Contents

Chapter 1

Outer Measure

1.1 Borel Sets

Definition 1.1.1. Let X be a set, we call $Q \subseteq \mathcal{P}(X)$ a ***σ-algebra*** of susbets of X if:

1. $\emptyset \in Q$.
2. $A \in Q \Rightarrow X \backslash A \in Q$.
3. $A_1, A_2, A_3, ... \in Q \Rightarrow \bigcup_{i=1}^{\infty} A_i \in Q$.

Remark 1.1.2. We note if Q is σ-algebra of X then:

1. $X \in Q$.
2. $A, B \in Q \Rightarrow A \cup B \in Q$.
3. $A_1, A_2, ... \in Q \Rightarrow \bigcap_{i=1}^{\infty} A_i \in Q$.
4. $A, B \in Q \Rightarrow A \cap B \in Q$.

Example 1.1.3. Note:

1. $\{\emptyset, X\}$ is σ-algebra.
2. $Q = \mathcal{P}(X)$ is σ-algebra.
3. The open sets of topological space is not a σ-algebra as $X \backslash U$ is closed if U is open.
4. The open and closed sets of topological space is not a σ-algebra as infinite union of closed sets may not be open nor closed.

Proposition 1.1.4. *Let X be a set, $C \subseteq \mathcal{P}(X)$. Then*

$$Q := \bigcap \{B : B \quad \sigma\text{-algebra}, C \subseteq B\}$$

is a σ-algebra.

Proof. Note $\mathcal{P}(X)$ is a σ-algebra contains C. Thus we get at least one σ-algebra that contains C. Note by definition $\emptyset \in Q$ as each B contains $\emptyset$. If $E \in Q$ then E is in every σ-algebra contains C, i.e. $X \backslash E$ is in every σ-algebra contains C by definition of σ-algebra and hence $X \backslash E \in Q$. Finally, if $A_1, A_2, ... \in Q$ then we see

$A_i \in B$ for all σ-algebra containing C, i.e. $\bigcup_{i=1}^{\infty} A_i \in B$ for all σ-algebra containing C and so the countable union is again in Q as desired. ♡

Definition 1.1.5. Let X be a topological space. We define $\mathcal{B}(X) = \mathcal{B}$ be the ***Borel σ-algebra*** as the smallest σ-algebra of X containing all the open subsets of X. Elements of $\mathcal{B}$ is called ***Borel sets***.

Remark 1.1.6. Consider $X = \mathbb{R}$, then:

1. Open and closed subsets of X are Borel sets.
2. Countable sets in X are Borel since each point in X is closed, i.e. $\mathbb{R}$ is Hausdorff.
3. $[a, b) = [a, b]\backslash\{b\} = [a, b] \cap (\mathbb{R}\backslash\{b\})$ is Borel.

Definition 1.1.7. We define the ***otter measure***, $|\cdot|$ or m^*, on $\mathbb{R}$: let $A \subseteq \mathbb{R}$, then

$$|A| = m^*(A) = \inf\{\sum_{k=1}^{\infty} \ell(I_k) : I_k \text{ are open intervals such that } A \subseteq \bigcup_{k\geq 1} I_k\}$$

where $\ell(I_k) = l(I_k) = b_k - a_k$ if $I_k = (a_k, b_k)$.

Example 1.1.8. Note $m^*(\emptyset) = 0$ and for all countable sets has measure 0 as well.

Proposition 1.1.9. *If A, B are subsets of $\mathbb{R}$ such that $A \subseteq B$. Then $|A| \leq |B|$.*

Proof. Say $\{I_k\}_{k\geq 1}$ contains B, then it also contains A. Thus

$$|A| \leq \sum_{k=1}^{\infty} l(I_k)$$

In particular, if we take inf the inequality preserves and so $|A| \leq |B|$. ♡

Proposition 1.1.10 (Countable Subadditivity)**.** *Let $\{A_k\}_{k\geq 1}$ be a sequence of subsets of $\mathbb{R}$. Then*

$$|\bigcup_{k=1}^{\infty} A_k| \leq \sum_{k=1}^{\infty} |A_k|$$

Proof. If $|A_k| = \infty$ for any one of A_k then the proof follows. Thus assume all $|A_k| < \infty$.

