Contents

永夜抛人何处去？绝来音。香阁掩，眉敛，月将沉。
争忍不相寻？怨孤衾。换我心，为你心，始知相忆深。

顾敻

1

So, let's start the course with some classical symmetric functions. In 1629, the idea of symmetric functions was first introduced by Girard. Then, the next big result is due to Cauchy, where he defined the Schur functions s_λ, in 1825. Over the next hundred years, we have the Jacobi-Trudi formula $s_\lambda = \det(S)$ for some matrix S. Next, in 1882, Kostka discovered $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$ where $K_{\lambda\mu} \in \mathbb{Z}$. In 1900, Frobenius invented the Frobenius map Frob, which sends irreducible representations of S_n, say χ^λ, to s_λ. The same year, Young invented Young tableaux and used $K_{\lambda\mu}$ to decompose into irreducible representations $h_\lambda = \sum K_{\lambda\mu} s_\mu$.

Then, in 1970, Gargia-Paul discovered $\tilde{H}_\lambda(x;q) = \sum_\lambda s_\lambda(x)\tilde{K_{\lambda\mu}}(q)$, which is called the modified Hall-Littlewood. On the left hand side, we have the Frobenius of characters of graded version of S_n, and the right hand side we have $\tilde{K}_{\lambda\mu}(q)$ is the multiplicity of χ^λ in kth component. It is natural to ask what is the combinatorial characterization of this.

In 1978, Lascoux and Schutscaburg invented "charges" to interpret the coefficient $\tilde{K_{\lambda\mu}}(q)$.

At the same time, we have the standard inner product $\langle \cdot, \cdot \rangle$ on Λ, which satisfy $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$. Then, s_λ is unique symmetric functions satisfies:

1. $s_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu$
2. $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

In 1954, Hall invented Hall algebra for Abelian groups, and the Hall-Littlewood polynomials $P_\lambda(x;t)$. In this case, consider the new inner product

$$\langle p_\lambda, p_\mu \rangle_t := z_\lambda \delta_{\lambda\mu} \prod_{i=1}^{\ell(\lambda)} \frac{1}{1-t^{\lambda_i}}$$

The polynomial $P_\lambda(x;t)$ then is defined to be the unique symmetric functions satisfies:

1. $P_\lambda(x;t) = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu}(t) m_\mu$
2. $\langle P_\lambda(x;t), P_\mu(x;t) \rangle = 0$ if $\lambda \neq \mu$.

Then, in 1985 comes Macdonald. He noticed something called trhe Jack basis, which appear naturally in statistical mechanics, which behaves pretty similar to

s_λ and has nice properties. Then, Stanley proved Jack polynomials have s_λ and Hall-Littlewood properties (i.e. this refers to the two following properties $f_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu$ and $\langle f_\lambda, f_\mu \rangle = 0$ if $\lambda \neq \mu$ for some polynomial f).

In 1988, Kadoll conjectured q-extension of Jack polynomials. Soon after this, Macdonald proved this conjecture by introudcing the Macdonald polynomials.

Macdonald introduced two new scalar products $\langle \cdot, \cdot \rangle_{qt}$ on $\Lambda_n(t)$ with

$$\langle p_\lambda, p_\mu \rangle_{qt} = z_\lambda \delta_{\lambda\mu} \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

Then, we have a unique family of polynomials $\{P_\lambda(x;q,t)\}$ such that they satisfy:

1. $P_\lambda(x;q,t) = m_\lambda(x) + \sum_{\mu<\lambda} c_{\lambda\mu}(q,t) m_\mu(x)$
2. $\langle P_\lambda(x;q,t), P_\mu(x;q,t) \rangle = 0$ if $\lambda \neq \mu$

This new family of polynomials $P_\lambda(x;q,t)$ generalizes Hall-Littlewood (by setting $q = 0$), s_λ (by setting $q = t$, or, just $q = t = 0$) and Jack polynomials (by $q = t^\alpha$ and let $t \to \infty$).

Macdonald conjectured that

$$\tilde{H}_\lambda(x;q,t) = \sum_\mu s_\mu(x) K_{\lambda\mu}(q,t)$$

with $K_{\lambda\mu}(q,t) \in \mathbb{Z}[q,t]$, where an algebra proof took about 30 years, and a combinatorial proof is still missing and hence open problem.

We also have the other side of the story. In 1993, Garsia and Haiman used diagonal harmonics in the attemp to prove the above conjecture.

This is about the history, and we gonna start some actual math. In particular, we are going to review some symmetric functions and set up notations.

We have

$$\Lambda_n = \mathbb{Z}[x_1, ..., x_n]^{S_n} = \bigoplus_{r\geq 0} \Lambda_n^r$$

with Λ_n^r the homogenous degree r piece of Λ_n. Since we have filter maps

$$\Lambda_{n+1} \to \Lambda_n, \quad x_{n+1} \mapsto 0$$

we get a projective limit

$$\Lambda^r := \varprojlim_n \Lambda_n^r$$

Formally, elements of Λ^r are families

$$\{(f_n)_{n\geq 0} : f_n \in \Lambda_n^r, f_{n+1}(x_1, ..., x_n, 0) = f_n(x_1, ..., x_n)\}$$

Then, we define

$$\Lambda := \bigoplus_{r\geq 0} \Lambda^r$$

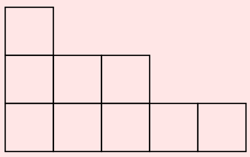
This is a vector space, so we are going to have bases. They will be parameterized by partitions. In particular, we will view λ and $\lambda 0$ as the same partition, i.e. $(\lambda_1, ..., \lambda_{\ell(\lambda)}) = (\lambda_1, ..., \lambda_{\ell(\lambda)}, 0)$ and so on. We will call parts of λ to be non-zero λ_i. We also have a frequency notation for partitions, which is given by $\lambda = 1^{m_1}2^{m_2}...$ with $m_j := \#\{i : \lambda_i = j\}$.

We have the dominance order on partitions. We say $\lambda \geq \mu$ if $\lambda_1 + ... + \lambda_j \geq \mu_1 + ... + \mu_j$ for all j. An example of this order is given by

$$\begin{tikzcd}
 & 5 \arrow[d] & \\
 & 41 \arrow[d] & \\
 & 32 \arrow[dl] \arrow[dr] & \\
311 \arrow[dr] & & 211 \arrow[dl] \\
 & 2111 \arrow[d] & \\
 & 11111 &
\end{tikzcd}$$

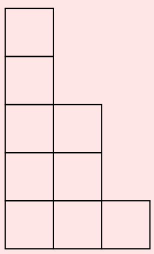
In this course, we will use French notation for Young diagrams... Thus, the tableaux will be bottom left justified.

For example, if $\lambda = 531$, then it correspond to



Formally, for λ we have the Young diagram is given by $D(\lambda) := \{(i, j) : 1 \leq j \leq \lambda_i\}$.

Then, the conjugate λ' of λ is the partition obtained by flip λ along the anti-diagonal. So, formally we have $D(\lambda') := \{(j, i) : 1 \leq j \leq \lambda_i\}$. For example, if $\lambda = 531$ the conjugate is given by



Now the bases of Λ.

We have monomial basis

$$m_\lambda = \sum_{\alpha \in S_\infty \cdot \lambda} x^\alpha$$

where $S_\infty \cdot \lambda$ is obtained by apply all suitable permutation to λ.

We have elementary basis

$$e_r = m_{(1^r)} = \sum_{i_1 < \ldots < i_r} x_{i_1} \ldots x_{i_r}$$

$$e_\lambda = \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}$$

One can prove e_j's are algebraically independent and hence $\{e_\lambda : \lambda \vdash r\}$ forms a basis of Λ^r. We have generating function

$$E(t) \coloneqq \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t)$$

We have homogenous basis

$$h_r = \sum_{\lambda \vdash r} m_\lambda = \sum_{i_1 \leq \ldots \leq i_r} x_{i_1} \ldots x_{i_r}$$

$$h_\lambda = \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i}$$

One can prove this is a basis. We have generating function

$$H(t) \coloneqq \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} \frac{1}{1 - x_i t}$$

Then, one see that

$$H(t)E(-t) = 1$$

which happends iff

$$\sum_{r=0}^{n} (-1)^r h_r e_{n-r} = 0$$

Now, since e_r's are algebraically independent, we can define maps on e_r and extend. In particular, define fundamental involution $\omega : \Lambda \to \Lambda$ by $\omega(e_\lambda) = h_\lambda$.

Then, observe

$$\sum_{r=0}^{n} (-1)^r \omega(h_r) \omega(e_r) = \sum_{r=0}^{n} (-1)^r \omega(h_r) h_{n-r} = 0$$

Or, we have

$$\sum_{r=0}^{n} (-1)^r h_r \omega(h_{n-r}) = 0$$

Thus, $\omega(h_{n-r})$ satisfies the same equation on e_{n-r} and hence $\omega(h_r) = e_r$. This implies $\{h_\lambda : \lambda \vdash r\}$ is a basis of Λ^r.

The last basis we need for now is the power sum. This is given by

$$p_r = m_r = \sum_{i \geq 1} x_i^r$$

$$p_\lambda = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}$$

This is defined only for $r \geq 1$.

In this case, we get generating function

$$P(t) = \sum_{r\geq 1} p_r t^{r-1} = \sum_{r\geq 1}\sum_{i\geq 1} x_i^r t^{r-1} = \sum_{i\geq 1} \frac{x_i}{1 - x_i t}$$

One can show

$$P(t) = \frac{H'(t)}{H(t)}$$

Thus we get

$$P(t)H(t) = H'(t) = \sum_{n\geq 0} \frac{d}{dt} h_n t^n = \sum_{n\geq 1} n h_n t^{n-1}$$

On the right hand side, we have

$$P(t)H(t) = \sum_{n\geq 1}\left(\sum_{r\geq 0}^{n} (p_r h_{n-r}\right) t^n$$

Thus

$$n h_n = \sum_{r\geq 0}^{n} p_r h_{n-r}$$

This concludes $h_n \in \mathbb{Q}[p_1, p_2, ...]$. Thus implies $\{p_\lambda : \lambda \vdash r\}$ spans $\Lambda^r \otimes_{\mathbb{Z}} \mathbb{Q}$.

Well, how does ω acts on p_r? To that end, consider

$$H(t)E(-t) = 1$$

and take derivative on both sides we get

$$H'(t)E(-t) - H(t)E'(-t) = 0$$

and hence

$$\frac{H'(t)}{H(t)} = \frac{E'(-t)}{E(-t)} \Rightarrow P(t) = \frac{E'(-t)}{E(-t)}$$

However, note

$$E'(-t)/E(-t) = \omega(\frac{H'(-t)}{H(-t)}) = \omega(P(-t))$$

In other word, when we expand $P(t) = \omega(P(-t))$, we get

$$\omega(p_r) = (-1)^{r-1} p_r$$

We also note, since $P(t) = \frac{H'(t)}{H(t)} = \frac{d}{dt}\log(H(t))$, we get

$$H(t) = \exp \int P(t)$$

Expand this out, we get

$$H(t) = \sum_{\lambda} \frac{p_\lambda}{z_\lambda} t^{|\lambda|}$$

and hence we conclude

$$h_n = \sum_{\lambda\vdash n} \frac{p_\lambda}{z_\lambda}$$

where we know $z_\lambda = \prod_{r\geq 1} r^{m_r} m_r!$ with $\lambda = 1^{m_1} 2^{m_2} \ldots$.

2

Today we consider the scalar product and Cauchy identity.

We will write $X = (x_1, x_2, \dots)$ and $Y = (y_1, y_2, \dots)$ to be set of variables, and $XY := (x_i y_j : i, j \geq 1)$. Then, $f(X) = f(x_1, x_2, x_3, \dots)$ and $f(XY) = f(x_1 y_1, x_1 y_2, \dots)$ for $f \in \Lambda$. Note in the definition of XY we didn't pick an order, this is because for all our purposes, i.e. for symmetric functions, the order does not matter.

Example 2.1. We have

$$p_n(XY) = \sum_{i,j} (x_i y_j)^n = \left(\sum_i x_i^n\right)\left(\sum_j y_j^n\right) = p_n(x) p_n(y)$$

$$p_\lambda(XY) = p_\lambda(X) p_\lambda(Y)$$

Definition 2.2. We define

$$\pi(X, Y) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

We note this looks similar, i.e. we have

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} \frac{1}{1 - x_i t}$$

and hence $H(1) = \prod_{i \geq 1} \frac{1}{1 - x_i} = \sum_{r \geq 0} h_r(X)$. Thus, we can write

$$\pi(X, Y) = H(XY; 1) = \sum_{r \geq 0} h_r(XY)$$

From last time, we derived $h_r(X) = \sum_\lambda \frac{p_\lambda(X)}{z_\lambda}$. Plug this in the above, we get

$$\pi(X, Y) = \sum_{r \geq 0} \sum_{\lambda \vdash r} \frac{p_\lambda(XY)}{z_\lambda} = \sum_\lambda \frac{p_\lambda(XY)}{z_\lambda} = \sum_\lambda \frac{p_\lambda(X) p_\lambda(Y)}{z_\lambda} \qquad \text{(Eq. 2.1)}$$

This is the first important identity about π.

Next, we note

$$H(y_j) = H(X_j y_j) = \sum_{r \geq 0} h_r(X) y_j^r = \prod_i \frac{1}{1 - x_i y_j}$$

Thus, we see

$$\begin{aligned}
\pi(X, Y) &= \prod_j \prod_i \frac{1}{1 - x_i y_j} \\
&= \prod_j \sum_{\alpha_j \geq 0} h_{\alpha_j}(X) y_j^{\alpha_j} \\
&= \sum_r \sum_{(\alpha_1, \alpha_2, \dots) \in \mathbb{Z}_{\geq 0}^r} \prod h_{\alpha_j}(X) y_j^{\alpha_j}
\end{aligned}$$

In the above, we see the $y_j^{\alpha_j}$ contributes to $m_\lambda(Y)$, and the $h_{\alpha_j}(X)$ contributes to $h_\lambda(X)$, hence we get the second important identity:

$$\pi(X,Y) = \sum_\lambda h_\lambda(X) m_\lambda(Y) \tag{Eq. 2.2}$$

Definition 2.3. We define a scalar product on λ, called ***Hall inner product***, to be

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$$

Theorem 2.4. *$\{u_\lambda\}$ and $\{v_\lambda\}$ are dual basis under $\langle \cdot, \cdot \rangle$ if and only if*

$$\pi(X,Y) = \sum_\lambda u_\lambda(X) v_\lambda(Y)$$

Proof. We see

$$u_\lambda(X) = \sum_\rho a_{\lambda\rho} h_\rho(X)$$
$$v_\nu(Y) = \sum_\sigma b_{\nu\sigma} m_\sigma(Y)$$

Suppose $\langle u_\lambda, v_\nu \rangle = \delta_{\lambda\nu}$, then we see

$$\langle u_\lambda, v_\nu \rangle = \sum_\rho \sum_\sigma a_{\lambda\rho} b_{\nu\sigma} \langle h_\rho, m_\sigma \rangle = \sum_\rho a_{\lambda\rho} b_{\mu\sigma} \delta_{\rho\sigma}$$

Thus this implies

$$\sum_\rho a_{\lambda\rho} b_{\mu\rho} = \delta_{\lambda\mu}$$

Next, suppose $\pi(X,Y) = \sum_\lambda u_\lambda(X) v_\lambda(Y)$. Expand this we get

$$\begin{aligned}
\sum_\lambda \sum_{\rho,\sigma} a_{\lambda\rho} h_\rho(X) b_{\lambda\sigma} m_\sigma(Y) &= \sum_{\rho,\sigma} h_\rho(X) m_\sigma(Y) \sum_\lambda a_{\lambda\rho} b_{\lambda\sigma} \\
&= \pi(X,Y) \\
&= \sum_\lambda h_\lambda(X) m_\lambda(Y)
\end{aligned}$$

However, the h_λ and m_λ are bases, hence the coefficient in the above expression must match up, i.e. we must have

$$\begin{aligned}
[h_\rho(X) m_\sigma(Y)] \sum_\lambda \sum_{\rho,\sigma} a_{\lambda\rho} h_\rho(X) b_{\mu\sigma} m_\sigma(Y) &= \sum_\lambda a_{\lambda\rho} b_{\lambda\sigma} \\
&= [h_\rho(X) m_\sigma(Y)] \sum_\lambda h_\lambda(X) m_\lambda(Y) \\
&= \delta_{\rho\sigma}
\end{aligned}$$

Hence, we see the two equations we obtained gives $A^T B = I$ and $AB^T = I$, where $A = (a_{\lambda\rho})_{\lambda,\rho \vdash n}$ and $B = (b_{\mu\sigma})_{\mu,\sigma \vdash n}$. However, by linear algebra, we get the two statements in the theorem are equivalent. ♡

This theorem is very powerful. It says satisfying the above condition, which is called the Cauchy condition, implies orthogonality under our Hall inner product.

Corollary 2.4.1. $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$

Proof. Just note
$$\sum_\lambda \frac{p_\lambda(X)p_\lambda(Y)}{z_\lambda} = \pi(X,Y)$$
♡

In particular, this corollary shows ω is an isometry, as $\omega(p_\lambda) = \pm p_\lambda$.

The next goal is to define Schur functions, which is the an important basis. We start with the Cauchy's formula.

Let $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}^n_{\geq 0}$ be a weak composition, and set $\delta := (n-1, n-2, \ldots, 1, 0)$. Then set
$$a_\alpha = \det(x_j^{\alpha_j}) = \sum_{w \in S_n} \mathrm{sgn}(w) x^{w \cdot \alpha} = \sum_{w \in S_n} \mathrm{sgn}(w) \prod_i x_i^{\alpha_{w(i)}}$$
In particular, we get
$$\alpha_\delta = \det(x_i^{j-1}) = \prod_{i>j} (x_i - x_j)$$
In particular, we note for any transposition (i,j), we get $\alpha_\delta = -(i,j) \cdot a_\delta$, i.e.
$$a_\delta(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = -a_\delta(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$$
This is an example of an alternating functions.

In particular, if we consider the module of alternating homogenous polynomials of degree n, which is the collection of f such that
$$f(x_1, \ldots, x_n) = \mathrm{sgn}(w) f(x_{w(1)}, \ldots, x_{w(n)})$$
Then, we have a basis
$$\{a_\alpha \in \mathbb{Z}^n_{\geq 0} : \alpha \text{ strictly decreasing}\}$$
for this module.

Definition 2.5. We define
$$s_\lambda(X) := \frac{a_{\lambda+\delta}(X)}{a_\delta(X)} \in \Lambda^n$$

We observe:

1. $a_{\lambda+\delta}$ is divisible by a_δ since it is divisible by $x_i - x_j$ for all i, j.
2. every alternating function is divisible by a_δ, thus we get an module isomorphism $\Lambda^n_r \to \mathrm{Alt}^n_{r+\binom{n}{2}}$ given by $f \mapsto f \cdot a_\delta$, where Alt^n_k is the degree k alternating homogenous polynomial of n variables.

In particular, the above definition extends to all α, i.e. we can define

$$s_\alpha := \frac{a_{\alpha+\delta}}{a_\delta}$$

In this case, we actually have

$$s_\alpha = \begin{cases} 0, & \alpha \text{ has repeated parts} \\ \operatorname{sgn}(w) s_\lambda, & \text{otherwise} \end{cases}$$

where in the above w is some permutation depends on α such that $w \cdot \alpha = \lambda + \delta$.

Theorem 2.6 (Triangularity). *We have*

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu$$

*where the $K_{\lambda\mu}$ are called the **Kostka numbers**.*

Proof. We see

$$\begin{aligned} m_\lambda a_\delta &= \sum_{\alpha \in S_n \cdot \lambda} x^\alpha \sum_{w \in S_n} x^{w \cdot \delta} \\ &= \sum_{\alpha \in S_n \cdot \lambda} \sum_{w \in S_n} \operatorname{sgn}(w) x^{w(\alpha+\delta)} \\ &= \sum_{\alpha \in S_n \cdot \lambda} a_{\alpha+\delta} \end{aligned}$$

Hence we conclude

$$m_\lambda = \sum_{\alpha \in S_n \cdot \lambda} s_\alpha$$

since

$$s_\alpha = \frac{a_{\alpha+\delta}}{a_\delta}$$

We claim that, if $\alpha \in S_n \cdot \lambda$ and $\alpha \neq \lambda$, then either $s_\alpha = 0$ or $s_\alpha = \pm s_\mu$ when $\mu < \lambda$. This is left as an exercise.

By the above claim, we see

$$m_\lambda = s_\lambda + \sum_{\mu < \lambda} s_\mu c_{\lambda\mu}$$

and hence we can inverse the matrix and get the desired claim

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu$$

and in particular since leading terms are all 1, $K_{\lambda\mu} \in \mathbb{Z}$. ♡

Corollary 2.6.1. *$\{s_\lambda : \lambda \vdash n\}$ forms a basis for Λ^n.*

Theorem 2.7.
$$\pi(X,Y) = \sum_\lambda s_\lambda(X)s_\lambda(Y)$$

We will prove this via Cauchy determinant. The Cauchy determinant is given by

$$\det\left(\frac{1}{x_i+y_j}\right)_{1\le i,j\le n}$$

and it satisfies the following equation.

