This note is by D.Dai for PMATH 351 at U of Waterloo with Prof Kenneth R. Davidson. No commercial uses are allowed. Please attend the lecture for best outcome as this note is only for supplementary purposes. This note definitely contain typos.

Latest update: **December 19, 2019**

Contents

Definition 0.0.1 (Ken Davidson).

1. Office: 5324
2. Office hour: Wed 10:30-11:30, can always make an appointment
3. Website: Click me
4. ***MIDTERM***: In class, monday, Oct. 21, right after the break week. It covers all the way to Arzela-Ascoli Theorem.
5. Crowdmark, learn, etc...
6. Textbook: nope, consider Marcoux's note or other people's note
7. Outline: this is left as an exercise
8. Assignments, 6-8 of them, 0.25. Midterm, TBA, late Oct, 0.25. Final, 0.5.

Chapter 1

Metric Spaces

1.1 Normed Vector Spaces

Definition 1.1.1. A ***normed vector space*** V is a vector space over $F \in \{\mathbb{R}, \mathbb{C}\}$ equipped with a norm $\|\cdot\| : V \to [0, \infty)$ with the following properties:

1. positive definite: $\forall x \in V, \|x\| = 0$ iff $x = 0$
2. positive homogeneity: $\forall \lambda \in F$ and $\forall x \in V$, $\|\lambda x\| = |\lambda| \cdot \|x\|$
3. triangle inequality: $\forall u, v \in V, \|v + u\| \le \|u\| + \|v\|$

Remark 1.1.2. If 1 does not hold in the above definition, then it is called ***semi-norm***

Theorem 1.1.3 (Cauchy-Schwarz inequality)**.** *Let V be an inner product space. Define $\|x\| = \sqrt{\langle x, x\rangle}$. For all $x, y \in V$, we have*

$$|\langle x, y\rangle| \le \|x\| \cdot \|y\|$$

Example 1.1.4.
1. $\mathbb{R}^n$ and $\mathbb{C}^n$ with Euclidean norm, i.e. $x = (x_1, ..., x_n)$ and $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$
2. Every inner product space $(V, \langle\cdot,\cdot\rangle)$ has a natural norm function, namely, $\|v\|_2 = \langle v, v\rangle^{1/2}$ by Cauchy-Schwarz inequality.
3. Let $X \subseteq \mathbb{R}^n$ be closed and bounded, then denote $C(X)$ and $C_{\mathbb{R}}(X)$ to be the spaces of continuous $\mathbb{C}$ or $\mathbb{R}$-valued functions. We define the norm as following: for all $f \in C(X)$, $\|f\|_\infty = \sup_{x\in X} |f(x)|$. We have $\|\cdot\|_\infty$ is always finite as X is closed and bounded (hence compact) and by Extreme Value Theorem(EVT). Moreover, if $X \subseteq \mathbb{R}^n$, $C_b(X)$ is the space of bounded continuous functions, then $\|\cdot\|_\infty$ is a norm for $C_b(X)$
4. We have $\ell_p^{(n)} = \{x \in F^n : \|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} < \infty\}$ is a normed vector space.
5. Let $1 \le p \le \infty$. We have

$$\ell_\infty = \{x = (x_1, x_2, ...) : x \text{ is bounded sequence in } F\}$$

with the norm $\|x\|_\infty = \sup_{n\geq 1} |x_n|$

For $1 \leq p < \infty$, we have $\ell_p = \{x = (x_1, x_2, ...) : \|x\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} < \infty\}$.

We will proof the triangle inequality as the two other properties are easy to verify. To do this, we frist exam the normed space $L^p \subseteq C[a,b]$ with the norm $\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}}$.

It is easy to see the first two properties, so again we exam the triangle inequality, to do this, we need a theorem.

Theorem 1.1.5 (Minkowski Inequality). *If $f, g \in L^p(a,b)$ or ℓ^p for $1 \leq p < \infty$, then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. Moreover, if $1 < p < \infty$, then equality holds iff f, g lie on the same ray ($f = \lambda g$ where λ is scalar or $g = 0$). If $p = 1$, the case is trivial, if $p = \infty$, then equality holds iff f, g have the same max.*

Proof. we will work with $(-\infty, \infty)$ as $L^p(a,b)$ is a subspace of $L^p(-\infty,\infty)$. Let $f, g \in L^p(-\infty, \infty)$, if $f = 0$ or $g = 0$ then we are done.

Let $A = \|f\|_p > 0$ and $B = \|g\|_p > 0$, let $f_0 = \frac{f}{A}$ and $g_0 = \frac{g}{B}$, then $\|f_0\|_p = \|g_0\|_p = 1$. Thus, we have

$$\frac{f+g}{A+B} = \frac{A}{A+B} f_0 + \frac{B}{A+B} g_0$$

Let $\phi(x) = x^p$, then the second derivative is always positive, so ϕ is convex. Hence $0 \leq x, y < \infty$ and $0 \leq t \leq 1$ imply

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

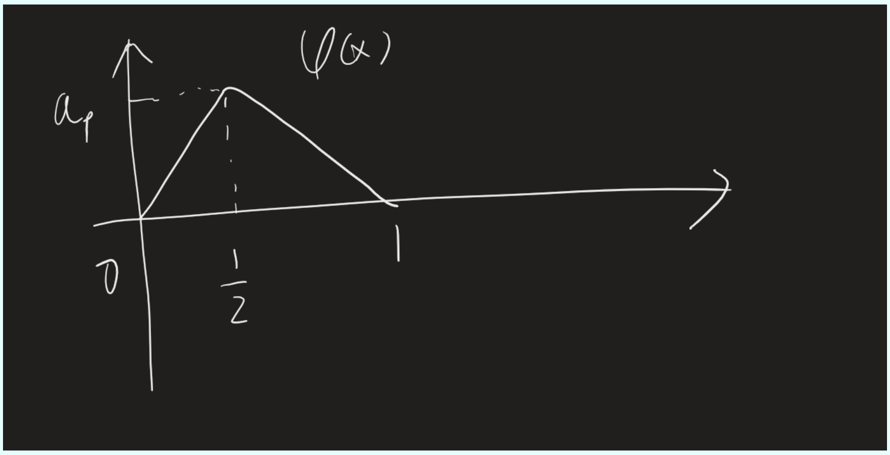
Then, we have

$$\begin{aligned}
\left(\frac{\|f+g\|_p}{A+B}\right)^p &= \frac{1}{(A+B)^p} \int |f(x) + g(x)|^p dx \\
&= \int \left| \frac{A}{A+B} f_0(x) + \frac{B}{A+B} g_0(x) \right|^p dx \\
&\leq \int \frac{A}{A+B} |f_0(x)|^p + \frac{B}{A+B} |g_0(x)|^p dx \quad \text{by convexity of } \phi \\
&= \frac{A}{A+B} \|f_0\|_p^p + \frac{B}{A+B} \|g_0\|_p^p = 1
\end{aligned}$$

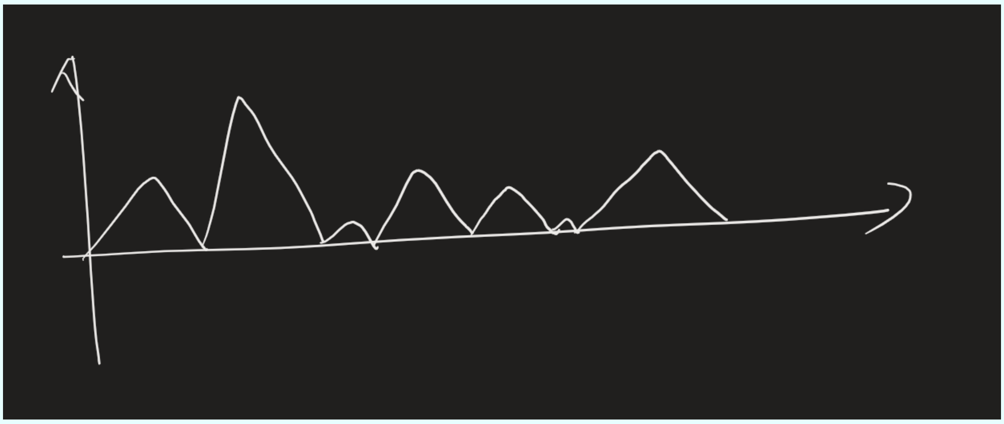
Thus, we have $\frac{\|f+g\|_p}{A+B} \leq 1$ as desired. ♡

Remark 1.1.6. With Minkowski inequality, the triangle inequality for ℓ_p follows from L^p as there is a linear isometry map $T : \ell_p \to L^p(\mathbb{R})$ so that $\|T(x)\|_p = \|x\|_p$.

To construct such T, we let $\phi \in L^p(\mathbb{R})$ be a continuous function on $(0,1)$ with $\|\phi\|_p = 1$, i.e.



Then we maps the sequence $s = (x_i)_{i=1}^{\infty}$ to the function on $L^p(\mathbb{R})$ to be $T_s(t) = \sum_{n=1}^{\infty} x_n \cdot \phi(t-n)$, i.e.

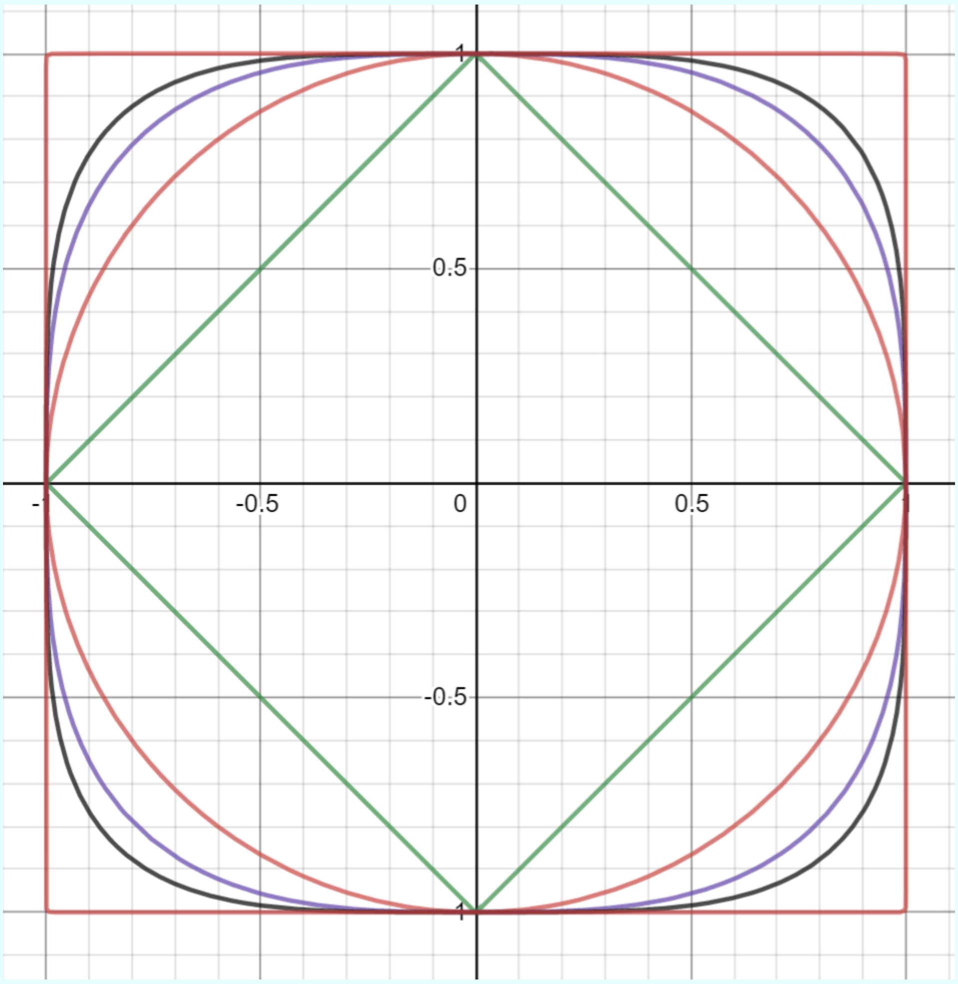


Then

$$\|T_s\|_p^p = \int_0^{\infty} |\sum_{n=1}^{\infty} x_n \phi(t-n)|^p dx = \sum_{n=0}^{\infty} \int_n^{n+1} |x_n|^p \cdot |\phi(t-n)|^p dx = \|s\|_p^p$$

Thus, Minkowski holds for ℓ_p as well as we embedded ℓ_p into $L^p(\mathbb{R})$.

Example 1.1.7. Here are some examples of unit circle of $\ell_p^{(2)}$ where $p = 1, 2, 3, 4, \infty$, where p increases from inside to outside (i.e. the green one is $p = 1$, the outmost red square is $p = \infty$).



1.2 Metric Spaces

Definition 1.2.1. A ***metric space*** (X, d) is a set X with a distance function $d : X \times X \to [0, \infty)$ such that, $\forall x, y, z \in X$,

1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Example 1.2.2.
1. If V is a normed vector space and $X \subseteq V$, then $d(x, y) = \|x - y\|$ is a metric.
2. Let X be any set, $d(x, y) = \begin{cases} 0, \text{if } x = y \\ 1, \quad \text{otherwise} \end{cases}$ is a metric on X.
3. Hamming metric on the power set $P([n])$ where $[n] = \{0, 1, ..., n-1\}$ where $d(A, B) = |A \triangle B| = |(A \cup B) \setminus (A \cap B)|$ is a metric.
4. Geodesic distance on any surface, e.g. $X = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$, $d(x, y)$ be the length of the shortest path from x to y. On S^2, the shortest path is following the great circle from x to y.
5. Hausdorff distance d_H. Fix a closed subset $Y \subseteq \mathbb{R}^n$. Let $\mathcal{H}(Y)$ denote the

collection of all closed bounded subsets of Y, let $A, B \in \mathcal{H}(Y)$, if $a \in A$, $d(a, B) = \inf_{b \in B} \|a - b\|$. Then, the Hausdorff distance is

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

We show Hausdorff distance is a metric. If $d_H(A, B) = 0$ then, if $a \in A$, $\exists b_n \in B$, $\|a - b_n\| \to 0$, since B is closed, $b_n \to a$ so $a \in B$. Thus $A \subseteq B$. By symmetry, we have $A = B$.
To show triangle inequality, let $A, B, C \in \mathcal{H}(Y)$. Take $a \in A$, then $d(a, C) = \inf_{c \in C} \|a - c\|$. Let $b \in B$ be arbitrary, then $d(a, C) \leq \inf_{c \in C} \|a - b\| + \|b - c\|$. Thus, $d(a, C) \leq \|a - b\| + d(b, C) \leq \|a - b\| + d_H(B, C)$. Thus, take sup of $d(a, C)$, we have $\sup_{a \in A} d(a, C) \leq \inf_{b \in B} \|a - b\| + d_H(B, C) = d(a, B) + d(B, C) \leq d(A, B) + d(B, C)$.
Thus, $\sup_{a \in A} d(a, C) \leq d_H(A, B) + d_H(B, C)$. Now, we reverse the role of A and C, we have $\sup_{c \in C} d(c, A) \leq d_H(C, B) + d(B, A)$ and so the triangle inequality follows.

6. Fix a prime p, put a norm on $\mathbb{Q}$ as follows: $\|0\|_p = 0$, and $\|x\|_p = p^{-a}$, where $x = p^a \frac{r}{s}$, $r, s, a \in \mathbb{Z}$, and $gcd(r, p) = gcd(s, p) = 1$. Then, the p-adic distance $d_p(x, y) = \|x - y\|_p$. It is easy to check the first two properties, we show triangle inequality.
We show $d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\}$. Suppose $x = p^a \frac{r}{s}$ and $y = p^b \frac{u}{v}$ as we described above. WLOG, suppose $a \leq b$, then

$$x - y = p^a(r/s - p^{b-a}u/v) = p^a(\frac{rv - p^{b-a}su}{sv})$$

Thus, we have $\|x - y\|_p = p^{-a} = \|x\|_p = d_p(x, 0)$. If $a = b$, then $\exists c \geq 0$ such that $rv - su = p^c w$ where $gcd(w, p) = 1$, thus $d(x, y) = p^{-a-c} < p^{-a}$. If $z \in \mathbb{Q}$, then $d_p(x, y) = d_p(x - z, y - z) \leq \max\{\|x - z\|_p, \|y - z\|_p\}$ and the proof follows.

Example 1.2.3. We can put a metric ρ on all the words in a dictionary by defining the distance between two distinct words to be 2^{-n} if the word agree for the first n letters and are different at $n+1$ one. In addition, we define $\rho(x, x) = 0$ for all words x. A whitespace is distinct from a letter. For example, we have $\rho(car, cart) = 2^{-3}$. We will show this is a metric and show that $\rho(w_1, w_3) = \max\{\rho(w_1, w_2), \rho(w_2, w_3)\}$ if w_1, w_2, w_3 are listed in alphabetical order.

Solution. $\rho(x, y) = \rho(y, x)$ trivially as the distinction between words would not change. In addition, if $\rho(x, y) = 0$ then we must have $x = y$ as if they were different then the value cannot be 0. Thus, it remains to show the triangle inequality.

Let A, B, C be words with finite letters. Suppose $\rho(A, C) = 2^{-n} > 0$ as $\rho(A, C) = 0$ then triangle inequality hold vacuously. Thus, the first n letters of A and C agrees, and the $n+1$ letter is different.

Suppose A and B agrees for the first m letters and are different at the $m+1$. Then, if $m > n$, we have $\rho(B, C) = 2^{-n}$ as A and C agrees for the first n then differ at the $n+1$ letter.Thus, we have $\rho(A, B) + \rho(B, C) = c + 2^{-n} > 2^{-n}$ where $c = \rho(A, B) > 0$.

If $m = n$ then $\rho(A, C) = 2^{-n} < 2^{-n} + 2^{-n} = \rho(A, B) + \rho(B, C)$ by the definition of m and n.

Suppose $m < n$, then we have $\rho(A, B) = 2^{-m}$, and $\rho(B, C) = 2^{-m}$ and in particular, we would have $\rho(A, B) + \rho(B, C) = 2^{-m} + 2^{-m} = 2^{-m+1}$. Since $m < n$, we have $-m + 1 > -n$, and so $2^{-m+1} > 2^{-n}$, which imply $\rho(A, B) + \rho(B, C) \geq \rho(A, C)$ as well. Thus, for all cases, we have the desired triangle inequality.

Next, suppose w_1, w_2, w_3 are in alphabetical order.

Say $\rho(w_1, w_2) = 2^{-n}, \rho(w_2, w_3) = 2^{-m}$, and $\rho(w_1, w_3) = 2^{-q}$. We have $2^{-q} \leq 2^{-n} + 2^{-m}$. In particular, if $2^{-n} = 2^{-m}$ then we must have

$$\rho(w_1, w_3) = \max\{\rho(w_1, w_2), \rho(w_2, w_3)\} = 2^{-n}$$

as $n = m$ imply they have the same first n letters and differ at $n + 1$ by the alphabetical order ensures that they must be disagree on the $n+1$th letters. Indeed, the $n + 1$th letters, a_1, a_2, a_3 for w_1, w_2, w_3, respectively, must be in order, i.e. $a_1 > a_2 > a_3$ as they are pairwise different and the words w_1, w_2, w_3 are in alphabetical order.

Suppose $2^{-n} < 2^{-m}$, then $n > m$. This means the first n letters of w_1 and w_2 are the same, and the first m letters of w_2 and w_3 are the same. Together with the alphabetical ordering property, we must have the $m + 1$th letter of w_1 and w_3 are different, hence $\rho(w_1, w_3) = 2^{-m} = \max\{\rho(w_1, w_2), \rho(w_2, w_3)\}$.

Suppose $2^{-n} > 2^{-m}$, then $n < m$. This means w_1 and w_3 only agrees on the first n letters by the same reasoning as above, so we have our desired result. ♠

Example 1.2.4. Let $X = 2^{\mathbb{N}} = \{(x_1, x_2, ...) : x_i \in \{0, 1\}\}$ and define $d(x, y) = 2\sum_{i\geq 1} 3^{-i}|x_i - y_i|$. Show this is a metric and show $f : X \to [0, 1]$ given by $f(x) = d(0, x)$ is a surjective mapping to the Cantor set and we have $\frac{1}{3}d(x, y) \leq |f(x) - f(y)| \leq d(x, y)$ for $x, y \in 2^{\mathbb{N}}$.

Solution. We note $d(\mathbf{x}, \mathbf{x}) = 0$ trivially. We note $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ as $|x-y| = |y-x|$ for all $x, y \in \{0, 1\}$. Thus, it suffice to show the triangle inequality. We remark all the following series are converging absolutely so that we can do things like splitting the infinite sums.

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in 2^{\mathbb{N}}$, then we have

$$\begin{aligned} d(\mathbf{x}, \mathbf{z}) = 2\sum_{i=1}^{\infty} 3^{-i}|x_i - z_i| &= 2\sum_{i=1}^{\infty} 3^{-i}|x_i - y_i + y_i - z_i| \\ &\leq 2\sum_{i=1}^{\infty} 3^{-i}(|x_i - y_i| + |y_i - z_i|) \\ &= 2\sum_{i=1}^{\infty} 3^{-i}|x_i - y_i| + 2\sum_{i=1}^{\infty} 3^{-i}|y_i - z_i| \\ &= d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \end{aligned}$$

Thus d is indeed a metric.

We note that $d(\mathbf{0},\mathbf{x}) = 2\sum_{i=1}^{\infty} 3^{-i}|x_i|$ where $x_i \in \mathbb{Z}_2$ and then I will give the definition of Cantor set we are using below.

$$C := [0,1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n-1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right)$$

Let $n \in \mathbb{N} \cup \{0\}$ be given and $0 \leq k \leq 3^n - 1$, we obtained a removed interval, i.e. not in the Cantor set. Let $x \in \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}}\right)$, then $x > \frac{3k+1}{3^{n+1}}$, i.e. $x/(\frac{3k+1}{3^{n+1}}) > 1$ so that in the ternary expression of x(we note $x < 1$), we have $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$ where $a_i \in \{0,1,2\}$ and a_k must be 1. Hence, if x has all its ternary expressions containing at least one 1, x is not in the Cantor set.

Thus, all the elements in the Cantor set are in the form of $\sum_{i=1} \frac{a_i}{3^i}$ where $a_i \in \{0,2\}$. In particular, this is precisely the range of f, $f(X)$ and hence we have the desired surjection.

Let $\mathbf{x}, \mathbf{y} \in 2^{\mathbb{N}}$ be given, we have

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{y})| &= \left| 2\sum_{i=1}^{\infty} 3^{-i}(|x_i| - |y_i|) \right| \\ &\leq 2\sum_{i=1}^{\infty} 3^{-i}\big||x_i| - |y_i|\big| \leq 2\sum_{i=1}^{\infty} 3^{-i}\big|x_i - y_i\big| = d(\mathbf{x},\mathbf{y}) \end{aligned}$$

On the other hand, if $f(\mathbf{x}) = f(\mathbf{y})$ then $\frac{1}{3}d(\mathbf{x},\mathbf{y}) \leq |f(\mathbf{x}) - f(\mathbf{y})|$ for trivial reason, i.e. $\mathbf{x}$ and $\mathbf{y}$ have the same ternary expression so they must be equal. Thus, we suppose that $f(\mathbf{x}) > f(\mathbf{y})$. Then, say $\mathbf{x} = (x_1, x_2, ...)$ and $\mathbf{y} = (y_1, y_2, ...)$, suppose $\mathbf{x}$ and $\mathbf{y}$ agree upon the first k terms where $k \in \{0\} \cup \mathbb{N}$, i.e. $\forall 1 \leq i \leq k, x_i = y_i$ (if $k = 0$ then $x_1 \neq y_1$). Since the first k terms are equal, we would have $f(\mathbf{x}) = \sum_{i=1}^{k} 2\frac{|x_i|}{3^i} + \frac{2|x_{k+1}|}{3^{k+1}} + \sum_{i=k+2}^{\infty} 2\frac{|x_i|}{3^i}$ and $f(\mathbf{y}) = \sum_{i=1}^{k} 2\frac{|y|}{3^i} + \frac{2|y_{k+1}|}{3^{k+1}} + \sum_{i=k+2}^{\infty} 2\frac{|y_i|}{3^i}$

In particular, we note $\frac{2}{3^{k+1}} > \sum_{i=k+2}^{\infty} 2\frac{|y_i|}{3^i}$ for all $y_i \in \{0,1\}$ as

$$\frac{2}{3^{k+1}} > \sum_{i=k+2}^{\infty} 2\frac{1}{3^i} = \frac{1}{3^{k+2}} \cdot \frac{2}{1 - 1/3} = \frac{1}{3^{k+1}} \geq \sum_{i=k+2}^{\infty} 2\frac{|y_i|}{3^i}$$

Hence, we would have $f(x) - y(x) > 0$ if and only if $|x_{k+1}| > |y_{k+1}|$ as the terms after this are inconsequential (suppose $x_{k+1} = 0$ then we must have $y_{k+1} = 1$ by definition of k, so the maximal possible value of $\sum_{i=k+2} \frac{2|x_i|}{3^i}$ would be less than $\frac{2}{3^{k+1}} = \frac{2|y_{k+1}|}{3^{k+1}}$).

Thus, we must have $x_{k+1} = 1$ and $y_{k+1} = 0$. In particular, this give us

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{y}) - \frac{1}{3}d(\mathbf{x}, \mathbf{y}) &= \frac{2}{3^{k+1}} + \left(2\sum_{i=k+2}^{\infty} \frac{|x_i| - |y_i|}{3^i}\right) - \frac{2}{3}\sum_{i\geq 1} \frac{|x_i - y_i|}{3^i} \\ &\geq \frac{2}{3^{k+1}} + \left(2\sum_{i=k+2}^{\infty} \frac{0-1}{3^i}\right) - \frac{2}{3}\sum_{i\geq 1}\frac{1}{3^i} \\ &= \frac{2}{3^{k+1}} - \frac{1}{3^{k+1}} - \frac{2}{3}\cdot\frac{1}{2\cdot 3^k} \\ &= 0 \end{aligned}$$

This established $f(\mathbf{x}) - f(\mathbf{y}) - \frac{1}{3}d(\mathbf{x}, \mathbf{y}) \geq 0$ if $f(\mathbf{x}) - f(\mathbf{y}) > 0$. If $f(\mathbf{x}) < f(\mathbf{y})$, the conclusion is established by symmetry.

Thus, we must have $\frac{1}{3}d(\mathbf{x}, \mathbf{y}) \leq |f(\mathbf{x}) - f(\mathbf{y})|$ as desired. ♠

1.3 The Topology of Metric Spaces

Remark 1.3.1. Some proofs of the theorems are from my MATH 247 notes rahter than from class.

Definition 1.3.2. Let (X, d) be a metric space. The ***open ball*** about x of radius $r > 0$ is

$$b_r(x) = \{y \in X : d(x, y) < r\}$$

We may also write $b_r(x) = B(x, r)$.

The ***closed ball*** about x of radius $r \geq 0$ is

$$\overline{b_r}(x) = \{y \in X : d(x, y) \leq r\}$$

We may also write $\overline{b_r}(x) = \overline{B}(x, r)$.

The ***punctured ball*** about x of radius $r > 0$ is

$$B^*(x, r) = \{y \in X : d(x, y) < r, y \neq x\}$$

A set $U \subseteq X$ is ***open*** if $\forall x \in U, \exists r > 0, b_r(x) \subseteq U$

A set $C \subseteq X$ is ***closed*** if $X \backslash C$ is open.

A ***neighborhood*** of x is a open set $N \subseteq X$ such that $\exists r > 0, b_r(x) \subseteq N$.

Remark 1.3.3. We note $\emptyset$ and X are both closed and open at the same time.

Proposition 1.3.4. *$b_r(x)$ is open and $\overline{b_r}(x)$ is closed.*

Proof. Say $y \in b_r(x)$, say $d(x, y) = s < r$. If $z \in b_{r-s}(y)$, then $d(x, z) \leq d(x, y) + d(y, z) < s + (r - s) = r$, so that $d_{r-s}(y) \subseteq b_r(x)$.

To show the closed ball is closed, we show the complement is open. If $d(x,y) = s > r$, let $z \in b_{s-r}(y)$ then $d(x,z) \geq d(x,y) - d(y,z) > s - (s-r) = r$. Thus ♡

Example 1.3.5.
1. $\{(x,y) \in \mathbb{R}^2 : xy > 1\}$ is open and $\{(x,y) \in \mathbb{R}^2 : xy \geq 1\}$ is closed.
2. Let $(\mathbb{N}, d_2)$ be the $2-adic$ metric, where

$$d_2(n,m) = \begin{cases} 0, n = m \\ 2^{-k}, & \text{if there exsits odd number } p, n - m = 2^k(p) \end{cases}$$

Then, we have the ball

$$\begin{aligned} b_{2^{-k}}(n) &= \{m : d_2(m,n) < 2^{-k}\} \\ &= \{m : d_2(m,n) \leq 2^{-k-1}\} \\ &= \{m : 2^{k+1} \mid m - n\} \\ &= \{n + 2^{k+1}\mathbb{Z}\} \cap \mathbb{N} \end{aligned}$$

Theorem 1.3.6 (Basic Properties of Open Sets)**.** *Let (X,d) be a metric space,*
1. *$\emptyset$ and X are open in X*
2. *Let K be index set, if u_k is an open set of each $k \in K$ then $\bigcup_{k\in K} u_k$ is open*
3. *If $U_1, U_2, ..., U_l$ are open sets in X then $\bigcap_{k=1}^{l} U_k$ is open*

Proof.
1. $\emptyset$ is open by convention, every statement start with $\forall x$ is true for $\emptyset$. To see X is open, we note $B(a,r) = \{x \in X : |x-a| < r\} \subseteq X$ for all $a \in X, r > 0$
2. Let u_k be open for each $k \in K$. Let $u = \bigcup_{k\in K} u_k$ and let $a \in u$. Since $a \in u$, we can choose an index k so that $a \in u_k$. Since u_k is open, we can choose $r > 0$, so that $B(a,r) \subseteq u_k$. Since $B(a,r) \subseteq u_k$, and $u_k \subseteq \bigcup u_k = u$, we have $B(a,r) \subseteq u$. Thus u is open.
3. Let $U_1, ..., U_l$ are open. Let $a \in \bigcap_{i=1}^{l} U_i$. For each index $1 \leq k \leq l$, since U_k is open, we can choose r_k such that $B(a, r_k) \subseteq U_k$. Let $r = \min\{r_1, ..., r_l\}$, then we have $r > 0$. Clearly we must have $B(a,r) \subseteq U_k$ for all $1 \leq k \leq l$. Thus it is open as a was arbitrary.

♡

Corollary 1.3.6.1 (Basic Properties of Closed Sets)**.**
1. *$\emptyset$ and X are closed in X*
2. *Let K_j is closed in X for each $j \in J$ where J is a set, then $\bigcap_{j\in J} K_j$ is closed*
3. *If $K_1, ..., K_l$ are closed in X then $\bigcup_{i=1}^{l} K_j$ is closed in X*

Definition 1.3.7. A sequence $(x_n)_{n\geq 1}$ in (X,d) converges to x if $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 1.3.8. If $A \subseteq X$ then
1. $x \in X$ is a ***limit point of*** A if there exists sequence (a_n) with $a_n \in A$ such that $\lim_{n\to\infty} a_n = x$.
2. $x \in X$ is a ***accumulation point of*** A if there exists sequence (a_n), $a_n \in A$ such that $a_n \neq x$ with $\lim a_n = x$.

3. $x \in A$ is an ***isolated point*** if there exists $r > 0$ such that $b_r(x) \cap A = \{x\}$.
4. Let $A \subseteq X$ be a set, then A' is the set of all accumulation points of A.

Lemma 1.3.9. *We have x is an accumulation point of A if and only if*

$$\forall \delta > 0, b^*_\delta(x) \cap A \neq \emptyset$$

Proof. We note $\forall \delta > 0, b^*_\delta(x) \cap A \neq \emptyset$ immediately imply there exists such a sequence, e.g. for each $n \in \mathbb{N}$, take the point $a_n \in A$ to be in $b^*_{1/n}(x) \cap A$, then we are done.

Conversely, if x is an accumulation point of A. Let $\{a_n\} \to x$ be a sequence in A where $a_i \neq x$ for all $i \in \mathbb{N}$. Then for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $\forall n > N, d(a_n, x) < \epsilon$. In particular, $a_n \in b^*_\epsilon(x) \cap A$ and the proof follows. ♡

Proposition 1.3.10. *A set $A \subseteq X$ is closed if and only if $A' \subseteq A$.*

Proof. We note

$$\begin{aligned} A' \subseteq A &\Leftrightarrow \forall a \in X, (a \in A' \Rightarrow a \in A) \\ &\Leftrightarrow \forall a \in X, (\forall r > 0, B^*(a, r) \cap A \neq \emptyset \Rightarrow a \in A) \end{aligned}$$

On the other hand,

$$\begin{aligned} A \text{ is closed} &\Leftrightarrow \forall a \in X, (a \in A^c \Rightarrow \exists r > 0, B(a, r) \subseteq A^c) \\ &\Leftrightarrow \forall a \in X, (a \notin A \Rightarrow \exists r > 0, B(a, r) \cap A = \emptyset) \end{aligned}$$

Since when $a \notin A$, we have $B(a, r) \cap A = B^*(a, r) \cap A$, we get

$$\Leftrightarrow \forall a \in X, (\forall r > 0, B^*(a, r) \cap A \neq \emptyset \Rightarrow a \in A)$$

The proof follows.

♡

Definition 1.3.11. The ***closure*** of A, $\overline{A}$, is the smallest closed set containing A. In particular, $\overline{A} = \cap_{\lambda \in \Lambda} C_\lambda$ where Λ is an index set and $\forall \lambda \in \Lambda, C'_\lambda \subseteq C_\lambda \wedge A \subseteq C$, i.e. C_λ is closed and containing A.

Proposition 1.3.12. *If $A \subseteq X$ then $\overline{A} = A \cup A'$. We also remark that $\overline{A}$ is also equal the set of all limit points of A and the set of all isolated points union A'.*

Proof. We shall show that $A \cup A'$ is the smallest closed set in X which contains A, that is $A \subseteq A \cup A'$, $A \cup A'$ is closed, and for every closed set $k \in X$ with $A \subseteq k$ we have $A \cup A' \subseteq k$. We shall show $(A \cup A')^c$ is open. Let $a \in (A \cup A')^c$. That is, $a \notin A$ and $a \notin A'$. Since $a \notin A'$, we can choose $r > 0$ so that $B^*(a, r) \cap A = \emptyset$. Note that since $a \notin A$, we have $B^*(a,r) \cap A = B(a,r) \cap A$ so we have $B(a,r) \cap A = \emptyset$. We claim that $B(a,r) \cap A' = \emptyset$. Suppose, for a contradiction, it is not. Choose $b \in B(a,r) \cap A'$. Since $B(a,r)$ is open, we can choose $s > 0$ so that $B(b,s) \subseteq B(a,r)$. Since $b \in A'$, we have $B^*(b,s) \cap A \neq \emptyset$ hence $B(a,r) \cap A \neq \emptyset$ since $B^*(b,s) \subseteq B(b,s) \subseteq B(a,r)$. This gives the desired contradiction hence $B(a,r) \cap A' = \emptyset$. Therefore $B(a,r) \cap (A \cup A') = \emptyset$. This shows that $(A \cup A')^c$ is open so $A \cup A'$ is closed.

It remains to show that for every closed set $k \in X$ with $A \subseteq k$, we have $A \cup A' \subseteq k$. Let k be any closed set in X with $A \subseteq k$. Since $A \subseteq k$, we have $A' \subseteq k'$ since when $a \in A'$, $\forall r > 0, B^*(a,r) \cap A \neq \emptyset$, hence $\forall r > 0$, $B^*(a,r) \cap k \neq \emptyset$. Since k is closed, $k' \subseteq k$ by Proposition 1.3.10. So we have $A' \subseteq k' \subseteq k$. Since $A \subseteq k$ and $A \subseteq k'$, we have $A \cup A' \subseteq k$. ♡

Proof From Lecture. We note $\overline{A}$ contains all other three (i.e. $A \cup A'$, all limit points of A, isolated points union A'), and the all other three are all equal. Thus, we need to show that the limit points of A is closed under taking limits, this would imply the set of limit points $\mathfrak{A}$ is closed and it is containing A so that $\overline{A} \subseteq \mathfrak{A}$ by definition. Then the two sets are equal as they are subset of each other.

Suppose (a_n) is a sequence of limit points of A such that $\lim\limits_{n\to\infty} a_n = x$. For each fixed n, there exists a sequence $(a_{n,k})_k$ in A such that it converges to a_n. We have $d(a_n, x) \to 0$, and for each n, pick $k_n \in \mathbb{N}$ such that $d(a_{n,k_n}, a_n) < d(a_n, x)$, this would give us a new sequence $(a_{n,k_n})_n$ so that

$$d(a_{n,k_n}, x) \leq d(a_{n,k_n}, a_n) + d(a_n, x) < 2d(a_n, x) \to 0$$

Hence, we have $\lim\limits_{n\to\infty} a_{n,k_n} = x$ so that x is a limit point of A and so that $\mathfrak{A}$ is closed and contained in $\overline{A}$ while containing $\overline{A}$, hence equal to $\overline{A}$. ♡

Example 1.3.13. Let d_2 be the 2-adic metric on $\mathbb{Q}$. Then,

1. Find $\lim_{n\to\infty} \frac{1-(-2)^n}{3}$ in $(\mathbb{Q}, d_2)$ and show this limit is in the closure of $\mathbb{N}$.
2. Find $\overline{\mathbb{N}}$ in $(\mathbb{Q}, d_2)$.

Solution. We claim $\lim\limits_{n\to\infty} \dfrac{1-(-2)^n}{3} = \dfrac{1}{3}$. Indeed, let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ to be so that

$$N > \frac{\ln(1/\epsilon)}{\ln(2)}$$

Then we would have $\forall n > N$, that $d_2(\frac{1-(-2)^n}{3}, \frac{1}{3}) = 2^{-n} = \frac{1}{2^n} < \epsilon$. Indeed, we have $\frac{1-(-2)^n}{3} - \frac{1}{3} = \frac{(2)^n(-1)^n}{-3}$ where $gcd((-1)^n, -3) = 1$. Thus, it converges to $\frac{1}{3}$ by definition.

Next, we show it is in the closure of $\mathbb{N}$. Consider the sequence $\{\frac{2^{2n+1}+1}{3}\}_{n=1}^{\infty}$. One see (by Euler's theorem/MATH 145) this sequence is indeed in $\mathbb{N}$ as we have

$$2n + 1 \equiv 1 \pmod{\phi(3)}$$

where $\phi(n)$ is the Euler's totient function, then

$$2^{2n+1} \equiv 2^1 \equiv -1 \pmod 3 \Rightarrow 2^{2n+1} + 1 \equiv 0 \pmod 3$$

Moreover, we note

$$\begin{aligned} d_2(\frac{2^{2n+1}+1}{3}, \frac{1}{3}) &= \left\| \frac{2^{2n+1}+1}{3} - \frac{1}{3} \right\|_2 \\ &= \left\| \frac{2^{2n+1}}{3} \right\|_2 = 2^{-n-1} \end{aligned}$$

Thus, we indeed have $(\frac{2^{2n+1}+1}{3})_{n=1}^{\infty} \to \frac{1}{3}$ as desired.

To show the second question, we claim the closure of $\mathbb{N}$ is $\{\frac{q}{p} : 2 \nmid p\}$. Let $\frac{q}{p} \in \mathbb{Q}$ be given, where $gcd(p,q) = 1$ and $p, q > 0$. We first suppose $2 \nmid p$ (as $2 \mid p$ would cause problem to our argument. In particular, Euler's theorem only holds when 2 and p are co-prime) and $p \neq q$ as $p = q$ is a trivial case. Moreover, we suppose $p \neq 1$ for the same reason.

Consider the sequence

$$\left\{ \frac{2^{\phi(p)\cdot n}(w) + q}{p} \right\}_{n=1}^{\infty}$$

where $w \in \{1, 2, ..., p\}$ is the smallest number such that $w \equiv -q \pmod p$.