Let $\epsilon > 0$ be arbitrary. For each $k \in \mathbb{Z}_{\geq 1}$ let $I_{1,k}, I_{2,k}, \ldots$ be open cover of A_k such that

$$\sum_{j=1}^{\infty} l(I_{j,k}) \leq \frac{\epsilon}{2^k} + |A_k|$$

This is allowed because $|A|$ is the infimum taken over all possible open covers, so there is always an open cover to make the sum less than or equal $x + |A|$ for any $x > 0$. Thus, we see (use geometric series to get that ϵ)

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} l(I_{j,k}) \leq \epsilon + \sum_{k=1}^{\infty} |A_k|$$

Next, we note $\bigcup_{j,k\geq 1} I_{j,k}$ contains $\bigcup_{k=1}^{\infty} A_k$ and so

$$\left|\bigcup_{k=1}^{\infty} A_k\right| \leq \epsilon + \sum_{k=1}^{\infty} |A_k|$$

and the proof follows. ♡

Proposition 1.1.11. *Let $a, b \in \mathbb{R}$ with $a < b$, then $|[a,b]| = b - a$.*

Proof. Clearly $|[a,b]| \leq b - a$ as we can take the cover $I = (a-\epsilon, b+\epsilon)$ and get $|[a.b]| \leq l(I) = b - a$ as desired. Now let $I_1, I_2, I_3, \ldots$ be arbitrary open cover of $[a,b]$, then there exists n, relabel if needed, such that $I_1, \ldots, I_n$ covers $[a,b]$. Thus we have $[a,b] \subseteq I_1 \cup \ldots \cup I_n$. If we can show $b - a \leq \sum_{k=1}^{n} l(I_k)$ then since $\sum_{k=1}^{n} l(I_k) \leq \sum_{k\geq 1} l(I_k)$ the proof follows as we get $b - a \leq |[a-b]|$.

We use induction on the value of n. If $n = 1$ then $[a,b] \subseteq I_1$ for some interval and so $b - a \leq l(I_1)$ as desired. Suppose it holds for $n-1$, we shall show the n case. Since $[a,b] \subseteq I_1 \cup \ldots \cup I_n$ there exists $1 \leq k \leq n$ so $b \in I_k$. Thus relabel $I_1, \ldots, I_n$ so that $b \in I_n$ with $I_n = (c,d)$. If $c \leq a$ then $l(I_n) \geq b - a$ and the induction follows. Thus say $a < c < b < d$. Then we see $[a,c]$ is covered by $I_1, \ldots, I_{n-1}$ and we can use our induction hypothesis as $\sum_{k=1}^{n} l(I_k) \geq c - a$. Hence

$$\sum_{k=1}^{n} l(I_k) \geq c - a + l(I_n) = c - a + d - c = d - a \geq b - a$$

♡

Corollary 1.1.11.1. *If $I \subseteq \mathbb{R}$ is an interval then $|I| = l(I)$.*

Proof. Suppose I is bounded with endpoint a, b. Then $I \subseteq [a,b]$ and hence $m^*(I) \leq b - a$. However note $[a + \frac{\epsilon}{2}, b - \frac{\epsilon}{2}] \subseteq I \Rightarrow b - a - \epsilon \leq m^*(I)$ and so $b - a \leq m^*(I)$.

Now suppose I is unbounded and it is easy to check $m^*(I) = \infty$. ♡

Proposition 1.1.12. *We have $m^*(x + A) = m^*(A)$.*

Proof. Clear. ♡

1.2 Measurable Sets

Definition 1.2.1. We say $A \subseteq \mathbb{R}$ is ***measurable*** iff $\forall E \subseteq \mathbb{R}$, we have

$$m^*(X) = m^*(X \cap A) + m^*(X \backslash A)$$

Remark 1.2.2. We always have $m^*(X) \leq m^*(X \cap A) + m^*(X \backslash A)$ and hence to prove measurable we just need to show the other direction of the inequality.

In particular, if $A \cap B = \emptyset$ where A is measurable, then we see $m^*(A \cup B) = m^*(X \cap A) + m^*(X \backslash A) = m^*(A) + m^*(B)$ where we set $X = A \cup B$. Thus we get additivity for measurable sets.