Theorem 2.8.
$$\det\left(\frac{1}{x_i+y_j}\right)_{1\le i,j\le n} = \frac{a_\delta(X)a_\delta(Y)}{\prod_{1\le i,j\le n}(x_i+y_j)}$$

Proof. Move the bottom of the RHS to the left, we see we get

$$LHS := \det\left(\frac{1}{x_i+y_j}\right)_{1\le i,j\le n} \cdot \prod_{1\le i,j\le n}(x_i+y_j)$$

In this case, we see this is the same as multiply each row of the determinant by the product $\prod_{1\le i,j\le n}(x_i+y_j)$. In particular, we see this is given by

$$\begin{matrix} \prod_{j=1}^n(x_1+y_j)\cdot \\ \prod_{j=1}^n(x_2+y_j)\cdot \\ \vdots \end{matrix} \begin{vmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \cdots \\ \vdots & \vdots & \vdots \end{vmatrix}$$

but this is a degree $n-1$ polynomial in each row and hence LHS is a polynomial with degree $n(n-1)$. Moreover, by write out the actual polynomial in each row, we see LHS is divisible by $x_i - x_j$ and $y_i - y_j$ for all $i < j$. Thus LHS is divisible by $a_\delta(X)a_\delta(Y)$, where the total degree of $a_\delta(X)a_\delta(Y)$ is also $n(n-1)$. Thus $LHS = c\cdot a_\delta(X)a_\delta(Y)$ for some constant c. Howeever, we can figure out the constant c by compare some particular value. Indeed, we see

$$[x^\delta y^\delta]\prod(x_i+y_j)\det\left(\frac{1}{x_i+y_j}\right) = 1$$

$$[x^\delta y^\delta]a_\delta(X)a_\delta(Y) = 1$$

which forces $c = 1$, and proves the theorem. ♡

Now we can prove the theorem 2.7.

Proof of Theorem 2.7. Let's try to rewrite our expressions so that it looks similar to Schur. So, say we have Cauchy determinant

$$\det\left(\frac{1}{x_i + y_j}\right)_{1\le i,j\le n}$$

We replace x_i with $-x_i^{-1}$, we see this means the entries become

$$\frac{1}{-x_i^{-1} + y_j} = \frac{-x_i}{1 - x_i y_j}$$

The a_δ becomes

$$a_\delta(-x_1^{-1}, -x_2^{-1}, \dots) = \prod_{i<j}(-x_i^{-1} + x_j^{-1}) = \prod_{i<j} \frac{x_i - x_j}{x_i x_j} = a_\delta(X) \cdot \frac{1}{\prod_i x_i^{n-1}}$$

The $x_i + y_j$ becomes

$$-x_i^{-1} + y_j = \frac{1 - x_i y_j}{-x_i}$$

and so

$$\prod_{i,j}(x_i + y_j) = \prod_{i,j}(1 - x_i y_j) \prod_i \frac{(-1)^n}{x_i^n}$$

Also, observe we have

$$\det(\frac{-x_i}{1 - x_i y_j}) = \left(\prod_i -x_i\right) \det(\frac{1}{1 - x_i y_j})$$

Now combine everything together, and use theorem 2.8, we get

$$\begin{aligned}
&(-1)^n \prod_i x_i \cdot \det\left(\frac{1}{1 - x_i y_j}\right) \\
&= a_\delta(Y) a_\delta(X) \cdot \prod_i \frac{1}{x_i^{n-1}} \cdot \left(\prod \frac{(-1)^n}{x_i^n}\right)^{-1} \cdot (-1)^n \prod \frac{1}{(1 - x_i y_j)}
\end{aligned}$$

This cancels everything out, and we conclude

$$\det(\frac{1}{1 - x_i y_j}) = a_\delta(X) a_\delta(Y) \pi(X, Y)$$

but $\det(\frac{1}{1 - x_i y_j})$ is equal

$$\sum_\lambda a_{\lambda+\delta}(X) a_{\lambda+\delta}(Y)$$

as we have $\frac{1}{1-z} = \sum_{k\ge0} z^k$. Indeed, we expand $\frac{1}{1-x_i y_j}$ as $\sum_{\alpha_{ij}\ge0}(x_i y_j)^{\alpha_{ij}}$ and hence the determinant is also equal

$$\det\begin{pmatrix} 1 + x_1y_1 + (x_1y_1)^2 + \dots & 1 + x_1y_2 + (x_1y_2)^2 + \dots & \dots \\ 1 + x_2y_1 + (x_2y_1)^2 + \dots & 1 + x_2y_2 + (x_2y_2)^2 + \dots & \dots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

Thus each term in expansion is $\prod_{i=1}^{n}(x_i y_{w(i)})^{\alpha_i}$ for some permutation w. Thus we get

$$\det(\frac{1}{1-x_i y_j}) = \sum_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n_{\geq 0}} x^d \underbrace{\sum_{w\in S_n} \mathrm{sgn}(w) y^{w\cdot\alpha}}_{=a_\alpha(Y)}$$

However, recall

$$a_\alpha(X) = \begin{cases} 0, & \text{has repeated parts} \\ \mathrm{sgn}(w) a_{\lambda+\delta}, & \text{otherwise} \end{cases}$$

and hence

$$\det(\frac{1}{1-x_i y_j}) = \sum_{\alpha\in\mathbb{Z}^n_{\geq 0}} x^\alpha \,\mathrm{sgn}(w)\alpha_{\lambda+\delta}(Y) = \sum_\lambda a_{\lambda+\delta}(X) a_{\lambda+\delta}(Y)$$

as desired. This concludes the proof. ♡

3

Today we will cover Schur functions, then talk about RSK, which lead us to Jacobi-Trudi formula, and hence we obtain a natural definition of semi-standard Young tableaux.

Last time, we see

$$\sum_\lambda s_\lambda(X) s_\lambda(Y) = \pi(X,Y),$$

which is proved by the Cauchy determinant.

Now we have:

1. $s_\lambda = m_\lambda + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu$. In particular, $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$ for all $\mu \leq \lambda$.
2. and in general we have $\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$.

The Jacobi identity expresses $s_\lambda(x)$ in terms of determinants.

Theorem 3.1.

$$s_\lambda(X) = \det(h_{\lambda_i+j-i})_{1\leq i,j\leq n}$$

where $h_m = 0$ if $m < 0$.

So, for example, we have

$$s_{411} = \begin{vmatrix} h_4 & h_5 & h_6 \\ h_0 & h_1 & h_2 \\ 0 & h_0 & h_1 \end{vmatrix}$$

Proof. We use Cauchy identity. So, we observe

$$H(t) = \sum_{r\geq 0} h_r(X) t^r = \prod_i \frac{1}{1-x_i t} \Rightarrow H(y_i) = \prod_i \frac{1}{1-x_i y_i}$$

Thus we get

$$\pi(X,Y) = \prod_j H(y_j)$$

Now multiply both side by $a_\delta(Y)$, we see

$$\pi(X,Y)a_\delta(Y) = \prod_j H(y_j)\det(y_i^{n-j})$$

and by linearity, this means we multiply jth row of the matrix by $H(X;y_j)$. Thus the RHS is equal to

$$\begin{aligned} RHS &= \det(H(y_i)y_i^{n-j})_{1\le i,j\le n} \\ &= \det((\sum_{\alpha_j\ge 0} h_{\alpha_j}(X)y_i^{\alpha_j})y_i^{n-j}) \\ &= \left(\sum_{\alpha_j\ge 0} h_{\alpha_j}(X)y_i^{\alpha_j+n-j}\right) \end{aligned}$$

Expand matrix along columns, we get the above is equal to

$$\sum_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{Z}^n_{\ge 0}} h_\alpha(X)\det(y_j^{\alpha_j+n-j})_{1\le i,j\le n}$$

but the $\det(y_j^{\alpha_j+n-j})_{1\le i,j\le n}$ is exactly equal to $a_{\alpha+\delta}(Y)$, which is equal 0 or $a_{\lambda+\delta}(Y)\operatorname{sgn}(w)$. Hence the above is equal to

$$\sum_\lambda \sum_{w\in S_n(\lambda+\delta)-\delta} h_{w(\lambda+\delta)}(X)a_{\lambda+\delta}(Y)\operatorname{sgn}(w)$$

Extract coefficients of $a_{\lambda+\delta}(Y)$, we get

$$[a_{\lambda+\delta}(Y)]RHS = \sum_{w\in S_n} h_{w(\lambda+\delta)-\delta}\operatorname{sgn}(w) = \det(h_{\lambda_j+n-i-(n-j)}) = \det(h_{\lambda_i+j-i})$$

From the LHS, we see

$$[a_{\lambda+\delta}(Y)]LHS = [a_{\lambda+\delta}(Y)]\pi(X,Y)a_\delta(Y) = s_\lambda$$

Thus we conclude the proof. ♡

We note this also relates to Lindstrom-Gessel-Viennot lemma.

Let G be a weighted direct graph with source $A + (a_1,\ldots,a_n)$ and sinks $B = (b_1,\ldots,b_n)$. Then suppose we have weight $\mathrm{wt}: E(G)\to R$, then for a path we define $\mathrm{wt}(p) = \prod_{e\in P}\mathrm{wt}(e)$. Next, we define $W_{ij} = \sum_{p:a_i\to b_j}\mathrm{wt}(p)$.

Then, the theorem says:

Theorem 3.2.

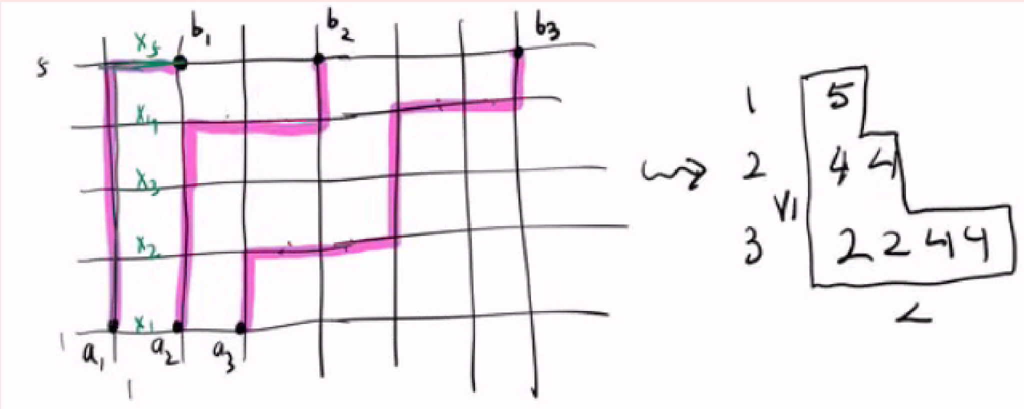
$$\sum_{\substack{(p_1,\ldots,p_n):A\to B \\ p_i:a_i\to b_i \\ \textit{paths non-crossing}}} \prod_{i=1}^n \mathrm{wt}(p_i) = \det(W_{ij})_{1\le i,j\le n}$$

Proof. ... ♡

In particular, we get a bijection between non-crossing path $(p_1, \ldots, p_k)$ and SSYT of shape λ.

Definition 3.3. An element of $\mathrm{SSYT}(\lambda)$ is a filling with $\mathbb{Z}_{\geq 0}$ of the diagram, $D(\lambda)$, of λ, that is weakly increasing left to right along the row and strictly increasing from bottom to top.

The bijection is given by the rows of $T \in \mathrm{SSYT}(\lambda)$, where row j of filling of $D(\lambda)$ records the y-coordinate of the horizontal step of p_{k-j+1}: see the following example



4

Today we are going to talk about Hall-Littlewood polynomials.

We start with $\Lambda_{\mathbb{Q}}[t]$. We define $\Lambda_n[t]$ to be symmetric polynomials with n variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{Z}[t]$. Take $n \to \infty$ we get $\Lambda[t]$ and tensor with $\mathbb{Q}$ give us $\Lambda_{\mathbb{Q}}[t]$.

Definition 4.1. For $|\lambda| = n$, we define

$$R_\lambda(x_1, \ldots, x_n; t) = \sum_{w \in S_n} w(x_1^{\lambda_1} \ldots x_n^{\lambda_n} \prod_{1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j})$$

First, we note the above definition only works for finitely number of variables.

Second, to see this above definition is not completely unmotivated, recall Schur function $s_\lambda(x_1, \ldots, x_n) = \sum_{w \in S_n / S_n^\lambda} w(x^\lambda \prod \frac{x_i}{x_i - x_j}) = \frac{a_{\lambda+\delta}}{a_\delta}$.

We note R_λ is symmetric (we are dividing by anti-symmetric factor), homogenous of degree $|\lambda|$.

Example 4.2. Take $n = 2$, $\lambda = (2)$ and $\mu = (1,1)$. Then

$$\begin{aligned} R_\lambda(x_1, x_2; t) &= x_1^2 x_2^0 \left(\frac{x_1 - t x_2}{x_1 - x_2} \right) + x_1^0 x_2^2 \left(\frac{x_2 - t x_1}{x_2 - x_1} \right) \\ &= \frac{1}{x_1 - x_2} (x_1^3 - t x_1^2 x_2 - x_2^3 + t x_1 x_2^2) \\ &= x_1^2 + x_2^2 + (1-t) x_1 x_2 \end{aligned}$$

Similarly, we get

$$\begin{aligned} R_{(1,1)}(x_1, x_2; t) &= x_1^1 x_2^1 \left(\frac{x_1 - t x_2}{x_1 - x_2} \right) + x_1^2 x_2^1 \left(\frac{x_2 - t x_1}{x_2 - x_1} \right) \\ &= (1+t) x_1 x_2 \end{aligned}$$

Theorem 4.3.

1. *$R_\lambda(x_1, \dots, x_n) = \sum_{\mu \le \lambda} u_{\lambda\mu}(t) S_\mu(x_1, \dots, x_n)$ with $u_{\lambda\mu}(t) \in \mathbb{Z}[t]$.*
2. *$u_{\lambda\lambda} = v_\lambda(t)$ where*
$$v_\lambda(t) = \prod_{i \ge 0} [m_i]_t!$$
where $\lambda = 0^{m_0} 1^{m_1} 2^{m_2} \dots$ and $[m]_t = 1 + \dots + t^{m-1} = \frac{1-t^m}{1-t}$ and $[m]_t! = [m]_t [m-1]_t \dots [1]_t$.

For example, if $\lambda = 311$, then

$$v_\lambda(t) = [2]_t! [1]_t! = (1+t)$$

On the other hand, if $\lambda = 31100$ (note the 0 matters here) we end up with

$$v_\lambda(t) = (1+t)^2$$

Motivated by the above theorem, we get the following definition.

Definition 4.4. We define the ***Hall-Littlewood polynomial*** to be

$$P_\lambda(X; t) = \frac{1}{v_\lambda(t)} R_\lambda(X; t) = \frac{1}{v_\lambda(t)} \sum_{w \in S_n} w \left(x^\lambda \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

The reason why we do this is that $R_\lambda(X; t)$ is not stable, in the sense that $R_\lambda(x_1, \dots, x_n, 0; t) \neq R_\lambda(x_1, \dots, x_n; t)$. However, $P_\lambda(X; t)$ is stable (since we divided out $v_\lambda(t)$).

An alternative definition of $P_\lambda(X; t)$ is given by

$$P_\lambda(X; t) = \sum_{w \in S_n / S_n^\lambda} w (x^\lambda \prod_{\lambda_i < \lambda_j} \frac{x_i - t x_j}{x_i - x_j})$$

where $S_n^\lambda = S_{m_0} \times S_{m_1} \times \dots$ where $\lambda = 0^{m_0} 1^{m_1} 2^{m_2} \dots$ is the Young subgroup. We remark that, if our sum is over S_n, then the index under the product should normally be $i < j$, but if our sum is over quotient S_n / S_n^λ, then the product should be $\lambda_i < \lambda_j$.

We remark some familiar specializations of P_λ:

1. $P_\lambda(X; 0) = s_\lambda(X)$
2. $P_\lambda(X; 1) = \sum_{w \in S_n / S_n^\lambda} w(X^\lambda) = \sum_{w \in S_n \lambda} x^{w\lambda} = m_\lambda(X)$

Thus, H-L interpolate between m_λ and s_λ, and we also have something we haven't seen before: $P_\lambda(x; -1)$ is what's called ***Schur p-function***.

Proposition 4.5. *$P_\lambda(x;t)$ is upper uni-triangular in s_μ basis.*

We also have another Hall-Littlewood polynomial:

Definition 4.6. For $\lambda = 0^{m_0} 1^{m_1} \dots$, we define

$$Q_\lambda(x;t) = b_\lambda(t) P_\lambda(x;t)$$

with

$$b_\lambda(t) = \prod_{i \geq 1} [m_i]_t! (1-t)^{m_i}$$

where we remark the product this time start with 1.

This Q_λ is also stable (it would not be the case if we miss the $(1-t)^{m_i}$).

Definition 4.7. Define $q_r(x_1, \dots, x_n; t) = Q_{(r)}(x_1, \dots, x_n; t)$.

We note from the definition we get

$$q_t(x_1, \dots, x_n; t) = (1-t) \sum_{i \geq 1} x_i^r \prod_{i \neq j} \frac{x_i - t x_j}{x_i - x_j} = (1-t) P_{(r)}(x_1, .., x_n; t)$$

Proposition 4.8. *Define*

$$Q(u) = \sum_{r \geq 0} q_r(x;t) u^r$$

Then

$$Q(u) = \prod_{i \geq 1} \frac{1 - x_i t u}{1 - x_i u} = \frac{H(x;u)}{H(x;tu)}$$

Theorem 4.9. *Define*

$$\begin{aligned} F(X; u_1, u_2, \dots) &:= \prod_{i \geq 1} \frac{H(u_i)}{H(tu_i)} \prod_{i<j} \frac{u_i - u_j}{u_i - t u_j} \\ &= \prod_{i \geq 1} Q(u_i) \prod_{i<j} \frac{u_i - u_j}{u_i - t u_j} \end{aligned}$$

Then, we have

$$[u^\lambda]F(X;u) = Q_\lambda(X;t)$$

Viz, $F(X;u) = \sum_\lambda u^\lambda Q_\lambda(X;t)$.

We can define those by raising and lowering operators as well.

In particular, for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m_{\geq 0}$, we define

$$R_{ij} : \mathbb{Z}^m_{\geq 0} \to \mathbb{Z}^m_{\geq 0}$$

by

$$R_{ij}(\alpha_1, \ldots, \alpha_m) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots \alpha_m)$$

Note the whole thing equal 0 if $\alpha_j = 0$.

Then, we define $R_{ij}h_\alpha = h_{R_{ij}\alpha}$ and $h_\alpha = \pm h_{dec(\alpha)}$ where $dec(\alpha) = \lambda \vdash n$ such that $\alpha \in S_n\lambda$.

Now, with our new definitions, we get

$$P_\lambda(X;t) = \frac{1}{v_\lambda(t)} = \prod_{i<j}(1 - tR_{ji})s_\lambda = \prod_{\lambda_i < \lambda_j}(1 - tR_{ji})s_\lambda$$

This comes from $q_\alpha(X;t) = \prod_{i \geq 0} q_{\alpha_i}(X;t)$ then we get the following theorem.

Theorem 4.10.

$$Q_\lambda(X;t) = \prod_{i<j} \frac{1 - R_{ij}}{1 - tR_{ij}} q_\lambda(X;t)$$

where $R_{ij}q_\alpha = q_{R_{ij}\alpha}$.

Next, we consider monomial expansions. Suppose $T \in \mathrm{SSYT}(\lambda)$, then we define $\lambda^{(i)}(T)$ be the subpartition of λ with cells of T with entries $\leq i$.

Example 4.11. Suppose T is

$$\begin{array}{|c|c|cccc}
\hline
4 & 5 & \multicolumn{1}{c}{} & & & \\
\hline
2 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} & & \\
\hline
1 & 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\
\hline
\end{array}$$

Then we get

$$\lambda^{(1)}(T) = 2$$

$$\lambda^{(2)}(T) = (3, 2)$$

$$\lambda^{(3)}(T) = (5, 3)$$

$$\lambda^{(4)}(T) = (6, 3, 1)$$

$$\lambda^{(5)}(T) = \lambda$$

We note by definition of SSYT, we see cells of $\lambda^{(i)}(T) \backslash \lambda^{(i-1)}(T)$ are in different columns, and it is exactly the set of cells containing i in T.

