)$ and on the RHS we have $F(x,\alpha)F(x,\beta)$, and we knew $F(x,\alpha+\beta) = F(x,\alpha)F(x,\beta)$.

This works because $F(x,\gamma)$ is exponential in γ, i.e. $F(x,\gamma) = \exp(\gamma A(x))$. In the first example, we have $A(x) = x$, and in the second example, we have $A(x) = \log(1+x)$.

Remark 1.5.5. Thus, can we prove the theorem using the same method? An educated guess is $F(x,\gamma) = \sum_{n\geq 0}\frac{\gamma(\gamma+n)^{n-1}}{n!}x^n$, and if we can show this is exponential, then we can use the same method.

Thus, we have reduced the Abel's identity to prove $F(x,\gamma)$ is exponential function. To do this, we want to find $A(x) \in \mathbb{Q}[[x]]_+$ such that

$$[x^n]e^{\gamma A(x)} = \frac{\gamma(\gamma+n)^{n-1}}{n!}$$

By working backwards, we see we must have $\frac{(\gamma+n)^{n-1}}{(n-1)!} = [x^{n-1}]e^{\gamma x}\phi(x)^n$ if the ϕ exists to make $A(x) = x\phi(A(x))$. Hence we see we must have $\phi(x) = e^x$. Therefore, we get

$$[x^n]A(x) = \frac{1}{n}[x^{n-1}]e^{nx} = \frac{1}{n}\cdot\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}$$

Hence, we see $F(x, \gamma) = \exp(\gamma A(x))$ where we just determined the $A(x)$ above and this concludes $F(x, \gamma)$ is exponential.

Thus, we see we have $F(x, \alpha + \beta) = F(x, \alpha)F(x, \beta)$ and so we have

$$\begin{aligned}
[x^n]F(x, \alpha + \beta) &= \sum_{k=0}^{n} ([x^k]F(x, \alpha)) \cdot ([x^{n-k}]F(x, \beta)) \\
&= \sum_{k=0}^{n} \frac{\alpha(\alpha + k)^{k-1}}{k!} \cdot \frac{\beta(\beta + n - k)^{n-k-1}}{(n-k)!} \\
&= \alpha\beta \sum_{k=0}^{n} \binom{n}{k} (\alpha + k)^{k-1} (\beta + n - k)^{n-k-1}
\end{aligned}$$

Chapter 2

Ordinary Generating Functions

2.1 Review

Remark 2.1.1 (Framework)**.** Let α be a set of combinatorial objects (finite or countably infinite). Let $w_1 : \alpha \to \mathbb{N}, w_2 : \alpha \to \mathbb{N}, ...$ with $\mathbb{N} = \{0, 1, 2, ...\}$ be one or more weight functions on α.

Then the OGF (ordinary generating function) of α (wrt $w_1, w_2, ...$) is

$$A(x_1, x_2, ...) = \sum_{a\in\alpha} \prod_i x_i^{w_i(a)}$$

In this case, we say variable x_i "marks" the weight function w_i. The point is that we have

$$\text{Card}\{a \in \alpha : w_1(a) = n_1, w_2(a) = n_2, ...\} = [x_1^{n_1} x_2^{n_2} ...]A(x_1, x_2, ...)$$

Remark 2.1.2 (OGF Technique)**.**

1. Figure out relevant set of objects and weight function(s).
2. Find bijections involving set of objects.
3. Convert bijections into OGF equations.
4. Simplify/solve equations.
5. Extract the answer to your problem.

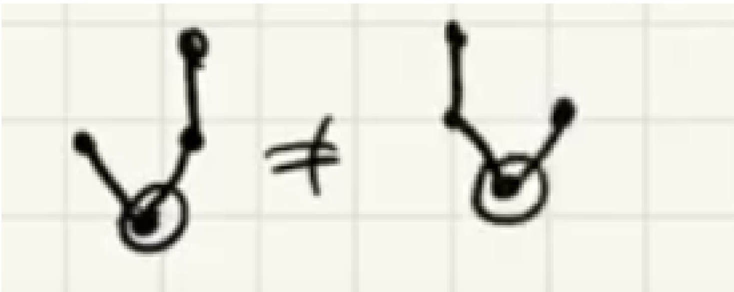
We remark that, in order to make everything to work, we must have the weight and bijection works together in certain way. In other word, we must have the weight of a composite object should be the sum of the weight of its components, i.e. we need to find weight-preserving bijections.

We will now introduce a series of operations on sets and their OGF correspondence.

Remark 2.1.3. In the following, we will set $\mathcal{A}, \mathcal{B}$ be two sets with weight function $w : \mathcal{A} \to \mathbb{N}$ and $w' : \mathcal{B} \to \mathbb{N}$ and their OGF are $A(x), B(x)$, respectively. Then we have

1. (***Disjoint Union***) If $\mathcal{A} \cup \mathcal{B}$ is disjoint union, then we get a new weight function w_0 given by $w_0(s) = w(s)$ if $s \in \mathcal{A}$ and $w_0(s) = w'(s)$ if $s \in \mathcal{B}$. Then we have the generating function for $\mathcal{A} \cup \mathcal{B}$ is $A(x) + B(x)$.
2. (***Products***) Consider $\mathcal{A} \times \mathcal{B}$ the product of sets. Set $w_0 : \mathcal{A} \times \mathcal{B} \to \mathbb{N}$ by $(a, b) \mapsto w(a) + w'(b)$ and we have the corresponding OGF for $\mathcal{A} \times \mathcal{B}$ is $A(x)B(x)$.
3. (***Star/Sequence***) Consider the set $\mathcal{A}^* = \bigcup_{k \geq 0} \mathcal{A}^k$. Then set $w_0(z_1, z_2, z_3, ...) = \sum_{k \geq 0} w(z_k)$ and we have the generating function for $\mathcal{A}^*$ is $\frac{1}{1-A(x)}$, if it is defined, i.e. we need $A(0) = 0$.
4. (***Composition***) Consider the set $\bigcup_{a \in \mathcal{A}} \{a\} \times \mathcal{B}^{w(a)}$. The ideal of composition: imagine $w : \mathcal{A} \to \mathbb{N}$ as the number of widgets, then start with object $a \in \mathcal{A}$ and we replace each widgets by an object in $\mathcal{B}$. We can define two different weights on this set, depends on whether we decide to count the weight of a or not.
   The first weight function is where a is dewidgetized, and we have $(a, b_1, b_2, ...) \mapsto \sum w'(b_i)$. In this case we have the OGF is $A(B(x))$.
   The second weight function is where we still want to take account of the weight of a, i.e. we want to make the widgets intact, and we have $(a, b_1, b_2, ...) \mapsto \sum w'(b_i) + w(a)$. In this case we have the OGF is $A(xB(x))$

Example 2.1.4 (Ordered Rooted Trees). An ordered rooted tree(ORT)/plane planted tree (PPT) is a rooted tree such that the childrens of any nodes are ordered (as 1st, 2nd,...). Note the order of children matters, and hence the following two trees are not equal:



The first question is, how many ORT with n nodes? We will use OGF to solve it. Let $\mathcal{Q}$ be the collection of all ORTs, then we define the weight function to be the number of nodes, i.e. given $q \in \mathcal{Q}$ we have $w(q)$ equal the number of nodes of q.

Next, we need to come up with some bijections. In this case, we see $\mathcal{Q} \Leftrightarrow \{\mathcal{O}\} \times \mathcal{Q}^*$ by sending q to the tuple $(\mathcal{O}, q_1, q_2, ..., q_k)$ where $q_1, ..., q_k$ are ORT that are childrens of q and $\mathcal{O}$ denote the root of q. Thus, we see for generating function we get

$$Q(x) = x\frac{1}{1 - Q(x)}$$

Thus use LIFT we get

$$[x^n]Q(x) = \frac{1}{n}[\lambda^{n-1}]\left(\frac{1}{1-\lambda}\right)^n = \frac{1}{n}\binom{2n-2}{n-1}, n \geq 1$$

This is the answer for $n \geq 1$.

Example 2.1.5 (Full Binary Trees). We say the ***updegree*** of a node equal the number of children of the node. We say a node is ***terminal*** if the node has updegree 0, i.e. it has no child. Then we define a ***full binary tree*** (FBT) to be a tree with each node has updegree 0 or 2.

So, how many FBTs with n terminals? Let $\mathcal{B}$ be all FBTs and let the weight function to be $w : \mathcal{B} \to \mathbb{N}$ to be $w(b)$ the number of terminals. Then we see (on the second union we are not consider the root node because we are counting the number of terminals and if the root node has two children then it is not a terminal)

$$\mathcal{B} \Leftrightarrow \{\mathcal{O}\} \bigcup \mathcal{B} \times \mathcal{B}$$

Then we see $B(x) = x + B(x)^2 \Rightarrow B(x) = \frac{x}{1-B(x)} \Rightarrow [x^n]B(x) = \frac{1}{n}\binom{2n-2}{n-1}$.

Example 2.1.6. How many ORTs with n nodes and k terminals. In this problem we have two parameters. Again let the set be $\mathcal{Q}$ the collection of ORTs. The two weight functions will be $m_1(q)$ equal the number of nodes and $m_2(q)$ the number of terminals.

Then we get

$$\mathcal{Q} \Leftrightarrow \{\mathcal{O}\} \cup \{\mathcal{O}\} \times \mathcal{Q} \times \mathcal{Q}^*$$

where we note this time on the second set in the union, we have $\mathcal{O}$ has weight $(1, 0)$. Thus we get

$$Q(x, y) = xy + xQ(x, y)\frac{1}{1 - Q(x, y)}$$

Now use LIFT we get

$$[x^n]Q(x, y) = \frac{1}{n}[\lambda^{n-1}](y + \frac{\lambda}{1-\lambda})^n$$

Thus immediately extract the coefficient of y^k we get

$$\begin{aligned}
[x^n y^k]Q(x, y) &= \frac{1}{n}[\lambda^{n-1}][y^k](y + \frac{\lambda}{1-\lambda})^n \\
&= \frac{1}{n}[\lambda^{n-1}]\binom{n}{k}(\frac{\lambda}{1-\lambda})^{n-k} \\
&= \frac{1}{n}[\lambda^{k-1}]\binom{n}{k}(1-\lambda)^{-(n-k)} \\
&= \frac{1}{n}\binom{n}{k}\binom{n-2}{k-1}
\end{aligned}$$

2.2 Partitions

Definition 2.2.1. A ***partition*** $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ is a weakly decreasing sequence of positive integers. The ***size*** of λ is $|\lambda| = \sum_{i=1}^{k} \lambda_i$. The ***length*** of λ is $\ell(\lambda) = k$. The ***width*** of λ is $width(\lambda) = \lambda_1$.

Example 2.2.2 (Counting Partitions by Size)**.** Let $\mathcal{P}$ be the set of all partitions, let weight function on $\mathcal{P}$ be the size of $\lambda \in \mathcal{P}$, i.e. $m(\lambda) = |\lambda|$. Then we want to compute OGF $P(\lambda)$. To this end, consider

$$\mathcal{P}_k := \{\lambda \in \mathcal{P} : width(\lambda) \leq k\}$$

Then we see

$$\mathcal{P}_k \Leftrightarrow \{k\}^* \times \{k-1\}^* \times ... \times \{1\}^*$$

and hence

$$P_k(x) = \frac{1}{1-x^k}\frac{1}{1-x^{k-1}}\cdots\frac{1}{1-x}$$

However, we note $P(x) = \lim_{k\to\infty} P_k(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$ and hence

$$[x^n]P(x) = [x^n]\prod_{i=1}^{\infty}\frac{1}{1-x^i}$$

Example 2.2.3 (Counting Partitions by Size and Length)**.** Let $\mathcal{P}$ be the set of all partitions, let $w_1(\lambda) = |\lambda|$ and $w_2(\lambda) = \ell(\lambda)$. Then we get

$$P_k(x,y) = \prod_{i=1}^{k}\frac{1}{1-x^i y}$$

with a similar bijection as before. Thus we see

$$P(x,y) = \prod_{i=1}^{\infty} P_k(x,y) = \prod_{i=1}^{\infty}\frac{1}{1-x^i y}$$

Example 2.2.4 ("At most" vs "Exactly")**.** The number of partitions of size n, length l and width k is equal to

$$[x^n y^l](P_k(x,y) - P_{k-1}(x,y))$$

On the other hand, the number of partitions with size n, length less than or equal l, and width less than or equal k is equal to

$$[x^n y^l]\frac{1}{1-y}P(x,y) = [x^n y^l]\prod_{i=0}^{k}\frac{1}{1-x^i y}$$

Example 2.2.5. Let $P_{k,l} = \{\lambda \in \mathcal{P} : \ell(\lambda) \leq l, width(\lambda) \leq k\}$. Then $|\mathcal{P}_{k,l}| = \binom{k+l}{k}$ by consider the paths from $(0,0)$ to (k,l) in $\mathbb{R}^2$.

Now we recall the q-analogous of binomial coefficients

$$\binom{k+l}{l}_q = \prod_{i=1}^{\infty}\frac{q^{l+i}-1}{q^i-1}$$

where those are called q-binomial coefficients. In particular we have a theorem says that

$$P_{k,l}(q) = \binom{k+l}{k}_q$$

To prove this, consider $\sum_{l\geq 0} P_{k,l}(q)y^l = \prod_{i=0}^{k} \frac{1}{1-q^i y}$, which is by the above example. Thus it suffices to show

$$\sum_{k\geq 0} \binom{k+l}{k} y^l = \prod_{i=0}^{k} \frac{1}{1-q^i y}$$

and this can be done by induction on k.

Theorem 2.2.6 (Euler's Pentagonal Number Theorem). *For $k \in \mathbb{Z}$, let $g_k = \frac{1}{2}k(3k+1)$. Then $P(x)^{-1} = \sum_{k\in\mathbb{Z}}(-1)^k x^{g_k}$*

Corollary 2.2.6.1. *$p(n) = \sum_{k\neq 0}(-1)^k p(n-g_k)$, where $p(x)$ is the number of partitions of x.*

Proof. Note $P(x)P^{-1}(x) = 1$ we see $[x^n]P(x)P^{-1}(x) = 0$ for $n \geq 1$ and hence $[x^n]\left(\sum_{n\geq 0} p(n)x^n\right)\left(\sum_{k\in\mathbb{Z}}(-1)^k x^{g_k}\right) = 0$. Now expand the left hand side and extract coefficients we get the result. ♡

Example 2.2.7. Compute the number of 01-strings of length n such that 011 is not a substring.

Let $\mathcal{A} = \{012-\text{strings where 011 is not a substring}\}$. We will define 3 weight functions, w_0, w_1, w_2 given by $w_i(\sigma)$ equal the number of i in σ. Then the OGF $A(x_0, x_1, x_2)$ is given by

$$A(x_0, x_1, x_2) = \sum_{\sigma\in\mathcal{A}} x_0^{w_0(\sigma)} x_1^{w_1(\sigma)} x_2^{w_2(\sigma)}$$

Then note

$$A(x, x, 0) = \sum_{\sigma\in\mathcal{A}} = \sum_{\substack{\sigma\in\mathcal{A} \\ w_2(\sigma)=0}} x^{w_0(\sigma)+w_1(\sigma)}$$

where we see $\sigma \in \mathcal{A}$ with $w_2(\sigma) = 0$ are exactly 01-strings without 011 as a substring. Hence the answer of our question is $[x^n]A(x, x, 0)$.

To compute $A(x_0, x_1, x_2)$, we use composition of $\mathcal{A}$ with $\mathcal{B} = \{2, 011\}$. Elements are then constructed as follows: for each $\sigma \in \mathcal{A}$, replace each 2 in σ by some string in $\mathcal{B}$. E.g. $\sigma = 2020$ then we have four strings we can construct by replacing elements from $\mathcal{B}$, namely $2020, 200110, 011020, 01100110$.

Then the result of such construction is equal to $\{0, 1, 2\}^*$, i.e. all 012-strings, as we can do this construction backwards as well. Thus we get

$$A(x_0, x_1, B(x_0, x_1, x_2)) = \frac{1}{1-(x_0+x_1+x_2)}$$

where we see

$$B(x_0, x_1, x_2) = x_2 + x_0 x_1^2$$

and hence

$$A(x_0, x_1, x_2+x_0x_1^2) = \frac{1}{1-(x_0+x_1+x_2)} \Rightarrow A(x_0, x_1, x_2) = \frac{1}{1-(x_0+x_1+x_2-x_0x_1^2)}$$

2.3 Transfer Matrix Method

Example 2.3.1. Let G be a directed graph, allow loops and parallel edges. Let any $u, v \in E$ we let:

1. $\mathcal{A}_{u,v} = \{\text{set of edges } u \to v\}$.
2. $\mathcal{T}_{u,v} = \{\text{all walks } u \to v\}$.
3. $\mathcal{T}_{u,v}^{(n)} = \{\text{all walks } u \to v \text{ with length } n\}$.
4. $\mathcal{T}_u := \{\text{all walks starting at } u\}$.

Then the TMM (transfer matrix method) theory is as follows: we assign a weight to each edge, say $e \mapsto wt(e)$. Then we obtain a generating function $A_{u,v}(x) = \sum_{e \in \mathcal{A}_{u,v}} x^{w(e)}$. Since for each u, v we have $A_{u,v}$, we get a matrix $A(x)$ with entries $\mathbb{Q}[[x]]$(or $\mathbb{Q}((x))$), just like we can get adjacency matrix from G.

For a walk w, we let the weight function to be $w \mapsto \sum_{\text{edges } e \text{ of } w} wt(e)$. Then we have the following propositions.

Proposition 2.3.2. *Use the notation as above, we have*

1. $T_{u,v}^{(n)}(x) = (A(x)^n)_{uv}$ *where for a matrix we use A_{ab} to denote the entry at ab place.*
2. $T_{u,v}(x) = ((I - A(x))^{-1})_{uv}$.
3. $T_u(x)$ *equal the u-component of $T(x)$, where $T(x)$ is the solution to $(I - A(x))T(x) = 1$ where 1 is a vector.*

Remark 2.3.3. The above proposition will work out nicely, provided $A(0)$ is nilpotent matrix.

Proof. Statement (1) is by induction on n. Statement (2) we note we have $T_{u,v}(x) = \sum_{n \geq 0} T_{u,v}^{(n)}(x) = \left(\sum_{n \geq 0} A(x)^n\right)_{uv}$ where if we assume $A(0)$ nilpotent then it always converges. In particular note $\sum A(x)^n$ is geometric series and hence we are done. For (3) we note $T_u(x) = \sum_v T_{u,v}(x)$ and hence by basic linear algebra we are done. ♡

Example 2.3.4 (TMM in Practice). We can use TMM in practice, when:

1. Our objects are build up recursively.
2. The options in recursive construction depends on cases.
3. Vertices correspond to cases.
4. Edges correspond to construction options.
5. Walks correspond to objects(we have to be very careful to make sure this is a bijection). Also we make to make sure this bijection is weight-preserving.

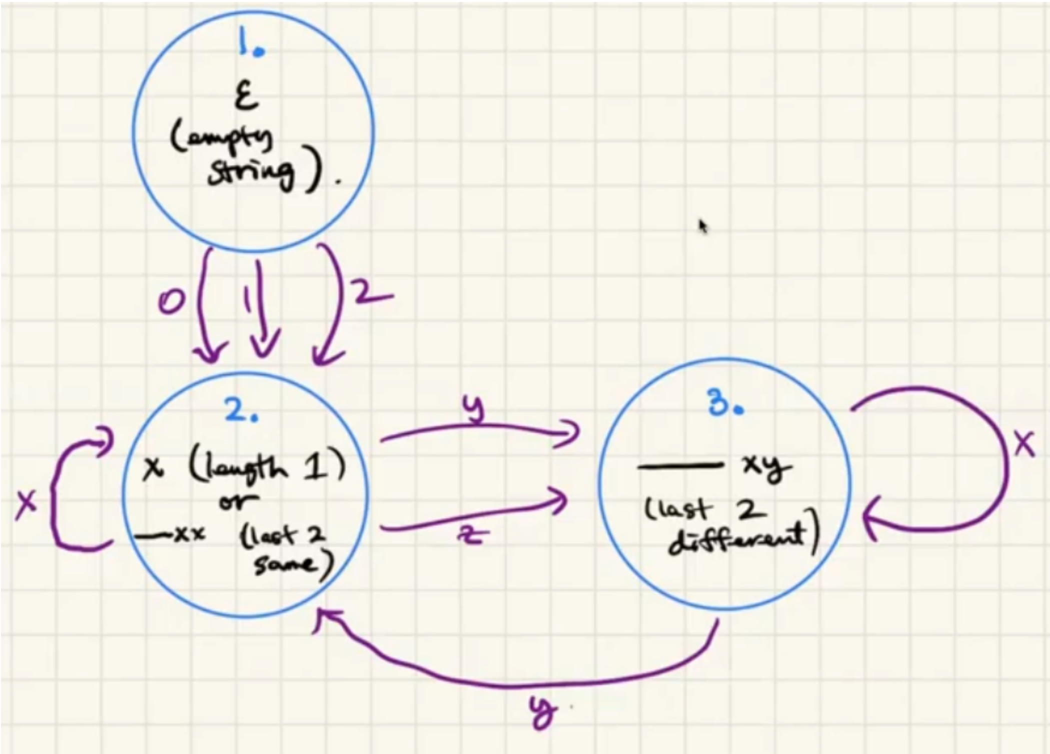
Example 2.3.5. Let $\mathcal{S} := \{012 - \text{strings, no 3 consecutive letters all different}\}$. For example, $01020 \notin \mathcal{S}$ and $01121 \in \mathcal{S}$. Then consider the weight function to be the length of the string. Compute $S(x)$.

We have three different cases:

1. Case 1: the empty string.
2. Case 2: string x with length 1 or $-xx$, i.e. strings end with the same letter twice.

3. Case 3: string $-xy$, i.e. at least length 2 and end with different letters.

Then we get the construction of $\mathcal{S}$ is the following graph



Then we get a bijection $\mathcal{S}$ and $\mathcal{T}_1$. Hence we get

$$A(x) = \begin{bmatrix} 0 & 3x & 0 \\ 0 & x & 2x \\ 0 & x & x \end{bmatrix}$$

and so

$$S(x) = T_1(x) = \frac{\det\left(\begin{bmatrix} 1 & -3x & 0 \\ 1 & 1-x & -2x \\ 1 & -x & 1-x \end{bmatrix}\right)}{\det(I - A(x))} = \frac{1+x+2x^2}{1-2x-x^2}$$

2.4 Exercises 2

Remark 2.4.1. For now, we will let $\mathcal{A}, \mathcal{B}$ be sets with weight function $wt : \mathcal{A} \to \mathbb{N}, wt' : \mathcal{B} \to \mathbb{N}$.

Example 2.4.2. Show $A(x) = \sum_{\sigma \in \mathcal{A}} x^{wt(\sigma)}$ is a convergent sum in $\mathbb{Z}[[x]]$ if and only if the question "how many elements of $\mathcal{A}$ have weight n" has a finite answer for all n.

Solution. Suppose $A(x)$ is a convergent sum in $\mathbb{Z}[[x]]$, this means $A(x) = \sum_{i \geq 0} a_i x^i$ with each $a_i \in \mathbb{Z}$. However we see $[x^n]A(x)$ is the answer to the question and hence we are done. Conversely we see $A(x)$ just converges to $\sum_{i \geq 0} a_i x^i$ with a_i being the answer to the question. ♠

Example 2.4.3. Show the number of objects in $\mathcal{A}$ of weight at most n is equal to $[x^n]\frac{A(x)}{1-x}$.

Solution. Just note $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ and hence $[x^n]\frac{A(x)}{1-x} = \sum_{i=0}^{n} a_i$ and so the proof follows. ♠

Example 2.4.4. A function $\phi : \mathcal{A} \to \mathcal{B}$ is said to be ***weight preserving*** if $wt' \circ \phi = wt$. Show if we can find weight preserving bijection then $A(x) = B(x)$.

Solution. Just note if we have such bijection then the number of objects in $\mathcal{A}$ with weight n is equal the number of objects in $\mathcal{B}$ with weight n and hence they must have the same generating function. ♠

Example 2.4.5. Let $L : \mathbb{N} \to \mathbb{N}$ be the linear function defined by $L(x) = ax + b$. If $A(x)$ is the generating function with respect to wt, what is the generating function of $\mathcal{A}$ with respect to the weight function $L \circ wt$?

Solution. We have the new generating function is

$$C(x) = \sum_{\sigma \in \mathcal{A}} x^{a \cdot wt(\sigma) + b} = x^b \sum_{\sigma \in \mathcal{A}} (x^a)^{wt(\sigma)} = x^b A(x^a)$$

♠

Example 2.4.6. Generalize the exercises above to the multivariate case.

Solution. The questions generalizes as follows:

1. The generating function $A(x, y)$ converges in $\mathbb{Z}[[x]][[y]]$ iff the question how many elements of $\mathcal{A}$ have weight n in x and weight m in y has a finite answer for all $(n, m) \in \mathbb{N}^2$.
2. The number of objects in $\mathcal{A}$ of first weight at most n is equal $[x^n]\frac{A(x,y)}{1-x}$.
3. You want $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{B} \times \mathcal{C}$ so they coordinate-wise are weight preserving.
4. Say we have $L : \mathbb{N}^2 \to \mathbb{N}^2$, then $A(x, y)$ with respect to $L \circ wt$ should be $x^a y^b A(x^c, y^d)$ for some $a, b, c, d \in \mathbb{N}$.

♠

Example 2.4.7. Prove the four operations in the review of OGF (disjoint, product, star, composition).

Solution. Review. ♠

Example 2.4.8. Show $\mathcal{A}^*$ has a convergent generating function iff $A(0) = 0$.

Solution. If $A(0) \neq 0$ then we see $1 + A(x) + A(x)^2 + ...$ does not converge. Suppose $A(0) = 0$, then clearly $1 + A(x) + A(x)^2 + ...$ converges because each coefficients are eventually constant. ♠

Example 2.4.9. Show the composition of $\mathcal{A}$ and $\mathcal{B}$ has a convergent generating function with respect to the weight $(a, b_1, ..., b_k) \mapsto \sum_{i=1}^{k} wt'(b_i)$ iff the composition $A(B(x))$ is defined.

Solution. Just note the set $\bigcup_{a\in\mathcal{A}}\{a\} \times \mathcal{B}^{wt(a)}$ is disjoint union and hence its generating function is $\sum_{a\in\mathcal{A}} B(x)^{wt(a)}$ and hence the proof follows. ♠

Example 2.4.10. Generalize the results above to multivariate case.

Solution. Nope. ♠

Example 2.4.11. Suppose you have a tree drawn in the plane, and one node is marked as the root. Why is this not the same thing as an ordered rooted tree?

Solution. It is unordered. ♠

Example 2.4.12. A full ternary tree is an ordered rooted tree in which each node has updegree 0 or 3. Show every FTT has an odd number of terminals, and the number of FTT with $2n + 1$ terminals is $\frac{1}{2n+1}\binom{3n}{n}$.

Solution. Let $\mathcal{F}$ be the set of FTT. Then we see

$$\mathcal{F} \Leftrightarrow \{\mathcal{O}\} \cup \mathcal{F}^3$$

and hence we get

$$F(x) = x + F(x)^3$$

and now apply LIFT we see

$$[x^n]F(x) = \frac{1}{n}[x^{n-1}](1 - x^2)^{-n}$$

and the claim follows as one should check. ♠

Example 2.4.13. Find a bijection between ordered rooted trees with n nodes and full binary trees with n terminals.

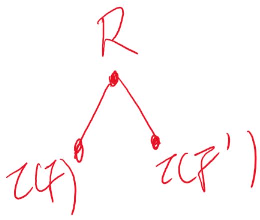
Solution. Consider the following construction:

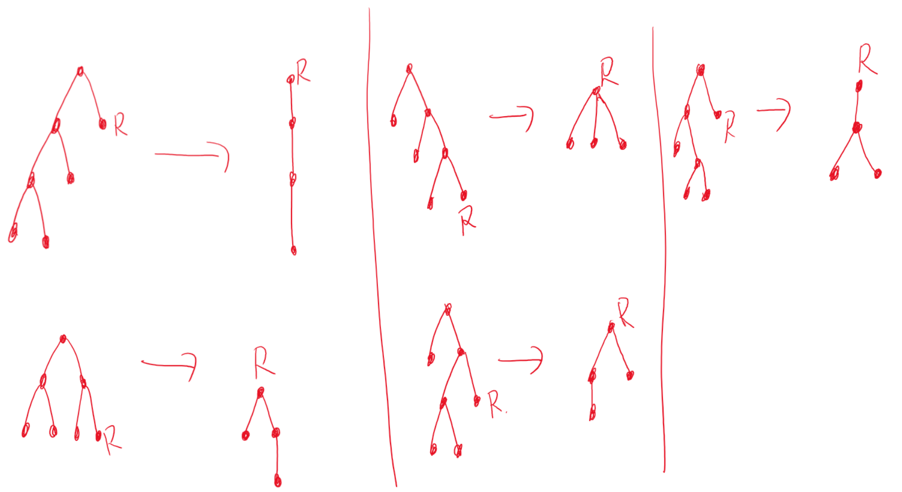
for $F, F' \in FBT$,
we let R be the root.

$\Rightarrow$

then run our transformation to F_n.

$\Rightarrow$ then we let





For any FBT, we start at the lowest right node, and make it the root of our ORT. Then for the tree at the left of the root, we run our transformation on the tree of smaller size and get the result and append to the root. Then we go up and keep appending. So in one word, once we decided the root, we keep append new subtrees to the right of the root.

One should try to show this is a bijection and it works as intended. ♠

Example 2.4.14. The height of a rooted tree is the maximum distance between the root and another node. Suppose we want to count the number of ORTs with n

nodes and height k. Following the general strategy we used so far, the obvious thing to consider is the set $\mathcal{Q}$ of all ORTs, and define two weight functions: the number of nodes and the height. Unfortunately this approach is totally doomed to failure. Explain why.

Solution. The problem is that when you try to decompose the trees, the second weight is not additive, i.e. if w_2 is the height weight function, then we see $w_2(T) = 1 + \max\{w_2(T_1), ..., w_2(T_n)\}$ if the PPT has $T_1, ..., T_n$ as its children. Thus we have no good bijections. ♠

Example 2.4.15. So you might be thinking, couldn't we just skip the whole business with $\mathcal{P}_k$ and say we have a bijection

$$\mathcal{P} \Leftrightarrow \prod_{i=1}^{\infty} \{i\}^*$$

Why is the alleged bijection above incorrect?

Solution. The bijection at each stage of $\mathcal{P}_k$ does not match up, hence we cannot just glue those pieces together to get a global bijection. ♠

Example 2.4.16. A strict partition is a partition $\lambda = (\lambda_1, ..., \lambda_l)$ such that $\lambda_1 > ... > \lambda_l$. Prove the number of strict partitions of size n is

$$[x^n] \prod_{i=1}^{\infty} (1 + x^i)$$

Solution. Note given such a partition, each part appear at most once. In other word, let $\mathcal{S}_k$ be strict partitions with width at most k, then we see

$$\mathcal{S}_k \Leftrightarrow \{\emptyset, k\} \times \{\emptyset, k-1\} \times ... \times \{\emptyset, 1\}$$

and hence we see

$$S_k(x) = (1 + x^k)(1 + x^{k-1})...(1 + x^1)$$

as desired. ♠

Example 2.4.17. Show the number of partitions of size n, width k and length l is equal to

$$[x^{n-k}y^l] \prod_{i=1}^{k-1} \frac{1}{1 - x^i y}$$

Solution. QAQ ♠

Example 2.4.18. Verify the following identities:

1. $\binom{k+l}{k}_q = \binom{k+l}{l}_q$

2. $\binom{k+l}{k}_q$ evaluated at $q=1$ is $\binom{k+l}{k}$
3. $\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$
4. $\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q$

Solution. Left as an exercise. ♠

Example 2.4.19. For each identity above, give a combinatorial interpretation.

Example 2.4.20. Use MATH 239 techniques to give a simple solution to the problem of counting 01-strings of length n that do not have 011 as a substring. Hence verify the answer we derived is correct.

Solution. Note we have the set of 01-string without 011 is just $\{1\}^*\{01,0\}^*$ and hence we see the generating function with respect to number of 0 as x and number of 1 as y is

$$A(x,y) = \frac{1}{1-y}\frac{1}{1-x-xy}$$

which is indeed equal what we obtained in the note. ♠

Example 2.4.21. Verify there is a bijection between the composition of $\mathcal{A}$ and $\mathcal{B}$ and the set of 012-strings, where $\mathcal{A}$ is the 01-strings without 011 and $\mathcal{B} = \{2, 011\}$.

Solution. We use $\mathcal{A}(\mathcal{B})$ to mean the composition. First we note given $\sigma \in \{0,1,2\}^*$, if σ does not contain 011 then it is in $\mathcal{A}(\mathcal{B})$. If it contains 011 then replace all 011 by 2 and run the construction we see it is in $\mathcal{A}(\mathcal{B})$. Thus every element in $\{0,1,2\}^*$ can be obtained by this construction in a unique way, i.e. $\{0,1,2\}^* \subseteq \mathcal{A}(\mathcal{B}) \subseteq \{0,1,2\}^*$ and hence there must be a bijection. ♠

Example 2.4.22. Why in the example $\mathcal{A}(\mathcal{B})$ above, it is essential that 011 is a string that cannot overlap with itself? Show the same technique work if 011 is replaced with any non-overlapping strings.

Solution. This is because if σ is a string that overlaps, we do not get a bijection. Consider the string 010, then 01010 can be obtained by both 012 and 210 from replacing 2 with elements in $\{2, 010\}$. ♠

Example 2.4.23. You roll an ordinary 6-sided die. If the outcome is k you roll the die k more times. Your score is the total of all $k+1$ rolls. Show the number of ways of getting a score of n is equal $[x^n]A(xA(x))$ where $A(x) = \sum_{i=1}^{6} x^i$.

Solution. Consider the set composition $\mathcal{A}(\mathcal{A})$ with the weight being $(a_1, b_1, \ldots) \mapsto \sum wt(b_i) + wt(a_i)$. Then we see its corresponding generating function is indeed $A(xA(x))$ and hence the proof follows. ♠

Example 2.4.24. Use transfer matrix method to give an solution to the forbiden string problem: find 01-strings without 011 of length n.

Solution. Consider the three cases:

1. String is $\epsilon, 1$ or ends in 11.
2. String ends in 0.
3. String ends in 01.

Then we are looking at $T_1(x)$ and it is given by

$$S(x) = \frac{\det \begin{bmatrix} 1 & -x & 0 \\ 1 & 1-x & -x \\ 1 & -x & 1 \end{bmatrix}}{\det \begin{bmatrix} 1-x & -x & 0 \\ 0 & 1-x & -x \\ 0 & -x & 1 \end{bmatrix}} = \frac{1}{(1-x)(1-x-x^2)}$$

as desired. ♠

2.5 Tutorial 2(Combinatorial LIFT)

Theorem 2.5.1 (LIFT). *Recall*

$$A(x) = x\phi(A(x)) \Rightarrow [x^n]A(x) = \frac{1}{n}[x^{n-1}]\phi(x)^n$$

Remark 2.5.2. How combinatorial proofs work:

1. To prove $a = b$:
   (a) Find a set $\mathcal{A}$ such that $|\mathcal{A}| = a$.
   (b) Find a set $\mathcal{B}$ such that $|\mathcal{B}| = b$.
   (c) Find a bijection $\mathcal{A} \to \mathcal{B}$.
2. To prove $a(x) = b(x)$:
   (a) Find a set $\mathcal{A}$ and weight function so $A(x) = a(x)$.
   (b) Find a set $\mathcal{B}$ and weight function so $B(x) = b(x)$.
   (c) Find a weight preserving bijection $\mathcal{A} \to \mathcal{B}$.