Proof. If $m^*(A) = 0$ then A is measurable. ♡

Proof. Let $X \subseteq \mathbb{R}$, then we see $m^*(X \cap A) + m^*(X \backslash A) = 0 + m^*(X \backslash A) \leq m^*(X)$ as $X \backslash A \subseteq X$. Hence the proof follows. ♡

Proposition 1.2.3. *$A_1, A_2, \ldots$ are measurable then $\bigcup_{i=1}^n A_i$ is measurable.*

Proof. We show $n = 2$ then the result follows by induction. Let $A, B \subseteq \mathbb{R}$ be measurable. Let $X \subseteq \mathbb{R}$, then $m^*(X) = m^*(X \cap A) + m^*(X \backslash A) = m^*(X \cap A) + m^*((X \backslash A) \cap B) + m^*((X \backslash A) \backslash B)$. We note $m^*((X \backslash A) \backslash B) = m^*(X \backslash (A \cup B))$ as the sets are equal. Thus we see

$$
\begin{aligned}
& m^*(X) \\
& = m^*(X \cap A) + m^*((X \backslash A) \cap B) + m^*((X \backslash A) \backslash B) \\
& = m^*(X \cap A) + m^*((X \backslash A) \cap B) + m^*(X \backslash (A \cup B)) \\
& \geq m^*(X \cap A \cup ((X \backslash A) \cap B)) + m^*(X \backslash (A \cup B)) \\
& = m^*(X \cap (A \cup B)) + m^*(X \backslash (A \cup B))
\end{aligned}
$$

and the proof follows. ♡

Proposition 1.2.4. *Let $A_1, \ldots, A_n$ be measurable and $A_i \cap A_j = \emptyset$ if $i \neq j$. Then if $A = A_1 \cup \ldots \cup A_n$ then $m^*(X \cap A) = \sum_i m^*(X \cap A_i)$.1*

Proof. We just prove $n = 2$. Let $A, B \subseteq \mathbb{R}$ be measurable with $A \cap B = \emptyset$. Let $X \subseteq \mathbb{R}$, then

$$m^*(X \cap (A \cup B)) = m^*((X \cap (A \cup B))) + m^*((X \cap (A \cup B)) \backslash A) = m^*(X \cap A) + m^*(X \cap B)$$

and the proof follows by induction. ♡

Corollary 1.2.4.1. *If $A_1, \ldots, A_n$ are measurable with $A_i \cap A_j = \emptyset$, then $m^*(A_1 \cup \ldots \cup A_n) = \sum m^*(A_i)$.*

Proof. Clear. ♡

Lemma 1.2.5. *Let $A_i \subseteq \mathbb{R}$ be measurable with $i \in \mathbb{N}$. If $A_i \cap A_j = \emptyset$ for $i \neq j$. Then $A = \bigcup_i A_i$ is measurable.*

Proof. Let $B_n = A_1 \cup ... \cup A_n$. Let $X \subseteq \mathbb{R}$ be arbitrary, then we see

$$\begin{aligned} m^*(X) &= m^*(X \cap B_n) + m^*(X \backslash B_n) \\ &\geq m^*(X \cap B_n) + m^*(X \backslash A) \\ &= \sum_{i=1}^{n} m^*(X \cap A_i) + m^*(X \backslash A) \end{aligned}$$

Now take $n \to \infty$ we get

$$m^*(X) \geq \sum_{i=1}^{\infty} m^*(X \cap A_i) + m^*(X \backslash A) \geq m^*(\bigcup (X \cap A_i)) + m^*(X \backslash A)$$

and the proof follows as $\bigcup (X \cap A_i) = X \cap A$. ♡

Proposition 1.2.6. *If $A \subseteq \mathbb{R}$ is measurable, then $\mathbb{R} \backslash A$ is measurable.*

Proof. Just note $m^*(X \cap (\mathbb{R} \backslash A)) + m^*(X \backslash (\mathbb{R} \backslash A)) = m^*(X \backslash A) + m^*(X \cap A)$. ♡