Definition 4.12. Define

$$J_r(T) = \left\{ \text{column indices j} : \begin{matrix} \lambda^{(r)}(T) \backslash \lambda^{(r-1)}(T) \text{ not in col } j \\ \lambda^{(r)}(T) \backslash \lambda^{(r-1)}(T) \text{ in col } j+1 \end{matrix} \right\}$$

Definition 4.13. Define

$$\psi_T(t) = \prod_{r \geq 1} \prod_{j \in J_r(T)} 1 - t^{m_j(\lambda^{(r-1)})}$$

where $m_j(\lambda^{(r-1)})$ is the multiplicity of j in the partition $\lambda^{(r-1)}(T)$.

Theorem 4.14.

$$P_\lambda(X; t) = \sum_{T \in \mathrm{SSYT}(\lambda)} \psi_T(t) x^T$$

where x^T is the content of T, i.e. $x^{\mathrm{wt}(T)} = \prod_{c \in T} x_c$.

Example 4.15. Suppose $\lambda = (2,1)$ and we consider the coefficient of $x_1^2 x_2$. First, consider

$$T = \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}$$

and we get $\lambda^{(0)} = \varnothing$, $\lambda^{(1)} = (2)$ and $\lambda^{(2)} = (2,1)$. Hence we get $J_1 = \varnothing$ and $J_2 = \varnothing$, which implies $\psi_T(t) = 1$, which is the coefficient of $x_1^2 x_2$.

Next, we consider the coefficient of $x_1 x_2^2$, which is supposed to be 1 as the polynomial should be symmetric. This time we get

$$T = \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

and similarly $J_1 = J_2 = \varnothing$.

Now let's look at $x_1 x_2 x_3$. This time we get two fillings

$$T_1 = \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}, \quad T_2 = \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

For T_1, we get

$$\lambda^{(0)} = \varnothing, \lambda^{(1)} = (1), \lambda^{(2)} = (2), \lambda^{(3)} = (2,1)$$

and hence $J_1 = \varnothing, J_2 = \{1\}, J_3 = \varnothing$ which implies $\psi_{T_1}(t) = 1 - t^{m_1(\lambda^{(1)})} = 1 - t$.

For T_2, we get

$$\lambda^{(0)} = \varnothing, \lambda^{(1)} = (1), \lambda^{(2)} = (1,1), \lambda^{(3)} = (2,1)$$

and get $J_1 = \varnothing, J_2 = \varnothing, J_3 = \{1\}$. This time we get $\psi_{T_2}(t) = 1 - t^{m_1(\lambda^{(2)})} = 1 - t^2$. Hence we conclude

$$P_{(2,1)}(x_1, x_2, x_3; t) = m_{21}(x_1, x_2, x_3; t) + (2 - t - t^2)m_{111}(x_1, x_2, x_3; t)$$

5

Last time we defined some polynomials

$$P_\lambda(X; t) = \frac{1}{v_\lambda(t)} \sum_{w \in S_n} w(x^\lambda \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}) = \sum_{w \in S_n/S_n^\lambda} w(x^\lambda \cdot \prod_{\lambda_i < \lambda_j} \frac{x_i - tx_j}{x_i - x_j})$$

$$Q_\lambda(X; t) = b_\lambda(t) P_\lambda(X; t)$$

with

$$b_\lambda = \prod_{i \geq 1} [m_i]_t! (1 - t)^{m_i}$$

We also defined

$$q_r(X; t) = Q_{(r)}(X; t) = (1 - t)P_{(r)}(X; t) = (1 - t) \sum_{i \geq 1} x_i^r \prod_{j \neq i} \frac{x_i - tx_j}{x_i - x_j}$$

and this gives us

$$Q(X; t; u) = \sum_{r \geq 0} q_r(X; t)u^r = \prod_{i \geq 1} \frac{1 - x_i tu}{1 - x_i u}$$

However, recall another useful generating function

$$H(X; z) = \sum_{r \geq 0} h_r(X) z^r = \prod_{i \geq 1} \frac{1}{1 - x_i z}$$

Therefore we get

$$Q(X; t; u) = \frac{H(X; u)}{H(X; tu)}$$

Next, like what we did with classical symmetric functions, we define a product.

Definition 5.1. We define $\pi(X, Y; t) := \prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j}$

Note we have $\pi(X, Y, 0) = \pi(X, Y)$ where on the right hand side is the one we used to prove the inner product for classical symmetric. We also note $\pi(X, Y, t) = \prod_{j \geq 1} \frac{H(X; y_j)}{H(X; ty_j)}$.

Definition 5.2. Define an inner product $\langle \cdot, \cdot \rangle_t$ by

$$\langle p_\lambda(X), p_\mu(X) \rangle_t = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1}{1 - t^{\lambda_i}}$$

Theorem 5.3. *The $\{u_\lambda(t)\}$ and $\{v_\lambda(t)\}$ are dual basis of $\Lambda^n(t)$ if and only if $\sum_\lambda u_\lambda(X;t)v_\lambda(Y;t) = \pi(X,Y;t)$.*

Next, we consider some identities:

1. $\pi(X,Y;t) = \sum_\lambda p_\lambda(X)p_\lambda(Y)z_\lambda(t)^{-1}$ where $z_\lambda(t) := z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1}{1-t^{\lambda_i}}$.

   To see this, we take log of $\pi(X,Y;t)$ and we get
$$\begin{aligned}\log \pi(X,Y;t) &= \sum_{i,j}(\log(1-tx_iy_j) - \log(1-x_iy_j))\\ &= \sum_{i,j}\sum_{m\geq 1}\frac{1}{m}(-(tx_iy_j)^m + (x_iy_j)^m)\\ &= \sum_{i,j}\sum_{m\geq 1}\frac{1-t^n}{m}(x_iy_j)^m\\ &= \sum_{m\geq 1}\sum_{i,j}\frac{1-t^n}{m}(x_iy_j)^m\\ &= \sum_{m\geq 1}\frac{1-t^m}{m}\sum_{i,j}(x_iy_j)^m\\ &= \sum_{m\geq 1}\frac{1-t^m}{m}\underbrace{p_m(X,Y)}_{=p_m(X)p_m(Y)}\\ &= \sum_{m\geq 1}\frac{1-t^m}{m}p_m(X)p_m(Y)\end{aligned}$$
   Take exp of both side, we get
$$\begin{aligned}\pi(X,Y;t) &= \exp(\sum_{m\geq 1}\frac{1-t^m}{m}p_m(X)p_m(Y))\\ &= \prod_{m\geq 1}\exp(\frac{1-t^m}{m}p_m(X)p_m(Y))\\ &= \prod_{i\geq 1}\sum_{m_i\geq 0}\frac{(1-t^i)^{m_i}}{i^{m_i}m_i!}p_i(X)^{m_i}p_i(Y)^{m_i}\\ &= \sum_\lambda z_\lambda(t)^{-1}p_\lambda(X)p_\lambda(Y)\end{aligned}$$
   where in the last equality we just used the frequency definition of partitions.
2. $\pi(X,Y;t) = \sum_\lambda q_\lambda(X;t)m_\lambda(Y)$.

   To see this, note
$$\begin{aligned}\pi(X,Y;t) &= \prod_{j\geq 1}\frac{H(y_j)}{H(ty_j)}\\ &= \prod_{j\geq 1}\sum_{a_j\geq 0}q_{a_j}(X;t)y_j^{a_j}\\ &= \sum_\lambda q_\lambda(X;t)m_\lambda(Y)\end{aligned}$$
   We note this implies $\langle q_\lambda(X;t), m_\mu(X)\rangle_t = \delta_{\lambda\mu}$.

3. $\pi(X,Y;t) = \sum_\lambda P_\lambda(X;t)Q_\lambda(Y;t) = \sum_\lambda P_\lambda(X;t)P_\lambda(Y;t)b_\lambda(t)$.

By the last equality we just proved, we get a corollary that P_λ and Q_μ are dual basis. Since they are basis, we can talk about structure coefficients.

We know $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$ where $c^\nu_{\lambda\mu}$ are Littlewood Robinson coefficients, i.e. $c^\nu_{\lambda\mu}$ is the number of SSYT of shape ν/λ with content μ such that the reverse reading word is a lattice (i.e. each prefix is a partition).

Similarly, we use $f^\nu_{\lambda\mu}$ to denote the structure coefficient of P_λ and P_μ, i.e. we set

$$P_\lambda(X;t)P_\mu(X;t) = \sum_\nu f^\nu_{\lambda\mu}(t)P_\nu(X;t)$$

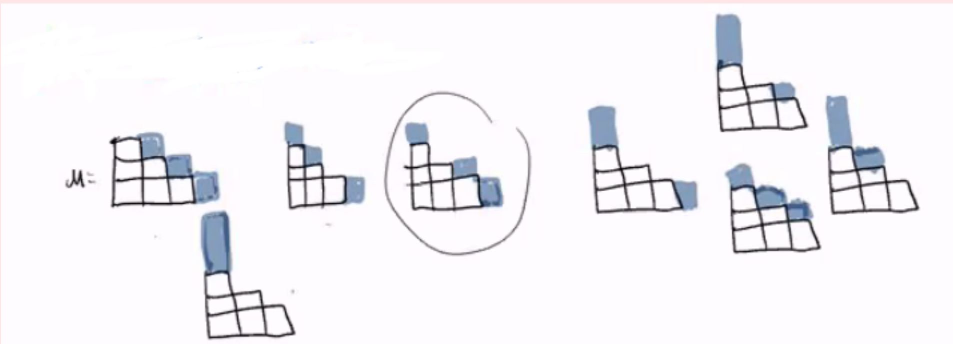
Sadly, these structure coefficients are not always positive.

However, in special cases, they are pretty nice. For example:

$$f^\lambda_{\mu,(1^r)}(t) = \prod_{j\geq 1} \binom{\lambda'_j - \lambda'_{j+1}}{\lambda'_j - \nu'_j}_t$$

if λ/μ is a vertical strip of length r, where the subscript t is t-analog.

Example 5.4. If $\mu = (3,2,1)$ and $r = 3$, then we have the following possible vertical strips



Thus if we take the circled one, we get $\lambda = (4,3,1,1,)$ and hence

$$f^{(4,3,1,1)}_{(3,2,1),(1^3)} = \binom{4-2}{4-3}_t \cdot \binom{2-1}{2-1}_t = \binom{2}{1}_t [1]_t = (1+t)\cdot 1 = 1+t$$

Next, we consider another structure coefficient

$$P_\lambda(X;t)q_r(X) = \sum_{\substack{\mu \\ \lambda/\mu \text{ horizontal strip}}} d^\lambda_{\mu,(r)}(t)P_\mu(X;t)$$

and this time we have a nice special case as follows:

$$d^\lambda_{\mu,(r)} = \prod_{\substack{\lambda'_j = \mu'_j + 1 \\ \lambda'_{j+1} = \mu'_{j+1}}} (1 - t^{m_j(\lambda)})$$

where we recall $m_j(\lambda)$ is the multiplicity of j in λ.

As a special case to the above, we have the Pieri rules for Schur functions

$$s_\lambda s_{(r)} = \sum_{\mu/\lambda \text{ horizontal strip}} s_\mu$$

$$s_\lambda s_{(1^r)} \sum_{\mu/\lambda \text{ vertical strip}} s_\mu$$

Theorem 5.5 (Konvalinka, Lauve,2012)**.**

$$P_\lambda(X;t)s_{(r)} = \sum f_{\mu/\lambda}(t)P_\mu(X;t)$$

where

$$f_{\mu/\lambda}(t) = t^{\sum_j \binom{\lambda_j-\mu_j}{2}} \prod_{j\geq 1} \binom{\lambda_i' - \mu_{j+1}'}{m_j(\mu)}_t$$

Next, we consider expansion of P_λ in s_μ and this is not positive, i.e. we have

$$P_\lambda(X;t) = \sum_\mu c_{\lambda\mu}(t)s_\mu$$

with $c_{\lambda\mu}(t) \in \mathbb{Z}[t]$. On the other hand,

$$s_\lambda = \sum_{\mu\geq\lambda} d_{\lambda\mu}(t)P_\mu(t) = \sum_{\mu\geq\lambda} K_{\lambda\mu}(t)P_\mu(t)$$

where $d_{\lambda\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$. The $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial given by

$$K_{\lambda\mu}(t) = \sum_{T\in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{charge}(T)}$$

where the charge we are going to define as follows.

Definition 5.6. The ***charge*** of $\sigma \in S_n$ is given by

$$\mathrm{charge}(\sigma) := \mathrm{maj}(\mathrm{rev}(\sigma^{-1}))$$

Example 5.7. If $\sigma = 314652$, then $\sigma^{-1} = 261354$, then $\mathrm{rev}(\sigma^{-1}) = 453162$ and hence $\mathrm{maj}(453162) = 2 + 3 + 5 = 10$.

We also have another way to think about charge is as follows. Write down 314652, we read from right to left in order $1, 2, 3, \dots, n$ for each i that we wrap, we add $n - i$ (i.e. we record the i, and if we trigger a warp, we add $n - i$).

For example, we get $(6 - 1) + (6 - 3) + (6 - 4) = 10$ as desired.

6

Last time we introduced charges. This time we will continue this topic, then define Kostka–Foulkes polynomials. In particular, the charge of w can defined as

$$\operatorname{charge}(w) = \sum_{j+1 \text{ right to } j} (n - j)$$

We can also compute charges as follows.

Example 6.1. Suppose we have 631425. Then we read from right to left and look for 1. This gives

$$63\hat{1}425$$

and then we put a bunch of 1's under all the terms besides the 1, where under 1 we put a mark −, i.e. we do

$$\begin{array}{cccccc} 6 & 3 & 1 & 4 & 2 & 5 \\ & & - & & & \end{array}$$

Next, we wrap around and back to the right most, and at this time, we put 1's under all the numbers that's not been marked, i.e. we get

$$\begin{array}{cccccc} 6 & 3 & 1 & 4 & 2 & 5 \\ 1 & 1 & - & 1 & 1 & 1 \end{array}$$

Now, we read from right to left again, and this time we look for 2 and 3 (as they are in order), and put − under the 2. This gives

$$\begin{array}{cccccc} 6 & 3 & 1 & 4 & 2 & 5 \\ 1 & 1 & - & 1 & 1 & 1 \\ & - & & & - & \end{array}$$

Now we wrap around again, and put 1's, i.e. we get

$$\begin{array}{cccccc} 6 & 3 & 1 & 4 & 2 & 5 \\ 1 & 1 & - & 1 & 1 & 1 \\ 1 & - & - & 1 & - & 1 \end{array}$$

Keep doing this, we end up with

$$\begin{array}{cccccc} 6 & 3 & 1 & 4 & 2 & 5 \\ 1 & 1 & - & 1 & 1 & 1 \\ 1 & - & - & 1 & - & 1 \\ 1 & - & - & - & - & 1 \end{array}$$

and so on. The point is, at the end of the day, we add all the 1's appear in the above, and that will be our charge.

Definition 6.2. For permutation $w \in S_n$, we define

$$\operatorname{cocharge}(w) := \binom{n}{2} - \operatorname{charge}(w)$$

Next, we can define charge on words with partition contents (i.e. the content of the word is a partition, i.e. we have λ_i many i in w, with $\lambda_i \geq \lambda_{i+1}$).

Definition 6.3. Given a word w, the charge of it is

$$\operatorname{charge}(w) = \sum_{w_i \text{standard subwords}} \operatorname{charge}(w_i)$$

Note in the above definition, we have what's called standard subwords. We say a word is standard if all letters only appear once. Then a standard subword is just a subword that's standard.

Example 6.4. Suppose we have $w = 2131114233122$. Then my first standard word is 3421 (what we do is start from the right, read to the left, and look for the first 1, then 2, then 3, then warp around and get the 4 and so on). Then the first one has charge $4 - 3$. The second standard subword is 213, which is obtained by cross out the 3421 subword and do ths process again. The whole list of standard subwords are given by $w_1 = 3421, w_2 = 213, w_3 = 123, w_4 = 12, w_5 = 1$.

Definition 6.5. For a SSYT $T \in \operatorname{SSYT}(\mu)$ with $\mu \vdash n$, the charge of T is given by

$$\operatorname{charge}(T) := \operatorname{charge}(w)$$

where w is the reading word of T obtained by read off T by scanning from top to bottom and left to right with each row.

Example 6.6. Say

$$T = \begin{array}{|c|cccc} \hline 3 & \multicolumn{1}{|c}{} & & & \\ \hline 2 & \multicolumn{1}{|c|}{2} & & & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{4} & \\ \hline \end{array}$$

Then the reading word of T is 3221114. Then $\operatorname{charge}(T) = 1$.

Definition 6.7. We define

$$K_{\lambda\mu}(t) = \sum_{T \in \operatorname{SSYT}(\lambda;\mu)} t^{\operatorname{charge}(T)}$$

Theorem 6.8.

$$s_\lambda = \sum K_{\lambda\mu}(t) P_\mu(x;t)$$

Example 6.9. Consider s_{22}. The SSYT are given by

$$T = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline\end{array} \Rightarrow \operatorname{charge} T = 0$$

$$T = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 1 \\ \hline\end{array} \Rightarrow \operatorname{charge} T = 1$$

$$T = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array} \Rightarrow \operatorname{charge} T = 4$$

$$T = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array} \Rightarrow \operatorname{charge} T = 2$$

Hence, we get

$$s_{22} = t^0 P_{22}(x;t) + t^1 P_{211}(x;t) + (t^4 + t^2) P_{1111}(x;t)$$

Proposition 6.10. *Charge is the unique statistics satisfies the following (let x, y be words with partition contents):*

1. $\operatorname{charge}(\varnothing) = 0$.
2. $\operatorname{charge}(yx) = \operatorname{charge}(xy) + 1$ *if* $x \neq 1$.
3. $\operatorname{charge}(y1^m) = \operatorname{charge}(y)$ *if* $1 \notin y$.
4. *If y, y' are Knuth-equivalent then* $\operatorname{charge}(y) = \operatorname{charge}(y')$.
5. *If σ is Lascoux-Schutzenberger involution, then* $\operatorname{charge}(y) = \operatorname{charge}(\sigma y)$

The next topic is transformed H-L polynomials. The motivation for this is that we want a Schur positive version of HL polynomials (which in term has nice interpretation in rep theory, comb, or stat mechanics). For that we need to recall the notion of plethysm.

Suppose $Y = y_1 + y_2 + \ldots = s_1(Y)$. Then for $f \in \Lambda^d$, the plethysm evaluation is given by

$$f[Y] = f(Y) = f(y_1, y_2, \ldots)$$

We also have

$$f[-Y] = (-1)^d (\omega f)[Y]$$

$$f[\frac{Y}{1-t}] = f[Y(1 + t + t^2 + \ldots)] = f(y_1, y_1 t, y_1 t^3, \ldots, y_2, y_2 t, \ldots)$$

Definition 6.11. We defined the ***transformed Hall-Littlewood polynomials*** $H_\lambda(X;t)$ to be

$$H_\lambda(X;t) := Q_\lambda[\frac{X}{1-t};t]$$

We define the ***modified Hall-Littlewood polynomials*** to be

$$\tilde{H}_\lambda(X;t) = t^{\kappa(\lambda)} H_\lambda(X;t^{-1})$$

with $k(\lambda)$ equal the max power of t in $H_\lambda = \sum_j \binom{\lambda'_j}{2}$.

Example 6.12. Well,

$$H_\lambda(X;0) = Q_\lambda(X;0) = s_\lambda(X)$$

$$H_\lambda(X;1) = h_\lambda(X)$$

Proposition 6.13.

$$H_\lambda(X;t) = \sum_\mu K_{\mu\lambda}(t) s_\mu(X)$$

Proof. Consider the Cauchy identity for P_λ, Q_λ. We get

$$\sum_\lambda Q_\lambda(X;t) P_\lambda(X;t) = \prod_{ij} \frac{1 - t x_i y_j}{1 - x_i y_j}$$

Next, we see we get

$$\begin{aligned}
\sum_\lambda H_\lambda(X;t) P_\lambda(Y;t) &= \sum_\lambda Q[\sum_i x_i \sum_i t^i] P_\lambda(Y;t) \\
&= \prod_{i,j,k} \frac{1 - t(x_i t^k) y_j}{1 - (x_i t^k) y_j} \\
&= \prod_{i,j} \prod_{k \geq 0} \frac{1 - x_i y_j t^{k+1}}{1 - x_i y_j t^k} \\
&= \prod_{i,j} \frac{1}{1 - x_i y_j} \\
&= \sum_\lambda s_\lambda(X) s_\lambda(Y)
\end{aligned}$$

Use

$$\begin{aligned}
s_\lambda &= \sum_{\mu \leq \lambda} K_{\lambda\mu}(t) P_\mu(X;t) \\
&= \sum_\lambda s_\lambda(X) \sum_{\mu \leq \lambda} K_{\lambda\mu}(t) P_\mu(Y;t)
\end{aligned}$$

Hence we see

$$\sum_\lambda H_\lambda(X;t) P_\lambda(Y;t) = \sum_\lambda P_\lambda(Y;t) \sum_{\mu \geq \lambda} K_{\mu\lambda}(t) s_\mu(X)$$

Now extract coefficients we get

$$H_\lambda(X;t) = \sum_{\mu \geq \lambda} K_{\mu\lambda}(t) s_\mu(X)$$

♡

Example 6.14. Let $\lambda = (2,1)$. Then $H_{(2,1)}(X;t) = t^0 s_{21} + t^1 s_3$.