First, we note this is indeed a sequence in $\mathbb{N}$, as by Euler's theorem, we have

$$2^{\phi(p)} \equiv 1 \pmod p \Rightarrow \forall n \in \mathbb{N}, 2^{n\phi(p)} \equiv 1^n \pmod p$$

Thus, we have, for all $n \in \mathbb{N}$,

$$2^{n\phi(p)} \cdot (w) \equiv w \equiv -q \pmod p \Rightarrow 2^{n\phi(p)} \cdot (w) + q \equiv 0 \pmod p$$

We note we have

$$\begin{aligned} d_2(\frac{2^{\phi(p)\cdot n}(w) + q}{p}, \frac{q}{p}) &= \left\| \frac{2^{\phi(p)\cdot n}(w) + q}{p} - \frac{q}{p} \right\|_2 \\ &= \left\| \frac{2^{\phi(p)\cdot n}(w)}{p} \right\|_2 \\ &\leq 2^{-\phi(p)n} \end{aligned}$$

We remark that in the final step, we have less than or equal to instead of equal is because $gcd(w,p)$ may not be equal one and we would have $\frac{w}{p} = 2^m \frac{q'}{p}$ where $gcd(q',p) = 1$ so $\left\| \frac{2^{\phi(p)\cdot n}(w)}{p} \right\|_2 = \left\| \frac{2^{\phi(p)\cdot n + m}(q')}{p} \right\|_2 = 2^{-\phi(p)n - m} \leq 2^{-\phi(p)n}$. Since $\phi(p) > 0$ and $n \to \infty$, we have $d_2(\frac{2^{\phi(p)\cdot n}(w)+q}{p}, \frac{q}{p}) \to 0$ as $n \to \infty$ by the upper bound we obtained, i.e. $2^{-\phi(p)n}$.

Now we consider the case where $2 \mid p$. Since $2 \mid p$, we have $p = 2^k \cdot z$ where z is odd number. Since $gcd(p, q) = 1$, we must have q is odd and $gcd(q, z) = 1$. Thus, we have $\frac{q}{p} = \frac{q}{2^k \cdot z}$. In particular, we note, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d_2(n, \frac{q}{p}) &= d_2(n, \frac{q}{2^k z}) \\ &= \left\| n - \frac{q}{2^k z} \right\|_2 = \left\| \frac{2^k zn - q}{2^k z} \right\|_2 = 2^{-0} = 1 \end{aligned}$$

Indeed, since q is odd, and $2^k zn$ is even, we have even minus odd is odd, hence we have $\frac{2^k zn - q}{2^k z} = (2)^0 \frac{2^k zn - q}{2^k z}$ is the only way to factor out any 2, so that the norm must be equal $2^{-0} = 1$.

Thus, if $2 \mid p$ then $\frac{q}{p}$ is not in the closure of $\mathbb{N}$.

Finally, we consider the case where $\frac{q}{p} < 0$. In particular, it suffice to consider $p > 0$ and $q < 0$. Then, consider the sequence $\left\{ \frac{2^{\phi(p) \cdot n}(w) - |q|}{p} \right\}_{n=n_0}^{\infty}$, where w is the same as we described above and n_0 is a number in $\mathbb{N}$ to make sure $2^{\phi(p) \cdot n}(w) - |q| > 0$ for all $n \geq n_0$. Then it converges to $\frac{q}{p}$ as we have $\frac{2^{\phi(p) \cdot n}(w) - |q|}{p} - \frac{q}{p} = \frac{2^{\phi(p) \cdot n}(w) - |q| + |q|}{p} = \frac{2^{\phi(p) \cdot n}(w)}{p}$ and the proof is almost the same. Thus, we indeed have

$$\overline{\mathbb{N}} = \{ \frac{q}{p} \in \mathbb{Q} : 2 \nmid p \wedge gcd(p, q) = 1 \}$$

♠

Definition 1.3.14. Let $A \subseteq X$, the ***interior of A***, $int(A)$, is the largest open set contained in A.

Remark 1.3.15. We note we have

$$int(A) := \bigcup_{b_r(x) \subset A} b_r(x)$$

Proposition 1.3.16. *We have $int(A) = A^{c-c} := (\overline{A^c})^c$*

Proof. We note $A^{c-} := \overline{A^c}$ is closed and it contains A^c. Hence A^{c-c} is open and disjoint from A^c, and so it is contained in A. Thus $A^{c-c} \subseteq int(A)$.

Conversely, if $x \in int(A)$ then $\exists r > 0$ such that $b_r(x) \subseteq A$ so that $\{y : d(x, y) \geq r\} \supseteq A^c$ and is closed, thus it contains A^{c-}. Thus, $A^{c-c} \supset b_r(x)$ and so $A^{c-c} \supseteq int(A)$. The proof follows. ♡

Definition 1.3.17. For $a \in A$, a is a ***boundary point*** of A when for all $r > 0$, $b_r(a) \cap A \neq \emptyset$ and $b_r(a) \cap A^c \neq \emptyset$. We write the collection of boundary points of A to be ∂A.

Definition 1.3.18. We will also denote $int(A) = A^o$ in this note.

Lemma 1.3.19. *$\partial A \cup A^o$ is closed for every set A.*

Proof. Let $B = \partial A \cup A^o$ and p be a accumulation point of B. If $p \in B$ then we are done. Suppose $p \notin B$. Hence, $\forall r > 0$, we have $b_r^*(p) \cap B \neq \emptyset$ as it is a accumulation point of B. Since $p \notin A^o$, that is, p is not in the interior of A, then we have $\forall r > 0$, $b_r(p)$ is not a subset of A. Indeed, if $\exists r > 0, b_r(p)$ is a subset of A, hence contained in A, then $p \in A^o$ by definition.

Hence $\forall r > 0$, $b_r(p) \cap A^c \neq \emptyset$. Since $p \notin \partial A$, we have $\exists r_1 > 0$ such that $b_{r_1}(p) \cap A = \emptyset$ or $b_{r_1}(p) \cap A^c = \emptyset$. $b_{r_1}(p) \cap A^c = \emptyset$ would contradict to the fact p is not in the interior of A. Thus we have

$$b_{r_1}(p) \cap A = \emptyset. \tag{1.1}$$

Since p is a limit point of B, we have $q_r \in B$ such that $q_r \neq p$ and $q_r \in b_{r_1}(p)$. If $q_r \in A^o$ then $q_r \in b_{r_1}(p)$ and A at the same time, which leads to contradiction by equation 1.1. Thus $q_r \in \partial A$. Hence, $\forall r > 0$, we have $b_r(q_r) \cap A \neq \emptyset$. In particular, note $q_r \in b_{r_1}(p)$, where $b_{r_1}(p)$ is a open set, hence $\exists \delta > 0$ such that $B(q_r, \delta) \subseteq b_{r_1}(p)$ since q_r is an element of an open set. Hence $b_\delta(q_r) \cap A \neq \emptyset$ and $B_\delta(q_r) \cap A = \emptyset$ at the same time and thus a contradiction. So, either $p \in B$ or p will lead to a contradiction and we have $p \in B$ for every limit point of B, thus B is closed. ♡

Lemma 1.3.20. *We have $\overline{A} \backslash A^o = \partial A$.*

Proof. We will show that $A^o \cup \partial A = A \cup A' = \bar{A}$.

Let $p \in A^o \cup \partial A$.

Case One Let $p \in A^o$, then $p \in A \subseteq A \cup A'$ by the definition of interior point.

Case Two Let $p \in \partial A$. Suppose p is not a limit point, then $\exists r > 0$ such that $\forall q \in A$, we have $q \notin b_r^*(p)$. However, for this r, we must also have $b_r(p) \cap A \neq \emptyset$ as $p \in \partial A$, hence we must have $p \in A$ as otherwise $b_r(p) \cap A = \emptyset$. Hence $p \in A \cup A'$. So p is either a limit point, or a point of A, which in both case, we have $p \in A \cup A'$.

Since in all case, p was arbitrary, we have $A^o \cup \partial A \subseteq A \cup A'$.

Let $p \in A$. Suppose p is not a boundary point, that is, $\exists r > 0$ such that $b_r(p) \cap A = \emptyset$ or $b_r(p) \cap A^c = \emptyset$. If $b_r(p) \cap A = \emptyset$ then in particular we have $p \notin A$, a contradiction. Thus it is impossible to have $b_r(p) \cap A = \emptyset$. Now, suppose $b_r(p) \cap A^c = \emptyset$, then $b_r(p) \subseteq A$. Hence $p \in A^o$. If p is a boundary point, then $p \in \partial A$. Therefore, $A \subseteq \partial A \cup A^o$. In addition, we learned in class that $\bar{A}$ is the smallest closed set such that contains A, so we must have $\bar{A} \subseteq \partial A \cup A^o$ if $\partial A \cup A^o$ is closed. It is closed by above Lemma so we are done.

Hence $\bar{A} = \partial A \cup A^o$ and therefore $\partial A = \bar{A} \backslash A^o$. ♡

Remark 1.3.21. For all set U, we have $(U')' \in U'$. We proved this in lecture.

Lemma 1.3.22. *Let A be open, then we have $\partial(\overline{A}) \subseteq A'$.*

Proof. Since A is open, we have $A \subseteq A'$. Indeed, let $x \in A$ then $\exists r_0 > 0$ so that $b_{r_0}(x) \subseteq A$ hence there exists some $y \neq x$ where $y \in b_{r_0}(x)$. In particular, we get a sequence $\{y_i\}_{i=1}^{\infty}$ where

$$x \neq y_i \in b_{\frac{r_0}{i}}(x)$$

Then this sequence converges to x and it is an accumulation point of A.

Thus, we have $\overline{A} = A \cup A' = A'$. Thus, we have $\partial(A \cup A') = \partial(\overline{A}) = \overline{A} \backslash \overline{A}^o \subseteq \overline{A} = A'$ ♡

Example 1.3.23 (Kuratowski's closure-complement problem)**.** With all the above lemmas and remarks, we can prove this closure-complement problem. In this question, we write A^- instead of $\overline{A}$. Consider the collection of sets obtained by repeated application of closure and complement inside (X, d), for example, A^{-c-c} and A^{c-}. Try to show the following:

1. Let U be open in X and define $B = U^-$, then $B = B^{c-c-}$
2. Starting with a set A, we have at most 14 possible sets, including A itself, obtained by repeated use of closure and complement.

Solution. We note $\overline{U} = U \cup U'$ where U' is the set of all accumulation points of U. Moreover, we have B^{c-c} is equal $int(B)$ by a proposition. Thus, it suffice to show, for all closed set B obtained by taking closure of open sets, we have $B = \overline{int(B)}$.

Suppose $B = \overline{U}$ where U is open.

We first exam the elements in $\overline{int(B)}$. Clearly, we have $\overline{int(B)} = int(B) \cup (int(B))'$. If $x \in int(B)$ then $x \in B$. Thus, if we can show $x \in (int(B))'$ imply $x \in B$, we would have $B \supseteq \overline{int(B)}$. We have

$$x \in (int(B))' \Leftrightarrow \forall \delta > 0, b^*_\delta(x) \cap int(B) \neq \emptyset$$

In particular, this and $int(B) \subseteq B$ imply

$$\delta > 0, b^*_\delta(x) \cap B \neq \emptyset \Rightarrow x \in B'$$

Hence, we have $B \supseteq \overline{int(B)}$.

Conversely, suppose $x \in B = \overline{U}$. Then, we have $B \backslash int(B) = \partial B$ as $B = \overline{B}$. First, suppose $x \in int(B)$, then $x \in \overline{int(B)}$. Next, suppose $x \notin int(B)$, then $x \in B \backslash int(B) = \partial B$. If we can show $x \in \partial B \Rightarrow x \in \overline{int(B)}$, then $\forall x \in B, x \in \overline{int(B)}$ and we are done.

In particular, we note $x \in \partial B \iff x \in \partial(U \cup U') \subseteq U' \Rightarrow x \in U'$. Next, we note $U \subseteq int(U \cup U')$ as U is open in $U \cup U'$ so that it must be contained in $int(U \cup U')$. Moreover, we must have $U' \subseteq int(U \cup U')'$, as every convergent sequences in U also in $int(U \cup U')$, so that every accumulation points of U must be also in $int(U \cup U')$ as an accumulation point. Thus, $U' \subseteq int(U \cup U')'$. Thus $x \in U'$ imply $x \in int(U \cup U')' \subseteq \overline{int(U \cup U')}$.

Hence, we have $x \in B\backslash int(B)$ then $x \in \partial B$, thus $x \in U'$, thus $x \in \overline{int(B)}$. In particular, this gives $\forall x \in B, x \in \overline{int(B)}$ and so $B \subseteq \overline{int(B)}$

Thus, $B = \overline{int(B)} = B^{c-c-}$ as desired.

To show the second question, we conduct proof by example.

Let A be given. We note $A^{--} = A^{-}$ and $A^{cc} = A$. We also note, by part 1, we have $(A^{c-c-})^{c-c-} = A^{c-c-}$ as A^{c-c} is the interior of A, hence a open set, so $(A^{c-c})^{-}$ is as we described in part 1 and so $(A^{c-c-})^{c-c-} = A^{c-c-}$.

Let A be the set

$$A := (0,1) \cup (1,2) \cup \{3\} \cup \{4 \leq x \leq 5 : x \in \mathbb{Q}\}$$

We will state the operation without mentioning the set. For instance, we would say the set is $c-c-c$ instead of A^{c-c-c}.

The first set we obtained is A, by taking cc.

The second set we obtained is $\overline{A}$, by taking $-$, namely $[0,1] \cup [1,2] \cup \{3\} \cup [4,5]$. The 3rd set we obtained is A^c, by taking c, namely

$$(-\infty, 0] \cup \{1\} \cup [2,3) \cup (3,4) \cup ((4,5)\backslash\mathbb{Q}) \cup (5,\infty)$$

The 4th set we obtained is $c-$, and we get

$$(-\infty, 0] \cup \{1\} \cup [2,\infty)$$

The 5th set we obtained is $-c$,

$$(-\infty, 0) \cup (2,3) \cup (3,4) \cup (5,\infty)$$

The 6th set we obtained is $c-c$, the interior of A, i.e. we have

$$(0,1) \cup (1,2)$$

The 7th is $-c-$, this is different from others as we consider, i.e.

$$(-\infty, 0] \cup [2,3] \cup [3,4] \cup [5,\infty)$$

The 8th set is $c-c-$, and we have

$$[0,1] \cup [1,2] := [0,2]$$

The 9th set is $-c-c$, and we have

$$(0,2) \cup (4,5)$$

The 10th set is $c-c-c$, and we have

$$(-\infty, 0) \cup (2,\infty)$$

The 11th set is $-c-c-$, and we have

$$[0,2] \cup [4,5]$$

The 12th set is $c-c-c-$, and we have

$$(-\infty,0] \cup [2,\infty)$$

The 13th set is $-c-c-c$ and we have

$$(-\infty,0) \cup (2,4) \cup (5,\infty)$$

The 14th set is $c-c-c-c$ and we have

$$(0,2)$$

Clearly those are different sets, so we have at least 14 different possible sets for A. Next, if we can show for any set A, we would have at most 14 possibilities, then we are done. Moreover, we note cc would result in the same set, so we cannot have cc appear in the string. We note $--$ would result in $-$, so we cannot have $--$ appear in the string. Therefore, all the valid actions are in the form of $c-c-...$ alternating or $-c-c...$ alternating.

In particular, if it is $c-c-$ alternating, then $c-c-c-c-$ is equal $c-c-$ by part 1. Hence, it jumps back to one of our 14 possibilities.

If it is $-c-c$ alternating, then $-c-c-c-$ is equal to $-c-$. Indeed, we note $-c$ must be a open set, so we can apply part 1. Let U be arbitrary set, then U^{-c} must be open, hence we have $U^{-c-} = U^{-c-c-c-}$. Therefore, $-c-c-c-$ jumps back to $-c-$.

Therefore, we indeed have 14 is the maximal possible number of distinct sets. ♠

1.4 Continuity

Definition 1.4.1. If (X,d) and (Y,p) are metric spaces and $f: X \to Y$, then

1. f is ***continuous*** at x if

$$\forall \epsilon > 0, \exists \delta > 0, d(x,x') < \delta \Rightarrow p(f(x), f(x')) < \epsilon$$

2. f is ***continuous*** if it is continuous at all points $x \in X$.
3. f is ***sequentially continuous*** if $(x_n) \to x$ is a sequence in X converges to x then $\lim\limits_{n\to\infty} f(x_n) = f(x)$

Theorem 1.4.2. *Let (X,d) and (Y,p) be metric spaces, $f: X \to Y$, then the following are equivalent:*

1. *f is continuous*
2. *f is sequentially continuous*
3. *$\forall V \subseteq Y$ that is open, we have $f^{-1}(V)$ is open in X.*

Proof. We first show $1 \Rightarrow 3$. Let V be open in Y, then there exists $x \in f^{-1}(V)$, i.e. $f(x) \in V$. Since V is open, there exists $b_\epsilon(f(x)) \subseteq V$. By continuity, $\exists \delta$ such that $f(b_\delta(x)) \subseteq b_\epsilon(f(x)) \subseteq V$. In particular, then we have $b_\delta(x) \subseteq f^{-1}(V)$ for all $x \in f^{-1}(V)$.

We show $3 \Rightarrow 2$ and 1. Let $x_n \to x$ in X, let $\epsilon > 0$ be given, take $V = b_\epsilon(f(x))$, we have V is open. By 3, we have $f^{-1}(V)$ is open, contains x. Thus, there exists $\delta > 0$ such that $b_\delta(x) \subseteq f^{-1}(V)$, so $f(b_\delta(x)) \subseteq b_\epsilon(f(x))$, i.e. f is continuous at x. This proves $3 \Rightarrow 1$.

Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that $n \geq N \Rightarrow d(x_n, x) < \delta$, which give us $p(f(x), f(x')) < \epsilon$. This proves $3 \Rightarrow 2$.

Now, we show $2 \Rightarrow 1$ by showing $\neg 1 \Rightarrow \neg 2$. Suppose f is not continuous at x. Therefore, there exists ϵ_0 so that $\forall \delta > 0$, $f(b_\delta(x)) \not\subseteq b_{\epsilon_0}(f(x))$. For $\delta = \frac{1}{n}$, pick $x_n \in b_\delta(x)$ such that $p(f(x), f(x_n)) \geq \epsilon_0$, then $x_n \to x$ but $f(x_n) \not\to f(x)$ ♡

Remark 1.4.3. The above theorem ensures that we can define continuity on any topological spaces, not just to the metric spaces.

Definition 1.4.4. Let (X, d) and (Y, p) are metric spaces, then

1. $f : X \to Y$ is an ***isometry*** if $p(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.
2. f is ***Lipschitz*** if $\exists c < \infty$ such that $p(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in X$.
3. f is ***biLipschitz*** if $\exists 0 < c \leq C < \infty$ such that $c \cdot d(x, y) \leq p(f(x), f(y)) \leq C \cdot d(x, y)$ for all $x, y \in X$
4. f is ***uniformly continuous*** if $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in X, d(x, y) < \delta \Rightarrow p(f(x), f(y)) < \epsilon$
5. f is a ***homeomorphism*** if f is continuous bijection and f^{-1} is continuous.

Lemma 1.4.5. *Lipschitz functions (with constant c) are uniformly continuous.*

Proof. If $c = 0$ then we are done as the function is constant. Let $\epsilon > 0$, let $\delta = \frac{\epsilon}{c}$, then $d(x, x') < \epsilon / c \Rightarrow p(f(x), f(x')) < c \cdot d(x, x') < \epsilon$. ♡

Proposition 1.4.6. *Let X, Y are metric spaces. If $f : X \to Y$ is biLipschitz and surjective, then f is a homeomorphism.*

Proof. We see f is injective, because $x \neq x'$ then $0 < c \cdot d(x, x') \leq p(f(x), f(x'))$ so that $f(x') \neq f(x)$. Thus, f is bijection, so that f^{-1} is a well-defined function.

We claim f^{-1} is Lipschitz with constant $1/c$. Let $y, y' \in Y$ where $y = f(x), y' = f(x')$, we have $cd(x, x') \leq p(f(x), f(x'))$, then we have

$$\frac{c}{c} d(f^{-1}(y), f^{-1}(y')) \leq \frac{1}{c} p(y, y')$$

and so f^{-1} is Lipschitz and hence continuous and so f is homeomorphism. $\heartsuit$

Definition 1.4.7. Two metric d and p on X are equivalent if $\exists 0 < C \leq D < \infty$ such that

$$Cd(x,y) \leq p(x,y) \leq Dd(x,y)$$

Two norm $|\cdot|$ and $\|\cdot\|$ on V are equivalent if

$$\exists 0 < C \leq D < \infty, C|v| \leq \|v\| \leq D|v|$$

Proposition 1.4.8. *If d_1, d_2 are equivalent metrics on X, then (X, d_1) and (X, d_2) have the same open sets.*

Proof. We note $f : (X, d_1) \to (X, d_2)$ by $f(x) = x$ is biLipschitz. Moreover, f is surjective, thus f is homeomorphism, i.e. f is continuous. Thus, for all open set V in (X, d_2), we must have $f^{-1}(V)$ is open in (X, d_1), i.e. V is open in (X, d_1). In addition, we note $g(x) := f^{-1}(x) = x$ is a continuous mapping from (X, d_2) to (X, d_1), so that $g^{-1}(V)$ is open in (X, d_2) for all open set $V \in (X, d_1)$. The proof follows. $\heartsuit$

Example 1.4.9. Note (X, d_1) and (X, d_2) have the same open sets does not imply the two metrics are equivalent.

Consider $\mathbb{N}$ with the discrete metric $d(x, y)$, then every subset of $\mathbb{N}$ is open. Consider $\mathbb{N}$ with the metric $p(x, y) = |x - y|$. We note this is a metric as absolute value as a metric holds for $\mathbb{R}$, $\mathbb{N} \subset \mathbb{R}$, and absolute value maps $\mathbb{N} \to \mathbb{N}$. This is a metric and every subset of $\mathbb{N}$ is open. Indeed, let $A \subseteq \mathbb{N}$ be given. Let $x \in A$, then take $b_{\frac{1}{2}}(x) = \{y \in \mathbb{N} : |x - y| < \frac{1}{2}\} = \{x\} \subseteq A$. Thus every set in $\mathbb{N}$ is open as desired.

However, let $0 \leq C \leq D < \infty$ be given as in the equivalent relation of metrics. Let $\lceil x \rceil$ be the ceiling function and $\lfloor x \rfloor$ be the floor function. Take $x = 2\lceil D \rceil + 10, y = \lceil D \rceil$, we would have $Dd(x, y) = D$ and $p(x, y) = \lceil D \rceil + 10$. In particular, this imply we cannot have $Cd(x, y) \leq p(x, y) \leq Dd(x, y)$ as $p(x, y) > Dd(x, y)$. Moreover, if we take $x = \lceil \frac{1}{C} + 1 \rceil + 1, y = 1$, then $Cp(x, y) = C(\lceil \frac{1}{C} + 1 \rceil) \geq C \cdot \frac{1}{C} + C > 1$. Thus $Cp(x, y) > d(x, y)$ and so $Cp(x, y) \leq d(x, y) \leq Dp(x, y)$ can never happen as well. In particular, this imply even $(\mathbb{N}, d)$ and $(\mathbb{N}, p)$ have the same open sets, they are not equivalent.

Remark 1.4.10. If $f : (X, d) \to (Y, p)$ is bijection and biLipschitz, then it is homeomorphism. Moreover, define $d_2(x, y) = p(f(x), f(y))$, then there exists $0 \leq C < D < \infty$ so that $Cd(x, y) \leq d_2(x, y) \leq Dd(x, y)$, which is a equivalent metric.

Proposition 1.4.11. *If $f : X \to Y$ is homeomorphism, then f maps the collection of all open sets of X onto the collection of all open sets of Y.*

Proof. Let $U \subseteq X$ be open, then $f(U) = (f^{-1})^{-1}(U) = V$ is open by continuity of f^{-1}. If $V \subseteq Y$ then $f^{-1}(V) = U$ is open because f is continuous and hence we obtained the surjective part. $\heartsuit$

Example 1.4.12. Let $X = [0, 1)$ with the usual metric[1] be a metric space. Let $f(x) = e^{2\pi ix}$, then f maps X to the unit circle S, where f is a bijection. Note f is surjective and Lipschitz. Indeed, note

$$|e^{2\pi ix} - e^{2\pi iy}| = |e^{2\pi i(x-y)} - 1| = 2\sin(\pi|x - y|) \leq 2\pi|x - y|$$

Even f is Lipschitz and bijective, f is NOT a homeomorphism. Indeed, consider

$$f(1 - \frac{1}{n}) = e^{2\pi i(1-\frac{1}{n})} \to 1$$

however, we have $f^{-1}(e^{2\pi i(1-1/n)}) = 1 - \frac{1}{n} \not\to f^{-1}(1) = 0$.

1.5 Completeness

Definition 1.5.1. Let (X, d) be a metric space. A sequence $(x_n)_{n\geq 1}$ in X is ***Cauchy sequence*** if

$$\forall \epsilon, \exists N \in \mathbb{N}, \forall n, m \geq N, d(x_n, x_m) < \epsilon$$

Definition 1.5.2. A metric space is ***complete*** if every Cauchy sequence converges in X.

Proposition 1.5.3. *If $\lim_{n\to\infty} x_n = x$ then (x_n) is Cauchy.*

Proof. Let $\epsilon > 0$, let N be so that $i > N$ imply $d(x_i, x) < \epsilon/2$. Then $n, m > N$ imply we would have $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \epsilon$. ♡

Proposition 1.5.4. *If $(x_n)_{n\geq 1}$ is a Cauchy sequence in X and there is a subsequence $(x_{n_l})_{l\geq 1}$ such that $\lim_{l\to\infty} x_{n_l} = x$, then $\lim_{n\to\infty} x_n = x$*

Proof. Let $\epsilon > 0$. Find N so that $n, m \geq N$ then $d(x_n, x_m) < \epsilon/2$. Since $\lim x_{n_l} = x$, there exists M such that if $i \geq M$, then $d(x_{n_i}, x) < \epsilon/2$.

Pick N_0 so that $n > N_0$ imply $n > n_M$ and let $N_1 = \max\{N, N_0\}$, if $n > N_1$, then there exists n_{i_0} such that $i_0 > M$ and $d(x_{n_{i_0}}, x) < \epsilon/2$ and $d(x_{n_{i_0}}, x_n) < \epsilon/2$ and so

$$d(x_n, x) \leq d(x_{n_{i_0}}, x_n) + d(x_{n_{i_0}}, x) < \epsilon$$

♡

Theorem 1.5.5. *If $S \subseteq \mathbb{R}$ is bounded, then S has a least upper bound and a greatest lower bound.*

Proof. I will skip this part... ♡

[1] If we are using the geodesic metric S then $\rho(e^{2\pi ix}, e^{2\pi iy}) = 2\pi|x - y|$ so ρ is equivalent to the usual metric from $\mathbb{R}^2$

Corollary 1.5.5.1. *If $x_n \in \mathbb{R}$, $x_1 \leq x_2 \leq \ldots$ is bounded above. Then $\lim x_n$ exists.*

Proof. Let $L = \sup\{x_n\}_{n \geq 1}$. Let $\epsilon > 0$, then $L - \epsilon$ is not an upper bound, so there exists $x_N > L - \epsilon$ so that $\forall n \geq N, L - \epsilon < x_N \leq x_n \leq L$ and the proof follows. ♡

Example 1.5.6.

1. Let X be a set with discrete metric. If $(x_n)_{n \geq 1}$ is Cauchy, take $\epsilon = 1$, then there exists N so that for all $n, m \geq N$ and $d(x_n, x_m) < 1$. Thus $x_n = x_m$, and so this sequence must be a constant sequence for n is large enough. Hence (x_n) is complete.
2. We have $\mathbb{R}^n$ is complete[1], $n \in \mathbb{N}$. If (x_n) is Cauchy in $\mathbb{R}$, then x_n is bounded. Indeed, let $\epsilon = 1$, there exists N so for all $n \geq N$, we have $|x_n - x_m| < 1$. In particular, $x_n < |x_n| + 1$. So, we have $|x_i| \leq \max\{|x_1|, \ldots, |x_N|, |x_N| + 1\}$ for all $i \in \mathbb{N}$ and hence it is bounded. Then, by split in half, pick one half with infinitely many points in (x_n), let a_i, b_i be the end points for each selection, we have $a_1 \leq a_2 \leq \ldots \leq b_2 \leq b_1$. Then, we have $\lim a_n = L = \lim b_n$ by Theorem 1.5.5 and so there is a subsequence of (x_n) converges. Then, by Proposition 1.5.4, we have every Cauchy in $\mathbb{R}$ converges.
   As we established the case of $\mathbb{R}$, we can show $\mathbb{R}^n$ is complete by considering the component sequences of the Cauchy sequence in $\mathbb{R}^n$.

Proposition 1.5.7. *If (X, d) is complete and $Y \subseteq X$ then (Y, d) is complete if and only if Y is closed.*

Proof. Suppose Y is complete. Suppose $\{y_n\}$ is a sequence in Y and $\lim y_n \to x$ where $x \in X$. Since $\{y_n\}$ converges, we have $\{y_n\}$ is Cauchy. Since Y is complete, we must have every Cauchy sequences converges in Y, i.e. $x \in Y$.

Suppose Y is closed. Let $\{y_n\}$ be a Cauchy sequence in Y. Thus $\{y_n\}$ is Cauchy in X, so $\{y_n\}$ converges in X, in particular, since Y is closed, $\{y_n\}$ must converge in Y. ♡

Proposition 1.5.8. *Let (X, d) be a complete metric space, and let Y be an open subset of Y. Then, there exists a metric ρ on Y which has the same open sets as (Y, d) and make (Y, ρ) complete.*

Proof. Let $f : Y \to \mathbb{R}_+$ to be

$$f(y) = \frac{1}{\inf_{x \in Y^c}(d(x, y))}$$

We remark that when $y \to Y^c$, we have $f(y) \to \infty$. Next, we show it is continuous.

For $y \in Y$, we denote $\overline{y} := f(y)^{-1} = \inf_{x \in Y^c}(d(x, y))$.

[1] This is because Bolzano–Weierstrass theorem, i.e. every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence

Let $\epsilon > 0$ and $y \in Y$ be given. Let

$$\delta < \min\{\frac{\epsilon \overline{y}^2}{1+\overline{y}\epsilon}, \overline{y}\}$$

In particular, we remark that

$$\begin{aligned} &\delta < \frac{\epsilon \overline{y}^2}{1+\overline{y}\epsilon} \\ &\Rightarrow \delta(1+\overline{y}\epsilon) < \epsilon \overline{y^2} \Rightarrow \delta < \epsilon \overline{y}^2 - \delta \overline{y}\epsilon \\ &\Rightarrow \frac{\delta}{\overline{y}(\overline{y}-\delta)} < \epsilon \end{aligned}$$

Then, suppose $d(x,y) < \delta$. We first note for all $t \in Y^c$, we have, by triangle inequality,

$$d(y,t) - d(x,y) \leq d(x,t) \leq d(t,y) + d(x,y)$$

In particular, when we take infimum over all $t \in Y^c$, we get

$$\overline{y} - \delta < \overline{y} - d(x,y) \leq \overline{x} \leq \overline{y} + d(x,y) < \overline{y} + \delta \Rightarrow |\overline{x} - \overline{y}| < \delta$$

Then, note both $\overline{x}, \overline{y}$ are greater than 0, we have $|\overline{xy}| = \overline{x} \cdot \overline{y}$ and since $\overline{y} - \delta < \overline{x}$, we have $(\overline{y} - \delta)\overline{y} < \overline{xy}$ and thus

$$\frac{1}{\overline{xy}} < \frac{1}{\overline{y}(\overline{y}-\delta)}$$

Together, we get

$$|f(x) - f(y)| = \left|\frac{\overline{y} - \overline{x}}{\overline{x} \cdot \overline{y}}\right| < \frac{\delta}{\overline{x} \cdot \overline{y}} < \frac{\delta}{\overline{y}(\overline{y}-\delta)} < \epsilon$$

Thus f is continuous on Y as y was arbitrary.

Next, define $\rho : Y \times Y \to \mathbb{R}_+$ to be $\rho(x,y) = d(x,y) + |f(x) - f(y)|$. We will show this is a metric, d and ρ form the same open sets, and then show (Y, ρ) is complete.

To show this is a metric, we note $d(x,y)$ is a metric and $|x - y|$ is also a metric (and hence both always greater than or equal to zero), so that ρ must be. Indeed, $x, y, z \in Y$ be given, we have $\rho(x,y) = \rho(y,x)$ trivially and $\rho(x,y) = 0$ if and only if $d(x,y) = |f(x) - f(y)| = 0$ if and only if $x = y$. Moreover,

$$\begin{aligned} \rho(x,z) &= d(x,z) + |f(x) - f(z)| \\ &\leq d(x,y) + d(y,z) + |f(x) - f(y)| + |f(y) - f(z)| \\ &= \rho(x,y) + \rho(y,z) \end{aligned}$$

Next, we show they have the same open sets. Let $\xi : (Y,d) \to (Y,\rho)$ to be $\xi(x) = x$. We will show ξ is continuous, and then since it is obviousely invertible, we must have both metrics give the same topology. Let $\epsilon > 0$ and $y \in Y$ be given. Since $f(x)$ is continuous on (Y, d), there exists $\delta_1 > 0$ such that $d(x, y) < \delta_1 \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2}$. Let $\delta = \min\{\frac{\epsilon}{2}, \delta_1\}$. Then, for all $x \in (Y, d)$ such that $d(x, y) < \delta$, we have

$$\rho(x, y) = d(x, y) + |f(x) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore d and ρ gives the same topology as ξ is continuous. Indeed, ξ pulls every open set in (Y, d) to the same open set in (Y, ρ) and (Y, ρ) pulls every open set in (Y, ρ) to the same open set in (Y, d).

Finally, we show (Y, ρ) is complete under this new metric.

Let $(y_n)_{n=1}^{\infty}$ be a Cauchy sequence in (Y, ρ). Then, for all $\epsilon > 0$, there exists $N > 0$ so that $n, m > N$ imply $\rho(y_n, y_m) < \epsilon$. Thence, $\forall n, m > N, d(y_n, y_m)+|f(y_n)-f(y_m)| < \epsilon$ and therefore $d(y_n, y_m) < \epsilon$. Thus $(y_n)_{n=1}^{\infty}$ is Cauchy in (Y, d). This apply to all Cauchy in (Y, ρ), i.e. every Cauchy under ρ is also Cauchy under d.

Note for each Cauchy in (Y, ρ), we have either (y_n) converges inside (Y, d) or inside $X \backslash Y$ as X is complete (so it must converge at least in X).

Suppose $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence in (Y, ρ) that converges to a point in (Y, d). Say it converges to $y \in (Y, d)$. We will show it also converges in (Y, ρ). Let $\epsilon > 0$, we have f is continuous on (Y, d) so that there exists $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon/2$. Moreover, since it is Cauchy in (Y, d), there exists N_1 so $d(y_n, y) < \epsilon/2$ for all $n, m > N_1$. Let N_2 be so that $d(y_n, y) < \delta$. Then, let $N = \max\{N_1, N_2\}$, we have for all $n > N$ that $\rho(y_n, y) = d(y_n, y)+|f(n)-f(m)| < \epsilon$ as desired.

Suppose $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence in (Y, ρ) that converges outside of (Y, d), i.e. $y_n \to x \in Y^c$ in (Y, d).

Next, let $\epsilon = 1$. For all $N \in \mathbb{N}$, let $n > N$ be so that $d(y_n, x) < \frac{1}{2}$, say $d(y_n, x) = \delta$, then take $m > N$ to be so that $d(y_m, x) < \frac{\delta}{2}$. We can do this because $y_n \to x$ in (Y, d). This enforces that $f(y_n) > \frac{1}{\delta} > 2$ and $f(y_m) > \frac{2}{\delta} > 4$.

Then, we have $\rho(y_n, y_m) = d(y_n, y_m) + |f(y_n) - f(y_m)| > d(y_n, y_m) + 2 > 1 = \epsilon$. Henec (y_n) is not Cauchy in (Y, ρ), a contradiction. Thus if (y_n) is Cauchy in (Y, ρ), then in (Y, d) it converges in Y, and so it converges in (Y, ρ) into Y. Thus Y is complete. ♡

Definition 1.5.9. If $A \subseteq (X, d)$, we say the ***diameter*** of A is

$$diam(A) = \sup_{x,y \in A} d(x, y)$$

Proposition 1.5.10. *A metric space (X, d) is complete if and only if, whenever A_n are nested closed sets, i.e. $A_n \supseteq A_{n+1}$, with $diam(A_n) \to 0$, then $\bigcap_{n \geq 1} A_n$ is not empty.*

Proof. Suppose (X, d) is complete, then every Cauchy sequence converges. Let $\{A_n\}_{n=1}^{\infty}$ be a collection of non-empty closed sets such that $A_n \supseteq A_{n+1}$ for $n \in \mathbb{N}$. Then, let $x_1 \in A_1$ such that $x_1 \in A_2$. Such x_1 exists as $A_1 \supseteq A_2$ and they are both non-empty. Let $x_2 \in A_2$ such that $x_2 \in A_3$, and recursively, we let $x_k \in A_k$ such that $x_k \in A_{k+1}$. We obtained a infinite sequence $(x_n)_{n=1}^{\infty}$ in A_1. Then, since $diam(A_n) \to 0$, for all $\epsilon > 0$, we have N so that $n > N$ imply $diam(A_n) < \epsilon$. In particular, for $n, m > N$, without lose of generality, assume $n > m > N$, then we have $x_n, x_m \in A_m$ and hence

$$d(x_n, x_m) \leq \sup_{x,y \in A_m} d(x, y) < \epsilon$$

as $m > N$. Thus (x_n) is a Cauchy sequence in X and so it must converges, i.e. say $x_n \to x$. We claim we must have $x \in \bigcap_{n \geq 1} A_n$. Let A_k be given, then we have $(x_n)_{n=k}^{\infty}$, which is a subsequence of $(x_n)_{n=1}^{\infty}$ and a sequence in A_k such that $(x_n)_{n=k}^{\infty} \to x$, since A_k is closed, we have $x \in A_k$. Since this holds for every $k \in \mathbb{N}$, $x \in \bigcap_{n \geq 1} A_n$ as desired.

Conversely, let $(x_n)_n$ be a Cauchy sequence in (X, d). Let $A_1 = \overline{\{x_i \in (x_n)_n : i \geq 1\}}$, let $A_2 = \overline{\{x_i \in (x_n)_n : i \geq 2\}}$ and inductively, $A_k = \overline{\{x_i \in (x_n)_n : i \geq k\}}$. We claim $A_n \supseteq A_{n+1}$ and $diam(A_n) \to 0$. We will show $diam(A_n) \to 0$ as $A_n \supseteq A_{n+1}$ follows immediately by definition. Let $\epsilon > 0$ be given. Since (x_n) is Cauchy, $\exists N \in \mathbb{N}$ such that $n, m > N$ imply $d(x_n, x_m) < \epsilon/2$. In particular, for $n > N$, we then have $\forall x, y \in A_n$ that

$$d(x, y) < \epsilon/2 \Rightarrow \sup_{x,y \in A_n} d(x, y) \leq \epsilon/2 < \epsilon \Rightarrow n \to \infty \text{ then } diam(A_n) \to 0$$

Hence, we have $\bigcap_{n \geq 1} A_n \neq \emptyset$, in particular, there exists $x \in X$ such that $x \in \bigcap_{n \geq 1} A_n$. We claim that $(x_n)_n \to x$. Indeed, since x in all of A_n, and $diam(A_n) \to 0$, we have $d(x, x_n) \to 0$ so that $(x_n)_n$ converges in X, hence complete. ♡

Definition 1.5.11. A complete normed vector space is called a ***Banach space***.

Proposition 1.5.12. *For $1 \leq p \leq \infty$, we have ℓ_p is complete.*

Proof. We first suppose $p = \infty$. Let $\{x_n\}_{n \geq 1}$ be a Cauchy sequence in ℓ_∞ where $x_n = (x_{n,1}, x_{n,2}, ...) \in \ell_\infty$ be Cauchy. Then, we have

$$\forall \epsilon > 0, \exists N, \forall n, m \geq N, \|x_n - x_m\|_\infty = \sup_{i \geq 1} |x_{n,i} - x_{m,i}| < \epsilon$$

Thus, $\{x_{n,l}\}_n$ is Cauchy for each l. So, by completeness of $\mathbb{R}$ or $\mathbb{C} \cong \mathbb{R}^2$, we have $\lim_{n\to\infty} x_{n,l} = a_l$ exists. We then let $A = (a_1, a_2, a_3, ...)$. Thus, recall

$$\forall n, m \geq N, |x_{n,l} - x_{m,l}| < \epsilon$$

Fix n, let $m \to \infty$, we get

$$|x_{n,l} - a_l| \leq \epsilon$$

Hence, we have

$$\|x_n - A\|_\infty = \sup_{l\geq 1} |x_{n,l} - a_l| \leq \epsilon$$

In particular, $A = x_n + (A - x_n) \in \ell_\infty$ and since ϵ was arbitrary, $\lim x_n = A$ in ℓ_∞ norm and thus ℓ_∞ is complete.

