Contents

First day

Definition 0.0.1 (Professor)**.** Stephen New, MC 5419

Definition 0.0.2 (Website)**.** www.math.uwaterloo.ca/$\sim$snew

Definition 0.0.3 (Grade)**.** We will have four term tests and we will have no final.

Test dates:

1. Monday, May 27th, in class, 12:30-1:20
2. Tuesday, June 18 from 4:30-5:20, MC 4021
3. Monday, July 8, from 12:30-1:20
4. Monday, July 29, from 12:30-1:20

Drop lowest assignments/tests and the final grade would be the

$$\max\{0.4A + 0.6T, 0.25A + 0.75T\}$$

Chapter 1

Topology in $\mathbb{R}^n$

1.1 Metric, Open, and Closed

Definition 1.1.1. For $x, y \in \mathbb{R}^n$, the ***dot product*** or the ***standard inner product*** of x and y is

$$x \cdot y = y^T x = \sum_{i=1}^{n} x_i y_i$$

Proposition 1.1.2 (Properties of the dot product)**.** *Let $x, y, z \in \mathbb{R}^n$, $t \in \mathbb{R}$,*

1. *(Bilinearity)*
   - $(x + y) \cdot z = x \cdot z + y \cdot z$
   - $(tx) \cdot y = t(x \cdot y)$
   - $x \cdot (y + z) = x \cdot y + x \cdot z$
   - $x \cdot (ty) = t(x \cdot y)$
2. *(Symmetry)* $x \cdot y = y \cdot x$
3. *(Positive Definiteness)* $x \cdot x \geq 0$, $x \cdot x = 0 \Leftrightarrow x = 0$

Definition 1.1.3. For $x \in \mathbb{R}^n$, we define the length of x, or the norm of x, to be

$$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Proposition 1.1.4 (Basic properties of norm)**.** *Let $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.*

1. *(Scaling)* $|tx| = |t||x|$
2. *(Positive Definiteness)* $|x| \geq 0$ *with* $|x| = 0$ *iff* $x = 0$
3. $|x \pm y|^2 = |x|^2 \pm 2x \cdot y + |y|^2$
4. *(Polarization Identity)* $x \cdot y = \frac{1}{2}(|x + y|^2 - |x|^2 - |y|^2) = \frac{1}{4}(|x + y|^2 - |x - y|^2)$
5. *(Cauchy-Schwarz Inequality)* $|x \cdot y| \leq |x||y|$ *with* $|x \cdot y| = |x||y|$ *iff* $\{x, y\}$ *is linear dependent*
6. *(Triangle Inequality)* $|x + y| \leq |x| + |y|$ *and* $||x| - |y|| \leq |x + y|$

Proof. (Proof of Cauchy-Schwarz)

If $\{x, y\}$ is linearly dependent, then one of x, y is a multiple of the other, say $y = tx$ with $t \in \mathbb{R}$. Then $|x \cdot y| = |x \cdot (tx)| = |t(x \cdot x)| = |t||x|^2$ and $|x||y| = |t||x|^2$ as desired.

Suppose $\{x, y\}$ is linearly independent, then $x + ty \neq 0$ for any $t \in \mathbb{R}$ by definition of linearly independent. So $|x + ty|^2 \neq 0$, hence

$$|x|^2 + 2tx \cdot y + t^2|y|^2 \neq 0$$

for any $t \in \mathbb{R}$.

Thus the discriminant of the quadratic $|y|^2t^2 + 2(x \cdot y)t + |x|^2$ must be negative, that is,

$$\begin{aligned} 4(x \cdot y)^2 - 4|x|^2|y|^2 < 0 \\ \Rightarrow |x \cdot y| < |x||y| \end{aligned}$$

♡

Proof. (Proof of Triangle Inequality)

For $x, y \in \mathbb{R}^n$, we have

$$\begin{aligned} |x + y|^2 &= |x|^2 + 2x \cdot y + |y|^2 \\ &\leq |x|^2 + 2|x \cdot y| + |y|^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 \text{ by Cauchy-Schwarz} \\ &= (|x| + |y|)^2 \end{aligned}$$

Hence $|x + y| \leq |x| + |y|$. ♡

Definition 1.1.5. For $a, b \in \mathbb{R}^n$, the ***distance*** between a, b in $\mathbb{R}^n$ is $dist(a, b) = d(a, b) = |b - a|$

Proposition 1.1.6. *(Basic Properties of Distances) Note we can use those properties to define a **metric** on any set as well. Let $a, b, c \in \mathbb{R}^n$ for now.*

1. *(Positive Definiteness) $d(a, b) \geq 0$ with $d(a, b) = 0$ iff $a = b$*
2. *(Symmetry) $d(a, b) = d(b, a)$*
3. *(Triangle Inequality) $d(a, b) + d(b, c) \geq d(a, c)$*

Definition 1.1.7. For $x, y \neq 0 \in \mathbb{R}^n$, the ***angle between*** x and y is defined to be $\theta(x, y) = \cos^{-1} \frac{x \cdot y}{|x||y|}$

Example 1.1.8. Let $0 \neq u, v, w \in \mathbb{R}^n$, where $0 \neq w = su + tv$ for some $s, t \geq 0$. Show that $\theta(u, v) = \theta(u, w) + \theta(w, v)$.

Solution. If $t > 0$ then we have

$$\theta(tu, v) = \cos^{-1} \frac{(tu) \cdot v}{|u||v|} = \cos^{-1} \frac{t(u \cdot v)}{t|u||v|} = \theta(u, v)$$

Similarly, we have $\theta(u, tv) = \theta(u, v)$. It follows that we only need to show our claim holds for $|u| = |v| = 1$ as we have $|\frac{u}{|u|}| = 1$ for all $u \neq 0$. Moreover, we can also consider $|w| = 1$ by noting $\theta(u, w) = \theta(u, w')$ where $w' = \frac{s}{|w|}u + \frac{t}{|w|}v$. Thus, let $x = u \cdot v$ we have

$$1 = |w|^2 = s^2 + 2stx + t^2$$

We have

$$\cos\theta(u, w) = s + tx$$

and

$$\sin\theta(u, w) = \sqrt{1 - \cos^2\theta(u, w)} = t\sqrt{1 - x^2}$$

Similarly, we have

$$\cos\theta(v, w) = t + sx$$

and

$$\sin\theta(v, w) = s\sqrt{1 - x^2}$$

Thus,

$$\begin{aligned}
\cos(\theta(u, w) + \theta(v, w)) &= (s + tx)(t + sx) - ts\sqrt{1 - x^2}\sqrt{1 - x^2} \\
&= st + s^2x + t^2x + stx^2 - st + stx^2 \\
&= (s^2 + t^2 + 2stx)x \\
&= x \\
&= \cos\theta(u, w)
\end{aligned}$$

In addition, we have

$$\begin{aligned}
\sin(\theta(u, w) + \theta(v, w)) &= \sin\theta(u, w)\cos\theta(v, w) + \cos\theta(u, w)\sin\theta(v, w) \\
&= t\sqrt{1 - x^2}(t + sx) + s(s + tx)\sqrt{1 - x^2} \\
&= (t^2 + stx + s^2 + stx)\sqrt{1 - x^2} \\
&= \sqrt{1 - x^2} \\
&= \sin\theta(u, v)
\end{aligned}$$

Therefore, we have $\theta(u, v) = \theta(u, w) + \theta(v, w)$ ♠

Definition 1.1.9. Let $a \in \mathbb{R}^n$ and $0 < r \in \mathbb{R}$, we define:

1. ***open ball*** centred at a of radius r to be the sets $B(a, r) = \{x \in \mathbb{R}^n : |x - a| < r\}$
2. the ***closed ball*** centred at a of radius r to be the sets $\bar{B}(a, r) = \{x \in \mathbb{R}^n : |x - a| \leq r\}$
3. the (open) ***punched ball*** centred at a of radius r to be the sets $B^*(a, r) = B(a, r)\backslash\{a\}$
4. ***sphere*** centred at a of radius r to be the sets $S(a, r) = \{x \in \mathbb{R}^n : |x - a| = r\}$

Definition 1.1.10. For $A \subseteq \mathbb{R}^n$ is ***open*** when for every $a \in A$ there exists $0 < r \in \mathbb{R}$ such that $B(a, r) \subseteq A$.

In addition, we say that A is ***closed*** (in $\mathbb{R}^n$) when the set $A^c = \mathbb{R}^n\backslash A$ is open, where A^c is called the ***complement*** of A.

Example 1.1.11. Show that for $a \in \mathbb{R}^n$ and $0 < r \in \mathbb{R}$, $B(a,r)$ is open (in $\mathbb{R}^n$) and $\bar{B}(a,r)$ is closed (in $\mathbb{R}^n$)

Solution. Let $a \in \mathbb{R}^n$ and $0 < r \in \mathbb{R}$. We claim that $B(a,r)$ is open. Let $b \in B(a,r)$ be arbitrary, so we have $|a-b| < r$. Let $s = r - |b-a|$, we have $0 < s$. Let $x \in B(b,s)$, so $|x-b| < s$ then

$$|x-a| = |x-b+b-a| \leq |x-b| + |b-a| < s + |b-a| = r$$

This complete the proof of the claim as $B(b,s) \subseteq B(a,r)$.

To show $\bar{B}(a,r)$ is closed, we need to prove the complement of that is open. This is similar to the proof of the claim, and we are done. ♠

Example 1.1.12. Show that $A = \{(x,y) \in \mathbb{R}^2 : 0 < x, 0 < y, xy < 1\}$ is open..

Solution. In order to show it is open, we want to find r for each $x = (a,b) \in A$ such that $B(x,r) \subseteq A$. It suffice to show that if we can find a open box $Q_x = \{(x,y) \in \mathbb{R}^2 : |x-a| < r, |y-b| < r\} \subseteq A$ for all $x = (a,b) \in A$, then since Q_x contains an open ball $B(x,r)$ centered at x, we are done. Note that if $r < a$ then

$$|x-a| < r \Rightarrow |x-a| < a \Rightarrow 0 < x < 2a \Rightarrow x > 0$$

Similarly, if $r < b$ then $|y-b| < r \Rightarrow y > 0$. Note $r < a$ and $r < b$ then $r < a+b$ and so $(a+r)(b+r) = ab + r(a+b) + r^2 < ab + r(a+b) + r(a+b) = ab + 2r(a+b)$ and we obtain $(a+r)(b+r) < 1$ by choosing $r < \frac{1-ab}{2(a+b)}$.

Let $(a,b) \in A$, then $0 < a, 0 < b$ and $ab < 1$. Let $r = \min\{a, b, \frac{1-ab}{2(a+b)}\}$. Let $(x,y) \in B((a,b),r)$, then we have

$$|x-a| = \sqrt{|x-a|^2} \leq \sqrt{|x-a|^2 + |y-b|^2} = |(x,y) - (a,b)| < r$$

Similarly, we have

$$|y-b| < r$$

Since $|x-a| < r \leq a$ we have $0 \leq a - r < x < a + r$ and since $|y-b| < r \leq b$ we have $0 \leq b - r < y < b + r$. Since $0 < x < a + r$, $0 < y < a + r$, $r < a + b$ and $r < \frac{1-ab}{2(a+b)}$ we have $xy < (a+r)(b+r) < 1$ as we showed above. Sinec $0 < x$ and $0 < y$, we have $(x,y) \in A$ and thus $B((a,b),r) \subseteq A$ as desired. ♠

Definition 1.1.13. Let $A \subseteq \mathbb{R}^n$, we define the ***interior*** of A in $\mathbb{R}^n$ to be

$$A^o := \bigcup_{u \in S} u$$

where S is the set of all open sets u in $\mathbb{R}^n$ with $u \subseteq A$. In addition, we define ***closure*** of A is the set

$$\bar{A} := \bigcap_{k \in T} k$$

where T is the set of all closed sets k in $\mathbb{R}^n$ with $A \subseteq k$

Theorem 1.1.14 (Basic Properties of Open Sets)**.**

1. *$\emptyset$ and $\mathbb{R}^n$ are open in $\mathbb{R}^n$*
2. *Let K be any set, if u_k is an open set of each $k \in K$ then $\bigcup_{k\in K} u_k$ is open*
3. *If $U_1, U_2, ..., U_l$ are open sets in $\mathbb{R}^n$ then $\bigcap_{k=1}^{l} U_k$ is open*

Proof.

1. $\emptyset$ is open because by convention, every statement start with $\forall x$ is true for $\emptyset$. To see $\mathbb{R}^n$ is open, we note $B(a,r) = \{x \in \mathbb{R}^n : |x-a| < r\} \subseteq \mathbb{R}^n$ for all $a \in \mathbb{R}^n, r > 0$
2. Let u_k be open for each $k \in K$. Let $u = \bigcup_{k\in K} u_k$ and let $a \in u$. Since $a \in u$, we can choose an index k so that $a \in u_k$. Since u_k is open, we can choose $r > 0$, so that $B(a,r) \subseteq u_k$. Since $B(a,r) \subseteq u_k$, and $u_k \subseteq \bigcup u_k = u$, we have $B(a,r) \subseteq u$. Thus u is open.
3. Let $U_1, ..., U_l$ are open. Let $a \in \bigcap_{i=1}^{l} U_i$. For each index $1 \leq k \leq l$, since U_k is open, we can choose r_k such that $B(a, r_k) \subseteq U_k$. Let $r = \min\{r_1, ..., r_l\}$, then we have $r > 0$. Clearly we must have $B(a,r) \subseteq U_k$ for all $1 \leq k \leq l$. Thus it is open as a was arbitrary.

♡

Corollary 1.1.14.1 (Basic Properties of Closed Sets)**.**

1. *$\emptyset$ and $\mathbb{R}^n$ are closed in $\mathbb{R}^n$*
2. *Let K_j is closed in $\mathbb{R}^n$ for each $j \in J$ where J is a set, then $\bigcap_{j\in J} K_j$ is closed*
3. *If $K_1, ..., K_l$ are closed in $\mathbb{R}^n$ then $\bigcup_{i=1}^{l} K_j$ is closed in $\mathbb{R}^n$*

Corollary 1.1.14.2. *For $A \subseteq \mathbb{R}^n$,*

1. *A^o is the largest open set which is contained in $A \subseteq \mathbb{R}^n$, i.e. A^o is open, $A^o \subseteq A$ and for every open set $u \subseteq \mathbb{R}^n$, if $u \subseteq A$ then $u \subseteq A^o$*
2. *$\bar{A}$ is the smallest closed set which contains A, i.e. $\bar{A}$ is closed, $A \subseteq \bar{A}$ and for all closed sets $u \subseteq \mathbb{R}^n$, if $A \subseteq u$ then $\bar{A} \subseteq k$*

Definition 1.1.15. Ler $A \subseteq \mathbb{R}^n$ and let $a \in \mathbb{R}^n$, we say

1. a is an ***interior point*** of A when $a \in A$ and $\exists r > 0$ such that $B(a,r) \subseteq A$
2. a is a ***limit point*** of A when for all $r > 0$, $B^*(a,r) \cap A \neq \emptyset$
3. a is a ***boundary point*** of A when for all $r > 0$, $B(a,r) \cap A \neq \emptyset$ and $B(a,r) \cap A^c \neq \emptyset$

Definition 1.1.16. The set of limit points of A is denoted by A' and the set of boundary points of A is denoted by ∂A and ∂A is called the boundary of A

Theorem 1.1.17 (Properties of interior points, limit points and boundary points)**.**

1. *The set of interior points of A is equal to A^o*
2. *A is closed if and only if $A' \subseteq A$*
3. *$\bar{A} = A \cup A'$*
4. *$\partial A = \bar{A} \backslash A^o$ or equivalently, $\bar{A} = A^o \cup \partial A$*

Proof.

1. Let B be the set of all interior points of A.
   Let $p \in A^o$ be given, then it must belong to at least one open set $u_p \in \mathbb{R}^n$ such that $u_p \subseteq A$. Since u_p is open, there exists $r > 0 \in \mathbb{R}$ such that $B(p, r) \subseteq u_p$. Hence $B(p, r) \subseteq A$ and it is an interior point. Since p was arbitrary, we have $A^o \subseteq B$.
   Let $q \in B$ be given. Then there exists $r > 0 \in \mathbb{R}$ such that $B(q, r) \subseteq A$. Since $B(q, r)$ is an open set as we proved it in class, we must have $B(q, r) \subseteq A^o$ as A^o is the union of all the open subsets of A, in particular, we have $q \in A^o$. Hence $B \subseteq A^o$.
   Hence $B = A^o$
2. $$\begin{aligned} A' \subseteq A &\Leftrightarrow \forall a \in \mathbb{R}^n (a \in A' \Rightarrow a \in A) \\ &\Leftrightarrow \forall a \in \mathbb{R}^n (\forall r > 0, B^*(a, r) \cap A \neq \emptyset \Rightarrow a \in A) \end{aligned}$$
   $$\begin{aligned} A \text{ is closed} &\Leftrightarrow \forall a \in \mathbb{R}^n (a \in A^c \Rightarrow \exists r > 0, B(a, r) \subseteq A^c) \\ &\Leftrightarrow \forall a \in \mathbb{R}^n (a \notin A \Rightarrow \exists r > 0, B(a, r) \cap A = \emptyset) \\ &\Leftrightarrow \forall a \in \mathbb{R}^n (a \notin A \Rightarrow \exists r > 0, B(a, r) \cap A = \emptyset) \end{aligned}$$
   Since when $a \notin A$, we have $B(a, r) \cap A = B^*(a, r) \cap A$, we get
   $$\Leftrightarrow \forall a \in \mathbb{R}^n (\forall r > 0, B(a, r) \cap A \neq \emptyset \Rightarrow a \in A)$$
   Note that $A' \subseteq A$ iff $\forall a \in \mathbb{R}^n (a \in A' \Rightarrow a \in A)$ iff $\forall a \in \mathbb{R}^n (\forall r > 0, B^*(a, r) \cap A \neq \emptyset \Rightarrow a \in A)$ and A is closed iff A^c is open iff $\forall a \in \mathbb{R}^n (a \in A^c \Rightarrow \exists r > 0, B(a, r) \subseteq A^c)$ iff $\forall a \in \mathbb{R}^n (a \notin A \Rightarrow \exists r > 0, B(a, r) \cap A = \emptyset)$ iff $\forall a \in \mathbb{R}^n (a \notin A \Rightarrow \exists r > 0, B(a, r) \cap A = \emptyset)$ since when $a \notin A$, $B(a, r) = B^*(a, r) \cap A$ iff $\forall a \in \mathbb{R}^n (\forall r > 0, B(a, r) \cap A \neq \emptyset \Rightarrow a \in A)$
3. We shall show that $A \cup A'$ is the smallest closed set in $\mathbb{R}^n$ which contains A, that is $A \subseteq A \cup A'$, $A \cup A'$ is closed, and for every closed set $k \in \mathbb{R}^n$ with $A \subseteq k$ we have $A \cup A' \subseteq k$. We shall show $(A \cup A')^c$ is open. Let $a \in (A \cup A')^c$. That is, $a \notin A$ and $a \notin A'$. Since $a \notin A'$, we can choose $r > 0$ so that $B^*(a, r) \cap A = \emptyset$. Note that since $a \notin A$, we have $B^*(a, r) \cap A = B(a, r) \cap A$ so we have $B(a, r) \cap A = \emptyset$. We claim that $B(a, r) \cap A' = \emptyset$. Suppose, for a contradiction, it is not. Choose $b \in B(a, r) \cap A'$. Since $B(a, r)$ is open, we can choose $s > 0$ so that $B(b, s) \subseteq B(a, r)$. Since $b \in A'$, we have $B^*(b, s) \cap A \neq \emptyset$ hence $B(a, r) \cap A \neq \emptyset$ since $B^*(b, s) \subseteq B(b, s) \subseteq B(a, r)$. This gives the desired contradiction hence $B(a, r) \cap A' = \emptyset$. Therefore $B(a, r) \cap (A \cup A') = \emptyset$. This shows that $(A \cup A')^c$ is open so $A \cup A'$ is closed.
   It remains to show that for every closed set $k \in \mathbb{R}^n$ with $A \subseteq k$, we have $A \cup A' \subseteq k$. Let k be any closed set in $\mathbb{R}^n$ with $A \subseteq k$. Since $A \subseteq k$, we have $A' = k'$ since when $a \in A'$, $\forall r > 0, B^*(a, r) \cap A \neq \emptyset$, hence $\forall r > 0$, $B^*(a, r) \cap k \neq \emptyset$. Since k is closed, $k' \subseteq k$ by Part 2. So we have $A' \subseteq k' \subseteq k$. Since $A \subseteq k$ and $A \subseteq k'$, we have $A \cup A' = K$.
4. Let $a \in \partial A$. We claim first that $a \in \bar{A}$. Since $\bar{A} = A \cup A'$, it suffice to show that either $a \in A$ or $a \in A'$. Suppose $a \notin A$. Let $r > 0$ be arbitrary. S icne $a \in \partial A$ we have $B(a, r) \cap A \neq \emptyset$. Since $a \notin A$ we have $B^*(a, r) \cap A = B(a, r) \cap A$ and so $B^*(a, r) \cap A \neq \emptyset$. Since $r > 0$ was arbitrary, we have $a \in A'$ as required.