Next, what happens if $t = 0$?

Well, we see $H_\lambda(X;t) = s_\lambda(X) = \sum_{\mu\geq\lambda} K_{\mu\lambda}(0) s_\mu(X)$ where we note $K_{\mu\lambda}(0) = 0$ for all $\mu \neq \lambda$.

We will give a combinatorial formula for $\tilde{H}_\lambda(X;t)$.

Definition 6.15. Let T be any filling of λ (not just SSYT). We define:

1. ***Major index***:

$$\operatorname{maj}(T) = \sum_{\substack{u\in T' \\ \text{such that} T'(u)>T'(\text{south}(u))}} \operatorname{leg} u + 1$$

   with $u = (r,j)$ where r is the row and j is the column (then $\text{south}(u) := (r-1,j)$), and

$$\operatorname{leg}(r,j) := \lambda_j - r$$

   (here diagrammatically, the leg of (r,j) is the portion in T' that's above (r,j)).
2. ***Standardization***: the standardization of T is the unique standard tableau τ such that the relative order of entries is preserved, i.e. $T(u) > T(v) \Rightarrow \tau(u) > \tau(v)$ and if $T(u) = T(v)$ and u come before v in the reading word of T, then $\tau(u) < \tau(v)$.

Example 6.16. Let's consider

$$\begin{array}{|c|cc}
\hline
3 \\ \cline{1-1}
2 \\ \cline{1-1}
4 \\ \hline
1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline
3 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline
\end{array}$$

Then for the top 3, it has leg = 0. For the 4 at third row, it has leg equal 2. On the other hand, for the 3 at the fourth row, it has leg 0. In total, the major index of T is 5. Note if the tableaux is semistandard, then the major index is 0.

Example 6.17. Let's consider the standardization of

$$T = \begin{array}{|c|cc}
\hline
3 \\ \cline{1-1}
2 \\ \cline{1-1}
4 \\ \hline
1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline
3 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline
\end{array}$$

To start, we look for our 1, which gives

$$T = \begin{array}{|c|c|c|}
\cline{1-1}
\ & \multicolumn{2}{c}{} \\
\cline{1-1}
\ & \multicolumn{2}{c}{} \\
\hline
1 & & \\
\hline
 & & \\
\hline
\end{array}$$

Then, we look for 2's in T, and the first 2 is still a 2, i.e.

$$T = \begin{array}{|c|c|c|}
\cline{1-1}
\ & \multicolumn{2}{c}{} \\
\cline{1-1}
2 & \multicolumn{2}{c}{} \\
\hline
1 & & \\
\hline
 & & \\
\hline
\end{array}$$

then the second 2 in the reading word becomes a 3, the third 2 in the reading word becomes a 4 and the fourth one is a 5, i.e. we get

$$T = \begin{array}{|c|c|c|}
\cline{1-1}
\ & \multicolumn{2}{c}{} \\
\cline{1-1}
2 & \multicolumn{2}{c}{} \\
\hline
1 & 3 & \\
\hline
 & 4 & 5 \\
\hline
\end{array}$$

Then the first 3 becomes 6, the second 3 becomes 7, the third 3 becomes 8, and the first 4 becomes 9, which gives the standardization as

$$\tau = \begin{array}{|c|c|c|}
\cline{1-1}
6 & \multicolumn{2}{c}{} \\
\cline{1-1}
2 & \multicolumn{2}{c}{} \\
\cline{1-1}
9 & \multicolumn{2}{c}{} \\
\hline
1 & 3 & 7 \\
\hline
8 & 4 & 5 \\
\hline
\end{array}$$

We have one last definition to make, which is the notion of inversion triples.

Definition 6.18. Let T be a filling, consider a ***triple*** of the form $t := \{x := (r, j), y := (r-1, j), z := (r, l) : j < l\}$. Then t is an ***inversion triple*** if we read counterclockwise start from one of the location the entries are increasing, i.e. $T(x) < T(y) < T(z)$, $T(y) < T(z) < T(x)$ or $T(z) < T(z) < T(y)$, and it is with respect to standardization (i.e. if we have repeated entries then we take standardization first).

Example 6.19. Well, say we have

$$T_1 = \begin{array}{|c|c|}
\hline
1 & 3 \\
\hline
2 & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}
\quad
T_2 = \begin{array}{|c|c|}
\hline
2 & 1 \\
\hline
3 & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}
\quad
T_3 = \begin{array}{|c|c|}
\hline
3 & 1 \\
\hline
2 & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}
\quad
T_4 = \begin{array}{|c|c|}
\hline
1 & 2 \\
\hline
3 & \multicolumn{1}{c}{} \\
\cline{1-1}
\end{array}$$

$$T_5 = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad T_6 = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \quad T_7 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Then T_1 is inversion (start with 1 and read CCT), T_2 is inversion (start with 1 and read CCT), T_3 is not an inversion (read CCT you cannot find increasing sequence), T_4 is not an inversion , T_5 is inversion, T_6 and T_7 are not inversion.

In general, if we have two a in the same column and b the other place, i.e. something like

$$\begin{array}{|c|c|}\hline a & b \\ \hline a \\ \cline{1-1}\end{array}$$

then it is inversion. If we have two a in different column, i.e. something like

$$\begin{array}{|c|c|}\hline b & a \\ \hline a \\ \cline{1-1}\end{array}$$

then it is not inversion.

Theorem 6.20.

$$\tilde{H}_\lambda(X;t) = \sum_{\substack{T \in \mathrm{Tab}(\lambda,n) \\ \mathrm{maj}(T)=0}} t^{\mathrm{inv}(T)} x^T$$

where we note $\mathrm{maj}(T) = 0$ *implies columns are weakly decreasing from bottom to top and* $\mathrm{Tab}(\mu, m)$ *is filling of shape* μ' *with entries at most* m.

7

Note last time we defined a bunch of statistics. In particular, we note the major index are defined in terms of T' instead of T, i.e. we need to take transpose.

Example 7.1. Suppose $\lambda = (3, 3, 2, 1)$ is a partition, then the shape of λ is the shape of the partition, i.e.

$$\mathrm{shape}(\lambda) := \begin{array}{|c|c|c|}\cline{1-1} \; \\ \cline{1-2} \; & \; \\ \hline \; & \; & \; \\ \hline \; & \; & \; \\ \hline\end{array}$$

On the other hand, in this course, we will say the diagram of λ as

$$dg(\lambda) := \begin{array}{|c|c|c|c|}\cline{1-2} \; & \; \\ \cline{1-3} \; & \; & \; \\ \hline \; & \; & \; & \; \\ \hline\end{array}$$

Then, we say σ is a filling of λ means a function $\sigma : dg(\lambda) \to \mathbb{Z}_+$

Definition 7.2. For a filling of λ, we define

$$\mathrm{des}(\sigma) := \{u \in dg(\lambda) : \sigma(u) > \sigma(\mathrm{south}(u))\}$$

We also had some concepts from last time, for example, we have

$$\mathrm{leg}((r,j)) = \lambda_j - r$$

and the notion of inversion triples. We note as a convention, we have $\sigma(0,j) = \infty$, i.e. at the most bottom, we say a pair of boxes x, z is inversion if $z < x$ (in such case we say it is degenerate).

Then, we defined the major index

$$\mathrm{maj}(\sigma) = \sum_{w \in \mathrm{des}(\sigma)} \mathrm{leg}(u) + 1$$

Definition 7.3. For a filling σ, we define

$$\mathrm{inv}(\sigma) := \# \text{ inversion triples}$$

and

$$\mathrm{coinv}(\sigma) = \kappa(\lambda) - \mathrm{inv}(\sigma)$$

where $\kappa(\lambda) = \sum_j \binom{\lambda'_j}{2}$.

Last time, we also defined one theorem

$$\tilde{H}_\lambda(X;t) = \sum_{\substack{\sigma : dg(\lambda) \to \mathbb{Z}_+ \\ \mathrm{maj}(\sigma) = 0}} t^{\mathrm{inv}(\sigma)} x^\sigma$$

We have another theorem:

Theorem 7.4.

$$\tilde{H}_\lambda(X;q) = \sum_{\substack{\sigma : dg(\lambda') \to \mathbb{Z}_+ \\ \mathrm{inv}(\sigma) = 0}} q^{\mathrm{maj}(\sigma)} x^\sigma$$

In particular, we note for the transformed HL, we simplify the above theorem as

$$H_\lambda(X;t) = \sum_{\substack{\sigma : dg(\lambda) \to \mathbb{Z}_+ \\ \mathrm{maj}(\sigma) = 0}} t^{\mathrm{coinv}(\sigma)} x^\sigma$$

Definition 7.5. We define the ***modified Macdonald polynomial***

$$\tilde{H}_\lambda(X;q,t) = \sum_{\sigma:dg(\lambda)\to\mathbb{Z}_+} q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)} x^\sigma$$

Its not hard to see if $t = 0$ we get back to $\tilde{H}_{\lambda'}(X;q)$ and if we set $q = 0$ then we get $\tilde{H}_\lambda(X;t)$.

Example 7.6. If $\lambda = (2,1,1)$, then $\lambda' = (3,1)$. Then in $\tilde{H}_\lambda(X;t)$, let's try to find coefficient of m_{211}. To compute this, we just need to pick an arbitrary content that would be in m_{211}, say content 1123. This gives 12 possibilities, but by the $\mathrm{maj}(\sigma) = 0$ restriction, we only have 7 that are good, namely

$$\begin{array}{|c|c|c|}\cline{1-1} 1 \\ \hline 1 & 2 & 3 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\cline{1-1} 1 \\ \hline 1 & 3 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\cline{1-1} 1 \\ \hline 2 & 1 & 3 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\cline{1-1} 1 \\ \hline 3 & 1 & 2 \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\cline{1-1} 1 \\ \hline 2 & 3 & 1 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\cline{1-1} 1 \\ \hline 3 & 2 & 1 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\cline{1-1} 2 \\ \hline 3 & 1 & 1 \\ \hline\end{array}$$

Thus, we get

$$[m_{211}]\tilde{H}_{211}(X;t) = 1 + 2t + 3t^2 + t^3$$

Next, let's compute the coefficient of m_{211} in $\tilde{H}_\lambda(X;q)$ using the second theorem. Again, the content should be 1123 and we get the following valid possibilities:

$$\begin{array}{|c|c|}\cline{1-1} 1 \\ \cline{1-1} 1 \\ \hline 2 & 3 \\ \hline\end{array}\quad \begin{array}{|c|c|}\cline{1-1} 1 \\ \cline{1-1} 1 \\ \hline 3 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|}\cline{1-1} 1 \\ \cline{1-1} 2 \\ \hline 1 & 3 \\ \hline\end{array}\quad \begin{array}{|c|c|}\cline{1-1} 1 \\ \cline{1-1} 3 \\ \hline 1 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|}\cline{1-1} 3 \\ \cline{1-1} 1 \\ \hline 1 & 2 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\cline{1-1} 2 \\ \cline{1-1} 1 \\ \hline 1 & 3 \\ \hline\end{array}\quad \begin{array}{|c|c|}\cline{1-1} 2 \\ \cline{1-1} 3 \\ \hline 1 & 1 \\ \hline\end{array}\quad \begin{array}{|c|c|}\cline{1-1} 3 \\ \cline{1-1} 2 \\ \hline 1 & 1 \\ \hline\end{array}$$

Next, by counting the major index, we get

$$[m_{211}]\tilde{H}_{211} = 1 + 2q + 3q^2 + q^3$$

In particular, we note we have $\tilde{H}_\lambda(X;0) = s_\lambda(X)$. This means we should have bijections between the set of $\sigma : dg(\lambda) \to \mathbb{Z}_+$ with $\mathrm{maj}(\sigma) = 0$, the set of $\sigma : dg(\lambda') \to \mathbb{Z}_+$ with $\mathrm{inv}(\sigma) = 0$, and the set of SSYT of λ.

Let's re-derive the charge formula using the inversion number theorem.

We claim that

$$\tilde{H}_\lambda(X;q) = \sum_{\substack{\sigma:dg(\lambda')\to\mathbb{Z} \\ \mathrm{inv}(\sigma)=0}} q^{\mathrm{maj}(\sigma)} x^\sigma = \sum_\mu s_\mu(x) \sum_{T\in \mathrm{SSYT}(\mu,\lambda)} q^{\mathrm{cocharge}(T)}$$

Let's recall some definitions, namely,

$$H_\lambda(X;t) = \sum_\mu s_\mu K_{\mu\lambda} \quad \tilde{H}(X;t) = \sum_\mu s_\mu \tilde{K}_{\mu\lambda}$$

$$K_{\mu\lambda} = \sum_{T\in\mathrm{SSYT}(\mu,\lambda)} t^{\mathrm{charge}(T)} \quad \tilde{K}_{\mu\lambda} = \sum_{T\in\mathrm{SSYT}(\mu,\lambda)} t^{\mathrm{cocharge}(T)}$$

The way we are going to prove our claim is that, we will come up with a way to write $\mathrm{inv}(\sigma)$ in terms of cocharges.

First, we ask, what structure do we have on fillings with $\mathrm{inv}(\sigma) = 0$.

Example 7.7. Suppose we have σ equal

$$\begin{array}{|c|c|c|c|c|}\cline{1-1} 2 \\ \cline{1-2} 2 & 1 \\ \cline{1-4} 3 & 2 & 1 & 1 \\ \hline 3 & 3 & 2 & 1 & 1 \\ \hline\end{array}$$

Then, we can sort this into a filling τ with $\mathrm{inv}(\tau) = 0$. In particular, we just switch from the bottom to top and eliminate all inversions. In our case, the τ is

$$\begin{array}{|c|c|c|c|c|}\cline{1-1} 2 \\ \cline{1-2} 1 & 2 \\ \cline{1-4} 2 & 3 & 1 & 1 \\ \hline 1 & 1 & 2 & 3 & 3 \\ \hline\end{array}$$

We then standardize this and get

$$\begin{array}{|c|c|c|c|c|}\cline{1-1} 6 \\ \cline{1-2} 1 & 7 \\ \cline{1-4} 8 & 10 & 2 & 3 \\ \hline 4 & 5 & 9 & 11 & 12 \\ \hline\end{array}$$

Next, we define the ***cocharge word*** of σ to be the word obtained by record the row indices of $(12, 11, 10, \ldots, 1)$ (where the bottom is row 1). In our case, we get the cocharge word of σ as

$$112123411223$$

Then we take standard subwords, and compute their cocharges, we would end up with the major index.

We claim $\mathrm{maj}(\sigma) = \mathrm{cocharge}(\mathrm{cw}(\sigma))$ where $\mathrm{cw}(\sigma)$ is the cocharge word of σ. This is because the ith standard subword is a permutation that order the entries in column i. Hence the major index of column i gives, i.e. $\mathrm{maj}(\text{column } i)$ is connected to the descent of the permutation.

Next, for a word w, we define the $RSK(w) := (P(w), Q(w))$ of some shape, where we call $Q(w)$ the recording tableau. If $P(w) = P'(w)$ then they have the same cocharge. Our goal is to show a bijection $\sigma \to (P(\sigma), Q(\sigma))$ where $P(\sigma) \in \mathrm{SSYT}(\mu, \lambda)$ and $Q(\sigma) \in \mathrm{SSYT}(\mu, \nu)$ where ν is the content of σ.

It turns out, if we take its cocharge word, we get the desired bijection, i.e. $\sigma \mapsto \mathrm{cw}(\sigma) \xrightarrow{RSK} (P(\sigma), Q(\sigma))$. Thus $\mathrm{maj}(\sigma) = \mathrm{cocharge}(\mathrm{cw}(\sigma)) = \mathrm{cocharge}(P(\sigma))$, $x^\sigma = x^{Q(\sigma)}$ and $P(\sigma) \in \mathrm{SSYT}(\mu, \sigma)$ for some μ.

Thus, we can rewrite our formula in terms of cocharge and get

$$\tilde{H}_\lambda(X;q) = \sum_{\substack{\sigma: dg(\lambda') \to \mathbb{Z}_+ \\ \mathrm{inv}(\sigma)=0}} q^{\mathrm{maj}(\sigma)} x^\sigma = \sum_\mu \left(\sum_{P \in \mathrm{SSYT}(\mu,\lambda)} q^{\mathrm{cocharge}(P)} \right) \left(\sum_{Q \in \mathrm{SSYT}(\mu)} x^Q \right)$$

where the left sum in product is equal $\tilde{K}_{\mu\lambda}(q)$ and the right sum in product is equal s_μ. Hence we conclude

$$\tilde{H}_\lambda(X;q) = \sum_\mu s_\mu \tilde{K}_{\mu\lambda}(q)$$

8

Today we introduce the full Macdonald polynomials, then we recall the modified Macdonald $\tilde{H}_\lambda(X;q,t)$, and prove they are symmetric via LLT polynomials.

Theorem 8.1.

$$\tilde{H}_\lambda(X;q,t) = \sum_{\sigma: dg(\lambda) \to \mathbb{Z}_+} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)} = \sum_{\sigma: dg(\lambda') \to \mathbb{Z}_+} x^\sigma q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)}$$

Before we show this, let's back up a little bit and define an inner product.

Definition 8.2. We define the q,t inner product

$$\langle \cdot, \cdot \rangle_{qt} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}$$

Definition 8.3. The usual ***Macdonald polynomials*** $P_\lambda(X;q,t)$ is uniquely characterized by:

1. $P_\lambda(X;q,t) = m_\lambda(X) + \sum_{\mu<\lambda} c_{\lambda\mu}(q,t) m_\mu(X)$ with $c_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t)$.
2. $\langle P_\lambda(X;q,t), P_\mu(X;q,t) \rangle_{qt} = 0$ if $\lambda \neq \mu$.

Definition 8.4. The ***modified Macdonald polynomials*** is given by

$$\tilde{H}_\lambda(X;q,t) := t^{n(\lambda)} J_\lambda[\frac{X}{1-t^{-1}};q,t^{-1}]$$

where $t^{n(\lambda)}$ is the maximal power of t in expression $J_\lambda[\frac{X}{1-t^{-1}};q,t^{-1}]$ and $J_\lambda(X;q,t) = f_\lambda(q,t)P_\lambda(X;q,t)$ is the ***integral form***, with $f_\lambda(q,t)$ some constant with variables q,t.

In particular, we have

$$\begin{aligned}
\tilde{H}_\lambda(X;q,t) &= t^{n(\lambda)} J_\lambda[\frac{X}{1-t^{-1}};q,t^{-1}] \\
&= \tilde{f}_\lambda(q,t) P_\lambda[\frac{X}{1-t^{-1}};q,t^{-1}] \\
&= \tilde{f}_\lambda(q,t) P_\lambda(x_1, x_1t^{-1}, x_1t^{-2}, ..., x_2, x_2t^{-1}, ...; q, t^{-1})
\end{aligned}$$

Proposition 8.5. *Here is some properties of $\tilde{H}$:*

1. $\tilde{H}_\lambda(X;q,t) = \sum_{\mu\le\lambda} c_{\lambda\mu}(q,t) m_\mu$ *with* $c_{\lambda\mu}(q,t) \in \mathbb{Z}_+(q,t)$
2. $\tilde{H}_\lambda(X;q,t) = \sum_{\mu\le\lambda} \tilde{K}_{\lambda\mu}(q,t) s_\mu$ *where the* $\tilde{K}_{\lambda\mu}$ *have no combinatorial formula at the moment (i.e. still open problem).*

Proposition 8.6. *The modified Macdonald polynomials $\tilde{H}_\lambda(X;q,t)$ are characterized by the following three axioms:*

1. $\tilde{H}_\lambda[X(1-q);q,t] = \sum_{\mu\ge\lambda} c_{\lambda\mu}(q,t) s_\mu$ *with* $c_{\lambda\mu} \in \mathbb{Q}(q,t)$
2. $\tilde{H}_\lambda[X(1-t);q,t] = \sum_{\mu\ge\lambda'} d_{\lambda\mu}(q,t) s_\mu$ *with* $d_{\lambda\mu} \in \mathbb{Q}(q,t)$
3. $\left\langle \tilde{H}_\lambda, s_{(n)} \right\rangle = 1j$

Why do these axioms imply $\langle P_\lambda, P_\mu \rangle_{qt} = 0$ for $\lambda \ne \mu$?