Remark 2.5.3. We work with specific $\phi(x)$ first. Let $\phi(x) = \frac{1}{1-x}$ and suppose $A(x) = x\phi(A(x))$, what is the combinatorial interpretation of $[x^n]A(x)$? Well, $A(x) = \frac{x}{1-A(x)}$ has coefficient as the $(n-1)$th Catalan number, i.e. $[x^n]A(x) = \frac{1}{n}\binom{2n-2}{n-1}$.

Recall one combinatorial interpretation for this $A(x) = x\phi(A(x))$ is the set of ordered rooted trees with n nodes.

Example 2.5.4. Give an example of a set/weight function where the OGF is $\phi(x) = \frac{1}{1-x}$.

Solution. Consider the set to be $\mathbb{N}$, with weight $wt(n) = n$. ♠

Example 2.5.5. What is the combinatorial interpretation of $[x^{n-1}]\phi(x)^n$.

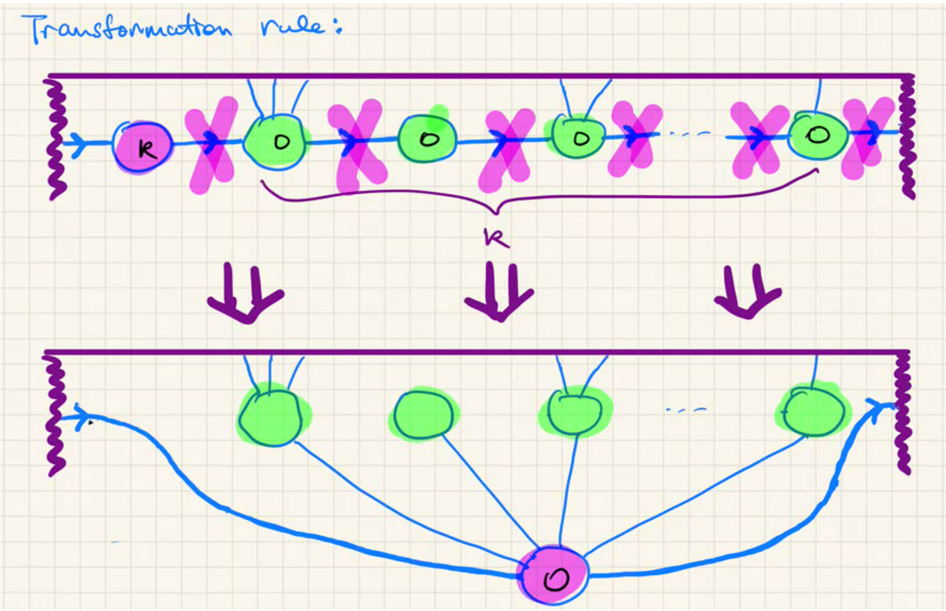
Solution. Well, $A(x)^n$ tell us we are probably looking at $\mathcal{A}^n$. Thus we are considering $\mathbb{N}^n$ here, with weight $(a_1, ..., a_n) = a_1 + ... + a_n$. In other word, a combinatorial interpretation of this is the number of $(a_1, ..., a_n) \in \mathbb{N}^n$ such that $a_1 + ... + a_n = n - 1$. ♠

Example 2.5.6. What about $\frac{1}{n}[x^{n-1}]\phi(x)^n$.

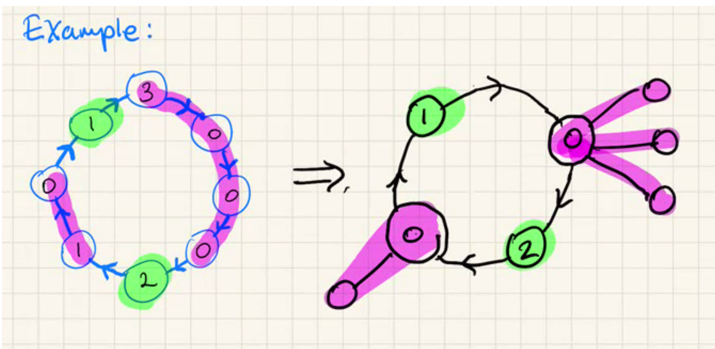
Solution. Define $\sim$ on $\mathbb{N}^n$ to be $(a_1, ..., a_n) \sim (b_1, ..., b_n)$ iff $\pi(a_1, ..., a_n) = (b_1, ..., b_n)$ where $\pi \in S_n$ is a cycle. Note this only works because $\gcd(n-1, n) = 1$. ♠

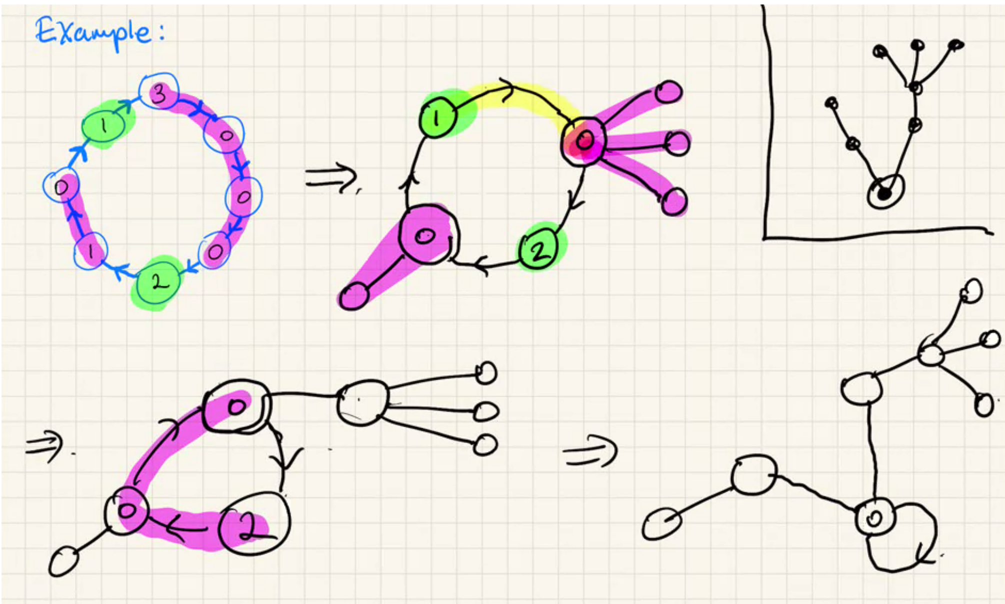
Remark 2.5.7. Thus, our goal right now is to prove LIFT for $\phi(x) = \frac{1}{1-x}$, where now we find combinatorial interpretations for both left hand side and right hand side. The left hand side of LIFT, we have $[x^n]A(x)$ with set $\mathcal{Q}_n$ (PPT with n nodes) and RHS is $\mathcal{C}_n$, which is the set of cycle of numbers with length n and sum equal $n - 1$.

Example 2.5.8 (A Bijection Between $\mathcal{Q}_n$ and $\mathcal{C}_n$)**.** The rules goes as follows:



and here is an example:





Remark 2.5.9. Now to the more general case, we add weight functions to our story with the same bijection.

What weight functions can we add to sets $\mathcal{Q}_n$ and $\mathcal{C}_n$ so that bijection always is weight preserving? On $\mathcal{Q}_n$, we have $wt_0 : \mathcal{Q}_n \to \mathbb{N}$ be the number of terminals, $wt_1 : \mathcal{Q} \to \mathbb{N}$ be the number of nodes of updegree 1,..., $wt_k : \mathcal{Q} \to \mathbb{N}$ be the number of nodes of updegree k. Then their corresponding weight functions on $\mathcal{C}_n$ is $wt'_0 : \mathcal{C}_n \to \mathbb{N}$ be the number of 0's, and $wt'_k : \mathcal{C}_n \to \mathbb{N}$ is the number of k's.

Thus, let $\phi(x) = \sum_{n\geq 0} y_i x^i$ with $A(x) = x\phi(A(x))$. We want to show that:

1. $[x^n]A(x) = \sum_{q\in\mathcal{Q}_n} y_0^{wt_0(q)} y_1^{wt_1(q)} \ldots$
2. $\frac{1}{n}[x^{n-1}]\phi(x)^n = \sum_{c\in\mathcal{C}_n} y_0^{wt'(c)} y_1^{wt'_1(c)} \ldots$

Chapter 3

Exponential Generating Functions

3.1 Species

Example 3.1.1 (Graphs). We note there is no set of all graphs, i.e. the class of all graphs form a proper class. Thus, we need to add definite constrains to make the proper class into a set.

In other word, given a finite set X, we can form the set $\mathcal{G}_X$ of graphs on X given by

$$\mathcal{G}_X = \{ \text{ all graphs } \Gamma : V(\Gamma) = X\}$$

. In particular, we see the following properties:

1. If $|X| = |Y| = n$ then $|\mathcal{G}_X| = |\mathcal{G}_Y| = 2^{\binom{n}{2}}$.
2. We have isomorphism: it is a bijection $f : V(\Gamma) \to V(\Gamma')$ such that $(u, v) \in E(\Gamma)$ if and only if $(f(u), f(v)) \in E(\Gamma')$.
3. If $f : X \to Y$ is a bijection and we have $\Gamma \in \mathcal{G}_X$, then there is a unique graph $\Gamma' \in \mathcal{G}_Y$, such that f is an isomorphism of graphs.
4. In other word, a bijection of sets $f : X \to Y$ induces a bijection $f_* : \mathcal{G}_X \to \mathcal{G}_Y$.

Definition 3.1.2. A ***species*** $\mathcal{A}$ is a rule that assigns:

1. To every finite set X, a (finite) set $\mathcal{A}_X$ called the ***set of $\mathcal{A}$-structures on*** X.
2. To every bijection $f : X \to Y$ of finite sets, we get a bijection $f_* : \mathcal{A}_X \to \mathcal{A}_Y$. This is called a ***transportation of*** A***-structures along*** f.

such that:

1. If $X \neq Y$ then $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$.
2. If $\mathrm{Id}_X : X \to X$ is the identity map, then the transportation $\mathrm{Id}_{X,*} : \mathcal{A}_X \to \mathcal{A}_X$ should be the identity.
3. If $f : X \to Y$ and $g : Y \to Z$ are bijections, then $(g \circ f)_* = g_* \circ f_*$.

Definition 3.1.3. Let $\mathcal{A}$ be a species, and $\alpha \in \mathcal{A}_X, \beta \in \mathcal{A}_Y$, then we say α and β are ***isomorphic*** if and only if there exists bijection $f : X \to Y$ such that $f_*(\alpha) = \beta$.

Definition 3.1.4. Let $\mathcal{A}$ be a species, assume $\mathcal{A}_X$ is finite for every finite set X. Then if $|X| = |Y| = n$, we must have $|\mathcal{A}_X| = |\mathcal{A}_X|$ and hence we define $|\mathcal{A}_n| := |\mathcal{A}_X|$

for any set X with $|X| = n$. Then we define the ***exponential generating function*** associated with $\mathcal{A}$ to be

$$A(x) := \sum_{n\geq 0} |\mathcal{A}_n| \frac{x^n}{n!}$$

Example 3.1.5. We have the following examples of species:

Name (Notation)	Structures on X	Transport along $f : X \to Y$	EGF
Sets ($\mathcal{E}$)	X, i.e. $\mathcal{E}_X = \{X\}$	$f_*(X) = Y$	$E(x) = \sum_{n\geq 0} 1 \frac{x^n}{n!} = e^x$
Linear Orders ($\mathcal{L}$)	$\mathcal{L}_X = \{(x_1, ..., x_n) : \lvert X\rvert = n, X = \{x_1, ..., x_n\}\}$	$f_*(x_1, ..., x_n) = (f(x_1), ..., f(x_n))$	$L(x) = \sum_{n\geq 0} n! \frac{x^n}{n!} = \frac{1}{1-x}$
Endofunctions ($\mathcal{N}$)	$\mathcal{N}_X = \{\alpha : X \to X\}$	$f_*(\alpha) = f \circ \alpha \circ f^{-1}$	$N(x) = \sum_{n\geq 0} n^n \frac{x^n}{n!}$
Permutations ($\mathcal{S}$)	$\mathcal{S}_X = \{\alpha : X \to X \text{ is bijection}\}$	$f_*(\alpha) = f \circ \alpha \circ f^{-1}$	$S(x) = \sum_{n\geq 0} n! \frac{x^n}{n!} = \frac{1}{1-x}$
Cyclic Permutations ($\mathcal{C}$)	$\mathcal{C}_X = \{\alpha : X \to X \text{ is a cycle}\}$	$f_*(\alpha) = f \circ \alpha \circ f^{-1}$	$C(x) = \sum_{n\geq 1} (n-1)! \frac{x^n}{n!} = \log(\frac{1}{1-x})$
Trees ($\mathcal{T}$)	$\mathcal{T}_X = \{\tau \in \mathcal{G}_X : \tau \text{ is a tree}\}$	same as graph	$T(x) = \sum_{n\geq 1} n^{n-2} \frac{x^n}{n!}$
Set Partitions (Π)	Π_X is set partitions of X	$f_*(\{X_1, .., X_k\}) = \{f(X_1), ..., f(X_k)\}$	$\Pi(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!}$

We remark for the last one, B_n are called the Bell numbers and Π_X is equal $\{\{X_1, ..., X_k\} : k \geq 1, X_i \subseteq X, X_i \cap X_j = \emptyset, X = \bigcup X_i\}$

Definition 3.1.6. Let $\mathcal{A}, \mathcal{B}$ be two species. A ***natural transformation*** $\Phi : \mathcal{A} \to \mathcal{B}$ is a rule which assigns to every finite set X a function $\Phi_X : \mathcal{A}_X \to \mathcal{B}_Y$, such that if $f : X \to Y$ is a bijection of sets, then the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}_X \arrow[r, "\Phi_X"] \arrow[d, "f_*"] & \mathcal{B}_X \arrow[d, "f_*"] \\ \mathcal{A}_Y \arrow[r, "\Phi_Y"] & \mathcal{B}_Y \end{tikzcd}$$

Example 3.1.7.

1. We have a natural transformation from trees to graphs by inclusion (if we have a sub-species then inclusion is always a natural transformation).
2. From directed graphs to graphs we have the forgetful natural transformation, i.e. we forget the direction of edges.
3. From endofunctions to directed graphs we have a natural transformation as follows. Given $\alpha : X \to X$, the edges are given by $(x, \alpha(x))$.
4. From any species to sets we get a natural transformation by forgetful natural transformation.
5. From graphs to graphs we have the graph complement as a natural transformation.

Lemma 3.1.8. *If $\alpha \in \mathcal{A}_X, \beta \in \mathcal{A}_Y$ are isomorphic, then $\Phi_X(\alpha)$ and $\Phi_Y(\beta)$ are isomorphic.*

Definition 3.1.9. Let Φ be a natural transformation. If $\Phi_X : \mathcal{A}_X \to \mathcal{B}_X$ is a bijection for all finite set X, then we say Φ is a ***natural equivalence*** and we say $\mathcal{A}$ and $\mathcal{B}$ are ***naturally equivalent*** and write $\mathcal{A} \equiv \mathcal{B}$.

Example 3.1.10.

1. The graph complement is an equivalence from graphs to graphs.
2. Let $\mathcal{G}$ be the species of graphs and $\hat{\mathcal{G}}$ be the species of pairs of equal graphs, i.e. $\hat{\mathcal{G}}_X = \{(\Gamma, \Gamma) : \Gamma \in \mathcal{G}_X\}$. Then the map $\Gamma \mapsto (\Gamma, \Gamma)$ is a natural equivalence.
3. We have an equivalence between $\mathcal{S}$ the species of permutations and the sub-species of $\mathcal{G}$ where the graphs has every vertex with in-degree 1 and out-degree 1.

Definition 3.1.11. We say two species $\mathcal{A}$ and $\mathcal{B}$ are ***numerically equivalent*** if $A(x) = B(x)$ and write $\mathcal{A} \approx \mathcal{B}$.

Remark 3.1.12. We note natural equivalence implies numerically equivalence but not vice versa.

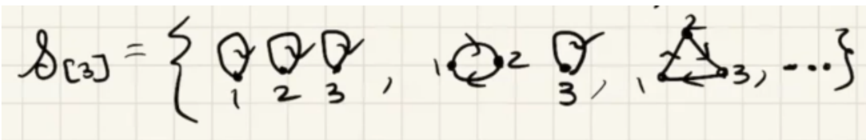
Example 3.1.13. We see $\mathcal{L}$ and $\mathcal{S}$ are numerically equivalent but $\mathcal{L} \not\equiv \mathcal{S}$.

Solution. Suppose $\Phi : \mathcal{L} \to \mathcal{S}$ is a natural transformation. Any two $\mathcal{L}$-structure of same order are isomorphic. But this is not true for $\mathcal{S}$ and hence Φ cannot be natural equivalence by the lemma above. ♠

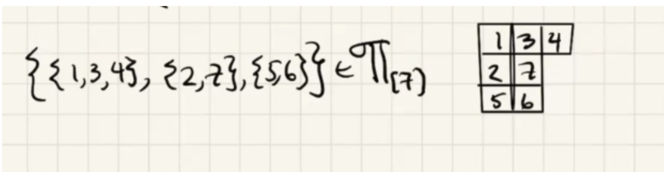
Definition 3.1.14. A species $\mathcal{A}$ is ***connected*** if and only if $\mathcal{A}_\emptyset = \emptyset$.

Remark 3.1.15 (Labelled & Unlabelled Diagrams)**.** Given a species $\mathcal{A}$, we can think $\mathcal{A}$-structures on X as diagrams with "widgets" that are labelled by elements of X.

For example, we can think $\mathcal{S}_{[3]}$ as graphs with each vertex has in-degree 1 and out-degree 1:



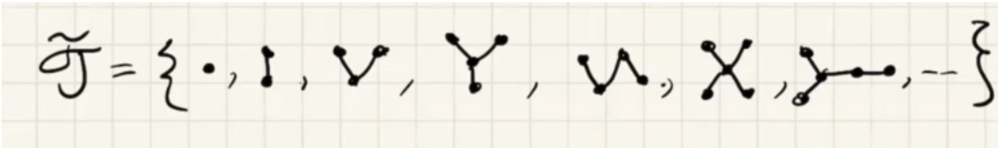
We could also have some examples of species that are not "graph-like" but still represented by diagrams. For instance, consider set partitions, and we have



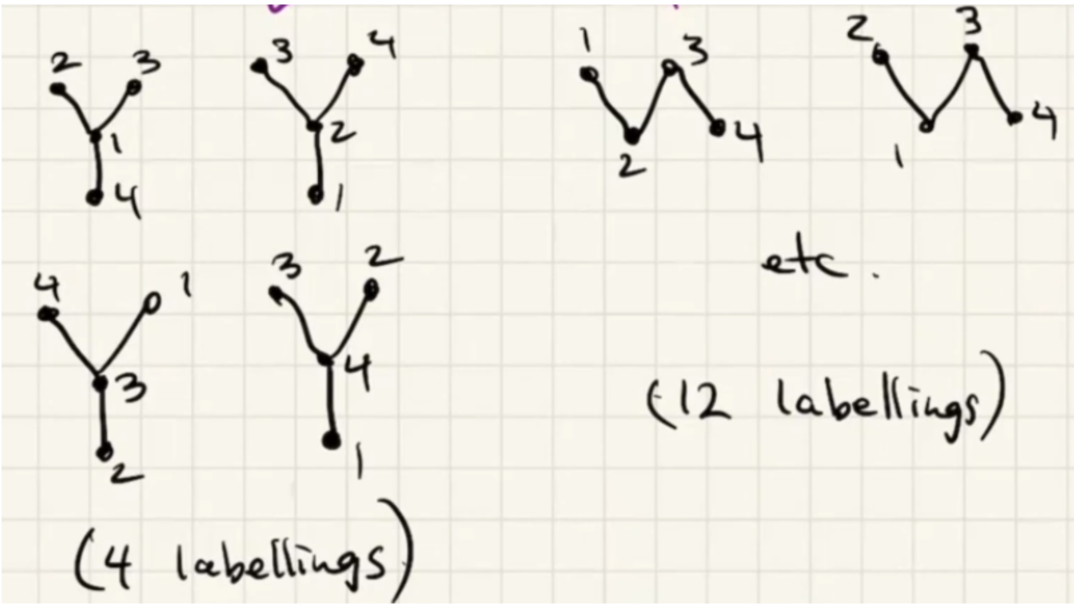
With this point of view of species, the transportations of $\mathcal{A}$-structures are always relabelling of our diagrammatic objects. Then the isomorphism types in this diagrammatic point of view are just unlabelled drawings and now we consider the widgets as "label receivers". Also, the "order" of species $\mathcal{A}$ in this context is just the number of label receivers.

We will write $\tilde{\mathcal{A}}$ for the set of isomorphism types of species $\mathcal{A}$. Then, we can just think species $\mathcal{A}$ as a set of isomorphism types plus a labelling operation.

For example, consider the species of trees, $\mathcal{T}$. Then we have the isomorphism types as



To form $\mathcal{T}_{[4]}$, we see we have two isomorphism types with 4 nodes, and we just need to label them with $[4] = \{1, 2, 3, 4\}$ and this gives us the set $\mathcal{T}_{[4]}$ as



3.2 Species Operations

Remark 3.2.1. In this section, $\mathcal{A}, \mathcal{B}$ will always be species.

Definition 3.2.2 (Sum and Difference)**.** We define $\mathcal{A}+\mathcal{B}$ as follows. The structures are given by

$$(\mathcal{A}+\mathcal{B})_X = \mathcal{A}_X \coprod \mathcal{B}_X = \{1\} \times \mathcal{A}_X \cup \{2\} \times \mathcal{B}_X$$

as the disjoint union of $\mathcal{A}_X$ and $\mathcal{B}_X$. This construction extends to infinite sums. In terms of isomorphism types, we have $\widetilde{\mathcal{A}+\mathcal{B}} = \tilde{\mathcal{A}} \coprod \tilde{\mathcal{B}}$.

Now assume $\mathcal{B} \subseteq \mathcal{A}$, i.e. $\mathcal{B}_X \subseteq \mathcal{A}_X$ for all X. Then we define $(\mathcal{A}-\mathcal{B})_X = \mathcal{A}_X \backslash \mathcal{B}_X$.

Definition 3.2.3 (Filtering by Order)**.** We define $\mathcal{A}_n, \mathcal{A}_+, \mathcal{A}_{even}, \mathcal{A}_{odd}$.

The definitions are given as follows

$$(\mathcal{A}_n)_X = \begin{cases} \mathcal{A}_X, & |X| = n \\ \emptyset, & \text{otherwise} \end{cases}$$

$$(\mathcal{A}_+)_X = \begin{cases} \mathcal{A}_X, & \text{if } |X| > 0 \\ \emptyset, & \text{otherwise} \end{cases}$$

$$(\mathcal{A}_{even/odd}) = \begin{cases} \mathcal{A}_X, & \text{if } |X| \text{ is even/odd} \\ \emptyset, & \text{otherwise} \end{cases}$$

Example 3.2.4. For example, we have $\mathcal{E}_+$ is the non-empty set operation. We also define $\mathcal{E}_1$ to be the ***species of singletons*** and denoted by $\mathcal{X}$. Finally we have $\mathcal{E}_0 := 1$.

Definition 3.2.5 (Products)**.** We define $\mathcal{K} \times \mathcal{A}, \mathcal{A} * \mathcal{B}, \mathcal{A} \boxtimes \mathcal{B}$.

The first one is ***Cartesian product*** with a set. Here $\mathcal{K}$ is a set, usually finite, and we define

$$(\mathcal{K} \times \mathcal{A})_X = \mathcal{K} \times \mathcal{A}_X$$

The second one is ***species product***. The species product is defined as follows

$$(\mathcal{A} * \mathcal{B})_X = \coprod_{S \subseteq X} \mathcal{A}_S \times \mathcal{B}_{X \setminus S}$$

We note the following:

1. To put an $\mathcal{A} * \mathcal{B}$ structure on X, we do the following:
   (a) partition X into $(S, X \setminus S)$.
   (b) put an $\mathcal{A}$ structure on S.
   (c) put a $\mathcal{B}$ structure on $X \setminus A$.
2. By default, $\mathcal{A}^m := \mathcal{A} * \mathcal{A} * ... * \mathcal{A}$.
3. 1 is the multiplicative identity of this multiplication.
4. We note $\widetilde{\mathcal{A} * \mathcal{B}} = \tilde{\mathcal{A}} \times \tilde{\mathcal{B}}$.

The third one is called ***superposition*** or ***Hadamard product***. It is defined as

$$(\mathcal{A} \boxtimes \mathcal{B})_X := \mathcal{A}_X \times \mathcal{B}_X$$

Definition 3.2.6 (Sequence)**.** We define the sequence $\mathcal{A}^*$ by

$$\mathcal{A}^* = \sum_{n \geq 0} \mathcal{A}^n$$

where $\mathcal{A}^0 = 1$. We note:

1. If $\mathcal{A}$ is connected and $\mathcal{A}_X$ is finite for all X, then $\mathcal{A}^*_X$ is connected and $\mathcal{A}_X$ is fintie.
2. $\tilde{\mathcal{A}}^* = (\tilde{\mathcal{A}})^*$

Definition 3.2.7 (Rooting and Derivatives)**.** We define $\mathcal{A}^\bullet$ and $\mathcal{A}'$.

The ***rooting*** $\mathcal{A}^\bullet$ (aka marking or pointing) is defined by

$$\mathcal{A}^\bullet_X = \mathcal{A}_X \times X$$

The ***derivative*** $\mathcal{A}'$ (aka mark and delete) is defined by

$$\mathcal{A}'_X = \mathcal{A}_{X \coprod \{*\}}$$

where $\{*\}$ is an element not in X.

We note:

1. $\tilde{\mathcal{A}}^\bullet$ is more or less $\tilde{\mathcal{A}}'$ and the difference is how we interprete them.
2. $\mathcal{A}^\bullet \equiv \mathcal{X} * \mathcal{A}'$

Definition 3.2.8 (Composition)**.** We define $\mathcal{A}[\mathcal{B}]$ (or $\mathcal{A} \circ \mathcal{B}$). We assume $\mathcal{B}$ is connected.

Then we define

$$\mathcal{A}[\mathcal{B}]_X = \coprod_{\{X_1,...,X_k\}\in\Pi_X} \mathcal{A}_{\{X_1,...,X_k\}} \times \mathcal{B}_{X_1} \times \mathcal{B}_{X_2} \times ... \times \mathcal{B}_{X_k}$$

We note:

1. To put an $\mathcal{A}[\mathcal{B}]$-structure on X, we do the following:
   (a) Form a set partition $\{X_1, ..., X_k\}$ of X.
   (b) Put an $\mathcal{A}$-structure on $\{X_1, ..., X_k\}$, which is a set with k elements.
   (c) Put a $\mathcal{B}$-structure on each X_i for $i = 1, ..., k$.
   (d) We note k is not fixed here.
2. Unlabelled $\mathcal{A}[\mathcal{B}]$-structure constructed as follows:
   (a) Start with unlabelled $\mathcal{A}$-structure α
   (b) Replace each label receiver in α by $\mathcal{B}$-structure β_i.
   (c) Label receiver of composite object are label receivers of β_i.

Example 3.2.9.

1. $\mathcal{E}[\mathcal{J}]$ is the species of forest.
2. $\mathcal{E}_2[\mathcal{J}]$ is the forest with 2 components. We note $\mathcal{E}_2[\mathcal{J}] \not\equiv \mathcal{J}^2$ as $|\mathcal{J}^2_X| = 2|\mathcal{E}_2[\mathcal{J}]_X|$
3. $\mathcal{E}[\mathcal{C}] = \mathcal{S}$.
4. $\mathcal{L}[\mathcal{B}] = \mathcal{B}^*$.
5. $\mathcal{N} \equiv \mathcal{S}[\mathcal{T}^\bullet]$.

Theorem 3.2.10. *Suppose we have $\mathcal{A}, \mathcal{B}$ be two species with $A(x) = \sum a_n \frac{x^n}{n!}$ and $B(x) = \sum b_n \frac{x^n}{n!}$, then*

Operation	Species	EGF
Sum/ Difference	$\mathcal{A} + \mathcal{B}$ $\mathcal{A} - \mathcal{B}$	$A(x) + B(x)$ $A(x) - B(x)$
Filters	$\mathcal{A}_n$ $\mathcal{A}_+$ $\mathcal{A}_{even}$ $\mathcal{A}_{odd}$	$a_n \frac{x^n}{n!}$ $A_+(x) = \sum_{n \geq 1} a_n \frac{x^n}{n!}$ $A_{even}(x) = \frac{1}{2}(A(x) + A(-x))$ $A_{odd}(x) = \frac{1}{2}(A(x) - A(-x))$
Products	$\mathcal{K} \times \mathcal{A}$ $\mathcal{A} \times \mathcal{B}$ $\mathcal{A} \boxtimes \mathcal{B}$	$\lvert\mathcal{K}\rvert A(x)$ $A(x)B(x)$ $\sum_{n \geq 0} a_n b_n \frac{x^n}{n!}$
Sequence	$\mathcal{A}^*$	$\frac{1}{1-A(x)}$
Derivatives/ Rooting	$\mathcal{A}'$ $\mathcal{A}^\bullet$	$A'(x) = \frac{d}{dx}A(x)$ $A^\bullet(x) = xA'(x)$
Composition	$\mathcal{A}[\mathcal{B}]$	$A(B(x))$

Proof. All of the proofs are elementary counting.

For example, we consider $\mathcal{A} * \mathcal{B}$. We see

$$\begin{aligned}
|(\mathcal{A} * \mathcal{B})_X| &= \sum_{S \subseteq X} |\mathcal{A}_S| \cdot |\mathcal{B}_{X \setminus S}| \\
&= \sum_{k=0}^{n} \binom{n}{k} |\mathcal{A}_k| \cdot |\mathcal{B}_{n-k}| \\
&= n! \sum_{k=0}^{n} \frac{1}{k!} a_k \cdot \frac{1}{(n-k)!} b_{n-k} \\
&= n! [x^n] A(x) B(x)
\end{aligned}$$

♡

3.3 Examples

Remark 3.3.1. We have some common notations. If $A(x) = \sum a_n \frac{x^n}{n!}$ then we write $[\frac{x^n}{n!}]A(x) = a_n$. We note we have $[\frac{x^n}{n!}]A(x) = n![x^n]A(x)$ and hence this operation is not linear.

Example 3.3.2 (Connected Graphs)**.** Let $\mathcal{G}$ be graphs and $\mathcal{G}^c$ be connected graphs. We know $\mathcal{G} \equiv \mathcal{E}[\mathcal{G}^c]$ and therefore we have

$$G(x) = \exp(G^c(x))$$

where we know $G(x) = \sum 2^{\binom{n}{2}} \frac{x^n}{n!}$. Hence we see

$$G^c(x) = \log(\sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!})$$

Example 3.3.3 (Permutations with even number of cycles)**.** The species we want to consider is $\mathcal{E}_{even}[\mathcal{C}]$. We see $E_{even}(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$ and $C(x) = \log(\frac{1}{1-x})$ and hence we see

$$E_{even}(C(x)) = \cosh\log(\frac{1}{1-x}) = \frac{1}{2}(\frac{1}{1-x} + 1 - x) = 1 + \frac{3}{2}\frac{x^2}{1-x}$$

Thus the number of permutations of $[n]$ with an even number of cycles is

$$[\frac{x^n}{n!}](1 + \frac{3}{2}\frac{x^2}{1-x}) = \begin{cases} 1, & n = 0 \\ 0, & n = 1 \\ \frac{1}{2}n!, & \text{otherwise} \end{cases}$$

Example 3.3.4 (Surjective Functions)**.** How many surjective functions $f : [m] \to [k]$?

Method 1: Fix m and let $\mathcal{A}, \mathcal{B}$ be the species $\mathcal{A}_X = \{\text{all functions } f : [m] \to X\}$ and $\mathcal{B}_X = \{\text{surjective functions } f : [m] \to X\}$. Then we have

$$\mathcal{A} \equiv \mathcal{B} * \mathcal{E}$$

where we are going to think $\mathcal{B}$ as surjective functions onto subset $S \subseteq X$ and $\mathcal{E}$ is just the set $X \backslash S$. Thus we see

$$A(x) = \sum_{n\geq 0} n^m \frac{x^n}{n!} = B(x)e^x$$

and hence

$$B(x) = e^{-x} \sum_{n\geq 0} n^m \frac{x^n}{n!}$$

and the answer is $[\frac{x^k}{k!}]B(x)$.

Method 2: Fix k and let $\mathcal{H}$ be the species given by

$$\mathcal{H}_X = \{\text{surjective functions } f : X \to [k]\}$$

Then we see

$$\mathcal{H} \equiv \mathcal{E}_+ * \mathcal{E}_+ * \ldots * \mathcal{E}_+ = (\mathcal{E}_+)^k$$

as k products, where given $f : X \to [k]$ we think the k products as $f^{-1}(1), \ldots, f^{-1}(k)$.

Thus we see

$$H(x) = (e^x - 1)^k$$

and hence the answer is $[\frac{x^m}{m!}]H(x)$.

Example 3.3.5 (Trees)**.** The species of trees have no natural decomposition. On the other hand, species of rooted trees does. In particular, we have

$$\mathcal{T}^\bullet \equiv \mathcal{X} * \mathcal{E}[\mathcal{T}^\bullet]$$

where $\mathcal{X}$ is the root and $\mathcal{E}[\mathcal{T}^\bullet]$ is the set of rooted tree branches. Thus we get

$$T^\bullet(x) = xe^{T^\bullet(x)}$$

and apply LIFT we get

$$[x^n]T^\bullet(x) = \frac{n^{n-1}}{n!}$$

and hence

$$T^\bullet(x) = \sum_{n\geq 1} \frac{n^{n-1}}{n!} x^n$$

However, note each tree has n ways to root the trees, and therefore we see

$$T(x) = \sum_{n\geq 1} \frac{n^{n-2}}{n!} x^n$$

and we conclude the number of trees of n nodes is n^{n-2}.

Example 3.3.6 (Trivalent trees)**.** Let $\mathcal{A}$ be the species of trees in which every vertex has degree 1 or 3. We will use the same trick and first look at the corresponding rooted structure.

However, note given a trivalent tree and if we break up, we do not get a set of trivalent trees and hence we we need to find something else in the recursive definition of $\mathcal{A}^\bullet$.