Next, suppose $1 \leq p < \infty$. Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence in ℓ_p where $x_n = (x_{n,1}, x_{n,2}, ...)$. Thus, we have

$$\forall \epsilon > 0, \exists N, \forall n, m \geq N, \|x_n - x_m\|_p^p < \epsilon^p$$

Next, note

$$|x_{n,l} - x_{m,l}|^p \leq \sum_{l=1}^{\infty} |x_{n,l} - x_{m,l}|^p$$

and hence we have $\{x_{n,l}\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ or $\mathbb{C}$ for each l. Let $a_l = \lim_{n\to\infty} x_{n,l}$.

Fix $J < \infty$, let $n, m \geq N$, then we have

$$\sum_{l=1}^{J} |x_{n,l} - x_{m,l}|^p \leq \sum_{l=1}^{\infty} |x_{n,l} - x_{m,l}|^p = \|x_n - x_m\|_p^p < \epsilon^p$$

Let $m \to \infty$, then we have

$$\sum_{l=1}^{J} |x_{n,l} - a_l|^p \leq \epsilon^p$$

Now, let $J \to \infty$, then we have

$$\sum_{l=1}^{\infty} |x_{n,l} - a_l|^p \leq \epsilon^p, \forall n \geq N$$

Set $A = (a_1, a_2, a_3, ...)$, then we have $A = x_n + (A - x_n) \in \ell_p$ and so

$$n \geq N \Rightarrow \|x_n - a\|_p \leq \epsilon$$

Thus, we get $\lim x_n = A$ and thus ℓ_p is complete as claimed. ♡

Example 1.5.13. Consider $\mathbb{C}[a, b]$ with the L^p norm, $1 \leq p < \infty$. This is not complete. Let $f = \begin{cases} 0, & a \leq x \leq \frac{a+b}{2} \\ 1, & \frac{a+b}{2} < x \leq b \end{cases}$. Let $g_n(x) = \begin{cases} 0, & a \leq x \leq \frac{a+b}{2} \\ linear, & inbetween \\ 1, & \frac{a+b}{2} + \frac{1}{n} \leq x \leq b \end{cases}$

Then, we have

$$\begin{aligned} \|f - g_n\|_p^p &= \int_a^b |f(x) - g_n(x)|^p dx \\ &= \int_{(a+b)/2}^{(a+b)/2+1/n} |f(x) - g_n(x)|^p dx \leq \int_{(a+b)/2}^{(a+b)/2+1/n} |1|^p dx = \frac{1}{n} \end{aligned}$$

Thus, $\lim_{n\to\infty} g_n = f$ in L^p norm, so $\{g_n\}$ is Cauchy. However, no continuous function g has $\|g - f\|_p = 0$ as this would force $g = 0$ on $[a, (a+b)/2]$ and $g = 1$ on $[(a+b)/2, b]$, i.e. g is not continuous.

Definition 1.5.14. If V is a normed vector space, we let

$$V^* = \{\phi \in Hom(V, F) : \phi \text{ is continuous}\}$$

be the ***dual space*** of V.

Proposition 1.5.15. *If $\phi \in Hom(V, \mathbb{F})$, the following are equivalent,*

1. *ϕ is continuous,*
2. *ϕ is continuous at 0*
3. *ϕ is Lipschitz, and $\|\phi\|_* = \sup_{\|V\| \leq 1} |\phi(v)| < \infty$. We remark $\|\phi\|_*$ is the Lipschitz constant.*

Proof. 3 imply 1 imply 2. We will show $\neg 3 \to \neg 2$.

If ϕ is Lipschitz, i.e. $\exists C$ so that $|\phi(v)| \leq C \|v\|$. We note $v = \|v\| \frac{v}{\|v\|}$, so that

$$|\phi(v)| = \|v\| \cdot |\phi(\frac{v}{\|v\|})| \leq \|\phi\|_* \cdot \|v\|$$

Conversely, we have $\|\phi\|_* = \sup_{\|v\| \leq 1} |\phi(v)| \leq \sup C \cdot 1 = C$. If $\|\phi\|_* = \infty$, pick $v_n \in V$ with $\|v_n\| \leq 1$ and $|\phi(v_n)| > n^2$. Then

$$\left\| \frac{1}{n} v_n \right\| \leq \frac{1}{n}, \quad \text{so } \frac{1}{n} v_n \to 0$$

However, we have $|\phi(\frac{1}{n} v_n)| > n$ so that $\phi(\frac{1}{n} v_n) \not\to 0 = \phi(0)$, thus ϕ is discontinuous at $v = 0$. ♡

Theorem 1.5.16. *If V is a normed vector space, then $(V^*, \|\cdot\|_*)$ is a Banach space.*

Proof. We first need to show it is a norm. The first two are easy, we check triangle inequality.

We have

$$\begin{aligned}
\|\phi + \psi\|_* &= \sup_{\|v\| \leq 1} |(\phi + \psi)(v)| \leq \sup_{\|v\| \leq 1} |\phi(v)| + |\psi(v)| \\
&\leq \sup_{\|v\| \leq 1} |\phi(v) + \sup_{\|v\| \leq 1} |\psi(v) \\
&= \|\phi\|_* + \|\psi\|_*
\end{aligned}$$

Hence, $\|\cdot\|_*$ is a norm. Next, we show it is complete.

Let $\{\phi_n\}$ be a Cauchy sequence in $\|\cdot\|_*$. Thus, for all $\epsilon > 0$, there exists $N > 0$ so that for all $n, m > N$, $\|\phi_n - \phi_m\|_* < \epsilon$. Thus, we have

$$|\phi_n(v) - \phi_m(v)| \leq \|\phi_n - \phi_m\|_* \|v\| < \epsilon \|v\|$$

Hence, $\{\phi_n(v)\}_{n\geq 1}$ is a Cauchy sequence. Define $\phi(v) = \lim_{n\to\infty}\phi_n(v)$ as we have F is complete so that $\{\phi_n(v)\}$ converges.

If $n, m \geq N$, let $m \to \infty$, then we have

$$|\phi_n(v) - \phi_m(v)| < \epsilon \|v\| \Rightarrow |\phi_n(v) - \phi(v)| \leq \epsilon \|v\|$$

Thus, we have $\|\phi_n - \phi\| \leq \epsilon$ if $n > N$. Thus, we have $\|\phi\|_* \leq \|\phi_n\|_* + \epsilon$, thus $\phi \in V^*$. Since ϵ was arbitrary, we have $\lim \phi_n = \phi$ and the proof follows. ♡

Proposition 1.5.17. *If (X, d) and (X, p) are equivalent metrics, then (X, d) is complete if and only if (X, p) is complete. The statement also holds for normed spaces.*

Proof. We first show that they have the same Cauchy sequences. Indeed, note $d(x, y) < \epsilon \Rightarrow p(x, y) < D\epsilon$ and $p(x, y) < \epsilon \Rightarrow d(x, y) < \frac{1}{C}\epsilon$. In addition, they also have the same convergent sequences for the same reason. Thus, we have complete in (X, d) if and only if (X, p). ♡

Theorem 1.5.18. *If $V = F^n$, then all norms on V are equivalent.*

Proof. Let $\|x\|_2 = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$ be the Euclidean norm. Let $|||\cdot|||$ be another norm on V. Then, we have

$$\begin{aligned}
|||x||| &= |||\sum_{i=1}^n x_i e_i||| \leq \sum_{i=1}^n |||x_i e_i||| \\
&= \sum_{i=1}^n |x_i| \cdot |||e_i||| \leq (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}} (\sum_{i=1}^n |||e_i|||^2)^{\frac{1}{2}}
\end{aligned}$$

So, we have $|||x||| \leq D\|x\|_2$ where $D = (\sum_{i=1}^n |||e_i|||^2)^{\frac{1}{2}}$ and so $f : V \to \mathbb{R}$ where $f(x) = |||x|||$ is continuous with respect to $\|\cdot\|_2$.

Let $S = \{x \in V : \|x\|_2 = 1\}$ be the unit sphere. Then S is closed, bounded and hence compact. By Extreme value theroem, we have f attains its minimum value on S. Thus, there exists $x_0 \in S$, such that $\|x_0\|_2 = 1$ where $|||x_0||| = \inf_{\|v\|_2=1} |||v|||$. Since $x_0 \neq 0$, we have $|||x_0|||$ is not zero. So if $v \in V$, then $\frac{v}{\|v\|_2} \in S$, and so $|||\frac{v}{\|v\|_2}||| \geq |||x_0|||$ and so $|||v||| \geq |||x_0||| \cdot \|v\|_2$. Hence they are equivalent as desired. ♡

Corollary 1.5.18.1. *All norms on F^n are equivalent and have the same topology.*

1.6 Compactness

Definition 1.6.1. If $A \subseteq (X, d)$, then an ***open cover*** of A is a collection $\{U_\lambda\}_{\lambda\in\Lambda}$ of open sets such that $A \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$.

Definition 1.6.2. A ***subcover*** of an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ is a collection $\{U_\lambda\}_{\lambda\in\Lambda'}$ where $\Lambda' \subseteq \Lambda$. In particular, a ***finite subcover*** is when Λ' is finite.

Definition 1.6.3. A set $A \subseteq (X, d)$ is ***compact*** if every open cover of A has a fintie subcover, i.e. if $A \subset \bigcup U_\lambda$ where U_λ is open, then $\exists \lambda_1, ..., \lambda_n$ such that $A \subset \bigcup U_{\lambda_i}$.

Definition 1.6.4. A set A is ***sequentially compact*** if every sequence $(a_n)_n$ in A has a subsequence $(a_{n_i})_i$ such that converges in A.

Definition 1.6.5. A collection of closed sets $\{F_\lambda\}_{\lambda\in\Lambda}$ has the ***finite intersection property***(FIP) if whenever $\lambda_1, ..., \lambda_n \in \Lambda$, we have $\bigcap_{l=1}^n F_{\lambda_l} \neq \emptyset$.

Definition 1.6.6. A set $A \subseteq X$ is ***totally bounded*** if $\forall \epsilon > 0, \exists x_1, ..., x_n \in A$ such that $A \subseteq \bigcup_{i=1}^n b_\epsilon(x_i)$.

Example 1.6.7. We have finite sets are compact. Moreover, $\mathbb{N}$ and $\mathbb{R}$ are not compact as they are not bounded.

Proposition 1.6.8. *Compact sets are bounded.*

Proof. Suppose (X, d) is compact and non-empty. Let $x_0 \in X$. Then we have $X = \bigcup_{n=1}^\infty b_n(x_0)$. Since X is compact, there exists a finite subcover, i.e. $n_1, ..., n_k$ such that

$$X \subseteq \bigcup_{i=1}^k b_{n_i}(x_0) = b_{\max\{n_1,...,n_k\}}(x_0)$$

So, we have $diam(X) \leq 2\max\{n_1, ..., n_k\}$. ♡

Example 1.6.9. We have $(0, 1)$ is not compact.

Example 1.6.10. Show that the closure $\overline{\mathbb{Z}}$ in $(\mathbb{Q}, d_2)$ is totally bounded but not complete.

Solution. Claim: a sequence in $(\mathbb{Q}, d_2)$ is Cauchy if and only if $d_2(x_n, x_{n+1}) \to 0$.

Suppose $(x_n)_{n\geq 1}$ in $(\mathbb{Q}, d_2)$ is Cauchy, then $d_2(x_n, x_{n+1}) \to 0$ by the definition of Cauchy sequence. Indeed, let $\epsilon > 0$ be given, then we have $N \in \mathbb{N}$ such that for all $n, m > N$, we have $d_2(x_n, x_m) < \epsilon$ and in particular $d_2(x_n, x_{n+1}) < \epsilon$ and so it goes to zero.

Conversely, let $(x_n)_{n\geq 1}$ be a sequence in $(\mathbb{Q}, d_2)$ and suppose $d_2(x_n, x_{n+1}) \to 0$. Recall that p-adic metric is ultrametric(this proof holds for all ultrametric, not just $2-$adic), i.e.

$$d_p(x, z) \leq \max\{d_p(x, y), d_p(y, z)\}$$

Let $\epsilon > 0$ be given. Since $d_2(x_n, x_{n+1}) \to 0$, there exists $N > 0$ so that for all $n > N$, we have $d_2(x_n, x_{n+1}) < \epsilon$. In particular, this imply

$$d_2(x_n, x_{n+2}) \leq \max\{d_2(x_n, x_{n+1}), d_2(x_{n+1}, x_{n+2})\} < \epsilon$$

as both $d_2(x_n, x_{n+1})$ and $d_2(x_{n+1}, x_{n+2})$ are less than ϵ. Therefore, we can show, inductively, $d_2(x_n, x_{n+p}) < \epsilon$, which is the same as, for all $m > n$, we have $d_2(x_n, x_m) <$

ϵ. Therefore, for all $m > n > N$, we have $d_2(x_n, x_m) < \epsilon$ and hence it is Cauchy. This proofs our claim.

Next, we answer the question.

Recall from Example 1.3.13, we have $\overline{\mathbb{N}} = \{\frac{p}{q} : 2 \nmid q \wedge q, p \in \mathbb{Z} \wedge gcd(p, q) = 1\}$. We claim $\overline{\mathbb{Z}} = \overline{\mathbb{N}}$. First, since $\mathbb{N} \subset \mathbb{Z}$ we have $\overline{\mathbb{N}} \subseteq \overline{\mathbb{Z}}$. Next, we note if $2 \mid p$ then we have $d_2(z, \frac{q}{p}) = 1$ if $z \in \mathbb{Z}$ and $gcd(p, q) = 1$. Indeed, $gcd(p, q) = 1$ and $2 \mid p$ so that q is odd, and hence $z - \frac{q}{p} = \frac{pz-q}{p}$. Note pz is even so $pz - q$ is odd and hence we cannot factor out any 2 from the expression $\frac{pz-q}{p}$ and hence the 2-adic distance between z and $\frac{q}{p}$ must be $2^0 = 1$. Therefore, it is not in the closure of $\mathbb{Z}$. Therefore, we see $\overline{\mathbb{Z}}$ is exactly the same as $\overline{\mathbb{N}}$.

We claim $\overline{\mathbb{Z}}$ is not complete. Let $x_1 = 3$, $x_2 = 3$ and $x_3 = 7$, we note we have $2^i \mid x_i^2 + 7$ for $i = 1, 2, 3$. Inductively, suppose $n > 3$, let $x_{n+1} = x_n$ if $2^{n+1} \mid x_n^2 + 7$ and if $2^{n+1} \nmid x_n^2 + 7$ then let $x_{n+1} = x_n + 2^{n-1}$. We claim that $2^{n+1} \mid x_{n+1}^2 + 7$. Indeed, if $2^{n+1} \mid x_n$ then we are done. Suppose the other case. Then, we have

$$x_{n+1}^2 + 7 = x_n^2 + 2^n x_n + 2^{2n-2} + 7$$

Since $2^n \mid x_n^2 + 7$ and $2^{n+1} \nmid x_n^2 + 7$ we have $x_n^2 + 7 = b2^n$ where b is odd (otherwise we must have $2^{n+1} \mid x_n^2 + 7$). In addition we remark that x_n must be odd as well, for otherwise we have $x_n^2 + 7$ is odd. Thus, we get

$$x_{n+1}^2 + 7 = 2^n(b + x_n) + 2^{2n-2}$$

where $2 \mid b + x_n$. Hence, we have $2^{n+1} \mid x_{n+1}^2 + 7$ as desired as $2^{n+1} \mid 2^{2n-2}$ for $n \geq 3$.

Thus, we get a sequence such that $2^n \mid x_n^2 + 7$ for all $n \in \mathbb{N}$. In particular, we claim this is Cauchy. Indeed, by part (a), we note $\|x_{n+1} - x_n\|_2 = 0$ if $x_n = x_{n+1}$ or $\|x_{n+1} - x_n\|_2 = \|2^{n-1}\|_2 = 2^{-n+1}$ and this clearly goes to 0 as $n \to \infty$. However, suppose this converges in $\mathbb{Q}$, then we have $x_n \to c \in \mathbb{Q}$ and note

$$\lim_{n\to\infty} x_n^2 + 7 = 0$$

where on the other hand we also have $\lim_{n\to\infty} x_n^2 + 7 = c^2 + 7$. Thus, we have $c^2 + 7 = 0$ in $\mathbb{Q}$, which is a contradiction.

To show $\overline{\mathbb{Z}}$ is totally bounded, let $\epsilon > 0$ be given, there exists n so that $2^{-n} < \epsilon$. Let $\frac{x}{y} \in \overline{\mathbb{Z}}$, then we have $gcd(x, y) = 1$ and $2 \nmid y$. Consider the ring $\mathbb{Z}_{2^n}$, we let $b = 1$, then let $a = y^{-1}x$ where y^{-1} is the invertible element in $\mathbb{Z}_{2^n}$, i.e. $y^{-1}y = 1$ where 1 is the multiplicative identity for $\mathbb{Z}_{2^n}$. Note we indeed have y^{-1} exists because $gcd(y, 2^n) = 1$ as $2 \nmid y$. Then, we get $ay - bx = y^{-1}xy - x = y^{-1}(xy - yx) = 0$ in $\mathbb{Z}_{2^n}$ as multiplication is commutative in $\mathbb{Z}_{2^n}$. Hence, we have $2^n \mid ay - bx$. In particular, back to $\mathbb{Q}$, this imply that $a - \frac{x}{y} = \frac{ay-x}{y}$ and since $2 \nmid y$ we must be able to factor out at least 2^n from the top and get $\frac{ay-x}{y} = 2^n \frac{x'}{y}$. Thus, we get $\left\|a - \frac{x}{y}\right\|_2 \leq 2^{-n} < \epsilon$.

Hence, let the set of points in $\overline{\mathbb{Z}}$ be $\{0, 1, 2, 3, ..., 2^n - 1\}$ where $2^{-n} < \epsilon$. Then, we have, for all $\frac{x}{y} \in \overline{\mathbb{Z}}$, there exists $a = y^{-1}x$ as we described above where $0 \leq a \leq 2^n - 1$. Hence, we would have $\left\|a - \frac{x}{y}\right\|_2 < \epsilon$ as desired so that $\overline{\mathbb{Z}}$ is totally bounded. ♠

Proposition 1.6.11. *We have $A \subseteq (X, d)$ is compact then A is closed.*

Proof. Let $x \in X \backslash A$. Then $a \in A$ imply $d(x, a) > 0$. Thus, $A \subseteq \bigcup_{n=1}^{\infty}\{y : d(x, y) > \frac{1}{n}\}$ is an open cover. So if A is compact, we have a fintie subcover and hence $A \subseteq \bigcup_{i=1}^{N}\{y : d(x, y) > \frac{1}{n_i}\} = \{y : d(x, y) > \frac{1}{\max\{n_1, \ldots, n_N\}}\}$. Thus $b_r(x) \cap A = \emptyset$ and so $X \backslash A$ is open and hence A is closed. ♡

Remark 1.6.12. Note sequentially compact imply closed. Indeed, if $a_n \in A$ and $a_n \to x$ then $\exists a_{n_l} \to y \in A$. Hence every limit points is in A.

Example 1.6.13. Note A is closed and bounded does not imply compact. Let X be an infinite set with the discrete metric. Then $diam(X) = 1$ and $X = \bigcup_{x \in X} b_1(x)$. If X is compact then $X \subseteq \bigcup_{i=1}^{n} b_i(x_i)$ and this imply $|X| < \infty$, a contradiction!

1.7 Borel-Lebesgue Theorem

Definition 1.7.1. An ***ϵ-net*** of X is a collection of points $\{x_i\} \subseteq X$ such that $X \subseteq \cup b_\epsilon(x_i)$.

Remark 1.7.2. X is totally bounded if for all $\epsilon > 0$, X has a finite ϵ-net.

Remark 1.7.3. We also have the ideal of a ***net***, which is the generalization of sequences. Let (X, τ) be a topological space, i.e. τ is a topology on X. Then, a ***directed set*** is a set Λ with a relation $\leq$ such that $\lambda \leq \lambda$; $\lambda_1 \leq \lambda_2, \lambda_2 \leq \lambda_3$ then $\lambda_1 \leq \lambda_3$; and if $\lambda_1, \lambda_2 \in \Lambda$ then $\exists \lambda_3 \in \Lambda$ such that $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$. This $\leq$ is called a ***direction*** on Λ.

Then, a net in X is a function $P : \Lambda \to X$ where Λ is a directed set. The point $P(\lambda)$ is denoted by x_λ and we write $(x_\lambda)_{\lambda \in \Lambda}$ to dentoe the net.

A ***subnet*** of a net $P : \Lambda \to X$ is the composition $P \circ \phi$ where $\phi : M \to \Lambda$ is an increasing ***cofinal*** function from a directed set M to Λ. Viz, we have $\phi(\mu_1) \leq \phi(\mu_2)$ if $\mu_1, \mu_2 \in M$ and $\mu_1 \leq \mu_2$, this is the increasing part, and for each $\lambda \in \Lambda$, there exists $\mu \in M$ so that $\lambda \leq \phi(\mu)$, this is the cofinal part.

For μ in M, we write x_{λ_μ} for $P \circ \phi(\mu)$ and we say the subnet is $(x_{\lambda_\mu})_\mu$.

Theorem 1.7.4 (Borel-Lebesgue)**.** *Let (X, d) be a metric space. Then the following are equivalent:*

1. *X is compact.*
2. *If $\{F_\lambda\}_{\lambda \in \Lambda}$ is a collection of closed set with FIP then $\bigcap_{\lambda \in \Lambda} F_\lambda \neq \emptyset$.*
3. *X is sequentially compact.*
4. *X is complete and totally bounded.*

Proof. We will first show $\neg 2 \to \neg 1$. Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be closed sets with FIP but $\bigcap_{\lambda\in\Lambda} F_\lambda = \emptyset$. Let $U_\lambda = F_\lambda^c$, then they are open. Moreover, we have $\bigcup_{\lambda\in\Lambda} U_\lambda = (\bigcap_{\lambda\in\Lambda} F_\lambda)^c = X$. Then, we have $(\bigcup_{i=1}^n U_{\lambda_i})^c = \bigcap_{i=1}^n F_{\lambda_i} \neq \emptyset$, hence we cannot have finite subcovers. Thus X is not compact.

We then show $2 \to 3$. Let $(x_n)_n$ be a sequence. Let $F_n = \overline{\{x_k : k \geq n\}}$ be closed. Moreover, we have $\bigcap_{i=1}^p F_{n_i} = F_{\max\{n_1,\ldots,n_p\}}$ is not empty as $x_{\max\{n_1,\ldots,n_p\}}$ is in the intersection. Hence, $\{F_n\}_n$ has FIP. Thus $\exists x \in \bigcap_{n=1}^\infty F_n$. In particular, we have $x \in F_1$. Pick $m_1 \geq 1$ such that $d(x_{m_1}, x) < 1$, then $x \in F_{m_1+1} = \{x_k : k > m_1\}$. Recursively, we let $m_k > m_{k-1}$ such that $d(x, x_{m_k}) < \frac{1}{k}$. Then, we have $x \in F_{m_k+1}$. In particular, we then have $\lim_{l\to\infty} x_{m_l} = x$ and so $(x_{m_l})_l$ is a convergent subsequence.

We show $3 \to 4$. Assume X is sequentially compact. Let $(x_n)_n$ be a Cauchy sequence. By sequential compactness, there is a subsequence x_{n_l} that converges in X. In particular, a Cauchy with convergent subsequence must converge. Thus we have $(x_n)_n$ converges and so X is complete. If X is not totally bounded, then there exists $\epsilon > 0$ such that X is not the union of finitely many $b_\epsilon(x)$'s. Suppose $x_1, \ldots, x_n$ are in X with $d(x_i, x_j) \geq \epsilon$ such that $i \neq j$ and $1 \leq i, j \leq n$. Then, we have $X \neq \cup b_\epsilon(x_i)$, pick $x_{n+1} \notin \bigcup_{i=1}^n b_\epsilon(x_i)$. Then, we have $d(x_{n+1}, x_i) \geq \epsilon$ for $1 \leq i \leq n$. Recursively, let $\{x_n, n \geq 1\}$ such that $d(x_q, x_i) \geq \epsilon$ if $i \neq q$, then $(x_n)_n$ is a sequence in X with no convergent subsequence. This is a contradiction and so X is totally bounded.

We show $4 \to 1$. Suppose X is compact and totally bounded and suppose $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover with no finite subcover. For each $k \geq 1$ let $x_{k,1}, \ldots, x_{k,p_k}$ be a 2^{-k}-net, i.e. $X = \bigcup_{i=1}^{p_k} b_{2^{-k}}(x_{k,i})$. We start with $k = 1$ and if each $\overline{b_{2^{-1}}(x_{1,i})}$ had a finite subcover from $\{U_\alpha\}$, then combining them would yield a finite subcover of X. That was not the case by our assumption, so that we pick i_1 such that $b_{2^{-1}}(x_{1,i_1})$ has no finite subcover where

$$b_{2^{-1}}(x_{1,i_1}) = b_{2^{-1}}(x_{1,i_1}) \cap (\bigcup_{i=1}^{p_2} \overline{b_{2^{-2}}(x_{2,i})}) = \bigcup_{i=1}^{p_2} \left(\overline{b_{2^{-1}}(x_{1,i_1})} \cap \overline{b_{2^{-2}}(x_{2,i})}\right)$$

Again, there must exists i_2 such that $A_2 = \overline{b_{2^{-1}}(x_{1,i_1})} \cap \overline{b_{2^{-2}}(x_{2,i_2})}$ had no finite subcover. Proceed recursively, suppose $i_1, \ldots, i_n$ are chosen so that

$$A_n = \bigcap_{k=1}^n \overline{b_{2^{-k}}(x_{k,i_k})}$$

had no finite subcover. Then,

$$A_n = A_n \cap \left(\bigcup_{i}^{p_{n+1}} \overline{b_{2^{-n-1}}(x_{n+1,i})}\right) = \bigcup_{i=1}^{p_{n+1}} A_n \cap \overline{b_{2^{-n-1}}(x_{n+1,i})}$$

so again, there must exists i_{n+1} such that $A_{n+1} = A_n \cap \overline{b_{2^{-n-1}}(x_{n+1,i_{n+1}})}$ has no finite subcover.

Since $diam(\overline{b_{2^{-n-1}}(x)}) \leq \frac{2}{2^{n+1}}$, we have $diam(A_n) \to 0$.

We claim that $(x_{n,i_n})_{n=1}^{\infty}$ is Cauchy. If $\epsilon > 0$ then there exists N such that $\epsilon > 2^{1-N}$. If $n > m \geq N$, then we have

$$\overline{b_{2^{-n}}(x_{n,i_n})} \cap \overline{b_{2^{-m}}(x_{m,i_m})} \supseteq A_n \neq \emptyset$$

Thus, $\exists y \in A_n$ as A_n is not empty so that

$$d(x_{n,i_n}, x_{m,i_m}) \leq d(x_{n,i_n}, y) + d(y, x_{m,i_m}) < \frac{1}{2^N} + \frac{1}{2^N} = \frac{1}{2^{1-N}} < \epsilon$$

Therefore, (x_{n,i_n}) is Cauchy and so the limit $\lim_{n\to\infty} x_{n,i_n} \to x_0$ exists. Therefore, we can pick $\lambda_0 \in \Lambda$ such that $x_0 \in U_{\lambda_0}$ and so there exists $r > 0$ so that $b_r(x_0) \subset U_{\lambda_0}$. Then, we find N such that $\frac{2}{2^N} < r$ and $\forall n \geq N, d(x_{n,i_n}, x_0) < \frac{r}{2}$. Therefore, $2^{-n} \leq 2^{-N} < \frac{r}{2}$. If $x \in b_{2^{-n}}(x_n, i_n)$, we have

$$d(x, x_0) \leq d(x, x_{n,i_n}) + d(x_{n,i_n}, x_0) < \frac{1}{2^n} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r$$

Therefore, we have $\overline{b_{2^n}}(x_{n,i_n}) \subseteq b_r(x_0) \subset U_{\lambda_0}$ where $A_N \subseteq b_{2^{-N}}(x_{N,i_N}) \subseteq U_{\lambda_0}$ so it does have a finite subcover!

This is a contradiction and so $\{U_\alpha\}$ must have a finite subcover and so X is compact.

♡

Remark 1.7.5. We remark in the above theorem 1.7.4, 1, 2 and 3 depends only on the topology, not on the metric. We can see trivially 1 and 2 only depends on the topology. For 3, we note sequentially compact is the same as saying, $\forall U$ that is open, $x_0 \in U$, there exists N so that $\forall n \geq N$, $x_n \in U$. Indeed, we note 3 means $\forall r > 0$, $\exists N$, $\forall n \geq N$, we have $x_n \in b_r(x_0)$, if U is open, then there exists x_0 such that $\exists b_r(x_0) \subset U$.

Condition 4 depends on the metric. We have $\mathbb{R}$ is homeomorphic to $(0, 1)$ via the mapping

$$x \mapsto \frac{\arctan(x) + \frac{\pi}{2}}{\pi}$$

We have $\mathbb{R}$ is complete but not totally bounded, and $(0, 1)$ is not complete but is totally bounded.

Lemma 1.7.6. *We have x is a limit point of A if and only if*

$$\forall \delta > 0, b_\delta(x) \cap A \neq \emptyset$$

Proof. We note $\forall \delta > 0, b_\delta(x) \cap A \neq \emptyset$ immediately imply there exists such a sequence in A converges to x, e.g. for each $n \in \mathbb{N}$, take the point $a_n \in A$ to be in $b_{1/n}(x) \cap A$, then we are done.

Conversely, if x is a limit point of A. Let $\{a_n\} \to x$ be a sequence in A. Then for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $\forall n > N, d(a_n, x) < \epsilon$. In particular, $a_n \in b_\epsilon(x) \cap A$ and the proof follows. ♡

Proposition 1.7.7. *A metric space (X,d) is compact if and only if, whenever A_n are nested closed sets, then $\bigcap_{n\geq 1} A_n$ is not empty.*

Proof. Suppose we have a collection of nested closed sets such that is non-empty. We have, for all $n \in \mathbb{N}$, $\lambda_1, ..., \lambda_n \in \mathbb{N}$ such that

$$\bigcap_{i=1}^{n} A_{\lambda_i} = A_{\min(\lambda_1,...,\lambda_n)} \neq \emptyset$$

Thus, if we have a nested non-empty closed sets $(A_n)_{n=1}^{\infty}$, then we have fintie intersection property (FIP).

Suppose (X,d) is compact. Let $(A_n)_{n=1}^{\infty}$ be a collection of non-empty nested closed sets, then it has FIP and by Borel Lebesgue, we have compact imiply FIP then intersection is not empty.

Conversely, suppose the converse holds. We will show the assumption give us sequential compactness of X, then by Borel Lebesgue we will be done. Let $(a_n)_{n=1}^{\infty}$ be a sequence in X. Let $A_k = \overline{\{a_i \in (a_n) : i \geq k\}}$ for $k \in \mathbb{N}$. Then, this is a nested closed sets, hence, the intersection $\bigcap_{n\geq 1} A_n$ is not empty and it has FIP property.

Thus, let $x \in \bigcap_{n\geq 1} A_n$, we have $x \in A_1$. Pick $m_k \in \mathbb{N}$ to be the smallest natural number such that $d(x, x_{m_k}) < \frac{1}{k}$ for all $k \in \mathbb{N}$. Such m_k indeed exists for all $k \in \mathbb{N}$. Let $S := \{x_i : i \in \mathbb{N}\}$, we have $x \in A_1 = \overline{S}$ so that x is a limit point of S and so for $\frac{1}{k}$, we have $b_{\frac{1}{k}}(x) \cap S$ is always not empty. Then, we just pick the one with smallest index in the set $b_{\frac{1}{k}}(x) \cap S$. Then, we would have $(x_{m_k})_{k=1}^{\infty}$ is a subsequence of $(x_n)_n$ that converges to x and so we have the sequential compactness as desired. Therefore, by Borel Lebesgue, we have (X,d) is compact. ♡

Example 1.7.8. A metric space can be closed and bounded yet not totally bounded. Consider ℓ_p. Let B_p be the closed unit ball of ℓ_p. This is closed and bounded as $diam(B_p) = 2$.

Consider $e_n = (0,0,...,0,1,0,...)$ where the 1 appear in the nth place. Then, we have $\|e_n - e_m\|_p = 2^{\frac{1}{p}} \geq 1$. A ball $b_{\frac{1}{2}}(x)$ can contain at most one e_n so that there is no finite $\frac{1}{2}$-net and so it is not totally bounded.

Example 1.7.9. The Hilbert cube $H := \{x = (x_1, x_2, ...) \in \ell_2 : \forall n \geq 1, 0 \leq x_n \leq \frac{1}{n}\}$ is compact.

Solution. We show H is totally bounded and complete. Note ℓ_2 is complete so we need to show H is closed then H is complete. Let $\chi \in \ell_2$ to an accumulation point of H, say $\chi = (x_1, x_2, ...)$, we will show $\chi \in H$. Since χ is an accumulation point, there exists a sequence $(\gamma_i)_{i=1}^{\infty}$ in H such that $\gamma_i \neq \chi$ and $\gamma_i \to \chi$. In particular, let $\gamma_i = (y_1^i, y_2^i, y_3^i, ...)$ where $0 \leq y_j^i \leq \frac{1}{j}$ for all $j \geq 1$. In particular, since $\gamma_i \to \chi$, we must have $y_j^i \to x_j$ as $i \to \infty$. Indeed, let $\epsilon > 0$, since $\gamma_i \to \chi$ we have

$$\exists N > 0, i > N \Rightarrow \|\gamma_i - x\|_2 < \epsilon$$

In particular, then we would have $|y^i_j - x_j|^2 \leq \sum_{j=1}^{\infty} |y^i_j - x_j|^2 < \epsilon$ for all $i > N$ and so $y^i_j \to x_i$ as desired.

Next, note since each $y^i_j \leq \frac{1}{j}$, the sequence $(y^i_j)_{i=1}^{\infty}$ is bounded by $\frac{1}{j}$ and hence x_j must be bounded by $\frac{1}{j}$ as well. Therefore, $\chi \in H$ as desired. Hence H is closed and hence complete in ℓ_2.

It suffice to show totally bounded. Let $\sqrt{\epsilon} > 0$ be given and arbitrary[1]. There exists N so that $\sum_{i=N+1}^{\infty} \frac{1}{i^2} < \epsilon/2$ as $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges. Next, let $E = \{(y_i)_{i=1}^{\infty} \in H : n \geq N \Rightarrow y_n = 0\}$, there should be a finite collection of points to make a ϵ-net as desired. Indeed, let $\chi = (x_i)_{i=1}^{\infty}$ be arbitrary element in H, let $\gamma = (y_i)_{i=1}^{\infty}$ in E, then we have

$$\|\chi - \gamma\|_2^2 = \sum_{i=1}^{N} |x_i - y_i|^2 + \sum_{i=N+1}^{\infty} |x_i|^2 \leq \sum_{i=1}^{N} |x_i - y_i|^2 + \sum_{i=N+1}^{\infty} \frac{1}{i^2} < \sum_{i=1}^{N} |x_i - y_i|^2 + \epsilon/2$$

Next, consider the finite product metric space $Q = \prod_{i=1}^{N} [0, \frac{1}{i}]$, by induction on N we have this is compact as each $[0, \frac{1}{i}]$ is. Therefore, there exists a $\sqrt{\frac{\epsilon}{2N}}$-net that covers $\prod_{i=1}^{N} [0, \frac{1}{i}]$. Say the $\sqrt{\frac{\epsilon}{2N}}$-net is $\bigcup_{i=1}^{k(N)} b_{\epsilon/2}(\mathbf{y}_i)$, where $\mathbf{y}_i = (v^i_1, ..., v^i_N)$, then for all $\mathbf{x} = (u_1, ..., u_N) \in Q$, there exists $\mathbf{y}_i$ such that $d_Q(\mathbf{y}_i, \mathbf{x}) < \epsilon/2N$, i.e.

$$\max\{|x_j - v^i_j| : 1 \leq j \leq N\} < \sqrt{\frac{\epsilon}{2N}}$$

In particular, this gives us, for all $\mathbf{x} \in Q$,

$$\sum_{j=1}^{N} |x_j - v^i_j|^2 \leq \sum_{j=1}^{N} \frac{\epsilon}{2N} = \frac{\epsilon}{2}$$

Let $\chi \in H$ be arbitrary, let $\gamma_i = (v^i_1, ..., v^i_N, 0, 0, ...) \in E$ for $1 \leq i \leq k(N)$. Then, for all $x \in H$, there exists i such that makes $\sum_{j=1}^{N} |x_j - v^i_j|^2 \leq \frac{\epsilon}{2}$ and hence

$$\|\chi, \gamma_i\|_2^2 < \epsilon/2 + \epsilon/2 = \epsilon$$

Therefore, we have $\chi \in b_{\sqrt{\epsilon}}(\gamma_i)$ and so $\gamma_1, ..., \gamma_{k(N)}$ is an $\sqrt{\epsilon}$-net of H and since ϵ was arbitrary, we have H is totally bounded and hence compact. ♠

Proposition 1.7.10. *If $A \subseteq (X, d)$, then A is a compact subset of X if and only if (A, d) is compact.*

Proof. If (A, d) is compact and $\{V_\alpha\}$ is collection of open sets in X such that covers A. We define $U_\alpha = A \cap V_\alpha$ for each index α and note this is open in (A, d). Thus, there exists a finite subcover $A \subset U_{\alpha_1} \cup ... \cup U_{\alpha_n} \subseteq V_{\alpha_1} \cup ... \cup V_{\alpha_n}$.

Conversely, if A is a compact subset of X, suppose $\{U_\alpha\}$ is open in A, $A \subseteq \cup U_\alpha$. FInd V_α open in X such that $V_\alpha \cap A = U_\alpha$. If $x \in U_\alpha$ then there exists $r_x > 0$

[1] Did not set up nicely but since ϵ was arbitrary, this is still valid proof

such that $b^A_{r_x}(x) \subset U_\alpha$. Let $V_\alpha = \bigcup_{x \in U_\alpha} b^X_{r_x}(X)$ and we have V_α is open in X. In particular, this gives us $V_\alpha \cap A = U_\alpha$. Hence, we have A compact on X and so that $A \subseteq V_{\alpha_1} \cup ... \cup V_{\alpha_n}$ which imply $A \subseteq \bigcup (V_{\alpha_i} \cap A) = U_\alpha \cup ... \cup U_{\alpha_n}$. Thus A is compact with respect to itself. ♡

Corollary 1.7.10.1. *Compactness is intrinsic to the set, independent of the larger universe.*

Theorem 1.7.11 (Heine-Borel)**.** *$A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. ($\Rightarrow$) Done, easy.

($\Leftarrow$) Note $\mathbb{R}^n$ is complete and A is a closed subset of $\mathbb{R}^n$ so that it is complete. Then, we claim A is totally bounded. Indeed, let $\sup_{x \in A} \|x\|_2 = R < \infty$. Then we have $A \subseteq \{x \in \mathbb{R}^n : -R \leq x_i \leq R, 1 \leq i \leq n\}$. Let ϵ be given, then set $\delta = \frac{\epsilon}{n}$. Choose k_i as follows

$$\{k_1\delta_1, k_2\delta, ..., k_n\delta, k_i \in \mathbb{Z}, |k_i| \leq \frac{R+1}{\delta}\}$$

If $x = (x_1, ..., x_n)$, then there exists k_i so that $|k_i\delta - x_i| < \delta$ then

$$|x - (k_1\delta, ..., k_n\delta)| < \sum |x_i - k_i\delta| < n\delta = \epsilon$$

This is an ϵ-net and so A is complete and totally bounded and thus compact. ♡

Theorem 1.7.12 (Bolzano–Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Remark 1.7.13. We can also show Heine-Borel by Bolzano–Weierstrass as we show $C = \{x \in \mathbb{R}^n : a_i \leq x \leq b_i, 1 \leq qi \leq n\}$ is sequentially compact. To do this, let $(X_n)_n$ be a sequence in C. Then, let $(x_{1,k})_{k=1}^\infty$ be a sequence in $[a_1, b_1]$ then we have a subsequence in $[a_2]$. Then, choose a subsubsequence and so on. Then, we would get C is sequentially compact and the proof follows.