Proposition 1.2.7. *Let $A_i \subseteq \mathbb{R}$ be measurable, then $A = \bigcup_{i=1}^{\infty} A_i$ is measurable.*

Proof. Let $B_1 = A_1$, $B_n = A_n \backslash (A_1 \cup ... \cup A_{n-1}) = A_n \cap ((A_1 \cup ... \cup A_{n-1})^c)$. Then we see B_n is measurable as A_n is measurable and $(A_1 \cup ... \cup A_{n-1})^c$ is measurable and intersection of measurable sets are measurable. Thus we get $\bigcup_i B_i = \bigcup_i A_i$ is measurable and the proof follows. ♡

Proposition 1.2.8. *Let $A_i \subseteq \mathbb{R}$ be measurable. If $A_i \cap A_j = \emptyset$ for $i \neq j$ then*

$$m^*(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m^*(A_i)$$

Proof. We have $m^*(\bigcup A_i) \leq \sum m^*(A_i)$ for free. On the other hand note

$$m^*(\bigcup_i A_i) \geq m^*(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} m^*(A_i)$$

and take $n \to \infty$ we are done. ♡

Proposition 1.2.9. *If $a \in \mathbb{R}$ then (a, ∞) is measurable.*

Proof. Let $X \subseteq \mathbb{R}$. We consider two cases.

Case 1: $a \notin X$. We show $m^*(X \cap (a, \infty)) + m^*(X \cap (-\infty, a)) \leq m^*(X)$ with $X_1 = X \cap (a, \infty)$ and $X_2 = X \cap (-\infty, a)$. Let I_i be a sequence of bounded open interval such that $X \subseteq \bigcup I_i$. Define $I_i' = I_i \cap (a, \infty)$ and $I_i'' = I_i \cap (-\infty, a)$. In particular we note $X_1 \subseteq \bigcup I_i'$ and $X_2 \subseteq \bigcup I_i''$. Thus

$$m^*(X_1) \leq \sum \ell(I_i'), \quad m^*(X_2) \leq \sum \ell(I_i'')$$

Then we get

$$\begin{aligned} m^*(X_1) + m^*(X_2) &\le \sum \ell(I'_i) + \sum \ell(I''_i) \\ &= \sum (\ell(I'_i) + \ell(I''_i)) \\ &= \sum \ell(I_i) \end{aligned}$$

Now take inf on both side we get

$$m^*(X_1) + m^*(X_2) \le m^*(X)$$

and the proof follows for case 1.

Case 2: suppose $a \in X$. Then consider $X' = X \backslash \{a\}$ and note $m^*(\{a\}) = 0$ and the proof should follows. ♡

Theorem 1.2.10. *Every Borel set is measurable.*

Proof. We see (a, ∞) is measurable by above lemma. Then we see $\bigcap_{n=1}^{\infty}(a - \frac{1}{n}, \infty) = [a, \infty)$ is measurable. Thus $\mathbb{R}\backslash[a, \infty) = (-\infty, a)$ is measurable. Hence we get $(a, b) = (a, \infty) \cap (-\infty, b)$ is measurable. However Borel sets are generated by open sets the proof follows. ♡

Definition 1.2.11. We call $m : \mathscr{L} \to [0, \infty]$ ***the Lebesgue measure*** given by $m(A) = m^*(A)$.

Proposition 1.2.12 (Excision Property)**.** *Let $A \subseteq B$, A measurable, and $m(A) < \infty$. Then $m^*(B\backslash A) = m^*(B) - m^*(A)$.*

Proof. Note $m^*(B) = m^*(B \cap A) + m^*(B\backslash A) = m(A) + m^*(B\backslash A)$ and the proof follows as $m(A) < \infty$ hence we can subtract. ♡

Theorem 1.2.13 (Continuity of Measure)**.**

1. *If $A_1 \subseteq A_2 \subseteq \ldots$ be measurable then $m(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} m(A_n)$.*
2. *If $B_1 \supseteq B_2 \supseteq \ldots$ be measurable and $m(B_1) < \infty$ then $m(\bigcap_{i=1}^{\infty} B_i) = \lim_{n\to\infty} m(B_n)$.*

Proof. (1): Since $m(A_k) \le m(\cup A_i)$ for all k we have $\lim m(A_n) \le m(\cup A_n)$. If one of the $m^*(A_k) = \infty$, then the proof is immediate. Hence we may assume each $m^*(A_k) < \infty$. Replace A_i with $D_i = A_i\backslash(A_{i-1})$ with $D_0 = \emptyset$, then we see D_i are measurable and $\bigcup D_i = \bigcup A_i$ and D_i are pair-wise disjoint.