What we want to use is $\mathcal{B}$, the rooted trees in which every vertex has updegree 0 or 2. Then we get

$$\mathcal{A}^\bullet \equiv \mathcal{X} * \mathcal{B} + \mathcal{X} * \mathcal{E}_3[\mathcal{B}]$$

$$\mathcal{B}^\bullet \equiv \mathcal{X} + \mathcal{X} * \mathcal{E}_2[\mathcal{B}]$$

By the above two relations, we get

$$A^\bullet(x) = xB(x) + x\frac{1}{6}B(x)^3$$

$$B(x) = x + \frac{1}{2}xB(x)^2$$

Therefore

$$[x^{2n}]A^\bullet(x) = [x^{2n}]x(B(x) + \frac{1}{6}B(x)^3) = [x^{2n-1}](B(x) + \frac{1}{6}B(x)^3)$$

Now use LIFT we see

$$[x^{2n}]A^\bullet(x) = \frac{1}{2n-1}[\lambda^{2n-2}]\frac{d}{d\lambda}(\lambda + \frac{1}{6}\lambda^3) \cdot (1 + \frac{1}{2}\lambda^2)^{2n-1} = \frac{1}{2n-1}\left(\frac{1}{2}\right)^{n-1}\binom{2n}{n-1}$$

Thus we see

$$|\mathcal{A}_{2n}^{\bullet}| = (2n)!\frac{1}{2n-1}\left(\frac{1}{2}\right)^{n-1}\binom{2n}{n-1}$$

and hence

$$|\mathcal{A}_{2n}| = \frac{1}{2n}|\mathcal{A}_{2n}^{\bullet}| = (2n-3)!!(2n)^{\underline{n-1}}$$

3.4 Exercises 3

Example 3.4.1. Let $\mathscr{A}$ be a species, suppose X, Y, Z are finite sets, $\alpha \in \mathcal{A}_X, \beta \in \mathcal{B}_X, \gamma \in \mathcal{A}_Z$. If α is isomorphic to β, β is isomorphic to γ, prove α is isomorphic to γ.

Solution. Say $f_*(\alpha) = \beta$ and $g_*(\beta) = \gamma$. Then we see $g_* f_*(\alpha) = \gamma$ where $g_* f_* = (g \circ f)_*$ and clearly $g \circ f$ is a bijection from X to Z. ♠

Example 3.4.2. Let $\mathcal{A}$ be two species, below are two definitions of a subspecies of $\mathcal{A}$:

1. $\mathcal{B}$ is a subspecies of $\mathcal{A}$ if $\mathcal{B}$ is a species and $\mathcal{B}_X \subseteq \mathcal{A}_X$ for every finite set X.
2. $\mathcal{B}$ is a subspecies of $\mathcal{A}$ if $\mathcal{B}$ is a rule that assigns to every finite set X a subset $\mathcal{B}_X \subseteq \mathcal{A}_X$ and $\mathcal{B}$ is closed under isomorphisms.

Solution. Just note in (2), by the fact it is closed under isomorphism we get $\mathcal{B}$ is a species, thus the two are equivalent. On the other hand, assume (1) we will get $\mathcal{B}$ is closed under isomorphism by definition of species. ♠

Example 3.4.3. Give 10 examples of subspecies of $\mathcal{G}$.

Solution. Simple graphs, connected graphs, trees, triagnle trees, binary trees (unordered children), ternary trees, cycles, graphs itself, forest, empty species. ♠

Example 3.4.4. The species "simple directed graph" assigns to each finite set X the set of all simple directed graphs with vertex set X. What is the exponential generating function for it?

Solution. Since it did not ask for a proof by species, just note given vertex set X with $|X| = n$, we see we have $\binom{n}{2}$ possible pair of vertices, and each pair would give 4 possibilities, i.e. they are adjacent with $u \to v$, $v \to u$ or both $u \to v$ and $v \to u$, or they are not adjacent. Thus we get the total number of simple directed graph is given by $4^{\binom{n}{2}}$. ♠

Example 3.4.5. For the species of $\mathcal{L}$ and $\mathcal{N}$, verify all conditions in the definition of a species are satisfied.

Solution. Say f is bijection, then see f_* is indeed a bijection. Given $(y_1, ..., y_n) \in \mathcal{A}_Y$, we see $f_*((f^{-1}(y_1), ..., f^{-1}(y_n))) = (y_1, ..., y_n)$. Clearly f_* is also injective. Thus f_* is bijection as desired. Clearly $X \neq Y$ then $X \cap Y = \emptyset$. Clearly Id_* is the identity. Clearly composition works. The other is left as exercise. ♠

Example 3.4.6. Say X, Y are finite sets, $|X| = |Y|$. If $\alpha \in \mathcal{L}_X$ and $\beta \in \mathcal{L}_Y$, prove α is isomorphic to β.

Solution. Say $\alpha = (x_1, ..., x_n)$ and $\beta = (y_1, ..., y_n)$. Then $f(x_i) = y_i$ is a bijection such that $f_*\alpha = \beta$. ♠

Example 3.4.7. Give an example of a finite set X and two permutations $\alpha, \beta \in \mathcal{S}_X$ such that α and β are not isomorphic.

Solution. Consider permutation (123) and (23). Clearly they are in different conjugacy classes and hence cannot be isomorphic. ♠

Example 3.4.8. Let $\alpha : X \to X$ be an endofunction, a fixed point of α is an element $x \in X$ such that $\alpha(x) = x$. An endofunction is fixed point free if it has no fixed points.

Let $\mathcal{N}^f$ be the species of fixed point free endofunctions. Show $N^f(x) = \sum_{n \geq 0}(n-1)^n \frac{x^n}{n!}$.

Solution. We have decomposition of endofunctions $\mathcal{E}(\mathcal{C}(\mathcal{T}^\bullet))$ and since we want fixed point free, we get $\mathcal{N}^f \equiv \mathcal{E}(\mathcal{C}_{\geq 2}(\mathcal{T}^\bullet))$. Thus we get

$$N^f(x) = \exp(\log(\frac{1}{1 - T^\bullet(x)}) - T^\bullet(x))$$

Recall $T^\bullet(x) = x\phi(T^\bullet(x))$ with $\phi(x) = e^x$. Thus we see

$$N^f(x) = \frac{1}{1 - T^\bullet(x)} e^{-T^\bullet(x)} = f(T^\bullet(x))$$

where $f(x) = \frac{1}{1-x}e^{-x}$. Now use alternate LIFT we get

$$[x^n]f(T^\bullet(x)) = [x^n]\frac{1}{1-x}e^{-x}(1-x)e^{nx} = [x^n]e((n-1)x) = \frac{(n-1)^n}{n!}$$

♠

Example 3.4.9. For a finite set X let $\mathcal{M}_X$ be the set of all functions $X \to \mathbb{N}$. For a bijection of finite sets $f : X \to Y$, and $\alpha \in \mathcal{M}_X$, let $f_*(\alpha) = \alpha \circ f^{-1}$. Show $\mathcal{M}$ satisfies all conditions in the definition of species, except the condition $\mathcal{M}_X$ is finite.

This is an example of ***infinite species***.

Solution. We just show $\mathcal{M}_X$ is not fintie. Note $f_n(x) = n$ would give us infinite many different functions, thus it is not finite. ♠

Example 3.4.10. Consider the species of directed graphs, with loops allowed but no parallel edges. Describe a subspecies that is equivalent to $\mathcal{N}$.

Solution. Just note the species of functional directed graphs of $\mathcal{N}$ is a subspecies of directed graphs with loop but no parallel edges. ♠

Example 3.4.11. Give an example of a natural transformation from $\mathcal{G}$ to Π and an example of a natural transformation from Π to $\mathcal{G}$.

Solution. Given a graph, we get a set of connected components and then forget their graph structure and thus obtain a set partition. This gives a natural transformation from $\mathcal{G}$ to Π. Conversely given a set partition $\{A_1, ..., A_k\}$, we can form the complete graph with each A_i and put them together to get a graph. ♠

Example 3.4.12. Prove the lemma that if $\alpha \in \mathcal{A}_X, \beta \in \mathcal{A}_Y$ are isomorphic, then $\Phi_X(\alpha)$ and $\Phi_Y(\beta)$ are isomorphic.

Solution. This is because the square commutes. ♠

Example 3.4.13. Show there are no natural transformation from $\mathcal{S} \to \mathcal{L}$.

Solution. $\mathcal{S}$ have more than one conjugacy classes, while $\mathcal{L}$ have only one. ♠

Example 3.4.14. Give an example of a natural transformation from $\mathcal{L}$ to $\mathcal{C}$.

Solution. For $(x_1, ..., x_n) \in \mathcal{L}_X$, send it to the cyclic function $x_1 \mapsto x_2, ..., x_n \mapsto x_1$. ♠

Example 3.4.15. For each species in the table about species, come up with a diagrammatic way to represent structures of the species.

Example 3.4.16. As noted, a partition diagram filled with elements of a set X can be interpreted as a drawing of a set partition. It can also be interpreted as a drawing of a linear order, simply by reading the entries in the boxes as a sequence in the usual reading order. Since the same diagram can represent both a set partition and linear orders, does that mean $\mathcal{L} \equiv \Pi$?

Solution. Just note if you swap two elements in the partition diagram on the same row, you get the same partition, but different linear order. ♠

Example 3.4.17. If $\mathcal{A}, \mathcal{B}$ are species and $f : X \to Y$ is a bijection of finite sets, how do you define f_*?

Solution. Clearly it should be $f_*(a) = f_*(a)$ if $a \in \mathcal{A}_X$ and $f_*(b)$ if $b \in \mathcal{B}_X$. ♠

Example 3.4.18. Show that $\mathcal{L} \equiv \mathcal{X}^*$.

Solution. Just note $\mathcal{X}_X^*$ contains elements of the form $(x_1, ..., x_n)$. ♠

Example 3.4.19. A permutation with no fixed points is called a derangement. Let $\mathcal{D}$ be the species of derangements. Show $\mathcal{D} * \mathcal{E} \equiv \mathcal{S}$.

Solution. Clearly we can think $\mathcal{E}$ as the identity and $\mathcal{D}$ as derangements. Thus we see every permutation is made up with one derangement and the identity. ♠

Example 3.4.20. Recall the species $1 = \mathcal{E}_0$ is the identity species for product operation. This means $\mathcal{A} \equiv 1 * \mathcal{A} \equiv \mathcal{A} * 1$ for all species $\mathcal{A}$. Which species is the identity species of superposition?

Solution. $\mathcal{E}$ is the identity. ♠

Example 3.4.21. Show that $\mathcal{X} * \mathcal{A}' \equiv \mathcal{A}^\bullet$.

Solution. Clearly on the left we just view the $\mathcal{X}$ as our root, thus the two are equivalent. ♠

Example 3.4.22. Show $\Pi \equiv \mathcal{E} \circ \mathcal{E}_+$.

Solution. Clearly set partition is just a set of non-empty sets. ♠

Example 3.4.23. Which species is the identity species for composition?

Solution. To make the EGF follow main theorem, $\mathcal{X}$ should be the identity. ♠

Example 3.4.24. Let $\mathcal{D}$ be the species of derangements. Show:

1. $\mathcal{D} \equiv \mathcal{E}[\mathcal{C} - \mathcal{C}_1]$.
2. If $\mathcal{N}^f$ is the species of fixed point free endofunctions, then show $\mathcal{N}^f \equiv \mathcal{D} \circ \mathcal{T}^\bullet$.

Solution. Clearly derangements has cycle decomposition with no fixed elements, i.e. the full permutations are given by $\mathcal{E}[\mathcal{C}]$ and to get rid of fixed elements, we just subtract a $\mathcal{C}_1$, i.e. $\mathcal{D} \equiv \mathcal{E}[\mathcal{C} - \mathcal{C}_1]$.

On the other hand, by consider the functional directed graph, we see $\mathcal{N} \equiv \mathscr{S}[\mathcal{T}^\bullet]$ and hence $\mathcal{N}^f \equiv \mathcal{D}[\mathcal{T}^\bullet]$. ♠

Example 3.4.25. Explain why $\mathcal{L}_+(\mathcal{T}^\bullet) \equiv \mathcal{T}^{\bullet\bullet}$.

Solution. We can decompose a tree with two roots using the unique path from the first root to the second root, then direct all edges not on the path towards the path, with each vertices on the path become a root. This decomposition is exactly $\mathcal{L}_+(\mathcal{T}^\bullet)$。 ♠

Example 3.4.26. If $A(x)$ is the exponential generating function of $\mathcal{A}$, prove that

1. $\frac{d}{dx}A(x)$ is the EGF of $\mathcal{A}'$.
2. $x\frac{d}{dx}A(x)$ is the EGF of $\mathcal{A}^\bullet$.

Solution. (1): Let $X = [n]$, then we see $\mathcal{A}'_X = \mathcal{A}_{X \amalg \{*\}}$. In other word, we have $|\mathcal{A}'_n| = |\mathcal{A}_{n+1}|$. Thus we have

$$A'(x) = \sum_{n\geq 0} |\mathcal{A}_{n+1}|\frac{x^n}{n!} = \frac{d}{dx}(\sum_{n\geq 0} |A_n|\frac{x^n}{n!})$$

(2): Let $X = [n]$, then $\mathcal{A}^\bullet_X = \mathcal{A}_X \times X$. Thus, we see $|\mathcal{A}^\bullet_n| = |\mathcal{A}_X \times X| = |X| \cdot |\mathcal{A}_X| = n|\mathcal{A}_n|$. Thus

$$A^\bullet(x) = \sum_{n\geq 0} n|\mathcal{A}_n|\frac{x^n}{n!} = \sum_{n\geq 0} |\mathcal{A}_n|\frac{x^n}{(n-1)!} = x\left(\sum_{n\geq 0} |\mathcal{A}_{n+1}|\frac{x^n}{n!}\right)$$

♠

Example 3.4.27. If $\mathcal{A}$ is not connected, show there are infinitely many $\mathcal{A}^*$ structure on every finite set, and hence the EGF of $\mathcal{A}^*$ is not defined.

Solution. Just note since $\mathcal{A}$ is not connected we have $\mathcal{A}_\emptyset$ is not emptyset, say it contains $A \in \mathcal{A}_\emptyset$. Then we see for all finite set X and all $n \geq 1$, we have

$$\mathcal{A}_X \times \mathcal{A}_\emptyset \times \mathcal{A}_\emptyset \times \ldots \times \mathcal{A}_\emptyset$$

is not empty. Thus it contains infinitely many structures because for each n we have at least one such structure. ♠

Example 3.4.28. Read a proof that the EGF of $\mathcal{A} \circ \mathcal{B}$ is $A(B(x))$.

Solution. First let $\mathcal{A} = \mathcal{E}$. We will show $\mathcal{E}\circ\mathcal{B}$ has EGF $\exp(B(x))$. In particular, note $\mathcal{E}\circ\mathcal{B} \equiv \sum_{k\geq 0} \mathcal{E}_k[\mathcal{B}]$. Since $\mathcal{B}$ is connected, we see if $X = [n]$ and $\{B_1, \ldots, B_k\} \in \mathcal{E}[\mathcal{B}]_X$, then we must have $k \leq n$ because each B_i uses at least one element from X. In other word, $\mathcal{E}_k[\mathcal{B}]_X = \emptyset$ if $k > n$.

We claim $\mathcal{E}_k[\mathcal{B}]$ has EGF $B(x)^k/k!$. Indeed, we see for any finite set $X = [n]$, we have $\mathcal{E}_k[\mathcal{B}]_X$ contains elements of the form $\{B_1, \ldots, B_k\}$ where $B_1, \ldots, B_k$ are $\mathcal{B}$-structures. On the other hand, we see there are $k!$ different permutations for us to get from $(B_1, \ldots, B_k)$ to $\{B_1, \ldots, B_k\}$, where the former one is actually given by $\mathcal{B}^k$. Viz, we see $|\mathcal{E}_k[\mathcal{B}]_X| = |\mathcal{B}^k_X|/k!$. Thus we get the EGF $B(x)^k/k!$ as desired. To conclude $\mathcal{E}[\mathcal{B}]$ has $\exp(B(x))$ as EGF, we just note $\sum_{k\geq 0} \mathcal{E}_k[\mathcal{B}]$ has EGF $\sum_{k\geq 0} \frac{B(x)^k}{k!}$.

Now to the general case where $\mathcal{A}$ is any species. We observe we have $|\mathcal{A}[\mathcal{B}]_X| = \sum_{k\geq 0}^{\infty} |\mathcal{E}_k[\mathcal{B}]_X| \cdot |\mathcal{A}_k|$ because it contains all possible partitions of X, with length from $k = 0$ to ∞, and with length k we have a k-set of $\mathcal{B}$-structures. Thus we get

$$\sum_{n\geq 0} |\mathcal{A}[\mathcal{B}]_n| \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{k\geq 0} |\mathcal{E}_k[B]_n||\mathcal{A}_k| \frac{x^n}{n!} = \sum_{k\geq 0} |\mathcal{A}_k| \sum_{n\geq 0} |\mathcal{E}_k[\mathcal{B}]_n| \frac{x^n}{n!} = A(B(x))$$

♠

Example 3.4.29. In the definition of composition $\mathcal{A} \circ \mathcal{B}$, we said $\mathcal{B}$ need to be connected. If $\mathcal{B}$ is not connected, then the EGF may not be defined. However, the problem with composition is deeper than this "undefined-ness". To see this, find an example so that:

1. $\mathcal{B}$ is not connected.
2. The composition of EGFs $\mathcal{A}(\mathcal{B})$ is defined.
3. There exists n so that $[\frac{x^n}{n!}]A(B(x))$ is not an integer.

Solution. Consider the species $\mathcal{B} = \mathcal{C} + \mathcal{E}_0$ with EGF $\log(\frac{1}{1-x}) + 1$. Then consider $\mathcal{E}_5$ and $\mathcal{B}$. Then we have

$$E_5(B(x)) = \frac{B(x)^5}{5!}$$

The Taylor expansion at $x = 0$ is given by

$$E_5(B(x)) = \frac{1}{120} + \frac{x}{24} + \frac{5}{48}x^2 + \frac{13}{72}x^3 + O(x^4)$$

Thus we see $[\frac{x^3}{3!}]E_5(B(x)) = 3!\frac{13}{72} = \frac{13}{12}$ is not integer. ♠

Example 3.4.30. Compute the first few terms of the formal power series $G^c(x)$, using $G^c(x) = \log G(x)$. Verify $[\frac{x^n}{n!}] \log G(x)$ is the number of connected graphs on vertex set $[n]$, for $n = 1, 2, 3$.

Solution. We see $\log(G(x)) = \log(1 + (G(x) - 1))$ where

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$$

Thus $\log(G(x)) = (G(x) - 1) - \frac{(G(x)-1)^2}{2} + \frac{(G(x)-1)^3}{3} + O(x^4)$ and we can compute this. ♠

Example 3.4.31. Verify the two formulas for the number of surjective functions $[m] \to [k]$ give the same answer.

Solution. On the one hand we have

$$B(x) = e^{-x} \sum_{n\geq 0} \frac{n^m}{n!} \frac{x^n}{1} = \sum_{k\geq 0} \left(\sum_{n+l=k} (-1)^l \frac{1}{l!} \frac{n^m}{n!} \right) x^k$$

Thus we see

$$[x^k]B(x) = \sum_{n=0}^{k} (-1)^{k-n} \frac{n^m}{n!(k-n)!}$$

and so

$$[\frac{x^k}{k!}] \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} n^m$$

Note this is just the Stirling number of second kind $S(m,k)$. On the other hand, note

$$(e^x - 1)^k = \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} e^{nx}$$

where $e^{nx} = \sum_{t\geq 0} \frac{n^t x^t}{t!}$ and hence

$$[x^m](e^x - 1)^k = [x^m] \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} \sum_{t\geq 0} \frac{n^t x^t}{t!}$$

This is equal to $\sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} n^m \frac{1}{m!}$. Thus we have

$$[\frac{x^m}{m!}](e^x - 1)^k = \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} n^m = S(m,k)$$

♠

Example 3.4.32.

1. Use decomposition $\mathcal{D} * \mathcal{E} \equiv \mathcal{S}$ to compute the EGF of $\mathcal{D}$.
2. Use decomposition $\mathcal{D} \equiv \mathcal{E}[\mathcal{C} - \mathcal{C}_1]$ to compute the EGF of $\mathcal{D}$.

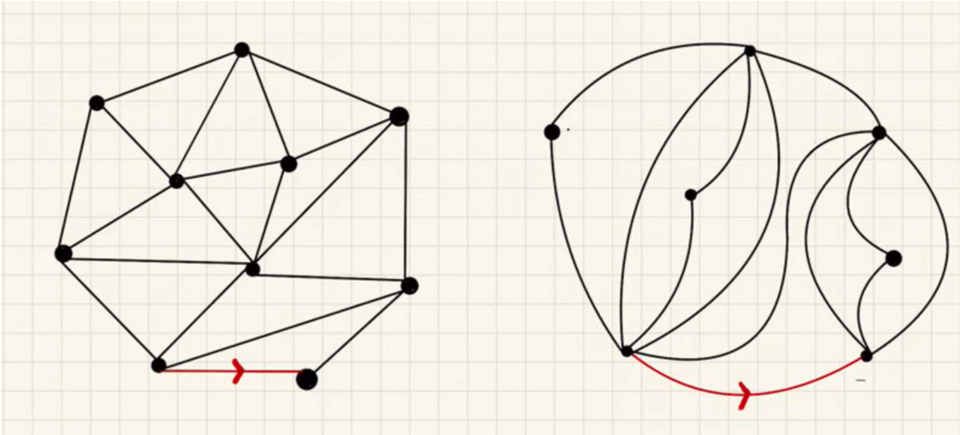
Solution. The first one gives us $D = \frac{e^{-x}}{1-x}$. The second one gives us $D = e^{\log(\frac{1}{1-x})-x} = \frac{e^{-x}}{1-x}$. ♠

Example 3.4.33. For these exercises, let $\mathcal{A}$ and $\mathcal{B}$ be species from the trivalent trees example.

1. Compute $|\mathcal{B}_{2n-1}|$.
2. Note a trivalent tree with $2n$ vertices has exactly $n+1$ leaves. Use this to show $(n+1)|\mathcal{A}_{2n}| = 2n|\mathcal{B}_{2n-1}|$.
3. Compute the number of trees with vertex set $[3n+2]$, such that every vertex has degree 1 or 4.
4. Use the equivalence $\mathcal{N} \equiv \mathcal{S} \circ \mathcal{T}^\bullet$ to prove $N(x)^{-1} = 1 - \sum_{n\geq 1} n^{n-1} \frac{x^n}{n!}$.

3.5 Tutorial 3 (Rooted Triangulations)

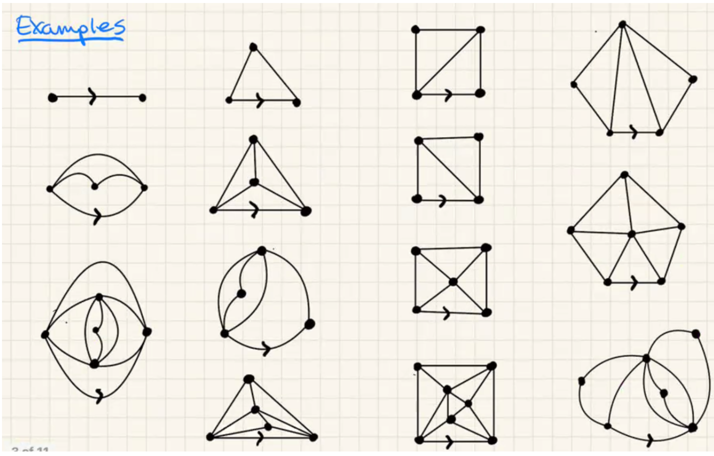
Example 3.5.1. The following is a rooted triangulation:



Remark 3.5.2. The rules for rooted triangulation:

1. They are planar drawings up to isotopy.
2. All internal faces have degree 3 (outer face has degree ≥ 2).
3. Parallel edges allowed.
4. No loops.
5. No cut vertices.
6. Root edge γ, oriented counter-clockwise on outer face.

Example 3.5.3. More examples here:



Definition 3.5.4 (Notations For This Section). We let $\mathcal{T}$ be all the rooted triangulations and $\mathcal{T}_k$ be the triangulations that has outer face with degree k. Then we put two weight functions wt_1 as the number of edges and wt_2 as the degree of outer face. Thus we get two generating functions

$$T(x, y) := \sum_{\tau \in \mathcal{T}} x^{wt_1(\tau)} y^{wt_2(\tau)}$$

and

$$T_k(x) = \sum_{\tau \in \mathcal{T}_k} x^{wt_1(\tau)} = [y^k]T(x, y)$$

Example 3.5.5. Give a recursive decomposition of $\mathcal{T}$.

Solution. We have three cases:

1. The base case, we only have one edge.
2. After removing the rooted edge we get a cut vertex, i.e. $\tau \mapsto \{\gamma\} \times \{\tau_1\} \times \{\tau_2\}$.
3. After removing the rooted edge we still get a triangulation (hence we note $\mathcal{T}_2$ are not inside this). In this case we have two rules to label the new root and hence we need to make a choice.

Thus we get the following decomposition:

$$\mathcal{T} \Leftrightarrow \{\gamma\} \cup \{\gamma\} \times \mathcal{T}^2 \cup \{\gamma\} \times (\mathcal{T} \backslash \mathcal{T}_2)$$

♠

Example 3.5.6. Use this decomposition to obtain an OGF equation.

Solution. By the above, we see we have

$$T(x,y) = xy^2 + xy^{-1}T(x,y)^2 + xy^{-1}(T(x,y) - y^2T_2(x))$$

Now rewrite the above equation we get

$$xT^2(x-y)T + xy^2(y - T_2(x)) = 0$$

However, we have a $T_2(x)$ inside our expression and how do we deal with this?

Recall if R is a ring and $a, b, c, z \in R$, we see $az^2 + bz + c = 0$ implies $b^2 - 4ac$ must be a square. Thus we see by the above functional equation we get

$$(x-y)^2 - 4x^2y^2(y - T_2(x)) \text{ is a square}$$

♠

Theorem 3.5.7. *If $G(x,y) = F(x,y)^2 \in \mathbb{Q}[[x,y]]$, let $G'(x,y) = \frac{\partial}{\partial y}G(x,y)$ and $G''(x,y) = \frac{\partial^2}{\partial y^2}G(x,y)$. Suppose $G(0,0) = 0$ and $G''(0,0)$ is invertible. Then there exists unique $g(x) \in \mathbb{Q}[[x]]_+$ such that*

$$G(x, g(x)) = G'(x, g(x)) = 0$$

Proof. Since $G(x,y) = F(x,y)^2$ we see $F(0,0) = 0$. However note $G'(x,y) = 2F(x,y)F'(x,y)$ and hence $G''(x,y) = 2F'(x,y)^2 + 2F(x,y)F''(x,y)$. Since by assumption $G''(0,0)$ is invertible we must have $2F'(x,y)^2 + 0$ is invertible and hence $F'(x,y)$ is invertible. Thus use Hensel's lemma on F we get $g(x)$ such that $F(x, g(x)) = 0$. This $g(x)$ will be the one we desired. ♡

Example 3.5.8. Prove $y^2 + x^4$ is not a square in $\mathbb{C}[[x,y]]$.

Solution. Suppose $G(x,y)$ is a square. Then we see $G''(x,y) = 2$ and hence it is invertible. Thus we must have the above theorem holds. However, $g(x)^2 + x^4 = 0$ and $2g(x) = 0$ cannot have a solution at the same time and hence it cannot be a square. ♠

Example 3.5.9. Find the unique series $T_2(x)$ such that $(y-x)^2 - 4x^2y^2(y-T_2(x))$ is a square in $\mathbb{Q}[[x,y]]$.

Solution. Let $G(x,y) = (y-x)^2 - 4x^2y^2(y-T_2(x))$ and since it is a square, by the above theorem there exists $g(x)$ such that $G(x,g(x)) = G'(x,g(x)) = 0$. Now solve for g and we will see $g(x) = x\phi(g(x))$ for some ϕ and thus use LIFT, then we get $T_2(x) = f(g(x))$ for some $f(x)$. Hence the proof follows. ♠

3.6 Tutorial 4 (Species Over Species)

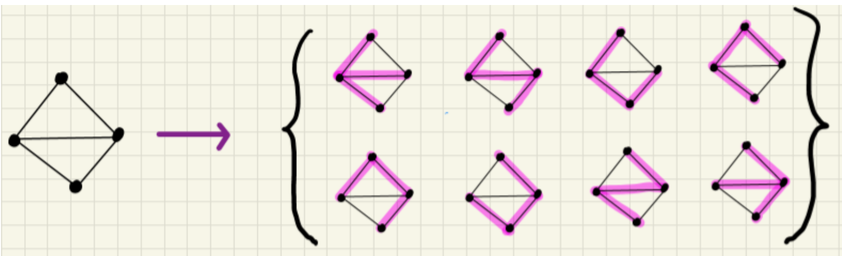
Definition 3.6.1. Let $\mathcal{H}$ be a species, a ***species over*** $\mathcal{H}$ or ($\mathcal{H}$-species) is a pair $(\mathcal{A}, \Phi)$ where $\mathcal{A}$ is a species and Φ is a natural transformation.

Remark 3.6.2. How it works:

1. To any $\mathcal{H}$-structure $\lambda \in \mathcal{H}_\lambda$, it assigns that set $\mathcal{A}_\lambda := \{\alpha \in \mathcal{A}_X : \Phi(\alpha) = \lambda\}$. This is the $\mathcal{A}$-structure on λ.
2. If $\lambda \in \mathcal{H}_X, \mu \in \mathcal{H}_Y$ and $f : X \to Y$ is an isomorphism, we get a bijection $f_* : \mathcal{A}_\lambda \to \mathcal{A}_\mu$. This is the transportation of $\mathcal{A}$-structure.

Remark 3.6.3 (Important Intuition). If $\mathcal{A}$ is $\mathcal{H}$-species, you should imagine $\mathcal{A}$ no longer accepts sets as input. It only accepts $\mathcal{H}$-structures.

Example 3.6.4. Consider $\mathcal{H}$ as connected graphs, then $\mathcal{A}_X = \{(\Gamma, \tau)\}$ where $\Gamma \in \mathcal{H}_X$ and τ is spanning tree of Γ.



Remark 3.6.5. Which concepts/definitions still make sense for $\mathcal{H}$-species in general?

Still Good

Equivalence

Sum/Difference

Superposition

$(\mathcal{A}\boxtimes\mathcal{B})_\lambda = \mathcal{A}_\lambda \times \mathcal{B}_\lambda$

Rooting

$\mathcal{A}^\bullet_\lambda = \mathcal{A}_\lambda \times X$

Maybe not.

Product

$(\mathcal{A}\cdot\mathcal{B})_X = \coprod_{S\subseteq X} \mathcal{A}_S \times \mathcal{B}_{X\setminus S}$

Sequence.

Derivative

$\mathcal{A}'_X = \mathcal{A}_{X\sqcup\{*\}}$

Composition

EGF

Example 3.6.6 (Linear Species/$\mathcal{L}$-species)**.** However, in the case when $\mathcal{H} = \mathcal{L}$, most of the things work out:

1. Inputs are ordered lists $\lambda = (x_1, ..., x_n)$ of distinct elements.
2. Order of the elements can be incorporated into definitions of structures.
3. Can also think of these as totally ordered finite sets $x_i \leq x_j \Leftrightarrow i \leq j$.
4. Canonical isomorphism with $(1, 2, 3, ..., n)$.

The intuition should be, we can always imagine the input is $(1, 2, 3, ..., n)$ for some $n \in \mathbb{N}$.

Example 3.6.7. Examples of linear species:

1. Alternating permutations:
$$\mathcal{A}_{(1,2,...,n)} = \{\sigma \in S_n : \sigma(1) < \sigma(2), \sigma(2) > \sigma(3), \sigma(3) < \sigma(4), ...\}$$
2. Increasing endofunctions:
$$\mathcal{A}_{(1,2,...,n)} = \{\alpha : [n] \to [n] : \alpha(i) \geq i\}$$

Example 3.6.8 (Operations with Linear Species)**.** Let $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-species. Then:

1. EGF will be
$$A(x) = \sum_{n\geq 0} |\mathcal{A}_{(1,2,...,n)}| \frac{x^n}{n!}$$
2. Product: Suppose $\lambda = (x_1, ..., x_h) \in \mathcal{L}_X$ and $S \subseteq X$. Then $\lambda|_S = (x_{i_1}, ..., x_{i_k})$ where $S = \{x_{i_1}, ..., x_{i_k}\}$. Then we can define products as
$$(\mathcal{A} * \mathcal{B})_\lambda = \coprod_{S\subseteq X} \mathcal{A}_{\lambda|_S} \times \mathcal{B}_{\lambda|_{X\setminus S}}$$
3. Derivatives: $\lambda = (x_1, ..., x_n)$, then we could define
$$\mathcal{A}'_\lambda = \mathcal{A}_{*,x_1,...,x_n}$$
or
$$\mathcal{A}'_\lambda = \mathcal{A}_{x_1,...,x_n,*}$$

Both works, but they are not the same!! Be sure to specify which one you are using here. We will work with the first one.

4. Composition: $\mathcal{E}$-species composed with $\mathcal{L}$-species is an $\mathcal{L}$-species.

With all the operations defined, the corresponding EGF operations are still true.

Example 3.6.9. Let $\mathcal{I}$ be the $\mathcal{L}$-species of "increasing rooted trees". Compute $I(x)$.

The decomposition is delete the root (this is the derivative of $\mathcal{I}$) and we get a set of increasing rooted trees. In other word, we get

$$\mathcal{I}' \equiv \mathcal{E}[\mathcal{I}]$$

and we get

$$\frac{d}{dx} I(x) = \exp(I(x))$$

This is a differential equation and we are done.

Chapter 4

Multivariate Generating Functions

4.1 Mixed Generating Functions

Definition 4.1.1. Let $\mathcal{A}$ be a species, a ***weight function*** $wt : \mathcal{A} \to \mathbb{N}$ is a rule that assigns to every $\mathcal{A}$-structure α a weight $wt(a) \in \mathbb{N}$ such that if α, β are isomorphic $\mathcal{A}$-structures then $wt(a) = wt(b)$.

Remark 4.1.2.

1. Weight functions on species are not functions from set theory because $\mathcal{A}$ is not a set. Nonetheless, its restriction to $\mathcal{A}_X$ is a indeed a weight function from $\mathcal{A}_X \to \mathbb{N}$.
2. Thus, it is not hard to see, a weight function $wt : \mathcal{A} \to \mathbb{N}$ is the same as specifying a weight function $\mathcal{A} \to \mathbb{N}$.
3. We will no longer require $\mathcal{A}_X$ to be finite.

Example 4.1.3.

1. For species of graphs, we can define a weight as the number of edges.
2. For trees, a weight is the number of leaves.
3. For endofunctions, a weight is the number of fixed points.
4. For permutations, a weight is the number of cycles.
5. For any species, a weight is the order itself.
6. For any species, a weight is the constant weight.

Definition 4.1.4. A ***weighted species*** is a species $\mathcal{A}$ with one or more weight functions $wt_1, ..., wt_n, ...$

Remark 4.1.5. For any finite set X, we can form OGF for $\mathcal{A}_X$ as

$$A_X(t) = A_X(t_1, t_2, ...) = \sum_{\alpha \in \mathcal{A}_X} t_1^{wt_1(\alpha)} t_2^{wt_2(\alpha)} ...$$

If $|X| = |Y| = n$ then $A_X(t) = A_Y(t\Psi$ and hence we will just write $A_n(t)$ for $|X| = n$.

Definition 4.1.6. Let $\mathcal{A}$ be a weighted species, then the ***mixed generating function*** (MGF) of $\mathcal{A}$ is

$$A(x;t) = \sum_{n\geq 0} A_n(t)\frac{x^n}{n!}$$

Example 4.1.7. Let $\mathcal{Y}$ be the species of graphs and weight functions to be the number of edges. Then we see $G_n(t) = (1+t)^{\binom{n}{2}}$ and hence

$$G(x;t) = \sum_{n\geq 0}(1+t)^{\binom{n}{2}}\frac{x^n}{n!}$$

Definition 4.1.8. Let $\mathcal{A}, \mathcal{B}$ be weighted species with weight functions $wt : \mathcal{A} \to \mathbb{N}$ and $wt' : \mathcal{B} \to \mathbb{N}$. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a natural transformation. We say Φ is ***weight preserving*** if $wt'(\Phi(\alpha)) = wt(\alpha)$ for any $\mathcal{A}$-structure α. If Φ is a weight preserving equivalence then $A(x;t) = B(x;t)$.