Proposition 1.7.14. *Let (X, d) be compact metric space. Let $A \subseteq X$ be closed. Then we have A is compact.*

Proof. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of A. Then $\{U_\lambda\} \cup \{A^c\}$ is an open cover of X. Since X is compact, so there exists $U_{\lambda_1}, ..., U_{\lambda_n}, A^c$ so that $A \subseteq (\bigcup U_{\lambda_i}) \cup A^c$ but $A \cap A^c = \emptyset$ so that $A \subseteq (\bigcup U_{\lambda_i})$. ♡

Example 1.7.15. Let X be a closed subset of $\mathbb{R}^n$, recall the Hausdorff metric d_H on the space $\mathcal{H}(X)$ of all non-empty closed bounded subsets of X. Show that

1. $\mathcal{H}(X)$ is complete
2. $\mathcal{H}(X)$ is compact if and only if X is compact.

Solution. Part 1. Let $(A_n)_{n\geq 1}$ be a Cauchy sequence in $\mathcal{H}(X)$. Let

$$A := \bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} A_k}$$

Note that for $\epsilon > 0$, there is an N so that $d_H(A_n, A_k) < \epsilon$ for $n, k \geq N$. In particular, $A_k \subset E_\epsilon(A_n)$ where $E_\epsilon(S) = \bigcup_{s\in S} b_\epsilon(s)$ for any set $S \subseteq X$. Thus $\bigcup_{k\geq n} A_k \subset \overline{E_\epsilon(A_n)}$ and therefore $A \subset \overline{E_\epsilon(A_n)}$. On the other hand, given $x \in A_n$ for $n \geq N$, there is a point $x_k \in A_k$ where $k \geq N$ with $\|x - x_k\| < \epsilon$. The sequence (x_k) is bounded and thus has an accumulation point x which lies in $\overline{\bigcup_{k\geq m} A_k}$ for all m, and thus belongs to A. This shows that $A_n \subseteq \overline{E_\epsilon(A)}$, hence $d_H(A_n, A) \leq \epsilon$ provided $n \geq N$. Therefore $\lim A_n = A$.

Part 2. Note $\mathcal{H}(X)$ is complete, then we will show it is totally bounded. Since X is compact, we can find a finite ϵ-net $F = \{x_1, ..., x_p\}$. Let $\mathcal{F}$ consist of all non-empty subsets of F, which is also fintie. Given $A \in \mathcal{H}(X)$, let $B = \{x_i \in F : d(x_i, A) < \epsilon\}$, then evidently $B \subset E_\epsilon(A)$. On the other hand, for each $a \in A$, t here is some $x_i \in F$ such that $d(a, x_i) < \epsilon$. So $x_i \in B$. This shows that $A \subset E_\epsilon(B)$. Thus $d_H(A, B) < \epsilon$. As $\mathcal{H}(X)$ is complete and totally bounded, we are done. ♠

Definition 1.7.16. If A is compact and $\{U_\lambda\}$ is an open cover of A, then the ***Lebesgue number*** of $\{U_\lambda\}$ is

$$\delta(\{U_\lambda\}) = \inf_{x\in A}(\sup\{r > 0 : \exists\lambda \in \Lambda, b_r(x) \subseteq U_\lambda\})$$

Theorem 1.7.17. *If A is compact, $\{U_\lambda\}$ is an open cover, then $\delta(\{U_\lambda\}) > 0$.*

Proof. For each $x \in A$, there exists λ_x so that $x \in U_{\lambda_x}$. Since U_{λ_x} is open, there exists $r_x > 0$ so that $b_{r_x}(x) \subset U_{\lambda_x}$.

Hence, we note

$$\{b_{\frac{r_x}{2}}(x) : x \in A\}$$

is an open cover of A and by compactness, there is a finite subcover

$$b_{\frac{r_{x_1}}{2}}(x_1), ..., b_{\frac{r_{x_n}}{2}}(x_n)$$

Let

$$r = \min\{\frac{r_{x_1}}{2}, ..., \frac{r_{x_n}}{2}\} > 0$$

Let $x \in A$, then there exists i so that $x \in b_{\frac{r_{x_i}}{2}}(x_i)$. If $y \in b_r(x)$ then

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < r + \frac{r_{x_i}}{2} \leq r_{x_i}$$

Hence,

$$b_r(x) \subseteq b_{r_{x_i}}(x_i) \subseteq U_{\lambda_{x_i}}$$

and so $\delta(\{U_\lambda\}) \geq r > 0$. ♡

Chapter 2

Metric Spaces II

2.1 More Topology

Definition 2.1.1. If (X, d) and (Y, p) are metric spaces, the ***product space*** $X \times Y = \{(x, y) : x \in X, y \in Y\}$ has a metric

$$D((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), p(y_1, y_2)\}$$

Remark 2.1.2. Keep the above notation, we have

$$b_r((x_0, y_0)) = \{(x, y) : D((x_0, y_0), (x, y)) < r\} = b_r(x_0) \times b_r(y_0)$$

where $d_r(x_0) = \{x : d(x_0, x) < r\}$ and $d_r(y_0) = \{y : p(y_0, y) < r\}$

In addition, if W is open in $X \times Y$ and (x_0, y_0) in W, then there exists $r > 0$ so that $b_r((x_0, y_0)) \subseteq W$.

Theorem 2.1.3. *If (X, d) and (Y, p) are compact metric, then $(X \times Y, D)$ is compact.*

Proof. Let $\{W_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $X \times Y$.

Fix $y_0 \in Y$, for each $x \in X$, there exists $\lambda(x)$ so that $(x, y_0) \in W_{\lambda(x)}$. Thus, there exists $r(x) > 0$ such that

$$b_{r(x)}((x, y_0)) = b_{r(x)}(x) \times b_{r(x)}(y_0) \subseteq W_{\lambda(x)}$$

Next, note $\{b_{r(x)}(x) : x \in X\}$ is an open cover of X and so since X is compact there is a finite subcover, say $x(i, y_0)$ where $1 \leq i \leq n(y_0)$. Then, we have

$$X \subseteq \bigcup_{i=1}^{n(y_0)} b_{r(x(i,y_0))}(x(i, y_0))$$

Let $r(y_0) = \min\{r(x(i, y_0)) : 1 \leq i \leq n(y_0)\}$ then we have

$$b_{r(y_0)}(x(i, y_0), y_0) = b_{r(y_0)}(x(i, y_0)) \times b_{r(y_0)}(y_0) \subseteq b_{r(x,y_0)}(x, y_0) \subseteq W_{\lambda(x)}$$

For each y_0, we get $r(y_0)$ so that $X \times b_{r(y_0)}(y_0)$ is covered by a finite set

$$W_{\lambda(x_1,y_0)}, ..., W_{\lambda(x_{n(y_0)},y_0)}$$

Note $\{b_{r(y_0)}(y_0) : y_0 \in Y\}$ is an open cover of Y and since Y is compact, we get a finite subcover

$$b_{r(y_1)}(y_1), ..., b_{r(y_m)}(y_m)$$

Hence, we have $X \times b_{r(y_i)}(y_i)$ is covered by

$$W_{\lambda(x(i,y_i),y_i)} \cup ... \cup W_{\lambda(x(n(y_i),y_i),y_i)}$$

Thus, $X \times Y$ is covered by $\{W_{\lambda(x(i,y_j),y_j)} : 1 \leq j \leq m, 1 \leq i \leq n(y_j)\}$ and hence $X \times Y$ is compact and the proof follows. ♡

Remark 2.1.4. The above theorem follows from Tychonoff theorem as well. Check the Appendix later.

Definition 2.1.5. A subset $S \subseteq (X, d)$ is ***dense*** if $\overline{S} = X$.

Definition 2.1.6. A space X is ***separable*** if it has a countable dense subset.

Proposition 2.1.7. *If (X, d) is compact, then X is separable.*

Proof. For each $n \geq 1$, there is a finite 2^{-n} net $x_1^{(n)}, ..., x_{k(n)}^{(n)}$. Let

$$S := \{x_i^{(n)} : n \geq 1, 1 \leq i \leq k(n)\}$$

We have S is countable. For each $x \in X$, let $\epsilon > 0$, then there exists n so that $2^{-n} < \epsilon$ and there exists $x_i^{(n)}$ such that $d(x, x_i^{(n)}) < \epsilon$. Hence S is dense. ♡

Definition 2.1.8. A metric space (X, d) is ***second countable*** if there is a countable family $\{U_n : n \geq 1\}$ of open sets that generates[1] the topology, i.e. each open set V satisfies $V = \bigcup_{U_b \subset V} U_n$.

Proposition 2.1.9. *A metric space (X, d) is second countable if and only if it is separable.*

Proof. Suppose (X, d) is second countable. Then, let $\{U_n : n \geq 1\}$ be a countable family that generates the topology. Let $D = \{x_n : x_n \in U_n, n \in \mathbb{N}\}$. Clearly D is countable and we claim $\overline{D} = X$. Indeed, we already have $\overline{D} \subseteq X$, so let $x \in X$ be arbitrary.

[1]Check Appendix for more info

Denote $\frac{1}{n}$ to be $\epsilon(n)$. Then there exists at least one $x_{k_1} \in D$ such that $x_{k_1} \in U_{k_1}$ where $U_{k_1} \subseteq b_{\epsilon(1)}(x)$ as $b_{\epsilon(1)}(x)$ is open. Next, we pick x_{k_2} be so that $x_{k_2} \in U_{k_2} \subseteq b_{\epsilon(2)}(x)$. Recursively, let x_{k_q} be so that $x_{k_q} \in U_{k_q} \subseteq b_{\epsilon(q)}(x)$. Denote this sequence be $(x_{k_n})_{n=1}^{\infty}$, then we have $x_{k_n} \to x$ as desired as $d(x_{k_n}, x) < \frac{1}{n}$.

Conversely, suppose (X, d) is separable. Let $\overline{S} = X$ where S is countable. Then, we have $S = \{x_1, x_2, x_3, ...\}$ as it is countable (and we keep S with that order). Then, let $U_{i,j} = b_{\frac{1}{j}}(x_i)$, consider the countable family (countable union of countable sets are still countable and we have $\bigcup_{j=1}^{\infty}\{U_{i,j}\}$ is clearly countable, hence D is)

$$D = \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{\infty}\{U_{i,j}\} = \{U_{i,j} : i, j \in \mathbb{N}\}$$

Next, let V be arbitrary open set in (X, d), let $x \in V$ be arbitrary. Since $\overline{S} = X$, there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of S such that converges to x. In particular, since V is open, we have $x \in b_\epsilon(x) \subseteq V$ and hence there exists n so that $\frac{1}{q} < \epsilon$. In particular, since $x_{n_i} \to x$, there exists x_{n_k} so that $d(x_{n_k}, x) < \frac{1}{q} < \epsilon$, thus we have $x \in U_{n_k, q} \subseteq b_\epsilon(x) \subseteq V$. Since for arbitrary $x \in V$, we can find $U_{i,j}$ such that $x \in U_{i,j} \subseteq V$, we must have $V = \bigcup_{U_{i,j} \subseteq V} U_{i,j}$. The proof follows. ♡

Example 2.1.10.

1. $\mathbb{R}^n$ is separable. We have $\mathbb{Q}^n$ is countable and dense.
2. We have ℓ_p is dense for $1 \leq p < \infty$. Consider

$$\begin{aligned} S &= \{\chi = (x_1, x_2, ...) : x_i \in \mathbb{Q} \wedge \exists n, i > n \Rightarrow x_i = 0\} \\ &= \bigcup_{n \geq 1}\{\chi = (x_1, ..., x_n, 0, 0, ...) : x_i \in \mathbb{Q}\} \\ &\cong \bigcup_{n \geq 1} \mathbb{Q}^n \end{aligned}$$

The countable union of countable sets is countable so S is countable. If $x \in \ell_p$, we have $\|x\|_p^p = \sum_{i=1}^{\infty} |x_i|^p < \infty$ so $x_i \to 0$. Given $\epsilon > 0$, pick N so that $\sum_{i=N+1}^{\infty} |x_i|^p < (\frac{\epsilon}{2})^p$, then we have

$$\|x - (x_1, ..., x_N, 0, 0, ...)\|_p < \epsilon/2$$

Moreover, there exists $y_i \in \mathbb{Q}$ such that

$$\|(x_1, ..., x_N, 0, 0, ...) - (y_1, ..., y_N, 0, ...)\|_p < \frac{\epsilon}{2}$$

Thus let $y = (y_1, ..., y_n, 0, ...)$, we have $\|x - y\|_p < \epsilon/2 + \epsilon/2 = \epsilon$.

3. We have ℓ_∞ is not separable. For $E \subseteq \mathbb{N}$, let x_E be the sequence $x_E = (x_E(1), x_E(2), ...)$ where $x_E(n) = 1$ if $n \in E$ and 0 if $n \notin E$. Then, we have $\|x_E - x_F\|_\infty$ and note $x_E - x_F = \begin{cases} 1, n \in E \backslash F \\ -1, n \in F \backslash E \\ 0, n \in (E \cap F) \cup (E^c \cap F^c) \end{cases}$ Thus, let $E \triangle F = (E \backslash F) \cup (F \backslash E)$, we have

$$\|x_E - x_F\|_\infty = \|x_{E \triangle F}\|_\infty = 1, \text{ if } E \neq F$$

By Cantor's theorem, we have $|P(X)| > |X|$ where $P(X)$ is the power set of X. We will show $P(\mathbb{N})$ is not countable. If it was countable, then there would exists a list $E_1, E_2, E_3, ...$ containing of all subsets of $\mathbb{N}$. Let $F = \{n : n \notin E_n\}$. If $m \in E_m$ then $m \notin F$, if $m \notin E_m$ then $m \in F$. Thus $F \neq E_m$ and so F is not in the list.
Hence, $\{x_E : E \in P(\mathbb{N})\}$ is an uncountable subset of ℓ_∞ and $\|x_E - x_F\|_\infty = 1$ which means they are all different.

Theorem 2.1.11. *Let (X, d) be a compact metric space, let (Y, p) be a metric space. Let $f : X \to Y$ be continuous. Then $f(X)$ is compact and f is uniformly continuous.*

Proof. We first show compact. We have $f(X) \subseteq Y$ and let $\{V_\lambda\}_{\lambda\in\Lambda}$ be an open cover of $f(x)$. Since f is continuous, we have $U_\lambda := f^{-1}(V_\lambda)$ is open in X. If $x \in X$, then $f(x) \in f(X)$, and so there exists λ such that $f(x) \in V_\lambda$, thus $x \in U_\lambda$ and so $\{U_\lambda\}$ is an open cover of X.

Let $U_{\lambda_1}, ..., U_{\lambda_n}$ be a finite subcover of X, then $f(X) = \bigcup_{i=1}^n f(U_{\lambda_i}) \subseteq \bigcup_{i=1}^n V_{\lambda_i}$. Therefore $f(X)$ is compact as desired.

Then, we show f is uniformly continuous. Fix $\epsilon > 0$, for each $x \in X$, by continuity of f, there is $\delta_x > 0$ such that $d(x, x') < \delta_x \Rightarrow p(f(x), f(x')) < \epsilon/2$. Then, we have

$$\{b_{\delta_x/2}(x) : x \in X\}$$

is an open cover of X. Let $b_{\delta_x(1)/2}(x_1), ..., b_{\delta_x(n)/2}(x_n)$ be a finite subcover. Let $\delta = \min\{\delta_{x(1)}/2, ..., \delta_{x(n)}/2\}$.

Suppose $z_1, z_2 \in X$, and $d(z_1, z_2) < \delta$, then we find i such that $d(z_1, x_i) < \delta_{x(i)}/2$, then we have $d(z_2, x_i) \leq d(z_2, z_1) + d(z_1, x_i) < \delta + \delta_{x(i)}/2 \leq \delta_{x(i)}$. Therefore, $p(f(z_1), f(x_i)) < \epsilon/2$ and $p(f(z_2), f(x_i)) < \epsilon/2$ and so

$$p(f(z_1), f(z_2)) < \epsilon/2 + \epsilon/2 = \epsilon$$

♡

Corollary 2.1.11.1 (Extreme Value Theorem)**.** *If (X, d) is compact, and $f : X \to \mathbb{R}$ is continuous. Then f attains its maximum and minimum value.*

Proof. $f(X)$ is compact in $\mathbb{R}$, so it is closed and bounded. Thus we have

$$\sup_{x\in X} f(x) = L < \infty$$

Thus $L \in \overline{f(X)} = f(X)$, thus there exists $x_0 \in X$ so that $f(x_0) = L$. ♡

Proposition 2.1.12. *If (X, d) is compact, $f : X \to Y$ is a continuous bijection, then f^{-1} is continuous and so f is a homeomorphism.*

Proof. Since f is a bijection, we have $f^{-1} : Y \to X$ is a function. We will show U is open in X then $(f^{-1})^{-1}(U) = f(U)$ is open in Y. Note since U is open in X we have U^c is closed and since X is compact we have U^c is compact and so $f(U^c)$ is compact and so $f(U^c)$ is closed in Y. Since f is bijection, we have $(f(U^c))^c = f(U)$ and hence open in Y. Hence f^{-1} is continuous. ♡

Remark 2.1.13. Recall that let (X, d) be a metric space, then $C^b_{\mathbb{R}}(X)$ and $C^b_{\mathbb{C}}(X)$ are the space of bounded continuous functions valued on $F = \{\mathbb{R}, \mathbb{C}\}$, respectively, with the metric $\|f\|_\infty = \sup_{x\in X} |f(x)|$.

We note if X is compact, by Extreme Value Theorem, $f \in C(X)$ so that $C_F(X) = C^b_F(X)$.

Theorem 2.1.14. *Let (X, d) be a metric space, then $C^b_F(X)$ is complete, i.e. the uniform limit of continuous functions is continuous.*

Proof. Let $(f_n)_n$ be a Cauchy sequence in $C^b_F(X)$. In particular, if (f_n) is converging uniformly then it is Cauchy with respect to $\|\cdot\|_\infty$.

For each $x \in X$, $(f_n(x))_{n=1}^\infty$ is Cauchy because $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty$. We know for all $\epsilon > 0$, there exists $N(\epsilon)$ such that $n, m > N(\epsilon)$ imply $\|f_n - f_m\|_\infty < \epsilon$. Note $\mathbb{R}$ or $\mathbb{C}$ is complete, so $f(x) = \lim_{n\to\infty} f_n(x)$ exists and hence we get pointwise convergence.

We claim f is bounded. Take $\epsilon = 1$, then $\|f_n - f_{N(1)}\|_\infty < 1$ and so $|f_n(x) - f_{N(1)}(x)| < 1$ and when $n \to \infty$ we get $|f(x) - f_{N(1)}(x)| \le 1$ and hence $\|f\|_\infty \le \|f_{N(1)}\|_\infty + 1 < \infty$. Therefore, for any $\epsilon > 0$, we have $n, m \ge N(\epsilon)$ and $|f_n(x) - f_m(x)| < \epsilon$ and so $\|f - f_m\|_\infty \le \epsilon$ for all $m \ge N(\epsilon)$, i.e. $f_m \to f$ uniformly.

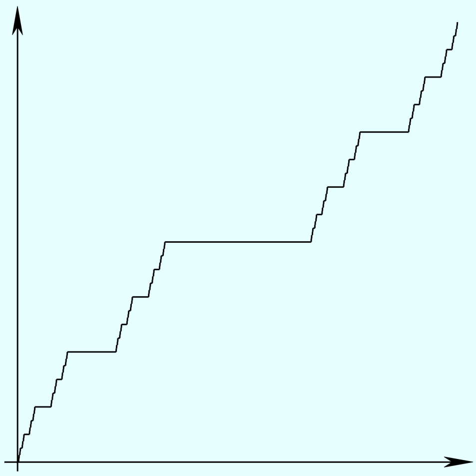
We claim f is continuous. Fix $x \in X$, $\epsilon > 0$. Let N be so that $\|f - f_N\|_\infty < \epsilon/3$. Since f_N is continuous, there exists $\delta > 0$ such that $d(x', x) < \delta \Rightarrow |f_N(x') - f_N(x)| < \epsilon/3$, then, we get

$$|f(x') - f(x)| \le |f(x') - f_N(x')| + |f_N(x') - f_N(x)| + |f_N(x) - f(x)| < \epsilon$$

♡

Theorem 2.1.15. *Let (X, d) be a compact metric space, then there is a surjective continuous function from the Cantor set C to X.*

Proof. For example, we have seen the Cantor function on $[0, 1]$, i.e.



Now, let X be compact, so it has a finite $\frac{1}{2}$-net, say $x_1^{(1)}, ..., x_{n(1)}^{(1)}$ and $n(1) \geq 2$. Split C into disjoint clopen (closed and open) subsets $C_1^{(1)}, ..., C_{n(1)}^{(1)}$ with $diam(C_i^{(1)}) < \frac{1}{3}$.

We define $f : C \to X$ to be $f_1|_{C_i^{(1)}} = x_i^{(1)}$ for $1 \leq i \leq n(1)$ where $f|_C$ means the restriction of f to the set C. Note f_1 is continuous and locally constant.

Then, note $X = \bigcup_{i=1}^{n(1)} b_{\frac{1}{2}}(x_i^{(1)})$ and note $\overline{b_{\frac{1}{2}}(x_i^{(1)})}$ is compact. Thus, we find a finite $\frac{1}{4}$-net $x_1^{(2,i)}, ..., x_{n(2,i)}^{(2,i)}$ for each i. Split $C_i^{(1)}$ into $n(2,i)$ clopen subsets of length less than or equal to $\frac{1}{3^2}$, i.e. $C_j^{(2,i)}$ where $1 \leq j \leq n(2,i)$. We define $f_2 : C \to X$ to be $f_2|_{C_j^{(2,i)}} = x_j^{(2,i)}$. We note f_2 is continuous.

We claim $\|f_1 - f_2\|_\infty \leq \frac{1}{2}$. Indeed, each $y \in C_j^{(2,i)} \subseteq C_i^{(1)}$ and so $f_2(y) = x_j^{(2,i)} \in \overline{b_{1/2}(x_i^{(1)})}$ and $f_1(y) = x_i^{(1)}$. Thus $d(f_2(y), f_1(y)) \leq \frac{1}{2}$.

Suppose at the k^{th} stage, I have a 2^{-k} net of X, $x_1^{(k)}, ..., x_{n(k)}^{(k)}$, a partition of C into disjoint clopen sets $C_1^{(k)}, ..., C_{n(k)}^{(k)}$. We define f_k where $f_k|_{C_i^{(k)}} = x_i^{(k)}$. We note $diam(C_i^{(k)}) \leq \frac{1}{3^k}$ and $\|f_i - f_{i+1}\|_\infty \leq \frac{1}{2^i}$ for $1 \leq i \leq k-1$. Thus, we can find a 2^{-k-1} net for $\overline{b_{2^{-k}}(x_i^k)}$, say $x_j^{(k+1,i)}$ for $j = 1, ..., n(k+1,i)$. We split $C_i^{(k)}$ into disjoint clopen sets $C_j^{(k+1,i)}$, then we have $diam(C_j^{(k+1,i)}) \leq \frac{1}{3^{k+1}}$. We define $f_{k+1}|_{C_j^{(k+1,i)}} = x_j^{(k+1,i)}$. Then f_{k+1} is continuous and $y \in C_j^{(k+1,i)} \subseteq C_i^{(k)}$ and so $\|f_{k+1} - f_k\|_\infty \leq \frac{1}{2^k}$.

Since $\|f_{k+1} - f_k\|_\infty \leq \frac{1}{2^k}$, where $\sum_{k=1}^{\infty} \|f_{k+1} - f_k\| < \infty$. Thus (f_n) is Cauchy and so there exists $f = \lim f_n$ is uniform limit and hence continuous function. This $f : C \to X$ is what we are looking for.

We claim f is onto(surjective). Let $x \in X$, then we have $x_{i_1}^{(1)}$ such that $d(x, x_{i_1}^{(1)}) < \frac{1}{2}$ and so $f_1|_{C_{i_1}^{(1)}} = x_{i_1}^{(1)}$. Thus, there exists $x_{i_2}^{(2,i_1)}$ such that $d(x, x_{i_2}^{(2,i_1)}) < \frac{1}{4}$. Recursively, we can find a decreasing sequence $C_{i_1}^{(1)} \supseteq C_{i_2}^{(2)} \supseteq C_{i_3}^{(3)} \supseteq ...$ such that $f_k|_{C_{i_k}^{(k)}} = x_{i_k}^{(k)}$

such that $d(x, x_{i_k}^{(k)}) < \frac{1}{2^k}$. Thus we have $\lim x_{i_k}^{(k)} = x$, let $y \in \bigcap_{k=1}^{\infty} C_{i_k}^{(k)} \neq \emptyset$ by FIP, then we have $f_k(y) = x_{i_k}^{(k)}$ and $f(y) = \lim x_{i_k}^{(k)} = x$. ♡

Definition 2.1.16. A ***path*** is a continuous image of $[0,1]$ in some metric space.

Definition 2.1.17. A ***Peano curve*** op a ***space filling curve*** is a path in $\mathbb{R}^n$, $n \geq 2$, such that the image(path) has interior.

Example 2.1.18. Consider the Hilbert curve:

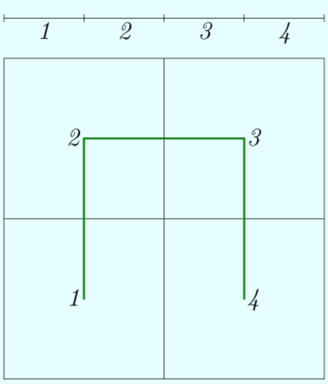


Fig. 1.

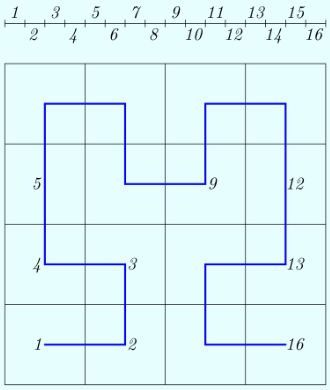


Fig. 2.

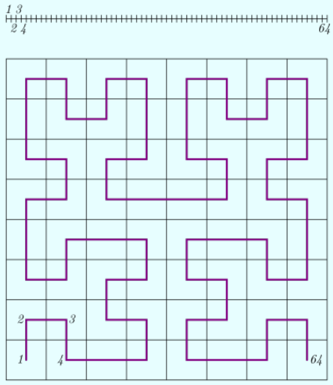
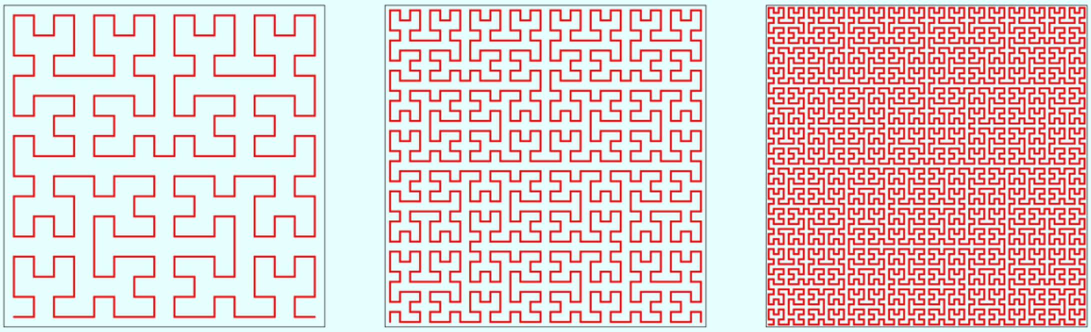


Fig. 3.



Definition 2.1.19. Let X be a subset of a vector space V over $\mathbb{R}$, then we say X is ***convex*** if for all $x, y \in X$, we have $tx + (1-t)y \subseteq X$ where t ranges over 0 to 1.

Theorem 2.1.20. *Let X be a compact convex subset of a normed vector space $(V, \|\|)$, then there is a continuous path $\gamma : [0,1] \to X$.*

Proof. By Theorem 2.1.15, there exists a continuous mapping $f : C \to X$ such that maps onto (surjectively) X. Write $[0,1] \backslash C = \bigcup_{n=1}^{\infty} (a_n, b_n)$ to be a disjoint union of intervals, then we have $b_n - a_n \to 0$ as $n \to \infty$. We define $g : [0,1] \to X$ by

$$g(x) = \begin{cases} f(x), x \in C \\ tf(a_n) + (1-t)f(b_n), \text{where } x = ta_n + (1-t)b_n \in (a_n, b_n) \end{cases}$$

Then g is linear on $[a_n, b_n]$ and we note $g \subseteq X$ by convexity of X, Thus g maps $[0,1]$ onto X as $f : C \to X$ is already surjective.

We then check g's continuity. Since f is continuous, let $\epsilon > 0$ be given, there exists $\delta_1 > 0$ such that $x, y \in C$, $|x - y| < \delta_1$ imply $\|f(x) - f(y)\| < \epsilon/3$. Let $F = \{n : b_n - a_n \geq \delta_1\}$ We have F is a fintie set as $b_n - a_n \to 0$. Let

$$D = \max_{n \geq 1} \|f(b_n) - f(a_n)\| = \max\{\max_{n \in F} \|f(b_n) - f(a_n)\|, \epsilon/3\}$$

Let $L = \min_{n \in F} b_n - a_n$ and let $\delta = \min\{\delta_1, \frac{\epsilon L}{3D}\}$. Let $x, y \in [0, 1]$ and suppose $|x - y| < \delta$.

Case 1: If $x, y \in C$ then we have $\|g(x) - g(y)\| = \|f(x) - f(y)\| < \epsilon/3$.

Case 2: If $x, y \in [a_n, b_n]$ where $n \in F$, then we note

$$\|g(x) - g(y)\| = \frac{|x - y|}{b_n - a_n} \cdot \|f(b_n) - f(a_n)\| < \frac{3L}{3D} \cdot \frac{1}{L} \cdot D = \epsilon/3$$

Case 3: If $x, y \in [a_n, b_n]$ where $n \notin F$, so $b_n - a_n < \delta_1$ imply $\|f(b_n) - f(a_n)\| < \epsilon/3$. Then, we have

$$\|g(x) - g(y)\| \leq \|f(b_n) - f(a_n)\| < \epsilon/3$$

Case 4: If $x \in C, y \in (a_n, b_n)$, then without lose of generality, suppose $x \leq a_n$ (we can see the same argument works for $x \geq b_n$).

Then, we have

$$\begin{aligned}
\|g(x) - g(y)\| &\leq \|f(x) - f(a_n)\| + \|f(a_n) - g(y)\| \\
&= \|f(x) - f(a_n)\| + \|f(a_n) - tf(a_n) - (1 - t)f(b_n)\| \\
&< \frac{\epsilon}{3} + \|(1 - t)(f(a_n) - f(b_n))\|, \quad (1) \\
&\leq \frac{\epsilon}{3} + \|(f(a_n) - f(b_n))\| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}
\end{aligned}$$

In (1), we have this because $x, a_n \in C$ and $|x - a_n| < \delta$ and by continuity of f.

Case 5: If $x \in (a_n, b_n)$ and $y \in (a_m, b_m)$, where $n \neq m$, then without lose of generality, suppose $b_n < a_m$. We have

$$\|g(x) - g(y)\| \leq \|g(x) - g(b_n)\| + \|g(b_n) - g(a_m)\| + \|g(a_m) - g(y)\|$$

We note $\|g(x) - g(b_n)\|$ and $\|g(a_m) - g(y)\|$ are Case 2 or 3 as $x, b_n \in [a_n, b_n]$ and $y, a_m \in [a_m, b_m]$. In addition, $\|g(b_n) - g(a_m)\|$ is Case 1, so both of them are bounded by $\epsilon/3$. Hence, we get

$$\|g(x) - g(y)\| < \epsilon$$

Therefore, in all cases, we have g is continuous and the proof follows. ♡

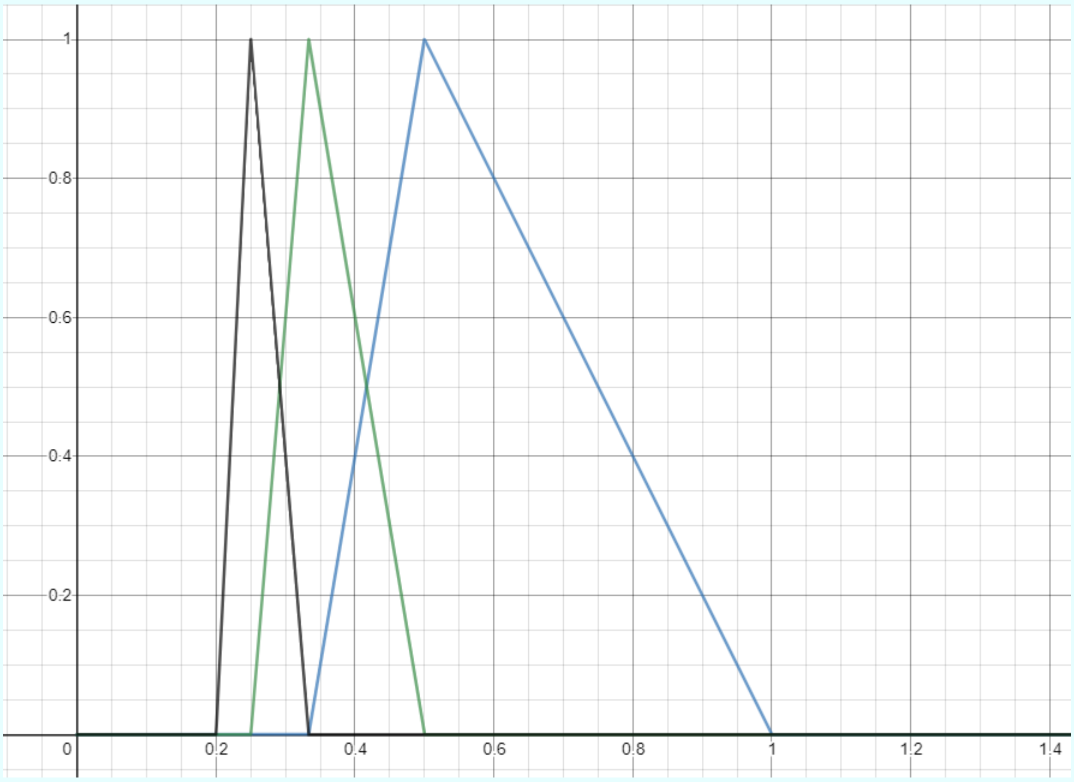
2.2 Arzela-Ascoli Theorem

Remark 2.2.1. Next, we consider the compactness of $C(X)$.

Example 2.2.2. Consider $C[0,1]$. We define

$$f_n(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{n+1} \\ (n^2+n)x - n, & \frac{1}{n+1} \le x \le \frac{1}{n} \\ (n-n^2)x + n, & \frac{1}{n} \le x \le \frac{1}{n-1} \\ 0, & x \ge \frac{1}{n-1} \end{cases}$$

where $n \in \mathbb{N}$. For example, we see in the following is when $n = 2, 3, 4$



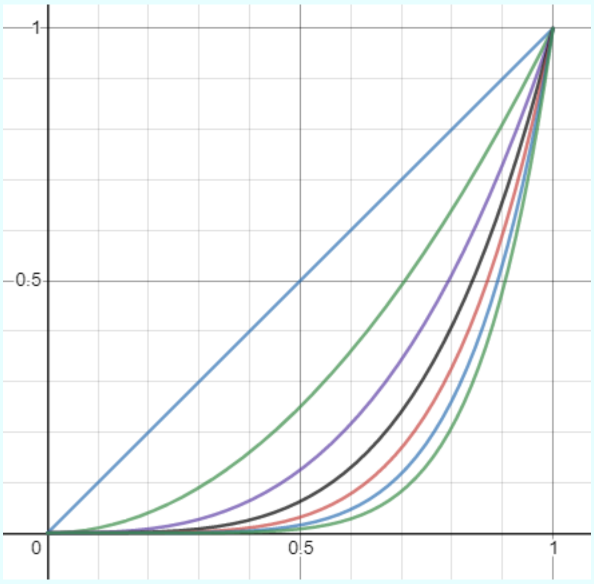
We have each f_n is continuous and $\|f_n\|_\infty = 1$. In addition, if $n \neq m$, then we have

$$\|f_n - f_m\|_\infty = \|f_n(x) - f_m(x)\|_\infty = 1$$

as f_n and f_m will never achieve 1 at the same time so max is still 1.

Thus $\{f_n\}$ is closed and bounded but not totally bounded and $(f_n)_{n=1}^\infty$ has no convergent subsequence (at least distances 1).

Example 2.2.3. Let the space be $C[0,1]$. Consider $f_n(x) = x^n$ where $n \in \mathbb{N}$. For example, below is when $1 \le n \le 7$:



$\{f_n : n \geq 1\}$ is closed and bounded. Suppose there exists a subsequence that converges, say $(f_{n_i})_i$. Then, we have

$$\lim_{i\to\infty} f_{n_i}(x) = \lim_{i\to\infty} x^{n_i} = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases} =: f(x)$$

Clearly f is not continuous so $f \notin C[0,1]$. Hence, $\{f_n : n \geq 1\}$ has no accumulation points and so all points are isolated and therefore it is closed and bounded but not compact.

Definition 2.2.4. A subset of $C(X)$ is ***equicontinuous*** if $\forall x \in X$ and $\epsilon > 0$, we have

$$\exists \delta > 0, d(x,y) < \delta \Rightarrow \forall f \in F, |f(x) - f(y)| < \epsilon$$

Lemma 2.2.5. *If $K \subset C(X)$ is compact subset, then K is equicontinuous.*

Proof. Fix $x \in X$ and $\epsilon > 0$. Since K is compact, it is totally bounded. Let $f_1, ..., f_n$ be a finite $\frac{\epsilon}{3}$ net of K. By continuity of f_i, there exists $\delta_i > 0$ such that

$$d(x,y) < \delta_i \Rightarrow |f_i(x) - f_i(y)| < \epsilon/3$$

Let $\delta = \min\{\delta_i : 1 \leq i \leq n\}$. Then, let $f \in K$, suppose $d(x,y) < \delta$, then there exists i_0 such that $\|f - f_{i_0}\|_\infty < \epsilon/3$ and hence

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{i_0}(x)| + |f_{i_0}(x) - f_{i_0}(y)| + |f_{i_0}(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

♡

Theorem 2.2.6 (Arzela-Ascoli)**.** *Let (X,d) be a compact metric space, then $K \subseteq C(X)$ is compact if and only if K is closed, bounded and equicontinuous.*

Proof. If it is compact then it is closed, bounded and equicontinuous.

Suppose K is closed bounded and equicontinuous. Since $K \subseteq C(X)$ and $C(X)$ is complete, we have K is complete. We will show K is totally bounded.

Suppose $\epsilon > 0$ is been given. Since K is equicontinuous, for each $x \in X$, there exists $\delta_x > 0$ such that $d(x,y) < \delta_x$ imply $|f(x) - f(y)| < \epsilon$. Note $\{b_{\delta_x}(x) : x \in X\}$ is an open cover, we have X is compact so there exists a finite subcover, say $b_{\delta_{x_1}}(x_1), ..., b_{\delta_{x_n}}(x_n)$.

Define $T : C(X) \to F^n$, $F \in \{\mathbb{C}, \mathbb{R}\}$, to be $T(f) = (f(x_1), ..., f(x_n))$. Since K is bounded in $C(X)$, so $T(K)$ is bounded in F^n. However, bounded subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ are totally bounded, so we pick $f_1, ..., f_m$ so that $T(f_1), ..., T(f_m)$ is an ϵ-net for $T(K)$.

We claim that $f_1, ..., f_m$ is a 4ϵ-net for K.

Let $f \in K$, $y \in X$, pick j such that $\|T(f) - T(f_j)\| = \sqrt{\sum_{i=1}^n (f(x_i) - f_j(x_i))^2} < \epsilon$ so we have $|f(x_i) - f_j(x_i)| < \epsilon$ for all $1 \leq i \leq n$.

Pick i such that $y \in b_{\delta_{x_i}}(x_i)$ and so

$$|f(y) - f_j(y)| < |f(y) - f(x_i)| + |f(x_i) - f_j(x_i)| + |f_j(x_i) - f_j(y_i)|$$

Note $y \in b_{\delta_{x_i}}(x_i)$ so $|f(y)-f(x_i)| < \epsilon$ and $|f_j(x_i)-f_j(y_i)| < \epsilon$, and $|f(x_i)-f_j(x_i)| < \epsilon$ because the choice of j. Hence, we have $|f(y) - f_j(y)| < 3\epsilon$.

Thus $\|f - f_j\|_\infty = \sup_{y \in X} |f(y) - f_j(y)| \leq 3\epsilon < 4\epsilon$.