Next, we claim that $a \notin A^o$. Suppose, for a contradiction, that $a \in A^o$. Then a is an interior point of A, so we can choose $r > 0$ so that $B(a,r) \subseteq A$. Since $B(a,r) \subseteq A$ we have $B(a,r) \cap A^c = \emptyset$. However, since $a \in \partial A$, we have $B(a,r) \cap A^c \neq \emptyset$ so we have obtained a contradiction. Thus $a \notin A^o$ as claimed. Hence we have $\partial A \subseteq \bar{A} \backslash A^o$.
Now, let $a \in \bar{A} \backslash A^o$. That is, let $a \in \bar{A}$ and $a \notin A^o$. We consider two cases. Case 1: suppose $a \in A$. Let $r > 0$ be arbitrary, Since $a \in A$ and $a \in B(a,r)$, we have $B(a,r) \cap A \neq \emptyset$. Since $a \notin A^o$, we have $B(a,r) \nsubseteq A$ and so $B(a,r) \cap A^c \neq \emptyset$. Thus $a \in \partial A$. Case 2: suppose $a \notin A$. Let $r > 0$ be arbitrary. Since $a \notin A$ and $a \in B(a,r)$, we have $B(a,r) \cap A^c \neq \emptyset$. Since $a \in \bar{A} = A \cup A'$ and $a \notin A$ we have $a \in A'$ and so $B^*(a,r) \cap A \neq \emptyset$ hence $B(a,r) \cap A \neq \emptyset$. Thus $a \in \partial A$.
In either case we find $a \in \partial A$. Thus $\partial A \subseteq \bar{A} \backslash A^o$.

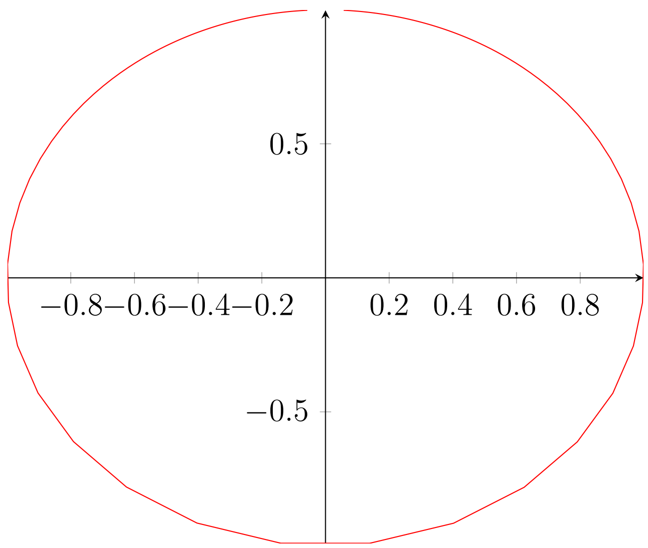
♡

Example 1.1.18. Show that

$$B = \{(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}) \in \mathbb{R}^2 : t \in \mathbb{R}\}$$

is not closed.

Solution. It suffice to show that $(0,1)$ is a limit point of B but not a point of B as we can see from the diagram.



Let $x(t) = \frac{2t}{t^2+1}$ and $y(t) = \frac{t^2-1}{t^2+1}$ and $f(t) = (x(t), y(t))$. Thus $B = \{f(t) : t \in \mathbb{R}\}$. We claim $a = (0,1)$ is a limit point of B but $a \notin B$. It is clear that $a \notin B$ as $f(t) = a$ imply $x(t) = 0$, thus $t = 0$, however then we have $y(t) = -1$. To show $a \in B'$, we first show $B(a,r) \cap B \neq \emptyset$ for all $r > 0$. Since $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 1$, we can choose $t \in \mathbb{R}$ such that $|x(t) - 0| < \frac{r}{2}$ and $|y(t) - 1| < \frac{r}{2}$. Then we have

$$f(t) - a = |(x(t), y(t)) - (0,1)| \leq |x(t)| + |y(t) - 1| < r$$

and so $f(t) \in B(a,r) \cap B$. This shows that for all $r > 0$ we have $B(a,r) \cap B \neq \emptyset$ and so $a \in B'$ as $a \notin B$ so $B(a,r) \cap B = B^*(a,r) \cap B$. Hence, B is not closed. ♠

Example 1.1.19. When $A = B(0,1)$ with $U = \{(cost, sint) : 0 \le t \le \pi\}$. We have $A^0 = B(0,1)$, $\partial A = S(0,1)$, $\bar{A} = \bar{B}(0,1)$ and $A' = \bar{a}$.

Example 1.1.20. Show that A' is closed for all $A \subseteq \mathbb{R}^n$.

Solution. Let (X, d) be a metric space and for $x, y \in X$ we write $d(x, y) = |x - y|$ sometimes.

Let A be a set in X. Let $p \in A'$ be a limit point of A. First, we claim if A is finite, then A has no limit points and $A' = \emptyset$ and there is nothing to prove as empty set is closed. Let $A = \{a_1, ..., a_n\}$, then let D be the set of all distances between all the points, namely, $D = \{x \in \mathbb{R} : \exists a_i, a_j \in A(x = d(a_j, a_i))\}$. Clearly D is finite and $|D| = \binom{n}{2}$, as we can define a bijection between D and a way to choose 2 elements from an n elements set, hence D has a minimum element $m := \min D$. Then let $r = \frac{m}{2}$, and let $a \in A$, we have $B^*(a, r) \cap A = \emptyset$ as m is the minimum distance between elements in A. Henec $A' \neq \emptyset$ only if A is infinite.

If $x \in A'$, then every punched ball centered at x must be intersected with A. We will show that every limit point of A' is a point in A', then it must be closed. If A' is finite, then it is closed as $(A')' = \emptyset \subseteq A'$.

Suppose A' is infinite. Let p be a limit point of A'. Since it is a limit point, every punched ball centered at p must be intersected with A'. Hence, $\forall r > 0(\exists q_r \in A'(q_r \in B^*(p, r)))$. Then, let $r > 0$ be given, we have $B^*(p, \frac{r}{2}) \cap A' \neq \emptyset$, hence $\exists q_r \in A'$ such that $q_r \in B^*(p, \frac{r}{2})$. Since $q_r \in A'$, we have $B^*(q_r, \frac{r}{4}) \cap A \neq \emptyset$. Hence $\exists a_r \in A$ such that $a_r \in B^*(q_r, \delta)$, where $\delta = \min\{\frac{r}{4}, \frac{|p - q_r|}{4}\}$. Note $p \neq q_r$ so $\delta > 0$ and $B^*(q_r, \delta)$ is a well-defined ball, moreover, the ball does not contain p as $\delta < |p - q_r|$. Hence,

$$d(p, a_r) = |p - a_r| = |p - q_r + q_r - a_r| \le |p - q_r| + |q_r - a_r| < \frac{r}{2} + \frac{r}{4} < r$$

Thus $a_r \in B^*(p, r)$ and clearly $a_r \neq p$ as $B^*(q_r, \delta)$ does not contain p. This shows that p is a limit point of A, as for every $r > 0$ we can find $a_r \in A$ such that $a_r \in B^*(p, r)$ so $B^*(p, r) \cap A \neq \emptyset$. Thus $p \in A'$ by definition. ♠

1.2 Connected Sets and Sets in Subsets of $\mathbb{R}^n$

Definition 1.2.1. Let $A \subseteq \mathbb{R}^n$. For sets $U, V \in \mathbb{R}^n$ we say U and V separate A when $U \cap A, V \cap A \neq \emptyset$ and $U \cap V = \emptyset$ and $A \subseteq U \cup V$. For $A \subseteq \mathbb{R}^n$, we say A is connected when there do not exists open sets U, V which separate A. If A is not connected when it is not connected.

Theorem 1.2.2. *The connected sets in* $\mathbb{R}$ *are the intervals in* $\mathbb{R}$.

Proof. We will show $E \subseteq \mathbb{R}$ is connected if and only if $x < y \in E$ and $x < z < y$ then $z \in E$.

Let $a < b \in E$ and suppose we have $a < c < b$ with $c \notin E$. Then we let $A = E \cap (-\infty, c)$ and $B = E \cap (z, \infty)$. Note $A \cap B = \emptyset$, A, B are both open, $A \cap E \neq \emptyset$, $B \cap E \neq \emptyset$, and $E = A \cup B$, thus A is disconnected.

Suppose E is disconnected. Let A, B separates E where $a \in A$ and $b \in B$. We have $a \neq b$ as $A \cap B = \emptyset$, suppose $a < b$. Let $u = \sup(A \cap [a, b])$, we have $u \neq a$ as A is open, thus $\exists r_a$ such that $B(a, r_a) \subseteq A$ and let $0 < r'_a = \min\{r_a, \frac{b-a}{2}\}$, then $B(a, r'_a) \subseteq B(a, r_a) \subseteq A$, and in particular, we have $[a, r'_a) \subseteq [a, b]$ so $[a, r'_a) \subseteq A \cap [a, b]$ so $u \geq a + r'_a > a$. Similarly, we cannot have $u = b$ as we can choose $r'_b > 0$ such that $(r'_b, b] \subseteq B \cap [a, b]$. Then we have $u \leq b - r'_b < b$ as $A \cap B = \emptyset$. Next, we have $u \notin A$, because if it is the case, then we can find r_u such that $B(u, r_u) \subseteq A \cap [a, b]$, which is a contradiction as $u = sup(A \cap [a, b])$. Moreover, $u \notin B$, because if it is the case, then we could find r'_u such that $B(u, r'_u) \subseteq B \cap [a, b]$ and this is impossible as well as $A \cap B = \emptyset$. Hence $a < u < b$ but $u \notin A$ and $u \notin B$, thus $u \notin A \cup B$ and $u \notin E$ as desired. ♡

Definition 1.2.3. For $A \subseteq P \subseteq \mathbb{R}^n$, A is open in P when $\forall a \in A$, $\exists r > 0$ such that $B_p(a, r) \subseteq A$ where $B_p(a, r) = B(a, r) \cap P$

Theorem 1.2.4. *For $A \subseteq P \subseteq \mathbb{R}^n$,*

1. *A is open in P iff $A = U \cap P$ for some open set U in P*
2. *A is closed in P iff $A = U \cap P$ for some closed set U in P*

Example 1.2.5. Let $A, B \subseteq \mathbb{R}^n$, show that if A is connected and $A \subseteq B \subseteq \bar{A}$ then B is connected.

Solution. Suppose A is connected and that $A \subseteq B \subseteq \bar{A}$. Suppose, for a contradiction, that B is disconnected. Choose open sets U, V which separate B, that is, $U \cap B \neq \emptyset$, $V \cap B \neq \emptyset$, $U \cap V = \emptyset$, and $B \subseteq U \cup V$. We claim that U and V also separate A. Sicne $A \subseteq B \subseteq U \cup V$, it suffices to prove that $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$. Since $U \cap B \neq \emptyset$, we can choose $b \in U \cap B$. Then we have $b \in B \subseteq \bar{A} = A \cup A'$ and so either $b \in A$ or $b \in A'$. If $b \in A$ then we have $b \in U \cap A$ and so $U \cap A \neq \emptyset$. Suppose $b \in A'$. Since $b \in U$ and U is open, we can choose $r > 0$ such that $B(b, r) \subseteq U$. Since $b \in A'$ we have $B(b, r) \cap A \neq \emptyset$ so we can choose $c \in B(b, r) \cap A$. Then $c \in B(a, r) \subseteq U$ and $c \in A$, hence $c \in U \cap A$ so $U \cap A \neq \emptyset$. Using similar method we obtain $V \cap A \neq \emptyset$ and so U and V separate A, which is a contradiction. ♠

Theorem 1.2.6. *Let $A \subseteq P \subseteq \mathbb{R}^n$, recall that A is connected when no disjoint open sets U, V that separate A, that is, $U \cap A, V \cap A \neq \emptyset$ and $A \subseteq U \cup V$. Define A to be connected in P when there do not exists disjoint open sets $E, F \subseteq P$ which separate A. Then A is connected in P iff A is connected in $\mathbb{R}^n$.*

Proof. Suppose that A is disconnected in $\mathbb{R}^n$. Let U, V be open disjoint sets in $\mathbb{R}^n$ which separate A. Let $E = U \cap P$, $F = V \cap P$. Note E and F are open in P. Then we are done as E and F separate A. Suppose A is disconnected in P. Choose $E, F \subseteq P$ which are open in P and separate A. Choose open sets U and V in $\mathbb{R}^n$ such that $E = U \cap P$ and $F = V \cap P$. We have $U \cap A \supseteq E \cap A \neq \emptyset$, $V \cap A \supseteq F \cap A = \emptyset$, and $A \subseteq E \cup F \subseteq U \cup V$, but we might not have $U \cap V \neq \emptyset$.

Thus, for each $a \in E$, choose $r_a > 0$ so that $B(a, 2r_a) \subseteq U$, we can do this since U is open in $\mathbb{R}^n$. Then let $U_0 = \bigcup_{a \in E} B(a, r_a)$. Then U_0 is open in $\mathbb{R}^n$ as it is a union of open sets and it contains E as every point of E is in U_0. Then, since $E \subseteq P$ so that $E \subseteq U_0 \cap P$ and $U_0 \subseteq U$, so $U_0 \cap P \subseteq U \cap U \cap P = E$, thus $E = U_0 \cap P$. Similarly, for each $b \in V$, we choose $s_b > 0$ so that $B(b, 2s_b) \subseteq F$ and let $V_0 = \bigcup_{b \in F} B(b, s_b)$ and then we have $V_0 \cap P = F$. If we can show the two are disjoint, then we are done. Suppose $U_0 \cap V_0$ is not empty, choose $c \in U_0 \cap V_0$. Since $c \in U_0 = \bigcup_{a \in E} B(a, r_a)$, we can choose $a \in E$ such that $c \in B(a, r_a)$. Likewise, we can choose $b \in F$ so that $c \in B(b, s_b)$. Suppose $r_a \leq s_b$. Then $|a - b| \leq |a - c| + |c - b| < r_a + s_b < 2s_b$, thus $a \in B(b, 2s_b) \subseteq V$, but then $a \in P$ and $a \in V$, so $a \in F$, and we also have $a \in E$ which contradicts the fact that $E \cap F = \emptyset$. ♡

Corollary 1.2.6.1. *For $A \subseteq \mathbb{R}^n$, A is connected iff the only subsets of A which are both open and closed in A are the sets $\emptyset$ and A*

Proof. Apply the last theorem to the case where $A = P$. Suppose $\exists E, F \subseteq A$ with $E \cap A, F \cap A \neq \emptyset$ and $E \cap F = \emptyset$ and $A \subseteq E \cup F$. Thus $A = E \cup F$, we have $E^c = F$ and $F^c = E$ so A is closed and open at the same time. ♡

1.3 Compact Sets

Definition 1.3.1. $A \subseteq \mathbb{R}^n$ is ***bounded*** when $A \subseteq B(a, r)$ for some $a \in \mathbb{R}^n$ and $r > 0$.

Proposition 1.3.2. *A is bounded iff $A \subseteq B(0, r)$ for some $r > 0$*

Proof. If A is bounded then $A \subseteq B(a, r)$. Let $\delta = |a| + 2r$, we have $B(a, r) \subseteq B(0, \delta)$.

If $A \subseteq B(0, r)$ then A is bounded. ♡

Definition 1.3.3. Let $A \subseteq \mathbb{R}^n$. An ***open cover*** of A is a set S of open sets in $\mathbb{R}^n$ such that $A \subseteq \bigcup S$. A ***subcover*** of an open cover S of A is a subset $T \subseteq S$ such that $A \subseteq \bigcup T$.

Definition 1.3.4. A set $A \subseteq \mathbb{R}^n$ is ***compact*** when every open cover A contains a finite subcover of A.

Example 1.3.5. Show $A = \{1/n : n \in \mathbb{Z}^+\}$ is not compact

Solution. For $n \in \mathbb{Z}^+$, let $U_n = B(\frac{1}{n}, \frac{1}{2n(n+1)}) = (\frac{1}{n} - \frac{1}{2n(n+1)}, \frac{1}{n} + \frac{1}{2n(n+1)})$. Then we have $\forall n \in \mathbb{Z}^+$, $1/n \in U_k \Leftrightarrow k = n$ and hence $A \subseteq \bigcup_{i=1}^{\infty} U_i$. However, A is not contained in the union of finitely many of the sets U_i. ♠

Example 1.3.6. Let $B = \bar{A} = \{1/n : n \in \mathbb{Z}^+\} \cup \{0\}$, show that B is compact.

Solution. Let S be an open cover of B. Choose $U_0 \in S$ with $0 \in U_0$. Choose $r > 0$ so that $B(0,r) \subseteq U_0$. Note $B(0,r)$ contains all of the points in B except $1, \frac{1}{2}, ..., \frac{1}{n}$ where $\frac{1}{n+1} < r$. For $1 \leq k \leq n$, choose $U_k \in S$ so that $\frac{1}{k} \in U_k$, then $T = \{U_0, ..., U_n\}$ is a finite subcover. ♠

Theorem 1.3.7 (The Nested Interval Theorem)**.** *Let $I_0, I_1, ...$ be nonempty closed bounded intervals in $\mathbb{R}$. Suppose that $I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$ then we have $\cap_{k=0}^{\infty} I_k \neq \emptyset$*

Proof. For each $k \geq 1$, let $I_k = [a_k, b_k]$ with $a_k < b_k$. For each k, we must have $a_k \leq a_{k+1} \leq b_{k+1} \leq b_k$. Since $a_k \geq a_{k+1}$, the sequence (a_k) is increasing and bounded by b_1. Thus it converges. Let $a = \lim_{n\to\infty} a_n$. Similarly, (b_k) is decreasing and bounded below by a_1 and so it converges. Say $b = \lim_{n\to\infty} b_n$. Note since $a_k \leq b_k$ for all k, we have $a < b$.