Remark 4.1.9. We can also extend species operations to weighted species using the principle: The weight of a composite object shoulded be the sum of the weight of its components.

Example 4.1.10. For example:

1. Consider $\mathcal{A} * \mathcal{B}$. Then the structures are (α, β) and hence the natural weight function on $\mathcal{A} * \mathcal{B}$ should be $wt''(\alpha, \beta) = wt(\alpha) + wt'(\beta)$.
2. Consider $\mathcal{A} \circ \mathcal{B}$. The structures are tuples $(\alpha, \beta_1, ..., \beta_k)$. Thus the weight should be $wt(\alpha) + \sum wt'(\beta_i)$.
3. For rooting $\mathcal{A}^\bullet$, the structure is (α, x) where α is a $\mathcal{A}$-structure and x is the root. Thus the weight function on $\mathcal{A}^\bullet$ is $(\alpha, x) \mapsto wt(\alpha)$.

Theorem 4.1.11 (Main Theorem). *The main theorem is true for MGFs/weighted species.*

Remark 4.1.12. A few things to note:

1. Composition is in x variable. In other word, the composition should be $A(B(x;t);t)$.
2. Derivatives are now become partial derivatives.
3. $\mathcal{A}^*$ no longer require $\mathcal{A}$ to be connected. Provided the mixed generating function is defined. On the other hand, $\mathcal{A} \circ \mathcal{B}$ still requires $\mathcal{B}$ connected.

Example 4.1.13. Compute the average number of cycles in a permutations of $[n]$. Take $\mathcal{S}$ with weight as the number of cycles, $\mathcal{E}$ with constant weight function 0, and $\mathcal{C}$ with constant weight function 1.

With these weight functions, we see $\mathcal{S} \equiv \mathcal{E}(\mathcal{C})$ becomes a weighted equivalent. Then we have

$$S(x;t) = \exp(C(x;t)) = \exp(t\log(\frac{1}{1-x})) = (1-x)^{-t}$$

In particular, we see the average number of cycles is then

$$\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_{[n]}} wt(\sigma) = \frac{1}{n}\left(\frac{d}{dt}S_n(t)\right)|_{t=1}$$

$$= [x^n]\frac{\partial}{\partial t}S(x;t)|_{t=1}$$

$$= [x^n]\log(\frac{1}{1-x})\frac{1}{1-x} = \sum_{k=1}^{n}\frac{1}{k}$$

Remark 4.1.14. In future, if the weight on a species was not specified, then it was assumed to be 0.

Example 4.1.15. Compute the average number of k-cycles among all permutations of $[n]$. Take $\mathcal{S}$ with weight function equal the number of k-cycles and $\mathcal{C}$ with weight 1 if the order is k and 0 otherwise.

Again, we have $\mathcal{S} \equiv \mathcal{E}(\mathcal{C})$ is a weighted equivalence. This time, we have

$$C(x;t) = x + \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{tx^k}{k} + \frac{x^{k+1}}{k+1} + ...$$

It is not hard to see this is the same as

$$C(x;t) = \log(\frac{1}{1-x}) + (t-1)\frac{x^k}{k}$$

and we can continue as before, and get the final answer to be $\frac{1}{k}$.

Example 4.1.16. If $\alpha : X \to X$ is an endofunction, we say $x \in X$ is a recurrent element, if $\alpha \circ \alpha ... \circ \alpha(x) = x$ where we composed k times for some $k \geq 1$.

Let $\mathcal{N}$ be endofunctions with weight equal the number of recurrent elements. We want to compute $N(x;t)$. Take $\mathcal{S}$ with weight as the order, then $S(x;t) = \frac{1}{1-tx}$. Then $\mathcal{N} \equiv \mathcal{S}(\mathcal{T}^\bullet)$ is a weighted equivalence and so

$$N(x;t) = S(T^\bullet(x);t) = \frac{1}{1-tT^\bullet(x)}$$

Since $T^\bullet(x) = xe^{T^\bullet(x)}$ and hence use LIFT, we get

$$N(x;t) = 1 + \sum_{n\geq 1}\sum_{k=1}^{n}\frac{kn^{n-k-1}}{(n-k)!}t^kx^n$$

Example 4.1.17. Consider $\mathcal{T}^\bullet$ with weight as the number of terminals. Consider $\mathcal{E}$ with weight function 1 if order is 0 and 0 otherwise. Then $E(x;t) = e^x + t - 1$ and $\mathcal{T}^\bullet \equiv \mathcal{X} * \mathcal{E}(\mathcal{T}^\bullet)$ becomes weighted equivalent. Thus we have

$$T^\bullet(x;t) = x(e^{T^\bullet(x;t)} + t - 1)$$

and use LIFT from here to obtain the solution.

Example 4.1.18. Consider rooted trees with weight function the degree of the root. In this example, we write $\mathcal{A}$ a species with weight function 0 and $\overline{\mathcal{A}}$ the same species with non-trivial weight.

In this example, we will consider $\mathcal{T}^\bullet$ and $\overline{\mathcal{T}^\bullet}$ with weight as the degree of the root. On the other hand, we put a non-trivial weight on $\mathcal{E}$ as $\overline{\mathcal{E}}$ with weight as the order.

We know $\mathcal{T}^\bullet \equiv \mathcal{X} * \mathcal{E}[\mathcal{T}^\bullet]$ and $\overline{\mathcal{T}^\bullet} \equiv \mathcal{X} * \overline{\mathcal{E}}(\mathcal{T}^\bullet)$. Thus $T^\bullet(x) = xe^{T^\bullet(x)}$ and

$$\overline{T^\bullet}(x;t) = x\exp(tT^\bullet(x))$$

Use LIFT, we get

$$\overline{T^\bullet}(x;t) = 1 + \sum_{n\geq 1}\sum_{k=1}^{n} \frac{1}{k!}\frac{kn^{n-k-1}}{(n-k)!}t^k x^{n-k}$$

4.2 Tutorial 5 (Averages)

Remark 4.2.1. Let $\mathcal{S}$ be a finite set of combinatorial objects with weight function $\phi : \mathcal{S} \to \mathbb{N}$. Then we see the average number of ϕ is

$$avg(\phi) = \frac{1}{S(1)}S'(1)$$

where $S(x)$ is the OGF of ϕ and $\mathcal{S}$.

Remark 4.2.2 (Elementary Method)**.** Suppose $\phi(s)$ counts the number of widgets in s, let W be the set of all possible widgets. If $s \in \mathcal{S}, w \in W$, then $\delta_{s,w} = 1$ if object s contains widget w and 0 otherwise. Thus we see

$$avg(\phi) = \frac{1}{|\mathscr{S}|}\sum_{s\in\mathcal{S}}\sum_{w\in W}\delta_{s,w} = \sum_{w\in W}\frac{1}{|\mathcal{S}|}\sum_{s\in\mathcal{S}}\delta_{s,w}$$

Thus we have two interpretations:

1. $avg(\phi)$ is $\sum_{w\in W}\frac{\text{number of objects containing } w}{\text{total number of objects}}$.
2. $avg(\phi)$ is $\sum_{w\in W}\mathbb{P}(w)$ where $\mathbb{P}(w)$ is the probability that w is contained in a randomly (uniform distribution) chosen object.

More generally, we can assign different probabilities to choosing objects.

Example 4.2.3. Let B_n be the set of 01-strings of length n with $n \geq 3$. For $\sigma \in B_n$, let $\phi(\sigma)$ equal the number of occurrences of 111 as a substring. Compute $avg(\phi)$.

Solution. The widgets in this case is $W = \{111----..., -111----..., --111----..., ..., ... ------111\}$. For each $w \in W$, what is $\mathbb{P}(w)$? It is just $\frac{1}{8}$. Thus we see the average $avg(\phi)$ is just $|W| \cdot \frac{1}{8} = (n-2)\frac{1}{8}$. ♠

Example 4.2.4. Let $\mathcal{G}_X$ be all graphs with vertex set X. For $\Gamma \in \mathcal{G}_X$ let $\phi(\Gamma)$ be the number of spanning trees of Γ, compute $avg(\phi)$.

Solution. Our set of widgets W is all trees with n vertices and we are looking for the probabilities Γ is having spanning tree of that tree. Thus $\mathbb{P}(w) = \frac{1}{2^{n-1}}$. ♠

Example 4.2.5. Let S_n be permutations of $[n]$ and fix $k \leq n$. For $\sigma \in S_n$, let $\phi(\sigma)$ be the number of k-cycles in σ. Compute $avg(\phi)$.

Solution. W in this case will be all possible k-cycles. Then $|W| = \frac{n!}{k(n-k)!}$ and $\mathbb{P}(w) = \frac{(n-k)!}{n!}$ and thus the average is $\frac{1}{k}$. ♠

Example 4.2.6. On each square of an n by n chess board you place a stone with probability p. What is the expected number of pairs of adjacent stones?

Solution. The widgets have $2n(n-1)$ elements and each has probability p^2. ♠

Remark 4.2.7. In the preceding examples, probabilities were same for all $w \in W$. When this is not the case, we can sometimes compute $avg(\phi)$ using generating function for w.

If there exists a weight function $wt : W \to \mathbb{N}$ and $0 < p < 1$, such that $\mathbb{P}(w) = p^{wt(w)}$, then

$$avg(\phi) = \sum_{w \in W} p^{wt(w)} = W(p)$$

where $W(x)$ is the OGF of W.

Example 4.2.8. Let $\mathcal{G}_X$ be the set of graphs with vertex set X. Let $\phi_X : \mathcal{G}_X \to \mathbb{N}$, $\phi_X(\Gamma)$ be the number of cycles in Γ.

Solution. The widgets are graphs where 1 component is a cycle and the rest are isolated vertices. Then $\mathbb{P}(w) = \frac{1}{2^{\ell(w)}}$ where $\ell(w)$ is the length of the cycle. Thus $p = \frac{1}{2}$ in this case.

Now we want to compute the mixed generating function $W(x;t)$ hence find the average. Note $W \equiv C * \mathcal{E}$ where C is species of cycles $C(x) = \sum_{n\geq 3} \frac{(n-1)!}{2} \frac{x^n}{n!}$. Thus for weight function, it is just equal the order and the mixed generating function is just $W(x,t) = C(xt)e^x$. ♠

4.3 Multi-Exponential Generating Functions

Definition 4.3.1. Let $\mathcal{A}$ be a set and R be a commutative ring. Then we define ***generalized weight function*** $Wt : \mathcal{A} \to R$. We define $|\mathcal{A}|_{Wt} = \sum_{a\in\mathcal{A}} Wt(a)$.

Example 4.3.2. If $wt : \mathcal{A} \to \mathbb{N}$ as a normal weight function, then let $R = \mathbb{Q}[[x]]$ and $Wt(\alpha) := x^{wt(\alpha)}$. Then $|\mathcal{A}|_{Wt} = A(x)$ is just the OGF of $\mathcal{A}$ with weight wt.

Remark 4.3.3. Under operations, our principle for the generalized weight of a composite object is that the weight should be the product of weights.

Example 4.3.4. Let $R = \mathbb{Q}$, consider weight $sgn : \mathcal{A} \to \{\pm 1\}$. Then $|\mathcal{A}|_{sgn} = |\{a \in \mathcal{A} : sgn(a) = 1\}| - |\{a \in \mathcal{A} : sgn(a) = -1\}|$.

Definition 4.3.5. Let $\mathcal{A}$ be a species and R a ring, we can define ***generalized weight function*** $Wt : \mathcal{A} \to R$ to be a rule that assigns $Wt(a) \in R$ for every $\mathcal{A}$-structure a. We want $Wt(a) = Wt(b)$ if a is isomorphic to b. Thus we will write $|\mathcal{A}_n|_{Wt} := |\mathcal{A}_X|_{Wt}$ for any set X of size n.

Definition 4.3.6. Given species $\mathcal{A}$ and generalized weight function Wt. Then we can define the ***generalized EGF*** to be

$$\sum_{n \geq 0} \frac{1}{n!} |\mathcal{A}_n|_{Wt} x^n$$

or

$$\sum_{n \geq 0} \frac{1}{n!} |\mathcal{A}_n|_{Wt}$$

Definition 4.3.7. We define ***multisort species***, it is like species, but input is a tuple of finite sets. We will focus on 2-sort species (or 2-species).

A 2-***species*** $\mathcal{A}$ is a rule which assigns:

1. To every ordered pair (X_1, X_2) of finite sets, we get a finite set $\mathcal{A}_{X_1,X_2}$ the ***$\mathcal{A}$-structures*** on (X_1, X_2).
2. To every pair of bijections $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ a bijection $(f_1, f_2)_* : \mathcal{A}_{X_1,X_2} \to \mathcal{A}_{Y_1,Y_2}$ called the ***transportation of structures*** such that.

Remark 4.3.8. We can also define isomorphisms, natural transformations and equivalence. The first big difference between 2-species and species is when we look at EGF.

Like before, we will write $|\mathcal{A}_{n,m}| = |\mathcal{A}_{X_1,X_2}|$ where $|X_1| = n$ and $|X_2| = m$. Then the EGF for 2-species is given by

$$A(x_1, x_2) = \sum_{m,n \geq 0} |\mathcal{A}_{n,m}| \frac{x_1^n}{n!} \frac{x_2^m}{m!}$$

Example 4.3.9.

1. Consider $\mathcal{F}$ as the 2-species of functions, where $\mathcal{F}_{X,Y}$ is the set of all functions from X to Y.
2. The 2-species of vertex and edges-labelled graphs. This assigns (X, Y) the set of all (Γ, γ) where Γ is a graph with vertex set X and γ is a bijection from the edges of the graph to Y.
3. $\Pi^{(2)}$, the 2-species of 2-set partitions. A 2-set partition of (X, Y) is a set of pairs $\{(S_1, T_1), ..., (S_k, T_k)\}$ where $X = S_1 \coprod ... \coprod S_k$ and $Y = T_1 \coprod ... \coprod T_k$ and $S_i \cup T_i \neq \emptyset$.

4. $\mathcal{X}_1, \mathcal{X}_2$ is the 2-species of singletons of 1st and 2nd sort. The definition is:

$$(\mathcal{X}_1)_{X_1,X_2} = \begin{cases} \{X_1\}, & \text{if } |X_1| = 1, |X_2| = \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

Similarly, we define

$$(\mathcal{X}_2)_{X_1,X_2} = \begin{cases} \{X_2\}, & \text{if } |X_1| = 0, |X_2| = 1 \\ \emptyset, & \text{otherwise} \end{cases}$$

Remark 4.3.10 (Diagrammatic perspective of 2-sort species)**.**

The unlabelled structures have "two sorts" of label receivers. The first sort receives labels from X_1 and the second receives labels from X_2.

For example, if we consider $\mathcal{F}$ as the 2-species of functions, then the unlabelled structures are two line of dots with some arrow from the first line to second. Then the first line would receive labels from X and second line receive labels from Y.

Remark 4.3.11. Now we talk about operations on 2-species. Suppose $\mathcal{A}, \mathcal{B}$ are two 2-species.

Definition 4.3.12 (Sums/Differences)**.** The sum and difference are more or less the same:

$$(\mathcal{A} + \mathcal{B})_{X,Y} = \mathcal{A}_{X,Y} \coprod B_{X,Y}$$

Similarly if $\mathcal{B}$ is a subspecies of $\mathcal{A}$ then $(\mathcal{A} - \mathcal{B})_{X,Y} = \mathcal{A}_{X,Y} \setminus \mathcal{B}_{X,Y}$.

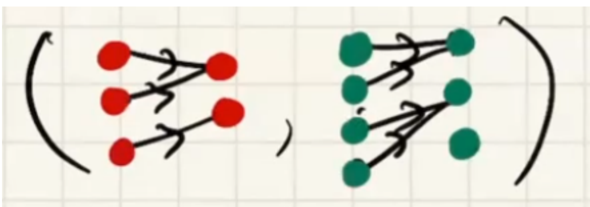
Remark 4.3.13. The filter operation is more or less the same and hence ommited.

Definition 4.3.14 (Products)**.**

1. If R is a set, then $(R \times \mathcal{A})_{X,Y} = R \times \mathcal{A}_{X,Y}$.
2. $$(\mathcal{A} \boxtimes \mathcal{B})_{X,Y} = \mathcal{A}_{X,Y} \times \mathcal{B}_{X,Y}$$
3. $$(\mathcal{A} * \mathcal{B})_{X,Y} = \coprod_{S \subseteq X} \coprod_{T \subseteq Y} \mathcal{A}_{S,Y} \times \mathcal{B}_{X \setminus S, Y \setminus T}$$

We note that the unlabelled structure is just $\tilde{\mathcal{A}} \times \tilde{\mathcal{B}}$ for $\mathcal{A} * \mathcal{B}$. Also, we note the multi-EGF of $\mathcal{A} * \mathcal{B}$ is just $A(x_1, x_2)B(x_1, x_2)$.

Example 4.3.15. Consider $\mathcal{F} * \mathcal{F}$ as the product of functions. We can think of this as the following:



For structures on $\mathcal{F} * \mathcal{F}$, we can view it as a single function ϕ where each elements of the domain/codomain is coloured red/green and $\phi(x)$ has the same colour as x.

Definition 4.3.16 (Sequences). We just define $\mathcal{A}^* = \sum_{n\geq 0} \mathcal{A}^n$.

Definition 4.3.17 (Derivatives). We have two derivatives $(\partial_1\mathcal{A}, \partial_2\mathcal{A})$.

1. $(\partial_1\mathcal{A})_{X,Y} = \mathcal{A}_{X\coprod\{*\},Y}$ where $* \notin X$.
2. $(\partial_2\mathcal{A})_{X,Y} = \mathcal{A}_{X,Y\coprod\{*\}}$ where $* \notin Y$.

Some remarks:

1. We note the rooting operations are $\mathcal{X}_1 * \partial_1\mathcal{A}$ and $\mathcal{X}_2 * \partial_2\mathcal{A}$ where in the first one we root at elements of the first sort and the second one we root at elements of second sort.
2. We note we can also consider "mark and change sort" operation. This is just $\mathcal{X}_2 * \partial_1\mathcal{A}$ and $\mathcal{X}_1 * \partial_2\mathcal{A}$. The first one is mark and change from 1st sort to 2nd sort and the second is mark and change 2nd to 1st.
3. The multi-EGF for $\partial_1\mathcal{A}$ is $\frac{\partial}{\partial x_1}A(x_1, x_2)$ and $\partial_2\mathcal{A}$ is $\frac{\partial}{\partial x_2}A(x_1, x_2)$.

Definition 4.3.18 (Diagonal). If $\mathcal{A}$ is 2-species, then we define the diagonal $\nabla\mathcal{A}$, which is a 1-species, as

$$(\nabla\mathcal{A})_X = \mathcal{A}_{X,X}$$

Example 4.3.19. We have $\nabla\mathcal{F} = \mathcal{N}$. If $A(x_1, x_2) = \sum_{m,n\geq 0} a_{m,n}\frac{x_1^m}{m!}\frac{x_2^n}{n!}$ then

$$(\nabla A)(x) = \sum_{n\geq 0} a_{n,n}\frac{x^n}{n!}$$

Remark 4.3.20. We can decompose k-species with k-tuples of l-species to a l-species. We will discuss $(k,l) = (1,2)$, $(2,1)$ and $(2,2)$.

Definition 4.3.21 (Composition when $(k,l) = (1,2)$). Let $\mathcal{A}$ be 1-species and $\mathcal{B}$ be connected 2-species, i.e. $B_{\emptyset,\emptyset} = \emptyset$. Then the unlabelled structures is defined by :

1. Start with unlabelled $\mathcal{A}$-structure α.
2. Replace its label receivers by unlabelled $\mathcal{B}$-structures $\beta_1, \beta_2, \ldots$
3. The label receivers of composite object come from β_i's only, which is a 2-sort species.

The the $\mathcal{A}(\mathcal{B})$ labelled structure consists of:

1. $P = \{(S_1, T_1), \ldots, (S_k, T_k)\} \in \Pi^{(2)}_{X,Y}$
2. $\alpha \in \mathcal{A}_P$.
3. $\beta_i \in B_{S_i,T_i}$ for $i = 1, \ldots, k$.

The multi-EGF of $\mathcal{A}[\mathcal{B}]$ is just $\mathcal{A}(B(x_1, x_2))$.

Example 4.3.22. Let $\mathcal{H}$ as the 2-species of vertex and edge labelled graphs and $\mathcal{H}^c$ be 2-species of connected graphs with vertex and edge labelled. Then $\mathcal{H} \equiv \mathcal{E}[\mathcal{H}^c]$.

Example 4.3.23. We can converting 1-species into 2-species: $\mathcal{A}[\mathcal{X}_1], \mathcal{A}[\mathcal{X}_2], \mathcal{A}[\mathcal{X}_1 + \mathcal{X}_2]$. The definition goes as:

$$\mathcal{A}[\mathcal{X}]_{X_1,X_2} = \begin{cases} \mathcal{A}_{X_1}, X_2 = \emptyset \\ \emptyset, X_2 \neq \emptyset \end{cases}$$

$$\mathcal{A}[\mathcal{X}_2]_{X_1,X_2} = \begin{cases} \mathcal{A}_{X_2}, X_1 = \emptyset \\ \emptyset, X_1 \neq \emptyset \end{cases}$$

$$\mathcal{A}[\mathcal{X}_1+\mathcal{X}_2]_{X_1,X_2}=\mathcal{A}_{X_1\coprod X_2}$$

Definition 4.3.24 (Composition when $(k,l)=(2,1)$)**.** Let $\mathcal{A}$ be 2-species, $\mathcal{B}_1,\mathcal{B}_2$ be connected 1-species. Then we can define $\mathcal{A}[\mathcal{B}_1,\mathcal{B}_2]$ as a 1-species.

The unlabelled structure is defined by:

1. Start with unlabelled $\mathcal{A}$-structure α.
2. Replace all label receivers of 1st sort in α by $\mathcal{B}_1$-structures $\beta_1,\beta_2,...$
3. Replace all label receivers of 2nd sort in α by $\mathcal{B}_2$-structures $\gamma_1,\gamma_2,...$
4. The label receivers of composite objects come from β_i's and γ_j's

The $\mathcal{A}[\mathcal{B}_1,\mathcal{B}_2]$-structure on X consists of:

1. $(P_1,P_2)\in(\Pi*\Pi)_X$, i.e. $P_1=\{S_{11},...,S_{1k}\}$ and $P_2=\{S_{21},...,S_{2l}\}$.
2. Then we have $\alpha\in\mathcal{A}_{P_1,P_2}$.
3. We have $\beta_i\in(\mathcal{B}_1)_{S_{1i}}$.
4. And $\gamma_j\in(\mathcal{B}_2)_{S_{2j}}$.

The multi-EGF of $\mathcal{A}[\mathcal{B}_1,\mathcal{B}_2]$ will be $A(B_1(x),B_2(x))$.

Example 4.3.25. We can convert 2-species into 1-species, i.e. we have $\mathcal{A}[\mathcal{X},\mathcal{X}]$, this forgets that there are two sorts of label receivers and treat all the same.

Formally, we have
$$\mathcal{A}[\mathcal{X},\mathcal{X}]_X=\coprod_{S\subseteq X}\mathcal{A}_{S,X\setminus S}$$
In this case the EGF is just $A(x,x)$.

Definition 4.3.26 (Composition when $(k,l)=(2,2)$)**.** Let $\mathcal{A},\mathcal{B}_1,\mathcal{B}_2$ be 2-species, then we define $\mathcal{A}[\mathcal{B}_1,\mathcal{B}_2]$ as a 2-species as follows.

The unlabelled structures is the same as the $(k,l)=(2,1)$ case.

The labelled structures is the same as the $(k,l)=(2,1)$ case but replace $\Pi*\Pi$ by $\Pi^{(2)}*\Pi^{(2)}$. The multi-EGF in this case is just $A(B_1(x_1,x_2),B_2(x_1,x_2))$.

Example 4.3.27. We consider the 2-species of functions. We have natural equivalence
$$\mathcal{F}\equiv\mathcal{E}[\mathcal{X}_2*\mathcal{E}[\mathcal{X}_1]]$$
Indeed, to form a function, for each elements in the codomain, which is $\mathcal{X}_2$, and we need to pair that with a set of elements of the first sort, i.e. $\mathcal{X}_2*\mathcal{E}[\mathcal{X}_1]$. We need to do this to every elements in the codomain, hence we need to put a $\mathcal{E}$ in front. The EGF of $\mathcal{F}$ is then
$$F(x_1,x_2)=\exp(x_2e^{x_1})$$

For surjective functions, we see this is just $\mathcal{E}(\mathcal{X}_2*\mathcal{E}_+(\mathcal{X}_1))$ with similar reasoning. On the other hand, the injective functions is $\mathcal{E}[\mathcal{X}_2*(1+\mathcal{X}_1)]$.

Example 4.3.28. Let X,Y be disjoint finite sets. We want to count the permutations of $X\coprod Y$ with the property that every cycle has at least 1 element of X and 1 element of Y.

Form the 2-species of $\mathcal{C}^{mix}$ of cycles, with at least one label receiver of each sort. We get

$$\mathcal{C}^{mix} \equiv \mathcal{C}[\mathcal{X}_1 + \mathcal{X}_2] - \mathcal{C}[\mathcal{X}_1] - \mathcal{C}[\mathcal{X}_2]$$

where $\mathcal{C}[\mathcal{X}_1 + \mathcal{X}_2]$ is cycles with labels of either sort, and $\mathcal{C}[\mathcal{X}_1] + \mathcal{C}[\mathcal{X}_2]$ are cycles without at least one label of each sort.

Thus we get

$$\begin{aligned}\mathcal{C}^{mix}(x_1, x_2) &= C(x_1 + x_2) - C(x_1) - C(x_2) \\ &= \log(\frac{1}{1 - x_1 - x_2}) - \log(\frac{1}{1 - x_1}) - \log(\frac{1}{1 - x_2})\end{aligned}$$

What we want is $|\mathcal{E}(\mathcal{C}^{mix})|_{x,y}$, which is

$$[\frac{x_1^m}{m!}\frac{x_2^n}{n!}]\exp(C(x_1, x_2))$$

Example 4.3.29. Determine the number of bipartite graphs such that:

1. m vertices in the first part.
2. n vertices in the second part.
3. no vertices of degree 0.

We let $\mathcal{B}$ be the 2-species of bipartite graphs, where $\mathcal{B}_{X,Y}$ is the set of all bipartite graphs with bipartition (X, Y). Let $\mathcal{H}$ be the subspecies of $\mathcal{B}$ with no vertices of degree 0.

Note we have $\mathcal{B} \equiv \mathcal{H} * \mathcal{E}[\mathcal{X}_1 + \mathcal{X}_2]$. From here, we see

$$B(x_1, x_2) = \sum_{m,n \geq 0} 2^{mn}\frac{x_1^m}{m!}\frac{x_2^n}{n!}$$

and hence

$$H(x_1, x_2) = e^{-x_1 - x_2}\sum_{m,n \geq 0} 2^{mn}\frac{x_1^m}{m!}\frac{x_2^n}{n!}$$

Thus the answer to the question is just $[\frac{x_1^m}{m!}\frac{x_2^n}{n!}]H(x_1, x_2)$.

4.4 Exercises 4

Example 4.4.1. Which of the following concepts correspond to weight functions on the species $\mathcal{T}^\bullet$ of rooted trees?

1. the number of terminals.
2. the number of leaves.
3. the label assigned to the root.
4. the degree of the root.
5. length of longest path.

Solution. One and two are the same and they are both weights. Three is not a weight as it may not even map to $\mathbb{N}$. Four is a weight. Five is a weight. ♠

Example 4.4.2. Let $\mathcal{A}$ be a species with EGF $A(x)$.

1. If we endow $\mathcal{A}$ with the "order" weight function (i.e. the weight on $\alpha \in \mathcal{A}_X$ is $|X|$), show the MGF is $A(xt)$.
2. If we endow $\mathcal{A}$ with constant weight k (i.e. the weight on $\alpha \in \mathcal{A}_X$ is k), show the MGF is $t^k A(x)$.

Solution. (1): We see for all $\alpha \in \mathcal{A}_n$ we have $wt(\alpha) = n$. In other word, we have $\sum_{\alpha \in \mathcal{A}_n} t^{wt(\alpha)} = |\mathcal{A}_n| t^n$. By definition we get

$$A(x;t) = \sum_{n \geq 0} |\mathcal{A}_n| t^n \frac{x^n}{n!} = A(xt)$$

(2): We see $\sum_{\alpha \in \mathcal{A}_n} t^{wt(\alpha)} = |\mathcal{A}_n| t^k$. Thus we see the MGF is indeed

$$\sum_{n \geq 0} |\mathcal{A}_n| t^k \frac{x^n}{n!} = t^k A(x)$$

♠

Example 4.4.3. Let $\mathcal{M}$ be species for which $\mathcal{M}_X$ be the set of all functions $X \to \mathbb{N}$. For $\alpha \in \mathcal{M}_X$, define $wt(\alpha) = \sum_{x \in X} \alpha(x)$.

1. Prove $wt : \mathcal{M} \to X$ is a weight function on $\mathbb{N}$.
2. Show $M_n(t) = (1-t)^{-n}$.
3. Show $M(x;t) = \exp(\frac{x}{1-t})$.

Solution. Say α and β are isomorphic. Then we see we have bijection $f : X \to Y$ and $\alpha \circ f^{-1} = \beta$. In particular, observe f is just a permutation of the elements, and hence $\sum_{y \in Y} \beta(y) = \sum_{f^{-1}(y) \in X} \beta(f(y)) = \sum_{x \in X} \alpha(x)$.

Next, observe $M_n(t) = \sum_{\alpha \in \mathcal{M}_{[n]}} t^{wt(\alpha)}$. In other word, for each k we are looking number of functions from $[n]$ that adds up to k. This is the same as count weak compositions of k into n parts. Thus $M_n(t) = (1-t)^{-n}$ as desired. Finally we see $M(x;t) = \sum_{n \geq 0} M_n(t) \frac{x^n}{n!} = \exp(\frac{x}{1-t})$ as desired. ♠

Example 4.4.4. Let $\mathcal{G}$ be the species of graphs, with weight function number of edges. Give an example of a natural equivalence $\Phi : \mathcal{G} \to \mathcal{G}$ that is not a weighted equivalence.

Solution. Graph complement. ♠

Example 4.4.5. Suppose $\mathcal{A}, \mathcal{B}$ are species, and $\Phi : \mathcal{A} \to \mathcal{B}$ is a natural transformation.

1. If $wt : \mathcal{B} \to \mathbb{N}$ is a weight function on $\mathcal{B}$, show there exists unique weight function on $\mathcal{A}$ so Φ is weight preserving.,
2. Give an example to show this does not work in reverse: If $wt : \mathcal{A} \to \mathbb{N}$ is a weight function, there may not exist a weight function on $\mathcal{B}$ such that Φ is weight preserving.

Solution. For $\alpha \in \mathcal{A}_X$, if $\Phi_X(\alpha) = \beta \in \mathcal{B}_X$, then we just define $wt'(\alpha) = wt(\beta)$. This is unqiue and indeed a weight function on $\mathcal{A}$ by definition. This makes Φ weight preserving as desired.

Conversely, if $wt : \mathcal{G} \to \mathbb{N}$ is the number of edges, then we see by the above example using graph complement, we cannot find a weight function on $\mathcal{G}$ which is weight preserving. ♠

Example 4.4.6. Summarize the main theorem for mixed generating function in a table.

Solution. Nah... ♠

Example 4.4.7. If $\mathcal{A}, \mathcal{B}$ are weighted species, write a complete proof of the fact the MGF of $\mathcal{A} * \mathcal{B}$ is $A(x;t)B(x;t)$.

Solution. Observe $(\mathcal{A} * \mathcal{B})_n(t) = \sum_{(a,b) \in \mathcal{A} * \mathcal{B}_n} t^{wt(a)+wt'(b)}$ but $\mathcal{A} * \mathcal{B}_n = \bigcup_{S \subseteq [n]} \mathcal{A}_S \cdot \mathcal{B}_{[n] \setminus S}$. Hence we see the sum is just $\sum_{k=0}^n \binom{n}{k} A_k(t) B_{n-k}(t)$ and the proof follows more or less like the proof for EGF. ♠

Example 4.4.8. Let $\mathcal{A}$ be a weighted species. Show the MGF of $\mathcal{A}^*$ is defined iff $\mathcal{A}$ has no structures of order zero and weight zero.

Solution. We see $\mathcal{A}^*$ should have MGF $\sum_{n \geq 0} A(x;t)^n$. However, we see $A(x;t)^n$ converges iff the $A_n(t)^n$ converges, which happends iff $\mathcal{A}$ has no structure of order zero and weight zero. ♠

Example 4.4.9. Consider $\mathcal{S}$ with weight function "number of cycles". In the first example, we made a weighted equivalence $\mathcal{S} \equiv \mathcal{E}[\mathcal{C}]$, by endowing $\mathcal{C}$ with constant weight 1 and leaving $\mathcal{E}$ unchanged.

There is another way to achieve this. If instead, we keep $\mathcal{C}$ unweighted, what weight function can we put on $\mathcal{E}$ to get the weighted equivalence $\mathcal{S} \equiv \mathcal{E}[\mathcal{C}]$?

Solution. The order weight. ♠

Example 4.4.10. Finish the second example. Show the average number of k-cycles among all permutations of $[n]$ is $\frac{1}{k}$, provided $n \geq k$.

Solution. Well, we already computed $C(x;t) = \log(\frac{1}{1-x}) + (t-1)\frac{x^k}{k}$, thus we get

$$S(x;t) = \exp(C(x;t)) = \frac{\exp((t-1)\frac{x^k}{k})}{1-x}$$

Thus take partial derivative gives us

$$[x^n]\frac{\partial}{\partial t}S(x;t) = [x^n]\frac{1}{1-x}\cdot\frac{x^k}{k}\exp((t-1)\frac{x^k}{k})$$

Now sub $t = 1$ we get

$$[x^n]\frac{1}{1-x}\cdot\frac{x^k}{k} = \frac{1}{k}$$

as desired (assume $n \geq k$ of course). ♠

Example 4.4.11. Verify the LIFT calculation in the third example (endofunctions, weight function number of recurrent elements).

Solution. Zzzzzz... ♠

Example 4.4.12. In each part below, a weight function on the species of endofunctions $\mathcal{N}$ is given. If we want to make $\mathcal{N} \equiv \mathcal{S}[\mathcal{T}^\bullet]$ a weighted equivalence, what weight function should we put on $\mathcal{S}$ and $\mathcal{T}^\bullet$.

1. Order on $\mathcal{N}$.
2. Number of fixed points.
3. Weight function on $\mathcal{N}$ is $wt(\alpha) = |\operatorname{Im}(\alpha)|$.

Solution. For (1) it should just be order on both $\mathcal{S}$ and $\mathcal{T}^\bullet$. For (2) we see fixed points means that cycle should be of length one and otherwise 0. Thus the weight on $\mathcal{S}$ should be the number of fixed points (of the permutation) and 0 on $\mathcal{T}^\bullet$.

For the last one, we see the weight on $\mathcal{S}$ should be order, and the weight on $\mathcal{T}^\bullet$ should be the number of non-leaf minus 1. ♠

Example 4.4.13. Let $\mathcal{M}$ be the weighted species, where $\mathcal{M}$ is set of all functions from $X \to \mathbb{N}$ and the weight is $wt(\alpha) = \sum_{x\in X}\alpha(x)$.