Note $P_\lambda[\frac{X}{1-t^{-1}};q,t^{-1}]$ is a scalar multiple of $\tilde{H}(X;q,t)$. Then we see $P_\lambda(X;q,t)$ is a scalar multiple of $\tilde{H}(X(1-t);q,t^{-1})$. Now recall a fact that

$$\langle f,g \rangle_{qt} = \left\langle f(X), g[X\frac{1-q}{1-t}] \right\rangle = \left\langle f[X\frac{1-q}{1-t}], g(x) \right\rangle$$

Use this fact, we see

$$\langle P_\lambda, P_\mu \rangle_{qt} = 0 \Leftrightarrow \left\langle P_\lambda(X;q,t), P_\mu[X\frac{1-q}{1-t};q,t] \right\rangle = 0$$

but $P_\mu[X\frac{1-q}{1-t};q,t]$ is a scalar multiple of $\tilde{H}_\mu[X(1-q);q,t]$. On the other hand, $P_\lambda(X;q,t)$ is a scalar multiple of $\tilde{H}_\lambda[X(1-t);q,t]$ and hence

$$\langle P_\lambda, P_\mu \rangle_{qt} = 0 \Leftrightarrow \langle \tilde{H}_\lambda[X(1-t);q,t], \tilde{H}_\mu[X(1-q);q,t] \rangle = 0$$

Now expand the $\tilde{H}_\lambda[X(1-t);q,t], \tilde{H}_\mu[X(1-q);q,t]$ by axiom (1) and (2), we are done.

Theorem 8.7 (Haglund, Haiman, Loehr, 2004).

$$\tilde{H}_\lambda(X;q,t) = C_\lambda(X;q,t)$$

where $C_\lambda(X;q,t) = \sum_{\sigma:dg(\lambda)\to\mathbb{Z}_+} x^\sigma t^{\mathrm{inv}(\sigma)} q^{\mathrm{maj}(\sigma)}$.

Theorem 8.8. $C_\lambda(X;q,t)$ *is symmetric in* x_i*'s.*

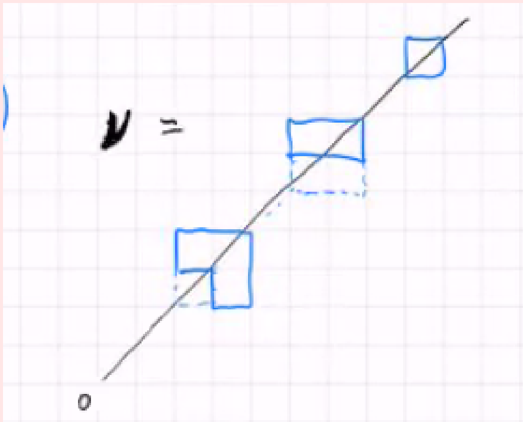
To prove the above theorem, we need to introduce LLT polynomials.

Definition 8.9 (Lascoux-Leclerc-Thibon). The LLT polynomials are indexed as

$$\mathcal{L}_{\boldsymbol{\nu}}$$

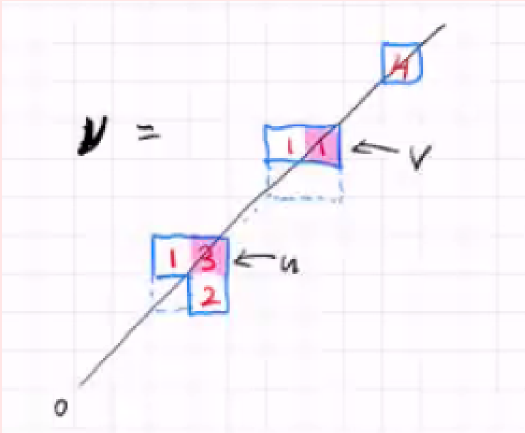
where $\boldsymbol{\nu} = (\lambda_1/\mu_1, ..., \lambda_\ell/\mu_\ell)$ is a tuple of skew shape (λ_i/μ_i). Then, for $\boldsymbol{\nu}$ we define $\mathrm{SSYT}(\boldsymbol{\nu}) = \mathrm{SSYT}(\lambda_1/\mu_1) \times ... \times \mathrm{SSYT}(\lambda_\ell/\mu_\ell)$ in $\mathbb{Z}^2$ via a particular arrangement defined as follows. Also, recall we use $u = (r,c)$ to denote a cell of SSYT where r is the y-coordinate and c the x-coordinate, then we define the diagonal $d(u) := r - c$. Then, the $\mathrm{SSYT}(\lambda_i/\mu_i)$ are arranged from south-west to north-east such that SW-most box of λ_i/μ_i is on diagonal zero.

Here is an example: $\boldsymbol{\nu} = ((2,2)/(1),(2,2)/(2),(2,2)/(2,1))$. Then the diagram of (ν) looks like



Then, $T \in \mathrm{SSYT}((\nu))$ is a semistandard filling of λ_i/μ_i, i.e. $T = (T_1, ..., T_\ell)$ with $T_i \in \mathrm{SSYT}(\lambda_i/\mu_i)$.

Next, we define inversions for $T \in \mathrm{SSYT}((\nu))$. Let $u \in \lambda_i/\mu_i$ and $v \in \lambda_j/\mu_j$ be two cells of T such that $T_i(u) > T_j(v)$. One example of (u,v) would be

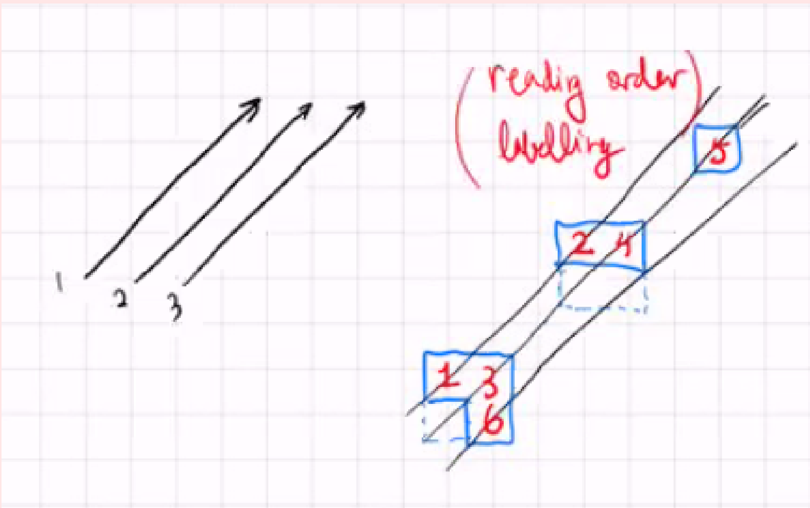


Then we say (u, v) form an ***inversion*** if:

1. $d(u) = d(v)$ and $i < j$, or,
2. $d(u) = d(v) + 1$ and $i > j$

In the example above, we see $(3,1)$ and $(4,2)$ are inversions and hence $\mathrm{inv}(T) = 2$.

Next, we define the reading word on T as reading from the first diagonals upwards to the last diagonal, e.g. we have



Finally, we define

$$\mathcal{L}_{(\nu)}(X;t) = \sum_{T \in \mathrm{SSYT}((\nu))} x^T t^{\mathrm{inv}(T)}$$

where x^T is given by the reading word on T.

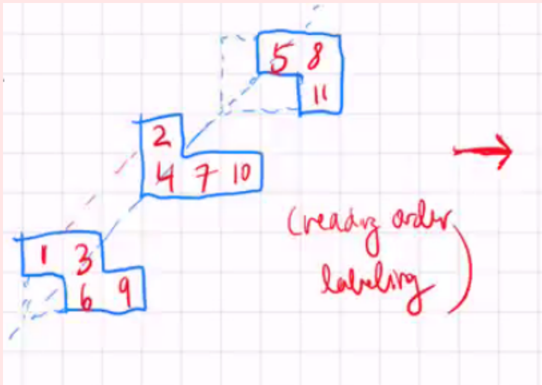
Proposition 8.10. *Some facts about LLT:*

1. $\mathcal{L}_{\boldsymbol{\nu}} \in \Lambda$
2. *they are Schur positive (due to Grojnowski-Haiman in 2007), but there are no known combinatorial expansion for most* $\boldsymbol{\nu}$
3. *we can expand* $\tilde{H}_\lambda(X;q,t)$ *in LLT polynomials.*

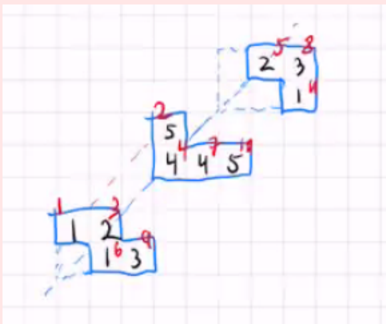
We will consider $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_\ell)$ where $\nu_j = \lambda_j/\mu_j$ are ribbons, i.e. they do not contain

$$(2,2) = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$$

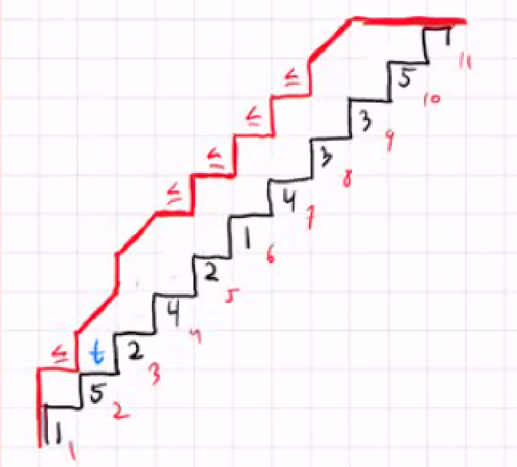
Example 8.11. Consider $\boldsymbol{\nu}$ with reading word labelling as follows



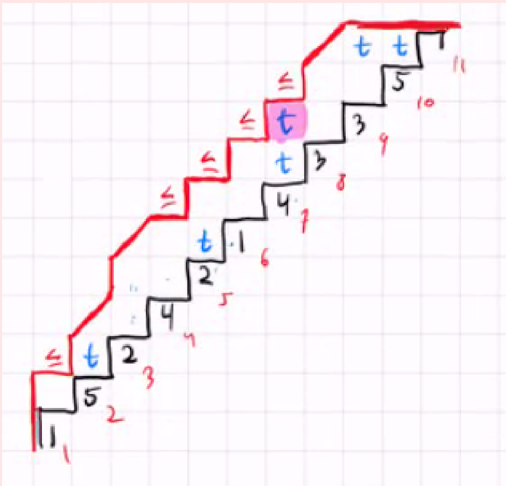
Now take an arbitrary filling



This has reading word 15242143351. Put all of this on the diagonal, we see we can convert $\boldsymbol{\nu}$ to schroder path and fillings become labelling of diagonal with certain restrictions. For example:

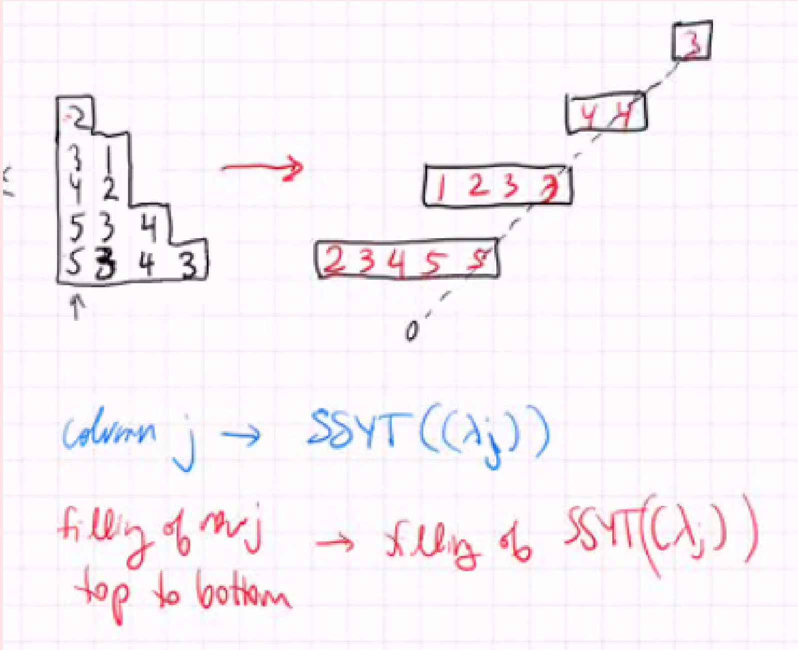


Then, we put a t if we have the bottom number is more than the top number, and we will get the number of t's equal the inversion of T. For example,



We claim that, filling $\sigma : dg(\lambda) \to \mathbb{Z}_+$ with $\mathrm{maj}(\sigma) = 0$ correspond to $\mathrm{SSYT}(\boldsymbol{\nu})$ for $\boldsymbol{\nu}(\nu_1, ..., \nu_k)$ with $\nu_j = (\lambda_j)$ (i.e. the partition with only one part) such that the rightmost box is on diagonal 0.

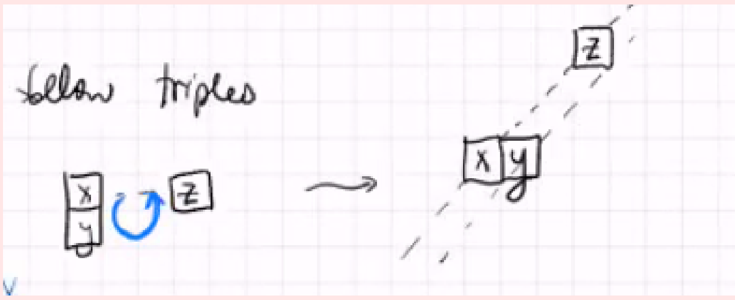
For example,



We denote this by $\boldsymbol{\nu}(\lambda)$ with λ a partition.

Proposition 8.12. *We have a bijection between* $D := \{\sigma : dg(\lambda) \to \mathbb{Z}_+ : \mathrm{maj}(\sigma) = 0\}$ *and* $\mathrm{SSYT}(\boldsymbol{\nu}(\lambda))$*, and we have* $\mathrm{inv}(\sigma) = \mathrm{inv}(T)$ *where* $\sigma \in D$ *and* $T \in \mathrm{SSYT}(\boldsymbol{\nu}(\lambda))$.

Well, we saw the bijeciton, and we look at the inversions. Say we have



Then we see x, y, z is inversion triple iff $z < x$ or $z > y$. But $\mathrm{maj}(\sigma) = 0$ implies $x \leq y$. We know we cannot have both $z < x$ and $z > y$ be true, hence we know exactly one of the (z, y) or (z, x) in the LLT polynomial (i.e. in $\mathrm{SSYT}(\boldsymbol{\nu}(\lambda))$) would form an inversion pair. On the other hand, if x, y, z are not inversion, then we get $z \geq x$ and $z \leq y$, hence neigher (z, y) or (z, x) form inversion in T. This concludes $\mathrm{inv}(\sigma) = \mathrm{inv}(T)$ as desired.

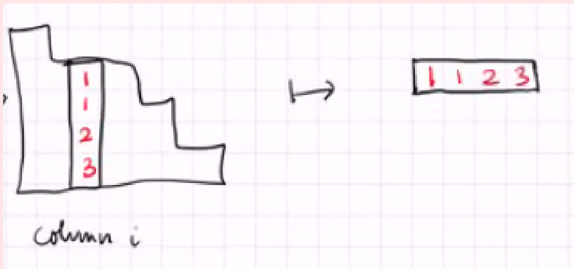
9

We have
$$C_\lambda(X;q,t) = \sum_{\sigma:dg(\lambda)\to\mathbb{Z}_+} x^\sigma q^{\text{maj}(\sigma)} t^{\text{inv}(\sigma)}$$
and we want to show:

1. C_λ is symmetric
2. $C_\lambda[X(1-t);q,t] = \sum_{\mu\geq\lambda'} a_{\mu\lambda}(q,t)s_\mu$
3. $C_\lambda[X(1-q);q,t] = \sum_{\mu\geq\lambda} b_{\mu\lambda}(q,t)s_\mu$
4. $\langle C_\lambda(X;q,t), s_{(n)}(x)\rangle = 1$

Last time we left off with $q=0$ case and get the set $\{\sigma : dg(\lambda)\to\mathbb{Z}_+ : \text{maj}(\sigma) = 0\}$ is in bijection with SSYT($\boldsymbol{\nu}(\lambda)$) (recall those are LLT fillings) and we have $\text{inv}(\sigma) = \text{inv}(T)$. From this we can conclude $C_\lambda(X;0,t) = \mathcal{L}_{(\nu)}(X;t)$.

Now we consinder general q case. When $q=0$, we see we get



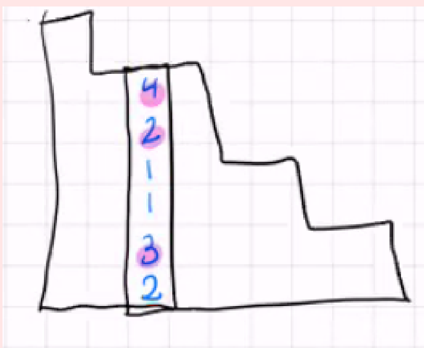
which lies inside SSYT((λ_i)). Next, say we have a triple

x	...	z
y		

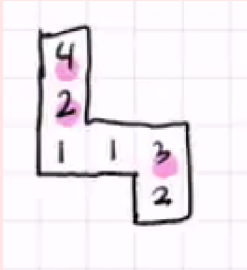
say $(x,y,z) = (a,b,c)$, then $(x,y,z) = (a,b,c)$ is a inversion implies $a > c$ or $c > b$. In particular if its not inversion then $a \leq c \leq b$.

Now let's see how to translate the above to general case.

In this case, suppose we have



we want to get a SSYT filling of a ribbon out of this. This can be obtained by doing the following:



where we go down if we have a descent (here we use $\mathrm{des}(\nu_i)$ to denote $\mathrm{des}(\nu_i) = \{u \in \nu_i : \mathrm{south}(u) \in \nu_i\}$).

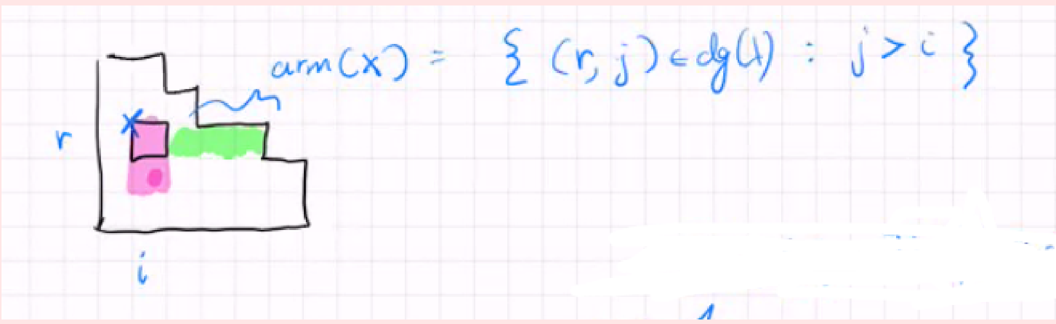
What about inversion triples (x, y, z) labelled as follows:

x	...	z
y		

Case 1: $x \notin \mathrm{des}(\sigma)$. Then (x, y, z) is inversions in σ implies $\{(x, z) \text{ or } (y, z)\}$ contribute 1 to $\mathrm{inv}(T)$. If its not inversion then it contributes 0.

Case 2: $x \in \mathrm{des}(\sigma)$. In this case (x, y, z) is an inversion, then we get $b < c < a$ and hence (x, z) and (y, z) both contribute to inversions and hence we get plus 2. On the other hand, if (x, y, z) is not inversion, then it contributes 1 to $\mathrm{inv}(T)$.

This means, we need to fix our number of inversions, and hence for every triple (x, y, z) with $x \in \mathrm{des}(\sigma)$, we need to subtract 1. However, the descents do not depend on the particular filling but the position. Thus if for x we define $arm(x) = \{(r, j) \in dg(\lambda) : j > i\}$, i.e.



then we get

$$\mathrm{inv}(\sigma) = \mathrm{inv}(T) - \sum_{x \in \mathrm{des}(\sigma)} arm(x)$$

In particular, this means $\mathrm{inv}(\sigma)$ is fixed for a fixed descent set. We use $Inv(\lambda, D)$ to denote $\sum_{x \in \mathrm{des}(\sigma)} arm(x)$.

Possible descent set are $D \subseteq \hat{dg}(\lambda)$, where $\hat{dg}(\lambda)$ is row ≥ 2 of $dg(\lambda)$.

Hence, we see we get

$$\sum_{\sigma \in dg(\lambda) \to \mathbb{Z}_+} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)} = \sum_{D \subseteq \hat{dg}(\lambda)} q^{\mathrm{maj}(\lambda, D)} \sum_{\sigma : dg(\lambda) \to \mathbb{Z}_+, \mathrm{des}(\sigma) = D} x^\sigma t^{\mathrm{inv}(\sigma)}$$

Therefore, we see that

$$C_\lambda(X; q, t) = \sum_{D \subseteq \hat{dg}(\lambda)} q^{\mathrm{maj}(\lambda, D)} t^{-Inv(\lambda, D)} \sum_{\substack{\sigma : dg(\lambda) \to \mathbb{Z} \\ \mathrm{des}(\sigma) = D}} x^\sigma t^{\mathrm{inv}(T(\sigma))}$$

where $\sum_{\substack{\sigma : dg(\lambda) \to \mathbb{Z} \\ \mathrm{des}(\sigma) = D}} x^\sigma t^{\mathrm{inv}(T(\sigma))}$ is equal $\mathcal{L}_{\nu(\lambda)}(X; t)$.