Therefore, K is totally bounded and complete hence compact. ♡

Lemma 2.2.7. *Let (X, d) be compact, K be equicontinuous of $C(X)$, then K is uniformly equicontinuous, i.e.*

$$\forall \epsilon > 0, \exists \delta > 0, d(x,y) < \epsilon \Rightarrow \forall f \in K, |f(x) - f(y)| < \epsilon$$

Proof. Ass 3 my asϵ. ♡

Remark 2.2.8. Now we give a second proof of Arzela-Ascoli.

Second Proof of Arzela-Ascoli. Suppose K is closed, bounded and equicontinuous. We will show K is sequentially compact. Let $(f_k)_{k=1}^\infty$ be a sequence in K, we pick $\epsilon_1 > \epsilon_2 > ...$ and $\epsilon_n \to 0$. We use Lemma 2.2.7, there exists δ_i such that $d(x,y) < \delta_i \Rightarrow \forall f \in K |f(x) - f(y)| < \epsilon_i$. Since X is compact, it has a finite δ_i-net, say $x_{i,1}, ..., x_{i,n_i}$. Then, relabel $x_{1,1}, ..., x_{1,n_1}, x_{2,1}, ..., x_{2,n_2}, ...$ to be $y_1, y_2, ...$. We have $(f_k(x))_k$ is a bounded sequence in $\mathbb{R}$ or $\mathbb{C}$, so it has a convergent subsequence.

Find subsequence of $(f_k(y_1))_k$ so that it converges to a_1, say $f_{1,1}, f_{1,2}, f_{1,3}, ...$. Similarly, there exists a subsequence of $(f_k(y_2))_k$ so that it converges to a_2, $f_{2,1}, f_{2,2}, ...$.

Recursively, we do this for each y_j, and we take the diagonal subsequence $(f_{i,i})_{i=1}^{\infty}$, then $\lim_{k\to\infty} f_{k,k}(y_j) = a_j$ exists for every $j \geq 1$.

We claim (f_{kk}) is a Cauchy sequence in $C(X)$. Fix $\epsilon > 0$, pick $\epsilon_i \leq \epsilon$, equicontinuous with $\delta_i > 0$. Thus, there exists a finite net $x_{i,1}, ..., x_{i,n_i} = y_{p+1}, ..., y_{p+n_i}$, since $\lim f_{kk}(x_{i,j})$ exists for $1 \leq j \leq n_i$, there exists N such that for all $k, l \geq N$, we have $|f_{kk}(x_{i,j} - f_{ll}(x_{i,j})| < \epsilon$. Now, for $y \in X$, there exists i so that $d(y, x_i) < \delta_i$ and hence we have

$$|f_{kk}(y) - f_{ll}(y)| \leq |f_{kk}(y) - f_{kk}(x_{i,j})| + |f_{kk}(x_{i,j}) - f_{ll}(x_{i,j})| + |f_{ll}(x_{i,j}) - f_{ll}(y)| < 3\epsilon$$

Thus we have $\|f_{kk} - f_{ll}\|_\infty \leq 3\epsilon < 4\epsilon$. Thus $(f_{kk})_k$ is Cauchy and since $C(X)$ is complete, we have $f = \lim f_{kk}$ exists. Since K is closed, $f \in K$. ♡

Example 2.2.9. Let $f_n \in C^b(\mathbb{R})$ to be

$$f_n(x) = \begin{cases} 0, x \leq n \\ x - n, n \leq x \leq n+1 \\ 1, x \geq n+1 \end{cases}$$

Let K be $\{f_n : n \in \mathbb{Z}\}$, we have $\|f_n - f_m\|_\infty = 1$. So K is discrete and so closed but not compact. Note $\|f_n\|_\infty = 1$, so K is bounded and note $Lip(f_n) = 1$ and so $|f_n(x) - f_n(y)| < |x - y|$ and hence K is uniformly equicontinuous.

Remark 2.2.10. Let (X, d) be a metric space, $Y \subseteq X$, then (Y, d) is a metric space, Then, every open set in Y, say U, has the form $Y \cap V$ where V is open in X. Indeed, let $y \in U$, then there exists $r_y > 0$ such that $b^Y_{r_y}(y) \subset U$, thus, let $V = \bigcup_{y\in U} b^X_r(y)$

In Cantor set C, we have $C_1 = C \cap [0, \frac{1}{3}]$ is closed but $C \cap (-1, \frac{1}{2})$ open in $\mathbb{R}$ and hence open in C. Thus C_1 is closed and open in C.

2.3 Connectedness

Definition 2.3.1. A set A is ***disconnected*** if there are open sets U and V such that

1. $A \subseteq U \cup V$
2. $A \cap U, A \cap V$ are non-empty
3. $U \cap V = \emptyset$

Definition 2.3.2. A set A is ***connected*** if it is not disconnected.

Example 2.3.3.
1. $[0, 1] \cup [2, 3]$ is disconnected, let $U = (-1, \frac{3}{2})$ and $V = (\frac{3}{2}, 4)$.
2. We have $\mathbb{Q}$ is disconnected, as we have $\mathbb{Q} = (Q \cap (-\infty, \pi)) \cup (Q \cap (\pi, \infty))$ and the two sets are disjoint.

Remark 2.3.4. A is connected if whenever U, V are disjoint open sets with $A \subseteq U \cup V$, then $A \subseteq U$ or $A \subseteq V$.

In a relative topology A, $U \cap A$ and $V \cap A$ are both open in A. In addition, we have $V \cap A = A \backslash U = A \cap U^c$ and so $V \cap A$ is open and closed relative to A. The same holds for U. Thus, disconnected sets have (relative) clopen subsets.

Theorem 2.3.5. *A subset of $\mathbb{R}$ is connected if and only if it is an interval(a subset of $\mathbb{R}$ with intermediate value property).*

Proof. An interval I is a subset of $\mathbb{R}$ such that $a, b \in I$, $a < c < b$ then $c \in I$. If $A \subseteq \mathbb{R}$ is not an interval, then there exists $a, b \in A$, such that $a < c < b$ and $c \notin A$, so

$$A = (A \cap (-\infty, c)) \cup (A \cap (c, \infty))$$

Hence A is not connected.

Suppose I is an non-empty interval, U and V are disjoint open sets in $\mathbb{R}$ and suppose $I \subseteq U \cup V$. Fix $a \in I$, WLOG, $a \in U$, let $b = \sup\{x \in \mathbb{R} : [a, x) \subseteq U\}$ so $[a, b) \subseteq U$. If $b \in I$ then, we have two cases. If $b \in U$ then $(b - \epsilon, b + \epsilon) \subseteq U$ then $[a, b + \epsilon) \subseteq U$, which would be a contradiction. If $b \in V$, then $(b - \epsilon, b + \epsilon) \subseteq V$, thus $U \cap V \supseteq (b - \epsilon, b)$, which would be a contradiction. Thus $b \notin I$.

Similarly, if $c = \inf\{x \in \mathbb{R} : (x, a] \subseteq U\}$, if $c \in I$ then we would get a contradiction. Thus $c \notin I$, so $I \subseteq (c, b) \subseteq U$, therefore $I \cap V = \emptyset$ and so I is connected. ♡

Theorem 2.3.6. *If A is connected, if $f : A \to Y$ is continuous, then $f(A)$ is connected.*

Proof. If $f(A)$ is not connected, then there exists disjoint open sets U, V in Y such that $f(A) \subseteq U \cup V$ and $f(A) \cap U, f(A) \cap V$ are not empty. Then, by continuity, we have $f^{-1}(U)$ and $f^{-1}(V)$ are open in A and they are disjoint. Moreover, we must have $f^{-1}(V), f^{-1}(U)$ are both non-empty as $f(A) \cap U, f(A) \cap V$ are not empty. Then, we see A is disconnected by $f^{-1}(U)$ and $f^{-1}(V)$. ♡

Example 2.3.7. Let $f : \mathbb{R} \to \mathbb{R}$ have the property that $f(\mathbb{Q}) \subseteq \mathbb{Q}^c$ and $f(\mathbb{Q}^c) \subseteq \mathbb{Q}$, determine if f is continuous or not.

Solution. It cannot be continuous. Note $f(\mathbb{R})$ must be countable. Indeed, $f(\mathbb{R}) = f(\mathbb{Q} \cup \mathbb{Q}^c) = f(\mathbb{Q}) \cup f(\mathbb{Q}^c)$ where $f(\mathbb{Q})$ is countable because $\mathbb{Q}$ is countable and $f(\mathbb{Q}^c)$ is countable because it is a subset of $\mathbb{Q}$.

Suppose, for a contradiction that f is continuous, then $f(\mathbb{R})$ must be connected as $\mathbb{R}$ is connected. Thus, we have either $f(\mathbb{R})$ is a singleton or an interval. It cannot be a singleton because it need to contain at least one rational number and one irrational number. It cannot be an interval because an interval of $\mathbb{R}$ is uncountable. The proof follows. ♠

Theorem 2.3.8 (Intermediate Value Theorem). *If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a)f(b) < 0$ then there exists $c \in (a, b)$ such that $f(c) = 0$.*

Proof. We have $[a, b]$ is connected, so $f([a, b])$ is connected, so $f([a, b])$ is an interval, and $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, and in either case, we must have c such that $f(c) = 0$. ♡

Proposition 2.3.9. *If $\{X_\alpha\}_{\alpha \in A}$ are connected sets and $x_0 \in \bigcap_{\alpha \in A} X_\alpha$. Then $\bigcup X_\alpha$ is connected.*

Proof. Suppose $X \subseteq U \cup V$ where U and V are disjoint open sets. WLOG, say $x_0 \in U_0$, since X_α is connected, we have $x_0 \in X_\alpha \cap U$ and so $X_\alpha \subseteq U$ and so $\bigcup_{\alpha \in A} X_\alpha \subseteq U$. ♡

Corollary 2.3.9.1. *If $x_0 \in A$, there is a largest connected subset of A containing x_0.*

Proof. Let $\{X_\alpha\}_{\alpha \in A}$ be the collection of all connected subsets containing x_0, then $X = \bigcup X_\alpha$ is the largest. ♡

Definition 2.3.10. The largest connected set containing x_0 is called the ***connected component*** of x_0.

Example 2.3.11.

1. What are the component of $\mathbb{Q}$? The answer is $\{r\}$. Any subset of $\mathbb{Q}$ with 2 or more points are clearly disconnected.
2. Consider C, the cantor set. The component of arbitrary point x in C is just the singleton $\{x\}$.

Definition 2.3.12. A set A is ***path connected*** if for any $a, b \in A$, there exists a path $f : [0, 1] \to A$ such that $f(0) = a$ and $f(1) = b$.

Lemma 2.3.13. *For $a, b \in A \subseteq V$ where V is normed vector space, then $a \sim b$ if and only if there exists a continuous path in A from a to b is an equivalence relation on A.*

Proof. Define $f : [0, 1] \to \{a\} \subseteq A$, and clearly this is continuous and thus $a \sim a$.

Suppose $a \sim b$, then we have $f : [0, 1] \to A$ is a continuous function such that $f(0) = a$ and $f(1) = b$. Next, we define $g(x) : [0, 1] \to A$ to be $g(x) = f(1 - x)$. We first note $g(x)$ is continuous as $t(x) = 1 - x$ is continuous on $[0, 1]$ and $g(x) = f(t(x))$ is the composition of two continuous functions. Note $g(0) = f(1) = b$ and $g(1) = f(0) = a$, so $b \sim a$ as desired.

Suppose $a \sim b$ and $b \sim c$. Then we have two continuous functions $f(x) : [0, 1] \to A$ and $g(x) : [0, 1] \to A$ where $f(0) = a, f(1) = b, g(0) = b, g(1) = c$. Then, define $h(x) : [0, 1] \to A$ to be $h(x) = \begin{cases} f(2x), & \text{if } x < \frac{1}{2} \\ b, & \text{if } x = \frac{1}{2} \\ g(2x - 1), & \text{otherwise} \end{cases}$

We immediately see $h(0) = f(0) = a$ and $h(1) = g(2-1) = c$. We show h is continuous on $[0,1]$. If $x < \frac{1}{2}$, then it is continuous as $f(2x)$ is continuous on $[0,1]$ and $2x \in [0,1]$. If $1 \geq x > \frac{1}{2}$ then it is continuous as $g(2x-1)$ is continuous on $[0,1]$ and $2x-1 \in [0,1]$. Hence, it suffice to show $h(x)$ is continuous at $x = \frac{1}{2}$. Let $\epsilon > 0$ be given. Note $f(x)$ is continuous at 1, so there exists $\delta_1 > 0$ such that for all $x \in [0,1]$, $|1-x| < \delta_1 \Rightarrow |f(1) - f(x)| < \epsilon$. Similarly, since g is continuous at 0, there exists $\delta_2 > 0$ such that for all $x \in [0,1]$, $|0-x| < \delta_2 \Rightarrow |g(0) - g(x)| < \epsilon$. Let $\delta = \frac{1}{2}\min\{\delta_1, \delta_2\}$.

Then, for $x \in [0, \frac{1}{2})$, $|\frac{1}{2} - x| = \frac{1}{2}|1-2x| < \delta$, so $|1-2x| < 2\delta < \delta_1$, and tehrefore, $|f(1) - f(2x)| = |h(\frac{1}{2}) - h(x)| < \epsilon$ as desired. For $x \in (\frac{1}{2}, 1]$, we have $|\frac{1}{2} - x| < \delta \Rightarrow |2x - 1 - 0| < 2\delta < \delta_2$, and so $|g(2x-1) - g(0)| = |h(x) - h(\frac{1}{2})| < \epsilon$. For $x = \frac{1}{2}$, we must have $|h(1/2) - h(x)| = 0 < \epsilon$. Hence $h(x)$ is continuous at $\frac{1}{2}$.

Therefore, $h : [0,1] \to A$ is a continuous mapping and hence $a \sim c$.

Thus $a \sim b$ is an equivalence relation on A. ♡

Lemma 2.3.14. *Let V be normed vector space with norm $\|\cdot\|$, let $v, u \in V$, suppose $v \neq 0$, then $f : [0,1] \to V$ given by $f(t) = (1-t)v + u$ is continuous.*

Proof. Let $\epsilon > 0$ be given. Then, let $\delta = \frac{\epsilon}{\|v\|}$. We have

$$\begin{aligned}\|f(x) - f(y)\| &= \|(1-x)v + u - (1-y)v - u\| = |x-y| \cdot \|v\| \\ &< \frac{\epsilon}{\|v\|} \cdot \|v\| = \epsilon\end{aligned}$$

♡

Proposition 2.3.15. *Let U be an open subset of a normed vector space V. Then, U is connected if and only if it is path connected.*

Proof. Suppose U is path connected. For the sake of contradiction, we suppose U is not connected, thus there exists open sets A and B such that $A \cap B = \emptyset$, $U \subseteq A \cup B$ and $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$. Let $a \in A \cap U$ and $b \in B \cap U$, we note $a \neq b$. Thus, let $\alpha : [0,1] \to U$ be a continuous path from a to b, i.e. $\alpha(a) = 0$ and $\alpha(b) = 1$, as U is path connected. We consider $\mathcal{A} = \alpha^{-1}(A')$ and $\mathcal{B} = \alpha^{-1}(B')$ where $A' = A \cap U$ and $B' = B \cap U$, they are both open as α is continuous. In addition, we remark that $U = B' \cup A'$.

First, note $0 \in \mathcal{A}$ and $1 \in \mathcal{B}$ so $[0,1] \cap \mathcal{A}$ and $[0,1] \cap \mathcal{B}$ are not empty. Next, suppose $x \in [0,1]$ such that $x \in \mathcal{A} \cap \mathcal{B}$, then we must have $\alpha(x) \in A'$ as $x \in \mathcal{A}$ and $\alpha(x) \in B'$ as $x \in \mathcal{B}$. Hence $x \in A' \cap B'$ but $A' \cap B' = \emptyset$, a contradiction. Thus $\mathcal{A} \cap \mathcal{B} = \emptyset$. Moreover, we must have $\alpha^{-1}(U) = [0,1]$ as each $x \in [0,1]$ must be mapped to U. Hence, we have $[0,1] = \alpha^{-1}(A' \cup B') = \alpha^{-1}(A') \cup \alpha^{-1}(B')$ as A' and B' are disjoint. Thus $[0,1] \subseteq \alpha^{-1}(A') \cup \alpha^{-1}(B') = \mathcal{A} \cup \mathcal{B}$. Thus $\mathcal{A}$ and $\mathcal{B}$ separates $[0,1]$, a contradiction as $[0,1]$ is connected.

Conversely, suppose A is an open set in V.

Suppose A is open and connected. We say $a \sim b$, where $a, b \in A$, when there exists a continuous path in A from a to b. In addition, we say $a \nsim b$ when there do not exist a continuous path from a to b.

Let $p \in A$. Then let $U = \{x \in A : x \sim p\}$ and let $V = \{x \in A : x \nsim p\}$. As every equivalence relation form partitions on the set (MATH 145), we have $U \cup V = A$ and $U \cap V = \emptyset$.

Next, we show U is open. Let $x \in U$, we have $x \sim p$. Since A is open, there exists $r > 0$ so $b_r(x) \subseteq A$. Next, note $\forall y \in b_r(x)$, there exists a continuous path between x and y, namely the "straight line" connecting y and x, that is, $f : [0, 1] \to A$ where $f(t) = (1 - t)y + x$. Note we can always find such a straight line if $x \neq y$ (one of them cannot be the zero vector, say y, then f must be continuous. If $y = 0$ then we consider $f(t) = (1 - t)x + y)$ and if $x = y$ then they are connected trivially. Hence $y \sim x$, however, since $p \sim x$, we have $y \sim p$ for all $y \in b_r(x)$ and thus $b_r(x) \subseteq U$. Thus U is open.

Suppose, for a contradiction, that V is not empty.

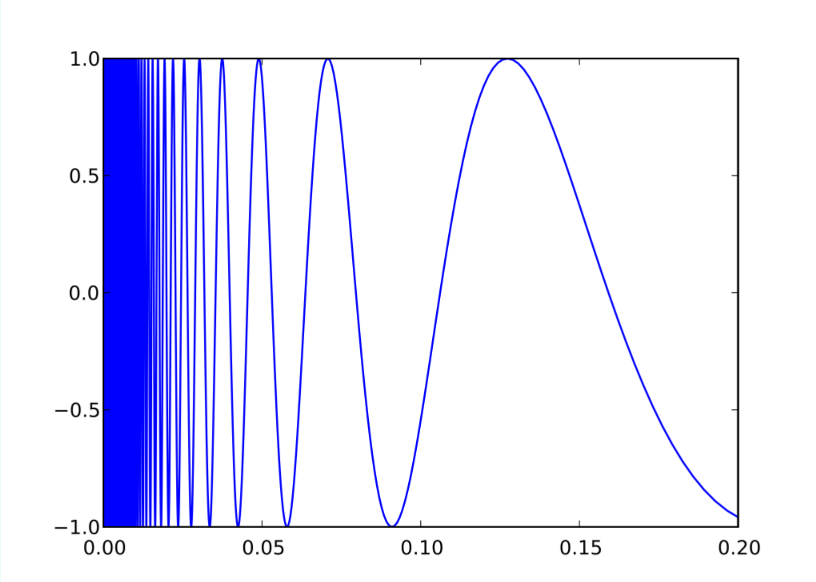
Now, if we can show V is open, then V, U saperates A as V are not empty (indeed, since in that case, we have they are non-empty, disjoint, and the union covers A, and both open).

Let $x \in V$, then x is not connected with p, and since A is open, we have $b_r(x) \subseteq A$. Suppose, for a contradiction, $y \in b_r(x)$ and $y \sim p$. Then since $y \sim x$ by the straight line, we have $x \sim p$, which is a contradiction. Hence, all the points in $b_r(x)$ are not connected with p and $b_r(x) \subseteq V$ as desired.

Thus, V, U saperates A, a contradiction and we must have $V = \emptyset$. Hence, A is path connected as for any point $p \in A$, we have the set of points that connected with p to be A, i.e. they are path-connected. ♡

Example 2.3.16 (Topologist's sine curve)**.** We provide a connected but not path connected sets. Consider the topological space (T, d) where $T = \{(x, \sin x) : x \in (0, 1]\} \cup \{0, 0\}$ and d is the Euclidean distance.

This is connected but not path connected because it includes the point (0,0) but there is no way to link the function to the origin so as to make a path.



2.4 Perfection

Definition 2.4.1. A subset A of a metric space is ***totally disconnected*** if all connected components are single points.

Example 2.4.2. The Cantor set and $\mathbb{Q}$ are both totally disconnected.

Definition 2.4.3. A closed set A is ***perfect*** if it has no isolated points, i.e. every point of A is an accumulation point of A.

Example 2.4.4. The Cantor set C is perfect and any connected closed set is perfect.

Lemma 2.4.5. *If X is compact metric space and totally disconnected, $x \in X$, $r > 0$ then there is a clopen set U, $x \in U \subseteq_r (x)$. In particular, the topology is generated by clopen sets.*

Proof. If $y \in \{y \in X : d(x,y) \geq r\} := B$, then x and y are in different connected components. Thus, there exists disjoint open sets U_y, V_y such that $X = U_y \cup V_y$ where $x \in U_y, y \in V_y$. Thus U_y, V_y are clopen as for example, we note $U_y = X \backslash V_y$ is closed. Thus $\{V_y, y \in B\}$ covers B, where B is closed in a compact set, hence compact. Thus, there exists $\{V_{y_1}, ..., V_{y_n}\}$ that covers B. In particular, we note $x \in U = U_{y_1} \cap ... \cap U_{y_n}$ where U is clopen and disjoint from $V_{y_1} \cup ... \cup V_{y_n} \supseteq B$. Hence, we have $U \subseteq B^c = b_r(x)$. Hence, if $W \subseteq X$ is open, then for all $x \in W$, there exists $r_x > 0$ such that $b_{r_x}(x) \subset W$. Hence, there exists clopen set U_x such that $x \in U_x \subseteq b_{r_x}(x) \subseteq W$. Thus $W = \bigcup_{x \in W} U_x$ and the proof follows. ♡

Corollary 2.4.5.1. *If X is compact, metric and totally disconnected. Let $\epsilon > 0$ then htere exists finitely many clopen sets $U_1, ..., U_n$ where $U_i \cap U_j = \emptyset$ if $i \neq j$, $diam(U_i) < \epsilon$, $X = U_1 \cup ... \cup U_h$.*

Proof. If $x \in X$, $x \in U_x$ be clopen, and $diam(U_x) < \epsilon$. Then there exists finite subcover $U_{x_1}, ..., U_{x_n}$ such that

$$V_S = (\bigcap_{i \in S} U_{x_i}) \cap (\bigcap_{j \notin S} U_{x_j}^c), S \in P\{1, 2, ..., n\}$$

Note V_S are clopen, pair wise disjoint, $diam < \epsilon$ and covers X. ♡

Remark 2.4.6. Note, the compact countable sets are much more complicated to classify.

Theorem 2.4.7. *If X is non-empty, compact, perfect, and totally disconnected metric space, then X is homeomorphic to C, the Cantor set.*

Proof. Tile X with finite number of clopen sets with diameter $< \frac{1}{2}$, say $x_1, x_2, ..., x_{n_1}$ with $n_1 \geq 2$. Split C into $C_1, ..., C_n$ with diameter $< \frac{1}{2}$. Tile each X_1 with finite number of clopen sets with diameter $< \frac{1}{4}$, say $X_{i_1}, ..., X_{i_{n(2,i)}}$ and split C_i into finite number of clopen sets $C_i, ..., C_{i_{n(2,i)}}$ with diameter $< \frac{1}{4}$.

Recursively, build two trees of clopen sets, at nth level pairwise disjoint and $diam < 2^{-n}$.

Each infinite path in the tree yields a nested sequence of clopen sets with $diam \to 0$, so intersection is non-empty by compactness and is a singleton.

For each n, at the nth level, we have $X_1^{(n)}, ..., X_{k(n)}^{(n)}$ and $C_1^{(n)}, ..., C_{k(n)}^{(n)}$. Pick $x_i^n \in X_i^{(n)}$ and $y_i^n \in C^{(n)}$. Define $f_n : X \to C$ and $g_n : C \to X$ by $f_n|_{X_i^{(n)}} = y_i^n$ and $g_n|_{C_i^{(n)}} = x_i^n$, so f_n and g_n are continuous.

Then, we have

$$g_n \circ f_n|_{X_i^{(n)}} = x_i^n$$

So, let Id_X be the identity function on X, we get

$$\begin{aligned} \|g_n \circ f_n - Id_X\|_\infty &= \max_{1 \leq i \leq k(n)} \sup_{x \in X_i^{(n)}} d(x, x_i^n) \\ &= \max_{1 \leq i \leq k(n)} diam(X_i^{(n)}) \leq 2^{-n} \end{aligned}$$

Similarly, we have $\|f_n \circ g_n - Id_C\|_\infty \leq 2^{-n}$

However, we also have

$$\|f_n - f_{n+1}\|_\infty = \max_{1 \leq i \leq k(n)} \max_j d(y_i^n, y_j^{n+1}) \leq diam(C_i^{(n)}) \leq 2^{-n}$$

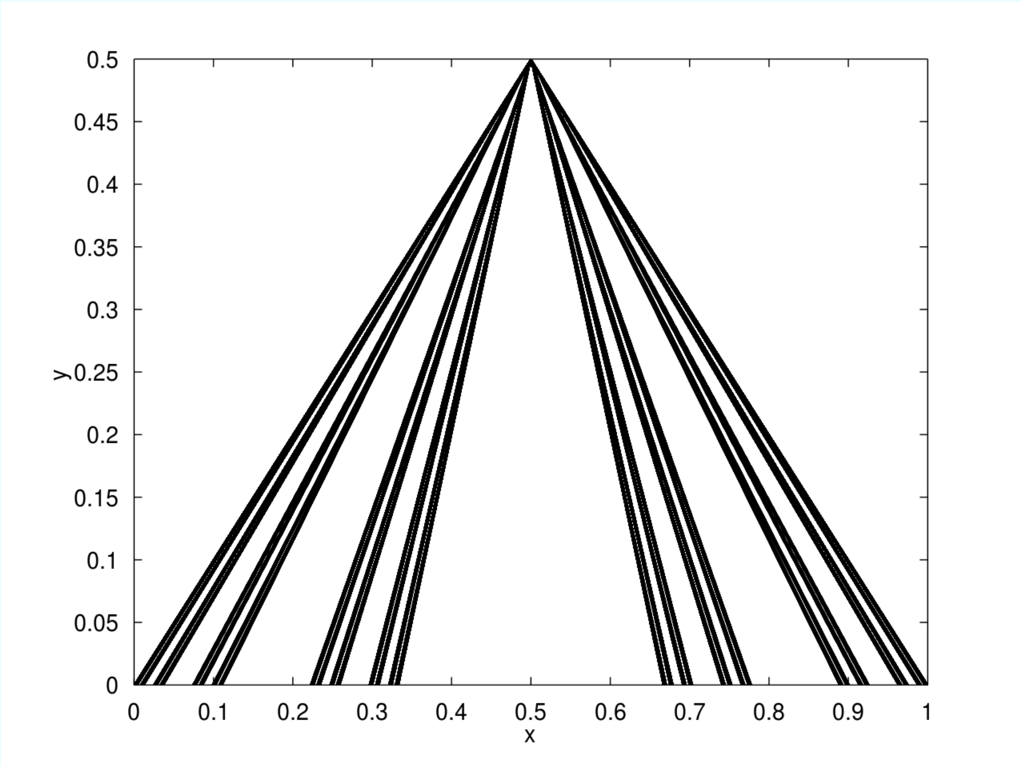
Next, note $X_i^{(n)} = X_1^{(n+1)} \cup ... \cup X_p^{(n+1)}$, so that $\|g_n - g_{n+1}\|_\infty \leq 2^{-n}$. Note $\sum 2^{-n} < \infty$ and so we have (f_n) and (g_n) are Cauchy. Let $f = \lim f_n$ and $g = \lim g_n$. Then, we have

$$g \circ f = \lim g_n \circ f_n = Id_X$$
$$f \circ g = \lim f_n \circ g_n = Id_C$$

So, f, g are continuous, bijective, inverses, so this is a homeomorphism. ♡

Remark 2.4.8. In the above proof, we used the property of perfect to ensure that we can always subdivide sets.

Example 2.4.9 (Weird!). The ***Knaster Kuratowski fan*** (or Cantor's triangle).



Let $c \in C$, the cantor set, let $L(c)$ be a line segment from $(c, 0)$ to $(\frac{1}{2}, \frac{1}{2})$.

Let $p = \{(\frac{1}{2}, \frac{1}{2})\}$. Next, if $c \in C$ has finite ternary expansion, we let

$$\mathcal{X}_c = \{(x, y) \in L(c) : y \in \mathbb{Q}\}$$

On the other hand, if $c \in C$ has infinite ternary expansion, we get

$$\mathcal{X}_c = \{(x, y) \in L(c) : y \notin \mathbb{Q}\}$$

$$F = p \cup \bigcup_{c \in C} \mathcal{X}_c$$

Then F is connected but $F \backslash \{(\frac{1}{2}, \frac{1}{2})\}$ is totally disconnected.

2.5 Baire's Category Theorem

Definition 2.5.1. A set $A \subseteq X$ is ***nowhere dense*** if $int(\overline{A}) = \emptyset$.

Example 2.5.2. The Cantor set is nowhere dense in $\mathbb{R}$.

Definition 2.5.3. A set $A \subseteq X$ is ***first category*** if A is a countable union of nowhere dense sets. A set $B \subseteq X$ is a ***residual*** if $X\backslash B$ is first category.

Remark 2.5.4. The following theorem, in short, will be called BCT.

Theorem 2.5.5 (Baire's Category Theorem)**.** *Let X be a complete metric space. Then X is not a first category. Viz, if $\{A_n\}_{n=1}^{\infty}$ are nowhere dense in X, then $X \neq \bigcup_{n\geq 1} A_n$ and $X\backslash \bigcup_{n\geq 1} A_n$ is dense.*

Proof. Let $x \in X$, $r > 0$, and $\{A_n\}_{n\geq 1}$ be all nowhere dense. We will find

$$y \in b_r(x)\backslash \bigcup_{n\geq 1} \overline{A_n}$$

Note $\overline{A_1}$ has no interior points, so $b_{r/2}(x)\backslash\overline{A_1}$ is a non-empty open set. Pick $x_1 \in b_{r/2}(x)\backslash\overline{A_1}$ and $0 < r_1 < r/2$ such that $\overline{b_{r_1}(x_1)} \subseteq b_{r/2}(x)\backslash\overline{A_1}$.

Recursively, choose x_n and $0 < r_n < r/2^n$ such that

$$\overline{b_{r_n}(x)} \subseteq b_{r_{n-1}}(x_{n-1})\backslash\overline{A_n}$$

Since X is complete and $diam(\overline{b_{r_n}(x_n)}) \leq 2r_n \to 0$ and $\overline{b_{r_n}(x_n)} \subseteq \overline{b_{r_{n-1}}(x_{n-1})}$ so that $\overline{b_{r_n}(x_n)}$ is a nested sequence, we have

$$\bigcap_{i=1}^{\infty} \overline{b_{r_n}(x_n)} \neq \emptyset$$

Therefore, $(x_n)_{n\geq 1}$ is Cauchy and so let $x_0 = \lim_{n\to\infty} x_n \in \bigcap_{n\geq 1} \overline{b_{r_n}(x_n)}$. However, $x_0 \in \overline{b_{r_n}(x_n)}$ for each n, and hence $x_0 \notin \overline{A_n}$. In particular, then we have

$$x_0 \notin \bigcup_{n\geq 1} A_n$$

Thus, the proof follows as this x_0 is the y we are looking for. ♡

Corollary 2.5.5.1. *If X is complete and $\{U_n\}_{n\geq 1}$ are dense open subsets, then $\bigcap_{n\geq 1} U_n$ is dense.*

Proof. Let U_n be dense open subsets, then we have $A_n = U_n^c$ is closed and nowhere dense, as we have $x \in int(A_n) \Rightarrow x \in \overline{U_n}^c$. Thus, we have $\bigcap_{n\geq 1} U_n = (\bigcup_{n\geq 1} A_n)^c$ is dense by BCT. ♡

Corollary 2.5.5.2. *Let X be complete and $\{A_n\}$ be closed sets and $X = \bigcup_{n\geq 1} A_n$. Then $\exists n_i \in \mathbb{N}$ and A_{n_i} such that $int(A_{n_i}) \neq \emptyset$.*

Proof. If $int(A_n) = \emptyset$ for all n, then $X \neq \bigcup_{n\geq 1} A_n$ by BCT. ♡

Example 2.5.6. Note $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}}\{q\}$ where each $\{q\}$ is nowhere dense. Thus, $\mathbb{Q}$ is a first category that is dense in $\mathbb{R}$.

Theorem 2.5.7. *Consider the subset ζ of $C[a,b]$, where ζ is the set of all nowhere differentiable functions. Then, ζ is a residual set, hence dense.*

Proof. Say f is Lipschitz at x if there exists $c < \infty$ such that $|f(x)-f(y)| \leq c|x-y|$ for all $y \in [a,b]$. The c above is called a Lipschitz constant.

Now, suppose f is differentiable at x, then $\lim_{y\to x} \frac{f(y)-f(x)}{y-x} = f'(x)$ exists.

Then, there exists $\delta > 0$ and

$$|x-y| < \delta \Rightarrow |\frac{f(y)-f(x)}{y-x} - f'(x)| \leq 1 \Rightarrow |f(y)-f(x)| \leq (|f'(x)|+1)|y-x|$$

If $|y-x| \geq \delta$, then

$$|f(y)-f(x)| \leq 2\left\|f\right\|_\infty \leq \left(\frac{2\left\|f\right\|_\infty}{\delta}\right)|y-x|$$

Therefore, $c = \max(|f'(x)|+1, \frac{2\|f\|_\infty}{\delta})$ is a Lipschitz constant for f.

Let $A_n = \{f \in C[a,b] : \exists x \in [a,b]$, f is Lipschitz at x with Lipschitz constant $\leq n\}$. Note $\cup A_n$ contains all functions differentiable at a point.

Claim: A_n is nowhere dense. Then, since $C[a,b]$ is complete, we have $\zeta \supseteq (\bigcup_{n\geq 1} A_n)^c$ is residual.

Suppose $f_k \in A_n$ for $k \geq 1$ and $f_k \to f$ uniformly. Then, there exists $x_k \in [a,b]$ such that

$$|f_k(y) - f_k(x)| \leq n|y-x_k|$$

By compactness, there exists a subsequence that converges, say $x_{k_i} \to x_0$. Then,

$$\begin{aligned}|f(y)-f(x_0)| &\leq |f(y)-f_{k_i}(y)| + |f_{k_i}(y)-f_{k_i}(x_{k_i})| \\ &\quad + |f_{k_i}(x_{k_i}) - f_{k_i}(x_0)| + |f_{k_i}(x_0)-f(x_0)| \\ &\leq \|f-f_{k_i}\|_\infty + n|y-x_{k_i}| + n|x_{k_i}-x_0| + \|f-f_{k_i}\|_\infty\end{aligned}$$

Let $k_i \to \infty$, then we have $\|f-f_{k_i}\|_\infty \to 0$ and $n|x_{k_i}-x_0| \to 0$, thus

$$|f(y)-f(x_0)| \leq n|y-x_0| \Rightarrow f \in A_n$$

Subclaim: A_n has no interior.

Let $f \in A_n$, and $\epsilon > 0$, we will find $g \notin A_n$ and $\|f-g\|_\infty < \epsilon$. This is the same as find piecewise linear function h with $\|f-h\|_\infty < \epsilon/2$.

There exists $\delta > 0$ such that $|x-y| < \delta$ imply $|f(x)-f(y) < \epsilon/4$. Let $a = x_0 < x_1 < ... < x_p = b$ with $|x_i - x_{i+1}| < \delta$. Let $h(x_i) = f(x_i)$ and linear in between. We see that $\|h-f\|_\infty < \epsilon/2$. Since h is linear, it is Lipschitz with constant $L = \max\{|\text{slope of } h|\}$

Let $M > \frac{4\pi}{\epsilon}(L+n)$ and let $g(x) = h(x) + \frac{\epsilon}{2}\sin(Mx)$. Then, we have

$$\|g-f\|_\infty \leq \|g-h\|_\infty + \|h-f\|_\infty < \epsilon$$

Subsubclaim: $g \notin A_n$. Let $x_0 \in [a,b]$, pick y such that $|y - x_0| < \frac{2\pi}{M}$ such that $\sin(My) = \begin{cases} 1, & \text{if } \sin(Mx_0) < 0 \\ -1, & \text{otherwise} \end{cases}$

Then, we have

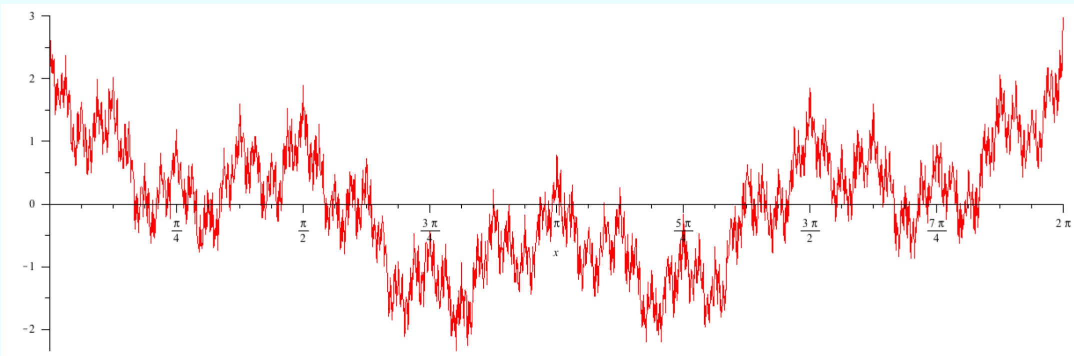
$$\begin{aligned}
|g(y) - g(x_0)| &\geq \frac{\epsilon}{2} \cdot |\sin(My) - \sin(Mx_0)| - |h(y) - h(x_0)| \\
&\geq \frac{\epsilon}{2} - L|y - x_0| \\
&\geq \frac{\epsilon}{2} \cdot \frac{M}{2\pi}|y - x_0| - L|y - x_0| \\
&= |y - x_0|\left(\frac{\epsilon M}{2\pi} - L\right) \\
&\geq n|y - x_0|
\end{aligned}$$

Thus $g \notin A_n$ and hence $g \notin \bigcup_{n\geq 1} A_n$ and the proof follows. ♡

Example 2.5.8. One example of nowhere differentiable function is

$$f(x) = \sum_{n\geq 0} (\frac{2}{3})^n \cos(4^n x)$$

The graph is:



Proposition 2.5.9 (Weierstrass M Test)**.** *If $f_n \in C(X)$ and $\sum \|f_n\|_\infty < \infty$ then $\sum f_n$ converges uniformly.*

Proof. Let $S_n(x) = \sum_{i=1}^N f_i(x)$. We note

$$|S_M(x) - S_N(x)| = |\sum_{k=N+1}^{M} f_k(x)| \leq \sum_{k=N+1}^{\infty} \|f_k\|_\infty < \epsilon$$

when N is large enough, thus the sequence is Caucyh and the proof follows. ♡

Example 2.5.10 (Weierstrass's nowhere differentiable function)**.** Consider $f(x) = \sum_{i=1}^{\infty} f_n(x)$ where $f_n(x) = 2^{-n}\sin(10^n \pi x)$. We have $f_n(x)$ is continuous and 1-periodic with $\|f_n\|_\infty = 2^{-n}$. Thus we have $\sum_{n\geq 1}\|f_n\|_\infty = \sum_{n\geq 1} 2^{-n} = 1 < \infty$. Hence f is continuous by Weierstrass M test.

We claim f is nowhere differentiable.

Let $x = 0.x_1x_2x_3... \in [0,1]$. For $n \geq 1$, let $y_n = 0.x_1x_2...x_n$ and $z_n = y_n + 10^{-n}$. We have

$$\begin{aligned}
f_n(y_n) &= 2^{-n}\sin(10^n\pi y_n) = 2^{-n}(-1)^{10^n y_n} = \pm 2^{-n} \\
f_n(z_n) &= 2^{-n}\sin(10^n\pi z_n) = 2^{-n}(-1)^{10^n y_n + 1} = \mp 2^{-n}
\end{aligned}$$

Thus, we have

$$|f_n(y_n) - f_n(z_n)| = 2^{1-n}$$

If $k > n$, we have

$$f_k(y_n) = 2^{-k}\sin(10^k\pi y_n) = 2^{-k}(-1)^{10^k y_n} = 2^{-k}$$

and

$$f_k(z_n) = 2^{-k}\sin(10^k\pi z_n) = 2^{-k}(-1)^{10^k z_n} = 2^{-k}$$

and so $f_k(y_n) - f_k(z_n) = 0$.