Fix $m \geq 1$, we have $a \geq a_m$ and $b \leq b_m$, hence $[a,b] \subseteq I_m$. Thus $[a,b] \subseteq \bigcap_{i=1}^{\infty} I_i$ and so it is not empty. ♡

Definition 1.3.8. A ***closed rectangle*** in $\mathbb{R}^n$ is a set of the form

$$\begin{aligned} R &= [a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n] \\ &= \{(x_1, ..., x_n) \in \mathbb{R}^n : \forall j, a_j \leq x_j \leq b_j\} \end{aligned}$$

Theorem 1.3.9 (Nested Rectangles)**.** *Let $R_1, R_2, ...$ be closed rectangles in $\mathbb{R}^n$ with $R_1 \supseteq R_2 \supseteq R_3 \supseteq ...$, then*

$$\bigcap_{i=1}^{\infty} R_k \neq \emptyset$$

Proof. Let $R_k = [a_{k,1}, b_{k,1}] \times [a_{k,2}, b_{k,2}] \times ... \times [a_{k,n}, b_{k,n}]$. Since $R_1 \supseteq R_2 \supseteq ...$ it follows that for each index j with $1 \leq j \leq n$ we have $[a_{1,j}, b_{1,j}] \supseteq [a_{2,j}, b_{2,j}] \supseteq ...$ By the nested interval theorem, for each index j we can choose $u_j \in \bigcap_{k=1}^{\infty} [a_{k,j}, b_{k,j}]$. Then for $u = (u_1, u_2, ..., u_n)$ we have $u \in \bigcap_{i=1}^{\infty} R_i$ ♡

Theorem 1.3.10 (Compactness of Closed Rectangles)**.** *Closed rectangles in $\mathbb{R}^n$ are compact*

Proof. Let R be a closed rectangle in $\mathbb{R}^n$.

Suppose for a contradiction that R is not compact. Choose an open cover S of R which does not contain a finite subcover. Let $R_1 = R = [a_{1,1}, b_{1,1}] \times [a_{1,2}, b_{1,2}] \times ... \times [a_{1,n}, b_{1,n}]$. Partition each interval $[a_{1,k}, b_{1,k}]$ into two equal subintervals

$$[a_{1,k}, \frac{a_{1,k} + b_{1,k}}{2}] \cup [\frac{a_{1,k} + b_{1,k}}{2}, b_k]$$

and in this way we partition R into 2^n equal-sized sub-rectangles. Note that since S covers $R = R_1$, it also covers each of the 2^n sub-rectangles and we can choose one of the 2^m smaller rectangles, which we call R_2 such that S has no finite subset which covers R_2.

We repeat the procedure to obtain closed rectangles $R = R_1 \supseteq R_2 \supseteq R_3 \supseteq ...$ so that for each $n \in \mathbb{Z}^+$, S does not have a finite subset which covers R_n. By the Nested Rectangles Theorem, we can choose q point $a \in \bigcap_{n=1}^{\infty} R_n$. Since $a \in R$ and S covers R, we can choose a open set $u \in S$ with $a \in u$. Then we can choose $r > 0$ so that $B * a, r \subseteq$ I ♡

Theorem 1.3.11. *Closed subsets of compact sets are compact. That is, let $A \subseteq P \subseteq \mathbb{R}^n$, if P is compact and A is closed in $\mathbb{R}^n$ then A is compact.*

Proof. Let S be an open cover of A. Since A is closed, A^c is open. Since S covers A, $S \cup \{A^c\}$ covers $\mathbb{R}^n$. Hence $S \cup \{A^c\}$ covers P. Since P is compact, we can choose a finite subset T of S such that $T \cup \{A^c\}$ covers P. It follows T covers A as if $a \in A$ then $a \in \bigcup(T \cup \{A^c\}) = A^c \cup \bigcup T$, so either $a \in \bigcup T$ or $a \in A^c$. However, $a \in A$ so $a \notin A$ hence $a \in \bigcup T$. Thus A is compact. ♡

Theorem 1.3.12. *[Heine-Borel] Let $A \subseteq \mathbb{R}^n$. A is compact if and only if A is closed and bounded.*

Proof. Suppose A is compact.

For $n \in \mathbb{Z}^+$, let $U_n = B(0, n)$. then $U_1 \subseteq U_2 \subseteq U_3 \subseteq ...$ and $\bigcup_{n=1}^{\infty} U_n = \mathbb{R}^n$. Thus the set $S = \{U_n : n \in \mathbb{Z}^+\}$ is an open cover of A. Since A is compact, we can choose $U_{n_1}, ..., U_{n_l}$ with $n_1 < n_2 < ... < n_l$, so that $\bigcup_{k=1}^{l} U_{n_k}$ covers A. Since $U_{n_1} \subseteq ... \subseteq U_{n_l}$, we have $\bigcup_{k=1}^{l} U_{n_k} = B(0, n_l)$ and hence A is bounded.

Suppose, for a contradiction, that A is not closed. Since A is not closed, we can choose $a \in A'$ but $a \notin A$. Since $a \in A'$, $\forall B^*(a, r) \cap A \neq \emptyset$. For each $n \in \mathbb{Z}^+$, let $U_n = (\bar{B}(a, 1/n))^c = \mathbb{R}^n \backslash \bar{B}(a, 1/n)$. Note that each U_n is open and $U_1 \subseteq U_2 \subseteq U_3 \subseteq ...$ and $\bigcup_{n=1}^{\infty} U_n = \mathbb{R}^n \backslash \{a\}$. Since $a \notin A$, we have $A \subseteq \mathbb{R}^n \backslash \{a\}$ and therefore, $S = \{U_n : n \in \mathbb{Z}^+\}$ is an open cover of A. Since A is compact, we can choose $n_1, n_2, ..., n_l$ with $n_1 < n_2 < ... < n_l$ such that $A \subseteq U_{n_1} \cup U_{n_2} \cup ... \cup U_{n_l}$. Since $U_{n_1} \subseteq U_{n_2} \subseteq ... \subseteq U_{n_l}$, we have the union of them to be equal to U_{n_l}. SO we have $A \subseteq U_{n_l} = (\bar{B}(a, \frac{1}{n_l}))^c$ so $A \cap \bar{B}(a, \frac{1}{n_l}) = \emptyset$. Since $a \in A'$, $B^*(a, \frac{1}{n_l}) \cap A \neq \emptyset$ but $B^*(a, \frac{1}{n_l}) \cap A \subseteq \bar{B}(a, \frac{1}{n_l}) \cap A$. So we have obtained the desired contradiction.

Thus we have proven that if A is compact then A is closed and bounded.

Suppose, conversely that A is closed and bounded. Since A is bounded, we can choose $r > 0$ such that $A \subseteq B(0, r)$. It follows that $A \subseteq R$ where R is closed rectangle $R = [-r, r] \times [-r, r] \times ... \times [-r, r]$. This is because, if $x \in B(0, r)$ then $|x| < r$ so $|x_k| < |x| < r$ for all k as $|x| = \sum |x_j|^2 \geq \sqrt{|x_k|^2} = |x_k|$, so $x \in R$. Since $A \subseteq R \subseteq \mathbb{R}^n$ and R is compact and A is closed, it follows that A is compact.

♡

Chapter 2

Intro to Vector Valued Functions

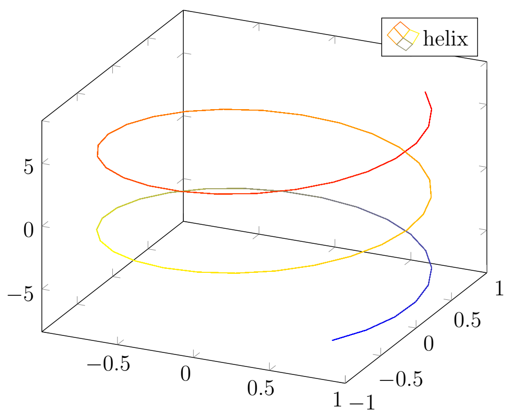
2.1 Overview

Definition 2.1.1.

1. For $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $Graph(f) = \{(x, y) : y = f(x)\} \subseteq \mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$ is given explicitly by the equation $y = f(x)$.
2. $Image(f) = Range(f) = \{y : y = f(x), x \in D\} \subseteq \mathbb{R}^m$ is given parametrically be the equation $y = f(x)$ and x or $()x_1, ..., x_n)$ is called the parameter (or parameters)
3. $Null(f) = f^{-1}(0) = \{x \in D : f(x) = 0\}$ and for $y \in \mathbb{R}^m$ and $f^{-1}(y) = \{x \in D : y = f(x)\}$ is given implicitly by the equation $y = f(x)$.

Example 2.1.2. For the circle $x^2 + y^2 = 1$, the top half is defined explicitly by $y = \sqrt{1 - x^2}$ (the top half-circle is the graph of the function $f : [-1, 1] \to \mathbb{R}$ given by $y = f(x) = \sqrt{1 - x^2}$). The entire circle can be given parametrically by the equation $(x, y) = (\cos t, \sin t)$, and the circle is the image (or the range) of the function $g : \mathbb{R} \to \mathbb{R}^2$ with $t \mapsto (\cos t, \sin t)$. The entire circle can be given implicitly by $x^2 + y^2 = 1$, and it is the Null set of the function $h : \mathbb{R}^2 \to \mathbb{R}$ where $(x, y) \mapsto x^2 + y^2 - 1$.

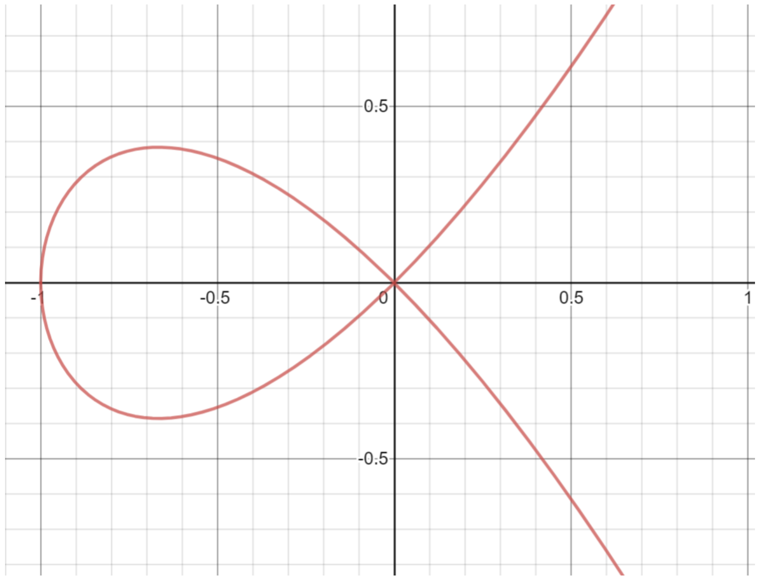
Example 2.1.3. The helix in $\mathbb{R}^3$ is given explicitly by $x = \cos z$ and $y = \sin z$. Thus $(x, y) = f(z) = (\cos z, \sin z)$ where $f : \mathbb{R} \to \mathbb{R}^2$.



The helix is given parametrically by $(x, y, z) = (\cos t, \sin t, t)$ and it is given implicitly by $x = \cos z, y = \sin z$ so the helix is the null set of $g : \mathbb{R}^3 \to \mathbb{R}^2$ where $g(x, y, z) = (x - \cos z, y - \sin z)$.

Example 2.1.4. The alpha curve is given implicitly by $y^2 = x^3 + x^2$.

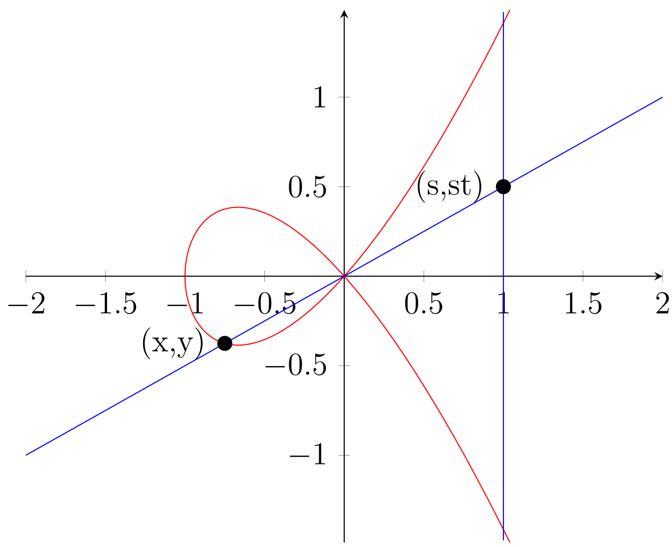
The top and bottom half are given by $y = \pm\sqrt{x^3 + x^2}$.



The line from $(0, 0)$ to $(1, t)$ is given parametrically by $(x, y) = s(1, t) = (s, st)$ where $s \in \mathbb{R}$ and $(x, y) = (s, st)$ lies on the alpha curve when

$$s^2t^2 = s^3 + s^2$$

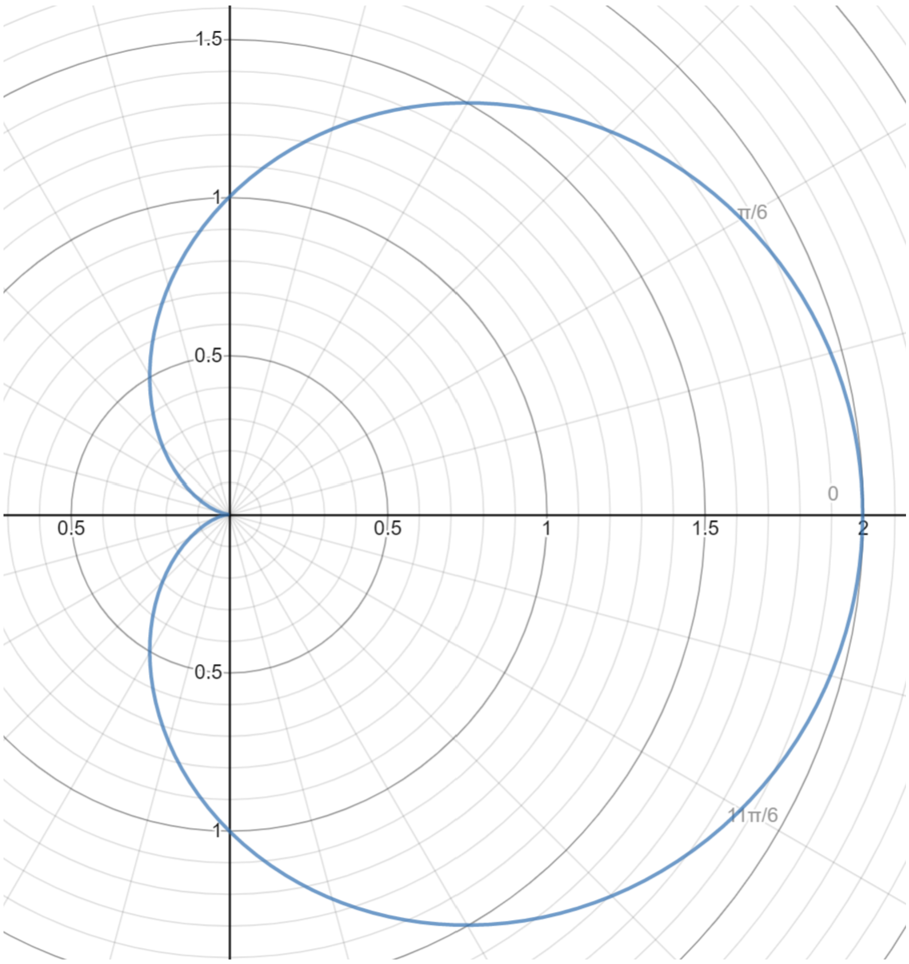
Thus $s = 0$ or $t^2 = s + 1$ which imply $s = t^2 - 1$. Hence, when $s = t^2 - 1$, we obtain the point $(x, y) = (s, st) = (t^2 - 1, t(t^2 - 1))$. Thus the alpha curve is given parametrically by $(x, y) = (t^2 - 1, t(t^2 - 1))$ for $t \in \mathbb{R}$.



Given (x, y) on the alpha curve, note that t is given by $t = \frac{y}{x}$ unless $x = 0$.

Note we can use the projection trick to find all pythagorean triple, if we used it on the unit circle.

Example 2.1.5. Consider $r = r(\theta) = 1 + \cos\theta$.



The cardioid is given parametrically in Cartesian coordinates by taking $t = \theta$ and using $x = r\cos\theta$, $y = r\sin\theta$. Thus $(x, y) = (r\cos\theta, r\sin\theta) = ((1+\cos t)\cos t, (1+\cos t)\sin t)$.

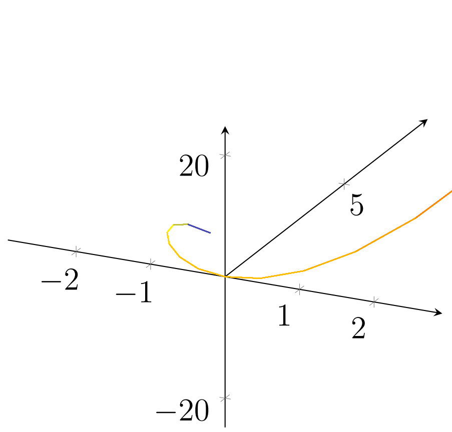
We can also obtain an implicit equation in Cartesian coordinates as follows:(we are using $x = r\cos\theta, y = r\sin\theta$, and $x^2 + y^2 = r^2$)

$$r = 1 + \cos\theta$$

$$r^2 = r + r\cos\theta$$

$$x^2 + y^2 = \sqrt{x^2 + y^2} + x \Rightarrow x^2 + y^2 = (x^2 + y^2 - x)^2$$

Example 2.1.6. The ***twisted cubic*** is given parametrically by $(x, y, z) = (t, t^2, t^3)$, (so it is the range of $f : \mathbb{R} \to \mathbb{R}^3$ where $f(t) = (t, t^2, t^3)$). x is given explicitly by $y = x^2, z = x^3$. x is given implicitly by $y = x^2, z = x^3$. So x is the null set of $g : \mathbb{R}^3 \to \mathbb{R}^2$ given by $(u, v) = h(x, y, z) = (y - x^2, z - x^3)$.

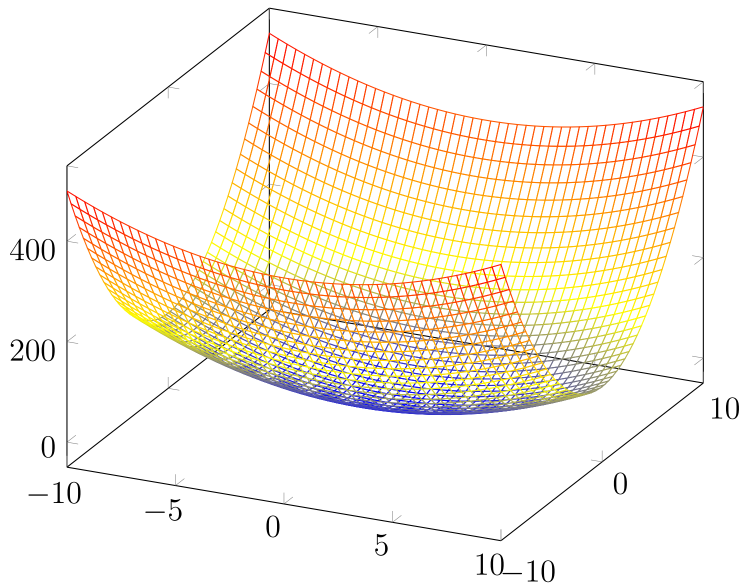


Next, let us verify that $Rang(f) = Null(h)$ where $f(t) = (t, t^2, t^3)$ and $h(x, y, z) = (y - x^2, z - x^3)$.