1. Explain why $\mathcal{M} \equiv \mathcal{E}[\mathbb{N} \times \mathcal{X}]$
2. Show if we endow $\mathbb{N}$ with weight $wt(n) = n$, the equivalence is weighted equivalence.
3. Use this to give an alternate derivation of the MGF of $M(x;t)$.

Solution. The equivalence is given by noting set of elements of the form (n, x) gives a function and vice versa. In particular, since the weight is given by $\sum_{x\in X} f(x)$, we see use weight on $wt(n) = n$ and zero weight on $\mathcal{E}$ we get the equivalence is indeed weighted equivalence.

By the weighted equivalence, we get $M(x;t) = \exp(N(x;t))$ where $N(x;t) = \sum_{n\geq 0} xt^n = \frac{x}{1-t}$. Thus the proof follows. ♠

Example 4.4.14. Let R be a ring, $\mathcal{A}, \mathcal{B}$ be finite sets with generalized weight functions $Wt : \mathcal{A} \to R$ and $Wt' : \mathcal{B} \to R$. Define $Wt'' : \mathcal{A} \times \mathcal{B} \to R$ by $Wt''(a, b) = Wt(a)Wt'(b)$. Then show

$$|\mathcal{A} \times \mathcal{B}|_{Wt''} = |\mathcal{A}|_{Wt} \cdot |\mathcal{B}|_{Wt'}$$

Solution. By definition, we see $|\mathcal{A} \times \mathcal{B}|_{Wt''} = \sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} Wt(a)Wt'(b)$. On the other hand we see

$$|\mathcal{A}|_{Wt}|\mathcal{B}|_{Wt'} = \left(\sum_{a\in\mathcal{A}} Wt(a)\right)\left(\sum_{b\in\mathcal{B}} Wt'(b)\right)$$

but that's exactly $\sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} Wt(a)Wt'(b)$. The proof follows. ♠

Example 4.4.15. Let $(\mathcal{E}_+)^*$ be the species of set compositions (also called ordered set partition). The structures on X are ordered tuples $(X_1, ..., X_k)$ where $X_i \neq \emptyset$ and $X_1 \cup ... \cup X_k = X$ as disjoint union.

Consider $(\mathcal{E}_+)^*$ with generalized weighted function $\text{sgn}(X_1, ..., X_k) = (-1)^k$. Show the generalized EGF is e^{-x}.

Solution. By definition, we see the generalized EGF is $\sum_{n\geq 0} |(\mathcal{E}_+)^*_{[n]}|_{\text{sgn}} \frac{x^n}{n!}$. Thus it suffices to determine $|(\mathcal{E}_+)^*_{[n]}|_{\text{sgn}}$.

However, note $(\mathcal{E}_+)^* = \sum_{n\geq 0} \mathcal{E}_+^n$. On $\mathcal{E}_+^n$, we see by the product lemma above, we get $|(\mathcal{E}_+)^k_{[n]}|_{Wt} = |(\mathcal{E}_+)_{[n]}|_{Wt} \cdot ... \cdot |(\mathcal{E}_+)_{[n]}|_{Wt} = (-1)^k$. Thus we get $|(\mathcal{E}_+)^*_{[n]}|_{Wt} = \sum_{k=0}^{n} (-1)^k = (-1)^n$. Hence we get the generalized EGF is

$$\sum_{n\geq 0} (-1)^n \frac{x^n}{n!} = \exp(-x)$$

♠

Example 4.4.16. Give two definitions of a subspecies of a 2-species, and prove they are equivalent.

Solution. Well, the first definition should be, $\mathcal{B}$ is a subspecies of $\mathcal{A}$ if $\mathcal{B}$ is a 2-species and $\mathcal{B}_{X,Y} \subseteq \mathcal{A}_{X,Y}$ for every finite sets X, Y. The second definition should be, $\mathcal{B}$ is a rule that assigns every finite sets X, Y a subset $\mathcal{B}_{X,Y} \subseteq \mathcal{A}_{X,Y}$, and $\mathcal{B}$ is closed under isomorphism. ♠

Example 4.4.17. For a 2-species $\mathcal{A}$, give a precise definition of $\tilde{\mathcal{A}}$, the set of isomorphism types of $\mathcal{A}$.

Solution. Let X, Y be finite sets with $|X| = n, |Y| = m$. The isomorphism type of $\alpha \in \mathcal{A}_{X,Y}$ is given by the set

$$\tilde{\alpha} = \{\gamma \in \mathcal{A}_{n,m} : \gamma \cong \alpha\}$$

♠

Example 4.4.18. If $\mathcal{A}$ is a 2-species, define the ***reversal*** of $\mathcal{A}$ to be the species $\mathcal{A}^r$, where $\mathcal{A}^r_{X,Y} = \mathcal{A}_{Y,X}$. Is it true that every species is equivalent to its reversal?

Solution. False. Consider the two-species $\mathcal{A} = \mathcal{L}(\mathcal{X}_1) + \mathcal{S}(\mathcal{X}_2)$. Then $\mathcal{A}_{X,Y} = \mathcal{L}_X \cup \mathcal{S}_Y$. On the other hand, $\mathcal{A}^r_{X,Y}$ is equal disjoint union of $\mathcal{L}_Y \cup \mathcal{S}_X$. If $\mathcal{A}^r$ is naturally equivalent to $\mathcal{A}$ then we should have $\mathcal{L}$ is naturally equivalent to $\mathcal{S}$. ♠

Example 4.4.19. What is the multi-EGF of:

1. the 2-species of functions?
2. the 2-species of vertex and edge labelled graphs?

Solution. We see there are $|Y|^{|X|}$ possible functions, thus we get the EGF for functions is given by

$$\sum_{m,n\geq 0} m^n \frac{x^n}{n!}\frac{y^m}{m!}$$

if $|X| = n$ and $|Y| = m$.

For the vertex and edge labelled graphs, say $|X| = n$ and $|Y| = m$. We see the number of edges range from 0 to $\binom{n}{2}$. However, to have a bijection from the number of edges to set Y, only those graphs with n vertices and m edges are left. In other word, we are counting the number of graphs with n vertices and m edges. This is given by $\binom{\binom{n}{2}}{m}$ because we want to choose m edges out of all possible of $\binom{n}{2}$ many. This concludes the EGF. ♠

Example 4.4.20. Give a bijection between $\Pi^{(2)}_{X,Y}$ and $\Pi_{X\cup Y}$. State both map and the inverse map.

Solution. We see $\Pi^{(2)}_{X,Y}$ contains 2-set partitions of (X, Y) of the form

$$\{(S_1, T_1), ..., (S_k, T_k)\}$$

Thus we see $S_1, T_1, ..., S_k, T_k$ form s a set partition of $X \cup Y$. Conversely, any set partition of $X \cup Y$, for any set T contains both elements from X and Y, we can divide this into T, S where $T \subseteq X$ nad $S \subseteq Y$, this would yield a element of $\Pi^{(2)}_{X,Y}$. ♠

Example 4.4.21. Note the concept of 2-species is similar to concept of $\mathcal{E}^2$-species. And yet, that was not the definition given, there is a reason for this. What is the only difference between the concept of a 2-species and an $\mathcal{E}^2$-species?

Solution. The order does not matter for $\mathcal{E}^2$ because $\{X, Y\} = \{Y, X\}$ as $\mathcal{E}^2$-structure (hence the output should be the same for $\mathcal{E}^2$-species). However, we seen reversal is not equivalence. Thus they are different. ♠

Example 4.4.22. If $\mathcal{A}$ and $\mathcal{B}$ are 2-species, prove the multi-EGF of $\mathcal{A} * \mathcal{B}$ is $A(x_1, x_2)B(x_1, x_2)$.

Solution. Just note

$$|(\mathcal{A} * \mathcal{B})X, Y| = \sum_{S \subseteq X} \sum_{T \subseteq Y} |\mathcal{A}_{S,T}| \cdot |\mathcal{B}_{X \setminus S, Y \setminus T}|$$

but we see there are $\binom{n}{s}\binom{m}{t}$ ways to choose and hence we get the above equation is equal to

$$\sum_{s=0}^{n} \sum_{t=0}^{m} \binom{n}{s}\binom{m}{t} |\mathcal{A}_{s,t}||\mathcal{B}_{n-s,m-t}|$$

The proof follows as we continue to mimic the proof for one-sort. ♠

Example 4.4.23. If $\mathcal{A}$ and $\mathcal{B}$ are 1-species, the tensor product of $\mathcal{A}$ and $\mathcal{B}$ is the 2-species $\mathcal{A} \otimes \mathcal{B}$, where

$$(\mathcal{A} \otimes \mathcal{B})_{X,Y} = \mathcal{A}_X \times \mathcal{B}_Y$$

Prove the multi-EGF of $\mathcal{A} \otimes \mathcal{B}$ is $A(x_1)B(x_2)$.

Solution. Note this is just $\mathcal{A}(\mathcal{X}_1) * \mathcal{B}(\mathcal{X}_2)$ and the proof follows. ♠

Example 4.4.24. Show $1 \otimes 1$ is the identity 2-species for $*$ and $\mathcal{E} \otimes \mathcal{E}$ is the identity 2-species of $\boxtimes$.

Solution. Just note $1 \otimes 1$ has EGF 1 and $\mathcal{E} \otimes \mathcal{E}$ has EGF

$$\sum_{n,m \geq 0} \frac{x_1^n}{n!} \frac{x_2^m}{m!}$$

and the proof follows. ♠

Example 4.4.25. Prove the multi-EGF of $\partial_1 \mathcal{A}$ is $\frac{\partial}{\partial x_1} A(x_1, x_2)$.

Solution. Zzzzzz. ♠

Example 4.4.26. What is the diagonal of the following 2-species:

1. surjective functions.
2. injective functions.
3. vertex and edge-labelled graphs.
4. $\mathcal{A} \otimes \mathcal{B}$ with $\mathcal{A}, \mathcal{B}$ 1-species.

Solution. Well, let $\mathcal{A}$ be the species of surjective functions, i.e. $\mathcal{A}_{X,Y}$ contains all surjective functions from $X \to Y$. Then we see $\nabla \mathcal{A}_X$ is the set of all surjective functions from $X \to X$. However, since X is finite, $\nabla \mathcal{A}_X$ is just the set of bijections from X to X, i.e. $\mathcal{S}_X$. For injective functions, the same reasoning shows the diagonal is also just the set of bijections from X to X.

For the vertex and edge-labelled graphs, we note we must have the number of edges equal the number of vertices. Those graphs are graphs contain a cycle as a subgraph.

For $\mathcal{A}\otimes\mathcal{B}$, we see this means we are looking at $(\mathcal{A}_X,\mathcal{B}_X)=(\mathcal{A}\boxtimes\mathcal{B})_X$. ♠

Example 4.4.27. Let $\mathcal{A},\mathcal{B}$ be 1-species, prove $\mathcal{A}\otimes\mathcal{B}=\mathcal{A}\mathcal{X}_1*\mathcal{B}[\mathcal{X}_2]$ and $\mathcal{E}\otimes\mathcal{E}\equiv\mathcal{E}[\mathcal{X}_1+\mathcal{X}_2]$.

Solution. Note $(\mathcal{A}[\mathcal{X}_1]*\mathcal{B}[\mathcal{X}_2])_{X,Y}$ would run over all (A,B) with $A\subseteq X$ and $B\subseteq Y$ and we look at $\mathcal{A}_{A,B}\times\mathcal{B}_{X\setminus A,Y\setminus B}$. However, $\mathcal{X}_1$ is only not empty if $B=\emptyset$ and $\mathcal{X}_2$ is not empty only when $A=\emptyset$. Thus we only have one tuple that works, which is $(X,\emptyset)$. This is then just $\mathcal{A}_X\times\mathcal{B}_Y$ as desired. ♠

Example 4.4.28. Let $\mathcal{A}$ be a 1-species. A 2-species $\mathcal{A}^s$ is a ***split*** version of $\mathcal{A}$ if $\mathcal{A}^s[\mathcal{X},\mathcal{X}]\equiv\mathcal{A}$. Verify the following:

1. $\mathcal{E}[\mathcal{X}_1]*\mathcal{E}[\mathcal{X}_2]$ is a split version of $\mathcal{E}^2$.
2. Let $\mathcal{T}^s$ be the 2-species in which $\mathcal{T}^s_{X,Y}$ is the set of trees on vertex set $X\coprod Y$, where the set X labels the leaves, and Y labels the non-leaf vertices. THen $\mathcal{T}^s$ is a split version of $\mathcal{T}$.
3. Let $\mathcal{N}^s$ be the 2-species in which the elements $\mathcal{N}^s_{X,Y}$ are surjective functions $\mathcal{X}\coprod\mathcal{Y}\to\mathcal{Y}$. Then $\mathcal{N}^s$ is a split version of $\mathcal{N}$.
4. $\Pi^{(2)}$ is not a split version of Π. WHat is the correct compositional statement that relates Π and $\Pi^{(2)}$?

Solution. Replace each $\mathcal{X}_i$ with $\mathcal{X}$ we indeed end up with $\mathcal{E}*\mathcal{E}=\mathcal{E}^2$. Yeah, this is just because $\mathcal{E}[\mathcal{X}_1]*\mathcal{E}[\mathcal{X}_2]$ has two label sorts of label receivers, and if we replace them with the same, we would end up with $\mathcal{E}^2$.

For $\mathcal{T}^s$, we just note if we replace the two sorts of label receivers to be into one sort, it is just a normal tree, i.e. $\mathcal{T}_X$.

For $\mathcal{N}^s$, we see $\mathcal{N}^s[\mathcal{X},\mathcal{X}]_X=\coprod_{S\subseteq X}\mathcal{N}^s_{S,X\setminus S}$ but this is jut the set of surjections from X to $X\setminus S$ where S is some subset of X, i.e. this is the set of functions from X to X if S is arbitrary.

The problem with $\Pi^{(2)}$ is that it is ordered tuple, i.e. $\Pi^{(2)}[\mathcal{X},\mathcal{X}]_X$ contains order tuple $(\{\pi_1\},\{\pi_2\})$ where π_1 is a partition of S and π_2 is a partition of $X\setminus S$. The correct compositional statement should be? ♠

Example 4.4.29. Let $\mathcal{A}$ be a 2-species, and $\mathcal{A}^r$ be the reversal of $\mathcal{A}$. Show the following:

1. $\mathcal{A}\equiv\mathcal{A}[\mathcal{X}_1,\mathcal{X}_2]$.
2. $\mathcal{A}^r\equiv\mathcal{A}[\mathcal{X}_2,\mathcal{X}_1]$.

Solution. Well, $\mathcal{A}\equiv\mathcal{A}[\mathcal{X}_1,\mathcal{X}_2]$ means the first sort of label receivers only take elements from the first set, and the second from the second, i.e. it is $\mathcal{A}_{X,Y}$. Similarly, we see $\mathcal{A}^r$ is just $\mathcal{A}[\mathcal{X}_2,\mathcal{X}_1]$ as desired. ♠

4.5 Tutorial 6 (Degrees In Trees)

Remark 4.5.1. We have the formula

$$T_n(t_1,...,t_n) = \sum_{\tau\in\mathcal{T}_{[n]}} \prod_{i=1}^{n} t_i^{\deg_\tau(i)} = t_1t_2...t_n(t_1+t_2+...+t_n)^{n-2}$$

In this tutorial we are going to prove this formula.

Remark 4.5.2 (Rooted Coloured Trees)**.** Let $K = \{r, b\}$ with r mean red and b mean blue. Let $\mathcal{T}_K^\bullet$ be the species of rooted trees, every vertex assigned a colour, then we get

$$\mathcal{T}_K^\bullet \equiv \mathcal{T}^\bullet[K\times\mathcal{X}]$$

However, this decomposition is not good and we want a recursive decomposition. To that end, we get

$$\mathcal{T}_K^\bullet \equiv (K\times\mathcal{X}) * \mathcal{E}[\mathcal{T}_K^\bullet] \equiv \{r\}\times\mathcal{X} * \mathcal{E}[\mathcal{T}_K^\bullet] + \{b\}\times\mathcal{X} * \mathcal{E}[\mathcal{T}_K^\bullet]$$

We now want to put four different weight functions:

1. $v_r(\tau)$ is the number of red vertices.
2. $p_r(\tau)$ is the number of vertices with red parents.
3. $v_b(\tau)$ the number of red vertices.
4. $p_r(\tau)$ the number of vertices with red parents.

Then $Wt(\tau) = s_r^{v_r(\tau)} t_r^{p_r(\tau)} s_b^{v_b(\tau)} t_b^{p_b(\tau)}$ is our generalized weight function on $\mathcal{T}_K^\bullet$.

Example 4.5.3. How can we add weight functions to the RHS of the decomposition

$$\mathcal{T}_K^\bullet \equiv \{r\}\times\mathcal{X} * \mathcal{E}[\mathcal{T}_K^\bullet] + \{b\}\times\mathcal{X} * \mathcal{E}[\mathcal{T}_K^\bullet]$$

to get a weighted equivalence?

Solution. Clearly the $\{r\}\times\mathcal{X}$ should have weight s_r and $\mathcal{E}$ have weight to be its order, i.e. t_r^{ord}. The same thing happends for the blue part as well. ♠

Example 4.5.4. For $|X| = n$, compute $|(\mathcal{T}_K^\bullet)_X|_{Wt}$.

Solution. By the above example, we get

$$T_K^\bullet(x) = xs_r\exp(t_rT_K^\bullet(x)) + xs_b\exp(t_bT_K^\bullet(x))$$

and we can use LIFT with $\phi(x) = s_r\exp(t_rx) + s_b\exp(t_bx)$. Then we get

$$\begin{aligned}[x^n]T_K^\bullet(x) &= \frac{1}{n}[x^{n-1}](s_r\exp(t_rx)+s_b\exp(t_bx))^n \\ &= \frac{1}{n}[x^{n-1}]\sum_{k=0}^{n}\binom{n}{k} s_r^k e^{kt_rx} s_b^{n-k} e^{(n-k)t_bx}\end{aligned}$$

♠

Remark 4.5.5. Now we generalizes to k colours. Let $K = \{1, ..., k\}$ be the set of colours and let $\mathcal{T}_K^\bullet$ be $\mathcal{T}^\bullet[K \times \mathcal{X}]$, and we consider the weight $v_i(\tau)$ be the number of vertices of colour i and $p_i(\tau)$ be the number of vertices with parents of colour i. Then consider generalized weight function $Wt(\tau) = \prod_{i=1}^k s_i^{v_i(\tau)} t_i^{p_i(\tau)}$.

Example 4.5.6. Now give a weighted recursive decomposition of $\mathcal{T}_K^\bullet$.

Solution. Clearly $\mathcal{T}_K^\bullet \equiv \sum\{i\} \times \mathcal{X} * \mathcal{E}[T_K^\bullet]$. ♠

Example 4.5.7. For $|X| = n$, compute $|(\mathcal{T}_K^\bullet)_X|_{Wt}$.

Solution. We have

$$T_K^\bullet(x) = x(\sum_{i=1}^k s_i \exp(t_i T_K^\bullet(x)))$$

and hence we can use LIFT and we get

$$[x^n]T_K^\bullet(x) = \frac{1}{n}[x^{n-1}](s_1 \exp(t_1 x) + ... + s_n \exp(t_n x))^n$$

♠

Remark 4.5.8. Now suppose $|X| = |K| = n$, consider the subset $\mathcal{U}_X \subseteq (\mathcal{T}_K^\bullet)_X$ of trees in which every vertex is a different colours.

Now we compute $U_n(t_1, ..., t_n) = \sum_{\tau \in U_X} t_1^{p_1(\tau)} ... t_n^{p_n(\tau)}$. In particular, we see $U_n(t_1, ..., t_n)$ is just $[s_1 s_2 ... s_n][\frac{x^n}{n!}]T_K^\bullet(x)$. We have

$$\begin{aligned}
[s_1 s_2 ... s_n][\frac{x^n}{n!}]T_K^\bullet(x) &= n!\frac{1}{n}[s_1 ... s_n][x^{n-1}](s_1 \exp(t_1 x) + ... + s_n \exp(t_n x))^n \\
&= n!\frac{1}{n}[x^{n-1}][s_1 ... s_n](s_1 \exp(t_1 x) + ... + s_n \exp(t_n x))^n \\
&= \frac{n!}{n}[x^{n-1}]\binom{n}{1, 1, ..., 1} \exp(t_1 x) ... \exp(t_n x) \\
&= \frac{n!}{n} n! [x^{n-1}] \exp((\sum t_i) x) \\
&= (n-1)! n! \cdot \frac{(\sum t_i)^{n-1}}{(n-1)!} = n!(\sum_{i=1}^n t_i)^{n-1}
\end{aligned}$$

Now we get rid of the labels!

1. We see we have a bijection between $\tilde{\mathcal{U}_n} \leftrightarrow \mathcal{T}_{[n]}^\bullet$.
2. For each unlabelled tree in $\tilde{\mathcal{U}_n}$ there are $n!$ labellings.
3. For $\tau \in \tilde{\mathcal{U}_n}$, $p_i(\tau) = updeg_\tau(i)$, which is the up-degree of unique vertex of colours i.

In this case, we get

$$T_n^\bullet(t_1, ..., t_n) = \sum_{\tau \in \mathcal{T}_{[n]}^\bullet} \prod_{i=1}^n t_i^{updeg_\tau(i)} = \frac{1}{n!} U_n(t_1, ..., t_n) = \tilde{U_n}(t_1, ..., t_n) = (t_1 + ... + t_n)^{n-1}$$

For the unrooted version, we have $T_n(t_1, ..., t_n) = \sum_{\tau \in \mathcal{T}_{[n]}} \prod t_i^{deg_\tau(i)}$ and one can show that

$$T_n^\bullet(t_1, ..., t_n) = \sum_{i=1}^{n} \frac{T_n(t_1, ..., t_n)}{\prod_{j \neq i} t_j}$$

which concludes the formula at the start.

4.6 Tutorial 7 (From the 2013 Putnam Competition)

Example 4.6.1 (The Problem)**.** Let $X = \{1, ..., n\}$ and $k \in X$. Show there are exactly $k \cdot n^{n-1}$ functions $f : X \to X$ such that for every $x \in X$ there is a $j \geq 0$ such that $f^{(j)}(x) \leq k$ where $f^{(j)}$ is the jth composition iteration, e.g. $f^{(0)}(x) = x$, $f^{(1)}(x) = f(x)$ and $f^{(2)}(x) = f(f(x))$.

Remark 4.6.2 ("Good" and "Bad" Components)**.** Many of the species we encountered can be described as the subspecies of "ambient species" where none of the components "are bad".

Example 4.6.3.

1. Derangements: it is subspecies of permutations where none of the components are fixed points.
2. 2-to-1 fixed point free endofunctions: subspecies of 2-to-1 endofunctions where none of the components have a fixed point.
3. 2-species of bipartition graphs where all vertices have degree at least 1: subspecies of bipartite graphs where none of the components to be isolated points.

Remark 4.6.4. In each case, we have the following relationships:

1. $\mathcal{A}$ is the ambient species.
2. $\mathcal{B}$ is subspecies of $\mathcal{A}$, all components are bad.
3. $\mathcal{G}$ is subspecies of $\mathcal{A}$, all components are good.

Then we get the relation

$$\mathcal{A} = \mathcal{B} * \mathcal{G}$$

We can also consider the connected case:

1. $\mathcal{A}^c$ is the connected $\mathcal{A}$-structures.
2. $\mathcal{B}^c$ is the connected $\mathcal{B}$-structures.
3. $\mathcal{G}^c$ is the connected $\mathcal{G}$-structures.

In this case, we have $\mathcal{A}^c = \mathcal{B}^c + \mathcal{G}^c$ and hence we have

$$\mathcal{A} = \mathcal{E}[\mathcal{B}^c + \mathcal{G}^c]$$

Example 4.6.5 (Restricting/Changing Sorts)**.** What is notation for the following 2-species?

1. Rooted trees, where root is the 1st sort, other vertices are of 2nd sort.
2. Permutations where fixed points are of the 1st sort, other points of 2nd sort.

3. Sets of rooted trees where roots are of 1st sort, other vertices of either sort.
4. Sets of trees where all vertices in the same components are of the same sort.

Solution.

1. $\mathcal{X}_1 * \mathcal{E}[\mathcal{T}^\bullet[\mathcal{X}_2]]$
2. $\mathcal{E}[\mathcal{C}_1[\mathcal{X}_1] + \mathcal{C}_{\geq 2}[\mathcal{X}_2]]$
3. $\mathcal{E}[\mathcal{X}_1 * \mathcal{E}[\mathcal{T}^\bullet[\mathcal{X}_1 + \mathcal{X}_2]]]$
4. $\mathcal{E}[\mathcal{T}^\bullet[\mathcal{X}_1] + \mathcal{T}^\bullet[\mathcal{X}_2]]$

♠

Example 4.6.6. Rephrase the problem in the start as a problem about 2-species with good and bad components.

Solution. We should make the 1st sort with $1, ..., k$ and 2nd sort correspond to $k+1, ..., n$.

Then the ambient space is $\mathcal{N}[\mathcal{X}_1 + \mathcal{X}_2]$. The bad components are cycles has only components of the second sort. Then the good components are cycles has at least one elements of the first sort. ♠

Example 4.6.7. Solve the problem for permutations instead of endofunctions.

Solution. In this case, ambient space is $\mathcal{S}[\mathcal{X}_1 + \mathcal{X}_2]$. Then $\mathcal{B}_{X_1, X_2} = \{\sigma \in \mathcal{S}_{X_1 \coprod X_2} :$ every cycle is bad$\}$ and $\mathcal{G}$ is permutations with every cycle good.

In particular, we see since $A(x_1, x_2) = B(x_1, x_2)G(x_1, x_2)$, we see

$$G(x_1, x_2) = \frac{A(x_1, x_2)}{B(x_1, x_2)} = \frac{1 - x_2}{1 - x_1 - x_2} = \frac{1}{1 - \frac{x_2}{1 - x_2}}$$

In particular, we want to compute

$$[\frac{x_1^k x_2^{n-k}}{k!(n-k)!}]G(x_1, x_2) = [\frac{x_2^{n-k}}{(n-k)!}]k!(1 - x_2)^{-k} = k!(n-k)!\binom{n-1}{n-k} = \frac{k}{n}n!$$

♠

Example 4.6.8. Now solve the problem as posed.

1. What is the ambient spaces?
2. What is the bad subspecies?
3. What is the multi-EGF for good subspecies?
4. Compute the coefficients of the multi-EGF from part (c).

Solution.

1. $\mathcal{N}(\mathcal{X}_1 + \mathcal{X}_2) = \mathcal{E}[\mathcal{C}[\mathcal{T}^\bullet[\mathcal{X}_1 + \mathcal{X}_2]]]$
2. $\mathcal{E}[\mathcal{C}[\mathcal{X}_2 * \mathcal{E}[\mathcal{T}^\bullet[\mathcal{X}_1 + \mathcal{X}_2]]]]$

3.

$$
\begin{aligned}
G(x_1, x_2) &= \frac{\frac{1}{1-T^\bullet(x_1+x_2)}}{\exp(\log(\frac{1}{1-x_2\exp(T^\bullet(x_1+x_2))}))} \\
&= \frac{\frac{1}{1-T^\bullet(x_1+x_2)}}{(\frac{1}{1-x_2\exp(T^\bullet(x_1+x_2))})} \\
&= \frac{1 - x_2\exp(T^\bullet(x_1+x_2))}{1 - T^\bullet(x_1+x_2)} \\
&= \frac{1 - x_2\exp(T^\bullet(y))}{1 - T^\bullet(y)}
\end{aligned}
$$

4. Method 1: We can make substitution $y = x_1 + x_2$, then we get

$$
G(x_1, x_2) = \frac{1 - (y - x_1)e^{T^\bullet(y)}}{1 - T^\bullet(y)} = 1 + \frac{x_1 e^{T^\bullet(t)}}{1 - T^\bullet(t)}
$$

Method 2: We see

$$
\begin{aligned}
N(x_1 + x_2)(1 - x_2 T'(x_1 + x_2)) &= N(x_1, x_2)(1 - \frac{x_2}{x_1 + x_2} T^\bullet(x_1, x_2)) \\
&= N(x_1, x_2)(1 - \frac{x_2}{x_1 + x_2}(1 - \frac{1}{N(x_1, x_2)})) \\
&= N(x_1 + x_2) - \frac{x_2}{x_1 + x_2} N(x_1 + x_2) + \frac{x_2}{x_1 + x_2} \\
&= \frac{x_1}{x_1 + x_2} N(x_1 + x_2) + \frac{x_2}{x_1 + x_2}
\end{aligned}
$$

♠

Chapter 5

Sieving Methods

5.1 Mobius Inversion and Inclusion-Exclusion

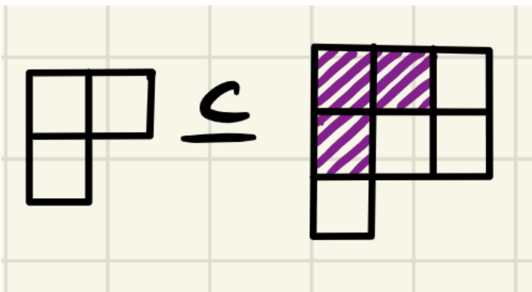
Definition 5.1.1. For poset $(P, \leq)$, for $\alpha, \beta \in P$ define ***interval*** $[\alpha, \beta] := \{\gamma \in P : \alpha \leq \gamma \leq \beta\}$.

Definition 5.1.2. We say poset $(P, \leq)$ is ***locally finite*** if all intervals are finite.

Remark 5.1.3. In this course we would assume the poset is always locally finite.

Example 5.1.4.

1. $(\mathbb{N}, \leq)$ is locally finite.
2. $(\mathbb{Z}_+, |)$ is locally finite where $|$ is divisibility, e.g. $3 \mid 6$ and $3 \nmid 5$.
3. Consider $(\mathcal{P}_f(X), \subseteq)$, where $\mathcal{P}_f(X)$ is the set of all finite subsets of X with X any fixed set.
4. Consider $(\mathcal{P}, \subseteq)$ where $\mathcal{P}$ is the set of all partitions, with $\subseteq$ the "diagram containment" relation. For example, we have



5. Consider $(\Gamma, \rightsquigarrow)$, where Γ is a directed acyclic graphs, and $u \rightsquigarrow v$ means there exists a directed path from u to v.

Definition 5.1.5 (Mobius Function). Let $(P, \leq)$ be a poset. Define $\mu : P \times P \to \mathbb{Z}$ recursively as follows:

$$\mu(\alpha, \beta) := \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \not\leq \beta \\ -\sum_{\gamma \in [\alpha, \beta] \setminus \{\beta\}} \mu(\alpha, \gamma), & \alpha < \beta \end{cases}$$

Example 5.1.6. Classical Mobius function $(\mathbb{Z}_+, |)$ given by

$$\mu(d,n) = \begin{cases} (-1)^{\text{number of prime factors of } n/d}, & \text{if } n/d \text{ is square-free integer} \\ 0, & \text{otherwise} \end{cases}$$

Since the RHS depends only on n/d, we write $\mu(\frac{n}{d}) = \mu(d,n)$.

Theorem 5.1.7. *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be two posets, then we can define poset $(P_1 \times P_2, \leq)$ by $(a,b) \leq (c,d)$ iff $a \leq_1 c, b \leq_2 d$. Then we have*

$$\mu_{P_1 \times P_2}((a,b),(c,d)) = \mu_{P_1}(a,c) \cdot \mu_{P_2}(b,d)$$

Theorem 5.1.8 (Mobius Inversion, Version I)**.** *Let R be a ring, $f, g : P \to R$ with $(P, \leq)$ a poset. The following are equivalent:*

1. $f(\alpha) = \sum_{\alpha \leq \beta} g(\beta)$ *for all* $\alpha \in P$.
2. $g(\alpha) = \sum_{\alpha \leq \beta} \mu(\alpha,\beta) f(\beta)$ *for all* $\alpha \in P$.

where we assume the sums are finite, or R is a valuation ring and sums are convergent.

Proof. We are going to prove the case when P is finite. Consider the poset incidence function $\zeta : P \times P \to \mathbb{Z}$ by

$$\zeta(\alpha,\beta) = \begin{cases} 1, & \alpha \leq \beta \\ 0, & \text{otherwise} \end{cases}$$

Then we can think μ, ζ as matrices where the rows and columns are indexed by P. Then the definition of μ is equivalent to $\mu\zeta = \mathrm{Id}$ is the identity matrix. Then we see (1) if and only if $f = \zeta g$ and (2) if and only if $g = \mu f$, where we consider f, g are vectors. Thus the proof follows. ♡

Corollary 5.1.8.1 (Mobius Inversion, Version II)**.** *With the same setting as above, the following are equivalent:*

1. $f(\beta) = \sum_{\alpha \leq \beta} g(\alpha)$ *for all* $\beta \in P$.
2. $g(\beta) = \sum_{\alpha \leq \beta} f(\alpha)\mu(\alpha,\beta)$ *for all* $\beta \in P$.

where we assume the sums are finite or R is valuation ring with sum convergent.

Proof. The same proof as above, where in this case we use ζ^T and μ^T, i.e. (1) iff $f = \zeta^T g$ and (2) iff $g = \mu^T f$. ♡

Remark 5.1.9 (Inclusion-Exclusion)**.** Now consider $(\mathcal{P}_f(X), \subseteq)$, we want to compute the Mobius inversion on this poset. It is not hard to see

$$\mu(\alpha,\beta) = \begin{cases} (-1)^{|\alpha|-|\beta|}, & \alpha \subseteq \beta \\ 0, & \text{otherwise} \end{cases}$$

Now we discuss the combinatorial set-up of this. Let $\mathcal{S}$ be a set of objects, $\mathcal{T} \subseteq \mathcal{S}$ be the set of "good" objects with $\mathcal{B}$ the set of "flaws" (that exclude an object from being "good").

For $S \in \mathcal{S}$, we can define $Flaws(S) = \{b \in \mathcal{B} : S \text{ has flaw } b\} \in \mathcal{P}_f(\mathcal{B})$.

For $\alpha \in P_f(\mathcal{B})$, we can define $\mathcal{F}_\alpha := \{S \in \mathcal{S} : Flaws(S) \supseteq \alpha\}$, which is the set of objects with flaws α, and possibly others. We can also define $\mathcal{G}_\alpha$ as $\{S \in \mathcal{S} : Flaws(S) = \alpha\}$. We note $\mathcal{F}_\emptyset = \mathcal{S}$ and $\mathcal{G}_\emptyset = \mathcal{T}$.

Then the basic formulation with $\mathcal{S}$ finite is that, $f, g : \mathcal{P}_f(\mathcal{B}) \to \mathbb{Z}$, with $f(\alpha) = |\mathcal{F}_\alpha|$, $g(\alpha) = |\mathcal{G}_\alpha|$. Then we have a theorem as follows:

$$g(\alpha) = \sum_{\alpha \subseteq \beta} (-1)^{|\beta| - |\alpha|} f(\beta)$$

Indeed, by construction, $f(\alpha) = \sum_{\alpha \subseteq \alpha} g(\beta)$ and so by Mobius inversion on $(\mathcal{P}_f(\mathcal{B}), \subseteq)$ the result follows.

In particular, we see $|\mathcal{T}| = g(\emptyset) = \sum_{\beta \subseteq B} (-1)^{|\beta|} f(\beta)$.