If $1 \leq k < n$, then we have $c \in (y_n, z_n)$

$$\begin{aligned}
|f_k(y_n) - f_k(z_n)| &= |f_k'(c)(y_n - z_n)|, \quad \text{by mean value theorem} \\
&= \|f_k'\|_\infty |y_n - z_n| \\
&= \left\|2^{-k}\cdot 10^k\pi\cos(10^k\pi x)\right\|_\infty \cdot 10^{-n} \\
&= \frac{5^k\pi}{10^n} = 2^{-n}\frac{\pi}{5^{n-k}}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|f(y_n) - f(z_n)| &\geq |f_n(y_n) - f_n(z_n)| - \sum_{k=1}^{n} |f_k(y_n) - f_k(z_n)| \\
&= 2^{1-n} - \sum_{k=1}^{n-1} 2^{-n}\pi\frac{1}{5^{n-k}} \\
&= 2^{-n}(2 - \pi\sum_{j=1}^{n-1}\frac{1}{5^j}) \\
&> 2^{-n}(2 - \pi\frac{1}{5}\cdot\frac{1}{1-\frac{1}{5}}) \\
&= 2^{-n}(2 - \frac{\pi}{4}) > 2^{-n}
\end{aligned}$$

So either we have $|f(y_n) - f(x)| > 2^{-n-1}$ or $|f(z_n) - f(x)| > 2^{-n-1}$. Call the point where its bigger w_n, then we have

$$\left|\frac{f(x) - f(w_n)}{x - w_n}\right| > \frac{2^{-n-1}}{10^{-n}} = \frac{5^n}{2} \to \infty$$

Thus, we have $\lim_{w \to x} |\frac{f(x)-f(w)}{x-w}|$ does not exists and so no where differentiable!

Definition 2.5.11. If $f : (X, d) \to (Y, \rho)$ is a function, the ***oscillation of f at*** x, $w_f(x)$, is defined as follows. For $\delta > 0$, consider

$$w_f(x, \delta) := \sup_{y,z \in b_\delta(x)} \rho(f(y), f(z))$$

then $w_f(x)$ is defined as $w_f(x) := \inf_{\delta > 0} w_f(x, \delta)$

Lemma 2.5.12. *f is continuous at x if and only if $w_f(x) = 0$.*

Proof. Haha ♡

Lemma 2.5.13. *Let $f : X \to Y$ be given. Let $\epsilon > 0$, then $\{x \in X : w_f(x) < \epsilon\}$ is open.*

Proof. Let $x \in U$, find $\delta > 0$ such that $w_f(x, \delta) < \epsilon$. If $y \in b_\delta(x)$, say $d(x, y) = r < \delta$. So, we have $b_{\delta-r} \subseteq b_\delta(x)$, so $w_f(y, \delta - r) \leq w_f(x, \delta) < \epsilon$.

Thus, $w_f(y) < \epsilon$ and so $b_\delta(x) \subseteq U$ and so U is open. ♡

Example 2.5.14. Note pointwise limit of continuous function need not be continuous.

Indeed, consider $f_n(x) = x^n$ for $0 \leq x \leq 1$.

Definition 2.5.15. A G_δ set is the countable intersection of open sets. An F_σ sets is the countable union of closed sets.

Theorem 2.5.16. *If $f_n \in C[a, b]$ and $f(x) = \lim_{n\to\infty} f_n(x)$ exists pointwise, then the set of points of continuity of f is a residual dense G_δ set.*

Proof. We have the set of points of continuity of f is the same as $\{x : w_f(x) = 0\}$, which is the same as $\bigcap_{n \geq 1}\{x \in X : w_f(x) < \frac{1}{n}\}$, we note this is G_δ set.

Then, the complement is $A := \bigcup_{n\geq 1} A_n$ where $A_n = \{x : w_f(x) \geq \frac{1}{n}\}$ is closed.

We will show that $int(A_n) = \emptyset$, and so A_n is nowhere dense and thus A is a first category and so A^c is a residual set, and in particular, it is dense.

Fix n. Let I be a small interval in $[a, b]$. Find $x \in I$ such that $x \notin A_n$, i.e. $w_f(x) < \frac{1}{n}$. Take $\epsilon = \frac{1}{3\epsilon}$. Let

$$E_n := \bigcap_{i,j \geq N} \{x : |f_i(x) - f_j(y)| \leq \epsilon\}$$

We note $\{x \in I : |f_i(x) - f_j(y)| \le \epsilon\}$ is closed as $f_i - f_j$ is continuous and so E_n is closed.

However, we have $\lim_{l\to\infty} f_l(x) = f(x)$ an dso $(f_l(x))_{l=1}^{\infty}$ is Cauchy and so there exists $N(x)$ such that $i, j \ge N(x)$ imply $|f_i(x) - f_j(x)| < \epsilon$ which imply $x \in E_{N(x)}$.

Thus $I = \bigcup_{N\ge1} E_N$. By BCT, there exists N_0 such that $int(E_{N_0}) \supseteq J \neq \emptyset$ where J is an interval. If $x \in J \subseteq I$, then f_{N_0} is uniform continuous, so $\exists\delta > 0$, such that $|x - y| < \delta$ imply $|f_{N_0}(x) - f_{N_0}(y)| < \epsilon$.

Choose δ small enough so that $(x - \delta, x + \delta) \subseteq J$, we have $|x - y| < \delta$ then

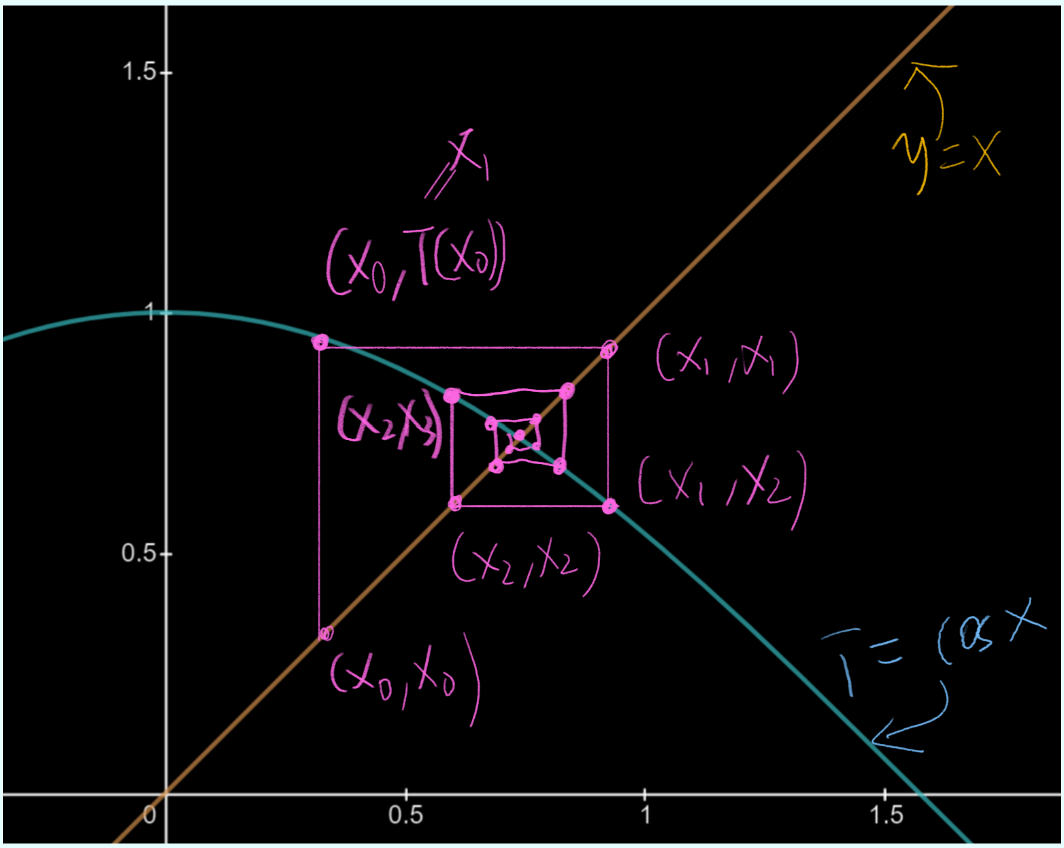
$$|f(x)-f(y)| \le |f(x)-f_{N_0}(x)|+|f_{N_0}(x)+f_{N_0}(y)|+|f(y)-f_{N_0}(y)| < \epsilon+\epsilon+\epsilon = 3\epsilon = \frac{1}{n}$$

Thus, we have $w_f(x,\delta) < \frac{1}{n}$ and so $w_f(x) < \frac{1}{n}$ and so $A_n^c \cap I \neq \emptyset$ and so A_n is nowhere dense. ♡

2.6 Contraction Mapping

Definition 2.6.1. A map $T : X \to X$ is a ***contraction*** if T is Lipschitz with a constant $c < 1$, i.e. $\forall x, y \in X, d(Tx, Ty) \le cd(x, y)$.

Example 2.6.2. We note $T(x) = \cos(x)$ is a contraction. Let $x_0 \in \mathbb{R}$ be arbitrary, let $x_n = T(x_{n-1})$ for $n \ge 1$.



As we can see the graph of $y = x$ and T intersects (at exactly one point), there exists $x_* = T(x_*)$. In particular, if $x, y \in [-1, 1]$, then, by mean value theorem, there exists $\theta \in [-1, 1]$ and we have

$$\left|\frac{Tx - Ty}{x - y}\right| = \left|\frac{\cos x - \cos y}{x - y}\right| = |\sin\theta| \leq \sin 1 = 1$$

In particular, then we have

$$\begin{aligned} |x_{n+1} - x_*| &\leq |\sin 1| \cdot |x_{n+1} - x_*| \\ &\leq |\sin 1|^n |x_0 - x_*| \to 0 \end{aligned}$$

Hence, if we have arbitrary point $x_0 \in [-1, 1]$ been picked, we would eventually converge to x_*.

Theorem 2.6.3. *Let (X, d) be a complete metric space, let $T : X \to X$ be a contraction with Lipschitz constant $c < 1$. Then, T has a unique fixed point x_* such that $x_* = T(x_*)$. In particular, if $x_0 \in X$ is arbitrary, and $x_{n+1} = T(x_n)$ for $n \geq 0$, then $x_* = \lim_{n\to\infty} x_n$ and*

$$d(x_n, x_*) \leq c^n d(x_0, x_*) \leq \frac{c^n}{1 - c} d(x_1, x_0)$$

Proof. We first note

$$d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) \leq cd(x_{n-1}, x_n) \leq ... \leq c^n d(x_0, x_1)$$

We will show $(x_n)_{n\geq 0}$ is a Cauchy sequence. Indeed, let $N \leq n \leq m$, then we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} c^k d(x_0, x_1) \\ &< \left(\sum_{k=N}^{\infty} c^k\right) d(x_0, x_1) = \frac{c^N}{1 - c} d(x_0, x_1) \end{aligned}$$

If we have N large enough, we indeed have $d(x_n, x_m) < \epsilon$ for all $\epsilon > 0$.

Thus, since X is complete, let $x_* = \lim_{n\to\infty} x_n$, we have our desired fixed point

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} \Rightarrow x_* = T(x_*)$$

Next, we observe

$$\begin{aligned} d(x_n, x_*) &= \lim_{m\to\infty} d(x_n, x_m) \\ &\leq \lim_{m\to\infty} \left(\sum_{k=n}^{m-1} c^k\right) d(x_0, x_1) \\ &= \frac{c^n}{1 - c} d(x_0, x_1) \end{aligned}$$

Finally, we show it is unique. Suppose $T(y) = y$, then we have

$$d(x_*, y) = d(T(x_*), T(y)) \leq cd(x_*, y) \Rightarrow d(x_*, y) = 0$$

♡

Example 2.6.4. Consider the function $T(x) = 108(x - x^3)$. Clearly T is not a contraction. Then, we would have three fixed points, namely 0, $\pm\frac{\sqrt{107/3}}{6}$ as we observe its graph.

However, if we select a point close enough to one of the three fixed points, we would still converge to that fixed point. Thus, we have the following theorem.

Theorem 2.6.5. *Suppose $T : [a, b] \to [a, b]$, $T(x_*) = x_*$, T is C^1 on $[a, b]$, and $|T'(x_*)| < 1$. Then, there is a $\delta > 0$ such that $T : b_\delta(x_*) \to b_\delta(x_*)$ is a contraction. Note $b_\delta(x_*)$ is just $[x_* - \delta, x_* + \delta]$.*

Proof. By continuity, there exists $\delta > 0$ such that

$$\sup_{|x - x_*| < \delta} |T'(x)| = c < 1$$

If $|x - x_*| \leq \delta$ then we have

$$\begin{aligned} |T(x) - x_*| &= |T(x) - T(x_*)| \\ &= |T'(\zeta)| \cdot |x - x_*|, \quad \text{by MVT} \\ &\leq c|x - x_*| \end{aligned}$$

♡

Example 2.6.6 (Fractals)**.** Let $T_1, ...T_n : \mathbb{R}^d \to \mathbb{R}^d$ be affine mappings, i.e. $T_i(x) = L_i(x) + y_i$ where L_i is linear and y_i is fixed. We want to find a set $X \subseteq \mathbb{R}^d$ such that

$$X = T_1(X) \cup ... \cup T_n(X)$$

Assume T_i are contractions, i.e. we have

$$\|L_i(x)\|_2 \leq c_i \|x\|_2, c_i < 1$$

If there is such an X, then we have X is similar to $T_i(X)$.

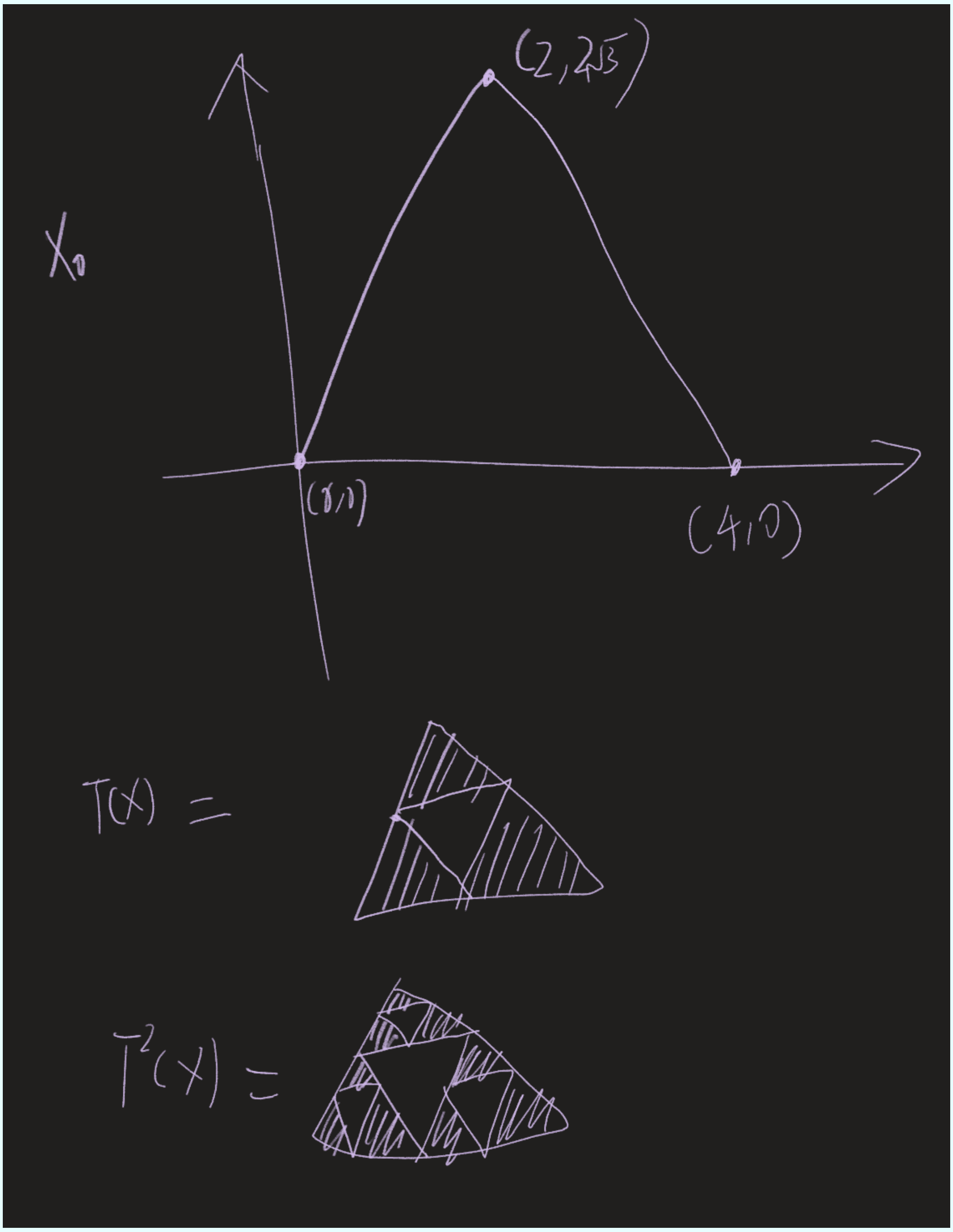
We provide a concrete example. Consider $T_l : \mathbb{R}^2 \to \mathbb{R}^2$ to be

$$T_0(x) = \frac{1}{2}x, T_1(x) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \frac{1}{2}x, T_2(x) = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} + \frac{1}{2}x$$

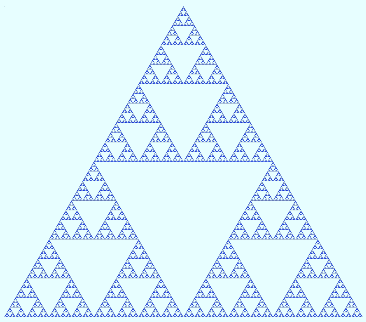
Then, we have $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is fixed for T_0, $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ is fixed for T_1 and $\begin{pmatrix} 2 \\ 2\sqrt{3} \end{pmatrix}$ for T_2.

Let $X \subseteq \mathbb{R}^2$ be compact and we define the mapping $T(X) = T_1(X) \cup T_2(X) \cup T_3(X)$.

Let X_0 be the solid triangle:



In particular, we have this limit to be the Sierpinski triangle, below is an example with a few mroe iterations



Lemma 2.6.7. *We have* $d_H(A_1 \cup ... \cup A_n, B_1 \cup ... \cup B_n) \leq \max\{d_H(A_i, B_i)\}$

Proof. Let RHS be r, then we have $d_H(A_i, B_i) \leq r$ means

$$A_i \subseteq E_r(B_i) := \{x : d(x, B_i) \leq r\} \wedge B_i \subseteq E_r(A_i)$$

So, we have

$$\bigcup A_i \subseteq E_r(\bigcup B_i) \wedge \bigcup B_i \subseteq E_r(\bigcup A_i) \Rightarrow d_H(\bigcup A_i, \bigcup B_i) \leq r$$

♡

Theorem 2.6.8. *Let* (X, d) *be complete, let* $T_1, ..., T_n : X \to X$ *be contractions. Recall* $H(X) := \{K \subset X : K \text{ is compact}\}$ *is a metric space with* d_H*, define* $T : H(X) \to H(X)$ *by*

$$T(Y) = T_1(Y) \cup ... \cup T_n(Y)$$

Then, T *is a contraction mapping, i.e. there exists unique compact set* $K_* \subseteq X$ *such that* $T(K_*) = K_*$.

Proof. We claim $d_H(T_i(A), T_i(B)) \leq c_i d_H(A, B)$.

Let $a \in A, \exists b \in B$ such that

$$\begin{aligned} &d(a, b) \leq d_H(A, B) \\ &\Rightarrow d(T_i(a), T_i(b)) \leq c_i d(a, b) \leq c_i d_H(A, B) \\ &\Rightarrow T_i(A) \subseteq E_{c_i d_H(A,B)}(T_i(B)) \end{aligned}$$

Repeat above to get $T_i(B) \subseteq E_{c_i d_H(A,B)}(T_i(A))$ then we have

$$d_H(T_i(A), T_i(B)) \leq c_i d_H(A, B)$$

Thus, T_i is a contraction in $H(X)$, and thus by Lemma 2.6.7, we have T is a contraction and so we have our desired unique K_*. ♡

Example 2.6.9. Note we must have Lipschitz constant less than 1.

1. Let $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x + 1$, this is a isometry so the Lipschitz constant is 1. Clearly, we have no fixed points.
2. Consider $S : [1, \infty] \to [1, \infty]$ given by $S(x) = x + \frac{1}{x}$. Then, we note

$$S(x) - S(y) = (x - y)(1 - \frac{1}{xy})$$

Thus, we have $|S(x) - S(y)| < |x - y|$ for all $x, y \in [1, \infty]$. However, S has no fixed point, i.e. $x \neq x + \frac{1}{x}$. Indeed, we have

$$\begin{aligned} S &= \sup_{1 \leq x < y < \infty} \frac{|S(x) - S(y)|}{|x - y|} \\ &= \sup_{1 \leq x < y < \infty} |1 - \frac{1}{xy}| = 1 \end{aligned}$$

and note $S(1) = 2$, $S^2(1) = 2+\frac{1}{2}$, $S^3(1) = 2+\frac{1}{2}-\frac{2}{5}$ and so on where $S^n(1) \to \infty$ as $n \to \infty$.

Example 2.6.10 (Newton's Method)**.** Let $f \in C^2$ with $f(x_*) = 0$.

Theorem 2.6.11. *If $f \in C^2$, $f(x_*) = 0$, and $f'(x_*) \neq 0$. Then there is exists $R > 0$ such that on $[x_* - R, x_* + R]$ the map $T(x) = x - \frac{f(x)}{f'(x)}$ is a contraction and if $x_0 \in [x_* - R, x_* + R]$, then $x_n = T^n(x_0) \to x_*$. Moreover, there is a constant M so that $|x_{n+1} - x_*| \leq M|x_n - x_*|^2$. The last part is called quadratic convergence.*

Proof. We note

$$\begin{aligned} T'(x) &= 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} \\ &= \frac{f'(x)^2 - f'(x)^2 + f(x)f''(x)}{f'(x)^2} \\ &= \frac{f(x)f''(x)}{f'(x)^2} \end{aligned}$$

and $T'(x_*) = 0$.

Pick R so that $|T'(x)| \leq \frac{1}{2}$ on $[x_* - R, x_* + R]$. Then, T is a contraction with $c \leq \frac{1}{2}$. Then, we have

$$\begin{aligned} |T(x) - T(y)| &= |T'(\zeta)| \cdot |x - y|, \zeta \in [x_* - R, x_* + R] \\ &\leq \frac{1}{2}|x - y| \end{aligned}$$

Therefore, we do have $T^n(x_0) \to x_*$ as x_* is a fixed point and the fixed point for contraction is unique. Indeed, note $T(x_*) = x_* - \frac{0}{f'(x_*)} = x_*$. We got our fixed point, now we estimate the bound.

Let $A = \sup_{|x - x_*| \leq R} |f''(x)|$ and $B = \sup_{|x - x_*| \leq R} |f'(x)|$ Let $M = \frac{A}{B}$ and we note $f(x_n) - f(x_*) = f'(\zeta_n)(x_n - x_*)$ where $\zeta_n \in (x_n, x_*)$. In addition, note $f(x_*) = 0$ so we get $f(x_n) = f'(\zeta_n)(x_n - x_*)$ and so $\frac{f(x_n)}{f'(\zeta_n)} = x_n - x_*$.

Then, we have

$$x_{n+1} = T(x_n) == x_n - \frac{f(x_n)}{f'(x_n)}$$

Therefore

$$\begin{aligned} x_{n+1} - x_* &= (x_{n+1} - x_n) + (x_n - x_*) \\ &= -\frac{f(x_n)}{f'(x_n)} + \frac{f(x_n)}{f'(\zeta_n)} \\ &= \frac{f(x_n)}{f'(\zeta_n)} \cdot \frac{f'(\zeta_n) - f'(x_n)}{f'(x_n)} \\ &= \frac{(x_n - x_*)}{f'(x_n)}(f'(\zeta_n) - f'(x_n)) \\ &= \frac{x_n - x_*}{f'(x_n)} f''(\psi_n)(\zeta_n - x_n), \psi_n \in (\zeta_n, x_n) \text{ by MVT} \end{aligned}$$

Hence, we have

$$x_{n+1} - x_* \leq \frac{f''(\phi_n)}{f'(x_n)}|x_n - x_*|^2 \leq \frac{A}{B}|x_n - x_*|^2 = M|x_n - x_*|^2$$

♡

Example 2.6.12. Compute $\sqrt{a}$, $a > 0$.

Solution. Let $f(x) = x^2 - a$, then let $T(x) = x - \frac{x^2-a}{2x} = \frac{x^2+a}{2x}$. Thus, we have $T(x) = \frac{x+a/x}{2}$. ♠

2.7 Metric Completion

Definition 2.7.1. If (X, d) is a metric space, a ***completion of*** (X, d) is a complete metric space (T, ρ) together with an isometry $J : X \to Y$ such that $J(X)$ is dense in Y.

Theorem 2.7.2. *Every metric space has a completion.*

Proof. Define $J : X \to C_b^{\mathbb{R}}(X)$ where $C_b^{\mathbb{R}}(X)$ is the Banach space of bounded continuous real-valued function on X.

Pick arbitrary $x_0 \in X$. Define, for $x \in X$, a function f_x given by $f_x(y) = d(x, y) - d(x_0, y)$ and we let $J(x) = f_x$.

We claim f_x is continuous. Note

$$f_x(y_1) - f_x(y_2) = d(x, y_1) - d(x_0, y_1) - d(x, y_2) + d(x_0, y_2)$$

and we should be able to deduct

$$|d(y, x| - d(y, x_0) \leq d(x, x_0) \Rightarrow \|f_x\|_\infty \leq d(x, x_0)$$

Then, we have

$$\begin{aligned} |f_x(y_1) - f_x(y_2)| &\leq |d(x, y_1) - d(x, y_2)| + |d(x_0, y_1) - d(x_0, y_2)| \\ &\leq 2d(y_1, y_2) \end{aligned}$$

Hence f_x is Lipschitz wiht constant less than or equal to 2. Hence continuous.

So $f_x \in C_b(X)$. In particular, we have

$$\begin{aligned} \|f_x - f_y\|_\infty &= \sup_{z \in X} |f_x(z) - f_y(z)| \\ &= \sup_{z \in X} |d(x, z) - d(y, z)| \\ &= d(x, y), \text{ by taking} z = y \end{aligned}$$

Hence, we have J is an isometry and let $Y = \overline{J(X)}$. Then Y is a closed subset of $C_b(X)$ and so $C_b(X)$ is complete, thus Y is complete. Thus $J(X)$ is dense by construction and so T is a completion of X. ♡

Remark 2.7.3. We will give a second proof that is more 'natural'.

Start with (X, d), let $\mathfrak{C} := \{(x_n)_{n=1}^{\infty} : \text{Cauchy sequences}\}$. Then, the two sequences (x_n) and (y_n) are trying to converge to the same point if and only if $\lim_{n\to\infty} d(x_n, y_n) = 0$. Next, we will show that $\lim_{n\to\infty} d(x_n, y_n) = 0$ determines a equivalence relation on $\mathfrak{C}$.

Second Proof. Let $\mathfrak{C} = \{(x_n) : (x_n) \text{ is Cauchy sequence in } (X, d)\}$. Define $(x_n) \sim (y_n)$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$. We claim $\sim$ is an equivalence relation.

1. We have $(x_n) \sim (x_n)$. Trivial.
2. We have $(x_n) \sim (y_n)$ iff $(y_n) \sim (x_n)$. Trivial.
3. We have $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$ then $(x_n) \sim (z_n)$. Trivial by triangle inequality.

Therefore, let $Y = \mathfrak{C}/\sim$ be the set of equivalence classes $\vec{x} = \{(y_n) : (x_n) \sim (y_n)\}$. Define $\rho(\vec{x}, \vec{y})$ on Y to be $\lim d(x_n, y_n)$. If $\epsilon > 0$, there exists N such that $n, m \geq N$ then $d(x_n, x_m) < \epsilon$ and $d(y_n, y_m) < \epsilon$. Therefore, we have

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) < d(x_m, y_m) + 2\epsilon$$

By symmetry, we also have $d(x_m, y_m) \leq d(X_n, y_n) + 2\epsilon$. Thus, we have

$$|d(x_n, y_n) - d(x_m, y_m)| < 2\epsilon$$

Thus, $(d(x_n, y_n))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$ and so $\lim d(x_n, y_n) = \rho(\vec{x}, \vec{y})$ exists. Thus the definition makes sense. Next, we check well-define. Let $[(x_n)] = [(x'_n)] = \vec{x}$ and $[(y_n)] = [(y'_n)] = \vec{y}$, then we have

$$d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n)$$

So, we have

$$\lim d(x'_n, y'_n) \leq \lim d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n) = \lim d(x_n, y_n)$$

By symmetry, two limit agree.

Next, we need to check ρ is a metric.

1. $\rho(\vec{x}, \vec{x}) = \lim d(x_n, x_n) = 0$, checked.
2. $\rho(\vec{y}, \vec{x}) = \rho\vec{x}, \vec{y}$, trivial.
3. $\rho\vec{x}, \vec{z} \leq \rho(\vec{x}, \vec{y}) + \rho(\vec{y}, \vec{z})$, trivial by triangle inequality in (X, d).

Define $J : X \to Y$ to be $J(x) = [(x)_{n=1}^{\infty}]$, which maps x to the constant sequence. Then, we have

$$\rho(J(x), J(y)) = \lim d(x, y) = d(x, y)$$

Therefore, J is isometry. Next, we claim $J(X)$ is dense in Y.

Let $\vec{y} = [(y_n)]$, let $\epsilon > 0$. There exists N so $n, m \geq N$ imply $d(y_n, y_m) < \epsilon$. Consider $\vec{x} = J(y_N)$, we have $\rho(\vec{x}, \vec{y}) = \lim_{n\to\infty} d(y_n, y_N) \leq \epsilon$. Thus $J(X)$ is indeed dense.

We claim (Y, ρ) is complete. Let $(\vec{y_k})_{k=1}^{\infty}$ be Cauchy in Y. Choose $x_k \in X$ such that $\rho(J(x_k), \vec{y_k}) < 2^{-k}$, where $k \geq 1$. We observe that $(x_k)_{k\geq 1}$ is Cauchy in X. If $\epsilon > 0$, there exists K so $k, l \geq K$ then $\rho(\vec{y_k}, \vec{y_l}) < \epsilon$ and $2^{-K} < \epsilon$. Let $k, l \geq K$, then we have

$$\begin{aligned} d(x_k, x_l) = \rho(J(x_k), J(x_l)) &\leq \rho(J(x_k), \vec{y_k}) + \rho(\vec{y_k}, \vec{y_l}) + \rho(\vec{y_l}, J(x_l)) \\ &< 2^{-K} + \epsilon + 2^{-K} < 3\epsilon \end{aligned}$$

Thus (x_k) is Cauchy and hence in $\mathfrak{C}$. We claim $\lim_{k\to\infty} \vec{y_k} = \vec{x} = [(x_k)]$.

We first show $\lim_{k\to\infty} J(x_k) = \vec{x}$. Note we have $\rho(J(x_k), \vec{x}) = \lim_{n\to\infty} d(x_k, x_n)$. If $\epsilon > 0$, then there exists N so $k, n \geq N$ imply $d(x_k, x_n) < \epsilon$. When $k \geq N$, we have $\rho(J(x_k), \vec{x}) \leq \epsilon$ and so $\lim J(x_k) = \vec{x}$ as desired.

If $k \geq N$, we have

$$\rho(y_k), \vec{x} \leq \rho(\vec{y_k}, J(x_k)) + \rho(J(x_k), \vec{x}) < 2^{-k} + \epsilon < 2\epsilon$$

when k is big enough.

Thus $\lim_{k\to\infty} \vec{y_k} = \vec{x}$ and so Y is complete. ♡

Theorem 2.7.4 (Extension Theorem)**.** *Let (X, d) be a metric space with completion (Y, ρ). Let (Z, σ) be another complete metric space. Suppose $f : X \to Z$ is uniformly continuous, then f extend uniquely to a uniform continuous funciton $\tilde{f} : Y \to Z$.*

Proof. We claim $[(f(x_n))]$ is well-defined equivalence classes of Cauchy sequences in Z.

Let $\epsilon > 0$ be given. By uniform continuity, there exists δ so $d(x, y) < \delta$ imply $\sigma(f(x), f(y)) < \epsilon$. Since (x_n) is Cauchy, there exists N so $\forall n, m \geq N$, we have $d(x_n, x_m) < \delta$. Thus, we have $\sigma(f(x_m), f(x_m)) < \epsilon$ and so $(f(x_m))_{m\geq 1}$ is Cauchy and so $\lim_{n\to\infty} f(x_n)$ exists.

Define $\tilde{f}\vec{x} = \lim_{n\to\infty} f(x_n)$ be our extension for any Cauchy sequence (x_n).

We first check well-defined. Suppose $(x_n) \sim (x'_n)$, then $(x_1, x'_1, x_2, x'_2, x_3, x'_3, ...)$ is Cauchy. Let $\epsilon > 0$, there exists N so $n, m \geq N$ so that

$$d(x_n, x_m) < \epsilon, d(x'_n, x'_m) < \epsilon, d(x_n, x'_n) < \epsilon \Rightarrow d(x_n, x'_m) < 2\epsilon$$

So $\lim f(x_n) = \lim f(x'_n) = \tilde{f}(\vec{x})$ is well-defined.

We claim $\tilde{f}$ is uniform continuous. Given $\epsilon > 0$, there exists $\delta > 0$, such that $d(x, x') < \delta \Rightarrow \sigma(f(x), f(x')) < \epsilon$. If $y, y' < Y$ and $\rho(y, y') = r < \delta$. Let $\vec{y} = \lim_{n\to\infty} J(x_n)$ and $\vec{y}' = \lim_{n\to\infty} J(x'_n)$. It is possible since $J(X)$ is dense in Y.

Then, we have $\rho(\vec{y}, \vec{y'}) = \lim_{n\to\infty} \rho(J(x_n), J(x'_n))$. If $\rho(\vec{y}, \vec{y}') < \delta$, then $\exists N$ so $n, m \geq n$ would give us $d(x_n, x'_n) = \rho(J(x_n), J(x'_n)) < \delta$ and so $\sigma(f(x_n), f(x'_n)) < \epsilon$.

Since (x_n) is Cauchy converge to y, we have $\tilde{f}(\vec{y}) = \lim f(x_n)$ and $\tilde{f}(\vec{y}') = \lim f(x_n')$. Hence, we have

$$\sigma(\tilde{f}(\vec{y}), \tilde{f}(\vec{y}')) = \lim \rho(f(x_n), f(x_n')) \leq \epsilon$$

So $\tilde{f}$ is uniform continuous. ♡

Theorem 2.7.5 (Uniqueness of Completion). *If (X, d) is a metric, and (Y, ρ) and (Z, σ) are two completion of X, i.e. $J : X \to Y$ and $K : X \to Z$ be their isometry. Then, there exists $h : Y \to Z$ such that $h(J(x)) = K(x)$, which is an isometric homeomorphism.*

Proof. Define $h_0 : J(X) \to K(X)$ given by $h_0(J(x)) = K(x)$. We note h_0 is an isometry, and hence uniform continuous. Extend h_0 to $h : Y \to Z$ be uniform continuous.

Then, we have

$$\sigma(h(\vec{y}), h(\vec{y}')) = \lim \sigma(h_0(J(x_n)), h_0(J(x_n'))) = \lim \rho(J(x_n), J(x_n')) = \rho(\vec{y}, \vec{y}')$$

To see onto, let $\vec{z} \in Z$ be arbitrary, then $\vec{z} = \lim K(x_n)$ where $(x_n)_{n\geq 1}$ is Cauchy. So $(J(x_n))_{n\geq 1}$ is Cauchy and $\vec{y} = \lim J(x_n) \in Y$, thus we have $h(\vec{y}) = \lim h(J(x_n)) = \lim K(x_n) = \vec{z}]$.

Since it is isometry, it is biLipschitz and hence h is homeomorphism.

To see uniqueness, if $y = \lim J(x_n)$, let h be continuous, then $h(y) = \lim h_0(J(x_n))$. ♡

2.8 P-Adic

Definition 2.8.1. Let p be a prime, recall the p-norm $|\cdot|_p$ on $\mathbb{Q}$ is $|p^n \frac{a}{b}|_p = p^{-n}$ with $|0|_p = 0$ where a, b, p are pairwise coprime.

Remark 2.8.2. We note we have

1. $|x|_p = 0 \iff x = 0$
2. $|xy|_p = |x|_p |y|_p$
3. $|x + y| \leq \max\{|x|_p, |y|_p\}$

Thus, $d_p(x, y) = |x - y|_p$ is a metric.

We let $\mathbb{Q}_p$ be the completion of $(\mathbb{Q}, d_p)$, then $\mathbb{Q}_p$ is the set of p-adic numbers.

Remark 2.8.3. In any balls in $\mathbb{Q}_p$, we have any point in that ball is the center.

Proposition 2.8.4. *If $(x_n)_{n\geq 1}$ is a d_p-Cauchy sequence in $\mathbb{Q}$ and $x = \lim x_n \neq 0$ in $\mathbb{Q}_p$, then $|x_n|_p$ is eventually constant.*

Proof. We claim

$$||x_n| - |x_m||_p \leq |x_n - x_m|_p \to 0$$

We have $|x_n|_p \in \{|r|_o : r \in \mathbb{Q}\} = \{p^n : n \in \mathbb{Z}\} \cup \{0\}$. Since x_n is Cauchy, so $\forall \epsilon > 0$ there exists N so $n, m \geq N$, we have $|x_n - x_m|_p < \epsilon$. Thus

$$||x_n|_p - |x_m|_p| < \epsilon \Rightarrow \lim_{n\to\infty} |x_n|_p \text{ exists}$$

Since $x_n \not\to 0$, so we have $|x_n|_p \not\to 0$ where $|x_n| = d_p(x_n, 0)$. In particular, we have $\lim |x_n|_p = p^k$. ♡

Theorem 2.8.5. *$\mathbb{Q}_p$ is a topologically complete field that contains $\mathbb{Q}$ as a dense subfield.*

Proof. $\mathbb{Q}_p$ is complete and $\mathbb{Q}$ is dense by construction.

We only need to check inverse. Let $x \in \mathbb{Q}_p$ with $x \neq 0$. We write $x = \lim x_n$, $x_n \in \mathbb{Q}$. Then,there exists N so for all $n \geq N$, we have $|x_n|_p = |x|_p \neq 0$.

So $x_n \neq 0$ for $n \geq N$. Let $y = \lim_{n\to\infty} \frac{1}{x_n}$, we have

$$|\frac{1}{x_n} - \frac{1}{x_m}|_p = |\frac{x_m - x_n}{x_n x_m}|_p = \frac{|x_m - x_n|_p}{|x_n|_p |x_m|_p} = \frac{|x_n - x_m|_p}{|x|_p^2}$$

Thus $\frac{1}{x_n}$ is Cauchy and we have $xy = \lim x_n \frac{1}{x_n} = 1$. ♡

Proposition 2.8.6. *In $\mathbb{Q}_p$, we have $\mathbb{Z}_p := \overline{\mathbb{Z}} = \{x \in \mathbb{Q}_p : |x| \leq 1\}$.*

Proof. RHS is closed and thus we have $\overline{\mathbb{Z}} \subseteq \{x \in \mathbb{Q}_p : |x| \leq 1\}$.