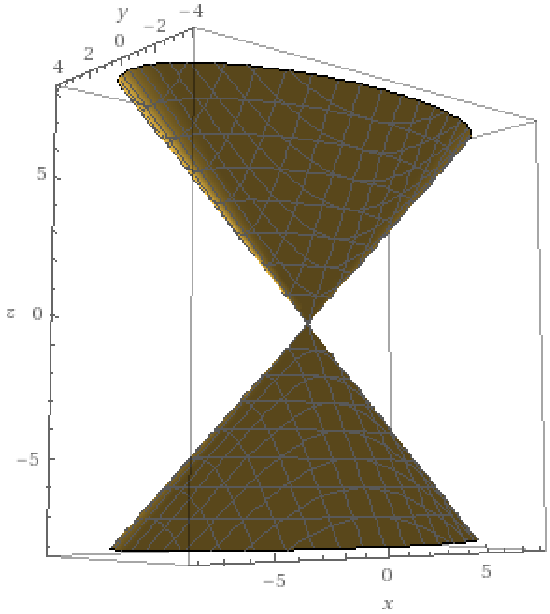
Choose $t \in \mathbb{R}$ such that $(x, y, z) = f(t)$. Then $y = t^2 = x^2$ and $z = t^3 = x^3$ so that $h(x, y, z) = (y - x^2, z - x^3) = (0, 0)$. Hence $(x, y, z) \in Null(h)$. Let $(x, y, z) = Null(h)$ so $(y - x^2, z - x^3) = (0, 0)$. So $y = x^2$ and $z = x^3$. Then choose $t = x$, we have $x = t$, $y = t^2$ and $z = t^3$ and then $f(t) = (t, t^2, t^3) = (x, y, z)$.

Example 2.1.7. Try to sketch $z = x^2 + 4y^2$, $z^2 = x^2 + 4y^2$, $z = x^2 - 4y^2$ and $(x, y, z) = (u, v, u^2 + 4v^2 - 3)$.

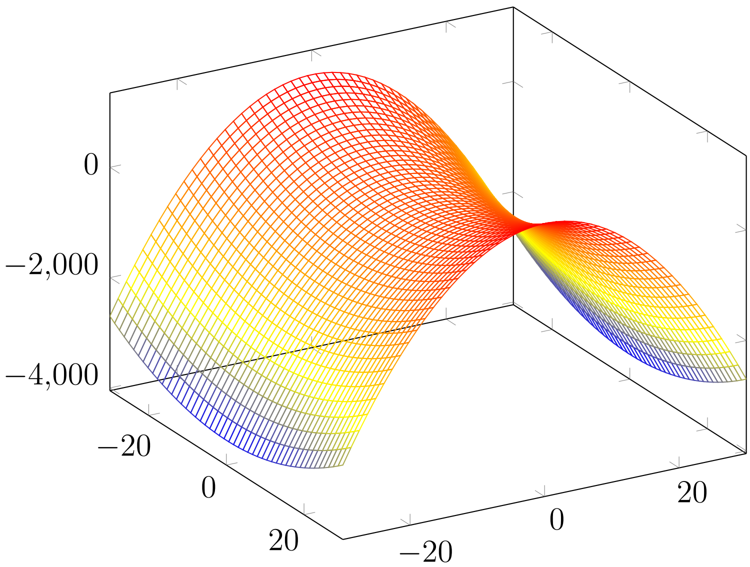
For $z = x^2 + 4y^2$, we have $z = 0$ then $x^2 + 4y^2 = 0$ means $x, y = 0$. If $z = 1$ then it is an ellipse. It is not hard to see this holds for all $z > 0$.



For $z^2 = x^2 + 4y^2$, it is not hard to see it is a cone since now we also have values when $z < 0$.



For $z = x^2 - 4y^2$, we have $z = 0$ then $x^2 - 4y^2 = 0$ and so $x = \pm y$. If $z = 1$, we have $x^2 - \frac{y^2}{1/4} = 1$, which is a hyperbola. Continue do so, it would always be an hyperbola, so we get the image.



For $(x, y, z) = (u, v, u^2 + 4v^2 - 3)$, we note this is the same as $z = x^2 + 4y^2 - 3$.

Example 2.1.8. Find a parametric equation for $x^2 + y^2 + z^2 = r^2$. We have $z = r\cos\phi$, $x = r\sin\phi\cos\theta$ and $y = r\sin\phi\cos\theta$

Chapter 3

Limits and Continuity

3.1 Sequences

Definition 3.1.1. A ***sequence*** in a set A is function of the form $a : \mathbb{Z}_{\geq p} \to A$ for some $p \in \mathbb{Z}$ where $\mathbb{Z}_{\geq p} = \{n \in \mathbb{Z} : n \geq p\}$. When $a \in \mathbb{Z}_{\geq p} \to A$ is a sequence, we write $a_p = a(p)$. Also, we write $a = (a_n)_{n\geq p} = (a_p, a_{p+1}, ...)$ to indicate that a is the sequence with $a(n) = a_n$ for $n \geq p$.

Definition 3.1.2. For a sequence $(x_n)_{n\geq p}$ in $\mathbb{R}^n$ and for $a \in \mathbb{R}^n$, we say $(x_n)_{n\geq p}$ ***converges to*** a and we write $x_n \to a$ or $lim_{n\to\infty} x_n = a$ when

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_{\geq p}, \forall n \in \mathbb{Z}_{\geq p}, (n \geq N \Rightarrow |x_n - a| < \epsilon)$$

Definition 3.1.3. When $x_n \to a$ for $a \in \mathbb{R}^n$ then we say $(x_n)_{n\geq p}$ ***converges***. Otherwise we say that $(x_n)_{n\geq p}$ diverges. We say that $(x_n)_{n\geq r}$ ***diverges to infinity***, and we write $x_n \to \infty$ or $lim_{n\to\infty} x_n = \infty$, when

$$\forall r > 0, \exists N \in \mathbb{Z}_{\geq p}, \forall n \in \mathbb{Z}_{\geq p}, (n \geq N \Rightarrow |x_n| \geq r)$$

Definition 3.1.4. We say $(x_n)_{n\geq r}$ is ***bounded*** when the set $\{x_n : n \in \mathbb{Z}_{\geq p}\}$ is bounded, that is, $\exists r > 0, |x_n| \leq r$ for all $n \in \mathbb{Z}_{\geq p}$.

Remark 3.1.5. Note when the sequence is unbounded, we do not have the sense of $+\infty$ and $-\infty$, recall the definition of diverges to infinity.

Definition 3.1.6. We say $(x_n)_{n\geq p}$ is Cauchy when

$$\forall \epsilon > 0, \exists N \in \mathbb{Z}_{\geq p}, \forall n, m \in \mathbb{Z}_{\geq p}(n, m \geq N \Rightarrow |x_n - x_m| < \epsilon)$$

Definition 3.1.7. A ***subsequence*** of a given sequence $(x_n)_{n\geq p}$ is a sequence of the form $(y_k)_{k\geq p} = (x_{n_k})_{k\geq q}$ when $n_q < n_{q+1} < n_{q+2} < ....$

Theorem 3.1.8. *Every convergent sequence in* $\mathbb{R}^n$ *is bounded.*

Theorem 3.1.9. *For a sequence* $(x_n)_{n\geq p}$ *in* $\mathbb{R}^n$, *if* $\lim_{n\to\infty} x_n$ *exists in* $\mathbb{R}^n \cup \{\infty\}$ *then the limit is unique.*

Theorem 3.1.10. *For a sequence $(x_n)_{n\geq p}$, if $\lim_{n\to\infty} x_n = u \in \mathbb{R}^n \cup \{\infty\}$ then for every subsequence $(x_{n_k})_{k\geq q}$ of $(x_n)_{n\geq p}$, we have $\lim_{k\to\infty} x_{n_k} = u$*

Remark 3.1.11. Because Theorem 3.1.10, the first finite many terms of the sequence do not affect whether the sequence converges or not, nor the value of the limit. Thus we often denote the sequence $(x_n)_{n\geq p}$ simply by (x_n).

Theorem 3.1.12. *Let (x_m) be a sequence in $\mathbb{R}^n$, for each $m \in \mathbb{Z}_{\geq 1}$, write $x_m = (x_{m,1}, x_{m,2}, ..., x_{m,n})$. Then we have*

1. *(x_m) is bounded $\Leftrightarrow$ $(x_m, k)_m$ is bounded for each index $1 \leq k \leq n$*
2. *$(x_m)_m$ converges $\Leftrightarrow$ $(x_{m,k})_m$ converges for each index $1 \leq k \leq n$ and let $u = (u_1, ..., u_n)$ then $\lim_{n\to\infty} x_n = u \Leftrightarrow \lim_{n\to\infty} x_{n,k} = u_k$ for all $1 \leq k \leq n$*
3. *$(x_m)_m$ is Caucyh $\Leftrightarrow$ $(x_{m,k})_m$ is Cauchy for all $1 \leq k \leq n$*

Proof. We will do a sample proof of (2).

Let $(x_n)_{n\geq p}$ be a sequence in $\mathbb{R}^m$ and let $u \in \mathbb{R}^m$. Suppose $\lim_{n\to\infty} x_n = u$. Let $1 \leq k \leq m$, let $\epsilon > 0$, choose $N \geq p$ so that $n \geq N \Rightarrow |x_n - x| < \epsilon$. Then for $n \geq N$, we have

$$|x_{n,k} - u_k| \leq |x_n - u| < \epsilon$$

Hence, $x_{n,k} \to u_k$, as we can use the fact that for $y \in \mathbb{R}^m$, $|y_k| = \sqrt{y_k^2} \leq \sqrt{\sum_{i=1}^m y_i^2} = |y|$.

Suppose, conversely, that $x_{n_k} \to u_k$ for all $1 \leq k \leq m$. Let $\epsilon > 0$, for each index k, we choose $N_k \geq p$ so that $n \geq N_k \Rightarrow |x_{n,k} - u_k| < \frac{\epsilon}{m}$. Let $N = max(N_1, ..., N_m)$. For $n \geq N$, we have $|x_{n,k} - u_k| < \frac{\epsilon}{m}$ for all k so $|x_n - u| \leq \sum |x_{n,k} - u_k| < m \cdot \frac{\epsilon}{m} = \epsilon$ ♡

Theorem 3.1.13 (Operations on limits)**.** *For sequences (x_n) and (y_n) in $\mathbb{R}^m$ and for $c \in \mathbb{R}$, if $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} y_n = v$, then we have*

$$\lim_{n\to\infty} cx_n = c \cdot u \in \mathbb{R}^m$$

$$\lim_{n\to\infty} x_n + y_n = u + v \in \mathbb{R}^m$$

$$\lim_{n\to\infty} x \cdot y = u \cdot v \in \mathbb{R}$$

$$\lim_{n\to\infty} |x_n| = |u|$$

In addition, if $m = 3$ and $\times : \mathbb{R}^3 \to \mathbb{R}^3$ is the cross product, then we have

$$\lim_{n\to\infty} x_n \times y_n = u \times v \in \mathbb{R}^3$$

Theorem 3.1.14. *Let $A \subseteq \mathbb{R}^m$ and let $a \in \mathbb{R}^m$ then a is a limit point of A if and only if there exists a sequence in $A \backslash \{a\}$ such that $x_n \to a$*

Theorem 3.1.15. *Let $A \subseteq \mathbb{R}^m$, then A is closed if and only if for every sequence (x_n) in A with $\lim_{n\to\infty} x_n = u \in \mathbb{R}^m$, we must have $u \in A$.*

Theorem 3.1.16 (Bolzano-Weierstrass)**.** *Every bounded sequence in $\mathbb{R}^m$ has a convergent subsequence*

Proof. Let $(x_n)_{n\geq p}$ be a bounded sequence in $\mathbb{R}^m$. Write $u \in \mathbb{R}^m$ to be $u = (u^1, u^2, ..., u^m)$. So, for each $n \geq p$, we have $x_n = (x_n^1, x_n^2, ..., x_n^m) \in \mathbb{R}^m$. Since (x_n) is bounded, thus $(x_n^1)_{n\geq p}$ is bounded since $|x_n^1| \leq |x_n|$. Thus, by Bolzano-Weierstrass theorem for real-valued sequence, we can choose a convergent subsequence $(x_{n_l}^1)_{l\geq 1}$ of (x_n^1). Use the same numbers $n_1 < n_2 < n_3 < ...$ to obtain the subsequence $(x_{n_l})_{l\geq 1}$ of $(x_n)_{n\geq p}$. Since $(x_n)_{n\geq p}$ is bounded, so is $(x_{n_l})_{l\geq 1}$. Hence so is $(x_{n_l}^2)_{l\geq 1}$. Since $(x_{n_l}^2)_{l\geq 1}$ is bounded we can choose a convergent subsequence $(x_{n_{l_k}}^2)_{k\geq 1}$ of $(x_{n_l}^2)_{l\geq 1}$, which is a subsequence of $(x_n)_{n\geq p}$. Use the same integers $n_{l_1} < n_{l_2} < ...$ to obtain the vector valued subsequence $(x_{n_{l_k}})_{k\geq 1}$ of $(x_{n_l})_{l\geq 1}$, which is a subsequence of $(x_n)_{n\geq p}$. We repeat this and obtain the desired result. ♡

Second proof. We first show that if E is an infinite subset of a compact set K, then E has a limit point in K.

Suppose no point of K were limit point of E. Let $p \in K$, $p \notin E'$ imply $\exists \epsilon_p > 0$ so that $B^*(p, \epsilon_p) \cap E = \emptyset$. Therefore, for all $p \in K$, we can find $\epsilon_p > 0$ so that $B(p, \epsilon_p) \cap E$ is either empty or contains one point, namely, p.

Next, let $V_p = B(p, \epsilon_p)$ so that $B(p, \epsilon_p) \cap E = \emptyset$ or $\{p\}$. Clearly V_p is open for all $p \in K$. Then, $\bigcup_{p\in K} V_p$ is an open cover of K. In addition, we note $\bigcup_{p\in K} V_p$ cannot finitely cover E as each V_p only covers at most one point in E and E is an infinite subset. However, $E \subseteq K$, and if there is a finite subcover of $\{V_p : p \in K\}$ such that covers K, it must also cover E, which is impossible. This lead to a contradiction as K is compact and that imply there must exists a finite subcover of $\{V_p : p \in K\}$.

Next, we show that every bounded infinite subset E of $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$. Indeed, since E is bounded, it is contained in a open ball and hence contained by a closed rectangle, denoted by R. Thus, E is a infinite subset of a compact set R. Thus, it has a limit point in R by our previous claim.

Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence, then $S := \{x_n : n \in \mathbb{N}\}$ is a infinite subset of $\mathbb{R}^n$ and it is bounded by the definition of bounded sequences. Hence, we have $u \in \mathbb{R}^n$ and $u \in S'$ by our previous claim. Then, there must exists a sequence (y_n) in $S\backslash\{u\}$ so that $\lim_{k\to\infty} y_k = u$ by a theorem. However, since every elements of y_n is an element of x_n, there must exists a reordering $\pi : \mathbb{N} \to \mathbb{N}$ such that makes (y_n) a subsequence of (x_n). Thus there exists a subsequence of (x_n) such that converges. ♡

Definition 3.1.17. A metric space X is ***complete***, if every Cauchy sequence converges.

Theorem 3.1.18. $\mathbb{R}^m$ *is complete. That is, for any sequence* $(x_n)_{n\geq p}$*, we have* (x_n) *converges if and only if* (x_n) *is Cauchy.*

3.2 Limit of Functions

Definition 3.2.1. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and let $a \in A'$ and $b \in \mathbb{R}^n$, we say $f(x)$ converges to b as x tends to a, and write $\lim_{x\to a} f(x)$ when

$$\forall \epsilon > 0, \exists \delta > 0, (\forall x \in A(0 < |x-a| < \delta \Rightarrow |f(x)-b| < \epsilon))$$

Remark 3.2.2. Try to show Definition 3.2.1 is the same when we write $0 < |x-a| \leq \delta \Rightarrow |f(x)-b| \leq \epsilon$.

Definition 3.2.3. We say $f(x)$ diverges to infinity as x tends to a, and we write $\lim_{x\to a} f(x) = \infty$ when

$$\forall \epsilon > 0, \exists \delta > 0(\forall x \in A(0 < |x-a| < \delta \Rightarrow |f(x)| \geq \epsilon))$$

Example 3.2.4. Guess and prove properties, analogues to properties of sequences.

Theorem 3.2.5. *Let $A \subseteq \mathbb{R}^n$ and let $a \in A'$ and let $f : A \to \mathbb{R}^m$. Let $u \in \mathbb{R}^m \cup \{\infty\}$, then $\lim_{x\to a} f(x) = u$ if and only if for every $(x_n)_{n\geq p}$ in $A\backslash\{a\}$ with $x_n \to a$, we have $\lim_{n\to\infty} f(x_n) = u$.*

Proof. Suppose $\lim_{x\to a} f(x) = u$. Let $(x_n)_{n\geq p}$ be a sequence in $A\backslash\{a\}$ with $x_n \to a$. Let $\epsilon > 0$, we can choose $\delta > 0$, such that $0 < |x-a| < \delta \Rightarrow |f(x)-u| < \epsilon$. Since $x_n \to a$, we can choose $N \geq p$ such that $\forall n > N$, we have $|x_n - a| < \delta$. Thus, for $n \geq N$, since x_n is never equal to a, so $0 < |x_n - a| < \delta$, and so $|f(x_n) - u)| < \epsilon$.

Now, suppose $\lim_{x\to a} f(x) \neq u$. Thus, by negating the definition of existence of limit, we have, $\exists \epsilon > 0$, such that $\forall \delta > 0$, $\exists x \in A$ such that $0 < |x-a| < \delta$ and $|f(x)-u| \geq \epsilon$. For each $n \in \mathbb{Z}_{\geq 1}$, we take $x_n \in A$ such that $0 < |x_n - a| < \frac{1}{n}$ and $|f(x)-u| \geq \epsilon$. Note that since $0 < |x_n - a|$, we can say $x_n \neq a$ and thus (x_n) is a sequence in $A\backslash\{a\}$. Thus, we have $x_n \to a$ by the way we constructed it (need to prove). However, we also have $f(x_n)$ cannot converge to u as $|f(x_n)-u| \geq \epsilon > 0$. Hence we have a contradiction.

The proof follows. ♡

Theorem 3.2.6 (Uniqueness of Limits of Functions)**.** *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, let $a \in A'$ and let $u, v \in \mathbb{R}^m \cup \{\infty\}$. If $\lim_{x\to a} f(x) = u$ and $\lim_{x\to a} f(x) = v$ then $u = v$.*

Theorem 3.2.7 (Local Determination of Limits of Functions)**.** *Let $A \subseteq \mathbb{R}^n$, let $a \in A'$, let $B = B^*(a,r) \cap A$ with $r > 0$. Let $f : A \to \mathbb{R}^m$ and let $g : B \to \mathbb{R}^m$ and suppose that $f(x) = g(x)$ for all $x \in B$. Then $\lim_{x\to a} f(x)$ exists in $\mathbb{R}^n \cup \{\infty\}$ if and only if $\lim_{x\to a} g(x)$ exists in $\mathbb{R}^m \cup \{\infty\}$ and, in this case, the limits are equal.*

Definition 3.2.8. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$. We can write

$$f(x) = (f_1(x), f_2(x), ..., f_m(x))$$

where $f_k : A \to \mathbb{R}$ for each index. Then the function f_k is called the kth component function of f. Note that $f_k = p_k \circ f$ where $p_k : \mathbb{R}^m \to \mathbb{R}$ is the k ***projection*** map given by $p_k(y_1, ..., y_m) = y_k$.