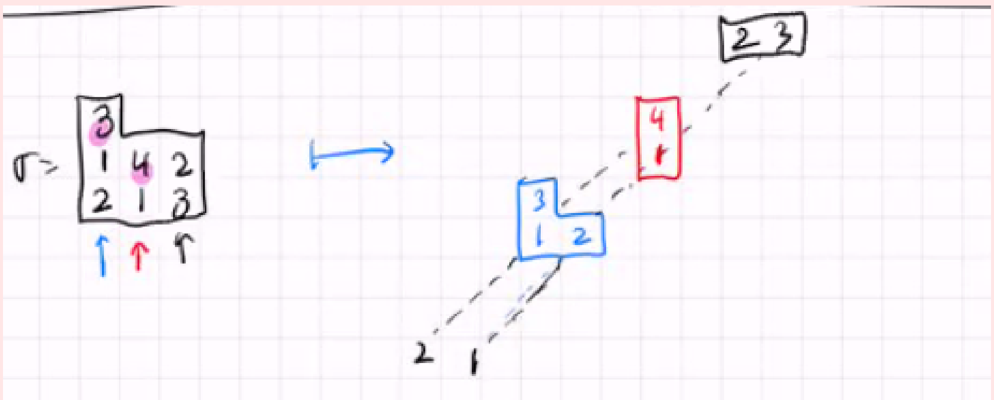
As a corollary, we get the following:

Proposition 9.1. *$C_\lambda(X; q, t)$ is symmetric.*

Example 9.2. If

$$\sigma = \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 1 & 4 & 2 \\ \hline 2 & 1 & 3 \\ \hline \end{array}$$

then it maps to the following



Then, the inversion for σ is equal 2, as $(4, 2)$, $(2, 1)$ and $(2, 1)$ are inversions, but we need to subtract 1 in our case.

Next, we talk aboout Cauchy identity for LLT polynomials:

$$\sum_\nu LLT_\nu(X; t) LLT_{rot(\nu)}(Y, t) = \prod_{ij} \prod_{m=0}^{k-1} \frac{1}{1 - x_i y_j t^m}$$

this is done via vertex model, and superization of non-crossing path.

Next, we talk about the two equations we want to prove:

$$C_\lambda[X(1-t);q,t] = \sum_{\mu \geq \lambda'} a_{\mu\lambda}(q,t)s_\mu$$

$$C_\lambda[X(1-q);q,t] = \sum_{\mu \geq \lambda} b_{\mu\lambda}(q,t)s_\mu$$

First, recall $s_\mu = \sum_{\mu \leq \lambda} m_\mu c_{\lambda\mu}$ and $\mu_\lambda = \sum_{\nu \leq \lambda} s_\lambda d_{\lambda\mu}$ and $\mu \leq \lambda \Leftrightarrow \lambda' \leq \mu'$.

Then, we note note the two equations above holds if and only if

$$C_\lambda[X(1-t);q,t] = \sum_{\mu \leq \lambda} a_{\mu\lambda}(q,t)s_{\mu'} = \sum_{\mu \leq \lambda} a_{\mu\lambda}(q,t) \sum_{\nu' \geq \mu'} c_{\mu'\nu'} m_{\nu'} = \sum_{\nu \leq \lambda} c_{\nu\lambda}(q,t) m_\nu(X)$$

$$C_\lambda[X(1-q);q,t] = \sum_{\nu \geq \lambda} f_{\nu\lambda}(q,t) m_\nu(t)$$

Combinatorially, we see

$$C_\lambda[X(q-1);q,t] = \sum_{\sigma : dg(\lambda) \to \mathbb{Z} \backslash 0} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)}$$

10

The goal for the following lectures will be try to prove the the two formulae

$$C_\lambda[X(q-1);q,t] = \sum_{\mu \leq \lambda'} c_{\lambda\mu} m_\mu$$

$$C_\lambda[X(t-1);q,t] = \sum_{\mu \leq \lambda} d_{\lambda\mu} m_\mu$$

The way we are going to prove it is what's called superization. This is a way to express plethysm using opeartions of tabloids.

We will use quasi-symmetric functions for this. Thus, we first recall some basic notions.

Note a composition of integer n is $\alpha \vdash n$ where $\sum \alpha_i = n$. It is a strong compostion if $\alpha_i > 0$ and it is a weak composition if $\alpha_i \geq 0$. In particular, for weak compostion α we use α^+ to denote the strong composition associated to α.

Then, we define

$$\mathrm{QSym} = \bigoplus_n \mathrm{QSym}^n$$

where $f \in \mathrm{QSym}^n$ means for all weak compostion $\alpha \vdash n$, we get $[x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k}]f$ are all the same, for all $i_1 < i_2 < \dots < i_k$, where k is the length of α.

Example 10.1. If $\alpha = (1,3,1)$ then

$$x_1x_2^3x_3 + x_1x_3^3x_4 + x_1x_2^2x_5 + \ldots \in \mathrm{QSym}^5$$

In particular, this means that $\{M_\alpha : \alpha \text{ weak compostion of } n\}$ will span QSym^n where $M_\alpha = \sum_{i_1<\ldots<i_k} x_{i_1}^{\alpha_1}\ldots x_k^{\alpha_k}$.

Next, we note there is a bijection $\alpha \vdash n$ and $S \subseteq [n-1]$. For example, (4) correspond to $\varnothing$, $(3,1)$ correspond to $\{3\}$, $(2,2)$ correspond to $\{2\}$, $(1,1,1,1)$ correspond to $\{1,2,3\}$, $(2,1,1)$ correspond to $\{2,3\}$, $(1,2,1)$ correspond to $\{1,3\}$ and so on. To think this bijection in the above example, we consider 4 dots, say $\bullet\bullet\bullet\bullet$, then the subset will be the postion of the stick we are putting, so that the corresponding partition of the four dots will be that compostion. For example, the compostion $(1,1,2)$ correspond to $\bullet|\bullet|\bullet\bullet$ and hence the set should be $\{1,2\}$. Similarly, we get $(1,2,1)$ correspond to $\bullet|\bullet\bullet|\bullet$ and hence the set should be $(1,3)$ as the two sticks are at position $1,3$.

This gives the fundamental QS basis

$$F_\alpha = \sum_{\beta\le\alpha} M_\beta = \sum_{\mathrm{set}(\alpha)\subseteq S\subseteq[n-1]} M_S$$

Here $\beta \le \alpha$ iff $\mathrm{set}(\beta) \supseteq \mathrm{set}(\alpha)$.

For example, we get

$$F_{13} = M_{13} + M_{121} + M_{112} + M_{1111}$$

where the above is the same as $F_{\{1\}}$.

Now, we prove

$$s_\lambda = \sum_{\sigma\in\mathrm{SSYT}(\lambda)} x^\sigma = \sum_{T\in\mathrm{SYT}(\lambda)} F_{n,D(t)}(x)$$

where $T \in \mathrm{SYT}(\lambda)$ then

$$D(T) = \mathrm{des}(T) = \{j \in [n-1] : j+1 \text{ is weakly right of } j\}$$

In the above, $F_{n,D(T)}(x)$ will be a new basis for quasi-symmetric.

Let's look at an example.

Example 10.2. Say $\lambda = (4,3,2)$ and

$$T = \begin{array}{|c|c|c|c|} \hline 5 & 8 \\ \cline{1-2} 4 & 7 & 9 \\ \cline{1-3} 1 & 2 & 3 & 6 \\ \hline \end{array}$$

Then, we see 3 is right to 4, 6 is right to 7, 4 is weakly right to 5, and 7 is weakly right to 8, hence they are all descents. Therefore, $D(T) = \{3,4,6,7\}$.

The way we get from SSYT to SYT is that we look at the preimage of standardization of a given SYT, i.e. our $F_{n,D(T)}$ will be all SSYT such that standardization is equal T. Viz,

$$F_{n,D(T)}(x) = \sum_{\sigma \in \mathrm{SSYT}(\lambda), \mathrm{std}(\sigma)=T} x^\sigma$$

Example 10.3. Consider

$$T = \begin{array}{|c|c|c} \hline 6 & 8 & \multicolumn{1}{|c}{} \\ \hline 2 & 5 & \multicolumn{1}{|c|}{7} \\ \hline 1 & 3 & \multicolumn{1}{|c|}{4} \\ \hline \end{array}$$

with $D(T) = \{1, 4, 5, 7\}$. Let's figure out all SSYT with standardization equal T. Well, after some basic enumeration, we get the answers are

$$\begin{array}{ccc} 4 & 5 & \\ 2 & 3 & 4 \\ 1 & 2 & 2 \end{array} \qquad \begin{array}{ccc} 5 & 6 & \\ 2 & 4 & 5 \\ 1 & 2 & 3 \end{array} \qquad \begin{array}{ccc} 4 & 6 & \\ 2 & 3 & 5 \\ 1 & 2 & 2 \end{array} \qquad \begin{array}{ccc} 5 & 7 & \\ 2 & 4 & 6 \\ 1 & 5 & 2 \end{array}$$

$$\begin{array}{ccc} 5 & 6 & \\ 2 & 4 & 5 \\ 1 & 3 & 3 \end{array} \qquad \begin{array}{ccc} 6 & 7 & \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{array} \qquad \begin{array}{ccc} 5 & 7 & \\ 2 & 4 & 6 \\ 1 & 3 & 3 \end{array} \qquad \begin{array}{ccc} 6 & 8 & \\ 2 & 5 & 7 \\ 1 & 3 & 4 \end{array}$$

The way you do it just build from the bottom row and consider all possible cells.

This gives weak compostions as

$$(1, 3, 1, 2, 1) \quad (1, 2, 1, 1, 2, 1) \quad (1, 3, 1, 1, 1, 1) \quad (1, 2, 1, 1, 1, 1, 1)$$

$$(1, 1, 2, 1, 1, 1, 1) \quad (1, 1, 1, 1, 1, 2, 1) \quad (1, 1, 2, 1, 1, 1, 1) \quad (1, 1, 1, 1, 1, 1, 1, 1)$$

We note a packed SSYT means all entries of $\{1, \ldots, \max(T)\}$ are included in the filling. Thus in the above example, all SSYT are packed. Therefore, we get

$$s_\lambda = \sum_{\sigma \in \mathrm{SSYT}(\lambda)} x^\sigma = \sum_{T \in \mathrm{SYT}(\lambda)} F_{n,D(T)}(x) = \sum_{\sigma \in \text{packed SSYT}(\lambda)} M_{\mathrm{content}(\sigma)}(x)$$

Example 10.4. Let's do an example with $\lambda = (2, 1)$. Then, the list of SYT are

$$\begin{array}{|c|c|} \hline 3 & \multicolumn{1}{|c}{} \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 2 & \multicolumn{1}{|c}{} \\ \hline 1 & 3 \\ \hline \end{array}$$

On the left, the descent set is $\{2\}$ and the right descent set is $\{1\}$.

Thus, the left SYT gives $F_{(2,1)} = F_{\{2\}} = M_{21} + M_{111}$ and right SYT gives $F_{(1,2)} = F_{\{1\}} = M_{12} + M_{111}$. THerefore we get $s_\lambda = (M_{12} + M_{21}) + 2M_{111}$. Here we note $m_\lambda = \sum_{p(\alpha)=\lambda} M_\alpha$ where $p(\alpha)$ means the partition correspond to weak compostion α.

Next, we start to consider superization.

Let $\mathcal{A} = \mathbb{Z}_+ \coprod \mathbb{Z}_+$ be the disjoint union of the two sets, say

$$\mathcal{A} = \{1, \overline{1}, 2, \overline{2}, \dots\}$$

We put two orders on $\mathcal{A} \cup \{0\}$, say $<_1$ and $<_2$. Here we get

$$0 <_1 1 < \overline{1} <_1 < 2 <_1 \overline{2} <_1 < \dots$$

$$0 <_2 < 1 <_2 2 <_2 3 < \dots <_2 \overline{3} <_2 \overline{2} <_2 \overline{1}$$

This can be thought as $\mathbb{Z}_+ \coprod \mathbb{Z}_-$.

Definition 10.5. We define $I(a,b) = \begin{cases} a > b, & \text{if } a \neq b \\ 0 & \text{if } a = b \in \mathbb{Z}_+ \\ 1 & \text{if } a = b \in \mathbb{Z}_- \end{cases}$

Note if $a \neq b$ then the expression $a > b$ is a boolean value, i.e. if the claim $a > b$ holds then we return 1 and 0 otherwise.

Therefore, we get a super version of the fundamental quasi-symmetric,

$$\tilde{F}_{n,D}(X,Y) = \sum_{\substack{i_1 \le \dots \le i_n \\ i_j = i_{j+1} \in \mathbb{Z}_+ \Rightarrow j \notin D \\ i_j = i_{j+1} \in \mathbb{Z}_- \Rightarrow j \in D}} z_1 \dots z_n$$

where

$$z_i = \begin{cases} X_i, & \text{if } j \in \mathbb{Z}_+ \\ X_j, & \text{if } j \in \mathbb{Z}_- \end{cases}$$

Well, then it is not hard to see $\tilde{F}(X,0) = F_{n,D}(X)$ and $\tilde{F}_{n,D}(0,Y) = F_{n,[n-1]\setminus D}(Y)$.

Definition 10.6. We define a ***semistandard super tableaux*** to be a filling of λ with $\mathcal{A}$ where for any pair $\boxed{a \mid b}$, we must have $I(a,b) = 0$ and for any pair $\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array}$ we must have $I(a,b) = 1$.

We note in the super case, $\sigma(r,i) = \sigma(r,j)$ then the values are in $\mathbb{Z}_+$ and if $\sigma(r_1,i) = \sigma(r_2,i)$ then the values are in $\mathbb{Z}_-$.

Example 10.7. Suppose

$$\sigma = \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} \overline{3} \\ \cline{1-2} 3 & \overline{3} \\ \cline{1-6} \overline{1} & \overline{3} & 4 & 4 & 4 & \overline{4} \\ \hline 1 & \overline{1} & 2 & 2 & 2 & 3 & \overline{3} \\ \hline \end{array}$$

where we are using the order $<_1$. Then STD would be

12						
7	11					
3	10	13	14	15	16	
1	2	4	5	6	8	9

Here the standardization is given by $\mathrm{STD}(\sigma) : \lambda \to [1, ..., n]$ where, for $a \in \mathbb{Z}_+$ then $\{u \in \lambda : \sigma(u) = a\}$ is increasing in reading order, and for $\overline{a} \in \mathbb{Z}_-$ then $\{u \in \lambda : \sigma(u) = \overline{a}\}$ is decreasing in reading order.

Then, the monomial associated to super fillings are given by

$$X^\mu Y^\nu$$

where $\mu = 1^{\mu_1} 2^{\mu_2} 3^{\mu_3} \ldots$ and $\nu = \overline{1}^{\nu_1} \overline{2}^{\nu_2} \ldots$. Hence, for the filling in the above example, we end up with

$$x_1 x_2^3 x_3^2 x_4^3 y_1^2 y_3^4 y_4$$

Next time we will see that we get

$$\tilde{s}_\lambda(X, Y) := \omega_Y(s_\lambda[X+Y]) = \sum_{T \in \widetilde{\mathrm{SSYT}}_{\mathcal{A}}(\lambda)} (X, Y)$$

Here ω_Y is the fundamental involution in Y-variables. In particular, the $s_\lambda[X+Y]$ is just $\sum_{\mu \subseteq \lambda} s_\mu(X) s_{(\lambda/\mu)'}(Y)$.

11

Last time we left off at definition of superization of f, which is $\tilde{f}(X, Y) := \omega_Y f[X+Y]$.

Proposition 11.1. *If $f(Z) = \sum_{D \subseteq [n-1]} c_D F_D(Z)$ in terms of fundamental quasi-symmetric, then $\tilde{f}(X) = \sum_{D \subseteq [n-1]} c_D \tilde{F}_D(X)$.*

Last time we also have $\tilde{s}_\lambda(X, Y) = \omega_Y s_\lambda[X+Y] = \sum_{\mu \leq \lambda} s_\mu(X) s_{(\lambda \setminus \mu)'}(X)$. In particular, this is the same as $\sum_{\sigma \in \widetilde{\mathrm{SSYT}}(\lambda)} x^\sigma = \sum_{T \in \mathrm{SYT}(\lambda)} \tilde{F}_{\mathrm{des}(T)}(X, Y)$.

Proof. The proof follows from expanding $\tilde{F}$ in terms of s_λ. ♡

Example 11.2. Let $\lambda = (2, 1)$. We compute $\tilde{s}_\lambda$. Well, we know $\tilde{s}_\lambda(X, Y) = \sum_{\widetilde{\mathrm{SSYT}}_{\mathcal{A}}(\lambda)} x^\sigma$. We want to go through ν, μ so that $|\nu| + |\mu| = 3$ and we compute $y^\nu x^\mu$.

If $\nu = \varnothing$, then we just get $s_{21}(X)$.

If $\nu = (1)$ and $\mu = (2)$. Then we get

$$\begin{array}{|c|c|}\cline{1-1} \bar{1} & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}$$

This correspond to $m_1(Y)m_2(X)$.

If $\nu = (1)$ and $\mu = (1,1)$. Then we get

$$\begin{array}{|c|c|}\cline{1-1} \bar{1} & \multicolumn{1}{c}{} \\ \hline 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & \bar{1} \\ \hline \end{array}$$

These two together correspond to $2m_1(Y)m_{11}(X)$.

If $\nu = (2)$ and $\mu = (1)$, we get

$$\begin{array}{|c|c|}\cline{1-1} \bar{1} & \multicolumn{1}{c}{} \\ \hline 1 & \bar{1} \\ \hline \end{array}$$

This correspond to $m_2(Y)m_1(X)$.

If $\nu = (1,1)$ and $\mu = (1)$, we get

$$\begin{array}{|c|c|}\cline{1-1} \bar{2} & \multicolumn{1}{c}{} \\ \hline 1 & \bar{1} \\ \hline \end{array} \qquad \begin{array}{|c|c|}\cline{1-1} \bar{1} & \multicolumn{1}{c}{} \\ \hline 1 & \bar{2} \\ \hline \end{array}$$

This correspond to $m_2(Y)m_1(X)$.

We also have $\nu = (1,1,1)$ and $\nu = (2,1)$.

Proposition 11.3. *For $\lambda \vdash n$, we have*

$$C_\lambda(X;q,t) = \sum_{\sigma : dg(\lambda)\to\mathbb{Z}_+} q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)} x^\sigma = \sum_{T\in\mathrm{std}(dg(\lambda)\to[n])} q^{\mathrm{maj}(T)} t^{\mathrm{inv}(T)} F_{\mathrm{des}(T)}(X)$$

We remark that, in this course we defined few notions of descent. FIrst, we have $D_{\mathrm{maj}}(T) = \{u \in dg(\lambda) : T(u) > T(\mathrm{south}(u))\}$ and in this case $\mathrm{maj}(T) = \sum_{u\in D_{\mathrm{maj}}(T)}(\mathrm{leg}(u)+1)$.

On the other hand, we have

$$\mathrm{des}(T) = \{j : j+1 \text{ precedes } j \text{ in reading word of } T\}$$

Recall we use $\mathrm{rw}(T)$ to denote the reading word of T.

We also remark that, $\sum_{T\in\mathrm{std}(dg(\lambda)\to[n])} q^{\mathrm{maj}(T)} t^{\mathrm{inv}(T)} F_{\mathrm{des}(T)}(X)$ is the same as a sum over permutations, i.e. we get

$$\sum_{T\in\mathrm{std}(dg(\lambda)\to[n])} q^{\mathrm{maj}(T)} t^{\mathrm{inv}(T)} F_{\mathrm{des}(T)}(X) = \sum_{\sigma\in S_n, \mathrm{rw}(T)=\sigma} q^{\mathrm{maj}(T)} t^{\mathrm{inv}(T)} F_{\mathrm{des}(T)}(X)$$

Example 11.4. Suppose $\lambda = (2,1)$. Then we look at the permutations. We have six permutations, $123, 213, 231, 132, 321, 312$. Their corresponding descents are

123	213	231	132	321	312
$\varnothing$	1	1	2	1, 2	2

This also gives their corresponding T as (in the above order)

$$\begin{array}{|c|c|}\hline 1 & \\ \hline 2 & 3 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 2 & \\ \hline 1 & 3 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 2 & \\ \hline 3 & 1 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 1 & \\ \hline 3 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 3 & \\ \hline 1 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|}\hline 3 & \\ \hline 1 & 2 \\ \hline\end{array}$$

Further computation gives their maj and inv as follows

	123	213	231	132	321	312
maj	0	1	0	0	1	1
inv	0	0	1	1	1	0

Hence we get their corresponding F as

123	213	231	132	321	312
$F_\varnothing$	$F_{\{1\}}(q+t)$	$F_{\{1\}}(q+t)$	$F_{\{2\}}(q+t)$	$qtF_{\{1,2\}}$	$F_{\{2\}}(q+t)$

Definition 11.5. We define

$$\tilde{C}_\lambda(X,Y;q,t) = \sum_{\sigma:dg(\lambda)\to\mathcal{A}} q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)} Z^\sigma$$

where $Z_j = \begin{cases} x_j, & j \in \mathbb{Z}_+ \\ y_j, & j \in \mathbb{Z}_- \end{cases}$.