Let $x \in \mathbb{Q}_p$ such that $|x|_p \leq 1$. Then, let $n \geq 0$, there exsits $r_n \in \mathbb{Q}$ such that $|x - r_n|_p \leq p^{-n}$ and so we have

$$|r_n|_p = |x - (x - r_n)|_p \leq \max\{|x_n|_p, |x - r_n|_p\} = 1$$

Thus, we have $r_n = p^k \frac{a}{b}$ where a, b, p are pairwise coprime and $k \geq 0$. Replace a with $a + p^n c$ so that $a + p^n c \equiv 0 \pmod{b}$. Solve for $c \in \mathbb{Z}$, since we have $gcd(p^n, b) = 1$. Let $s_n = p^k(\frac{a}{b} + \frac{p^n c}{b}) = p^k(\frac{a+p^n c}{b}) \in \mathbb{Z}$. Thus,

$$|r_n - s_n|_p = |p^k \cdot \frac{p^n c}{b}|_p = p^{-k-n} \leq p^{-n}$$

Therefore, we have

$$|x - s_n|_p \leq \max\{|x - r_n|_p, |r_n - s_n|_p\} \leq p^{-n}$$

Therefore, we have $\lim s_n = x$ and so $x \in \mathbb{Z}_p$. The proof follows. ♡

Remark 2.8.7 (P-adic expansion). We let $x \in \mathbb{Z}_p$, we claim there exists unique

$$x_0 \in \{0, 1, ..., p-1\}$$

such that $|x - x_0|_p \leq \frac{1}{p}$. Pick $k \in \mathbb{Z}$ such that $|x - k| \leq \frac{1}{p}$, then let $k \equiv x_0 \pmod{p}$, then we have $x_0 \in \{0, ..., p-1\}$. We have $|k - x_0| \leq \frac{1}{p}$. Thus, we have $|x - x_0| \leq \max\{|x-k|_p, |k-x_0|_p\} \leq \frac{1}{p}$.

To see uniqueness, suppose $y_0 \in \{0, ..., p-1\}$ such that $x_0 \neq y_0$. Then $|x_0 - y_0| = 1$ and $p \nmid x_0 - y_0$, thus we have

$$1 = |x_0 - y_0|_p = |(x_0 - x) + (x - y_0)|_p \leq \max\{|x_0 - x|_p, |x - y_0|\}$$

This forces $|x - y_0| \geq 1$.

Then, we claim $\forall n \geq 0$, there exists unique $x_n \in \{0, 1, ..., p-1\}$ such that

$$|x - \sum_{i=0}^{n} x_i p^i| \leq \frac{1}{p^{n+1}}$$

We already have this for $n = 0$. Suppose it holds up to $n - 1$. Let

$$y = x - \sum_{i=0}^{n-1} x_i p^i$$

then $|y|_p \leq \frac{1}{p^n}$. Then, we have $|p^{-n}y|_p = |p^{-n}|_p|y|_p \leq 1$. By the $n = 1$ case, there exists unique $x_n \in \{0, 1, ..., p-1\}$ such that $|p^{-n}y - x_n| \leq \frac{1}{q}$. Therefore, we have

$$\begin{aligned} |y - x_n p^n|_p &= |p^n(p^{-n}y - x_n)|_p \\ &\leq p^{-n} \cdot \frac{1}{p} = \frac{1}{p^{n+1}} \end{aligned}$$

Thus, $|x - \sum x_i p^i| \leq \frac{1}{p^{n+1}}$ and so

$$x = \lim_{n \to \infty} \sum_{i=0}^{n} x_i p^i = \sum_{i \geq 0} x_i p^i$$

If $x \in \mathbb{Q}_p$, with $|x|_p = p^n$

2.9 Real Numbers

Definition 2.9.1. A field $\mathbb{F}$ is ***ordered*** if there is a subset P (for positive) such that

1. $\mathbb{F} = P \cup \{0\} \cup -P$
2. $P + P \subseteq P$, i.e. $x, y \in P \to x + y \in P$

3. $P \cdot P \subseteq P$, i.e. $x, y \in P \to xy \in P$

We say $x < y$ if $y - x \in P$.

Definition 2.9.2. An ordered field $\mathbb{F}$ has **least upper bound property** (LUBP) if every nonempty $S \subseteq \mathbb{F}$ with an upper bound $x \in \mathbb{F}$, i.e. $s \leq x$ for all $s \in S$ has a least upper bound $y = \sup S$ such that, for all upper bound x of S, we have $y \leq x$.

Definition 2.9.3. An ordered field $\mathbb{F}$ is ***Archimedean*** if $x > 0 \Rightarrow \exists n \in \mathbb{N}, \frac{1}{n} < x$, where $\mathbb{N}$ is an isomorphic copy of natural number in $\mathbb{F}$.

Definition 2.9.4. An ordered field $\mathbb{F}$ is complete if every Cauchy sequence converges.

Proposition 2.9.5. *Let $\mathbb{F}$ be an ordered field, then*

1. $\mathbb{Q} \subseteq \mathbb{F}$
2. *$\mathbb{F}$ has LUBP iff $\mathbb{F}$ is complete*
3. *$\mathbb{F}$ has LUBP then $\mathbb{F}$ is Archimedean*
4. *$\mathbb{F}$ is Archimedean, then we have $x < y \Rightarrow \exists r \in \mathbb{Q}, x < r < y$*

Proof.

1. Note $0, 1 \in \mathbb{F}$ and so $\mathbb{Z} \in \mathbb{F}$ (as an isomorphic copy). Then, take the field of fraction of this $\mathbb{Z}$, we obtained $\mathbb{Q}$ as desired, provided that the characterstic of $\mathbb{F}$ is zero.
2. LUBP imply completeness is by Bolzano–Weierstrass. Completeness imply LUBP has been done before.
3. Let $J = \{x : x > 0, nx < 1 \forall n \geq 1\} = \text{“}infinitesimal\text{''}$. Suppose J is nonempty, then 1 is an upper bound, if $x, y \in J$, then $x + y, nx + my \in J$ for $n, m \geq 1$.
   If $\mathbb{F}$ has LUBP, let $y = \sup J$, let $x_0 \in J$, then for all $x \in J$, we have $x + x_0 \in J$ so $x + x_p \leq y \Rightarrow x \leq y - x_0$ so y is not the sup, which is a contradiction. Thus, LUBP imply $J = \emptyset$ and hence Archimedean.
4. We have $x < y$ then $y - x > 0$ and so there exists n such that $0 < \frac{1}{n} < y - x$ and so there exists $k \in \mathbb{Z}$ such that $\frac{k}{n} \leq x < \frac{k+1}{n} \leq x + \frac{1}{n} < y$ where $\frac{k+1}{n} \in \mathbb{Q}$.

♡

Definition 2.9.6. An ***ordered embedding*** of an ordered field $\mathbb{F}$ into an ordered field $\mathbb{K}$ is an order preserving ring homomorphism.

Proposition 2.9.7. *Let $\mathbb{K}$ be an Archimedean ordered field and $\mathbb{K}$ be a complete ordered field, then there is an ordered embedding $\sigma : \mathbb{F} \to \mathbb{K}$.*

Proof. Note $\sigma(0) = 0$ and $\sigma(1) = 1$, thus $\sigma(n) = n$ and $\sigma(r) = r$ where $\forall n \in \mathbb{Z}$ and $\forall r \in \mathbb{Q}$. If $x \in \mathbb{F}$, let $S_x = \{r \in \mathbb{Q} : r < x\}$. Thus, there exists $n \in \mathbb{N}$ such that $x < n$ by Archimedean and so S_x is bounded above. THus, we have $\sigma(S_x)$ is bounded above in $\mathbb{K}$ by $\sigma(n)$. Define $\sigma(x) = \sup \sigma(S_x)$.

Note that $S_x + S_y = S_{x+y}$. Next, let $r, s \in \mathbb{Q}$, with $r < x$ and $s < y$, then $r + s < x + y$. If $t \in \mathbb{Q}$ with $t < x + y$ and so $x + y - t > \frac{1}{n}$ since $n \in \mathbb{N}$. Choose $r, q \in \mathbb{Q}$ so that

$x - \frac{1}{2n} < r < x$ and $y - \frac{1}{2n} < s < x$. Hence, we have

$$t < x + y - \frac{1}{n} < r + s < x + y$$

and so $t = r + (t - r)$ with $r \in S_x$ and $t - r < s$ with $t - r \in S_y$. Thus, $\sigma(x) + \sigma(y) = \sigma(x + y)$.

If $x > 0$ and $y > 0$, we have $S_x = \{r \leq 0\} \cup \{0 < r < x\}$ and $S_y = \{r \leq 0\} \cup \{0 < s < y\}$, then, we should check

$$S_{xy} = \{r \leq 0\} \cup \{rs : 0 < r < x, 0 < s < y\} \Rightarrow \sigma(xy) = \sigma(x)\sigma(y)$$

♡

Theorem 2.9.8. *There is a unique complete ordered field $\mathbb{R}$.*

Proof. Let $\mathbb{K}$ and $\mathbb{L}$ be two complete ordered fields, then they are Archimedean. By the second proposition, there exists $\sigma : \mathbb{K} \to \mathbb{L}$ order homomorphism. Also, there exists $\delta : \mathbb{L} \to \mathbb{K}$ order homomorphism, with both identity on $\mathbb{Q}$.

Thus, $\sigma\delta : \mathbb{K} \to \mathbb{K}$ and $\delta\sigma(r) = r$. Thus $\delta\sigma(x) = \sup S_x = x$ and so $\delta\sigma = I_{\mathbb{K}}$. Similarly, we have $\sigma\delta = I_{\mathbb{L}}$. Thus they are isomorphic. ♡

Remark 2.9.9 (Construction from $\mathbb{Q}$)**.** In the following, we talks about construction of $\mathbb{R}$ from $\mathbb{Q}$.

Example 2.9.10 (Cantor's)**.** Let $\mathbb{R} = \mathfrak{C}/\sim$ where $\mathfrak{C}$ is the Cauchy sequences in $\mathbb{Q}$ and $\sim$ means their limit goes to zero.

It has all the well-defined operations, we only need to check order. We define $[(y_n)] > [(x_n)]$ if there exists N so $x_n < y_n$ for all $n \in\geq N$. Let $[x_n] \neq [y_n]$ then there exists N such that for all $n, m \geq N$, we have $|x_n - x_m| < \frac{1}{10^N}$, $|y_n - y_m| < \frac{1}{10^N}$ and so $|x_n - y_n| \geq \frac{3}{10^N}$. Hence we indeed have order here.

Example 2.9.11 (Dedikind Cuts)**.** Consider $\mathfrak{S} = \{S : S \subseteq \mathbb{Q}, S \text{ is a cut}\}$. We say S is a cut if $S \neq \emptyset$ and $S \neq \mathbb{Q}$ and $x \in S$ imply $\forall r \leq x, r \in S$. Then, S has upper bound, S has no largest element, and we define $S > 0$ if there exists $r \in S$ such that $r > 0$.

Example 2.9.12. Let $R = \{x = a_0 a_1 ... : a_i \in 0, 1, ..., d - 1\}$ to be the base d expression. We see it has LUBP, and all the things we need.

Then we mod out by $\sim$ to get $\mathbb{R} = R/\sim$, to make sure $0.4999... = 0.50000...$

Remark 2.9.13. We are finally done with this!!

2.10 Polynomial Approximation

Remark 2.10.1. In this section, we try to approximate continuous function f on $[a, b]$. We want to find $g \in \mathbb{R}[x]$ such that $\|f - p\|_\infty < \epsilon$.

Example 2.10.2 (Interpolation)**.** The first method we are going to consider is interpolation.

We divide it to n pieces and let $x_i = a + i\frac{b-a}{n}$, then we are looking for a polynomial p such that $p(x_i) = f(x_i)$ for $0 \leq i \leq n$. Let

$$q_i(x) = \frac{\prod_{0\leq j\leq n, j\neq i} x - x_j}{\prod_{j\neq i} x_i - x_j}$$

Thus q_i has degree n and we have $q_i(x_j) = \delta_{ij}$. Then, let $p(x) = \sum_{i=0}^{n} f(x_i)q_i(x)$, then $degree(p) \leq n$ and so $p(x_i) = f(x_i)$ for all $0 \leq i \leq n$.

If $r(x)$ is another polynomial with degree $r \leq n$ and $r(x_i) = f(x_i)$ then $(r-p)(x_i) = 0$ for all $0 \leq i \leq n$ and so $degree(r-p) \leq n$ with $n+1$ roots. Thus $r - p = 0$ and hence they are equal.

However, Runge in 1901 shared that these interpolation do not always converge uniformly to f on $[a, b]$.

Example 2.10.3. But what about Taylor polynomial? We do not want to assume the existence of derivative!

Theorem 2.10.4 (Weierstrass Approximation)**.** *$\mathbb{R}[x]$ is dense in $C_{\mathbb{R}}[a, b]$.*

Remark 2.10.5. We remark that it holds for complex case as well. Note $f = Re(f) + i \cdot Im(f)$ and so we can find two polynomials to approximate $Re(f)$ and $Im(f)$, say p and q, then $p + iq$ approximates f.

In addition, to conduct the proof, it suffice to prove the theorem for $[0, 1]$ as we consider the change of variable $x \mapsto \frac{x-a}{b-a}$.

Let $f \in C[a, b]$, let $g(t) = f(a+(b-a)t)$. Find $g'(t) \in P_n[x]$ so that $\|g - g'\|_{[0,1]} < \epsilon$ where $g'(t) = \sum a_i t^i$. Then, $f'(x) = g'(\frac{x-a}{b-a}) = \sum a_i(\frac{x-a}{b-a})^i$. So f' is a polynomial with degree n. In particular, we have

$$\|f - f'\|_{[a,b]} = \sup_{a\leq x\leq b} |f(x) - f'(x)| = \sup_{0\leq t\leq 1} |g(t) - g'(t)| < \epsilon$$

Bernstein's Proof. We have

$$1 = (x + (1-x))^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$

We let $P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. Then, we have

$$\begin{aligned} P'_{n,k}(x) &= \binom{n}{k} \left(kx^{k-1}(1-x)^{n-k} + (n-k)x^k(1-x)^{n-k-1}\right) \\ &= \binom{n}{k} x^{k-1}(1-x)^{n-k-1}(k(1-x) - (n-k)x) \end{aligned}$$

Thus, we have $P'_{n,k}(x) = 0 \Leftrightarrow k - nx = 0 \Leftrightarrow x = \frac{k}{n}$.

Note
$$B_nf(x) = \sum_{k=0}^{n} f(\frac{k}{n})P_{n,k}(x) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}$$
We have B_nf is a polynomial with $degree(B_nf) \le n$.

Then, we claim $B_n : C[0,1] \to \mathbb{R}[x]$ has the following properties:

1. $B_n(sf + tg) = sB_nf + tB_ng$ where $f, g \in C[0,1]$ and $s, t \in \mathbb{R}$
2. If $f \ge 0$ then $B_nf \ge 0$
3. $f \le g \Rightarrow B_nf \le B_ng$ and $|f| \le g \Rightarrow |B_nf| \le B_ng$

To see linearity, note
$$\begin{aligned} B_n(sf + tg) &= \sum_{k=0}^{n}(sf(k/n) + tg(k/n))P_{n,k}(x) \\ &= sB_nf + tB_ng \end{aligned}$$

To see positivity, we have $P_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k} \ge 0$. So if $f \ge 0$ then $f(k/n) \ge 0$ so $B_nf = \sum f(k/n)P_{n,k}(x)$ where $P_{n,k}(x)$ is positive and $f(k/n)$ is positive and so $B_nf \ge 0$.

Next, to see $f \le g \Rightarrow B_nf \le B_ng$, we note $f \le g \Rightarrow g - f \ge 0 \Rightarrow 0 \le B_n(g - f) = B_ng - B_nf$ and so $B_nf \le B_ng$. In additino, note $|f| \le g$ then $-g \le f \le g$ and so $-B_ng \le B_nf \le B_ng$ and so $|B_nf| \le B_ng$.

We claim the following

1. $B_n(1) = 1$
2. $B_n(x) = x$
3. $B_n(x^2) = \frac{n-1}{n}x^2 + \frac{x}{n}$. Hence, we would have $\|B_n(x^2) - x^2\|_\infty = \left\|\frac{x-x^2}{n}\right\|_\infty = \frac{1}{4n}$ and so it goes to zero.

To prove 1, note $(B_n1)(x) = \sum_{k=0}^{n} 1 \cdot \binom{n}{k}x^k(1-x)^{n-k} = (x + (1-x))^n = 1$.

To prove 2, we have
$$\begin{aligned} (B_nx)(x) &= \sum_{k=0}^{n} \frac{k}{n}\frac{n!}{k!(n-k)!}x^k(1-x)^{n-k} \\ &= x\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{(n-1)-(k-1)} \\ &= x\sum_{k=1}^{n} \binom{n-1}{k-1}x^{k-1}(1-x)^{(n-1)-(k-1)} \\ &= x\sum_{j=0}^{n-1} \binom{n-1}{j}x^j(1-x)^{n-1-j} \\ &= xB_n(1) = x \end{aligned}$$

To prove 3, we have

$$
\begin{aligned}
(B_n x^2)(x) &= \sum_{k=0}^{n} \frac{k^2}{n^2} \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \\
&= \sum_{k=1}^{n} \frac{k}{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\
&= \frac{1}{n} \sum_{k=1}^{n} (k-1+1) \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\
&= \frac{1}{n} \sum_{k=2}^{n} (k-1) \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\
&\quad + \frac{1}{n} \sum_{k=1}^{n} 1 \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\
&= \frac{1}{n} \sum_{k=2}^{n} \frac{(n-1)!}{(k-2)!(n-k)!} x^k (1-x)^{n-k} + \frac{x}{n} \\
&= \frac{x^2}{n} \sum_{k=2}^{n} (n-1) \frac{(n-2)!}{(k-2)!(n-k)!} x^{k-2} x^{n-k} + \frac{x}{n} \\
&= \frac{n-1}{n} x^2 \sum_{j=0}^{n-2} \frac{(n-2)!}{j!(n-2-j)!} x^j x^{n-2-j} + \frac{x}{n} \\
&= \frac{n-1}{n} x^2 + \frac{x}{n}
\end{aligned}
$$

Therefore, we have $B_n x^2 - x^2 = \frac{x-x^2}{n}$ and so

$$
\left\| B_n x^2 - x^2 \right\|_\infty = \frac{1}{n} \sup_{0 \le x \le 1} |x - x^2| = \frac{1}{4n}
$$

Thus, it indeed goes to 0 as claimed.

We remark $B_n(x^2)$ goes to x^2 uniformly as $n \to \infty$ and $B_n(x^2) = x^2 + \frac{x-x^2}{n}$.

Given $\epsilon > 0$, by uniform continuity, there exists $\delta > 0$ such that

$$
|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon
$$

For $a \in [0,1]$, we have $|f(x) - f(a)| \le \epsilon$ if $|x - a| < \delta$. On the other hand, if $|x - a| \ge \delta$, we have

$$
|f(x)| - f(a) \le 2 \|f\|_\infty \le \frac{2 \|f\|_\infty}{\delta^2} (x-a)^2
$$

Therefore, we have

$$
f(x) - f(a) \le \epsilon \cdot 1 + \frac{2 \|f\|_\infty}{\delta^2} (x-a)^2
$$

Hence, we have

$$\begin{aligned}|(B_nf)(x)| &= |(B_nf)(x) - f(a)\cdot(B_n1)(x)| \\ &\leq \epsilon B_n1 + \frac{2\|f\|_\infty}{\delta^2}B_n((x-a)^2) \\ &= \epsilon B_n1 + \frac{2\|f\|_\infty}{\delta^2}B_n(x^2-2ax+a^2\cdot 1) \\ &= \epsilon + \frac{2\|f\|_\infty}{\delta^2}((x-a)^2+\frac{x-x^2}{n})\end{aligned}$$

Plug in $x = a$, we have

$$|B_nf(a) - f(a)| \leq \epsilon + \frac{2\|f\|_\infty}{\delta^2}(0+\frac{a-a^2}{n}) \leq \epsilon + \frac{\|f\|_\infty}{2\delta^2 n}$$

Choose $n \geq ceiling(\frac{\|f\|_\infty}{2\delta^2\epsilon})$, then we have

$$|B_nf(a) - f(a)| \leq 2\epsilon$$

This holds for all $a \in [0,1]$ and so

$$\|B_nf - f\|_\infty \leq 2\epsilon$$

♡

Proposition 2.10.6. *If $f \in C[a,b]$, $n \in \mathbb{N}$, then there exists a polynomial $p \in P_n[a,b]$ such that*

$$\|f-p\|_\infty = dist(f, P_n[a,b]) := \inf_{h\in P_n[a,b]}\|f-h\|_\infty$$

Proof. Note P_n is a subspace with $dim(P_n) = n+1 < \infty$. Then, P_n is complete and hence closed. The closest polynomial are at least as close as 0 (since 0 is also a polynomial). So they lies in $P_n[a,b] \cap \overline{b_{\|f\|_\infty}(f)} := K_n$ and K_n is compact. Define

$$\rho : K_n \to [0,\infty], \rho(p) = \|f-p\|_\infty$$

This is continuous and so by extreme value theorem, ρ attains minimum and the proof follows. ♡

Example 2.10.7.

1. Let $S = \{f \in C[a,b] : f(0) = 0\}$, then $dist(1,S) = \inf_{f\in S}\|1-f\|_\infty = \inf\|1-0\| = 1$ and so we have lots of closest points.
2. I did not get the example two :(
3. And this one as well... :(

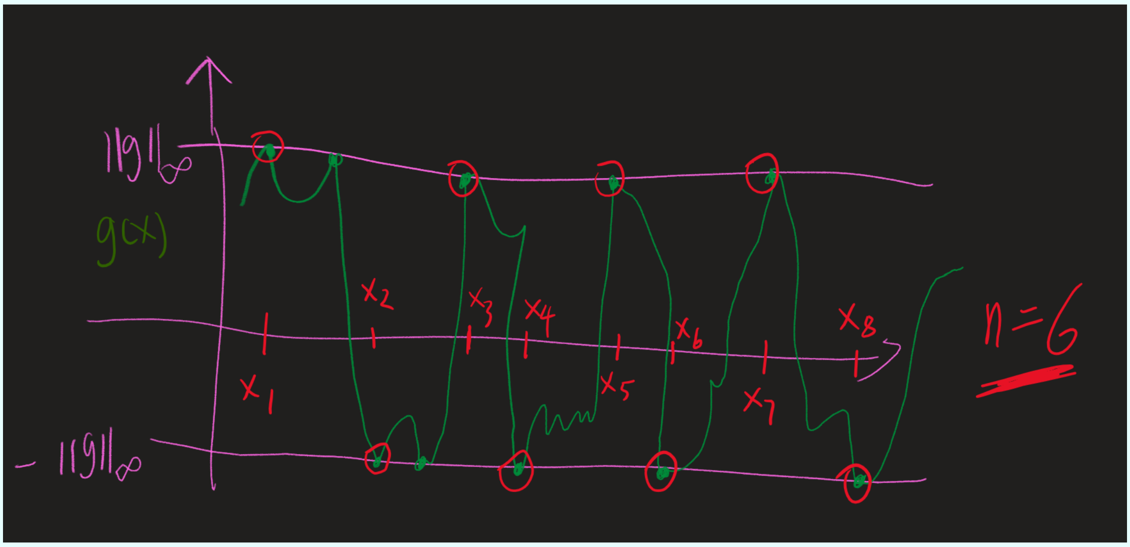
Definition 2.10.8. Let $g \in C_{\mathbb{R}}[a,b]$. We say g satisfies ***equioscillation of degree*** n if

$$\exists a \leq x_1 < x_2 < ... < x_{n+2} \leq b$$

such that

$$g(x_i) = (-1)^i \|g\|_\infty \text{ or } g(x_i) = (-1)^{i+1} \|g\|_\infty$$

Example 2.10.9. Consider the following example when $n = 6$,



Lemma 2.10.10. *If $f \in C_{\mathbb{R}}[a,b]$ and $p \in P_n[a,b]$ and $f - p$ satisfies equioscillation of degree n, then $\|f - p\|_\infty = dist(f, P_n[a,b])$*

Proof. If there is a polynomial $q \in P_n$ such that

$$\|f - p\|_\infty - \delta = \|f - p - q\|_\infty < \|f - p\|_\infty$$

where $\delta > 0$

Let $a \leq x_1 < ... < x_{n+2} \leq b$ exhibit the equioscillation. Let $g = f - q$ and we have

$$\|g\|_\infty - \delta \geq |g(x_i) - q(x_i)| = |\pm \|g\|_\infty - q(x_i)|$$

Thus $q(x_i)$ has the same sign as $g(x_i)$. Indeed, observe that if they have different sign, say $g(x_i) > 0$ and $q(x_i) < 0$ (the other case is similar), then $|g(x_i) - q(x_i)| = \|g\|_\infty - q(x_i) > \|g\|_\infty$, but we have $|g(x_i) - q(x_i)| \leq \|g\|_\infty - \delta$, which is a contradiction.

Thus $q(x)$ changes sign on $[x_i, x_{i+1}]$ for $1 \leq i \leq n+1$. By intermediate value theorem, there exists $y_i \in [x_i, x_{i+1}]$ such that $q(y_i) = 0$. Thus q has at least $n+1$ roots and $deg(q) \leq n$. Thus $q = 0$, which is a contradiction. ♡

Lemma 2.10.11. *Let $f \in C_{\mathbb{R}}[a,b]$, $p \in P_n$ such that $\|f - p\|_\infty = dist(f, P_n[a,b])$. Then r satisfies equioscillation of degree n.*

Proof. WLOG, suppose $r := f - p \neq 0$. By uniform continuity of r, there exists $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\|r\|_\infty}{2}$. Divide $[a, b]$ into intervals of length less than δ. Label the intervals I_i at which r attains values $\pm \|r\|_\infty$ to be $\overline{I_i}$.

If $r(x_i) = \|r\|_\infty$ on I_i, then $|I_i| < \delta$ imply we have $y \in \overline{I_i}$ such that $|r(x) - r(y)| < \frac{\|r\|_\infty}{2}$ and so $r(y) \geq \frac{\|r\|_\infty}{2}$. Similarly, if $r(x_i) = -\|r\|_\infty$ on I_i, then for all $y \in \overline{I_i}$, we have $r(y) \leq -\frac{\|r\|_\infty}{2}$.

Pick $x_i \in I_i$ such that $r(x_i) = \pm \|r\|_\infty$. Define $\epsilon_i = sign(r(x_i)) \in \{-1, 1\}$. Group I_i into adjacent groups with a common sign, call it J_j. We have $J_1, ..., J_k$.

If $k \geq n + 2$, pick x_i in each J_j and obtain the equioscillation as desired.

So suppose $k \leq n + 2$, pick points $a_1, ..., a_{k-1}$ where $J_j < a_j < J_{j+1}$, let $q(x) = \prod_{j=1}^{k-1}(x - a_j)$, we have $deg(q) = k - 1 \leq n$ and $q \in P_n$. If necessary, replace q by $-q$ so that $sign(q(x)) = \epsilon_i$ on I_i. So $q \neq 0$ on J_j. Let $m = \min_{x \in \cup J_j} |q(x)| > 0$ by EVT and we note $\cup J_j \supseteq \cup I_i$.

Let $L = \cup \overline{I_i}$ and $M = \overline{[a, b] \setminus L}$, M is the union of the interval on which r does not attain $\pm \|r\|_\infty$ and we remark M is closed.

So $\sup_{x \in M} |r(x)| = \|r\|_\infty - d \leq \|r\|_\infty$. Thus, let $s = \frac{d}{2} \frac{q}{\|q\|_\infty} \in P_n$. Then, we have

$$\begin{aligned}
\|f - (p + s)\|_\infty = \|r - s\|_\infty &= \max\{\sup_L |r - s|, \sup_M |r - s|\} \\
&\leq \max\{\|r\|_\infty - \frac{dm}{2\|q\|_\infty}, \|r\|_\infty - d + \|s\|_\infty\} \\
&= \max\{\|r\|_\infty - \frac{dm}{2\|q\|_\infty}, \|r\|_\infty - \frac{d}{2}\} \\
&< \|r\|_\infty
\end{aligned}$$

This contradicts $\|f - p\| = dist(f, P_n)$ and so r does satisfies equioscillation. ♡

Theorem 2.10.12 (Chebyshev)**.** *If $f \in C_{\mathbb{R}}[a, b]$ and $n \in \mathbb{N}$, there is a unique polynomial $p \in P_n[a, b]$ such that*

$$\|f - p\|_p = dist(f, P_n)$$

In addition, it is characterized by the fact that $f - p$ satisfies equioscillation of degree n.

Proof. The two lemmas show that $p \in P_n$ is a closest polynomial of degree less than or equal to n if and only if $f - p$ satisfies equioscillation of degree n. By a proposition, there exists at least one such closest polynomial.

Suppose p, q are two closest polynomial, then we have

$$\begin{aligned}\left\|f-\frac{p+q}{2}\right\|_\infty &= dist(f,P_n) \leq \left\|\frac{1}{2}(f-p)\right\|_\infty + \left\|\frac{1}{2}(f-q)\right\|_\infty \\ &\leq \frac{1}{2}dist(f,P_n) + \frac{1}{2}dist(f,P_n) \\ &= dist(f,P_n)\end{aligned}$$

Thus $f - \frac{p+q}{2}$ satisfies equioscillation, so there exists $a \leq x_1 < x_2 ... < x_{n+2} \leq b$ such that $(f - \frac{p+q}{2})(x_i) = \pm D$ which alternates sign.

Thus, suppose $f - \frac{p+q}{2}(x_i) = D$, then

$$\begin{aligned}(f-\frac{p+q}{2})(x_i) = D &= \frac{1}{2}(f-p(x_i)) + \frac{1}{2}((f-q)(x_i)) \\ &\leq \frac{1}{2}D + \frac{1}{2}D = D\end{aligned}$$

Similarly, if $f - \frac{p+q}{2}(x_i) = -D$ then

$$\frac{1}{2}(f-p(x_i)) + \frac{1}{2}((f-q)(x_i)) \geq -\frac{1}{2}D - \frac{1}{2}D = -D$$

So $f(x_i) - p(x_i) = \pm D = f(x_i) - q(x_i)$ for $1 \leq i \leq n+2$. So $(p-q)(x_i) = 0$ for $1 \leq i \leq n+2$ with $deg(p-q) \leq n$. Hence $p - q = 0$ and so $p = q$. Thus p is unique. ♡

Example 2.10.13. Let $f(x) = \cos x \in C_{\mathbb{R}}[-\pi/2, \pi/2]$. Try to find the closed cubic.

Solution. If p is closed, let $\tilde{p}(x) = p(-x)$. Then, we have

$$|(f-\tilde{p})(x)| = |f(x) - p(-x)| = |f(-x) - p(-x)| \leq \|f-p\|_\infty$$

Thus $\|f-\tilde{p}\|_\infty = \|f-p\|_\infty$ and so $p = \tilde{p}$ and so p is even.

Let $p(x) = ax^2 + b$. Let $r(x) = f - p$, then $r'(x) = -\sin(x) - 2ax$ and $r''(x) = -\cos x - 2a$. Then r'' has at most 2 zeros, which imply $-\frac{1}{2} < a < 0$.

Thus r' has at most 3 zeros, so r satisfies equioscillation of degree 3. So 5 times, take $\pm \|r\|_\infty$, we have $\pm\frac{\pi}{2}, 0, \pm x_0$. Then $r''(0) = -1 - 2a < 0$ so 0 is a max, so as $\pm\frac{\pi}{2}$.

Then, let $d = \|f-p\|_\infty$, we have

$$\begin{aligned}d &= \cos(0) - p(0) = 1 - b \\ &= \cos(\pi/2) - p(\pi/2) = -a\pi^2/4 - b\end{aligned}$$

Also, we have $-d = \cos(x_0) - ax_0^2 - b$ and $0 = r'(x_0) = -\sin x_0 - 2ax_0$ and thus $a = -\frac{4}{\pi^2}$, and so $\sin(x_0) = \frac{8x_0}{\pi^2}$ and we solve this and we are done. ♠

2.11 Stone-Weierstrass

Definition 2.11.1. Let F be a field, then a ***F-algebra*** is a ring R with a bilinear scalar multiplication, i.e. $\alpha \in F, x, y \in R$ then $(\alpha \cdot x)y = x(\alpha \cdot y) = \alpha(xy)$.

Definition 2.11.2. A ***subalgebra*** of $C(X)$ or $C_R(X)$ is a subspace which is closed under multiplication.

A ***vector (sub)lattice*** of $C_R(X)$ is a subspace A such that for all $f, g \in A$, we also have $f \vee g = \max(f, g) \in A$ and $f \wedge g = \min(f, g) \in A$.

Example 2.11.3. We have

$$A = \{f(x) = a_0 + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)\} \subseteq C[-\pi, \pi]$$

is a subalgebra.

If $f \in A$, then $f(-\pi) = f(\pi)$ so it is not dense in $C_R[-\pi, \pi]$, i.e. does not approximate those g such that $g(\pi) = g(-\pi)$. For example, $g(x) = x$.

Definition 2.11.4. A subset $A \subseteq C(X)$ are called ***separates points*** if

$$\forall x \neq y \in X, \exists f \in A, f(x) \neq f(y)$$

Example 2.11.5. Let $A = \{p(x) = \sum_{k=1}^{n} a_k x^k : a_k \in R\} \subseteq C[0, 1]$ then we cannot approximate $f(x) = 1$.

Definition 2.11.6. A subset $A \subseteq C(X)$ ***vanishes*** at $x_0 \in X$ if $\forall f \in A, f(x_0) = 0$.

Lemma 2.11.7. *If A is a subalgebra of $C_{\mathbb{R}}(X)$, then $\overline{A}$ is a subalgebra and vector lattice.*

Proof. We note $\overline{A}$ is a vector space by assignment question.

If $f, g \in \overline{A}$ then there exists $f_n, g_n \in A$ such that $f_n \to f$ and $g_n \to g$ uniformly. Then, $f \wedge g_n \to fg$ uniformly, and so $fg \in A$.

$$f \vee g = \frac{f+g}{2} + \left|\frac{f-g}{2}\right|, f \wedge g = \frac{f+g}{2} - \left|\frac{f-g}{2}\right|$$

So, we need to show $f \in A$ imply $|f| \in \overline{A}$. Choose $p_n(x) \in P_n$ such that $p_n \to |x|$ uniformly on $[-\|f\|_\infty, \|f\|_\infty]$. Let $q_n(x) = p_n(x) - p_n(0) = \sum_{k=1}^{n} a_k x^k$. Then, we have

$$\begin{aligned} \||x| - q_n(x)\|_\infty &\leq \||x| - p_n\|_\infty + \|p_n - q_n\|_\infty \\ &\leq 2\||x| - p_n\|_\infty \to 0 \end{aligned}$$

Then, $q_n(f) = \sum_{k=1}^{n} a_k(f)^k \in A$ and note

$$\begin{aligned} \||f| - q_n(f)\|_\infty &= \sup_{x\in X} ||f(x)| - q_n(f(x))| \\ &\leq \sup_{t\in[-\|f\|_\infty, \|f\|_\infty]} |(|f| - q_n(f))(t)| \\ &= \||x| - q_n\|_\infty \to 0 \end{aligned}$$

Therefore, we have $|f| \in \overline{A}$ and so $f \wedge g$ and $f \vee g$ are both in $\overline{A}$. ♡

Lemma 2.11.8. *Let (X, d) be compact and $A \subseteq C_{\mathbb{R}}(X)$ be a subalgebra which separates points and does not vanish at any point. If $x \neq y \in X$, $\alpha, \beta \in \mathbb{R}$, then $\exists h \in A$ such that $h(x) = \alpha$ and $h(y) = \beta$.*

Proof. Since A separates points so there exists $f \in A$ such that $a = f(x) \neq b = f(y)$. WLOG, suppose $b \neq 0$.

Case One: If $a \neq 0$, then we look for $h = uf + vf^2$. Let $\alpha = h(x) = ua + va^2$ and $\beta = h(y) = ub + vb^2$. Then since a, b not equal 0, the vandermonde determinant for $\begin{bmatrix} a & a^2 \\ b & b^2 \end{bmatrix}$ is not zero. Thus there exists a solution (u, v) such that

$$\begin{bmatrix} a & a^2 \\ b & b^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Then we are done.

Case Two: If $a = 0$. Then A does not vanish at x, so choose $g \in A$ such that $g(x) \neq 0$. Let $h = uf + vg$, so $h(x) = vg(x)$ and $h(y) = ub + vg(y)$. Solve for this we get

$$v = \frac{\alpha}{g(x)}, u = \frac{\beta - \frac{\alpha}{g(x)} g(y)}{b}$$

The proof follows. ♡

Theorem 2.11.9 (Stone-Weierstrass)**.** *Let (X, d) be compact and $A \subseteq C_{\mathbb{R}}(X)$ be a subalgebra which separates points and does not vanish at any point. Then A is dense in $C_{\mathbb{R}}(X)$.*

Proof. Fix $f \in C_{\mathbb{R}}(X)$ and let $\epsilon > 0$. Then $\overline{A}$ is a vector lattice. Fix $a \in X$, let $a \neq x \in X$. Choose $h_x \in A$ such that $h_x(a) = f(a)$ and $h_x(x) = f(x)$.

Let $U_x = \{y \in X : h_x(y) > f(y) - \epsilon\}$. We note U_x is open as $U_x = (h_x - f)^{-1}(-\epsilon, \infty)$ is the preimage of a continuous function.

So $\{U_x : x \neq a\}$ is an open cover of X. Choose a finite subcover $U_{x_1}, ..., U_{x_n}$ of X, let $g_0 = \max(h_{x_1}, ..., h_{x_n}) \in \overline{A}$. Now, $g_0(a) = f(a)$ and if $x \neq a$, then $\exists U_{x_i}$ such that $h_{x_i}(y) > f(y) - \epsilon$ and hence

$$g_0(y) > f(y) - \epsilon$$

Let $V_a = \{x \in X : g_0(x) < f(x) + \epsilon\}$, note $a \in V_a$, so $\{V_a : a \in X\}$ is an open cover. Choose $X \subseteq V_{a_1} \cup ... \cup V_{a_m}$ and let $g = \min\{g_{a_1}, ..., g_{a_m}\} = ((g_{a_1} \wedge g_{a_2}) \wedge g_{a_3}) ... \wedge g_{a_m} \in \overline{A}$. Let $x \in X$, then $g_{a_i}(x) > f(x) - \epsilon$ for all $1 \leq i \leq m$ by the way we choose g_0.

Thus $g(x) > f(x) - \epsilon$. However, $\exists j$ such that $x \in V_{a_j}$, $g_{a_j}(x) < f(x) + \epsilon$ imply $g(x) < f(x) + \epsilon$ and so $f - \epsilon < g < f + \epsilon$ and so $\|f - g\|_\infty \leq \epsilon$.

Therefore, $\overline{A} = C_{\mathbb{R}}(X)$. ♡

Corollary 2.11.9.1. *Let $X \subseteq \mathbb{R}^n$ be compact. Then the algebra A of polynomials in $x_1, ..., x_n$ is dense in $C_{\mathbb{R}}(X)$.*

Proof. Note $1 \in A$ so does not vanish. Next, note the projection maps $\chi_1, ..., \chi_n$ separates points. Thus $\overline{A} = C_{\mathbb{R}}(X)$. ♡

Corollary 2.11.9.2. *Let X, Y be compact metric space. Then*

$$A = \{\sum_{i=1}^{n} f_i(x) g_i(y) : f_i \in C_{\mathbb{R}}(X), g_i \in C_{\mathbb{R}}(Y), n \in \mathbb{N}\}$$

is dense on $C_{\mathbb{R}}(X \times Y)$

Proof. Put metric on $X \times Y$ to be $\rho((a, b), (c, d)) = d_x(a, c) + d_y(b, d)$. Then the identity mapping is in A, so A does not vanish points.

Next, if $(x_1, y_1) \neq (x_2, y_2)$, if $x_1 \neq x_2$, then let $f(x) = d_x(x_1, x)$ and $g(y) = 1$. We have $f(x_1)g(y_1) = 0$ and $f(x_2)g(y_2) = d(x_1, x_2) \neq 0$. If $x_1 = x_2$ then use similar argument, we get A separates points.

By Stone Weierstrass we are done. ♡

Remark 2.11.10. Note complex polynomials are dense in $C(X)$ as $f \in C(X)$, we can write $f = Re(f) + Im(f)$. Find $p, q \in A$ such that $\|Re(f) - p\|_\infty < \epsilon/2$ and $\|Im(f) - q\|_\infty < \epsilon/2$ then $\|f - (p + qi)\|_\infty < \epsilon$.

Remark 2.11.11. Missing notes for one day after the Stone Weierstrass. Will try to make that up as soon as possible.

2.12 ODE

Example 2.12.1. Consider $y' = 1 + x - y$ with $|x| < \frac{1}{2}$ and $y(0) = 1$. Then,

$$\begin{aligned} y(x) = y(0) &= \int_0^x y'(t)dt \\ &= 1 + \int_0^x 1 + t - y(t)dt \\ &= 1 + x + \frac{x^2}{2} - \int_0^x y(t)dt \end{aligned}$$

Define $T : C[-1/2, 1/2] \to C[-1/2, 1/2]$ to be

$$Tf(x) = 1 + x + \frac{x^2}{2} - \int_0^x f(t)dt$$

If $y = f(x)$ solves the ODE, then $Tf = f$ is a fixed point. Conversely, if $Tf = f$, then $f(0) = 1$ and $f'(x) = 1 + x - f(x)$ by fundamental theorem of calculus.