In particular we get

$$\begin{aligned} m^*(\cup A_i) = m(\cup D_i) &= \sum m(D_i) \\ &= \sum (m(A_i) - m(A_{i-1})) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} (m(A_i) - m(A_{i-1})) \\ &= \lim_{n\to\infty} m(A_n) - m(A_0) \end{aligned}$$

and the proof follows as $m(A_0) = 0$.

(2): Let $N_k = B_1 \backslash (B_k)$, then we see N_k are measurable and $N_1 \subseteq N_2 \subseteq \ldots$. Thus $m(\cup N_i) = \lim m(N_i)$ where we see $\cup N_i = B_1 \backslash (\cap B_i)$ and so

$$\lim m(N_i) = m(\cup N_i) = m(B_1 \backslash (\cap B_i)) = m(B_1) - m(\cap B_i)$$

However,

$$\lim_{n\to\infty} m(N_n) = \lim_{n\to\infty} m(B_1) - m(B_n) = m(B_1) - \lim_{n\to\infty} m(B_n)$$

Thus we get

$$m(B_1) - m(\cap B_i) = m(B_1) - \lim m(B_i)$$

and the proof follows as $m(B_1) < \infty$. ♡

1.3 Examples

Lemma 1.3.1. *Let $A \subseteq \mathbb{R}$ be bounded and measurable and $\Lambda \subseteq \mathbb{R}$ is bounded and countably infinite. If $\lambda + A$ are pairwise disjoint for all $\lambda \in \Lambda$, then $m(A) = 0$.*

Proof. Consider $U = \bigcup_{\lambda\in\Lambda} \lambda + A$, we see U is bounded and measurable as it is countable union of measurable sets. In particular, since $\lambda + A$ are pair-wise disjoint, we see

$$m(\bigcup_{\lambda} \lambda + A) = \sum_{\lambda} m(\lambda + A) = |\Lambda| \cdot m(A)$$

but since it is bounded it must be finite. The only possible value of $m(A)$ is then 0. ♡

Example 1.3.2. Start with $\emptyset \neq A \subseteq \mathbb{R}$, consider $a \sim b$ if and only if $a - b \in \mathbb{Q}$. Then $\sim$ is an equivalence relation. Let C_A be a single choice of equivalence class representatives for A relative to $\sim$.

Then we see $\lambda + C_A$ for $\lambda \in \mathbb{Q}$ are pairwise disjoint. Indeed, $x \in (\lambda_1 + C_A) \cap (\lambda_2 + C_A)$ then $x = \lambda_1 + a = \lambda_2 + b$ and hence $a - b = \lambda_2 - \lambda_1 \in \mathbb{Q}$ and hence $a \sim b$. Then by definition of C_A we must have $\lambda_1 = \lambda_2$ and $a = b$.

Theorem 1.3.3 (Vitali). *Every set $A \subseteq \mathbb{R}$ with $m^*(A) > 0$ contains a non-measurable subset.*

Proof. By Quiz 1, we may assume A is bounded, say $A \subseteq [-N, N]$ for some $N \in \mathbb{N}$. Then we claim C_A is non-measurable where C_A is defined above. For a contradiction, assume C_A is measurable. Let $\Lambda \subseteq \mathbb{Q}$ be bounded and infinite. By the lemma and remark, $m(C_A) = 0$. Let $a \in A$, then $a \sim b$ for some $b \in C_A$ and so $a = b + \lambda$ for some $\lambda \in \mathbb{Q}$. Moreover, we see $\lambda \in [-2N, 2N]$. Thus taking $\Lambda_0 = \mathbb{Q}[-2N, 2N]$, then we see

$$A \subseteq \bigcup_{\lambda\in\Lambda_0} (\lambda + C_A) \Rightarrow m(A) \leq \sum_{\lambda\in\Lambda_0} m(\lambda + C_A) = 0$$