We have some basic properties need to check:

1. $\tilde{C}_\lambda(Xq, -X; q,t) = \sum_{\mu\le\lambda'} c_{\lambda\mu} m_\mu$.
2. $\tilde{C}_\lambda(Xt, -X; q,t) = \sum_{\mu\le\lambda} d_{\lambda\mu} m_\mu$.

To this end, we define

$$I(a,b) = \begin{cases} 0, & a < b \text{ or } a = b \in \mathbb{Z}_+ \\ 1, & a > b \text{ or } a = b \in \mathbb{Z}_- \end{cases}$$

Then we get

$$D_{\mathrm{maj}}(\sigma) := \{u \in dg(\lambda) : I(\sigma(u), \sigma(\mathrm{south}(u))) = 1\}$$

and we also have $\mathrm{maj}(\sigma) = \sum_{u\in D_{\mathrm{maj}}} \mathrm{leg}(u) + 1$. We also define inversion $\mathrm{inv}(\sigma)$. Suppose we have

$$\begin{array}{|c|c|c|}\hline x & \dots & z \\ \hline y & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

Then we say this triple is inversion if $I(x,z) + I(z,y) - I(x,y) = 1$. On the other hand, if it is equal 0 then we say it is coinversion.

Example 11.6. For example, if

$$\begin{array}{|c|c|c|c|}\hline 3 & \dots & 2 & \\ \hline & 1 \\ \cline{1-2}\end{array}$$

Then we get $1 + 1 - 1 = 1$, hence it is inversion. On the other hand,

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$$

gives $0 + 1 - 1 = 0$, which is not inversion.

Next, consider

$$\begin{array}{|c|c|}\hline 3 & \overline{2} \\ \hline \overline{2} \\ \cline{1-1}\end{array}$$

Pick the ordering $<_2$: $1 < 2 < 3 < \dots < \overline{2} < \overline{1}$. Then the above one has $0 + 1 - 0 = 1$, which shows it is inversion. On the other hand, consider

$$\begin{array}{|c|c|}\hline \overline{2} & \overline{2} \\ \hline \overline{2} \\ \cline{1-1}\end{array}$$

This gives $1 + 1 - 1 = 1$, so it is also inversion.

Thus, let $\tilde{C}_\lambda(X, Y) = \sum_\sigma q^{\mathrm{maj}} t^{\mathrm{inv}} Z^\sigma$. We define $p(\sigma) = \{u : \sigma(u) \in \mathbb{Z}_+\}$ and $m(\sigma) = \{u : \sigma(u) \in \mathbb{Z}_-\}$.

Then
$$\tilde{C}_\lambda(Xq, -X) = \sum_{\sigma:\mathrm{dg}(\lambda)\to\mathcal{A}} q^{\mathrm{maj}(\sigma)+p(\sigma)} t^{\mathrm{inv}(\sigma)} (-1)^{m(\sigma)} x^{|\sigma|}$$

Similarly we get

$$\tilde{C}(tX; -X) = \sum_{\sigma:\mathrm{dg}(\lambda)\to\mathcal{A}} q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)+p(\sigma)} (-1)^{m(\sigma)} x^{|\sigma|}$$

We will define an involution that kills everything.

Consider order $1 < \overline{1} < 2 < \overline{2} < \dots$. Then we say two cells (u, v) are attacking cells if v is to the right of u, or v is to the right of u and one row above u.

Then we define ϕ as $\phi(\sigma) = \sigma$ if there does not exists (u, v) such that $|\sigma(u)| = |\sigma(v)|$ (i.e. they are non-attacking).

Suppose the content of σ is $1^{\mu_1} 2^{\mu_2} \dots$, with $\mu_i \geq \mu_{i+1}$. Then we see $\mu_1 \leq \lambda_1$, $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$ and so on. Thus $\mu \leq \lambda$.

Otherwise, if we have attacking cells, take $a \in \mathbb{Z}_+$ minimal so there exists (u, v) with $|\sigma(u)| = |\sigma(v)| = a$. Let V be last cell in reading order in attacking pair, which equal $\pm a$. Let x be the last cell in reading word that attacks V. Then we define $\phi(\sigma)$ as

$$\begin{cases} \sigma(x) \mapsto \overline{\sigma(x)} & \\ \sigma(y) \mapsto \sigma(y) & \text{if } y \neq x \end{cases}$$

Example 11.7. Suppose we have

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \hline 2 & \bar{1} & \bar{1} & 3 \\ \hline \bar{1} & 3 & 2 & 2 \\ \hline \end{array} \xrightarrow{\phi} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & \bar{\mathbf{1}}3, \bar{1}2 & 33 \\ \hline \end{array}$$

Here the second 1 in the second row is x in the above definition, and v is is the $\bar{1}$ in the bottom-most row and left-most col.

We want to show that

$$q^{\mathrm{maj}(\phi(\sigma))} t^{\mathrm{inv}(\phi(\sigma)+p(\phi(\sigma)))} = q^{\mathrm{maj}(\sigma)} t^{\mathrm{inv}(\sigma)+p(\sigma)}$$

where

$$p(\phi(\sigma)) = p(\sigma) + \begin{cases} 1, & \sigma(x) \in \mathbb{Z}_- \\ -1, & \sigma(x) \in \mathbb{Z}_+ \end{cases}$$

Major index: say we have

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline c \\ \hline \end{array}$$

with b be x in the definition. Then this maps to

$$\begin{array}{|c|}\hline b \\ \hline \bar{a} \\ \hline c \\ \hline \end{array}$$

If $|b| \neq |a|$, we get $I(b,a) = I(b,\bar{a})$, $I(a,a) = 0 = I(a,\bar{a})$, $I(a,c) = I(\bar{a},c)$ and $I(\bar{a},a) = 1 = I(\bar{a},\bar{a})$. This shows the major index is preserved. The proof for inversion is similar.

To prove the other equality about $\tilde{C}_\lambda(Xq,-X)$, we define involution $\psi(\sigma)$ as follows. We let $\psi(\sigma) = \sigma$ if there does not exists $(r,j) \in \mathrm{dg}(\lambda)$ such that $|\sigma(r,j)| < r$. In this case, if content of σ is $1^{\mu_1} 2^{\mu_2} \ldots$, then we see $\mu_1 \leq \lambda_1'$, $\mu_1 + \mu_2 \leq \lambda_1' + \lambda_2'$ and so on. Thus $\mu \leq \lambda'$.

Otherwise, we let $a \in \mathbb{Z}_+$ be minimal so $|\sigma(r,j)| = a$ and $r > a$. Let x be the 1st in reading order with $|\sigma(x)| = a$ and we define $\psi : \sigma(x) \mapsto \overline{\sigma(x)}$.

An example would be that

$$\psi : \begin{array}{|c|}\hline b \\ \hline a \\ \hline c \\ \hline \end{array} \mapsto \begin{array}{|c|}\hline b \\ \hline \overline{a} \\ \hline c \\ \hline \end{array}$$

where x is the middle cell. In this case, $I(b,a) = 1 - I(b,\overline{a})$ if $|b| \neq |a|$. Thus we will lose one descent. In other word, before b is a descent and now $\overline{a}$ is a descent. This means the major index is changed by one.

This concludes the proof.

12

This time we will use $\tilde{C}_\lambda$ to study P_λ. Recall that we defined inner product $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$. We note that if $q = 0$ then we get back $\langle \cdot, \cdot \rangle_t$, and if $q = t$ then we get back $\langle \cdot, \cdot \rangle$.

Then, our P_λ's are defined via:

1. $P_\lambda(X; q, t) = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu}(q,t) m_\mu$ where $C_\lambda(q,t) \in \mathbb{Q}(q,t)$.
2. $\langle P_\lambda, P_\mu \rangle = 0$ if $\lambda \neq \mu$.

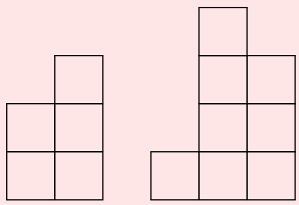
Now, because how $\langle \cdot, \cdot \rangle$ specializes when $q = 0$ and $q = t$, we get $P_\lambda(X; q, q) = s_\lambda(X)$ and $P_\lambda(X, 0, t) = P_\lambda(X; t)$ where the latter one is Hall-Littlewood.

We also have the integral form, which is

$$J_\lambda(X; q, t) = b_\lambda(q,t) P_\lambda(X; q, t)$$

where $b_\lambda(q,t)$ is some constant.

To get this, we note that if $\alpha \vDash n$ is a composition (not just partition), then $dg(\alpha)$ is a diagram with column size α_i, from left to right. For example

$dg(230143) =$ 

Then:

1. $leg((r,j)) = \alpha_j - r$ is the boxes above (r,j).
2. $arm((r,j)) = |\{(r,j') : j' > j, \lambda'_j \leq \lambda_j\}| + |\{(r-1, j'') : j'' < j, \lambda''_j < \lambda_j\}|$

Then, if $\alpha \vdash n$, $u \in dg(\alpha)$, we define $arm(u)$ to be the right arm. Then we get

$$b_\lambda(q,t) = \prod_{u \in dg(\lambda)} (1 - q^{leg(u)} t^{arm(u)+1})$$

Then, we get

$$\tilde{H}_\lambda(X;q,t) = t^{n(\lambda)} J_\lambda[X/(1-t^{-1});q,t^{-1}]$$

where $n(\lambda) = \sum_{i=1}^{\ell(\lambda)} (i-1)\lambda_i$.

Then, we get

$$J_\lambda(X;q,t) = t^{n(\lambda)} \tilde{H}_\lambda[X(1-t);q,t^{-1}] = t^{n(\lambda)+n} \tilde{H}_\lambda[X(t^{-1}-1),q,t^{-1}]$$

where $\tilde{H}_\lambda[X(t^{-1}-1),q,t^{-1}]$ is just superfillings $dg(\lambda) \to \mathcal{A}$ with $<_1$: $1 < \overline{1} < \ldots$ and entries non-attacking.

Thus we get the following combinatorial formula

$$J_\lambda(X;q,t) = (1-t)^n \sum_{\substack{\sigma: dg(\lambda) \to \mathbb{Z}_+ \\ \text{non-attacking}}} x^\sigma t^{n(\lambda)-\text{inv}(\sigma)} q^{\text{maj}(\sigma)} \prod_{\substack{u \in dg(\lambda) \\ \sigma(u) = \sigma(\text{south}(u))}} \frac{1 - q^{leg(u)+1} t^{arm(u)+1}}{1-t}$$

We compute $P_{(2,1)}(X;q,t) = m_{21} + c m_{111}$. The coefficient for m_{21} is given by

$$\begin{array}{|c|c}\hline 1 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \qquad \begin{array}{|c|c}\hline 1 & \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline\end{array}$$

Note we have another way to compute coinversion, which is that, if we have $\begin{array}{|c|c|c|}\hline x & \ldots & z \\ \hline y & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$, then it is a coinversion if we get $y < x < z$, or $x < z < y$, or $z < y < x$.

Therefore, we see the $\begin{array}{|c|c}\hline 1 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$ gives $\frac{t(1-qt)}{(1-t)}$ and $\begin{array}{|c|c}\hline 1 & \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline\end{array}$ gives 1. Also, we get

$$b_{21}(q,t) = (1-t)^2(1-qt^2)$$

and hence, the coefficient is equal to

$$\frac{(1-t)^3(\frac{t(1-qt)}{(1-t)} + 1)}{b_{21}(q,t)} = 1$$

Next, we do the coefficient of m_{111}. We get 6 possibilities, say

$$\begin{array}{|c|c}\hline 1 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 1 & \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 2 & \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}$$

Hence, we get the coefficient of m_{111} should equal to

$$\frac{(1-t)^3}{(1-t)^2(1-qt^2)} \cdot (t + 1 + qt + 1 + q + qt) = \frac{(2+q+t+2qt)(1-t)}{1-qt^2}$$

This computation is not very efficient. Hence, we want better formula. To get a compressed formula, we need to define non-symmetric Macdonald polynomials.

Definition 12.1. Let $\alpha \vDash n$ be a composition. Then define $E_\alpha \in Q[X](q,t)$ by the following:

1. $E_\alpha = x^\alpha + \sum_{\beta<\alpha} c_{\alpha\beta} x^\beta$
2. $\langle E_\alpha, E_\beta \rangle_{qt} = 0$

In particular, this E_α is related to P_λ as follows:

$$P_\lambda(x_1, \dots, x_n; q, t) = E_{(\lambda, 0^n)}(x_1, \dots, x_n, 0, \dots, 0; q, t)$$

We can also compute P_λ using symmetrization of E_α: we get

$$P_\lambda(x_1, \dots, x_n; q, t) = c_\lambda(q,t) \sum_{\alpha \in S_n \lambda} \frac{E_\alpha(X; q^{-1}, t^{-1})}{\prod_{u \in dg(\alpha)} 1 - q^{leg} t^{arm}}$$

THen, we define $NAF(\alpha)$ to be non-attacking fillings of $dg(\alpha)$. Then, for $\alpha \vDash n$ and $\tau \in S_n$ we define $NAF_\tau(\alpha) = NAF(\alpha)$ with basement τ (recall basement is just an extra row at the bottom of the diagram).

Theorem 12.2 (HHL).

$$E_\alpha^T(X; q, t) = \sum_{\sigma \in NAF_\tau(\alpha)} x^\sigma q^{\mathrm{maj}(\sigma)} t^{\mathrm{coinv}(\sigma)} \prod_{\substack{x \in dg(\sigma) \\ \sigma(x) \neq \sigma(\mathrm{south}(x))}} \frac{(1-t)}{(1 - q^{leg(x)+1} t^{arm(x)+1})}$$

Note in the above, x is in the diagram, not in the basement.

Example 12.3. We compute E_{012}^{123}. In this case we get

$$\begin{array}{ccc} & & 1 \\ & 2 & 3 \\ 1 & 2 & 3 \end{array} \qquad \begin{array}{ccc} & & 3 \\ & 2 & 3 \\ 1 & 2 & 3 \end{array}$$

Hence, we see

$$E_{012}^{123} = x_2 x_3^2 + \frac{(1-t)}{1 - qt^2} x_1 x_2 x_3$$

We compute E_{120}^{321}. In this case we get

$$\begin{array}{ccc} & 1 & \\ 3 & 1 & \\ 3 & 2 & 1 \end{array} \qquad \begin{array}{ccc} & 2 & \\ 3 & 1 & \\ 3 & 2 & 1 \end{array} \qquad \begin{array}{ccc} & 2 & \\ 1 & 2 & \\ 3 & 2 & 1 \end{array} \qquad \begin{array}{ccc} & 3 & \\ 1 & 2 & \\ 3 & 2 & 1 \end{array}$$

$$
\begin{ytableau} \none & 3 \\ 2 & 1 \\ 3 & 2 & 1 \end{ytableau}
\qquad
\begin{ytableau} \none & 1 \\ 2 & 1 \\ 3 & 2 & 1 \end{ytableau}
\qquad
\begin{ytableau} \none & 2 \\ 3 & 2 \\ 3 & 2 & 1 \end{ytableau}
\begin{ytableau} \none & 1 \\ 3 & 2 \\ 3 & 2 & 1 \end{ytableau}
$$

We observe that,

$$P_\lambda = \sum_{\sigma \in NAF(\lambda)} \mathrm{wt}(\sigma) = \sum_{\substack{\sigma \in NAF(\alpha) \\ \alpha \in S_n \lambda}} \mathrm{wt}(\alpha)$$

Therefore,

$$P_\lambda(X;q,t) = \sum_{\substack{\tau \in S_n : \tau \cdot \alpha = inc(\lambda) \\ \tau \text{ longest} \\ \alpha \in S_n \lambda}} E^{\tau}_{inc(\lambda)}(X;q,t)$$

In particular, we just want those permutations so that if we have equal height column of the diagram then entries of the permutation must be decreasing.

13

We have our $\langle \cdot, \cdot \rangle_{qt}$ which is defined in terms of power sum basis $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda \prod \frac{1-q^\lambda}{1-t^\mu}$. However, we forget the define the Cauchy identity...

Definition 13.1. We define

$$\pi(X,Y;q,t) = \prod_{ij} \frac{(tx_i y_j : q)_\infty}{(x_i y_j ; q)_\infty}$$

where we define $(a;b)_r = \prod_{k=0}^{\infty}(1-ab^k)$ and if $r = \infty$ we just take infinite product.

In particular, we see:

1. $(tx_i y_j ; q)_\infty = \prod_{k \geq 0}(1 - tx_i y_j q^k)$
2. $(tx_i y_j ; 0)_\infty = \prod_{k \geq 0}(1 - tx_i y_j 0^k) = 1 - tx_i y_j$
3. $(x_i y_j ; 0) = 1 - x_i y_j$

Theorem 13.2. *Two bases $\{u_\lambda\}, \{v_\lambda\}$ for $\Lambda \otimes \mathbb{Q}(q,t)$ satisfy $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$ if and only if $\sum_\lambda u_\lambda(X) v_\lambda(Y) = \pi(X,Y;q,t)$.*

Theorem 13.3.

$$\sum_\lambda \frac{p_\lambda(X) p_\lambda(Y)}{z_\lambda(q,t)} = \pi(X,Y;q,t)$$

where $z_\lambda(q,t) = z_\lambda \prod \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$.

Theorem 13.4.

$$\sum_\lambda P_\lambda(X;q,t)Q_\lambda(Y;q,t) = \pi(X,Y;q,t)$$

where $Q_\lambda(X;q,t) = b_\lambda(q,t)^{-1}P_\lambda(X;q,t)$ *and*

$$b_\lambda(q,t) = \langle P_\lambda(X;q,t), P_\lambda(X;q,t)\rangle_{qt} = \prod_{u\in dg(\lambda)} \frac{1-q^{leg(u)+1}t^{arm(u)}}{1-q^{leg(u)}t^{arm(u)+1}}$$

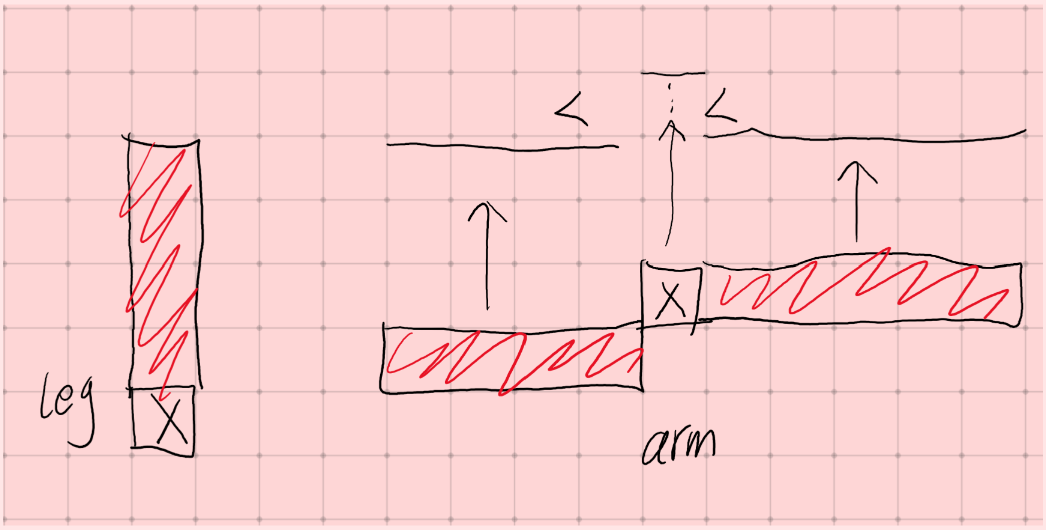
Last time we also had a definition which we will recall now:

Definition 13.5. We define E_α via the following properties:

1. $E_\alpha = x^\alpha + \sum_{\beta <_{st} \alpha} c_{\alpha\beta}x^\beta$
2. $\langle E_\alpha, E_\beta\rangle_{qt} = 0$ if $\alpha \neq \beta$.

Note here $<_{st}$ is Bruhat order.