Note

$$\begin{aligned}
|Tf(x) - Tg(x)| &= \left| \int_0^x g(t) - f(t)dt \right| \\
&\leq \int_0^{|x|} \|f - g\|_\infty \, dt \\
&= |x| \, \|f - g\|_\infty \\
&\leq \frac{1}{2} \|f - g\|_\infty
\end{aligned}$$

Therefore, $\|Tf - Tg\|_\infty \leq \frac{1}{2} \|f - g\|$ and so T is a contraction.

So $Tf = f$ has a unique solution $h(x)$. Let $f_0 = 1$, then $T^n f_0 = f_n \to h$ and we suppose

$$f_{n-1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!} + x$$

Proceed inducetively, we have

$$\begin{aligned}
f_n(x) = 1 + x + x^2/2 - \int_0^x 1 + \sum_{k=2} \frac{(-1)^k t^k}{k!} dt \\
= \sum_{k=0}^{n+1} \frac{(-1)^k x^k}{k!} + x
\end{aligned}$$

Therefore, $f_n(x) \to e^{-x} + x$ and so the solution is $e^{-x} + x$. In fact, power series converges on $(-\infty, \infty)$ and satisfied the ODE.

Remark 2.12.2 (General Setup)**.** Consider nth ODE with

1. relationship between $x, f(x), f'(x), ..., f^{(n)}(x)$
2. Initial condition, usually at a point x_0

The ***standard form*** is

$$f^{(n)}(x) = \Phi(x, f(x), f'(x), ..., f^{(n-1)}(x))$$

$$\Phi : [a, b] \times \mathbb{R}^n \to \mathbb{R}, \Phi \text{ will be continuous, maybe better}$$

along with initial datas about $f(a) = \gamma_0, ..., f^{(n-1)}(a) = \gamma_{n-1}$.

We convert this to a first order vector valued ODE. Set

$$F(x) = (f(x), f'(x), ..., f^{(n-1)}(x))$$
$$F : [a, b] \to \mathbb{R}^n$$

Note F is differentiable because each coordinate is. Therefore, suppose

$$F'(x) = (f'(x), ..., f^{(n)}(x)) = \Psi(x, F(x))$$

with $\Psi : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Psi(x, y_0, y_1, ..., y_{n-1}) = (y_1, y_2, ..., y_{n-1}, \Phi(x, y_0, y_1, ..., y_{n-1}))$$

This is a first order ODE.

Suppose $F(x) = (f_0(x), f_1(x), ..., f_{n-1}(x))$ is a solution, then

$$F'(x) = \begin{bmatrix} f_0'(x) \\ f_1'(x) \\ \vdots \\ f_{n-1}'(x) \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_{n-1}(x) \\ \Phi(x, f_0(x), ..., f_{n-1}(x)) \end{bmatrix}$$

with $f_1 = f_0'$, $f_2 = f_1' = f_0^{(2)}$,..., and

$$f_{n-1}'(x) = f^{(n)}(x) = \Phi(x, ..., f_0^{(n-1)}(x))$$

Then,

$$F(a) = \begin{bmatrix} f_0(a) \\ \vdots \\ f_{n-1}(a) \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_{n-1} \end{bmatrix} := \Gamma$$

Next, we convert it to an integral equation. If F is a solution, then

$$F(x) = F(a) + \int_a^x F'(t)dt = \Gamma + \int_a^x \Phi(t, F(t))dt$$

Define $T : C([a, b], \mathbb{R}^n) \to C([a, b], \mathbb{R}^n)$ to be that $F(x) = (f_0(x), ..., f_{n-1}(x))$ then $TF = \Gamma + \int_c^x \Phi(t, F(t))dt$.

If F solves the DE, then $T(F) = F$ is a fixed poitn of T. Conversely, if $TF = F$ then $T(F(a)) = \Gamma$ and so $F(x) = \Gamma + \int_0^x \Phi(t, F(t))dt$. By FTC, we would have $F'(x) = \Phi(x, F(x))$ satisfies the DE with the initial value.

Definition 2.12.3. We say that $\Phi(x, y_0, ..., y_{n-1})$ where $\Phi : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is ***(globally) Lipschitz in y variable*** if there exists $L < \infty$ such that

$$\|\Phi(x, \vec{y}) - \Phi(x, \vec{z})\|_2 \leq L \|\vec{y} - \vec{z}\|$$

for all $x \in [a, b]$ and $\vec{y}, \vec{z} \in \mathbb{R}^n$.

Example 2.12.4.

1. Suppose Φ is C^1 in y-variable. We have

$$\nabla_y \Phi = (\frac{\partial \Phi}{\partial y_0}, ..., \frac{\partial \Phi}{\partial y_{n-1}})$$

By MVT, there exists $\zeta \in [\vec{y}, \vec{z}]$ such that $\Phi(x, \vec{y}) - \Phi(x, \vec{z}) = (\nabla\Phi(\zeta) \cdot (\vec{y} - \vec{z}))$. Therefore, we have

$$\|\Phi(x,\vec{y}) - \Phi(x,\vec{z})\|_2 \le \|\nabla\Phi\|_\infty \cdot \|\vec{y} - \vec{z}\|_2 \,, \|\nabla\Phi\|_\infty := \sup_{\zeta\in[\vec{y},\vec{z}]} \|\nabla\Phi(\zeta)\|_2$$

If

$$\|\nabla\Phi\|_\infty = \sup_{\substack{\vec{y}\in\mathbb{R}^n \\ x\in[a,b]}} \|\nabla\Phi(x,\vec{y})\|_2 < L < \infty$$

Then Φ is Lipschitz in y.

2. Linear ODE's have the form $\phi(x, y(x), ..., y^{(n-1)}(x))$ with

$$y^{(n)}(x) = a_0(x)y(x) + \sum_{i=1}^{n-1} a_i(x)y^{(i)}(x) + b(x)$$

where all of the above functions are linear in $y, ..., y^{(n-1)}$. We have

$$\begin{aligned}
\Phi(x, y_0, ..., y_{n-1}) &= (y_1, ..., y_{n-1}, \sum_{i=0}^{n-1} a_i(x)y_i + b(x)) \\
&= \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \\ a_0(x) & 0 & 0 & ... & a_{n-1}(x) \end{bmatrix} \cdot \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b(x) \end{bmatrix} \\
&= A(x)\vec{y} + B(x)
\end{aligned}$$

where $A \in M_n(C[a,b])$ and $B \in C([a,b], \mathbb{R}^n)$.
Then,

$$\nabla_y\Phi(x,\vec{y}) = (A(x)\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, A(x)\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, ..., A(x)\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix})$$

which is independent of y.
Thus, we have

$$\|\nabla\Phi\|_\infty = \sup_{\substack{\vec{y}\in\mathbb{R}^n \\ x\in[a,b]}} \|\nabla\Phi(x,y)\|_2 = \sup_{x\in[a,b]} \|\nabla\Phi(x,y)\| < \infty \text{ by EVT}$$

Indeed,

$$\Phi(x,\vec{y}) - \Phi(x,\vec{z}) = \begin{bmatrix} y_1 - z_1 \\ \vdots \\ y_{n-1} - z_{n-1} \\ \sum_{i=1}^n a_i(x)(y_i - z_i) \end{bmatrix}$$

Therefore, we have

$$\begin{aligned}\|\Phi(x,\vec{y})-\Phi(x,\vec{z})\|_2^2 &= \sum_{i=1}^{n-1}|y_i-z_i|^2+|\sum a_i(x)(y_i-z_i)|^2\\ &\le \|y-z\|_2^2+((\sum a_i(x)^2)^{1/2}(\sum|y_i-z_i|^2)^{1/2})^2\\ &\le \|y-z\|_2^2\,(1+\sup_x\sum a_i(x)^2):=L<\infty\end{aligned}$$

Lemma 2.12.5. *Suppose Φ is Lipschitz in y with constant L. Suppose $T(F)=\Gamma+\int_a^x\Phi(t,F(t))dt$. Suppose $F,G\in C([a,b],\mathbb{R}^n)$ satisfies*

$$\|F(x)-G(x)\|_2\le\frac{M|x-c|^k}{k!}$$

then

$$\|T(F)-T(G)\|_2\le\frac{LM|x-a|^{k+1}}{(k+1)!}$$

Proof.

$$\begin{aligned}\|TF-GF\|_2 &= \left\|\Gamma+\int_a^x\Phi(t,F(t))dt-\Gamma-\int_a^x\Phi(t,G(t))dt\right\|_2\\ &\le\left|\int_a^x\|\Phi(t,F(t))-\Phi(t,G(t))\|_2\,dt\right|\\ &\le\left|\int_a^x L\,\|F(t)-G(t)\|_2\,dt\right|,\ \text{by Lipschitz constant}\\ &\le\left|\int_a^x LM\frac{|t-a|^k}{k!}dt\right|\\ &=\frac{LM|x-a|^k}{(k+1)!}\end{aligned}$$

♡

Theorem 2.12.6 (Global Picard Theorem)**.** *Consider DE*

$$y^{(n)}(x)=\phi(x,y(x),...,y^{(n-1)}(x))$$

where $x\in[a,b]$ with initial data $y(c)=\gamma_0,...,y^{(n-1)}(c)=\gamma_{n-1}$. Assume $\Phi(x,\vec{y})=(y_1,...,y_{n-1},\phi(x,y_0,...,y_{n-1}))$ is Lipschitz in y with constant L.

Then the ODE has a unique solution on $[a,b]$. Moreover, if $TF=\Gamma+\int_c^x\Phi(t,F(t))dt$ is the solution, then $F(x)=\lim_{n\to\infty}T^n(\Gamma)$ uniformly.

Proof. Set $F_0(x)=\Gamma$ be constant. Let $F_k(x)=T^kF_0$. Then, we have

$$\begin{aligned}\|F_1(x)-F_0(x)\|_2 &= \left\|\int_c^x\Phi(t,\Gamma)dt\right\|_2\\ &\le|x-c|\cdot\sup_{a\le t\le b}\|\Phi(t,\Gamma)\|_2\\ &=\frac{M|x-c^1|}{1!}\end{aligned}$$

Claim: We have

$$\|F_{k-1}(x) - F_k(x)\|_2 \leq \frac{ML^{k-1}|x-c|^k}{k!}$$

for all $k \geq 1$.

If $k = 1$ we are done. Assume it holds for k. Then,

$$\begin{aligned} \|F_{k+1}(x) - F_k(x)\|_2 &= \|TF_k(x) - TF_{k-1}(x)\|_2 \\ &\leq \frac{L(ML^{k-1})|x-c|^{k+1}}{(k+1)!} \end{aligned}$$

Thus, $\|F_k - F_{k-1}\|_\infty \leq \frac{ML^k(b-a)^{k+1}}{(k+1)!}$. However, note

$$\sum_{k=1}^{\infty} \frac{ML^k(b-a)^{k+1}}{(k+1)!} = \frac{M}{L} \sum \frac{(L(b-a))^{k+1}}{(k+1)!} = \frac{M}{L}(e^{L(b-a)} - 1 - L(b-a)) < \infty$$

If $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{k \geq N} \frac{ML^k(b-a)^{k+1}}{(k+1)!} < \epsilon$ and so $N \leq n < m$ we have

$$\|F_m - F_n\|_\infty = \sum_{k=n}^{m-1} \|F_{k+1} - F_k\|_\infty < \infty$$

Thus F_n is Cauchy and so $F_*(x) = \lim F_n(x)$ exists and we have

$$F_*(x) = \lim F_n(x) = \lim F_{n+1}(x) = \lim TF_n(x) = T(F_*(x))$$

So $F_*'(x) = \Phi(x, F_*(x))$ by FTC and $F_*(c) = \Gamma$. So $F_* = (f_{*0}, ..., f_{*n-1})$ solves $F'(x) = \Phi(x, F(x))$ and $F(c) = \Gamma$. We need to show uniqueness next.

Suppose $TG = G$ with $G(c) = \Gamma$. Then,

$$|F(x) - G(x)| \leq \|F - G\|_\infty = \sup_{a \leq x \leq b} \|F(x) - G(x)\|_2 \leq \|F - G\|_\infty \frac{|x-c|^0}{0!}$$

Therefore,

$$\begin{aligned} \|F(x) - G(x)\|_2 = \|T^n F(x) - T^n G(x)\|_2 & \\ &\leq \|F - G\|_\infty \frac{L^n|x-c|^n}{n!} \text{ by lemma} \\ &\to 0 \end{aligned}$$

Thus $F = G$ and the solution is unique. ♡

Definition 2.12.7. Let $\Phi : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$, then we say Φ is ***Lipschitz in y on*** $[c, d] \times K$ where $K \subseteq \mathbb{R}^n$ is compact, if

$$\exists L < \infty, \|\Phi(x, y) - \Phi(x, z)\|_2 \leq L \|y - z\|_2$$

for $x \in [c, d]$ and $y, z \in K$

Definition 2.12.8. We say Φ above is ***locally Lipschitz*** in y if $\forall(x,y) \in [a,b]\times\mathbb{R}^n$ there exists $\epsilon > 0$ such that Φ is Lipschitz in y on $[x-\epsilon, x+\epsilon] \times b_\epsilon(y)$.

Lemma 2.12.9. *If Φ is C^1 in y variable then Φ is Lipschitz in y on $[a,b] \times \overline{b_R(\Gamma)}$ for any $0 < R < \infty$.*

Proof. Note $\nabla_y(\Phi)$ is continuous so $\sup \|\nabla_y \Phi\|_2 = L > \infty$ by EVT on $[a,b] \times \overline{b_R(\Gamma)}$. Consider $y, z \in \overline{b_R(\Gamma)}$, there exists $\zeta \in [y,z] \subseteq \overline{b_R(\Gamma)}$. So

$$\|\Phi(x,y) - \Phi(x,z)\|_2 = \|\nabla_y(\Phi)(\zeta) \cdot (y-z)_2\| \leq L\,\|y-z\|_2$$

♡

Lemma 2.12.10. *If Φ is locally Lipschitz in y on $[a,b] \times \mathbb{R}^n$, then Φ is Lipschitz in y on $[a,b] \times K$ for any compact $K \subseteq \mathbb{R}^n$.*

Proof. Replace K with $\overline{conv(K)}$. WLOG, K is convex.

For each $(x,y) \in [a,b] \times K$, find $\epsilon > 0$, $L_{x,y}$, such that

$$\|\Phi(x',y') - \Phi(x',z')\|_2 \leq L_{x,y}\,\|y'-z'\|_2$$

for $x' \in [x-\epsilon, x+\epsilon] \cap [a,b]$, $y', z' \in \overline{b_\epsilon(y)}$. Consider $U_{x,y} = (x-\epsilon, x+\epsilon) \times b_\epsilon(y)$ is open. $\{U_{x,y} : (x,y) \in [a,b] \times K\}$ is covers $[a,b] \times K$.

By compactness, there exists finite subcover $U_{x_1,y_1}, \ldots, U_{x_p,y_p}$.

Let $L = \max\{L_{x_i,y_i} : 1 \leq i \leq p\}$. Then $x \in [a,b], y, z \in K$ we have $\{x\} \times [a,b]$ is covered by $\{U_{x_i,y_i}\}_{i=1}^p$.

Pick $y = z_0, z_1, \ldots, z_m = z$ on $[y,z]$ such that $[z_{i-1}, z_i] \subseteq U_{x_j,y_j}$. The proof follows. ♡

Example 2.12.11. Consider $y' = y^2$, $y(0) = 1$, $x \in [0,2]$. Then $\phi(x,y_0) = y_0^2$ and $y'(x) = \phi(x,y)$. Thus $\frac{\partial\phi}{\partial y} = 2y$ and

$$|\phi(x,y) - \phi(x,z)| = |y^2 - z^2| = |y-z| \cdot |y+z|$$

No global Lipschitz but there is a local one.

On $[0,2] \times [-R,R]$, we have Lipschitz constant $2R$.

Start with $y(0) = 1$. Consider

$$\frac{y'}{y^2} = 1 \Rightarrow \int_0^x \frac{y'(t)}{y^2(t)}dt = \int_0^x 1dt \Rightarrow y(x) = \frac{1}{1-x}$$

However, this blows up on $[0,2]$ as we have an asymptote at $x = 1$.

Start with $y(2) = c$. Then, we have

$$\int_2^x \frac{y'}{y}dt = \int_2^x 1dt = x - 2 \Rightarrow y(x) = \frac{c}{1+2(c-x)}$$

This blows up when $x = \frac{1+2c}{2}$. This has nothing to do with the solution on $[0,1)$.

Theorem 2.12.12 (Local Picard Theorem). *Suppose* $F'(x) = \Phi(x, F(x))$, $\Phi : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$, $F(c) = \Gamma$ *where* $c \in [a,b]$. *Suppose* Φ *is locally Lipschitz on* y. *Then there is* $h > 0$ *so that the DE has a unique solution on* $[c-h, c+h] \cap [a,b]$.

Proof. Let $R > 0$. And Φ has a Lipschitz condition on y on $[a,b] \times \overline{b_R(\Gamma)}$. Let $M = \|\Phi\|_{[a,b]\times\overline{b_R(\Gamma)}}$. We will show $h = \frac{R}{M}$ works. Note h does not depends on the Lipschitz condition.

Proof is the same as for global Picard. Take $F_0 = \Gamma$ and $F_{n+1} = TF_n$ where $TF(x) = \Gamma + \int_c^x \Phi(t, F(t))dt$.

Works provided that $F_n(x) \in \overline{b_R(\Gamma)}$. Then

$$\|F_{n+1}(x) - \Gamma\| = \left\| \int_c^x \Phi(t, F(t))dt \right\|$$

♡

Remark 2.12.13 (Uniqueness with Local Picard Theorem). Found solution F such that $F'(x) = \Phi(x, \Phi(x))$ with initial condition $F(c) = \Gamma$, suppose there is another solution G. Let $d = \sup\{t \in [c, c+t] : G|_{[c,c+t]} = F|_{c,c+t}\}$. By continuity, it this d is not zero and say we have $F(d) = G(d) = \Delta$. Both solution gives us

$$F'(x) = \Phi(x, F(x)), F(d) = \Delta$$

$$G'(x) = \Phi(x, G(x)), G(d) = \Delta$$

Find an $r > 0$, $k > 0$, both $F(x)$ and $G(x)$ stay inside $[d, d+k] \times \overline{b_r(\Delta)} \subseteq [c, c+h] \times \overline{b_r(\Gamma)}$.

The argument of the Global Picard Theorem apply here because solution stay in the ball. Then,

$$\|F(x) - G(x)\| = \|T^n F(x) - T^n G(x)\| \leq \|F - G\|_{[d,d+k]} \cdot \frac{(x-d)^n}{n!} \to 0$$

Thus, $G = F$ in $[d, d+k]$, a contradiction. Therefore, we must have $G = F$ on $[c, c+h]$. The same works on $[c-h, c]$.

Theorem 2.12.14 (Continuation Theorem). *Let* $F'(x) = \Phi(x, F(x))$, $F(c) = \Gamma$, *and* $\Phi : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ *is locally Lipschitz in* y. *Suppose* $c \in [a,b]$. *Then either*

1. $F(x)$ *extend to a solution on* $[c,b]$

or,

2. $F(x)$ *extend to a solution on* $[c,d)$, $d \leq b$ *and* $\lim_{x\to d} \|F(x)\| = +\infty$

Proof. Let
$$d = \sup\{t \geq c : \text{DE has a solution on}[c,t]\}$$

By the local Picard Theorem, $d > c$.

If $F_1(x)$ is a solution on $[c, c+t_1]$ and $F_2(x)$ is a solution on $[c, c+t_2]$ with $t_1 \le t_2$.

Claim: $F_2|_{[c,c+t_1]} = F_1$.

Let $d_1 = \sup\{t : F_2|_{[c,c+t]} = F_1|_{[c,c+t]}\}$, with $F_1(d_1) = F_2(d_1) = \Delta$. If $d_1 = t_1$ then we are done. Suppose that's not the case.

Both F_1, F_2 are solution of $F'(x) = \Phi(x, F(x))$ where $F(d_1) = \Delta$. By Local Picard, there exists $h > 0$, $F_1|_{[d,d_1+h]} = F_2|_{[d_1,d_1+h]}$, contradiction of the minimality of d_1. So $d_1 = t_1$.

Let $F^*(x)$ be the unique solutino on $[c, d]$ obtained by piecing together solution on $[c, t]$ for $t < d$.

If $d = b$ and F continue by continuity to b and we are in case 1.

However, otherwise, maybe $\lim_{x \to d} \|F(x)\| = \infty$, i.e. case 2 holds.

Third possibility: F_* failed to continue to d, yet there exists $x_n \to d$, $\|F_*(x_n)\| \le K$. Then, Φ is locally Lipschitz in y and than is Lipschitz in y on $[c, d] \times \overline{b_{K+1}(0)} \supseteq [x_n, b] \times \overline{b_1(F_*(x_n))}$.

Local Picard Theorem provide a solution on $[x_n, x_n+h]$ where $h = \min\{b - x_n, \frac{n}{\|\Phi\|_\infty}\}$. But we have

$$\|\Phi\|_\infty = \sup_{[c,d] \times \overline{b_{K+1}(0)}} \|\Phi(x, y)\| = M < \infty$$

Choose x_n such that $d - x_n < \frac{1}{2M}$ then solution extends to $[x_n, x_n + \frac{1}{M}] \supseteq [x_n, d + \frac{1}{2M}]$ where both of them are subset of $[c, d]$. This contradicts the definition of d, and if $d = b$ and F is bounded on $x_n \to b$, then shows the solution extends to be continuous at b. Either 1 or 2 holds. ♡

Corollary 2.12.14.1. *If $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, and $F'(x) = \Phi(x, F(x))$ and $F(c) = \Gamma$. Then we have a solution*

1. *on all of $\mathbb{R}$*

or 2. *on $(-\infty, d)$, with $\lim_{x \to d} \|F(x)\| = \infty$*

2'. *on (d, ∞), with $\lim_{x \to d} \|F(x)\| = \infty$*

3. *on (d, d_2), with $\lim_{x \to d} \|F(x)\| = \infty$*

Example 2.12.15. Consider $x^4 y'' + 2x^3 y' + y = 0$, with $y(\frac{2}{\pi}) = 1$ and $y'(\frac{2}{\pi}) = 0$.

We have $F(x) = (f_0(x), f_1(x))$ standard form to be

$$y'' = \frac{-2x^3 y' - y}{x^4}$$

Then,

$$F'(x) = \Phi(x, F(x)) = (f_1(x), \frac{-2x^3 f_1(x) - f_0(x)}{x^4})$$

where

$$\Phi(x, y_0, y_1) = (y_1, \frac{-x^3 y_1 - y_0}{x^4})$$

on $[\epsilon, \infty)$, because it is a linear ODE with continuous coefficient, it globally Lipschitz on $[\epsilon, R]$.

So there is a solution on $(0, \infty)$ by the Continuation Theorem.

The solution is $f_0(x) = \sin(\frac{1}{x})$ and $f_1(x) = \frac{-1}{x^2}\cos(\frac{1}{x})$. But note $F(x_n) = (\pm 1, 0)$ where $x_n = \frac{1}{(2n+1)\pi/2}$. Then $x_n \to 0$, and $\|F(x_n)\| = 1$ is bounded. However, F does not extend to 0 and so no local Lipschitz condition on $(0, \frac{2}{\pi})$.

Example 2.12.16 (Existence of Solutions Without Lipschitz Condition)**.** Consider $y' = y^{2/3}$ with $y(0) = 0$. We have $\frac{y'}{y^{2/3}} = 1$. Intergral, we have

$$\int_0^x \frac{y'(t)}{t(t)^{2/3}} dt = \int_0^x 1 dt = x$$

This has a lot of solutions, for example, $y = \frac{x^3}{27}$ and $y = 0$.

2.13 Peano's Theorem

Theorem 2.13.1 (Peano)**.** *Let $\Phi : [a, b] \times \overline{b_R(\Gamma)} \to \mathbb{R}^n$ be continuous. Consider the DE $F'(x) = \Phi(x, F(x))$ with $F(a) = \Gamma$. Then there is at least one solution on $[a, a+h]$ where $h = \min\{b - a, \frac{R}{\|\Phi\|_\infty}\}$.*

Proof. We have $TF(x) = \Gamma + \int_a^x \Phi(t, F(t))dt$, we are looking for a fixed point. Define $F_n(x)$ on $[a, a+h]$ by

$$F_n(x) = \begin{cases} \Gamma, & a \le x \le a + \frac{1}{n} \\ \Gamma + \int_a^{x-\frac{1}{n}} \Phi(t, F_n(t))dt, & a + \frac{1}{n} \le x \le a + h \end{cases}$$

For each x, $F_n(t)$ is already defined on $[a, x - \frac{1}{n}]$, so the integral makes sense. We have

$$\|F_n(x) - \Gamma\| = \begin{cases} 0, & a \le x \le a + \frac{1}{n} \\ \text{otherwise:} \\ \le \int_a^{x-\frac{1}{n}} \|\Phi(t.F_n(t))\| dt \le \|\Phi\|_\infty \cdot h \le \|\Phi\|_\infty \cdot \frac{R}{\|\Phi\|_\infty} = R \end{cases}$$

This is needed so that F_n is defined.

Therefore, we have

$$TF_n(x) - F_n(x) = \begin{cases} \Gamma + \int_a^x \Phi(t, F_n(t))dt - \Gamma, & a \le x \le a + \frac{1}{n} \\ \Gamma + \int_a^x \Phi(t, F_n(t))dt - \Gamma - \int_a^{x-1/n} \Phi(t, F_n(t))dt, & \text{else} \end{cases}$$

Since we are looking for the norm, we have

$$\begin{aligned}
\|TF_n(x) - F_n(x)\| &= \begin{cases} \left\|\Gamma + \int_a^x \Phi(t, F_n(t))dt - \Gamma\right\|, & a \le x \le a + \frac{1}{n} \\ \left\|\Gamma + \int_a^x \Phi(t, F_n(t))dt - \Gamma - \int_a^{x-1/n} \Phi(t, F_n(t))dt\right\|, & \text{else} \end{cases} \\
&\le \begin{cases} \int_a^x \|\Phi(t, F_n(t))\| \, dt \\ \int_{x-1/n}^x \|\Phi(t, F_n(t))\| \, dt \end{cases} \\
&\le \frac{1}{n} \|\Phi\|_\infty
\end{aligned}$$

The set $\{F_n, n \ge 1\}$ is bounded, since $\|F_n\|_\infty = \sup \|F_n(x)\| \le \|\Gamma\| + R$ as $F_n(x) \in \overline{b_R(\Gamma)}$. We then show it is equicontinuous. We will only do the second case, i.e. $a + 1/n \le x \le a + h$.

Let $a \le x_1 < x_2 \le a + h$, then $F_n(x_2) - F_n(x_1) = \int_{x_1-1/n}^{x_2-1/n} \Phi(t, F_n(t))dt$ and so

$$\|F_n(x_2) - F_n(x_1)\| \le \int_{x_1-1/n}^{x_2-1/n} \|\Phi(t, F_n(t))\| \, dt \le \|\Phi\|_\infty \cdot ||x_2 - x_1|$$

Given $\epsilon > 0$, take $\delta = \frac{\epsilon}{\|\Phi\|_\infty}$. We have

$$|x_1 - x_2| < \delta \Rightarrow \|F_n(x_1) - F_n(x_2)\| < \|\Phi\|_\infty \frac{\epsilon}{\|\Phi\|_\infty} = \epsilon$$

So $\overline{\{F_n\}}$ is compact by Ascoli theorem.

Therefore, there exists a sequence $n_1 < n_2 < n_3 < \ldots$ such that $F_{n_i} \to F(x)$ uniformly. We have $\|F - \Gamma\|_\infty \le R$ since $\|F_n - \Gamma\| \le R$. We also have

$$TF - F = TF - TF_{n_i} + TF_{n_i} - F_{n_i} + F_{n_i-F}$$

Note

$$\begin{aligned}
\|TF - F\|_\infty &\le \frac{\|\Phi\|_\infty}{n_i} + \|F_{n_i} - F\|_\infty + \left\|\int_a^x \Phi(t, F(t))dt - \int_a^x \Phi(t, F_n(t))dt\right\|_\infty \\
&\le \frac{\|\Phi\|_\infty}{n_i} + \|F_{n_i} - F\|_\infty + \int_a^x \|\Phi(t, F(t)) - \Phi(t, F_{n_i}(t))\|
\end{aligned}$$

Note Φ is uniformly continuous on $[a, b] \times \overline{b_R(\Gamma)}$. Given $\epsilon > 0$, there exists δ, such that $\|y_1 - y_2\| < \delta \Rightarrow \|\Phi(t, y_1) - \Phi(t, y_2)\| < \epsilon$ for any $t \in [a, b]$. Pick n_i big enough so $\|F_{n_i} - F\|_\infty < \min\{\delta, \epsilon\}$. Therefore,

$$\int_a^x \|\Phi(t, F(t)) - \Phi(t, F_{n_i}(t))\| \, dt \le \int_a^x \epsilon dt \le \epsilon h$$

Hence,

$$\begin{aligned}
\|TF - F\| &\le \frac{\|\Phi\|_\infty}{n_i} + \|F_{n_i} - F\|_\infty + \epsilon h \\
&< \epsilon + \epsilon + \epsilon h, \quad \text{if } n_i \text{ big enough} \\
&= (2 + h)\epsilon
\end{aligned}$$

This holds for arbitrary ϵ, so $TF = F$ and so F is a solution. ♡

Chapter 3

Appendix I, General Topology

3.1 Even More Topology

Definition 3.1.1. A ***topology*** on a set X is a collection ,denoted as τ, of subsets of X having the following properties:

1. $\emptyset$ and X are in τ
2. The (countable or uncountable) union of the elements of τ is in τ.
3. The finite intersection of elements in τ is still in τ.

Definition 3.1.2. A ***topological space*** (X, τ) is a set X with a topology τ on X. In addition, we call elements in τ the ***open sets*** of (X, τ).

Definition 3.1.3. Let τ, τ' be two topologies on X. If $\tau \subseteq \tau'$ then we say τ' is ***finer***than τ, and if the inclusion is strict, we say τ' is ***strictly finer***. In this case we also say τ is ***coarser*** (***strictly coarser***) than τ'.

Example 3.1.4. $P(X)$ is a topology on X, this is called the discrete topology. The collection $\{\emptyset, X\}$ is a topology on X, this is called the trivial topology.

Example 3.1.5. Let X be a set, let τ_f be the collection of all subsets U of X such that $X \backslash U$ is either finite or is all of X. Then τ_f is a topology on X. This is called the finite complement topology.

Moreover, let τ_c be the collection of all subsets U of X such that $X \backslash U$ is either countable or is all of X, then τ_c is a topology on X.

Definition 3.1.6. Let X be a set, then a ***basis*** B of a topology is a collection of subsets (they are called ***basis elements***) of X such that

1. For each $x \in X$, there is at least one basis element β containing x.
2. If x belongs to the intersection of two basis element β_1, β_2, then there is a basis element β_3 containing x such that $\beta_3 \subset \beta_1 \cap \beta_2$.

In X, given a basis B, we define the ***topology generated by*** B as follows: A subset U of X is said to be in τ if for each $x \in U$, there is a basis element $\beta \in B$ such that $x \in B$ and $B \subseteq U$.

Proposition 3.1.7. *Let B be a basis in X and let τ be the topology generated by B, then τ is a topology on X.*

Proof. First, $\emptyset \in \tau$ vacuously. Next, $X \in \tau$ because for each $x \in X$, we have $x \in \beta \subseteq X$ and so $X \in \tau$. Now, let J be an index set, and let $U_j \in \tau$ for all $j \in J$. We will show $U := \bigcup_{j \in J} U_j \in \tau$.

Given $x \in U$, we have $x \in U_\alpha$ for at least one $\alpha \in U$. Since U_α is open, there exists a basis element β such that $x \in \beta \subseteq U_\alpha$ and so $x \in B \subseteq U$. Thus U is in τ by definition.

Next, consider $U_1 \cap U_2$ where $U_1, U_2 \in \tau$. Let $x \in U_1 \cap U_2$, we have $\beta_1, \beta_2 \in B$ such that $\beta_1 \subseteq U_1$ and $\beta_2 \subseteq U_2$. Thus there exists $\beta_3 \in B$ such that $x \in \beta_3 \subset \beta_1 \cap \beta_2 \subseteq U_1 \cap U_2$.

Finally, to show any finite union is still in τ, we use a simple induction and the proof follows. ♡

Definition 3.1.8. If τ is a topology generated by B, we say B is a ***basis*** of τ.

Lemma 3.1.9. *Let X be a set, let B be a basis of the topology τ on X. Then τ is equal the collection of all unions of elements of B.*

Proof. A collection of elements $\{\beta_i\}_{i \in J}$ of B is also a collection of elements of τ. Since τ is a topology, the union of β_i is also in τ.

Conversely, given $U \in \tau$, choose for each $x \in U$ an element β_x in B such that $x \in \beta_x$. Then $U = \bigcup_{x \in U} \beta_x$ and the proof follows. ♡

Lemma 3.1.10. *Let (X, τ) be a topological space. Suppose C is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element $\sigma \in C$ such that $x \in \sigma \subseteq U$, then C is a basis for the topology τ.*

Proof. Given $x \in X$, since X is open, there is $\sigma \in C$ such that $x \in \sigma \subseteq X$.

Next, let $x \in \sigma_1 \cap \sigma_2$, where $\sigma_1, \sigma_2 \in C$. Since both σ_1, σ_2 are open, we have $\sigma_1 \cap \sigma_2$ is also open. Therefore, we must have $\sigma_3 \in C$ such that $x \in \sigma_3 \subseteq \sigma_1 \cap \sigma_2$. Thus C is a basis.

Next, we must show the topology generated by C, denoted by τ', is equal τ. Note if $U \in \tau$ and if $x \in U$, then by hypothesis there exists $\sigma \in C$ such that $x \in \sigma \subseteq U$. Thus $U \in \tau'$. Conversely, if W is in τ', then W equal a union of elements of C by the Lemma before this one. Since each element of C belongs to τ and τ is a topology, we have W also in τ. The proof follows. ♡

Definition 3.1.11. Let X, Y be topological space, the ***product topology*** on $X \times Y$ is the topology generated by the basis B of all sets of the form $U \times V$ where U is an open subset of X and V is an open subset of Y.

Remark 3.1.12. We see B is indeed a basis. $X \times Y$ is in B trivially. Next, note the intersection of any two basis element $U \times V$ and $A \times B$ is another basis element. This is because

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

where $U \cap A$ and $V \cap B$ are both open.

Theorem 3.1.13. *If B is a basis for the topology of X and C is a basis for the topology of Y, then the collection*

$$D = \{\beta \times \sigma : \beta \in B, \sigma \in C\}$$

is a basis for the topology of $X \times Y$.

Proof. We use Lemma 3.1.10. Let W be open in $X \times Y$ and a point $\{x, y\} \in W$. Then, there exists a basis element $U \times V$ such that $\{x, y\} \in U \times Y \subseteq W$. Since B and C are bases for X and Y, respectively, we can choose $\beta \in B$ such that $x \in B \subseteq U$ and $\sigma \in C$ such that $y \in \sigma \subseteq C$. Then $\{x, y\} \in \beta \times \sigma \subseteq W$. Thus the collection D meets the criterion of Lemma 3.1.10 and the proof follows. ♡

Definition 3.1.14. A ***subbasis*** S for a topology on X is a collection of subsets of X whose union equals X.

Definition 3.1.15. The ***topology generated by the subbasis*** S is defined to be the collection τ of all unions of finite intersection of elements S.

Remark 3.1.16. We see that τ generated by the subbasis S is indeed a topology. We will show that the collection B of all finite intersections of elements of S is a basis, then by Lemma 3.1.9 we are done. Given $x \in X$, it belongs to an element of S and hence to an element of B, this is the first condition for a basis.

Let $B_1 = S_1 \cap \ldots \cap S_m$ and $B_2 = S_1' \cap \ldots \cap S_n'$ be two element of B. Their intersection

$$B_1 \cap B_2 = (S_1 \cap \ldots \cap S_m) \cap (S_1' \cap \ldots \cap S_n') \in B$$

Thus B is indeed a basis of τ.

Definition 3.1.17. Let $\pi_1 : X \times Y \to X$ be $(x, y) \mapsto x$ and $\pi_2 : X \times Y \to Y$ be $(x, y) \mapsto y$. The maps π_1 and π_2 are called the ***projection*** of $X \times Y$ onto its first and second factors, respectively.

Remark 3.1.18. If U is an open subset of X, then $\pi_1^{-1}(U)$ is precisely the set $U \times Y$, which is open in $X \times Y$. Similarly, we have $\pi_2^{-1}(V) = X \times V$ is open in $X \times Y$ if V is open in Y.

In particular, the intersection of these two sets is the set $U \times V$, and hence, we have the following theorem.

Theorem 3.1.19. *Consider (X, τ_1), (Y, τ_2) and the product topology $(X \times Y, \tau)$, then*

$$S = \{\pi_1^{-1}(U) : U \in \tau_1\} \cup \{\pi_2^{-1}(V) : V \in \tau_2\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let τ' be the topology generated by S. Because every element of S belongs to τ, so do arbitrary unions of finite intersections of elements of S. Thus $\tau' \subseteq \tau$. On the other hand, every basis element $U \times V$ for the topology τ is a finite intersection of element of S, since

$$U \times V = \pi_1^{-1}(U) \times \pi_2^{-1}(V)$$

Therefore, $U \times V$ belongs to τ', so that $\tau \subseteq \tau'$ as well. ♡

3.2 Even Even More Topology

Definition 3.2.1. Let $x \in X$, a ***neighborhood*** of x is an open set containing x.

Definition 3.2.2. A topological space X is called a ***Hausdorff space*** if for each distinct pair $x_1, x_2 \in X$, there exist neighborhood U_1 and U_2, of x_1 and x_2, respectively, that are disjoint.

Remark 3.2.3. Next, we consider the more general product topology. Say we have Cartesian product of sets

$$X_1 \times X_2 ... \times X_n \quad \text{and} \quad X_1 \times X_2 \times X_3 \times ...$$

Now, we have two way to proceed from here to define a topology. One way is to take the basis as all sets of the form $U_1 \times ... \times U_n$ in the first case and of the form $U_1 \times U_2 \times U_3 \times ...$, where U_i is an open set of X_i for each i. This procedure does indeed define a topology on the Cartesian product, and we call it ***the box topology***.

Another way to proceed is to generalize the subbasis formulation of the definition, recall Theorem 3.1.19. In this case, we take as a subbasis all sets of the form $\pi_i^{-1}(U_i)$, where i is any index and U_i is an open set of X_i. We shall call this topology ***the product topology***.

The natural question to ask is, how do these topologies differ? For finite products, the two topologies are the same. Indeed, let $U_1 \times U_2 \times ... \times U_n$ be a basis element of the box topology, then we have

$$\begin{aligned}\pi_1^{-1}(U_1) \cap ... \cap \pi_1^{-1}(U_n) &= (U_1 \times X_2 \times ... \times X_n) \cap (X_1 \times U_2 \times ...) \cap ... \\ &= U_1 \times ... \times U_n\end{aligned}$$

Conversely, consider the finite intersection of $\pi_i^{-1}(U_i)$, then clearly they are the Cartesian product of $U_1 \times ... \times U_n$.

For infinite products, the two topological spaces are different.

Definition 3.2.4. If X is a topological space, X is said to be ***metrizable*** if there exists a metric d on X that induces the topology of X.

Example 3.2.5. $\mathbb{R}$ and $\mathbb{R}^n$ and $\mathbb{R}^\infty$ are all metrizable.

Chapter 4

Acknowledgements

Remark 4.0.1. If anyone in the Acknowledgements sees their name and wish to be removed from the list, email me.

Thanks for the following list of people to point out my mistakes.

1. For Proposition 1.3.10, J.Lin pointed out the existence of an otiose line in the proof, then I killed it (not the person who pointed out the problem but the problem itself).
2. For Theorem 1.5.18, fixed $\int_{\|v\|_2=1}$ to $\inf_{\|v\|_2=1}$ thanks to H.Shi.
3. For Theorem 2.1.3, thans to H.Shi to provide some notes to help me finish
4. Thanks my friend A and D for their notes for Oct 2nd's update.