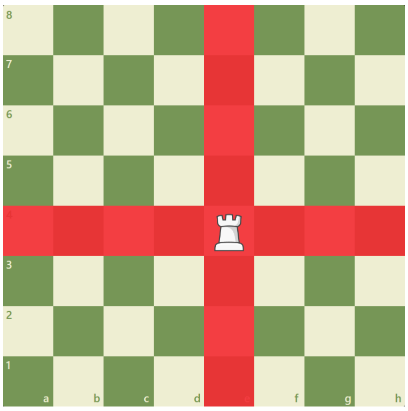
Example 5.1.10 (Derangements)**.** Let $\mathcal{S} = S_n$ be permutations of $[n]$, $\mathcal{T} \subseteq \mathcal{S}$ be the set of derangements. Then the flaws are $\sigma(1) = 1, \sigma(2) = 2, ..., \sigma(n) = n$. In other word, we can think the set $\mathcal{B} = \{1, ..., n\}$. Then $Flaws(\sigma) = \{i \in [n] : \sigma(i) = i\}$. Then

$$\mathcal{F}_\alpha = \{\sigma \in S_n : \forall i \in \alpha, \sigma(i) = i\}$$

and hence $f(\alpha) = |\mathcal{F}_\alpha| = (n - |\alpha|)!$ Thus, we get

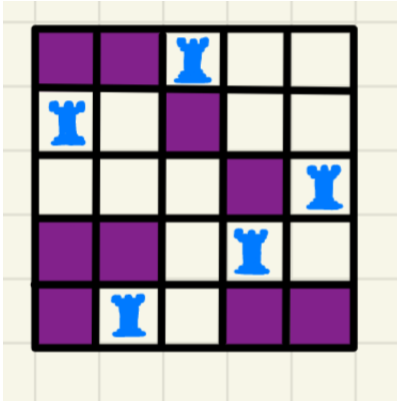
$$|\mathcal{T}| = \sum_{\beta \subseteq [n]} (-1)^{|\beta|} (n - |\beta|)! = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Example 5.1.11 (Rook Polynomials)**.**



Consider $[n] \times [n]$ as the squares of a n by n chess board, then a permutation $\sigma \in S_n$ is a placement of n non-attacking rooks on the chess board.

Now we want to look at rook placements with some blocks disallowed, for example,



maybe we want non-attacking rook placements with none of the rooks placed on the purple blocks.

Formally, let $B \subseteq [n] \times [n]$ be the set of disallowed squares, let $\mathcal{D}_B = \{\sigma \in S_n : \forall (i,j) \in B, \sigma(i) \neq j\}$ be the set of placements of non-attacking rooks avoiding B. Then we want to compute $|\mathcal{D}_B|$.

For this purpose, we define the rook polynomial $R_B(x) = \sum_{k=0}^{n} r_k x^k$ where r_k is the number of ways to place k non-attacking rooks within squares of B. Then we have a theorem

$$|\mathcal{D}_B| = [x^n] R_B(-x) \sum_{j \geq 0} j! x^j$$

To prove this, we apply inclusion-exclusion to S_n with B the set of "flaws". For $\alpha \subseteq B$, let $\mathcal{F}_\alpha = \{\sigma \in S_n : \forall (i,j) \in \alpha, \sigma(i) = j\}$. Then

$$f(\alpha) = |\mathcal{F}_\alpha| = \begin{cases} (n - |\alpha|)!, & \text{if } \alpha \text{ is an arrangment of non-attacking rooks} \\ 0, & \text{otherwise} \end{cases}$$

Then we see

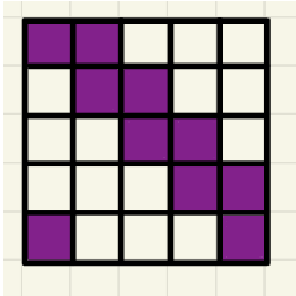
$$|\mathcal{D}_B| = \sum_{\beta \subseteq B} (-1)^{|\beta|} f(\beta) = \sum_{k=0}^{n} (-1)^k r_k (n-k)! = [x^n] R_B(-x) \sum_{j \geq 0} j! x^j$$

Example 5.1.12 (Menage Problem)**.** The set-up is:

1. We have n male-female couples.
2. Seat them around a round table.
3. No person is next to someone of the same sex.
4. No person is next to their partner.

Then we want to compute the number of such seatings.

Fix a seating of men in every other seat. Then the number of ways to seat the women is equivalent to count $\sigma \in S_n$, such that $\sigma(i) \neq i, i+1 \pmod{n}$. This is just the following disallowed squares configuration:



Then by elementary counting, we can show $r_k = \frac{2n}{2n-k}\binom{2n-k}{k}$ and use the theorem about rook polynomial to get the answer.

Remark 5.1.13 (GF Formulation)**.** Now consider the generating function formulation of the inclusion-exclusion principle. Let $\mathcal{S}, \mathcal{T}, \mathcal{B}, \mathcal{F}_\alpha, \mathcal{G}_\alpha$ as before.

We introduce a weight function on this set-up. Let R be a ring, $Wt : \mathcal{S} \to R$ be a generalized weight function with $f(\alpha) = |\mathcal{F}_\alpha|_{Wt} = \sum_{s\in\mathcal{F}_\alpha} Wt(s)$ and $g(\alpha) = |\mathcal{G}_\alpha|_{Wt} = \sum_{s\in\mathcal{G}_\alpha} Wt(s)$. Then our usual goal is to compute $|\mathcal{T}|_{Wt} = g(\emptyset)$.

Then we consider the objects with "marked flaws". Let $\mathcal{F} = \{(s, \alpha) : s \in \mathcal{F}_\alpha\}$ and $\mathcal{G} = \{(s, \alpha) : s \in \mathcal{G}_\alpha\}$. We can think of (s, α) as object with marked flaws, where not all flaws need to be marked. Then $\mathcal{G} \subseteq \mathcal{F}$ is the subset in which all flaws are marked. We note we have bijection $\mathcal{G} \Leftrightarrow \mathcal{S}$.

Then we have the inclusion-exclusion principle, full version, as follows: let $\underline{t} = \{t_b : b \in B\}$ be the set of variables. For $\alpha \in \mathcal{P}_f(\mathcal{B})$, we write $\underline{t}^\alpha = \prod_{b\in\alpha} t_b$. Then we define

$$\underline{F}(\underline{t}) := \sum_{(s,\alpha)\in\mathcal{F}} Wt(s)\underline{t}^\alpha$$

$$\underline{G}(\underline{t}) = \sum_{(s,\alpha)\in\mathcal{G}} Wt(s)\underline{t}^\alpha$$

Then the theorem says

$$\underline{F}(\underline{t} - \underline{1}) = \underline{G}(\underline{t})$$

where $\underline{t} - \underline{1}$ means substitute t_b by $t_b - 1$ for all $b \in B$.

Proof: We see

$$\underline{F}(\underline{t}) = \sum_{\alpha\in\mathcal{P}_f(\mathcal{B})} f(\alpha)\underline{t}^\alpha$$

and

$$\underline{G}(\underline{t}) = \sum_{\alpha\in\mathcal{P}_f(\mathcal{B})} g(\alpha)\underline{t}^\alpha$$

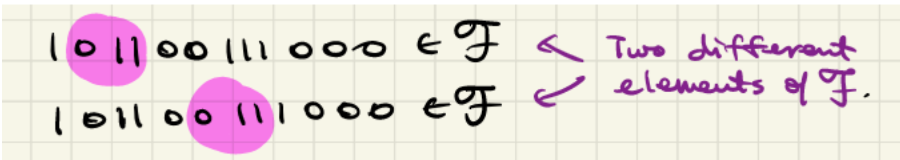
Then we see

$$\begin{aligned}
\underline{F}(\underline{t}-\underline{1}) &= \sum_{\beta\in\mathcal{P}_f(\mathcal{B})} f(\beta)(\underline{t}-\underline{1})^{\beta} \\
&= \sum_{b\in\mathcal{P}_f(\mathcal{B})} f(\beta)\sum_{\alpha\subseteq\beta}(-1)^{|\beta|-|\alpha|}\underline{t}^{\alpha} \\
&= \sum_{\alpha\in\mathcal{P}_f(\mathcal{B})} \underline{t}^{\alpha}\left(\sum_{\alpha\subseteq\beta}(-1)^{|\beta|-|\alpha|}f(\beta)\right) \\
&= \sum_{\alpha\in\mathcal{P}_f(\mathcal{B})} \underline{t}^{\alpha}g(\alpha) = \underline{G}(\underline{t})
\end{aligned}$$

We also have the inclusion exclusion principle, simplified version. In this version, we just set all variables $t_b = t$. Then we get $F(t) = \sum_{(s,\alpha)\in\mathcal{T}} Wt(s)t^{|\alpha|}$ and $G(t) = \sum_{(s,\alpha)\in\mathcal{G}} Wt(s)t^{|\alpha|}$. Then we have $G(t) = F(t-1)$ and $|\mathcal{T}|_{Wt} = F(-1)$.

In some applications, $\mathcal{B}$ can be a set of "features" rather "flaws". In this case, since $\mathcal{G}$ has bijection to $\mathcal{S}$, $G(t)$ is a generating function for $\mathcal{S}$, where t marks the number of features.

Example 5.1.14. Let $\mathcal{J}$ be the set of 01-strings where 011 is not a substring. Then $\mathcal{S}$ is the set of all 01-strings. Let $\mathcal{F}$ be the set of 01-strings with 011 as a substring, may be marked. For example, we have

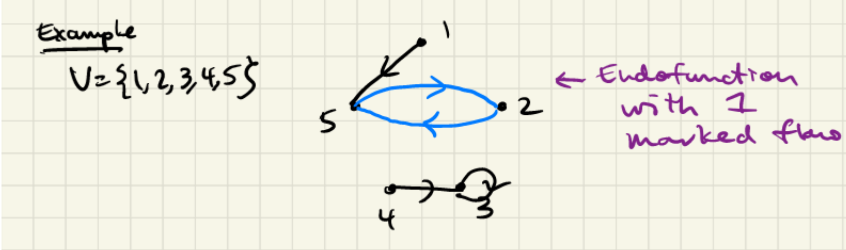


Then we see $\mathcal{F} = \{0, 1, (011)\}^*$ where we use (011) to mean 011 is marked. Then $F(x,t) = \frac{1}{1-(2x+x^3t)}$ where x is mark the length, and t marks the number of marked 011. Finally, by inclusion exclusion, we get

$$T(x) = F(x,-1) = \frac{1}{1-2x+x^3}$$

Remark 5.1.15 (Matrix Tree Theorem)**.** The inclusion-exclusion set-up:

1. V is a finite set.
2. $\mathcal{S}$ is the set of all endofunctions from $V \to V$.
3. $\mathcal{T}$ is the set of endofunctions from V to V, where every recurrent element is fixed. In other word, if $\phi\circ\phi\circ\ldots\circ\phi(v) = v$ then $\phi(v) = v$. Equivalently, no cycles of length greater than or equal 2.
4. Then $\mathcal{F}$ contains (ϕ, α), where $\phi : V \to V$ is endofunction and α a set of "marked flaws", i.e. cycles of length ≥ 2. For example,



5. Let K_V be the complete graph with vertex set V. Then we consider variable x_v for all $v \in V$, and y_e for $e \in E(K_V)$. Then for $\phi \in \mathcal{S}$, we define

$$Wt(\phi) = \prod_{v=\phi(v)} x_v \prod_{v \neq \phi(v)} y_{\{v,\phi(v)\}}$$

With the above set-up, we can state the matrix tree theorem as follows.

Theorem 5.1.16. *Let M be the matrix with rows and columns indexed by V, then we define*

$$M_{uv} = \begin{cases} x_v + \sum_{w \neq v} y_{vw}, & \text{if } u = v \\ -y_{vw}, & \text{if } u \neq v \end{cases}$$

Then $|\mathcal{T}|_{Wt} = \det(M)$.

Proof. By inclusion-exclusion, we see $|\mathcal{T}|_{Wt} = F(-1)$. We need to show $\det(M) = F(-1)$.

Expanding $\det(M)$. First level expansion we get

$$\det(M) = \sum_{\sigma \in S_V} \operatorname{sgn}(\sigma) \prod_{v \in V} M_{v,\sigma(v)}$$

Then each of the product we can expand

$$\prod_{v \in V} M_{v,\sigma(v)} = \sum \text{Terms}$$

The terms in the expansion of $\det(M)$ is the same as we make a choice of permutations $\sigma : V \to V$ plus a choice of terms from each $M_{v,\sigma(v)}$.

For example, we have

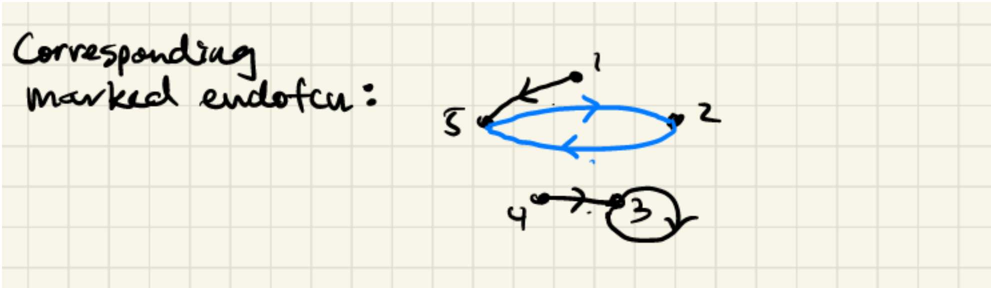
Example. $V = \{1, 2, 3, 4, 5\}$.

$$\begin{vmatrix} x_1+y_{12}+y_{13}+y_{14}+y_{15} & -y_{12} & -y_{13} & -y_{14} & -y_{15} \\ -y_{12} & x_2+y_{12}+y_{23}+y_{24}+y_{25} & -y_{23} & -y_{24} & -y_{25} \\ -y_{13} & -y_{23} & x_3+y_{13}+y_{23}+y_{34}+y_{35} & -y_{34} & -y_{35} \\ -y_{14} & -y_{24} & -y_{34} & x_4+y_{14}+y_{24}+y_{34}+y_{45} & -y_{45} \\ -y_{15} & -y_{25} & -y_{35} & -y_{45} & x_5+y_{15}+y_{25}+y_{35}+y_{45} \end{vmatrix}$$

where the yellow underline is our first choice, i.e. we choose a permutation $\sigma = 15342$. Then we also need to pick up one term from each of the yellow terms, marked by purple. In other word, the choice we end up with is

$$\operatorname{sgn}(15342)(y_{15})(-y_{25})(x_3)(y_{34})(-y_{25})$$

However, this correspond to a marked endofunction in a very natural way:



where the cycle between 2 and 5 is marked because those y_{25} are coming from off-diagonal.

It turns out, terms T in the expansion of $\det(M)$ are in bijection to $(\sigma, \alpha) \in \mathcal{F}$. Moreover, $T = (-1)^{|\alpha|} Wt(\phi)$ and hence $\det(M) = \sum T = \sum_{(\sigma,\alpha)\in\mathcal{T}} (-1)^{|\alpha|} Wt(\phi) = F(-1)$. ♡

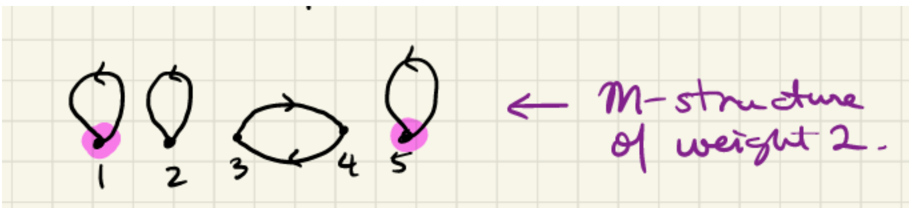
5.2 Marked and Virtual Species

Remark 5.2.1 (Weighted Species Techniques). The set-up:

1. Let $\mathcal{A}$ be a species, which is structures that can have flaws.
2. Let $\mathcal{B}$ be the subspecies of $\mathcal{A}$ which are good.
3. Let $\mathcal{M}$ be the weighted species of $\mathcal{A}$-structures with marked flaws with weight as the number of marks.

Then by inclusion exclusion we get $B(x) = M(x; -1)$.

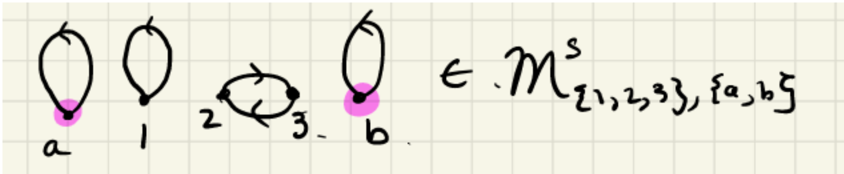
Example 5.2.2. We will re-do the example of derangements: let $\mathcal{S}$ be permutations with flaws being fixed points. Then $\mathcal{D}$ is going to be the subspecies of derangements. Then let $\mathcal{M}$ be the permutation with marked fixed points. For example, we have



Then we have $\mathcal{M} \equiv \mathcal{E} * \mathcal{S}$ where $\mathcal{E}$ is the set of marked fixed points and $\mathcal{S}$ is a permutation of the rest. Then the weight on $\mathcal{E}$ is just the order, to match our weight on $\mathcal{M}$. Thus the MGF of $\mathcal{M}$ is $E(x;t)S(x) = e^{tx}\frac{1}{1-x}$. On the other hand, we would have $D(x) = M(x,-1) = \frac{e^{-x}}{1-x}$.

Remark 5.2.3 (2-Species Technique)**.** Let $\mathcal{A}$ and $\mathcal{B}$ as before. We assume: for $\sigma \in \mathcal{A}_X$, we have $Flaws(\sigma) \subseteq X$. Then we define $\mathcal{M}^S$ be the 2-species of "split" $\mathcal{A}$-structure with marked flaws. The unmarked labels are the 1st sort, and the marked labels are the second sort. Then $B(x) = M^S(x, -x)$.

Example 5.2.4. Let us do the derangement example again. An example of $\mathcal{M}^S$ is



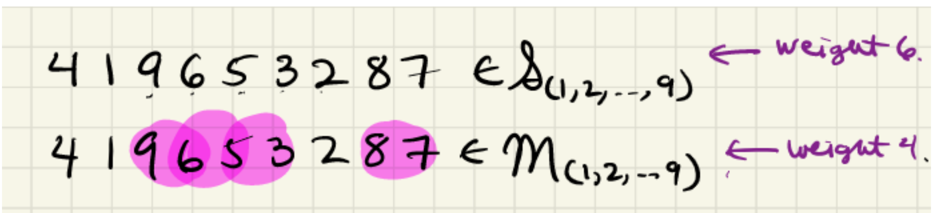
Then we have $\mathcal{M}^S = \mathcal{S}[\mathcal{X}_1] * \mathcal{E}[\mathcal{X}_2]$ and hence the EGF is $M^S(x_1, x_2) = \frac{1}{1-x_1}e^{x_2}$ and hence we are done.

Example 5.2.5 (Feature, Not Bugs!)**.** In this example we consider an example where we have features, instead of flaws. Let $\mathcal{A}$ be a weighted species, weight function equal the number of widgets. Let $\mathcal{M}$ be the weighted species of $\mathcal{A}$-structures with marked widgets with weight function equal the number of marks. Then $A(x;t) = M(x;t-1)$.

The example we are going to consider is permutations and descents. If $\sigma \in S_n$, and $\sigma(i) > \sigma(i+1)$, then we say i is a descent of σ. The question is, how many permutations of S_n with k descents?

Here we consider the linear species of permutations $\mathcal{S}$. In other word, the input should be an ordered list instead of a set and we should produce the set of permutation of the given list. Let the weight function on $\mathcal{S}$ is the number of descents. Let $\mathcal{M}$ be the linear species of permutations with marked descents with weight function equal the number of marks.

For example,



Then let $\mathcal{D}$ be the linear species of decreasing sequences of length ≥ 2 with weight function being the order minus 1. For example, we have $\mathcal{D}_{(3,5,6,9)} = \{(9652)\}$ and (9652) has weight 3. Then the MGF of $\mathcal{D}$ should be

$$D(x;t) = \sum_{n\geq 2} t^{n-1}\frac{x^n}{n!} = \frac{e^{xt} - xt - 1}{t}$$

Then we see we get decomposition $\mathcal{M} \equiv (\mathcal{X} + \mathcal{D})^*$ and it is a weighted equivalence. Thus we get

$$M(x;t) = \frac{1}{1-(x+D(x;t))} = \frac{1}{1-x-\frac{e^{xt}-xt-1}{t}} = \frac{-t}{e^{xt}-t-1}$$

Finally, we get

$$S(x;t) = M(x;t-1) = \frac{1-t}{e^{x(t-1)}-t}$$

Definition 5.2.6. Let R be a ring with 2 not a zero-divisor. Let $\mathcal{S}, \mathcal{S}'$ be two sets with generalized weight functions $Wt : \mathcal{S} \to R, Wt' : \mathcal{S}' \to R$. Then suppose $\Phi : \mathcal{S} \to \mathcal{S}'$ is a bijection, then we say Φ is ***sign-reversing bijection*** (SRB) if $Wt'(\Phi(s)) = -Wt(s)$ for all $s \in \mathcal{S}$.

Remark 5.2.7. If a SRB exists between $\mathcal{S}$ and $\mathcal{S}'$ then $|\mathcal{S}|_{Wt} = -|\mathcal{S}'|_{Wt'}$.

Now we talk about the common scenarios of SRB.

Proposition 5.2.8. *Let $\mathcal{T} \subseteq \mathcal{S}$, suppose there exists SRB $\Phi : \mathcal{S}\backslash\mathcal{T} \to \mathcal{S}\backslash\mathcal{T}$ then $|\mathcal{T}|_{Wt} = |\mathcal{S}|_{Wt}$.*

Proof. Since SRB exists, we have $|\mathcal{S}\backslash\mathcal{T}|_{Wt} = -|\mathcal{S}\backslash\mathcal{T}|_{Wt}$ and hence we must have $|\mathcal{S}\backslash\mathcal{T}|_{Wt} = 0$. However, we see $|\mathcal{S}|_{Wt} = |\mathcal{S}\backslash\mathcal{T}|_{Wt} + |\mathcal{T}|_{Wt} = |\mathcal{T}|_{Wt}$. ♡

Remark 5.2.9. If $\Phi \circ \Phi = \text{Id}$, then Φ is called a ***sign-reversing involution*** (SRI).

Example 5.2.10 (Euler's Pentagonal Number Theorem)**.** Let $\mathcal{S}$ be the set of strict partitions, i.e. $\lambda = (\lambda_1, ..., \lambda_l)$ with $\lambda_1 > \lambda_2 > ... > \lambda_l$. Then we have weight $Wt : \mathcal{S} \to \mathbb{Q}[[x]]$ be

$$Wt(\lambda) = (-1)^{length(l)} x^{|\lambda|}$$

Then let $\mathcal{T}$ be the pentagonal partitions, which is equal

$$\{\epsilon, (1), (2), (3, 2), (4, 3), (5, 4, 3), (6, 5, 4), (7, 6, 5, 4), (8, 7, 6, 5), ...\}$$

Then there exists a SRI $\Phi : \mathcal{S}\backslash\mathcal{T} \to \mathcal{S}\backslash\mathcal{T}$. Thus we have

$$\prod_{i\geq 1}(1-x^i) = |\mathcal{S}|_{Wt} = |\mathcal{T}|_{Wt} = \sum_{k\in\mathbb{Z}}(-1)^k x^{g_k}$$

where g_k is the pentagonal numbers.

Remark 5.2.11 (What is Virtual Species)**.** We consider the analogy:

1. The species are like natural numbers $\mathbb{N}$ (we have operations like $+, -, \cdot$).
2. Then generalized weighted species are like $\mathbb{N}^X$, which is functions from X to $\mathbb{N}$ (we also have operations like $+, -, \cdot$).
3. Then, virtual species are like integers, as we note subtraction for natural numbers are only partially defined, e.g. $3 - 5$ is not defined in $\mathbb{N}$.

Thus, to define virtual species, we recall the construction of $\mathbb{Z}$:

1. We start with the set $\mathbb{N}^{\{+,-\}}$. In other word, this is the set of functions from set $\{+, -\}$ map to $\mathbb{N}$. We can write these functions as pairs, i.e. $f = (f(+), f(-)) \in \mathbb{N}^{\{+,-\}}$.

2. Then we can add interpretations of those functions to think them as integers. The idea is that function f represents integer $f(+)-f(-)$. However, $f(+)-f(-)$ does not make sense yet. To achieve this, we define an equivalence relation on $\mathbb{N}^{\{+,-\}}$, which is $f \equiv_{\mathbb{Z}} g$ if and only if $f(+)+g(-)=f(-)+g(+)$.
3. Then we have an embedding $\mathbb{N} \to \mathbb{Z}$. In other word, if we have natural number $m \in \mathbb{N}$, then we can identify m as the function $(m, 0)$. For $m, n \in \mathbb{N}$, we have $m \equiv_{\mathbb{Z}} n$ if and only if $m = n$.
4. Negation of integers. For integer $f = (f(+), f(-))$, we can define $-f$ as the function $(f(-), f(+))$.
5. Addition of integers. We can define $f+g$ as addition as functions.
6. We can also subtract integers. In other word, we just define $f-g$ as $f+(-g)$.
7. Thus we have defined the additive group $\mathbb{Z}$.
8. We still have one operation left, which is multiplication. This is not just the product of functions, the reason is that integer multiplication must respect the equivalence relation. If $f \equiv_{\mathbb{Z}} f'$ and $g \equiv_{\mathbb{Z}} g'$, then we must have $fg \equiv_{\mathbb{Z}} f'g'$. The function multiplication does not have this property. The correct formula is
$$fg = f(+)g(+) - f(+)g(-) - f(-)g(+) + f(-)g(-)$$
This is the unique formula satisfying distributive property and rules about negation.
9. New definitions of $+, -, \cdot$ consists with the old definitions when we consider $\mathbb{N}$ embedded into $\mathbb{Z}$.

Definition 5.2.12 (Virtual Species)**.** A ***virtual species*** $\mathcal{V}$ is a generalized weighted species, with generalized weight function $\operatorname{sgn}_V : \mathcal{V} \to \{+1, -1\}$. We can also think $\mathcal{V}$ as a pair of ordinary (unweighted) species, called $\mathcal{V}^+, \mathcal{V}^-$. The $\mathcal{V}_X^+$ is defined to be $\{\alpha \in \mathcal{U}_X : \operatorname{sgn}_V(\alpha) = 1\}$ and similarly $\mathcal{V}_X^- = \{\alpha \in \mathcal{V}_X : \operatorname{sgn}_V(\alpha) = -1\}$.

Definition 5.2.13. Suppose $\mathcal{V}, \mathcal{W}$ be two virtual species, we say $\mathcal{V}, \mathcal{W}$ are ***virtually equivalent*** and write $\mathcal{V} \equiv_{vir} \mathcal{W}$ if and only if $\mathcal{V}^+ + \mathcal{W}^- \equiv \mathcal{W}^+ + \mathcal{V}^-$.

Remark 5.2.14. Every ordinary species $\mathcal{A}$ can be turned into a virtual species by giving it constant sign function $+1$. If $\mathcal{A}, \mathcal{B}$ are ordinary species, then $\mathcal{A} \equiv_{vir} \mathcal{B}$ if and only if $\mathcal{A} \equiv \mathcal{B}$.

Definition 5.2.15. Let $\mathcal{V}$ be a virtual species, we define the ***negation*** of a virtual species $-\mathcal{V}$ as $(-\mathcal{V})_X = \mathcal{V}_X$ with $\operatorname{sgn}_{-V} = -\operatorname{sgn}_V$.

Definition 5.2.16. Let $\mathcal{V}, \mathcal{W}$ be virtual species, then ***addition*** is defined as $\mathcal{V}+\mathcal{W}$ as addition of generalized weighted species. Then ***subtraction*** is defined to be $\mathcal{V} - \mathcal{W} := \mathcal{V} + (-\mathcal{W})$.

Remark 5.2.17. Under such definitions, we get $\mathcal{V} \equiv_{vir} \mathcal{V}^+ - \mathcal{V}^-$. Thus what about other operations? For most, same definitions as generalized weighted species. For others, the GWS definitions does not respect virtual equivalence.

Finally, our new definitions will be consistent with the old definitions.

Remark 5.2.18 (Why Virtual Species?)**.**

1. We have seen some species with no known decomposition but can still get EGF by inclusion exclusion.

2. However, IE is based on numerical equivalence, not natural equivalence.
3. Thus to go beyond EGFs, this is not good enough.
4. Sometimes, when we do not have decomposition, we can come up some virtual decomposition.

Definition 5.2.19. Let $\mathcal{V}$ be a virtual species, then the EGF of $\mathcal{V}$ is just $V(x) = \sum_{n\geq 0} |\mathcal{V}_n|_{\mathrm{sgn}_V} \frac{x^n}{n!}$ where

$$|\mathcal{V}_n|_{\mathrm{sgn}_V} = \sum_{\alpha \in \mathcal{V}_X} \mathrm{sgn}(\alpha)$$

where $|X| = n$.

Remark 5.2.20. A virtual species operation is good, if it respects the virtual equivalence relation. For example, if R is a virtual species operation, with $\mathcal{V} \equiv_{vir} \mathcal{V}'$ and $\mathcal{W} \equiv_{vir} \mathcal{W}'$ then we must have $\mathcal{V}R\mathcal{W} \equiv_{vir} \mathcal{V}'R\mathcal{W}'$.

Definition 5.2.21. The virtual species operations for filters, products (all three), sequences and derivatives and rootings are just the generalized weighted species operation.

Remark 5.2.22. In other word, the GWS definitions for the above operations respect virtual equivalence. We note:

1. Perform operations on bases species (base $\mathcal{V} = \mathcal{V}^+ + \mathcal{V}^-$).
2. Add sign functions according the rules: sign of composite objects is the product of the signs of its components.

In this case, the EGF is just given by the main theorem.

We note the concept of "base species" does not respect virtual equivalence, i.e. $\mathcal{V} \equiv_{vir} \mathcal{U}$ does not imply the base of $\mathcal{V}$ is equivalent to the base of $\mathcal{U}$. But nonetheless, these operations do respect virtual equivalence.

Example 5.2.23 (Products)**.** We have $(\mathcal{V} * \mathcal{W})_X = ((\mathcal{V}^+ + \mathcal{V}^-) * (\mathcal{W}^+ + \mathcal{W}^-))_X$. However, this is just

$$(\mathcal{V}^+ * \mathcal{W}^+)_X \cup (\mathcal{V}^+ * \mathcal{W}^-)_X \cup (\mathcal{V}^- * \mathcal{W}^+)_X \cup (\mathcal{V}^- * \mathcal{W}^-)_X$$

Then we look at the signs and we get

$$\mathcal{V} * \mathcal{W} := \mathcal{V}^+ * \mathcal{W}^+ - \mathcal{V}^+ * \mathcal{W}^- - \mathcal{V}^- * \mathcal{W}^+ + \mathcal{V}^- * \mathcal{W}^-$$

Definition 5.2.24. If $\mathcal{V} * \mathcal{W} \equiv_{cir} 1$ then we say $\mathcal{W}$ is a ***multiplicative inverse*** of $\mathcal{V}$.

Theorem 5.2.25.

1. *$\mathcal{V}$ has a multiplicative inverse if and only if $\mathcal{V}_0 \equiv_{vir} 1$.*
2. *Assume $\mathcal{V}$ has multiplication inverse $\mathcal{V}^{-1}$, then it is unique up to virtual equivalence.*
3. *If $\mathcal{V}_0 \equiv_{vir} 1$ then $\mathcal{V}^{-1} \equiv_{vir} (-\mathcal{V}_+)^*$.*
4. *The EGF of $\mathcal{V}^{-1}$ is $V(x)^{-1}$.*

Proof. We will only talk about (3). We note we have $\mathcal{V} * \mathcal{V}^{-1} \equiv_{vir} (1 + \mathcal{V}_+) * (1 - \mathcal{V}_+ + \mathcal{V}_+^2 - \mathcal{V}_+^3 + \ldots) = 1$. ♡

Example 5.2.26. We have

$$\begin{aligned}\mathcal{E}^{-1} &= (-\mathcal{E}_+)^* \\ &= \sum_{n\geq 0}(\mathcal{E}_+)^{2n} - \sum_{n\geq 0}(\mathcal{E}_+)^{2n+1}\end{aligned}$$

where we see the left part is just ordered set partitions with even number of parts and the right sum is ordered set partitions with odd number of parts.

Remark 5.2.27. We finally talk about virtual multisort species. Everything so far extends to multisort species. Moreover, the diagonal operation ∇ on generalized weighted 2-species also respect virtual equivalence.

Remark 5.2.28 (Compositions). We will not give a definition of virtual species composition. Instead, we will give a set of rules so that we can compute virtual species compositions.

Let $\mathcal{A}, \mathcal{B}$ be ordinary connected 1-species or 2-species. Let $\mathcal{V}$ be virtual 1-species and $\mathcal{Z}$ be virtual 2-species.

Then we have:

1. $\mathcal{V}[\mathcal{A}] \equiv_{vir} \mathcal{V}^+[\mathcal{A}] - \mathcal{V}^-[\mathcal{A}]$.
2. $\mathcal{Z}[\mathcal{A}, \mathcal{B}] \equiv_{vir} \mathcal{Z}^+[\mathcal{A}, \mathcal{B}] - \mathcal{Z}^-[\mathcal{A}, \mathcal{B}]$.
3. $\mathcal{V}[-\mathcal{X}] \equiv_{vir} \mathcal{V} \boxtimes \mathcal{E}^{-1}$.
4. $\mathcal{Z}[\mathcal{X}_1, -\mathcal{X}_2] \equiv_{vir} \mathcal{Z} \boxtimes (\mathcal{E}[\mathcal{X}_1] * \mathcal{E}^{-1}[\mathcal{X}_2])$.
5. $\mathcal{V}[-\mathcal{B}] \equiv_{vir} \mathcal{V}[-\mathcal{X}][\mathcal{B}]$.
6. $\mathcal{Z}[\mathcal{A}, -\mathcal{B}] \equiv_{vir} \mathcal{Z}[\mathcal{X}_1, -\mathcal{X}_2][\mathcal{A}, \mathcal{B}]$.
7. $\mathcal{V}[\mathcal{A} - \mathcal{B}] \equiv_{vir} \mathcal{V}[\mathcal{X}_1 + \mathcal{X}_2][\mathcal{A}, -\mathcal{B}]$.

In each of the case, the rule is a slight generalizations of an identity for ordinary species. For example, for $k \in \mathbb{N}$, we can define $k\mathcal{X} = \mathcal{X} + \mathcal{X} + ... + \mathcal{X}$. Then $\mathcal{A}[k\mathcal{X}] \equiv \mathcal{A} \boxtimes \mathcal{E}^k$. Now let $\mathcal{A}$ be virtual and $k = -1$, then we get (3) in the above rule.

Example 5.2.29. If $\mathcal{W}$ is connected virtual species, then $\mathcal{E}[\mathcal{W}] \equiv_{vir} \mathcal{E}[\mathcal{W}^+] * \mathcal{E}^{-1}[\mathcal{W}^-]$.