Last time we also had formulae using tableaux. In particular, recall we defined *leg* and *arm* for arbitrary compostions. In terms of drawing, we get



where on the right, the arm only cares about cells that has lower column than itself.

Then, we have type A inversion, which is, for $\lambda_i \leq \lambda_j$ we read the following

$$\begin{array}{|c|c|c|}\hline x & \dots & z \\ \hline y & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

clockwise, and for $\lambda_i < \lambda_j$ we read the following

$$\begin{array}{ccc} \multicolumn{2}{c|}{} & \multicolumn{1}{c|}{x} \\ \hline \multicolumn{1}{|c|}{z} & \dots & \multicolumn{1}{c|}{y} \\ \hline \end{array}$$

counterclockwise. It is a coinversion if cyclically we have $x < z < y$.

Then, the formula for E_α, with α weak composition, is given by

$$E_\alpha(x_1, ... x_m; q, t) = \sum_{\substack{\sigma \in NAF(\alpha,[n]) \\ \text{basement=Id}}} x^\sigma q^{\operatorname{maj}\sigma} t^{\operatorname{coinv}\sigma} \prod_{\substack{u \in dg(\alpha) \\ \sigma(u) \neq \sigma(\text{south}(u))}} \frac{1-t}{1-q^{leg(u)+1} t^{arm(u)+1}}$$

In particular,

$$P_\lambda = \sum_{\substack{\sigma \in NAF(\alpha,[n]) \\ \text{basement=Id}}} x^\sigma q^{\operatorname{maj}\sigma} t^{\operatorname{coinv}\sigma}$$

and,

$$P_\lambda = \left(\sum_{\text{sort}(\alpha)=\lambda} \frac{E_\alpha}{\prod_{u \in dg(\alpha)} 1 - q^{leg(u)} t^{arm(u)+1}} \right) \prod_{u \in dg(\lambda)} (1 - q^{leg(u)} t^{arm(u)+1})$$

Theorem 13.6. *For all $\alpha \in S_n \cdot \lambda$, we have*

$$P_\lambda(x_1, ..., x_n; q, t) = E_{(\alpha, 0^n)}(x_1, ..., x_n, 0, ..., 0; q, t) \cdot \frac{\prod_{x \in \alpha} 1 - q^{leg(x)+1} t^{arm(x)+1}}{\prod_{x \in \alpha} 1 - q^{leg(x)} t^{arm(x)+1}}$$

In particular, if we set $q, t = 0$, then we get Demazure atoms, i.e. $E_\alpha(X; 0, 0) = \mathcal{A}_\alpha(X)$. For $E_\alpha^\sigma(X; q, t)$, if we set $\sigma = \omega_0 = (n, n-1, ..., 1)$, $q = t = 0$ and $x_j \mapsto x_{n-j+1}$, then we get the key polynomial $K_\alpha(X)$.

We also have $\sum_{\text{sort}(\alpha)=\lambda} E_\alpha^{\omega_0}(X; 0, 0) = s_\lambda$.

We also define $\partial_i = \frac{1-s_i}{x_i - x_{i+1}}$ and then $T_i = (1-t)(\partial_i x_i - 1) + s_i = t - \frac{t x_i - x_{i+1}}{x_i - x_{i+1}}$. This satisfies $(T_i - 1)(T_i - t) = 0$ and the braid relation $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$.

Then, T_i permutes the basement by

$$T_i E_\alpha^\sigma = E_\alpha^{\sigma s_i} \times \begin{cases} t \\ 1 \end{cases}$$

14

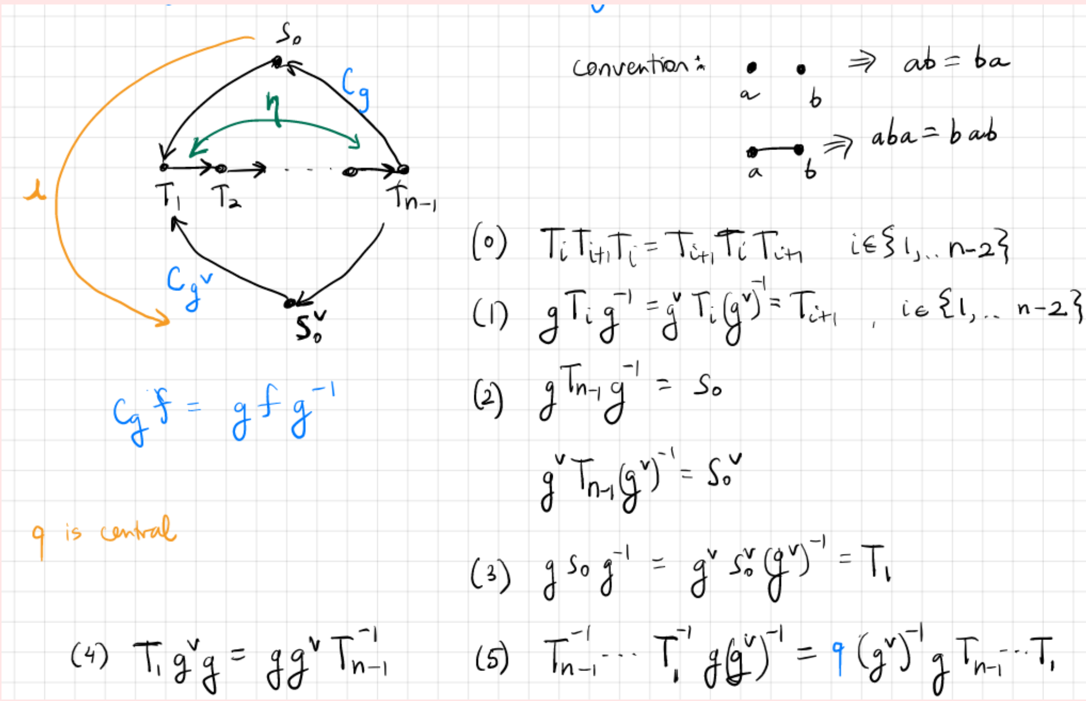
This class we start with “doubly affine” stuff(i.e. DAART and DAHA).

We start with doubly affine Artin group (DAART), then we talk about doubly affine Hecke algebra (DAHA). The point of this is that DAHA has polynomial generators $X_1, ..., X_n$ and $Y_1, ..., Y_n$ where the Y_i are Cherednik-Dunki operators, then the E_μ's are simultaneous eigenvectors of the Y_i's.

We note the all the things we are going to define are of type GL_n.

We start with DAART. We will write it as $\tilde{B}_n$ or $\tilde{B}_{\mathrm{GL}_n}$. We are going to define it in terms of generators. It is defined as:

1. The generators are $q, g, g^\vee, s_0, s_0^\vee, T_1, \ldots, T_{n-1}$.
2. The relation is given by



15

Last time we have DAWG (doubly affine Weyl group) to DAART to DAHA. This is how we construct a Hecke algebra.

We recall that Weyl groups are groups generated by $s_1, \ldots, s_n$ with relations $s_i^2 = 1$ and $s_is_js_is_js_i\ldots = 1$ where we have $m(ij) = m(i,j) \geq 2$ terms in the product, with $m(i,j)$ a function with inputs i, j.

For $w \in W$, we always have reduced expression $w = s_{i_1} \ldots s_{i_k}$ for some s_i.

Given Weyl group W, we get associated Braid group (ART). This is generated by $T_1, \ldots, T_{n-1}$ with relation $T_w = T_{i_1} \ldots T_{i_k}$ for $w \in W$ where $w = s_{i_1} \ldots s_{i_k}$ is reduced expression. Then we define $\ell(T_w) = \ell(w) = k$ and we see $T_{ww'} = T_wT_{w'}$ if and only if $\ell(w) + \ell(w') = \ell(ww')$. We denote this group by B_W.

In particular, we have affine Weyl group (AWG), which is given by $W := \mathbb{Z}^n \rtimes S_n$. It has some standard actions:

1. W acts on $\mathbb{Z}^n$ on the left: $\sigma \in S_n$ permutes the coordinate and $u \in \mathbb{Z}^n$ acts as $t_u(x) = x + u$ where $x \in \mathbb{Z}^n$.
2. We have W acts on $\mathbb{Z}$ on the left. This is n-periodic permutation, i.e. for $w \in W$ we have $w(i+n) = w(i) + n$. The action is given by $\sigma \in S_n$ permutes the columns and t_u slides jth column up by u_j for $j = 1, \ldots, n$. Here we think

$\mathbb{Z}$ as n columns of numbers, with the first column contains all elements inside $[1] \in \mathbb{Z}/\mathbb{Z}_n$ and so on.

Then, we get DAWG for $L, L^\vee = \mathbb{Z}^n$ by $L \rtimes S_n \ltimes L^\vee$.

To get from DAART to DAHA, we have a general construction of Hecke algebra associated to ART.

That is we choose $\tau : I \to F$, then $H_\tau := B_n / \langle (T_i - \tau_i)(T_i + \tau_i^{-1}) = 0 \rangle$ for $i \in I$. In our case, we have $I = \{1, \ldots, n-1\}$ and the relation is $(T_i - t)(T_i + 1) = 0$, i.e. we get $\tau_i = t^{1/2}$.

In particular, recall we get another representation of DAHA via X_i and Y_i. We note g acts on X^μ with $\mu = (\mu_1, \ldots, \mu_n)$ by $gX^\mu = q^{\mu_n} X^{\pi(\mu)}$ where $\pi(\mu) := (\mu_n, \mu_1, \ldots, \mu_{n-1})$.

Theorem 15.1. *f is symmetric in X_i and X_{i+1} if and only if $T_i f = tf$.*

Definition 15.2. E_μ is unique polynomial with $E_\mu = X^\mu + \sum_{\alpha<\mu} c_{\alpha\mu} X^\alpha$ such that $Y_i E_\mu = y_i(\mu) E_\mu$.

If μ is a anti-partition, then $y_i(\mu) = q^{\mu_i} t^k$ where $k = \#\{j < i : \mu_i = \mu_j\} - \#\{j > i : \mu_j = \mu_i\}$.

For example, take $\mu = 012$. Then $E_{012} = x_1^2 x_2 + \frac{q(1-t)}{1-qt^2} x_1 x_2 x_3$. One check $Y_2 E_{012} = q^1 t^0 E_{012}$. Indeed, note $Y_2 = T_1^{-1} g T_2$. Note we get T_2 correspond to $t - \frac{tx_2 - x_3}{x_2 - x_3}(1 - s_2)$ and hence we see $T_2 E_{012} = t x_1^2 x_2 + \frac{qt(1-t)}{1-qt^2} x_1 x_2 x_3 - \frac{(1-s_2)}{x_2 - x_3} E_{012}$ where $\frac{1-s_2}{x_2-x_3} E_{012} = \frac{x_1^2 x_2 - x_1^2 x_3}{x_1 - x_3} = x_1^2$. The rest is left as exercise.

Theorem 15.3. *If $\ell(\sigma s_i) < \ell(\sigma)$, then*

$$T_i E_\mu^\sigma = E_\mu^{\sigma s_i} \times \begin{cases} t, & \mu_{\sigma_i^{-1}} \le \mu_{\sigma_{i+1}^{-1}} \\ 1, & \textit{otherwise} \end{cases}$$

Else,

$$T_i^{-1} E_\mu^\sigma = E_\mu^{\sigma s_i} \times \begin{cases} t, & \mu_{\sigma_i^{-1}} \le \mu_{\sigma_{i+1}^{-1}} \\ 1, & \textit{otherwise} \end{cases}$$

For example, take E_{012}^{213}. Then $T_2 E_{012}^{312} = E_{012}^{213} \cdot 1$. On the other hand, $T_2^{-1} E_{012}^{213} = E_{012}^{312}$.

16

Today we consider $q = t = 0$ specializations of $E_\alpha(X, q, t)$.

In particular, we consider Demazure atoms. To define this, for $F = \mathbb{Z}[x_1, ..., x_n]$ and $s_i \in S_n$ we define $\partial_i := \frac{1-s_i}{x_i - x_{i+1}}$. Then we let $\pi_i = \partial_i x_i$ and $\pi_\omega = \pi_{i_1} ... \pi_{i_k}$ where $\omega \in S_n$ has reduced word $s_{i_1} ... s_{i_k}$.

Then we get the key polynomials/Demazure characters

$$K_\omega(X) = \pi_\omega X^\lambda$$

where $\lambda = (\lambda_1, ..., \lambda_n)$ is a partition. The π_ω is called Demazure operator.

For example, if $n = 3, \lambda = 21, \omega = s_2$, then $\pi_\omega = \pi_2 = \frac{1-s_2}{x_2-x_3} x_2$. Hence $K_\omega(x) = \pi_\omega(x_1^2 x_2) = \frac{1-s_2}{x_2-x_3}(x_1^2 x_2^2) = x_1^2(x_2 + x_3)$. Next, let $\omega = s_1 s_2$, then we get $\pi_\omega(x_1^2 x_2)$ is equal

$$\partial_1(x_1 \pi_2 x^\lambda) = \partial_1(x_1^3 x_2 + x_1^3 x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$$

In particular, note we always have $\pi_i f = f + (\text{something else})$. Hence, this motivates the definition of $\theta_i := \pi_i - 1$, which get rid of the copy of f in the above.

Then, we define $A_\omega(X) = \theta_\omega X^\lambda$ to be the Demazure atom.

We note the λ is any partition, i.e. it is part of the definition, but we just don't label it. Alternatively, we can denote those by $A_{\omega,\lambda}(X)$ and $K_{\omega,\lambda}(X)$.

We also recall $T_i = t - \frac{tx_i - tx_{i+1}}{x_i - x_{i+1}}(1 - s_i)$ and hence if we set $t = 0$ we get $\frac{x_{i+1}(1-s_i)}{x_i - x_{i+1}}$ which is θ_i.

Theorem 16.1 (Haglund-Lvoto-Mason)**.**

$$A_\alpha(X) = E_\alpha(X; 0, 0)$$

Theorem 16.2 (Macdonald-Cherednik)**.**

$$\hat{E}_{\mathrm{rev}(\alpha)}(X_n, ..., X_1; q^{-1}, t^{-1}) = E_\alpha(X_1, ..., X_n, q, t)$$

For us here, $Y_i = T_{i-1}^{-1} ... T_1^{-1} g T_{n-1} ... T_i$ defines our E_α, and $\hat{Y}_i := T_i ... T_{n-1} g T_1^{-1} ... T_{i-1}^{-1}$ defines our $\hat{E}_\alpha$. Those are just two different conventions, and we follow the Alexanderson 2016 and use E_α.

We have

$$\pi_\omega X^\lambda = \sum_{\sigma \leq \omega} \theta_\sigma X^\lambda$$

where $\sigma \leq \omega$ is Bruhat order.

Now, define $QS_\gamma = \sum_{\alpha^+ = \gamma} A_\alpha$, we get the following result

$$s_\lambda = \sum_{\lambda(\gamma)=\lambda} QS_\gamma = \sum_{\lambda(\alpha)=\lambda} A_\alpha = \sum_{\lambda(\alpha)=\lambda} E_\alpha(X; 0, 0)$$

which is due to Mason in 2009.

17

We have $\mathbb{C}[X^{\pm}] = Ind_{AHA}^{DAHA}(\mathrm{Id})$. We defined

$$T_i f = t^{-\frac{1}{2}}(tf - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(f - s_i f))$$

$$\tau_i = T_i + \frac{t^{-1/2}(1 - tY_i^{-1}Y_{i+1})}{1 - Y_i^{-1}Y_{i+1}}$$

for $i = 1, ..., n-1$. We also have $\tau_\pi = g^\vee = X_1 T_1 ... T_{n-1}$. Then, for element w in AWG, we define $\tau_w = \tau_{i_1} ... \tau_{i_k}$ where $i_j \in \{1, ..., n-1, \pi\}$.

Then, we define $E_\mu = t^{-\frac{1}{2}(\ell(v_\mu))} \cdot \tau_{u_\mu}(\mathrm{Id})$ where $t_\mu = u_\mu \cdot v_\mu$. Here v_μ is a permutation so that $v_\mu \cdot \mu = \mathrm{sort}(\mu)$.

For example, if $\mu = (1,0,2)$, then $v_\mu = s_1$. On the other hand, u_μ should be $s_2 s_1 \pi\pi s_2 s_1 \pi$. This is because, if we start with $(0,0,0)$, then we want to apply π to get $(1,0,0)$, then $s_1 s_2$ to get $(0,0,1)$, then π to get $(2,0,0)$, π again to get $(1,2,0)$, now apply $s_2 s_1$ to get $(1,0,2)$.

We want to ask what is the invariant subgroup of $\mathbb{C}[X^{\pm}]$ under span of E_μ. We see $\tau_i E_\mu = t^* E_{s_i \mu}$. We see $E_{w\mu}$ is in the subspace generated by $E_\alpha = \mathrm{Span}\{E_\mu : \mu \in \mathbb{Z}^n\}$. Then we see $P_\lambda \in \mathbb{C}[X]^\lambda = \mathrm{Span}(E_{w\lambda} : w \in S_n)$.

Next, we define symmetrizers.

We let $1_0 = t^{-\ell(w_0)/2} \sum_{\sigma \in S_n} t^{\ell(\sigma)/2} T_\sigma$.

We see $1_0 T_i = T_i 1_0 = t^{\frac{1}{2}} 1_0$ for $i = 1, ..., n-1$.

We also see this is almost idopotent, as we have $1_0^2 = t^* 1_0$. Here, t^* is an exponent of t, namely $t^* = t^{-\ell(w_0)/2} \sum_{\sigma \in S_n} t^{\ell(\sigma)}$.

We see the image of 1_0 is $\{1_0 f : T_i f = t^{\frac{1}{2}} f\}$. We also note $T_i(1_0 f) = t^{\frac{1}{2}}(1_0 f)$.

Thus, we define

$$P_\lambda = \frac{t^{\ell(w_0)}}{w_\lambda(t)} 1_0 E_\lambda$$

where

$$w_\lambda(t) = \sum_{\sigma : \sigma\lambda = \lambda} t^{\ell(\sigma)}$$

We have few properties:

$$P_\lambda \in \mathbb{C}[X]^\lambda \cap \mathbb{C}[X]^{s_n}$$

$$P_\lambda = \frac{1}{w_\lambda(t)} \sum_{\sigma \in S_n} t^{\ell(\sigma)/2} T_\sigma E_\lambda$$

We get $P_\lambda = \sum c_\gamma E_\gamma$ where the sum is over eigenvalues of Y_i.

Proposition 17.1. *P_λ is an eigenvector of any $f \in \mathbb{C}[Y]^{S_n}$.*

Theorem 17.2.

$$P_\lambda = \frac{1}{w_\lambda(t)} \sum_{\sigma \in S_n} E^\sigma_{inc(\lambda)}$$

$$P_\lambda = \sum_{\substack{u \in S_n \lambda \\ \sigma \text{ longest } \sigma \cdot u = inc(\lambda)}} E^\sigma_{inc(\lambda)}$$

18

We start with how to compute P_λ.

We have $E_\mu = t^{-\frac{1}{2}\ell(v_\mu)} \tau_{u_\mu} 1$ where $t_n = u_\mu v_\mu$ where $v_\mu \in W^{fin}$ and $u_\mu \in W/W^{fin}$.

Thus, we see $E_{s_i \gamma} = t^{\frac{1}{2}} \tau_i E_\gamma$ if $\gamma_i > \gamma_{i+1}$. Hence, $E_\gamma = t^{\frac{1}{2}} \tau_i E_{s_i\gamma}$ if $\gamma_i < \gamma_{i+1}$. Hence, $\tau_i E_\gamma = t^{\frac{1}{2}} \tau_i^2 E_{s_i \gamma}$.

Now we figure out what τ_i^2 is. Recall $\tau_i = T_i + \frac{t^{-1/2}(1-t)}{1 - Y_i Y_{i+1}}$ and hence

$$\tau_i^2 E_\gamma = \frac{t(1 - tY_i^{-1} Y_{i+1})(1 - tY_i Y_{i+1}^{-1})}{(1 - Y_i^{-1} Y_{i+1})(1 - Y_i Y_{i+1}^{-1})}$$

Since E_γ is eigenvector of τ_i, we see $\tau_i^2 E_\gamma = \text{scalar} \cdot E_\gamma$.

Next, we have $\mathbb{C}[X]^\lambda = \text{Span}\{E_\gamma : \gamma \in S_n \cdot \lambda\}$ which is invariant under τ_i, Y_i, T_i. In particular, $T_i E_\gamma \in \text{Span}(E_\gamma, E_{s_i \gamma})$.

The point is, we get

$$P_\lambda = \frac{t^{\ell(w_0)/2}}{w_\lambda(t)} 1_0 E_\lambda = \frac{1}{w_\lambda(t)} \sum_{\sigma \in S_n} t^{\frac{1}{2}\ell(\sigma)} T_\sigma E_\lambda = \frac{1}{w_\lambda(t)} \sum_{\sigma \in S_n} E^\sigma_{inc(\lambda)}$$

19