Theorem 3.2.9 (Comparison Theorem)**.** *Let $f, g : A|subseteq\mathbb{R}^n \to \mathbb{R}$ with $f(x) \leq g(x)$ for all $x \in A$ and let $a \in A'$.*

1. *If $\lim_{x\to a} f(x) = u \in \mathbb{R}\cup\{\pm\infty\}$ and $\lim_{x\to a} g(x) = v \in \mathbb{R}\cup\{\pm\infty\}$ then $u \leq v$.*
2. *If $\lim_{x\to a} f(x) = \infty$ then $\lim_{x\to a} g(x) = \infty$*
3. *If $\lim_{x\to a} g(x) = -\infty$ then $\lim_{x\to a} f(x) = -\infty$*

Theorem 3.2.10 (Squeeze Theorem)**.** *Let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ with $f(x) \leq g(x) \leq h(x)$ for all $x \in A$ and let $u \in \mathbb{R} \cup \{\pm\infty\}$. If $\lim_{x\to a} f(x) = u = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = u$.*

3.3 Continuity

Definition 3.3.1. For $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $a \in A$, we say that f is ***continuous at*** a when

$$\forall\epsilon > 0, \exists\delta > 0, \forall x \in A(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)$$

In addition, we say f is continuous (in or on A) when f is continuous at every point $a \in A$.

Remark 3.3.2. ***Uniform continuity*** can be defined in a similar way, namely, we say f is uniformly continuous when $\forall\epsilon > 0, \exists\delta > 0, \forall x, y \in A(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon)$

Theorem 3.3.3. *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and let $a \in A$. Then f is continuous at a if and only if for all sequences $(x_n)_{n\geq p}$ in A with $x_n \to a$, we have $f(x_n) \to f(a)$.*

Theorem 3.3.4. *We also have many similar theorems:*

1. *Components of continuous functions*
2. *Local determination of continuity*
3. *Operations of continuous functions, i.e. $f, g : A \to \mathbb{R}^m$ be continuous and $c \in \mathbb{R}$ then $cf, f \pm g, f \cdot g, |f|$ are all continuous. In addition, when $m = 3$, $f \times g$ is continuous. If $m = 1, 2$ then fg and f/g are continuous when $g(x) \neq 0$.*

Remark 3.3.5. For $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $a \in A$. If $a \in A'$ then f is continuous at a iff $\lim_{x\to a} f(x) = f(a)$. If $a \notin A'$ then f is continuous (vacuously) at a.

Theorem 3.3.6. *Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $g : B \subseteq \mathbb{R}^m \to \mathbb{R}^l$. Suppose that $C = A \cap f^{-1}(B) \neq \emptyset$ and let $h = g \circ f : C \to \mathbb{R}^l$ so $h(x) = g(f(x))$ for $x \in C$. Let $a \in C'$ (hence $a \in A'$) and let $b \in B'$. Suppose $\lim_{x\to a} f(x) = b$ and $\lim_{y\to b} g(y) = c \in \mathbb{R}^l \cup \{\infty\}$. Then*

1. *If $f(x) \neq b$ for all $x \in C\backslash\{a\}$ then $\lim_{x\to a} h(x) = c$*
2. *If $b \in B$ and g is continuous at b then $\lim_{x\to a} h(x) = \lim_{x\to a} g(f(x)) = c$*

proof of (1). Suppose $f(x) \neq b$ for any $x \in C\backslash\{a\}$. Let $\epsilon > 0$. Since $\lim_{y\to b} g(y) = c$, we can choose $\delta_1 > 0$ so that for all $y \in B$, $0 < |y - b| < \delta_1 \Rightarrow |g(y) - c| < \epsilon$. Since

$\lim_{x\to a} f(x) = b$, we can choose $\delta > 0$ such that for all $x \in A$, $0 < |x-a| < \delta \Rightarrow |f(x)-b| < \delta_1$. Then for $x \in C = A \cap f^{-1}(B)$ with $0 < |x-a| < \delta$, we have $|f(x)-b| < \delta$ and since $0 < |x-a|$ we have $x \in C\backslash\{a\}$, so $0 < |f(x)-b| < \delta_1$ and hence, by the choice of δ_1, we have $|g(f(x))-c| < \epsilon$. ♡

Corollary 3.3.6.1. *If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $g : B \subseteq \mathbb{R}^m \to \mathbb{R}^l$, and $C = A \cap f^{-1}(B) \neq \emptyset$ and $h = g \circ f : C \to \mathbb{R}^l$. Then if f is continuous at $a \in C$ and g is continuous at $b = f(a) \in B$ then h is continuous at a. If f and g are continuous then so is g.*

Corollary 3.3.6.2. *All elementray functions $f : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ are continuous.*

An elementray function $f : A \to \mathbb{R}^m$ is any function which can be obtained from the basic elementray functions: c, x, x^n, e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$, $\arcsin x$, $\arccos x$, $\arctan x$, the kth inclusion map: $I_k : \mathbb{R} \to \mathbb{R}^m$ given by $I_k(t) = (0, ..., t, ..., 0) = te_k$ where e_k is the standard basis of $\mathbb{R}^m$ and the kth projection map given by $p_k : \mathbb{R}^n \to \mathbb{R}$, $p_k(x_1, ..., x_n) = x_k$.

Example 3.3.7. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-2y^2}{x^2+y^2}$ does not exists.

Solution. Let $f(x,y) = \frac{x^2-2y^2}{x^2+y^2}$. If $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, then by considering $(x,y) = (t,0)$ we have $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} \frac{t^2}{t^2} = 1$ and by considering $(x,y) = (0,t)$, we have $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} \frac{0-2t^2}{0+t^2} = -2$. Thus it cannot exists. ♠

Theorem 3.3.8. *Let $f : A \subseteq \mathbb{R}^n \to B \subseteq \mathbb{R}^m$. Then f is continuous on A if and only if*

1. *$f^{-1}(E)$ is open in A for every open set E in B*
2. *$f^{-1}(F)$ is closed in A for every closed set F in B*

Proof. We will prove part one. Suppose f is continuous in A. Let E be a open set in B. Let $a \in f^{-1}(E)$ so $f(a) \in E$. Since E is open, we can choose $\epsilon > 0$ so that $B_B(f(a), \epsilon) \subseteq E$. Since f is continuous at a, we can choose $\delta > 0$ so that $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \epsilon$. We then claim $B_A(a,\delta) \subseteq f^{-1}(E)$. Let $x \in B_A(a,\delta)$, then $x \in A$ and $|x-a| < \delta$. So $|f(x)-f(a)| < \epsilon$ by the choice of δ, hence $f(x) \in B(f(a),\epsilon)$. Since $x \in A$ and $f : A \to B$, we have $f(x) \in B$ for any x, and hence $f(x) \in B_B(f(a),\epsilon) \subseteq E$ and hence $x \in f^{-1}(E)$. Thus $B_A(a,\delta) \subseteq f^{-1}(E)$ and hence it is open in A.

Suppose conversely, we have $f^{-1}(E)$ is open in A for every open set $E \in B$. Let $a \in A$, let $\epsilon > 0$ be given. Let $E = B_B(f(a),\epsilon)$, we have E is open in B so $f^{-1}(E)$ is open in A. Since $f(a) \in E$ we have $a \in f^{-1}(E)$. Choose $\delta > 0$ so that $B_A(a,\delta) \subseteq f^{-1}(E)$. For $x \in A$ with $|x-a| < \delta$, we have $x \in B_A(a,\delta)$ so $x \in p^{-1}(E)$ and so $x \in f^{-1}(E)$ so $f(x) \in E = B_B(f(a),\epsilon)$ hence $|f(x)-f(a)| < \epsilon$. ♡

Theorem 3.3.9. *Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. Let $f : A \to B$ be continuous. Then,*

1. *A is connected then $f(A)$ is connected,*
2. *if A is compact then so is $f(A)$,*

3. *if A is compact then f is uniformly continuous,*
4. *if A is compact and $m = 1$ then f attains its maximum and minimum values on A,*
5. *if A is compact and f is bijective then f^{-1} is continuous.*

Proof.

1. Suppose $f(A)$ is not connected. Choose two open sets $U, V \in \mathbb{R}^m$ which separate $f(A)$, then $U \cap f(A), V \cap f(A) \neq \emptyset$, $U \cap V = \emptyset$ and $f(A) \subseteq U \cup V$. By Theorem 3.3.8, we have $f^{-1}(U)$ and $f^{-1}(V)$ are open in A and verify that $f^{-1}(U), f^{-1}(V)$ separate A in A.
2. Suppose A is compact. Let S be an open cover of $f(A)$. Say $S = \{U_k : k \in K\}$ with each U_k open in $\mathbb{R}^m$. Let $T = \{f^{-1}(U_k) : k \in K\}$, and note that each set $f^{-1}(U_k)$ is open in A and we have T covers A. Sine A is compact, we can choose a finite subset $J \subseteq K$ so that $A \subseteq \bigcup_{j \in J} f^{-1}(U_j)$, then verify $\{U_j : j \in J\}$ covers $f(A)$, and hence $f(A)$ is compact.
3. Skip
4. Suppose A is compact and $m = 1$. That is, $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$. Since A is compact, we have $f(A)$ is compact and so $f(A)$ is closed and bounded (by Heine-Borel, Theorem 1.3.12). Since $f(A)$ is nonempty and bounded, $f(A)$ has a supremum and an infimum in $\mathbb{R}$, let $u = sup(f(A))$. We can choose $x_n \in A$ so that $u - \frac{1}{n} < f(x_n) \leq u$ for each $n \in \mathbb{Z}_{\geq 1}$. Thus, $u - \frac{1}{n} < f(x_n) \leq u$ for all n, we have $f(x_n) \to u$. Since $(f(x_n))_{n \geq 1}$ is a sequence in $f(A)$ with $f(x_n) \to u$, it follows that u is a limit point of $f(A)$. Since $f(A)$ is closed, it contains its limit points and hence $u \in f(A)$ and hence u is the maximum element of $f(A)$. Hence there exists $a \in A$ so that $u = f(a)$. Similarly we can show the infimum is obtained in A.
5. Skip

♡

Definition 3.3.10. Let $A \subseteq \mathbb{R}^n$. We say that A is ***convex*** when for all $a, b \in A$, the line segment $[a, b] \subseteq A$ where $[a, b] = \{a + t(b - a) : 0 \leq t \leq 1\}$.

We say A is ***path-connected*** when for all $a, b \in A$, there exists a continuous map $\alpha : [0, 1] \subseteq \mathbb{R} \to A$ with $\alpha(0) = a$ and $\alpha(1) = b$. We call such a continuous map α a ***path*** from a to b in A.

Theorem 3.3.11. *Let A be in $\mathbb{R}^n$, then*

1. *if A is path-connected, then A is connected,*
2. *if A is open then path-connected is the same as connected.*

Proof. We only sketch proof part 1. Suppose A is path connected and suppose for contradiction A is not connected. Choose open sets U, V in $\mathbb{R}^n$ which separate A. Choose $a \in U \cap A$ and $b \in V \cap A$. Note we have $a \neq b$. Let $\alpha : [0, 1] \to A$ be continuous with $\alpha(0) = a$ and $\alpha(1) = b$. Verify that $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ separate $[0, 1]$. However, $[0, 1]$ is connected. ♡

Example 3.3.12. Show that $B(a, r)$ is convex, hence path-connected, hence connected, where $a \in \mathbb{R}^n$ and $r > 0$.

Solution. Let $b, c \in B(a, r)$. Let $\alpha(t) = b + t(c - b) = (1 - t)b + tc$ for $0 \leq t \leq 1$.

For $0 \leq t \leq 1$, we have

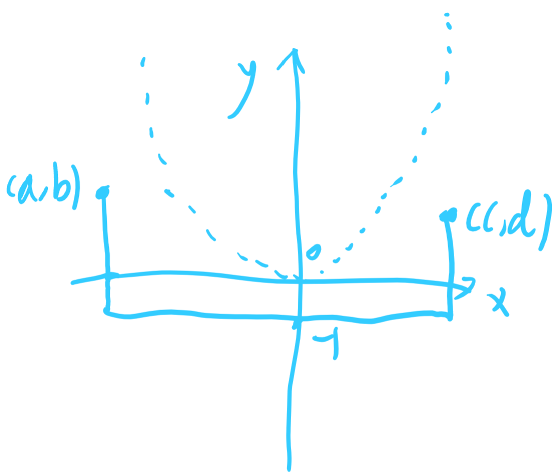
$$\begin{aligned}
|\alpha(t) - a| &= |(b + t(c - b) - a| \\
&= |(b - a) + t((c - a) - (b - a))| \\
&= |(1 - t)(b - a) + t(c - a)| \\
&\leq (1 - t)|b - a| + t|c - a| \text{by Triangle Inequality} \\
&< (1 - t)r + tr = r
\end{aligned}$$

Hence, $\alpha(t) \in B(a, r)$ for all $0 \leq t \leq 1$. Thus $B(a, r)$ is convex as desired. ♠

Example 3.3.13. Let $A = \{(x, y) \in \mathbb{R}^2 : y < x^2\}$, we will show A is open and connected.

Solution. Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = y - x^2$. Then $A = \{(x, y) : f(x, y) < 0\} = f^{-1}((-\infty, 0))$. Thus A is open as $f(x, y)$ is continuous and $(-\infty, 0)$ is open.

To show it is connected, we show it is path connected. Given $(a, b) \in A$, so $b < a^2$. We define a map $\alpha(t) = (a, b) + t((a, -1) - (a, b)) = (a, b) + t(0, -1 - b) = (a, (1 - t)b - t)$, this map is continuous with $\alpha(0) = (a, b)$ and $\alpha(1) = (a, -1)$. Next, we check $\alpha(t) \in A$. Say $(x, y) = \alpha(t) = (a, (1 - t)b - t)$, we have $y = b - t(b + 1) \leq b < a^2 = x^2$ so $(x, y) \in A$. Hence $\alpha(t)$ is a continuous path from (a, b) to $(a, -1)$ in A. Similarly, we have $\beta(t) = (a, -1) + t((c, -1) - (a, -1)) = (a + t(c - a), -1)$ is a continuous path from $(a, -1)$ to $(c, -1)$. Also, as above, we have a continuous path from $(c, d) \in A$ to $(c, -1)$. It follows that there is a continuous path from (a, b) to (c, d).



♠

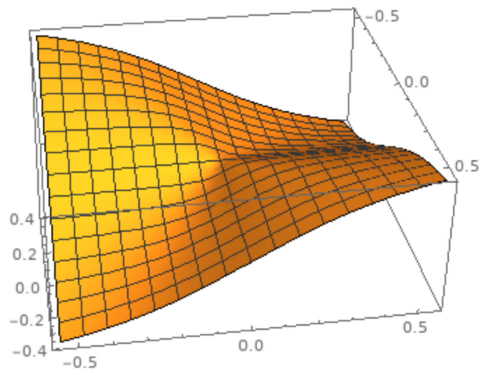
Example 3.3.14. For each of the following functions $g(x, y)$, find $\lim\limits_{(x,y)\to(0,0)} g(x, y)$, if it exists.

1. $\frac{3x^2y}{x^2+2y^2}$
2. $\frac{xy}{\sqrt{x^2+y^2}}$

3. $\frac{x^2-2y^2}{x^2+y^2}$
4. $\frac{xy}{x^2+y^2}$
5. $\frac{xy^2}{x^2+y^2}$

Solution.

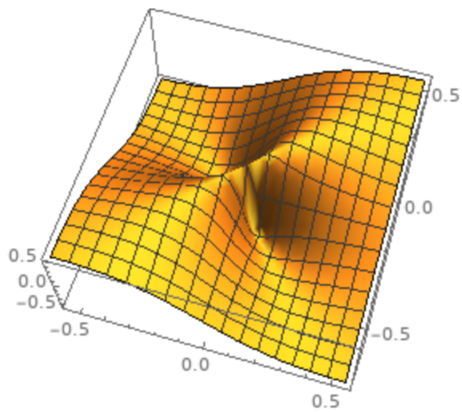
1. $|g(x,y)-0| = \left|\frac{3x^2y}{x^2+2y^2}\right| = \frac{3x^2|y|}{x^2+y^2} \leq \frac{3x^2|y|}{x^2} = 3|y| \to 0$ as $(x,y) \to (0,0)$. Indeed, given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{3}$ and hence for $|(x,y)-(0,0)| < \delta$ and thus $\sqrt{x^2+y^2} < \delta = \frac{\epsilon}{3}$. Thus note $|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} < \frac{\epsilon}{3}$ and therefore $|g(x,y)-0| \leq 3|y| < \epsilon$.
2. $|g(x,y)-0| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq \frac{0.5(x^2+y^2)}{\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2} \to 0$ as $(x,y) \to (0,0)$. Since $0 \leq (|x|-|y|)^2 = x^2 - 2|x||y| + y^2$ so $2|xy| \leq x^2+y^2$. Thus $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$. Indeed, given $\epsilon > 0$, choose $\delta = 2\epsilon$, then $|(x,y)-(0,0)| < \delta = 2\epsilon$ we have $\sqrt{x^2+y^2} < 2\epsilon$ hence $|g(x,y)-0| \leq \frac{1}{2}\sqrt{x^2+y^2} < \epsilon$.



3. For $g(x,y) = \frac{x^2-2y^2}{x^2+y^2}$, if we define $\alpha : \mathbb{R} \to \mathbb{R}^2$ by $\alpha(t) = (t,0)$, then if $\lim_{(x,y)\to(0,0)} g(x,y) = u \in \mathbb{R} \cup \{\infty\}$. Then by the limits and composites theorem, we have

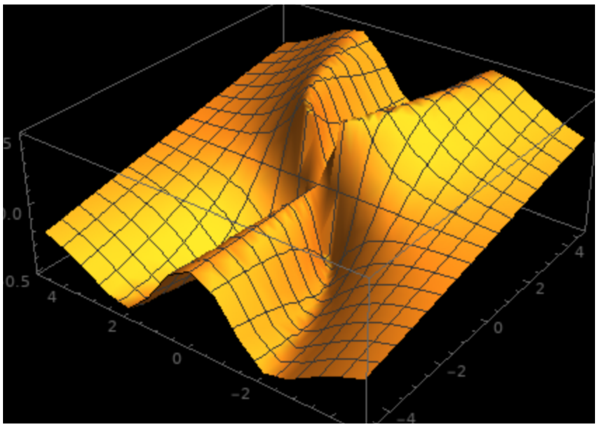
$$u = \lim_{(x,y)\to(0,0)} g(x,y) = \lim_{t\to0} g(\alpha(t)) = \lim_{t\to0} g(t,0) = \lim_{t\to0} 1 = 1$$

But if we define $\beta = \mathbb{R} \to \mathbb{R}^2$ by $\beta(t) = (0,t)$ then $u = \lim_{(x,y)\to(0,0)} g(x,y) = \lim_{t\to0} \frac{-2t^2}{t^2} = -2$. Thus $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exists as it must be unique.



Alternatively, we can use two different sequences, namely $x_n = (\frac{1}{n}, 0)$ and $y_n = (0, \frac{1}{n})$ and if the limit exists it must equal. However, they are not.