THis is a contradiction to the fact $m(A) > 0$. ♡

Corollary 1.3.3.1. *There exists $A, B \subseteq \mathbb{R}$ such that $A \cap B = \emptyset$ and $m^*(A \cup B) < m^*(A) + m^*(B)$.*

Proof. Let C be non-measurable set. Then there exists $X \subseteq \mathbb{R}$ such that

$$m^*(X) < m^*(X \cap C) + m^*(X \backslash C)$$

as if otherwise we must have C be measurable. The proof follows. ♡

Remark 1.3.4. Recall the definition of Cantor set C. Also recall C is uncountable and closed.

Proposition 1.3.5. *The Cantor set C is Borel and has measure 0.*

Proof. Since Cantor set is closed we see it is Borel. To see C has measure 0 we see $m(C) = \lim_{n\to\infty} m(\bigcap_{i=1}^n C_i) = m(\lim_{n\to\infty} C_n) = 0$ where the C_n's are intervals removing the middle third at nth iteration of the construction of Cantor set. ♡

Example 1.3.6 (Cantor-Lebesgue Function)**.**

Step 1: For $k \in \mathbb{N}$, let U_k be the union of open intervals deleted in the process of constructing $C_1, C_2, ..., C_k$, i.e. $U_k = [0,1] \backslash C_k$.

Step 2: Define $U = \bigcup_{k=1}^\infty U_k$, i.e. $U = [0,1] \backslash C$.

Step 3: Say $U_k = I_{k,1} \cup I_{k,2} \cup ... \cup I_{k,2^k-1}$. Define $\phi : U_k \to [0,1]$ by $\phi|_{I_{k,i}} = \frac{i}{2^k}$.

Step 4: Define $\phi : [0,1] \to [0,1]$ by, for $0 \neq x \in C$, let

$$\phi(x) = \sup\{\phi(t) : t \in U \cap [0,x]\}$$

and $\phi(0) = 0$. This is the Cantor-Lebesgue function.

Remark 1.3.7. Things to know about ϕ:

1. ϕ is increasing.
2. ϕ is continuous. Indeed, we see ϕ is continuous on U. For $x \in C$ with $x \neq 0, 1$, for large k there exists $a_k \in I_{k,i}$ such that $b_k \in I_{k,i+1}$ such that $a_k < x < b_k$. But $\phi(b_k) - \phi(a_k) = \frac{i+1}{2^k} - \frac{i}{2^k} = \frac{1}{2^k} \to 0$, i.e. there is no jump in the function and hence continuous.
3. $\phi : U \to [0,1]$ is differentiable and $\phi' = 0$.
4. ϕ is surjective.

Example 1.3.8. Next, we are going to construct a measurable set which is not Borel. Let ϕ be the Cantor-Lebesgue function and define $\psi : [0,1] \to [0,2]$ by

$$\psi(x) = x + \phi(x)$$

Then we see:

1. ψ is strictly increasing.

2. ψ is continuous.
3. ψ is onto.

Thus we see ψ is invertible.

Proposition 1.3.9.

1. *$\psi(C)$ is measurable and has positive measure.*
2. *ψ maps a particular measurable subset of C to a non-measurable set.*

Proof. By A1, ψ^{-1} is continuous and hence $\psi(C) = (\psi^{-1})^{-1}(C)$ is closed as it is inverse image of closed sets. Thus $\psi(C)$ is measurable and Borel. Note that $[0,1] = C \cup U$ where C is Cantor set and U is $[0,1]\backslash C$. Then we see

$$[0,2] = \psi(C) \cup \psi(U) \Rightarrow 2 = m(\psi(C)) + m(\psi(U))$$

Thus to show $\psi(C)$ has positive measurable and so to show $\psi(C)$ has positive measure it suffices to show $m(\psi(U)) = 1$.