Proof: We see $\mathcal{E}[\mathcal{W}] \equiv_v \mathcal{E}[\mathcal{W}^+ - \mathcal{W}^-]$. Then by rule (7), we get this is the same as

$$\mathcal{E}\mathcal{W} \equiv_v \mathcal{E}[\mathcal{X}_1 + \mathcal{X}_2][\mathcal{W}^+, -\mathcal{W}^-]$$

Now apply rule (6), we get this is

$$\mathcal{E}[\mathcal{W}] \equiv \mathcal{E}[\mathcal{X}_1 + \mathcal{X}_2][\mathcal{X}_1, -\mathcal{X}_2][\mathcal{W}^+, \mathcal{W}^-]$$

Now use rule (4), we get

$$\mathcal{E}[\mathcal{W}] \equiv \left(\mathcal{E}[\mathcal{X}_1 + \mathcal{X}_2] \boxtimes (\mathcal{E}[\mathcal{X}_1] * \mathcal{E}^{-1}[\mathcal{X}_2])\right)[\mathcal{W}^+, \mathcal{W}^-]$$

However, we note the multiplicative identity for the operation superposition is $\mathcal{E}[\mathcal{X}_1 + \mathcal{X}_2]$, and hence we get

$$\mathcal{E}[\mathcal{X}_1 + \mathcal{X}_2] \boxtimes (\mathcal{E}[\mathcal{X}_1] * \mathcal{E}^{-1}[\mathcal{X}_2]) = \mathcal{E}[\mathcal{X}_1] * \mathcal{E}^{-1}[\mathcal{X}_2]$$

Thus by use rule (2) and (1) a few times, we would get

$$\mathcal{E}[\mathcal{W}] \equiv_{vir} \mathcal{E}[\mathcal{W}^+] * \mathcal{E}^{-1}[\mathcal{W}^-]$$

Theorem 5.2.30.

1. *Rule* (1) *to* (7) *can be used to compute* $\mathcal{V}[\mathcal{W}]$ *for any virtual species* $\mathcal{V}, \mathcal{W}$ *with* $\mathcal{W}$ *connected*
2. *Virtual species composition respects* $\mathcal{E}_{vir}$.
3. *EGF of* $\mathcal{V}[\mathcal{W}]$ *is* $V(W(x))$.

Remark 5.2.31. Recall 2-species inclusion exclusion technique:

1. $\mathcal{A}$ is a species with structures have flaws $\subseteq X$.
2. $\mathcal{B}$ is the species of good $\mathcal{A}$-structures.
3. $\mathcal{M}^S$ is the 1-species of split $\mathcal{A}$-structures with marked flaws.

In this set-up, we have the following theorem.

Theorem 5.2.32. $\mathcal{B} \equiv_{vir} \mathcal{M}^S[\mathcal{X}, -\mathcal{X}]$.

Proof. Define a split version of $\mathcal{A}$, defined to be $\mathcal{A}^S$, as the subspecies of $\mathcal{M}^S$, where all flaws are marked. Then $\mathcal{M}^S \equiv \mathcal{A}^S[\mathcal{X}_1, \mathcal{X}_1 + \mathcal{X}_2]$ where the $\mathcal{X}_1 + \mathcal{X}_2$ is like unmark subset of the flaws. Thus we get $\mathcal{B} \equiv \mathcal{A}^S[\mathcal{X}, 0]$, which is no marks, which is the same as no flaws. Thus we get

$$\mathcal{B} \equiv_{vir} \mathcal{A}^S[\mathcal{X}, \mathcal{X} + (-\mathcal{X})] \equiv_{vir} \mathcal{M}^S[\mathcal{X}, -\mathcal{X}]$$

♡

5.3 Exercise 5

Example 5.3.1. Let $(P, \leq)$ be a finite poset. The Hasse diagram of P is the directed graph Γ where $V(\Gamma) = P$, and we have a directed edge from α to β if $\alpha < \beta$ and there does not exist $\gamma \in P$ so $\alpha < \gamma < \beta$.

1. Draw the Hasse diagram for $\mathcal{P}_{fin}(\{1, 2\})$ and $\mathcal{P}_{fin}(\{1, 2, 3\})$
2. Show $\alpha \leq \beta$ in P iff there is a directed path from α to β in the Hasse diagram.

Example 5.3.2. For a poset $\mathbb{N}$, show

$$\mu(a, b) = \begin{cases} 1, & b = a \\ -1, & b = a + 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution. Well, $b = a$ then it is 1. If $b = a + 1$ then it is -1. Now suppose $b < a$, then it is zero. Thus now suppose $a \leq b$ with $b = a + n$ where $n \geq 2$. In this case, we do induction. If $n = 2$ then we get $1 - 1 = 0$. Suppose induction holds for all values less than or equal n, i.e. $\mu(a, b - 1) = 0$. Then we see

$$\mu(a, b) = -\sum_{a \leq y < b} \mu(a, y) = -(1 - 1 + \text{ sum where induction holds}) = 0$$

♠

Example 5.3.3. Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be locally finite posets. We say P_1, P_2 are isomorphic posets if there exists order preserving bijection. Suppose $a_1, b_1 \in P_1, a_2, b_2 \in P_2$ and suppose the interval $[a_1, b_1]$ and $[a_2, b_2]$ be isomorphic. Show $\mu_{P_1}(a_1, b_1) = \mu_{P_2}(a_2, b_2)$.

Solution. Well, if $a_1 = b_1$, then we see b_1 must equal b_2 as its isomorphic intervals. Hence $\mu(a_1, b_1) = \mu(a_2, b_2) = 1$. Similarly if $a_1 > b_1$ then $a_2 > b_2$. Thus suppose $a_1 < b_1$, we see it has a bijection between $[a_1, b_1]$ and $[a_2, b_2]$ that perserves order, thus $\sum_{\gamma \in [a_1, b_1)}$ has the same number of terms as $\sum_{\gamma \in [a_2, b_2)}$ and by induction we see the claim holds. ♠

Example 5.3.4. Let $(P, \leq)$ be a poset. Show μ is the Mobius function of P iff:

1. $\mu(a, b) = 0$ if $a \not\leq b$.
2. $\sum_{\gamma \in [a,b]} \mu(a, b) = \delta_{a,b}$ for all $a, b \in P$.

Solution. Suppose μ is the Movius function, then we just need to prove (2) as (1) is by definition. To that end, suppose $x < z$, then we see $\mu(x, z) = -\sum_{x \leq y < z} \mu(x, y)$ by definition. Thus we see

$$\mu(x, z) - \mu(x, z) = -\sum_{x \leq y < z} \mu(x, y) - \mu(x, z) = -\sum_{x \leq y \leq z} \mu(x, y) = 0 = \delta_{x,z}$$

Conversely, say the two conditions hold. Then we see $\mu(a, b) = 1$ if $a = b$, equal 0 if $a \not\leq b$. Next, say $a < b$, then we see $\mu(a, b) = 0$, which equal the Mobius function's output. Thus it is the Movius function as desired. ♠

Example 5.3.5. Let $(P, \leq)$ be a poset. Show the conditions (1) and (2) in the above is equivalent to the statement that $\mu\zeta = I$ as matrices. More precisely, this means

$$\sum_{\gamma \in P} \mu(a, \gamma)\zeta(\gamma, b) = \delta_{\alpha,\beta}$$

Example 5.3.6. Use above examples to prove the theorem about product of posets.

Example 5.3.7. Prove the Mobius function for $P_f(X)$ is

$$\mu(a, b) = \begin{cases} (-1)^{|b|-|a|}, & a \subseteq b \\ 0, & \text{otherwise} \end{cases}$$

Solution. Suppose $a \subseteq b$ first. If $a = b$ then we are done. Thus say $a \subsetneq b$, and we see $\mu(a, b) = \sum_{a \subseteq c \subsetneq b} \mu(a, c)$. Well, we use induction. If $|b| - |a| = 1$ then we see there is only one $a \subseteq c \subsetneq b$, which is a itself. Hence $\mu(a, b) = -(\mu(a, a)) = -1$ as desired. Now suppose the induction holds for $|b|-|a|<n$. Say $|b|-|a|=n$ now and we see

$$-\sum_{c\in[a,b)}\mu(a,c)=-\sum_{a\subseteq c\subsetneq b}(-1)^{|b|-|c|}=-\sum_{i=1}^{n-1}\binom{n}{i}(-1)^i=((-1+1)^n+\binom{n}{n}(-1)^n)$$

This concludes the proof. ♠

Example 5.3.8. Use inclusion-exclusion to count fixed point free endofunctions. Here $\mathcal{S}$ is the set of all endofunctions $\phi:[n]\to[n]$, and the rest of the set-up is similar to the derangements example, i.e. $B=[n]$, and for $b\in B$, $\phi\in S$ has flaw iff $\phi(b)=b$.

Solution. Well, $\mathcal{S}$ is set of endofunctions, the set of flaws will be $[n]$, then ϕ has a flaw if $\phi(b)=b$. Thus $Flaws(\phi)=\{b\in B:\phi(b)=b\}$. Hence $F_\alpha=\{\phi\in\mathcal{S}:Flaws(\phi)\supseteq\alpha\}$ and $G_\alpha=\{\phi:Flaws(\phi)=\alpha\}$. Now set $f(\alpha)=|F_\alpha|, g(\alpha)=|G_\alpha|$, we get, by Movius inversion that $|T|=g(\emptyset)=\sum_{\beta\subseteq B}(-1)^{|\beta|}f(\beta)$. To this end, we see $F_\alpha=\{\phi:\forall i\in\alpha,\phi(i)=i\}$ and hence we get $f(\alpha)=n^{n-|\alpha|}$. Thus we see

$$g(\emptyset)=\sum_{k=0}^{n}\binom{n}{k}(-1)^k n^{n-k}=(n-1)^n$$

♠

Example 5.3.9. Use inclusion-exclusion to count surjective functions $\phi:[m]\to[n]$. The set-up is follows: let S be the set of all functions $\phi:[m]\to[n]$. Let $B=[n]$, for $b\in B$, $\phi\in S$ has flaw b iff $b\notin\text{Im}(\phi)$.

Solution. We are looking at $F_\alpha=\{\phi:\forall a\in\alpha,\alpha\notin\text{Im}(\phi)\}$. Well, say $|\alpha|=k$, then we see we have the image should have at most $n-k$ many elements, i.e. we are looking at set of functions from $[m]$ to $[n-k]$. In other word, we get

$$|\mathcal{T}|=\sum_{k=0}^{n}\binom{n}{k}(-1)^k(n-k)^m$$

♠

Example 5.3.10. Use IE to count derangement again. Let S be the set of fixed point free endofunctions $\phi:[n]\to[n]$. THen $\phi\in S$ is a derangement iff ϕ is surjective (derangements are fixed point free bijection and surjection from $[n]$ to $[n]$ is just bijection). Hence let $B=[n]$ and for $b\in B$, we say ϕ has flaw b iff $b\notin\text{Im}(\phi)$. Compute the number of derangement in this set-up.

Solution. Well, in this let $|\alpha|=k$, then we see $F_\alpha=\{\phi:\forall i\in[n],\phi(i)\neq i,\forall j\in\alpha,j\notin\text{Im}(\phi)\}$. Then say $[n]=\{a_1,...,a_k\}\cup\{b_1,...,b_{n-k}\}$, we see for each b_i we have

$(n-k-1)$ choices to map to because they cannot be mapped to any of a_i and it cannot be mapped to itself. For a_i we see we have $n-k$ choices to map to. Thus

$$|F_\alpha| = (n-k)^k(n-k-1)^{n-k}$$

This concludes

$$|T| = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^k(n-k-1)^{n-k} = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

♠

Example 5.3.11. For the problem of counting derangements in S_n, show the rook polynomial is $(1+x)^n$. Verify the formula

$$[x^n]R_B(-x)\sum_{j\geq 0} j!x^j$$

gives the correct answer.

Solution. Well, derangements are permutations with the diagonal disallowed. In other word, $B \subseteq [n]\times[n]$ should be $B = \{(a,a) : 1 \leq a \leq n\}$. Then $\mathcal{D}_B$ would be the set of derangements. In particular, we see $R_B(x) = \sum_{k=0}^{n} r_k x^k$ with $r_k = \binom{n}{k}$ and so we get $R_B(x) = (1+x)^n$. Then we see

$$R_B(-x)\sum_{j\geq 0} j!x^j = (1-x)^n\sum_{j\geq 0} j!x^j$$

Thus we get

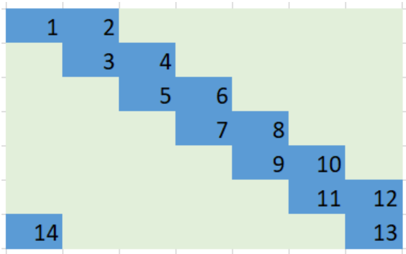
$$\begin{aligned}
[x^n]R_B(-x)\sum_{j\geq 0} j!x^j &= [x^n]\sum_{k\geq 0}(-1)^k\binom{n}{k}x^k\sum_{j\geq 0} j!x^j \\
&= \sum_{k+j=n}(-1)^k\binom{n}{k}j! \\
&= \sum_{k=0}^{n}(-1)^k\frac{n!}{(n-k)!k!}(n-k)! \\
&= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}
\end{aligned}$$

♠

Example 5.3.12. Let B be the configuration in the Menage problem.

1. Show arrangments of k non-attacking rooks are in bijection with k-subsets $S \subseteq [2n]$, such that $x, y \in S$, $x \neq y \pm 1 \pmod{2n}$.
2. Count these arrangments by elementary methods. Hint: Break up the count into two cases: $2n \in S$ and $2n \notin S$.

Solution. Well, one example of such bijection is given by



In other word, we just put the $2n$ number on the disallowed blocks the way we did for the $n = 7$ case above, then k non-attacking rooks would give a k subset of $[2n]$ and vice-versa. In particular, the condition $x \neq y \pm 1 \pmod{2n}$ is exactly the non-attacking condition.

Thus we need to count number of subsets now. annnnd I'm not sure how to do it. ♠

Example 5.3.13. Compute the number of 012 strings of length n without substring 0112.

Solution. Well, we use marking. Say T is the set of 012 strings without 0112, S is the set of all 012 strings, let F be the set of 012 strings with 0112 as substring, may or may not be marked. Then we see $F = \{0, 1, 2, (0112)\}^*$. Hence we get

$$F(x,t) = \frac{1}{1 - (3x + x^4 t)}$$

and so $T(x) = \frac{1}{1-3x+x^4}$ ♠

Example 5.3.14. Say we want to count 01-strings with respect to two weight functions: the length and the number of occurrences of 011 as substring. What is the ordinary generating function?

Solution. Note $G(t) = F(t-1)$ and hence it should just be $F(x, t-1) = \frac{1}{1-(2x+x^3(t-1))}$ ♠

Example 5.3.15. Compute the number of 01-strings of length n that do not have 011 or 0001 as substring.

Solution. Well, in this case $\mathcal{F} = \{0, 1, (011), (0001), (00(01)1)\}^*$ and hence the proof follows from here. ♠

Example 5.3.16. As we already see above, if forbiden substrings overlap, things become trickier. Say $\mathcal{S}$ is the set of all 01-strings, $\mathcal{T}$ subset of strings don't have 00 as substring. In this case, $\mathcal{F} = \{0, 1, (00), (0(0)0), (0(0)(0)0), ...\}^*$ and its not hard to compute $F(x,t) = \frac{1}{1-(2x+\frac{x^2 t}{1-xt})}$.

Example 5.3.17. By direct computation, verify the statement of the matrix tree theorem in the cases where $|V| = 2$ and $|V| = 3$.

Solution. Well, say $V = \{1, 2\}$. Then we see we have 4 endo functions in total, $f_1 = \mathrm{Id}$, $f_2 = (1 \to 2, 2 \to 2)$, $f_3 = (1 \to 1, 2 \to 1)$ and $f_4 = (1 \to 2, 2 \to 1)$. In particular, three of them are in $\mathcal{T}$ because the longest possible cycle on two vertices is 2. Thus $|\mathcal{T}|_{Wt} = \sum Wt(f_i) = x_1x_2 + y_{12}x_2 + x_1y_{12}$. On the other hand, we see

$$M = \begin{bmatrix} x_1 + y_{12} & -y_{12} \\ -y_{12} & x_2 + y_{12} \end{bmatrix}$$

Thus we get $\det(M) = x_1x_2 + y_{12}x_2 + x_1y_{12}$ as desired. ♠

Example 5.3.18. In the case where we can use the "2-species technique", show that:

1. $\mathcal{M}^s$ is a split version of $\mathcal{M}$, i.e. $\mathcal{M}^s[\mathcal{X}, \mathcal{X}] = \mathcal{M}$.
2. $M(x; t) = M^s(x, xt)$.

Example 5.3.19. Define $\mathcal{A}^s$ to be the 2-species which the subspecies of $\mathcal{M}^s$ with all flaws marked.

1. Show $\mathcal{A}^s$ is a split version of $\mathcal{A}$.
2. Explain why $\mathcal{M}^s = \mathcal{A}^s[\mathcal{X}_1, \mathcal{X}_1 + \mathcal{X}_2]$.

Solution. Well, note $\mathcal{X}_1 + \mathcal{X}_2$ means we can place both set X as label or Y as label on the second sort of label receivers, i.e. those marked flaws can come from first set or second set, i.e. meaning its exactly some flaws being marked and some not. ♠

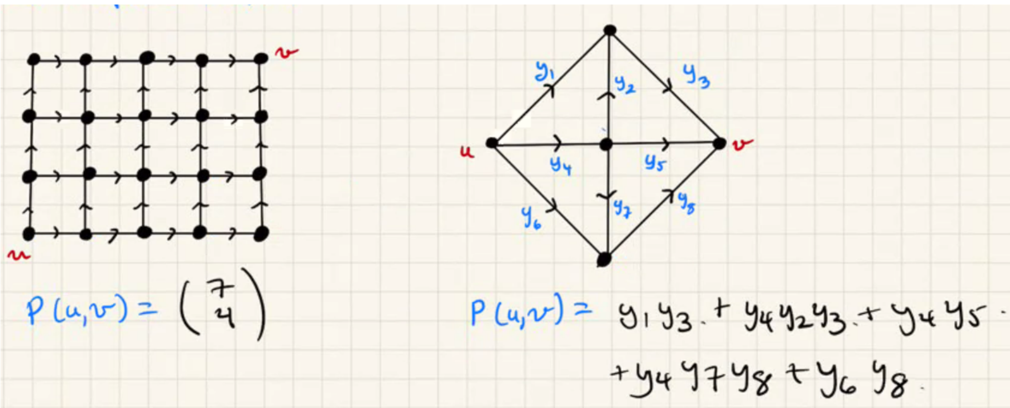
5.4 Tutorial 8 (Non-Intersecting Paths)

Remark 5.4.1. Let Γ be acyclic directed graph, R a ring, for each edge $e \in E(\Gamma)$, we let $Wt(e) \in R$. For each subgraph $Q \subseteq \Gamma$, we let $Wt(Q) = \prod_{e \in E(Q)} Wt(e)$.

Remark 5.4.2. For $u, v \in V(\Gamma)$, we define the path enumerator

$$P(u, v) = \sum_{Q \text{ a path from } u \to v} Wt(Q)$$

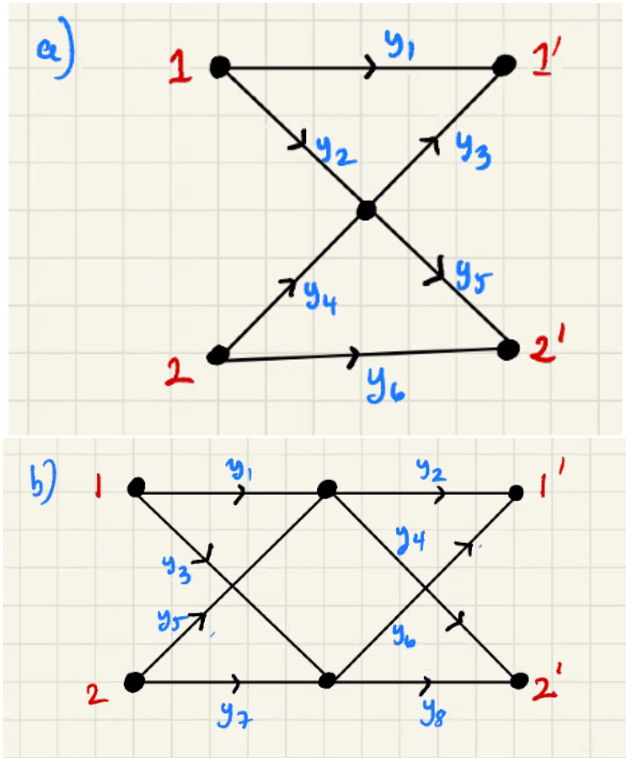
For example, we have



Example 5.4.3. Compute

$$\det \begin{bmatrix} P(1,1') & P(1,2') \\ P(2,1') & P(2,2) \end{bmatrix}$$

for the following graphs. Take note of which terms cancel and which terms don't. The graphs are given by



Solution. For (a) we get

$$\det \begin{bmatrix} y_1 + y_2y_3 & y_2y_5 \\ y_3y_4 & y_6 + y_4y_5 \end{bmatrix} = y_1y_6 + y_1y_4y_5 + y_2y_3y_6 + y_2y_3y_4y_5 - y_2y_3y_4y_5$$

For (b) we get

$$\det \begin{bmatrix} y_1y_2 + y_3y_4 & y_1y_6 + y_3y_8 \\ y_5y_2 + y_7y_4 & y_5y_6 + y_7y_8 \end{bmatrix}$$

Compute this we get

$$y_{1278} + y_{3456} - y_{1647} - y_{3825}$$

In both cases, we observe the terms cancelled involves intersecting paths. ♠

Definition 5.4.4. Let $1, 2, ..., n$ and $1', 2', ..., n' \in V(\Gamma)$ with Γ acyclic graph. Define an ***n-path*** to be $Q : (1, 2, ..., n) \to (1', ..., n')$ is an $(n+1)$-tuple $(\sigma_1, Q_1, ..., Q_n)$ where $\sigma \in S_n$ and $Q_i : i \to \sigma(i)'$ is directed path. Then we define

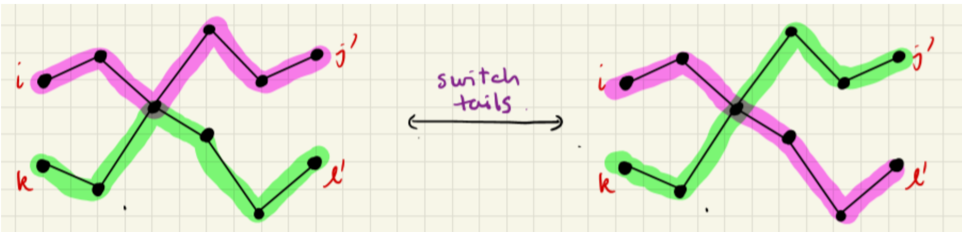
$$Wt(Q) = \operatorname{sgn}(\sigma) \prod_{i=1}^{n} Wt(Q_i)$$

Definition 5.4.5. We say n-path Q is ***non-intersecting*** iff $Q_1, ..., Q_n$ pairwise vertex disjoint. We define $\mathcal{T}$ be the set of non-intersecting n-paths.

Lemma 5.4.6 (Lindstrom-Gessel-Viennot Lemma)**.** *We have*

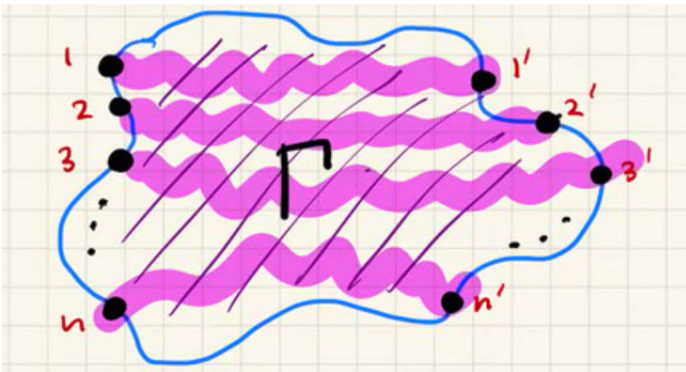
$$\det \begin{bmatrix} P(1,1') & \dots & P(1,n') \\ \vdots & \ddots & \vdots \\ P(n,1') & \dots & P(n,n') \end{bmatrix} = |\mathcal{T}|_{Wt}$$

Proof. Let $\mathcal{J}$ be the set of all n-paths, not just the non-intersecting n-paths. Expanding the determinant naturally yields $\det(P(i,j')) = |\mathcal{J}|_{Wt}$. To prove the result, we need a sign-reversing involution $\Phi : \mathcal{J} \backslash \mathcal{T} \to \mathcal{J} \backslash \mathcal{T}$. The main idea is:



This is sign-reversing, with some care, one can get an involution. ♡

Remark 5.4.7. Assume Γ embedded in the plane, let $1, ..., n, 1', ..., n'$ be on the outer face, in this order. For example, we have:



In this case, every $Q \in \mathcal{T}$ is of the form $Q = (\mathrm{Id}, Q_1, ..., Q_n)$ with $Q_i : i \to i'$. As a result of this, we see $|\mathcal{T}|_{Wt} = +1$ in the planar case.

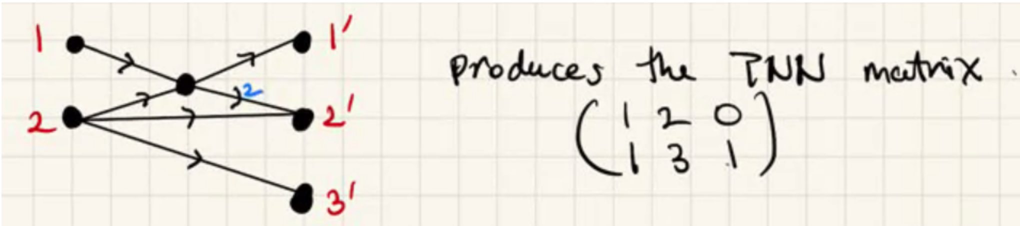
Remark 5.4.8. Now we talk about some applications, involves TNN matrices, weakly non-crossing lattice paths,

Definition 5.4.9. A totally non-negative matrices is a matrix with all minors ≥ 0.

Example 5.4.10. For example, we see $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ is TNN, since all entries are non-negative and all 2 by 2 minors are non-negative.

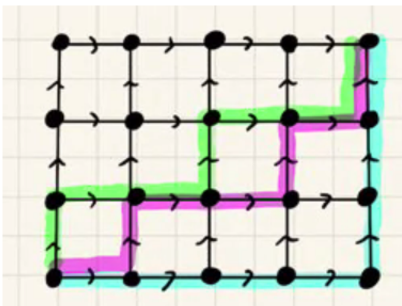
Theorem 5.4.11. *Let $M \in M_{m\times n}(\mathbb{R})$, then M is TNN if and only if there exists a directed acyclic planar graph Γ with edge weight function $Wt : E(\Gamma) \to \mathbb{R}_+$ and vertices $1, ..., n, 1', ..., n'$ such that $M_{i,j} = P(i, j')$.*

Example 5.4.12. The example above continued:

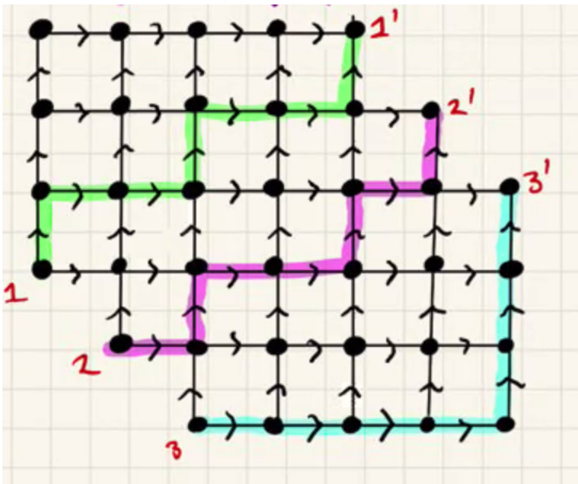


Definition 5.4.13. A ***non-crossing*** paths can intersect, but upper path must stay weakly above the lower path.

Example 5.4.14. For example, the following is three non-crossing paths:

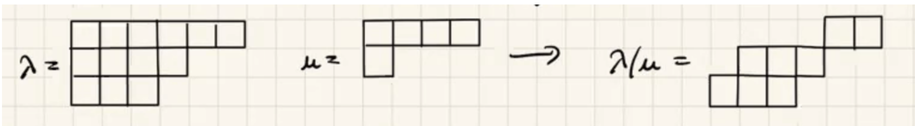


In particular, we can transform non-crossing to non-intersecting by shifting paths as follows:



Definition 5.4.15. A ***skew partition*** is λ/μ is a pair of partitions where $\mu \subseteq \lambda$, depicted as the "difference of diagrams"

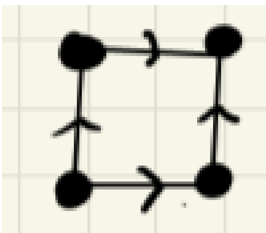
Example 5.4.16. For example, we have



Try to determine the number of skew partitions inside k by l rectangles.

Solution. It is just the above example. ♠

Example 5.4.17. By consider n non-crossing paths in the following diagram:



Try to compute the following:

$$
\det \begin{pmatrix}
2 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 2 & 1 & 0 & & 0 & 0 \\
0 & 1 & 2 & 1 & & 0 & 0 \\
0 & 0 & 1 & 2 & & 0 & 0 \\
\vdots & & & & \ddots & & \vdots \\
0 & 0 & 0 & 0 & & 2 & 1 \\
0 & 0 & 0 & 0 & \cdots & 1 & 2
\end{pmatrix}
$$

Solution. Do the path shift and by counting non-intersecting from that graph we are done. This is an Cartan matrix. ♠

Example 5.4.18. How many ways to pack with cubes in the corner of an $n \times n \times n$ cube?

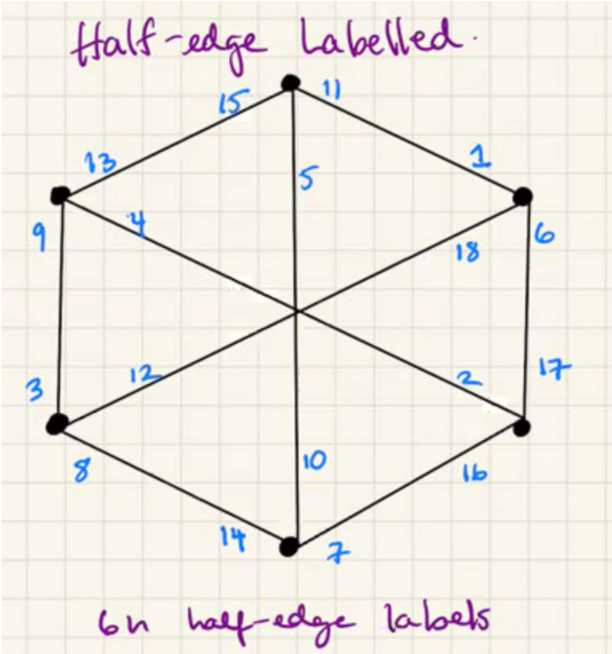
Solution. To solve this, we think each layer as a path, then we are counting n non-crossing paths in $n+1$ by $n+1$ lattice. ♠

5.5 Tutorial 9 (3-Regular Graphs)

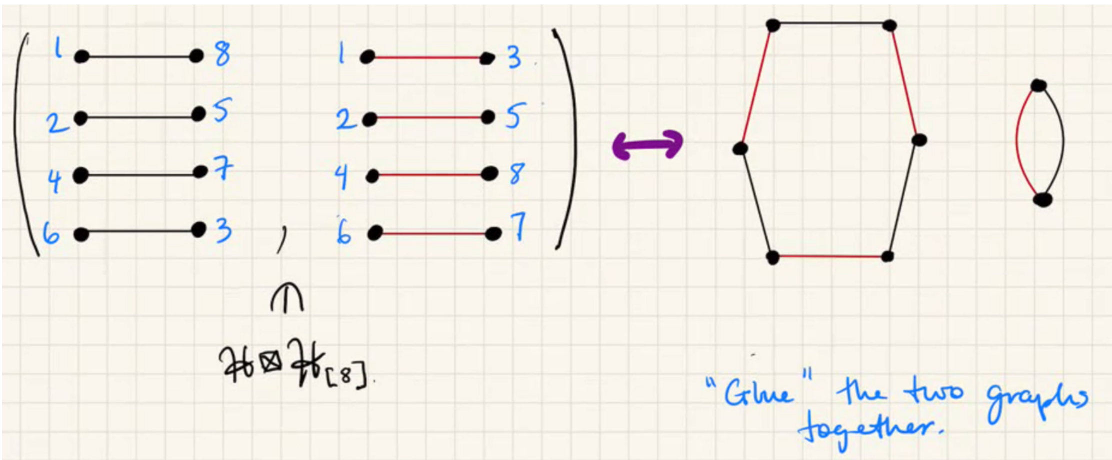
Definition 5.5.1. A k-***regular*** graph is a graph with each vertex has degree k.

Remark 5.5.2 (Half-Edge VS Vertex Labelling)**.** Our goal is to count vertex labelled graphs, but we will count half-edge labelled graphs instead of vertex labelled overcount by some factor (the factor should be $\frac{(6n)!}{(2n)!}$).

The following is a graph with half-edges labelled:



Remark 5.5.3 (Superposition As "Gluing"). Let $\mathcal{H}$ be the species of 1-regular graphs. Then we see



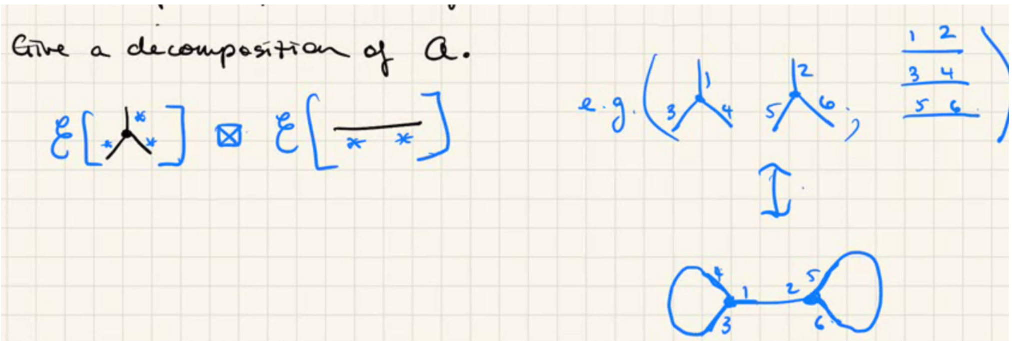
Remark 5.5.4 (Species Defined By Diagram). We use

1. $*$ as label receivers of 1st sort.
2. $\#$ as label receivers of 2nd sort.
3. $\to$ as directed edges.
4. $\rightsquigarrow$ as marking.

Example 5.5.5. Let $\mathcal{A}$ be the species of half-edge labelled 3-regular graphs, loops and parallel edges allowed.

1. Give a decomposition of $\mathcal{A}$.
2. Compute $\#\mathcal{A}_{6n}$.

Solution. We see



and hence

$$|\mathcal{A}_{6n}| = [\frac{x^{6n}}{(6n)!}] \exp(\frac{x^3}{3!}) \cdot [\frac{x^{6n}}{(6n)!}] \exp(\frac{x^2}{2}) = \frac{((6n)!)^2}{(3!)^{2n}(2n)! \cdot (2!)^{3n}(3n)!}$$

Then divide the overcount factor, we are done. ♠

Remark 5.5.6. Now suppose $n = 1$, then we get the answer is

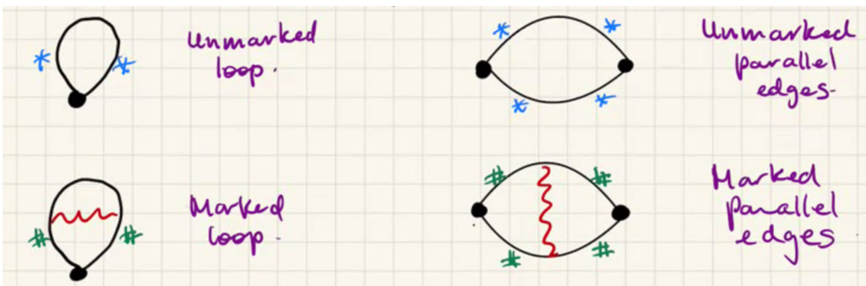
$$\frac{6!}{3^3 \cdot 2^2 \cdot 3!} = \frac{10}{9}$$

What is wrong? Well, the overcounting factor we computed only works for simple graphs, while we allowed loops and parallel edges when defining $\mathcal{A}$. Thus we need to get rid of the loops and parallel edges, using inclusion exclusion.