4. Let $g(x,y) = \frac{xy}{x^2+y^2}$. Define $\alpha : \mathbb{R} \to \mathbb{R}^2$ by $\alpha(t) = (t, 0)$ and $\beta : \mathbb{R} \to \mathbb{R}^2$ by $\beta(t) = (t,t)$. Then if the limit exists and it is u, then $u = \lim_{t\to 0} g(\alpha(t)) = 0$ and $u = \lim_{t\to 0} g(\beta(t)) = \frac{1}{2}$ and hence it cannot exists.
5. Let $g(x,y) = \frac{xy^2}{x^2+y^4}$. For fixed $y \neq 0$ and for $h(x) = \frac{xy^2}{x^2+y^4}$, we have $h'(x) = \frac{y^2(y^4-x^2)}{(y^4+x^2)^2}$ so $h'(x) = 0 \Leftrightarrow x = \pm y^2$. When $x = \pm y^2$ and $h(x) = \frac{\pm y^4}{2y^4} = \pm\frac{1}{2}$.



For $\alpha(t) : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (0, t)$ and $\beta = \mathbb{R} \to \mathbb{R}^2$ given by $\beta(t) = (t^2, t)$. If $\lim_{(x,y)\to(0,0)} g(x,y) = u$, then we must have $u = \lim_{t\to 0} g(\alpha(t)) = 0$ and $u = \lim_{t\to 0} g(\beta(t)) = \lim_{t\to 0} g(t^2, t) = \frac{1}{2}$. Thus the limit does not exist.

♠

Remark 3.3.15. We will give some algebraic interpretation of the graph of the functions above in Example 3.3.14.

For $z = \frac{xy}{x^2+y^2}$, it will be helpful to use polar coordinate. In particular, let $x = r\cos\theta, y = r\sin\theta$ then we have $z = \frac{1}{2}\sin 2\theta$. Similarly, for $z = \frac{xy}{\sqrt{x^2+y^2}}$, we have $z = \frac{1}{2} r \sin 2\theta$.

Chapter 4

Derivatives

4.0 Motivation

Remark 4.0.1. Recall that for $f : U \subseteq \mathbb{R} \to \mathbb{R}$ where U is an open interval and $a \in U$, we say f is differentiable at a iff $\exists m \in \mathbb{R}$ and $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = m$. This happen iff $\exists m \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, \forall x \in U, 0 < |x-a| < \delta \Rightarrow |\frac{f(x)-f(a)}{x-a} - m| < \epsilon$ iff $\exists m \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, \forall x \in U, 0 < x-a| < \delta \Rightarrow |f(x)-f(a)-m(x-a| < \epsilon|x-a|$ iff $\exists m \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, \forall x \in U, 0 < x-a| \leq \delta \Rightarrow |f(x)-f(a)-m(x-a| \leq \epsilon|x-a|$.

m is unique, we denote $m = f'(a)$ so when f is differentiable at a, $\forall \epsilon > 0, \exists \delta > 0, \forall x \in U, |x-a| < \delta \Rightarrow |f(x) - (f(a) - f'(a)(x-a))| \leq \epsilon|x-a|$, the function $l(x) = f(a) + f'(a)(x-a)$ is the linearization of f at a.

Thus, when $x \cong a$, we have $f(x) \cong l(x)$.

4.1 Intro

Definition 4.1.1. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, where U is open in $\mathbb{R}^n$, $a \in U$, the kth derivative of f at a is $\frac{\partial f}{\partial x_k} = g'(0) = h'(a_k)$ where $g(t) = f(a + te_k) = f(a_1, ..., a_{k-1}, a_k + t, ..., a_n)$ and $h(t) = f(a + (t - a_k)e_k) = f(a_1, ..., a_{k-1}, t, a_{k+1}, ..., a_n)$

Definition 4.1.2. For $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, given by $f = (f_1, ..., f_m)$, with U open in $\mathbb{R}^n$, and $a \in U$, the ***derivative matrix*** or ***Jacobian matrix*** of f at a is the $m \times n$ matrix

$$Df(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & ... & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & ... & \frac{\partial f_2}{\partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & ... & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix}$$

if each partial derivative exists.

In addition, the ***linearization*** of f at a is the affine map $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(x) = f(a) + Df(a)(x - a) \in \mathbb{R}^m$.

Definition 4.1.3. For $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with U open in $\mathbb{R}^n$ and for $a \in U$, we say f is differentiable at $a \in U$ when $\exists A \in M_{m\times n}(\mathbb{R}), \forall \epsilon > 0, \exists \delta > 0, \forall x \in U, |x - a| \leq \delta \Rightarrow |f(x) - (f(a) + A(x - a))| \leq \epsilon|x - a|$

Definition 4.1.4. For $f : U \subseteq \mathbb{R}^1 \to \mathbb{R}^m$, and we have $f(t) = (f_1(t), ..., f_m(t))^T$ then $Df(a) = f'(a) = \begin{bmatrix} f_1'(a) \\ \vdots \\ f_m'(a) \end{bmatrix}$

Definition 4.1.5. For $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in U$, the gradient of f at a is the vector

$$\nabla f(a) = Df(a)^T = (\frac{\partial f}{\partial x_1}(a), ..., \frac{\partial f}{\partial x_n}(a))^T$$

Proposition 4.1.6.

1. *If f is differentiable at a then the matrix A is unique. The partial derivatives all exists and A is the derivative matrix.*
2. *If f is differentiable at a then f is continuous at a.*
3. *If the partial derivatives all continuous at a then f is differentiable at a.*

Definition 4.1.7. For $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where f is differentiable at $a \in U$, the ***affine tangent space*** to the graph $y = f(x)$ at the point $(a, f(a))$ is the graph $y = L(x)$ where $L(x) = f(a) + Df(A)(x - a)$ is the linearization of f at a.

When $f(a) = k$, the ***affine tangent space*** to the level set $f(x) = k$ (that is the set $\{x \in U : f(x) = k\}$) at the point a is the level set $L(x) = k$.

The ***affine tangent space*** to the image $x = f(t)$ (that is the set $\{f(x) \in \mathbb{R}^m : x \in U\}$) at the point $f(a)$ is the image $x = L(t)$.

Definition 4.1.8. The ***dimension*** of the graph $y = f(x)$ at the point $(a, f(a))$ is the dimension of the graph $y = L(x)$ (namely, n).

The ***dimension*** of the level set $f(x) = k$ at the point a is by definition, the dimension of the level set of the linearization $L(x) = k$ (namely, we have that equal to $dim(Null(Df(a)))$.)

The ***dimension*** of the image $x = f(t)$ at the point $f(a)$ is the dimension of the image $x = L(t)$ (namely, we have this equal to $rank(Df(a))$).

Example 4.1.9. For $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^1$ where $z = f(x, y)$, we have $Df(a, b) = (\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b))$ and thus

$$L(x, y) = f(a, b) + Df(a, b)\begin{pmatrix} x - a \\ y - b \end{pmatrix} = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

Example 4.1.10. For $f : U \subseteq \mathbb{R} \to \mathbb{R}^3$ writing $(x, y, z)^T = f(t) = (x(t), y(t), z(t))^T$. We have $Df(a) = f'(a) = (x'(a), y'(a), z'(a))^T$.

Example 4.1.11. For $f : U \subseteq \mathbb{R}^3 \to \mathbb{R}$ written as $u = f(x, y, z)$. We have $Df(a, b, c) = (\frac{\partial f}{\partial x}(a, b, c), \frac{\partial f}{\partial y}(a, b, c), \frac{\partial f}{\partial z}(a, b, c)) = \nabla f(a, b, c)^T$.

Example 4.1.12. For $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, writing $(x, y, z)^T = f(s, t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix}$.

Then, we have

$$Df(a, b) = \begin{bmatrix} \frac{\partial x}{\partial s}(s,t) & \frac{\partial x}{\partial t}(s,t) \\ \frac{\partial y}{\partial s}(s,t) & \frac{\partial y}{\partial t}(s,t) \\ \frac{\partial z}{\partial s}(s,t) & \frac{\partial z}{\partial t}(s,t) \end{bmatrix}$$

Example 4.1.13. Find the tangent line at $(1, 1, 0)$ to the curev of intersection of $z = 2 - x^2 - y^2$ and $y = \sqrt{x^2 + z^2}$ or equivalently, $y^2 = x^2 + z^2$ with $y > 0$.

Solution. For $g : \mathbb{R}^3 \to \mathbb{R}^2$ where $\begin{bmatrix} u \\ v \end{bmatrix} = g(x, y, z) = \begin{bmatrix} x^2 + y^2 - z \\ x^2 - y^2 + z^2 \end{bmatrix}$, the curve is the level set $\begin{bmatrix} u \\ v \end{bmatrix} = g(x, y, z) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. We have $g(1, 1, 0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $Dg(x, y, z) = \begin{bmatrix} 2x & 2y & 1 \\ 2x & -2y & 2z \end{bmatrix}$ and so $Dg(1, 1, 0) = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -2 & 0 \end{bmatrix}$ and the tangent line is the level set $L(x, y, z) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ where $L(x, y, z) = g(1, 1, 0) + Dg(1, 1, 0) \begin{bmatrix} x - 1 \\ y - 1 \\ z - 1 \end{bmatrix}$. That is, we are solving

$$\begin{cases} 2(x-1) + 2(y-1) + z = 0 \\ 2(x-1) - 2(y-1) = 0 \end{cases} \Rightarrow \begin{cases} 2x + 2y + z = 4 \\ x - y = 0 \end{cases}$$

Hence, we have $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} t$

We can also solve this by find the tangent space to the graph $z = 2 - x^2 - y^2$ and the tangent space to the graph $y = \sqrt{x^2 + z^2}$ then looking for intersection.

We provide a third solution by finding the parametric equation for the intersection first, then take derivative. ♠

Theorem 4.1.14 (Chain Rule)**.** *Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ and $g : V \subseteq \mathbb{R}^m \to \mathbb{R}^k$. Let $h = g \circ f$ such that $h : U \subseteq \mathbb{R}^n \to \mathbb{R}^l$. Suppose f is differentiable at $a \in U$ and g is differentiable at $b = f(a)$, then h is differentiable at $a \in U$ and*

$$Dh(a) = Dg(f(a)) \cdot Df(a)$$

Explanation. For $x \cong a$ and $y = f(x) \cong f(a) = b$, we konw $f(x) \cong f(A) + Df(a)(x - a)$ and $h(x) = g(f(x)) = g(y) \cong g(b) + Dg(b)(y - b) = g(f(a)) + Dg(f(a))(f(x) - f(a)) \cong h(a) + Dg(f(a))Df(a)(x - a)$. ♡

Definition 4.1.15. Let f be a function from $U \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in U$ and let $v \in \mathbb{R}^n$. We define the ***directional derivatives*** $D_v f(a)$ of f at a with respective to u as follows: choose a differentiable function $\alpha : (-\epsilon, \epsilon) \subseteq \mathbb{R} \to U \subseteq \mathbb{R}^n$ such that $\alpha() = a$ and $\alpha'(0) = u$. We let $g : (-\epsilon, \epsilon) \to \mathbb{R}$ given by $g(t) = f(\alpha(t))$ and we define $D_u f(a) = g'(0)$.

Remark 4.1.16. By the chain rule, we have the above directional derivatives $D_u f(a) = g'(t) = Df(\alpha(t))D\alpha(t) = Df(\alpha(t))\alpha'(t)$ so

$$D_u f(a) = g'(0) = Df(\alpha(0))\alpha'(0) = Df(a)u = \nabla f(a) \cdot u$$

Theorem 4.1.17. *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ where U is open be differentiable at $a \in U$. Then $\nabla f(a)$ is perpendicular to the level set $f(x) = b$ and $\nabla f(a)$ points in the direction in which $f(x)$ increases most rapidly and $|\nabla f(a)|$ is the rate of change of f in that direction.*

Proof. Let $\alpha(t)$ be any curve in the level set $f(x) = b$ with $\alpha(0) = a$. We wish to show that $\nabla f(a) \perp \alpha'(0)$. Since $\alpha(t)$ lies in the level set $f(x) = b$, we have $f(\alpha(t)) = b$ for all t. Take the derivative of both sides to get $Df(\alpha(t))\alpha'(t) = 0$, and put $t = 0$ we get $Df(a)\alpha'(t) = 0$, that is $\nabla f(a) \cdot \alpha'(0) = 0$. Thus we have $\nabla f(a)$ is perpendicular to the level set $f(x) = b$.

Next, let u be a unit vector, then $D_u f(a) = \nabla f(a) \cdot u = |\nabla f(a)| \cos\theta$ where θ is the angle between u and $\nabla f(a)$. So the maximum possible value of $D_u f(a)$ is $|\nabla f(a)|$ and this occurs when $\cos\theta = 1$, that is when $\theta = 0$, which happens when u is in the direction of $\nabla f(a)$. ♡

Example 4.1.18. For $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}$, find whether $f(x,y)$ is differentiable at $(0,0)$ or not.

Solution. This is not continuous, and thus cannot be differentiable.

However, note $\frac{\partial f}{\partial x}(0,0) = 0$ and $\frac{\partial f}{\partial y}(0,0) = 0$ as well, thus, the partial derivatives well-defined, but not differentiable at $(0,0)$. ♠

Example 4.1.19. Suppose $f(x,y) = |xy|$, find whether $f(x,y)$ is differentiable at $(0,0)$ or not.

Solution. We claim f is differentiable at $(0,0)$ and $Df(0,0) = (0,0)$. Let $\epsilon > 0$, we choose $\delta = 2\epsilon$, then let (x,y) be $|(x,y)| = \sqrt{x^2+y^2} < \delta = 2\epsilon$, then we have

$$\begin{aligned} |f(x,y) - f(0,0) - (0,0)(x-0, y-0)^T| = |f(x,y)| = |xy| &\leq \frac{1}{2}\sqrt{x^2+y^2}\sqrt{x^2+y^2} \\ &= \epsilon\sqrt{x^2+y^2} = \epsilon|(x,y) - (0,0)| \end{aligned}$$

Thus, $Df(0,0) = (0,0)$ as claimed. ♠

Example 4.1.20. Let $f(x,y) = \sqrt{|xy|}$, determine whether $f(x,y)$ is differentiable at $(0,0)$ or not.

Solution. If $f(x, y)$ was differnetiable at $(0, 0)$ then $D_{(0,0)}f(0, 0)$ would exist. For $g(t) = f(t, t) = \sqrt{t^2} = |t|$, we would have $D_{(0,0)}f(0, 0) = g'(0)$. However, $g'(0)$ does not exist.

We remark that the partial derivative both exists and are zero. ♠

Example 4.1.21. Try to determine whether $f(x, y) = \begin{cases} \frac{x^3}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ is differentiable at $(0, 0)$ or not.

Example 4.1.22. Try to determine whether $f(x, y) = \begin{cases} \frac{x^2-3xy^2}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ is differnetiable at $(0, 0)$ or not.

Solution. Suppose $f(x, y)$ at $(0, 0)$ is differnetiable. We have $g(t) = f(t, 0) = t^2/t^2 = t$ so $g'(t) = 1$ for all t and thus $\frac{\partial f}{\partial x}(0, 0) = g'(0) = 1$.For $g(t) = f(0, t) = 0$ and thus $g'(0) = 0$ so $\frac{\partial f}{\partial y}(0, 0) = 0$.

If f was differnetiable at $(0, 0)$ then we would have $D_{u,v}f(0, 0) = Df(0, 0) = \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = u$ ♠

4.2 Differentiability

Theorem 4.2.1. *If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differnetiable at $a \in U$ then $A \in M_n(\mathbb{R})$ in the definition of Differentiability is unique, the partial derivative all exists, and A is equal to the derivative matrix.*

Proof. Let A be a matrix in the definition. Then $\forall \epsilon > 0, \exists \delta, \forall x \in U$, we have $|x - a| \leq \delta \Rightarrow |f(x) - f(a) - A(x - a)| \leq \epsilon(x - a)$. For indices k, l with $1 \leq k \leq n$ and $\leq l \leq m$, let $g(t) = f_k(a + te_l)$ so that $\frac{\partial f_k}{\partial x_l}(a) = g'(0)$, if $g'(0)$ exists. We need to show $g'(0)$ exists and $g'(0) = A_{k,l}$.

Let $\epsilon > 0$ be arbitrary and we choose $\delta > 0$ so that for all $x \in U$ so that

$$|x - a| \leq \delta \Rightarrow |f(x) - f(a) - A(x - a)| \leq \epsilon|x - a|$$

Then, for $x = a + te_l$ with $|t| \leq \delta$. Note $|x - a| = |te_l| = |t| \leq \delta$. Therefore, $|f(x) - f(a) - A(x - a)| \leq \epsilon|x - a|$ and, note $\forall u \in \mathbb{R}^m$, we have $|u_k| \leq |u|$, so we have

$$\begin{aligned}
|g(t) - g(0) - A_{k,l}t| &= |f_k(a + te_l) - f_k(a) - (A(te_l))_k| \\
&\leq |f(a + te_l) - f(a) - A(te_l)| \\
&= |f(x) - f(a) - A(x - a)| \\
&\leq \epsilon|x - a| = \epsilon|t|
\end{aligned}$$

and that proves $g'(0) = A_{kl}$ so $A_{kl} = \frac{\partial f_k}{\partial x_l}(a)$. ♡

Definition 4.2.2. For $A \in M_{m\times n}(\mathbb{R})$, we define the ***matrix norm*** of A to be $\|A\| = \max\{|Ax| : x \in \mathbb{R}^n \wedge |x| = 1\}$.

Remark 4.2.3. Note the maximum does exists by the extreme value theorem, since the sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is compact (indeed, $S^{n-1} = g^{-1}(\{1\})$ where $g(x) = |x|$ is continuous) and since the function $f : S^{n-1} \to \mathbb{R}$ given by $f(x) = |Ax|$ is continuous.

Proposition 4.2.4. *Let $A \in M_{m\times n}(\mathbb{R})$, then*

1. $|Ax| \leq \|A\| \cdot |x|$ *for all* $x \in \mathbb{R}^n$,
2. $\|A\| \leq \sum_{i=1}^m \sum_{j=1}^n |A_{ij}|$

Proof.

1. When $x = 0 \in \mathbb{R}^n$, then $|Ax| = 0$ and $\|A\| \cdot |x| = 0$. When $0 \neq x \in \mathbb{R}^n$ then $|Ax| = |(|x|A(\frac{x}{|x|}))| = |x| \cdot |A(\frac{x}{|x|})| \leq |x| \, \|A\|$
2. When $x \in \mathbb{R}^n$ with $|x| = 1$, then $|x_k| \leq |x| \leq 1$ for all k, so

$$Ax = A(\sum_{l=1}^n x_l e_l) = \sum_{l=1}^n x_l A e_l$$

Thus,

$$|Ax| \leq \sum_{l=1}^n |x_l A e_l| = \sum_{l=1}^n |x_l||Ae_l| \leq \sum_{l=1}^n |Ae_l|$$

and Ae_l is the lth column of A. Next, $Ae_l = \sum_{k=1}^m A_{kl} e_k$ so $|Ae_l| \leq \sum_{k=1}^m |A_{kl}|$ ♡

Remark 4.2.5. For $A \in M_{m\times n}(\mathbb{R})$, $\|A\|$ is equal to the square root of the largest eigenvalue of $A^T A$. Also, $\|A + B\| \leq \|A\| + \|B\|$.

Theorem 4.2.6. *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where U is open in $\mathbb{R}^n$ with $a \in U$. If f is differentiable at a then it is continuous at x.*

Proof. Let $\epsilon > 0$, choose $\delta > 0$ with $\delta < \frac{\epsilon}{1+\|Df(a)\|}$, then we have $|f(x) - f(a) - Df(a)(x-a)| \leq 1 \cdot |x - a|$ and hence

$$\begin{aligned}|f(x) - f(a)| &\leq |f(x) - f(a) - Df(a)(x-a)| + |Df(a)(x-a)| \\ &\leq |x-a| + \|Df(a)\| \, |x-a| = (1 + \|Df(a)\|)|x-a| \\ &\leq (1 + \|Df(a)\|)\delta \leq \epsilon\end{aligned}$$

♡

Theorem 4.2.7. *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ with U open in $\mathbb{R}^n$ and $a \in U$. Suppose that all partial derivatives $\frac{\partial f_k}{\partial x_l}(x)$ exist at every $x \in U$ and are continuous at a. Then f is differnetiable at a.*

Proof. Recall that $\frac{\partial f_k}{\partial x_l}(a) = g'(a_l)$ where $g(t) = f(a_1, ..., a_{l-1}, t, a_{l+1}, ..., a_n)$ hence $\frac{\partial f_k}{\partial x_l}(a_1, ..., a_{l-1}, t, a_{l+1}, ..., a_n) = g'(t)$ for all t. That is, $\frac{\partial f_k}{\partial x_l}(\alpha(t)) = g'(\alpha(t))$ where $\alpha(t) = (a_1, ..., a_{l-1}, t, a_{l+1}, ..., a_n)$. Let $\epsilon > 0$, choose $\delta > 0$ so that $\overline{B}(a, \delta) \subseteq U$ and so that $|\frac{\partial f_k}{\partial x_l}(y) - \frac{\partial f_k}{\partial x_l}(a)| \leq \frac{\epsilon}{nm}$ for all $y \in \overline{B}(a, \delta)$ and we want to show that $\forall x \in U, |x - a| \leq \delta \Rightarrow |f(x) - f(a) - Df(a)(x - a)| \leq \epsilon |x - a|$.

Let $u_l = (x_1, ..., x_l, a_{l+1}, ..., a_n)$ for $0 \leq l \leq n$ with $u_0 = a$, $u_n = x$ and note that each $u_l \in \overline{B}(a, \delta)$. For $1 \leq l \leq n$, let $\alpha_l(t) = (x_1, ..., x_{l+1}, t, a_{l+1}, ..., a_n)$ for t between a_l and x_l so that $\alpha_l(a_l) = u_{l-1}$ and $\alpha_l(x_l) = u_l$. For $1 \leq l \leq n$, $1 \leq k \leq m$, let $g_{kl}(t) = f_k(\alpha_l(t))$ so that $g'_{kl}(t) = \frac{\partial f_k}{\partial x_l}(\alpha_l(t))$ for t between a_l, x_l.

By the mean value theorem (applied to $g_{kl}(t)$ for t between a_l and x_l), we can choose s_{kl} between a_l and x_l so that $(x_l - a_l)g'_{kl}(s_{kl}) = g_{kl}(x_l) - g_{kl}(a_l)$, that is, $\frac{\partial f_k}{\partial x_l}(\alpha_l(s_{kl}))(x_l - a_l) = f_k(u_l) - f_k(u_{l-1})$. So, we have $f_k(x) - f_k(a) = f_k(u_n) - f_k(u_0) = \sum_{l=1}^n f_k(u_l) - f_k(u_{l-1}) = \sum_{l=1}^n \frac{\partial f_k}{\partial x_l}(\alpha_l(s_{kl}))(x_l - \alpha_l)$.

Thus,

$$f_k(x) - f_k(a) - (Df(a)(x - a))_k = \sum_{l=1}^n \frac{\partial f_k}{\partial x_l}(\alpha_l(s_{kl}))(x_l - a_l) - \sum_{l=1}^n \frac{\partial f_k}{\partial x_l}(a)(x_l - a_l)$$

which is equal to $(B(x - a))_k$ where B is the matrix with entry $B_{kl} = \frac{\partial f_k}{\partial x_l}(\alpha_l(s_{kl})) - \frac{\partial f_k}{\partial x_l}(a)$. Thus, we have

$$\begin{aligned}
|f(x) - f(a) - Df(a)(x - a)| &= |B(x - a)| \leq \|B\| \cdot |x - a| \\
&\leq (\sum_{k,l} |B_{kl}|)|x - a| \\
&= \left(\sum_{k,l} |\frac{\partial f_k}{\partial x_l}(\alpha_l(s_{kl})) - \frac{\partial f_k}{\partial x_l}(a)| \right) |x - a| \\
&\leq (\sum_{k,l} \frac{\epsilon}{nm})|x - a| \\
&= \epsilon |x - a|
\end{aligned}$$

♡

Corollary 4.2.7.1. *If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is C^1 in U, which means all partial derivatives exists and continuous in U, then f is differnetiable in U.*

Theorem 4.2.8. *Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$, let $g : V \to \mathbb{R}^l$ and let $h(x) = g(f(x))$. If f is differentiable at a and g is differentiable at $f(a)$ then h is differentiable at a and $Dh(a) = Dg(f(a))Df(a)$.*

Proof. Let $y = f(x)$ and $b = f(a)$. We have

$$\begin{aligned}
&|h(x) - h(a) - Dg(f(a))Df(a)(x - a)| = |g(y) - g(b) - Dg(b)Df(a)(x - a)| \\
&\quad = |g(y) - g(b) - Dg(b)(y - b) + Dg(b)(y - b) - Dg(b)Df(a)(x - a)| \\
&\quad \leq |g(y) - g(b) - Dg(b)(y - b)| + \|Dg(b)\| \cdot |y - b - Df(a)(x - a)| \\
&\quad = |g(y) - g(b) - Dg(b)(y - b)| + (1 + \|Dg(b)\|)|f(x) - f(a) - Df(a)(x - a)|
\end{aligned}$$

In addition, note

$$\begin{aligned}|y-b| &= |f(x)-f(a)| \\ &= |f(x)-f(a)-Df(a)(x-a)+Df(a)(x-a)| \\ &\leq |f(x)-f(a)-Df(a)(x-a)| + \|Df(a)\| \cdot |x-a|\end{aligned}$$

Thus, let $\epsilon > 0$ be given. Since g is differentiable at b, we can choose $\delta_0 > 0$ so that

$$|y-b| \leq \delta_0 \Rightarrow |g(y)-g(b)-Dg(b)(y-b)| \leq \frac{\epsilon}{2(1+\|Df(a)\|)}|y-b|$$

Since f is continuous at a, we can choose δ_1 so that

$$|x-a| \leq \delta_1 \Rightarrow |y-b| = |f(x)-f(a)| \leq \delta_0$$

Since f is differentiable at a, we can choose $\delta_2 > 0$ so that

$$|x-a| \leq \delta_2 \Rightarrow |f(x)-f(a)-Df(a)(x-a)| \leq |x-a|$$

and we can choose $\delta_3 > 0$ so that

$$|x-a| \leq \delta_3 \Rightarrow |f(x)-f(a)-Df(a)(x-a)| \leq \frac{\epsilon}{2(1+\|Dg(a)\|)}|x-a|$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then for $|x-a| \leq \delta$, we have

$$\begin{aligned}|y-b| &\leq |f(x)-f(a)-Df(a)(x-a)| + |Df(a)(x-a)| \\ &\leq |x-a| + \|Df(a)\|\,|x-a| \\ &= (1+\|Df(a)\|)|x-a|\end{aligned}$$

Therefore, we have

$$|g(y)-g(b)-Dg(b)(y-b)| \leq \frac{\epsilon}{2(1+\|Df(a)\|)}|y-b| \leq \frac{\epsilon}{2}|x-a|$$

and

$$(1+\|Dg(b)\|)|f(x)-f(a)-Df(a)(x-a)| \leq \frac{\epsilon}{2}|x-a|$$

Hence,

$$|h(x)-h(a)-Dg(f(a))Df(a)(x-a)| \leq \frac{\epsilon}{2}|x-a| + \frac{\epsilon}{2}|x-a| = \epsilon|x-a|$$

and thus h is differnetiable at a with derivative $Dh(a) = Dg(f(a))Df(a)$ as desired. ♡

4.3 Higher Derivatives

Lemma 4.3.1. *Let I and J be intervals in $\mathbb{R}$ with $a \in I$ and $b \in J$. Let $f : (I \times J)\backslash\{(a,b)\} \to \mathbb{R}$, suppose $\lim_{v \to b} f(u,v)$ exists in $\mathbb{R}$ for each $u \in I\backslash\{a\}$ and suppose $\lim_{(u,v)\to(a,b)} f(u,v)$ exists in $\mathbb{R}$. Then $\lim_{u\to a}\lim_{y \to b} f(x,y)$ exists and is equal to $\lim_{(u,v)\to(a,b)} f(u,v)$.*

Proof. Let $g(u) = \lim_{v\to b} f(u,v)$ for all $u \in I\backslash\{a\}$. Let $\lim_{(u,v)\to(a,b)} f(u,b) = l \in \mathbb{R}$.

Let $\epsilon > 0$ be given. Choose $\delta > 0$ so that for all $(u,v) \in (I \times J)\backslash\{(a,b)\}$ and $0 < |(u,v)-(a,b)| < 2\delta \Rightarrow |f(u,v)-l| < \epsilon$. Let $u \in I\backslash\{a\}$ with $|u-a| < \delta$ and $v \in J$ with $|v-b| < \delta$. Then, we have $0 < |(u,v)-(a,b)| < 2\delta$. Hence, $|f(u,v)-l| < \epsilon$. So, we have $|g(u)-l| \leq |g(u)-f(u,v)| + |f(u,v)-l| < |g(u)-f(u,v)| + \epsilon$. Since this is true for all $v \in J$, we can take the limit as $v \to b$ on both side to get $|g(u)-l| \leq |g(u)-g(u)| + \epsilon = \epsilon$. Thus $\lim_{u\to a} g(u) = l$. That is, $\lim_{u\to a}\lim_{v\to b} f(u,v) = l$ as desired. ♡

Theorem 4.3.2. *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ where U is open and $a \in U$. Let $1 \leq k \leq n$ and $1 \leq l \leq n$. Suppose that $\frac{\partial f}{\partial x_k}(x)$ exists and is continuous at every $x \in U$. Suppose that $\frac{\partial^2 f}{\partial x_k \partial x_l}(x)$ and is continuous at a. Then $\frac{\partial^2 f}{\partial x_l \partial x_k}(a)$ exists and is equal to $\frac{\partial f}{\partial x_k \partial x_l}(a)$.*

Proof. Choose $r > 0$ so that $\overline{B}(a,2r) \subseteq U$. For $u,v \in \mathbb{R}$ with $|u| < r$ and $|v| < r$, we have $a + ue_k + ve_l \in \overline{B}(a,2r) \subseteq U$. Let $g(u,v) = f(a+ue_k+ve_l) - f(a+ue_k) - f(a+ve_l) + f(a)$. By the mean value theorem applied to the function $f(a+ue_k+ve_l) - f(a+ve_l)$ using variable v (with u fixed), we can choose t (which is dependent on u and v) between 0 and v such that

$$\begin{aligned} v \cdot \frac{\partial f}{\partial x_l}(a+ue_k+te_l) - \frac{\partial f}{\partial x_l}(a+te_l) &= (f(a+ue_k+ev_l) - f(a+ve_l)) \\ &\quad - (f(a+ue_k) - f(a)) \\ &= g(u,v) \end{aligned}$$

By the mean value theorem applied to the function $\frac{\partial f}{\partial x_l}(a+ue_k+te_l)$ as a function of u with t fixed, we can choose s between 0 and u such that

$$u \cdot \frac{\partial^2 f}{\partial x_k \partial x_l}(a+se_k+te_l) = \frac{\partial f}{\partial x_l}(a+ue_k+te_l) - \frac{\partial f}{\partial x_l}(a+te_l)$$

so we have $uv \cdot \frac{\partial^2 f}{\partial x_k \partial x_l}(a+se_k+te_l) = g(u,v)$.

Apply the mean value theorem again to the function $f(a+ue_k+ve_l) - f(a+ue_k)$ using the variable u, we can choose r between 0 and u such that

$$\begin{aligned} u \cdot \frac{\partial f}{\partial x_k}(a+re_k+ve_l) - f(a+re_k) &= g(u,v) \\ &= uv \cdot \frac{\partial^2 f}{\partial x_k \partial x_l}(a+se_k+te_l) \end{aligned}$$

For $0 < |u| < r, 0 < |v| < r$, we have

$$\frac{\frac{\partial f}{\partial x_k}(a+re_k+ve_l) - \frac{\partial f}{\partial x_k}(a+re_k)}{v} = \frac{\partial^2 f}{\partial x_k \partial x_l}(a+se_k+te_l)$$

Since $\frac{\partial^2 f}{\partial x_k \partial x_l}(x)$ is continuous at a and since $|s| \leq |u|$, $|t| \leq |v|$, we have

$$\lim_{(u,v)\to(0,0)} \frac{\partial^2 f}{\partial x_k \partial x_l}(a + se_k + te_l) = \frac{\partial^2 f}{\partial x_k \partial x_l}(a)$$

Since $\frac{\partial f}{\partial x_k}(x)$ is continuous everywhere (hence at a and $a + ve_l$), we have

$$\lim_{u\to 0} \frac{\frac{\partial f}{\partial x_k}(a + re_k + ve_l) - \frac{\partial f}{\partial x_k}(a + re_k)}{v} = \frac{\frac{\partial f}{\partial x_k}(a + ve_l) - \frac{\partial f}{\partial x_k}(a)}{v}$$

By the above lemma, $\lim_{v\to 0} \frac{\frac{\partial f}{\partial x_k}(a + ve_l) - \frac{\partial f}{\partial x_k}(a)}{v}$ exists and is equal to $\frac{\partial^2 f}{\partial x_k \partial x_l}(a)$. Thus $\frac{\partial^2 f}{\partial x_l \partial x_k}(a) = \frac{\partial^2 f}{\partial x_k \partial x_l}(a)$ ♡

Remark 4.3.3. In particular, if f is C^2, then $\frac{\partial f}{\partial x_k \partial x_l}(x) = \frac{\partial f}{\partial x_l \partial x_k}(a)$ for all $x \in U$.

Example 4.3.4. Show that for $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \text{otherwise} \\ 0 & (x,y) = (0,0) \end{cases}$, $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Definition 4.3.5. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, where U is open in $\mathbb{R}^n$ and $a \in U$. For $l \in \mathbb{N}$, the l^{th} ***total differential*** of f at a is the function $D^l f(a) : \mathbb{R}^n \to \mathbb{R}$ given by

$$D^l f(a)(u) = \sum_{\substack{1\leq k_1 \leq n \\ 1 \leq k_2 \leq n \\ \vdots \\ 1 \leq k_l \leq n}} \frac{\partial^l f}{\partial x_{k_1} \partial x_{k_2} ... \partial x_{k_l}}(a) \cdot u_{k_1} u_{k_2} ... u_{k_l}$$

provided that all the lth partial derivatives exists.

Theorem 4.3.6 (Taylor's Theorem)**.** *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ where U is open. Suppose that f is C^m in U. Then for all $a, x \in U$, such that $[a,x] \subseteq U$, there exists $c \in [a,x]$ such that $f(x) = \sum_{l=0}^{m-1} \frac{1}{l!} D^l f(a)(x-a) + \frac{1}{m!} D^m f(c)(x-a)$.*

Proof. Let $a, x \in U$ with $[a,x] \subseteq U$. Let $\alpha(t) = a + t(x-a)$. Note $\alpha(t) \in U$ for all $0 \leq t \leq 1$(or for all $-\delta < t < 1 + \delta$ for sufficiently small δ). Let $g(t) = f(\alpha(t))$.

By the chain rule,

$$\begin{aligned} g'(t) = Df(\alpha(t)) \cdot \alpha'(t) &= \begin{bmatrix} \frac{\partial f}{\partial x_1}(\alpha(t)) & ... & \frac{\partial f}{\partial x_n}\alpha(t) \end{bmatrix} \begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{bmatrix} \\ &= \sum_{1 \leq k \leq n} \frac{\partial f}{\partial x_k} \alpha(t)(x_k - a_k) \\ &= D'f(\alpha(t))(x-a) \end{aligned}$$

By chain rule again, we have

$$\begin{aligned}
g''(t) &= \sum_{1\le k\le n} \frac{\partial f}{\partial x_k}(\alpha(t))(x_k - a_k) \\
&= \sum_{1\le k\le n} \left(\sum_{1\le l\le n} \frac{\partial}{\partial x_l}(\frac{\partial f}{\partial x_k})(\alpha(t))(x_l - a_l) \right)(x_k - a_k) \\
&= \sum_{\substack{1\le k\le n\\ 1\le l\le n}} \frac{\partial^2 f}{\partial x_k \partial x_l}(\alpha(t))(x_k - a_k)(x_l - a_l) \\
&= D^2 f(\alpha(t))(x-a)
\end{aligned}$$

By induction, we have

$$g^{(l)}(t) = \sum_{\substack{1\le k_1\le n\\ 1\le k_2\le n\\ \vdots \\ 1\le k_l\le n}} \frac{\partial^l f}{\partial x_{k_1}\partial x_{k_2}...\partial x_{k_l}}(\alpha(t))(x-a) = D^l f(\alpha(t))(x-a)$$

By Taylor's theorem for single variable functions applied to $g(t)$ on $[0,1]$, we have $g(1) = \sum_{l=1}^{m-1} \frac{1}{l!} g^{(l)}(0)1^l + \frac{1}{m!} g^{(m)}(s)1^m$ for some $s \in [0,1]$. Since $g(t) = f(\alpha(t))$ with $\alpha(t) = a + t(x-a)$, this gives

$$f(x) = \sum_{l=0}^{m-1} \frac{1}{l!} D^l f(a)(x-a) + \frac{1}{m!} D^m f(c)(x-a)$$

♡

Definition 4.3.7. For $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ where U is open in $\mathbb{R}^n$ with $a \in U$, the mth Taylor polynomial of f centered at a is the polynomial $T^m f(a)(x) = \sum_{l=0}^{m-1} \frac{1}{l!} D^l f(a)(x-a)$, provided all the mth partial derivatives exists.

Example 4.3.8. For $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ with $(a,b) \in U$,

$$D^0 f(a,b)(u,v) = f(a,b)$$

$$D' f(a,b)(u,v) = \frac{\partial f}{\partial x}(a,b)u + \frac{\partial f}{\partial x}(a,b)v$$

$$D^2 f(a,b)(u,v) = \frac{\partial^2 f}{\partial x^2}(a,b)u^2 + \frac{\partial^2 f}{\partial x \partial y}(a,b)uv + \frac{\partial^2 f}{\partial y \partial x}(a,b)vu + \frac{\partial^2 f}{\partial y^2}(a,b)v^2$$

Assume f is C^3, then

$$D^3 f(a,b)(u,v) = \frac{\partial^3 f}{\partial x^3}(a,b)u^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}(a,b)u^2 v + 3\frac{\partial^3 f}{\partial x \partial y^2}(a,b)uv^2 + \frac{\partial^3 f}{\partial y^3}(a,b)v^3$$

Thus, we have the Taylor polynomial to be

$$\begin{aligned}T^3f(a,b)(x,y) &= f(a,b) + (\frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial x}(a,b)(y-b)) \\ &+ \frac{1}{2!}(\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2) \\ &+ \frac{1}{3!}(\frac{\partial^3 f}{\partial x^3}(a,b)(x-a)^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}(a,b)(x-a)^2(y-b) \\ &+ 3\frac{\partial^3 f}{\partial x \partial y^2}(a,b)(x-a)(y-b)^2 + \frac{\partial^3 f}{\partial y^3}(a,b)(y-b)^3)\end{aligned}$$

Example 4.3.9. More generally, for $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, with U open in $\mathbb{R}^n$ and $a \in U$, and f be C^2, we have

$$T^0f(a)(x) = f(a)$$

$$T^1f(a) = f(a) + Df(a)(x-a)$$

$$T^2f(a)(x) = f(a) + Df(a)(x-a) + \frac{1}{2!}(x-a)^T Hf(a)(x-a)$$

where $Hf(a)$, the Hessian matrix of f at a, is defined as the (k,l) entry of $Hf(a)$ is $\frac{\partial^2 f}{\partial x_k \partial x_l}(a)$.