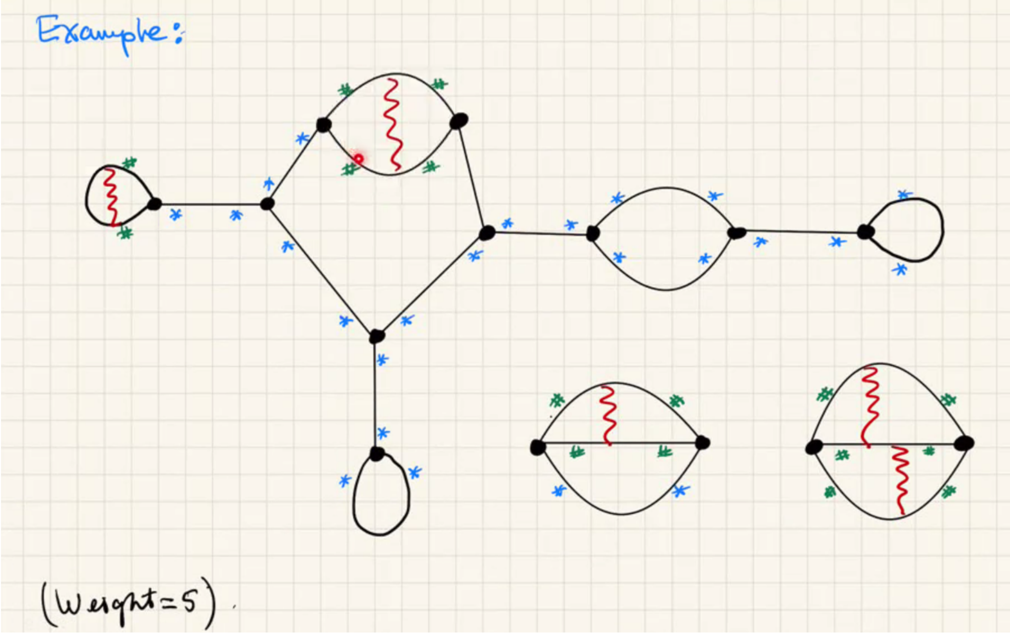
Let $\mathcal{M}^s$ be the split and weighted species of half edge labelled graphs, with marked flaws.

1. Loops and pairs of parallel edges may be marked.
2. Half-edges not touching a mark are of 1st sort. Half-edges touching a mark are of 2nd sort.
3. Weight function equal the number of marking.

For example, we have



The following is also an example:

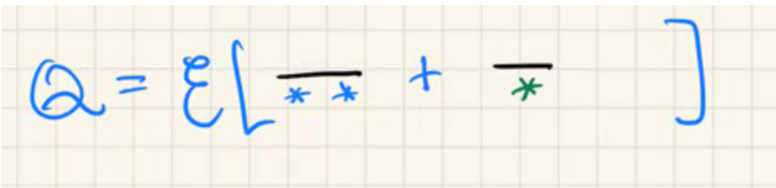


Now the goal is to compute $\mathcal{M}^s$.

Example 5.5.7. Give a decomposition of $\mathcal{M}^s$.

Solution. Well, we are going to glue things together, say $\mathcal{M}^s = \mathcal{P} \boxtimes \mathcal{Q}$, where $\mathcal{P}$ is local structures and $\mathcal{Q}$ is the gluing edges. In particular, we see

Then our $\mathcal{Q}$ should join the local structures using first sorts and not glue the second sort label receivers. In other word, we have



♠

Example 5.5.8. Compute $M^s(x_1, x_2; t)$.

Solution. We see

$$P(x_1, x_2; t) = \exp\left(\frac{x_1^3}{3!} + \frac{tx_1x_2^2}{2!} + \frac{tx_1^2x_2^4}{4} + \frac{t^2x_2^6}{4} + \frac{t^3x_2^6}{2 \cdot 3!}\right)$$

and

$$Q(x_1, x_2; t) = \exp(\frac{x_1^2}{2!} + x_2)$$

♠

Remark 5.5.9. To finish the question, apply the weighted species inclusion exclusion technique. Let $\mathcal{M}$ be the species of $\mathcal{A}$-structures with marked flaws. Then we see

$$\mathcal{M} = \mathcal{M}^s[\mathcal{X}, \mathcal{X}] = (\mathcal{P} \boxtimes \mathcal{Q})[\mathcal{X}, \mathcal{X}]$$

Then we see

$$M(x; t) = \sum_{m,n \geq 0} \left([\frac{x_1^m x_2^n}{m!n!}] P(x_1, x_2; t) [\frac{x_1^m x_2^n}{m!n!}] Q(x_1, x_2; t) \right) \frac{x^{m+n}}{m!n!}$$

and hence the EGF for simple 3-regular half-edge labelled graphs is

$$M(x; -1)$$

5.6 Counting With Automorphism

Definition 5.6.1. Let G be a finite group, Y a finite set of "points" with a G-action. Then we define the ***orbit*** of $y \in Y$ is given by

$$\tilde{y} = G \cdot y := \{g \cdot y : g \in G\}$$

We write the set of all orbits as $\tilde{Y} = Y/G = \{\tilde{y} : y \in Y\}$. We define the ***stabilizer*** of $y \in Y$ is given by

$$G_y := \{g \in G : gy = y\}$$

We define the fixed points of $g \in G$ by

$$Y^g := \{y \in Y : gy = y\}$$

Remark 5.6.2. We recall the orbit-stabilizer theorem, which states $|G_y| \cdot |\tilde{y}| = |G|$. We also recall the orbit counting lemma, which states

$$|\tilde{Y}| = \frac{1}{|G|} \sum_{g \in G} |Y^g|$$

Now we prove this. We see

$$\begin{aligned}
\frac{1}{|G|}\sum_{g\in G}|Y^g| &= \frac{1}{|G|}\sum_{\substack{(g,y)\in G\times Y\\ gy=y}} 1\\
&= \frac{1}{|G|}\sum_{y\in Y}|G_y|\\
&= \sum_{y\in Y}\frac{|G_y|}{|G|}\\
&= \sum_{y\in Y}\frac{1}{|\tilde{y}|}\\
&= \sum_{\tilde{y}\in\tilde{Y}}\sum_{y\in\tilde{y}}\frac{1}{|\tilde{y}|}\\
&= \sum_{\tilde{y}\in\tilde{Y}} 1\\
&= |\tilde{Y}|
\end{aligned}$$

Example 5.6.3. How many ways can we colour n-gon up to rotation? Well, let K be the set of colours, say $|K| = k$, and let our group be $C_n = (\mathbb{Z}_n, +)$, which we can think as sides of n-gon.

We want to count $Y = K^{\mathbb{Z}_n}$, which we can think as functions from $\mathbb{Z}_n$ to K, which correspond to a way of colour the n-gon. The group action will be, for function $f : \mathbb{Z}_n \to K$ and $c \in C_n$, we define C_n-action

$$cf : \mathbb{Z} \to K, \quad i \mapsto f(i-c)$$

Then our goal is to count the orbits. Hence we need to look at fixed points. For $c \in C_n$, we see

$$\begin{aligned}
Y^c &= \{f : \mathbb{Z}_n \to K : \forall i \in \mathbb{Z}_n, f(i-c) = f(i)\}\\
&= \{f : \mathbb{Z}_n \to K : \forall i \in \mathbb{Z}_n, f(i-d) = f(i), d = \gcd(c,n)\}
\end{aligned}$$

Thus we see $|Y^c| = k^{\gcd(c,n)}$ and hence

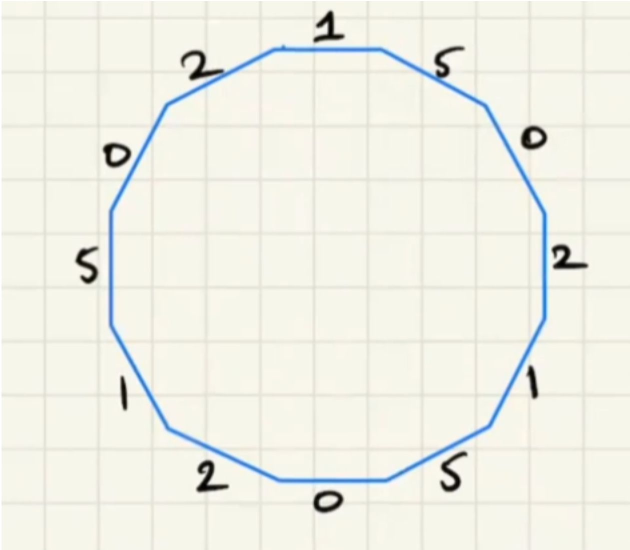
$$|\tilde{Y}| = \frac{1}{n}\sum_{c\in C_n} k^{\gcd(c,n)} = \frac{1}{n}\sum_{d|n}\phi(\frac{n}{d})k^d$$

where ϕ is the Totient function.

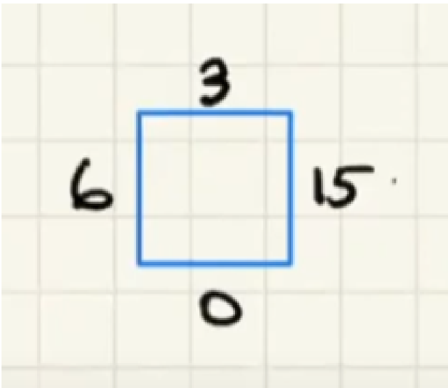
Example 5.6.4. Let $\mathscr{M}_m := \{f : \mathbb{Z}_n \to \mathbb{N} : \sum_{i\in\mathbb{Z}_n} f(i) = m\}$. How many C_n-orbits? Again, this C_n-action would given by $cf : \mathbb{Z}_n \to \mathbb{N}$ with $i \mapsto f(i-c)$. To use generating function, we let $\mathcal{S} = \mathbb{N}^{\mathbb{Z}_n} = \{f : \mathbb{Z}_n \to \mathbb{N}\}$, then we let weight function be $wt : \mathcal{S} \to \mathbb{N}$ with $wt(f) = \sum_{i\in\mathbb{Z}_n} f(i)$. Note for $S(x) = (1-x)^{-n}$ and hence

$$|\mathcal{S}_m| = [x^m]S(x)$$

Now we want to generalize this to count fixed points. For $c \in C_n$, we see $\mathcal{S}^c$ is in bijection with $(\frac{n}{d}\mathbb{N})^{\mathbb{Z}_d}$ where $d = \gcd(c,n)$. For example, if we have



This is a function from $\mathbb{Z}_{12}$ to $\mathbb{N}$ fixed by 4, which have a bijective correspondence to some functions from $\mathbb{Z}_4$ to $3\mathbb{N}$, as $\gcd(4, 12) = 3$. The above picture correspond to the following picture



Thus, we see $S^c(x) = (1 - x^{h/d})^{-d}$ and hence

$$|\mathcal{S}_m^c| = [x^m]S^c(x)$$

Now by orbit counting lemma, we get

$$|\tilde{\mathcal{S}}_m| = \frac{1}{n}\sum_{c \in C_n} |\mathcal{S}_m^c| = [x^m]\frac{1}{m}\sum_{d|n}\phi(\frac{n}{d})(1 - x^{n/d})^{-d}$$

Remark 5.6.5 (Type Generating Function)**.** Let $\mathcal{A}$ be a species, S_n the symmetric group of $[n]$. Then we have a S_n-action of sets of structures on $\mathcal{A}_{[n]}$. For $\alpha \in \mathcal{A}_{[n]}$ and $\sigma \in S_n$, we have

$$\sigma\alpha = \sigma_*(\alpha)$$

where σ_* is transportation of $\mathcal{A}$-structures along σ. In particular, the isomorphism type of order n are exactly the S_n-orbits on $\mathcal{A}_{[n]}$.

In this case, we can define type generating functions. For an isomorphism type $\tilde{\alpha} \in \tilde{\mathcal{A}}_n$, write $ord(\tilde{\alpha}) = n$. Then the type generating function for $\mathcal{A}$ is the OGF for $\tilde{\mathcal{A}}$ with respect to ord. In other word,

$$\tilde{A}(x) = \sum_{\tilde{\alpha} \in \tilde{\mathcal{A}}} x^{ord(\tilde{\alpha})} = \sum_{n \geq 0} |\tilde{\mathcal{A}}_n| x^n$$

Example 5.6.6.

1. Consider the species $\mathcal{E}$. We see we have 1 isomorphism type of each order, thus
$$\tilde{E}(x) = \sum_{n \geq 0} 1x^n = \frac{1}{1-x}$$
2. Consider the rooted linear order $\mathcal{L}^{\bullet}$. We see $\mathcal{L}$ has 1 isomorphism type of each order n, thus the rooted version has n non-isomorphic ways to root. Thus
$$\tilde{L^{\bullet}}(x) = \sum_{n \geq 0} nx^n = \frac{x}{(1-x)^2}$$
3. Consider $\mathcal{S}$, the species of permutations. The cycle type of a permutation $\sigma \in \mathcal{S}_X$ is the partition $\lambda = (\lambda_1, ..., \lambda_l)$ such that $\lambda_1, ..., \lambda_l$ are the lengths of the cycles of σ. Then we see two permutations are isomorphic iff they have the same cycle type. In other word
$$\tilde{\mathcal{S}}(x) = \prod_{i \geq 1} \frac{1}{1-x^i}$$

Proposition 5.6.7. *Let $\mathcal{A}, \mathcal{B}$ be two species, then*

1. *TGF of $\mathcal{A} + \mathcal{B}$ is $\tilde{A}(x) + \tilde{B}(x)$.*
2. *TGF of $\mathcal{A} * \mathcal{B}$ is $\tilde{A}(x)\tilde{B}(x)$.*

Proof. This is because $\widetilde{\mathcal{A} + \mathcal{B}} = \tilde{\mathcal{A}} \cup \tilde{\mathcal{B}}$ and $\widetilde{\mathcal{A} * \mathcal{B}} = \tilde{\mathcal{A}} \times \tilde{\mathcal{B}}$. ♡

Remark 5.6.8. However, this is all we get. For other operations, TGF of output species cannot be determined from TGF of input. For example, $\mathcal{E}$ and $\mathcal{L}$ have the same TGF, but $\mathcal{E}^{\bullet}$ and $\mathcal{L}^{\bullet}$ have different TGF.

Definition 5.6.9. Let G be a finite group, $\mathcal{A}$ a species. Then a ***natural G-action on*** $\mathcal{A}$ is a rule that assigns to each group element a natural equivalence $\Phi^g : \mathcal{A} \to \mathcal{A}$ such that for every finite set X, we get a G-group action on $\mathcal{A}_X$ given by

$$g\alpha = \Phi^g_X(\alpha)$$

Definition 5.6.10. Let G acts naturally on $\mathcal{A}$. We define the ***species of orbits*** $\mathcal{A}/G$ to be

$$(\mathcal{A}/G)_X = \mathcal{A}_X / G$$

Example 5.6.11. We see S_n acts naturally on $\mathcal{A}^n$. For $(\alpha_1, ..., \alpha_n) \in \mathcal{A}_X$ and $\sigma \in S_n$, we define

$$\sigma(\alpha_1, ..., \alpha_n) = (\alpha_{\sigma(1)}, ..., \alpha_{\sigma(n)})$$

This is a natural S_n-action and

$$\mathcal{A}^n / S_n \equiv \mathcal{E}_n[\mathcal{A}]$$

Remark 5.6.12 (Speciesification)**.** Let G be a subgroup of S_n, $\mathcal{Y}$ be a finite set with G-action. The natural action of G on $Y \times \mathcal{X}^n$ will be

$$g(y, (x_1, ..., x_n)) = (gy, (x_{g(1)}, ..., x_{g(n)}))$$

Then the speciesification $\mathcal{Y}^{sp}$ will be defined as

$$\mathcal{Y}^{sp} := (\mathcal{Y} \times \mathcal{X}^n)/G$$

Proposition 5.6.13. *Let G be a subgroup of S_n, $\mathcal{Y}$ a finite set with G-action. Then there is a bijection*

$$\widetilde{\mathcal{Y}^{sp}} \leftrightarrow \tilde{\mathcal{Y}}$$

Example 5.6.14. Say $\mathcal{Y} = K^{\mathbb{Z}_n}$ be the colourings of sides of n-gon with C_n-action. Then $\mathcal{Y}^{sp} \equiv \mathcal{C}[K \times \mathcal{X}]$.

Remark 5.6.15 (Cycle Index Functions)**.** Let $\mathcal{A}$ be a species, we are going to construct the cycle index function $Z_{\mathcal{A}} \in \mathbb{T} = \mathbb{T}(\mathbb{Q}[[x]])$, which is a algebraic transformation.

First, for a species $\mathcal{A}$ we can construct the species of automorphisms as follows. For $\alpha \in \mathcal{A}_X, \sigma \in \mathcal{S}_X$, σ is an automorphism of α iff $\sigma_*(\alpha) = \alpha$. Then $\operatorname{Aut}(\mathcal{A})$ is a subspecies of $\mathcal{A} \boxtimes \mathcal{S}$, given by

$$\operatorname{Aut}(\mathcal{A})_X = \{(\alpha, \sigma) : \alpha \in \mathcal{A}_X, \sigma \in \mathcal{S}_X, \sigma_*(\alpha) = \alpha\}$$

Then the TGF of species of automorphisms is given by

$$|\tilde{\mathcal{A}}_n| = \frac{1}{n!} |\operatorname{Aut}(\mathcal{A})_n|$$

using

$$|\tilde{\mathcal{Y}}| = \frac{1}{|G|} \sum_{(s,y) \in G \times Y, gy = y} 1$$

Thus we get

$$\tilde{A}(x) = \sum_{n \geq 0} |\operatorname{Aut}(\mathcal{A})|_n \frac{x^n}{n!}$$

However, this may not be helpful if $\operatorname{Aut}(\mathcal{A})$ is hard to compute. Thus, we consider the power evaluation maps for $k = 1, 2, 3, ...$. In particular, $p_k := ev_{x^k} \in \mathbb{T}_+$ is given by $p_k[u(x)] = u(x^k)$. Then for a partition $\lambda = (\lambda_1, ..., \lambda_l)$, we define the algebraic transformation

$$p_\lambda := p_{\lambda_1} ... p_{\lambda_l}$$

as

$$p_\lambda[u(x)] = \prod_{i=1}^{l} u(x^{\lambda_i})$$

In particular, if $|\lambda| = 0$ then $p_\lambda = 1 \in \mathbb{T}$.

For permutation σ, with the cycle type $cyc(\sigma) = \lambda$. Then we write $p_\sigma = p_\lambda$. Now we can define the ***cycle index function*** (which is an algebraic transformation) as follows

$$Z_{\mathcal{A}} = \sum_{n\geq 0} \frac{1}{n!} \sum_{(\alpha,\sigma)\in \operatorname{Aut}(\mathcal{A})_n} p_\sigma$$

We can think this as generalized EGF for $\operatorname{Aut}(\mathcal{A})$ with $Wt(\alpha,\sigma) = p_\sigma$.

Proposition 5.6.16.

$$\tilde{A}(x) = Z_{\mathcal{A}}[x]$$

Proof. For all $\sigma \in S_n$, we see $p_\sigma[x] = x^n$. Thus

$$Z_{\mathcal{A}}[x] = \sum_{n\geq 0} \frac{1}{n!} \sum_{(\alpha,\sigma)\in \operatorname{Aut}(\mathcal{A})_n} p_\sigma[x] = \sum_{n\geq 0} |\operatorname{Aut}(\mathcal{A})_n| \frac{x^n}{n!} = \tilde{\mathcal{A}}(x)$$

♡

Definition 5.6.17. We define $\Lambda \subseteq \mathbb{T}$ be the closed subring of $\mathbb{T}$ generated over $\mathbb{Q}$ in $p_1, p_2, p_3, \ldots$. In other word,

$$\Lambda = \{f \in \sum_{\lambda\in p} f_\lambda p_\lambda : f_\lambda \in \mathbb{Q}\}$$

where p is the set of integer partitions. We also define $\Lambda_+ := \Lambda \cap \mathbb{T}_+$.

Theorem 5.6.18. *The set of $\{p_\lambda : \lambda \in p\}$ is $\mathbb{Q}$-linearly independent.*

Remark 5.6.19 (Implications). Some implications of the above theorem includes:

1. For $f = \sum_{\lambda\in p} f_\lambda p_\lambda$, we can define

$$[p_\lambda]f := f_\lambda$$

2. We can also define partial derivative $\frac{\partial}{\partial p_k} : \Lambda \to \Lambda$

Proposition 5.6.20. *Let $f \in \Lambda$ and $g \in \Lambda_+$. Then*

1. $p_k \circ p_l = p_{kl}$ *where* $k, l \in \mathbb{N}_{\geq 1}$.
2. $p_k \circ g = p \circ p_k$ *where* $k \in \mathbb{N}_{\geq 1}$.
3. $\circ$ *(composition) is left homomorphic, i.e. if* $f = \sum_\lambda f_\lambda p_\lambda$, *then*

$$f \circ g = \sum_\lambda f_\lambda \prod_{i=1}^{length(\lambda)} (p_{\lambda_i} \circ g)$$

Corollary 5.6.20.1. *Λ is closed under composition.*

Example 5.6.21. We have

$$\begin{aligned}(p_2^3 + p_3) \circ (2p_2 - p_5) &= ((p_2 \circ (2p_2 - p_5)))^3 + (p_3 \circ (2p_2 - p_5)) \\ &= ((2p_3 - p_5) \circ p_2)^3 + ((2p_2 - p_5) \circ p_5) \\ &= (2p_2 \circ p_2 - p_5 \circ p_2)^3 + (2p_2 \circ p_3 - p_5 \circ p_3) \\ &= (2p_4 - p_{10})^3 + (2p_6 - p_{15})\end{aligned}$$

Remark 5.6.22 (Dual Basis)**.** Let $\lambda \in p$, define $z_\lambda \in \mathbb{Z}_+$ using one of the three definitions:

1. If $\sigma \in S_n$, with cycle type $cyc(\sigma) =: \lambda$, then $z_\lambda = |Cent_{S_n}(\sigma)|$ where $Cent_{S_n}(\sigma)$ means the centralizer of σ.
2. If $|\lambda| = n$, then
$$z_\lambda = \frac{n!}{|\{\sigma \in S_n : cyc(\sigma) = \lambda\}|}$$
3. If $n_j(\lambda)$ equal the number of parts of size j in λ, then
$$z_\lambda = \prod_{j \geq 1} (j^{n_j(\lambda)} \cdot n_j(\lambda)!)$$

We often prefer to write elements of Λ in the form

$$f = \sum_{\lambda \in p} f_\lambda \frac{p_\lambda}{z_\lambda}$$

and we will write $[\frac{p_\lambda}{z_\lambda}]f := f_\lambda$. The analogy is $A(x) = \sum_{n \geq 0} a_n x^n$ as to $f = \sum f_\lambda p_\lambda$ and $A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$ as to $f = \sum f_\lambda \frac{p_\lambda}{z_\lambda}$.

Remark 5.6.23 (Cycle Index Functions In Dual Basis)**.** Say $\mathcal{A}$ is a species, and

$$Z_{\mathcal{A}} = \sum_{\lambda \in p} a_\lambda \frac{p_\lambda}{z_\lambda}$$

Then for any permutation $\sigma \in \mathcal{S}_X$ with cycle type $cyc(\sigma) = \lambda$, we have

$$a_\lambda = \#\{\alpha \in \mathcal{A}_X : \sigma_*(\alpha) = \alpha\}$$

Remark 5.6.24 (Symmetric Functions)**.** Let $f = \sum_{\lambda \in p} f_\lambda p_\lambda \in \Lambda$, we associate symmetric function $F(x_1, ..., x_m) \in \mathbb{Q}[[x_1, ..., x_m]]$ defined as follows:

$$F(x_1, ..., x_m) = \sum_{\lambda \in p} f_\lambda \prod_{i=1}^{length(\lambda)} (x_1^{\lambda_i} + ... + x_m^{\lambda_i})$$

Since each expression is symmetric in variables $x_1, ..., x_m$, the overall expression is symmetric.

On the other hand, we have equivalent description of $F(x_1, ..., x_m)$, which is the unique FPS with the property

$$F(x^{k_1}, ..., x^{k_m}) = f[x^{k_1} + x^{k_2} + ... + x^{k_m}]$$

for all tuples $(k_1, ..., k_m) \in \mathbb{Z}_+^m$.

Then, use symmetric function theory, we can express $[p_\lambda]f$ in terms of F provided $m \geq length(\lambda)$

Example 5.6.25. Let $\mathcal{L}$ be the species of linear orders. Since linear orders have no non-trivial automorphisms, we see

$$\mathrm{Aut}(\mathcal{L})_X = \mathcal{L}_X \times \{\mathrm{Id}_X\}$$

Hence we get

$$Z_{\mathcal{L}} = \sum_{n\geq 0} \frac{1}{n!}(n! p_{\mathrm{Id}_n}) = \sum_{n\geq 0} p_1^n = \frac{1}{1-p_1}$$

Next, consider $\mathcal{E}$ be the species of sets. Then every permutation is going to be an automorphism. Thus $\mathrm{Aut}(\mathcal{E}) \equiv \mathcal{S}$ where the generalized weight function is $Wt(\sigma) = p_\sigma$. We see $\mathcal{S} \equiv \mathcal{E}[\mathcal{C}]$ and the weight on $\mathcal{C}$ should be p_k where k is the order of $\alpha \in \mathcal{C}$ and the weight on $\mathcal{E}$ is the trivial one. Then the generalized EGF for $\mathcal{C}$ is $\sum_{k\geq 1} \frac{p_k}{k}$ and hence $Z_{\mathcal{S}} = \exp(\sum_{k\geq 1} \frac{p_k}{k})$.

Next, we look at $\mathcal{S}$. We see $\mathrm{Aut}(\mathcal{S})_X = \{(\alpha, \sigma) \in \mathcal{S}_X \times \mathcal{S}_X : \alpha\sigma = \sigma\alpha\}$. Write

$$Z_{\mathcal{S}} = \sum_{\lambda \in p} s_\lambda \cdot \frac{p_\lambda}{z_\lambda}$$

Then for any $\sigma \in \mathcal{S}_X$, $cyc(\sigma) = \lambda$, we have $s_\lambda = \#\{\alpha \in \mathcal{S}_X : \alpha\sigma = \sigma\alpha\} = z_\lambda$. Therefore,

$$Z_{\mathcal{S}} = \sum_{\lambda \in p} p_\lambda = \prod_{k\geq 1} \frac{1}{1-p_k}$$

Finally, let's look at $\mathcal{C}$. For $\alpha \in \mathcal{C}_X$, let $\langle\alpha\rangle \subseteq \mathcal{S}_X$ be the cycle subgroup generated by α. Then $\mathrm{Aut}(\mathcal{C})_X = \{(\alpha, \sigma) \in \mathcal{C}_X \times \mathcal{S}_X : \sigma \in \langle\alpha\rangle\}$. If $\alpha \in \mathcal{C}_{[n]}$, then $\langle\alpha\rangle$ has $\phi(\ell)$ permutations of cycle type $(\ell, \ell, ..., \ell)$ for all $\ell \mid n$. Thus

$$\begin{aligned}
Z_{\mathcal{C}} &= \sum_{n\geq 1} \frac{1}{n!} \#\mathcal{C}_n \sum_{\ell | n} \phi(\ell) p_\ell^{n/\ell} \\
&= \sum_{n\geq 1} \sum_{\ell | n} \frac{1}{n} \phi(\ell) p_\ell^{n/\ell} \\
&= \sum_{d\geq 1} \sum_{\ell\geq 1} \frac{1}{\ell \cdot d} \phi(\ell) p_l^d = \sum_{\ell \geq 1} \frac{\phi(\ell)}{\ell} \log(\frac{1}{1-p_\ell})
\end{aligned}$$

Theorem 5.6.26 (Main Theorem). *Let $\mathcal{A}, \mathcal{B}$ be two species, with $Z_{\mathcal{A}} = \sum a_\lambda \frac{p_\lambda}{z_\lambda}$ and $Z_{\mathcal{B}} = \sum b_\lambda \frac{p_\lambda}{z_\lambda}$. Then:*

1. *$\mathcal{A} \pm \mathcal{B}$ has cycle index function $Z_{\mathcal{A}} \pm Z_{\mathcal{B}}$.*
2. *$K \times \mathcal{A}$ has cycle index function $|K| \cdot Z_{\mathcal{A}}$.*
3. *$\mathcal{A} * \mathcal{B}$ has CIF $Z_{\mathcal{A}} Z_{\mathcal{B}}$.*
4. *$\mathcal{A} \boxtimes \mathcal{B}$ has CIF $Z_{\mathcal{A}} \boxtimes Z_{\mathcal{B}} := \sum_{\lambda \in p} a_\lambda b_\lambda \frac{p_\lambda}{z_\lambda}$.*
5. *$\mathcal{A}^*$ has CIF $(1 - Z_{\mathcal{A}})^{-1}$ provided its defined.*
6. *$\mathcal{A}'$ has CIF $\frac{\partial}{\partial p_1} Z_{\mathcal{A}}$.*
7. *$\mathcal{A}^\bullet$ has CIF $p_1 \frac{\partial}{\partial p_1} Z_{\mathcal{A}}$.*
8. *$\mathcal{A} \circ \mathcal{B}$ has CIF $Z_{\mathcal{A}} \circ Z_{\mathcal{B}}$ if B is connected.*

Corollary 5.6.26.1. *The TGF of $\mathcal{A} \circ \mathcal{B}$ is $Z_{\mathcal{A}}[\tilde{B}(x)]$.*

Proof. We have $Z_{\mathcal{A}\circ\mathcal{B}}[x] = Z_{\mathcal{A}} \circ Z_{\mathcal{B}}[x] = Z_{\mathcal{A}}[Z_{\mathcal{B}}[x]] = Z_{\mathcal{A}}[\tilde{B}(x)]$ ♡

Example 5.6.27. We revisit the colouring of n-gon. Previously, we see $Y = K^{\mathbb{Z}_n}$ where K is the set of colours with C_n-action. Then $Y^{sp} \equiv \mathcal{C}_n[K \times \mathcal{X}] \equiv \mathcal{C}[K \times \mathcal{X}]$. Then $|\tilde{Y}| = [x^n]\tilde{Y^{sp}}(x) = [x^n]Z_{\mathcal{C}}[kx]$. The cycle index function for $\mathcal{C}$ evaluated at kx is

$$\left(\sum_{\ell\geq 1} \frac{\phi(\ell)}{\ell} \log(\frac{1}{1-p_\ell})\right)[kx] = \sum_{\ell\geq 1} \frac{\phi(\ell)}{\ell} \log(\frac{1}{1-kx^\ell})$$

We can just expand this and get

$$\sum_{\ell\geq 1} \frac{\phi(\ell)}{\ell} \log(\frac{1}{1-kx^\ell}) = \sum_{\ell\geq 1}\sum_{d\geq 1} \frac{\phi(\ell)}{\ell}\frac{k^d x^{dl}}{d} = \sum_{n\geq 1}\left(\frac{1}{n}\sum_{d|n}\phi(\frac{n}{d})k^d\right)x^n$$

Example 5.6.28 (Rooted Trees). Consider rooted trees. We see $\mathcal{T}^\bullet \equiv \mathcal{X} * \mathcal{E}[\mathcal{T}^\bullet]$. Thus

$$\tilde{T^\bullet}(x) = x \cdot Z_{\mathcal{E}}[\tilde{T^\bullet}(x)]$$

Write $\tilde{T^\bullet}(x) = \sum_{m\geq 1} t_m x^m$ then

$$Z_{\mathcal{E}}[\tilde{T^\bullet}(x)] = \prod_{m\geq 1}(1-x^m)^{-t_m}$$

Compare coefficients of x^n on both sides, we get

$$t_n = [x^{n-1}]\prod_{m=1}^{n-1}(1-x^m)^{-t_m}$$

which is a recurrence relation which defines t_n in terms of $t_1, ..., t_{n-1}$.

Example 5.6.29 (Endofunctions). We see $\mathcal{N} \equiv \mathcal{S}[\mathcal{T}^\bullet]$, then $\mathcal{Z}_{\mathcal{S}} = \prod_{k\geq 1}\frac{1}{1-p_k}$. Then

$$\tilde{N}(x) = Z_{\mathcal{S}}[\tilde{T^\bullet}(x)] = \prod_{k\geq 1}\frac{1}{1-\tilde{T^\bullet}(x^k)}$$

Example 5.6.30 (Unrooted Trees). There is no general way to get $\tilde{A}(x)$ from $\tilde{A}^\bullet(x)$, but for trees there is a trick.

In particular, we have

$$\tilde{T}(x) = ([p_1 - \frac{1}{2}p_2^2 + \frac{1}{2}p_2])[\tilde{T^\bullet}(x)]$$

To see this, we note there are two types of unlabelled trees:

1. Trees for which every automorphism has a fixed vertex.
2. Trees which have a fixed point free automorphism.

For $\tau \in \tilde{\mathcal{T}}$, let v_τ be the number of ways to root at vertex up to isomorphism. We also let e_τ be the number of ways to root at edge up to isomorphism.

For type 1, we have $v_\tau - e_\tau = 1$ and for type 2 we get $v_\tau - e_\tau = 0$. Next, note the number of unrooted trees is equal the number of vertex rooted trees minus number of edge rooted trees plus number of type 2 trees.

5.7 Tutorial 10 (Cyclic Sieving Phenomenon)

Definition 5.7.1 (Sieving Polynomial)**.** Let C_n be the nth cyclic group, Y a finite set with C_n-action. Let $wt : Y \to \mathbb{N}$ be a weight function with $Y(q) := \sum_{y \in Y} q^{wt(y)}$. Let $\zeta = e^{\frac{2\pi i}{n}}$ be the primitive nth root of unity. Then $Y(q)$ is called a ***Sieving polynomial*** for (Y, C_n) if

$$|Y^c| = Y(\zeta^c)$$

for all $c \in C_n$.

Remark 5.7.2.

1. This is weird: getting cardinality of a set by evaluating a polynomial at a complex number.
2. For most weight functions on Y, $Y(q)$ is not a Sieving polynomial (unless $n = 1$).
3. Theorem: Sieving polynomials exists.
4. Though in practice, they can be hard to determine. We often have to guess. Important clue: $Y(1) = |Y|$.
5. What's so phenomenal? In many examples, the "first thing you might guess" actually works (q-analogous).

Example 5.7.3 (2-colouring of n-gon)**.** Let Y be the set of colourings of sides of regular n-gon with k sides red, $n-k$ sides blue. Let C_n acts on Y by rotating the n-gon.

Then $|Y| = \binom{n}{k}$ and we would guess

$$Y(q) := \binom{n}{k}_q = \prod_{i=0}^{k-1} \frac{q^{n-i} - 1}{q^{k-i} - 1}$$

However, this is indeed true if we consider the combinatorial interpretation of $\binom{n}{k}_q$.