Definition 4.3.10. Let $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, then for $a \in A$, we say f has a ***local maximum*** value at a when $\exists r > 0$ such that $f(x) \leq f(a)$ for all $x \in B(a,r) \cap A$. In addition, f has a ***global maximum value*** at a when $f(x) \leq f(a)$ for all $x \in A$.

The definition for local minimum and global minimum is similar.

Example 4.3.11. Show that when $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and $a \in A^o$, then if $Df(a)$ exists and $Df(a) \neq 0$, then f does not have a local max/min value at a.

Remark 4.3.12. Recall that for $A \in M_n(\mathbb{R})$, if $A^T = A$, then all the eigenvalues of A are real and A is orthogonally diagonalizable(meaning $\exists P \in M_n(\mathbb{R})$ with $P^TP = I$ (or say $P \in O_n(\mathbb{R})$) such that $P^TAP = D = diag(\lambda_1, ..., \lambda_n)$).

Also recall that A is positive-definite if and only if $u^TAu > 0$ for all $0 \neq u \in \mathbb{R}^n$ if and only if all the eigenvalues of A are positive.

Recall A is positive-semidefinite if and only if $u^TAu \geq 0$ for nonzero $u \in \mathbb{R}^n$ if and only if all eigenvalues are non-negative.

The definition for negative-definite/semidefinite are similar.

Recall that A is indefinite if and only if $\exists u, v \in \mathbb{R}^n$ and $u^TAu > 0$, $v^TAv < 0$ if and only if A has at least positive and one negative eigenvalue.

Recall that $F(u,v)$ is bilinear form when F is linear in both u and v. A quadratic form is $Q(u) := F(u,u)$ for some bilinear form F. Note $F(u,v)$ can be written as v^TAu and $Q(u) = u^TAu$ for some matrix A.

Theorem 4.3.13. *Let $A \in M_n(\mathbb{R})$ be symmetric. Let $A^{(k)}$ be the upper-left $k \times k$ submatrix of of A. Then*

1. *A is positive-definite if and only if $det(A^{(k)}) > 0$ for all $1 \le k \le n$,*
2. *A is negative-definite if and only if $(-1)^k \cdot det(A^{(k)}) > 0$ for all $1 \le k \le n$.*

Proof. See notes ♡

Theorem 4.3.14 (the Second Derivative Test)**.** *Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ where U is open in $\mathbb{R}^n$ and $a \in U$. Suppose f is C^2 in U. Suppose $Df(a) = 0$ then*

1. *if $Hf(a)$ is positive definite then f has a local min*
2. *if $Hf(a)$ is negative definite then f has a local max*
3. *if $Hf(a)$ is indefinite then f has neither a local max or local min at $x = a$ and we call it a saddle point at a.*

Proof.

1. Suppose $Hf(a)$ is positive definite. We have $det(Hf(a)^{(k)}) > 0$ for $1 \le k \le n$. Note that $det(Hf(x)^{(k)})$ is a continuous function of x.
Choose $r > 0$ such that for all $x \in B(a, r) \subseteq U$, we have $det(Hf(x)^{(k)}) > 0$. Then $Hf(x)$ is positive definite for all $x \in B(a, r)$. Given $x \in B(a, r)$, we apply Taylor's theorem and $f(x) = f(a) + Df(a)(x - a) + \frac{1}{2!}(x - a)^T Hf(c)(x - a)$. Since $Df(a) = 0$, we have $f(x) = f(a) + \frac{1}{2!}(x - a)^T Hf(c)(x - a)$ for some c between a and x. Then $f(x) > f(a)$ for all $a \ne x \in B(a, r)$ since $Hf(c)$ is positive definite so $(x - a)^T Hf(c)(x - a) > 0$.
2. similar to above.
3. Suppose $Hf(a)$ is indefinite, choose $r > 0$ so that $B(a, r) \subseteq U$. Since $Hf(a)$ is indefinite, we can choose $u, v \in \mathbb{R}^n$ so that $u^T Hf(a)u > 0$ and $v^T Hf(a)v < 0$. By scaling u, v, if necessary, we can assume that $a+u \in B(a, r), a+v \in B(a, r)$. Let $\alpha(t) = a + tu$ and $g(t) = f(\alpha(t))$ for $0 \le t \le 1$(or for $-\delta < t < 1 + \delta$ for some $\delta > 0$). As in the proof of Taylor's theorem, the chain rule gives $g'(t) = Df(\alpha(t))\cdot\alpha'(t) = Df(\alpha(t))u$ and $g''(t) = D^{(2)}f(\alpha(t))u = u^T Hf(\alpha(t))u$ so we have $g'(0) = 0$ and $g''(0) = u^T Hf(a)u > 0$. By MATH 147, $g(t)$ has a local minimum at $t = 0$ with $g(t) > g(0)$ for small t that not equal to zero. Thus, $f(x)$ takes positive values at $x = \alpha(t) = a+tu$ for small $t \ne 0$. Similarly, using v instead of u, we have $f(x)$ takes negative values at $x = a + tv$.

♡

Example 4.3.15. For $f(x, y) = \frac{4}{2+x+y^2}$, defined when $2 + x + y^2 \ne 0$. Find the second degree Taylor polynomial of $f(x, y)$ at $(0, 0)$ as a function of x, y.

Solution. We have $\frac{\partial f}{\partial x} = \frac{-4}{(2+x+y^2)^2}$, $\frac{\partial f}{\partial y} = \frac{-8y}{(2+x+y^2)^2}$. In addition, $\frac{\partial^2 f}{\partial x^2} = \frac{8}{(2+x+y^2)^3}$ and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{16y}{(2+x+y^2)^3}$ and $\frac{\partial^2 f}{\partial y^2} = \frac{-8(2+x+y^2)^2+32y^2(2+x+y^2)}{(2+x+y^2)^4}$ ♠

Example 4.3.16. Find and classify the critical points of $f(x, y) = x^3 + 3x^2y - 6y^2$

Solution. The two critical points are $(0, 0)$ and $(-2, 1)$. The derivative test does not give information on $(0, 0)$, and $(-2, 1)$ is a saddle point.

However, let $\alpha(t) = (0, t)$ and $g(t) = f(\alpha(t))$. We have $g(t) = -6t^2$, so $g(t)$ has a local max at $t = 0$. In addition, for $\beta(t) = (t, 0)$, we have $h(t) = f(\beta(t))$ and $h(t) = f(t, 0) = t^3$. Thus, $h(t)$ has neither a local max nor local min. Hence, $(0, 0)$ is a saddle point. ♠

Example 4.3.17. Find the absolute max and min values of $f(x, y) = 4xy - x^4 - 2y^2$ on the compact set $\overline{B}((0, 0), 2)$.

Chapter 5

Integration: Calculation

5.1 Intro

Theorem 5.1.1 (Fubini's theorem). *If $D = \{(x, y) : a \le x \le b, g(x) \le y \le h(x)\}$ and $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ where f, g, h are continuous, then*

$$\int_D f = \int \int_D f(x) dV = \int \int_D f(x, y) dx dy$$

is given by

$$\int_D f = \int_a^b \left(\int_{g(x)}^{h(x)} f(x, y) dy \right) dx$$

Theorem 5.1.2 (Fubini's theorem for $\mathbb{R}^3$). *For $D = \{(x, y, z) : a \le x \le b, g(x) \le y \le h(x), k(x, y) \le z \le l(x, y)\}$ with f, g, h, k, l all continuous, then*

$$\int_D f = \int \int \int_D f dV = \int \int \int_D f(x, y, z) dx dy dz$$

is given by

$$\int_D f = \int_a^b \left(\int_{g(x)}^{h(x)} \left(\int_{k(x,y)}^{l(x,y)} f(x, y, z) dz \right) dy \right) dx$$

Example 5.1.3. Let D be the triangle with vertices at $(0, -1)$, $(2, 1)$, and $(2, 3)$ and let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = 2xy$. Find $\int_D f$.

Solution. Note $D = \{(x, y) : 0 \le x \le 2, g(x) \le y \le h(x)\}$ where $g(x) = x - 1$ and

$h(x) = 2x - 1$. Thus,

$$
\begin{aligned}
\int_D f &= \int_0^2 \int_{x-1}^{2x-1} 2xydydx \\
&= \int_0^2 [xy^2]_{x-1}^{2x-1} dx \\
&= \int_0^2 x(2x-1)^2 - x(x-1)^2 dx \\
&= \int_0^2 3x^3 - 2x^2 dx \\
&= [\frac{3}{4}x^4 - \frac{2}{3}x^3]_0^2 \\
&= \frac{20}{3}
\end{aligned}
$$

Alternatively, we can integrate x first, then y. In particular, we note $D = D_1 \cup D_2$ where $D_1 = \{(x,y) : -1 \leq y \leq 1, \frac{y+1}{2} \leq x \leq y+1\}$ and $D_2 = \{(x,y) : 1 \leq y \leq 3, \frac{y+1}{2} \leq x \leq 2\}$. Then, we have

$$\int_D f = \int_{D_1} f + \int_{D_2} f = \int_{-1}^{1} \int_{(y+1)/2}^{y+1} 2xydxdy + \int_1^3 \int_{(y+1)/2}^{2} 2xydxdy$$

$\spadesuit$

Example 5.1.4. Find the volume of the region D in $\mathbb{R}^3$ which lies above the paraboloid $z = x^2 + y^2$ and below the plane $z = 2x$.

Solution. Points on the curve of intersection satisfy $z = x^2 + y^2 = 2x$ and thus $(x-1)^2 + y^2 = 1$. Therefore, the given solid D is the set

$$
\begin{aligned}
D &= \{(x,y,z) : (x-1)^2 + y^2 \leq 1, x^2 + y^2 \leq z \leq 2x\} \\
&= \{(x,y,z) : 0 \leq x \leq 2, -\sqrt{2x - x^2} \leq y \leq \sqrt{2x - x^2, x^2 + y^2 \leq z \leq 2x}\}
\end{aligned}
$$

The volume is

$$
\begin{aligned}
V &= \int\int\int_D 1dV = \int_0^2 \int_{\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \int_{x^2+y^2}^{2x} 1dzdydx \\
&= \int_0^2 \int_{\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (2x - x^2 - y^2)dydx = \int_0^2 \left[(2x-x^2)y - \frac{1}{3}y^3\right]_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} dx \\
&= \int_0^2 \frac{2}{3}(2x-x^2)^{3/2} + \frac{2}{3}(2x-x^2)^{3/2}dx = \int_{-1}^{1} \frac{4}{3}(1-u^2)^{3/2}du \\
&= \int_{-\pi/2}^{\pi/2} \frac{4}{3}(\cos\theta)^3 \cos\theta d\theta = \int_0^{\pi/2} \frac{8}{3}(\cos\theta)^3 \cos\theta d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} (\frac{1+\cos 2\theta}{2})^2 d\theta = \frac{2}{3}\int_0^{\pi/2} (1 + 2\cos 2\theta + \cos^2 2\theta)d\theta \\
&= \frac{2}{3}(\pi/2 + 0 + \pi/4) = \frac{\pi}{2}
\end{aligned}
$$

♠

Example 5.1.5. Find the mass of the tetrahedron with vertices at $(0,0,0)$, $(2,0,0)$, $(2,2,0)$ and $(2,2,2)$ given that the density is given by $\rho(x,y,z) = 2xy(3-z)$.

Solution. Note that the tetrahedron is the set $D = \{(x,y,z) : 0 \le x \le 2, 0 \le y \le x, 0 \le z \le y\}$. Thus, the mass is

$$\begin{aligned} M &= \int_0^2 \int_0^x \int_0^y \rho(x,y,z)dzdydx \\ &= \int_0^2 \int_0^x \int_0^y 2xy(3-z)dzdydx \\ &= \int_0^2 \int_0^x \left[6xyz - xyz^2\right]_0^y dydx \\ &= \int_0^2 \int_0^x 6xy^2 - xy^3 dydx \\ &= \int_0^2 \left[2xy^3 - 1/4xy^4\right]_0^x dx \\ &= \left[2/5x^5 - 1/24x^6\right]_0^2 = \frac{152}{15} \end{aligned}$$

♠

Definition 5.1.6. Let $C = \overline{U} \subseteq \mathbb{R}^n$ and $D = \overline{V} \subseteq \mathbb{R}^n$ where U and V are open in $\mathbb{R}^n$. A ***change of coordinates map*** from C to D is a continuous map $f : C \to D$ such that $f(U) \subseteq V$ and the map $f : U \to V$ is bijective and f and its inverse $f^{-1} : V \to U$ are both C^1 with $Df(a)$ is invertible for all $a \in U$.

Remark 5.1.7. If U is connected in the above definition, then either $det(Df(x)) > 0$ for all $x \in U$ or $det(Df(x)) < 0$ for all $x \in U$. This is because, for the map $\Phi(x) = det(Df(x))$, $\Phi : U \to R\backslash\{0\} = (-\infty, 0) \cup (0, \infty)$ is continuous and $\Phi(U)$ is connected, either $\Phi(U) \subseteq (-\infty, 0)$ or $\Phi(U) \subseteq (0, \infty)$.

Definition 5.1.8. When $def(Df(x)) > 0$ for all $x \in U$ in the above definition, we say f preserves orientation and when $det(Df(x)) < 0$ for all $x \in U$, we say f reverses orientation.

Example 5.1.9 (The Polar Coordinates)**.** The polar coordinates map $g : \overline{U} \subseteq \mathbb{R}^2 \to \overline{V} \subseteq \mathbb{R}^2$ where $U = \{(r,\theta) : r > 0, \alpha < \theta < \alpha + 2\pi\}$ and $V = \{(r\cos\alpha, r\sin\alpha) : r \ge 0\}$ is given by $(x,y)^T = g(r,\theta) = (r\cos\theta, r\sin\theta)^T$. We have $Dg(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$ and so $det(Dg(r,\theta)) = r > 0$ for all $(r,\theta) \in U$ so the polar coordinates map is orientation perserving.

Example 5.1.10 (The Cylindrical Coordinates)**.** The cylindrical coordinates is given by $(x,y,z) = g(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$, we have

$$Dg(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that $det(Dg(r,\theta,z)) = r$.

Example 5.1.11 (The Spherical Coordinates)**.** The spherical coordinates map is given by $(x, y, z) = g(r, \phi, \theta) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi)$.

We have

$$Dg = \begin{bmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{bmatrix}$$

and

$$\begin{aligned} det(Dg) &= \cos\phi(r^2\sin\phi\cos\phi\cos^2\phi + r^2\sin\phi\cos\phi\sin^2\theta) \\ &\quad + r\sin\phi(r\sin^2\phi\cos^2\theta + r\sin^2\phi\sin^2\theta) \\ &= \cos\phi(r^2\sin\phi\cos\phi) + r\sin\phi r\sin^2\phi \\ &= r^2\sin\phi \end{aligned}$$

Theorem 5.1.12 (Change of Variable)**.** *When U, V are open in $\mathbb{R}^n$ and $C = \overline{U}$, $D = \overline{V}$ and $g : C \to D$ is a change of coordinates map and $f : D \to \mathbb{R}$ is continuous, we have*

$$\int_D f = \int_C (f \circ g)|detDg|$$

Example 5.1.13. For $U, V \subseteq \mathbb{R}^3$, $f : D \to \mathbb{R}$, let $(x, y, z) = g(u, v, w)$, we have

$$\int\int\int_D f(x, y, z)dxdydz = \int\int\int_C f(g(u, v, w))|detDg(u, v, w)|dudvdw$$

For $U = (c, d)$, $C = \overline{U} = [c, d]$ and $V = (a, b)$ and $D = \overline{V} = [a, b]$, let $f : D = [a, b] \to R$ and $x = g(u)$ and $u = u(x) = g^{-1}(x)$. We have

$$\int_D f(x)dx = \int_C f(g(u))|detDg(u)|du$$

That is,

$$\int_a^b f(x)dx = \int_c^d f(g(u))|g'(u)|du = \int_{u(a)}^{u(b)} f(g(u))g'(u)du$$

Example 5.1.14. Find the area inside the cardioid $r = 2 + 2\cos\theta$.

Solution. The area is

$$\begin{aligned} A &= \int\int_D 1dxdy = \int\int_C 1|detDg(r,\theta)|drd\theta \\ &= \int\int_C rdrd\theta \\ &= \int_0^{2\pi}\int_0^{2+2\cos\theta} rdrd\theta \\ &= \int_0^{2\pi} \left[\frac{1}{2}r^2\right]_0^{2+2\cos\theta} d\theta \\ &= \int_0^{2\pi} 2 + 4\cos\theta + 2\cos^2\theta d\theta \\ &= 6\pi \end{aligned}$$

♠

Example 5.1.15. Find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 - 2x + y^2 = 0$.

Solution. Consider

$$\begin{aligned} V &= \int_D 1 \\ &= 4 \int_0^2 \int_0^{\sqrt{2x-x^2}} \int_0^{\sqrt{4-(x^2+y^2)}} 1 dz dy dx \end{aligned}$$

Or, we use cylindrical coordinate, and so we have

$$\begin{aligned} V &= \int\int\int_C r dz dr d\theta \\ &= 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} \int_{z=0}^{\sqrt{4-r^2}} r dz dr d\theta \\ &= 4 \int_0^{\pi/2} \int_0^{2\cos\theta} r\sqrt{4-r^2} dr d\theta \\ &= 4 \int_0^{\pi/2} \left[-\frac{1}{3}(4-r^2)^{3/2} \right]_0^{2\cos\theta} d\theta \\ &= \frac{4}{3} \int_0^{\pi/2} 8 - (4 - 4\cos^2\theta)^{3/2} d\theta \\ &= \frac{16\pi}{3} - \frac{32}{3} \int_0^{\pi/2} \sin^3\theta d\theta \\ &= \frac{16\pi}{3} - (32/3)\frac{2}{3} \\ &= \frac{16(3\pi - 4)}{9} \end{aligned}$$

♠

Example 5.1.16. Find the mass of the ball $x^2 + y^2 + z^2 \leq 4$ where the density is given by $\rho(x, y, z) = 1 - \frac{1}{2}\sqrt{x^2 + y^2 + z^2}$.

Solution. For $D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 4\}$, and g, the spherical coordinate map given by $(x, y, z) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\theta)$, we have $D = g(C)$ where $C = \{(r, \phi, \theta) : r \leq 2, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$. The mass is given by

$$\begin{aligned} M &= \int\int\int_D \rho(x, y, z) dx dy dz \\ &= \int\int\int_C (1 - \frac{1}{2}r)|detDg(r, \phi, \theta)| dr d\phi d\theta \\ &= \int_{r=0}^2 \int_\phi^\pi \int_{\theta=0}^{2\pi} (1 - \frac{1}{2}r) r^2 \sin\phi d\theta d\phi dr \\ &= 2\pi \int_0^2 (2r^2 - r^3) dr = 2\pi(\frac{16}{3} - 4) = \frac{8\pi}{3} \end{aligned}$$

♠