Note $\psi(U) = \bigcup_{i=1}^{\infty} \psi(I_i)$ and hence

$$m(\psi(U)) = \sum m(\psi(I_i))$$

Note that for all $i \in \mathbb{N}$, $\exists r \in \mathbb{R}$ such that $\psi(x) = r$ for all $x \in I$. In particular $\psi(x) = x + r$ for all $x \in I_i$ and so $\psi(I_i) = r + I_i$ and hence

$$m(\psi(U)) = \sum m(I_i) = m(\bigcup I_i) = m(U)$$

Since $[0,1] = U \cup C$ we see $1 = m(U) + m(C) = m(U) + 0$ and hence we concluded $\psi(U)$ has measure 0 and hence $m(\psi(C)) = 1 > 0$ as desired.

For (2), by Vitali we see $\psi(C)$ contains a subset $A \subsetneq \psi(C)$ which is non-measurable. Let $B = \psi^{-1}(A) \subsetneq C$, then $\psi(B) = A$ is non-measurable. ♡

Theorem 1.3.10. *The Cantor set contains an element of measurable set that is not Borel.*

Proof. Note $B \subsetneq C$ implies B is measurable as it has measure 0. However, we know $\psi(B)$ is non-measurable for some $B \subsetneq C$. By A1, if B is Borel, then $\psi(B)$ is Borel, which is a contradiction as $\psi(B)$ is not measurable. Thus B cannot be Borel and the proof follows. ♡

Chapter 2

Integration

2.1 Measurable Functions

Definition 2.1.1. Let $A \subseteq \mathbb{R}$ be measurable, we say $f : A \to \mathbb{R}$ is ***measurable*** iff for all open $U \subseteq \mathbb{R}$, we have $f^{-1}(U)$ is measurable.

Proposition 2.1.2. *If $A \subseteq \mathbb{R}$ is measurable and $f : A \to \mathbb{R}$ is continuous, then f is measurable.*

Proof. If $U \subseteq \mathbb{R}$ then $f^{-1}(U)$ is open hence measurable. ♡

Definition 2.1.3. Let $A \subseteq \mathbb{R}$, we define $\chi_A : \mathbb{R} \to \mathbb{R}$ to be

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$$

Proposition 2.1.4. *Let $A \subseteq \mathbb{R}$ be measurable, then χ_A is measurable.*

Proof. We see $\chi_A^{-1}(U) \in \{\mathbb{R}, A, \mathbb{R}\backslash A, \emptyset\}$ and they are all measurable. ♡

Proposition 2.1.5. *Let $A \subseteq \mathbb{R}$ be measurable, $f : A \to \mathbb{R}$. Then TFAE:*

1. *f is measurable.*
2. *$\forall a \in \mathbb{R}, f^{-1}(a, \infty)$ is measurable.*
3. *$\forall a < b, f^{-1}(a, b)$ is measurable.*

Proof. We see (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3): Let $b \in \mathbb{R}$, so that $f^{-1}(b, \infty)$ is measurable. Thus $\mathbb{R}\backslash f^{-1}(b, \infty) = f^{-1}(\mathbb{R}\backslash(b, \infty)) = f^{-1}((-\infty, b])$ is measurable. Thus we see

$$(-\infty, b) = \bigcup_{n=1}^{\infty}(-\infty, b - \frac{1}{n}] \Rightarrow f^{-1}(-\infty, b) = \bigcup_{n=1}^{\infty} f^{-1}((-\infty, b - \frac{1}{n}])$$

and we see $f^{-1}(-\infty, b)$ is measurable.

Finally, for $a < b$ we see $(a,b) = (a,\infty) \cap (-\infty, b)$ which concludes

$$f^{-1}(a,b) = f^{-1}(a,\infty) \cap f^{-1}(-\infty, b)$$

is measurable.

(3) $\Rightarrow$ (1): We see every open set is countable union of intervals thus it is trivial. ♡

Proposition 2.1.6. *Let $A \subseteq \mathbb{R}$ and $f, g : A \to \mathbb{R}$ are measurable, then $af + bg$ and fg are measurable where $a, b \in \mathbb{R}$.*

Proof. Let $a \in \mathbb{R}$, for $\alpha \in \mathbb{R}$, we see $(af){-}1 = \{x \in A : af(x) > \alpha\}$. If $a > 0$ then $(af)^{-1}(\alpha, \infty) = f^{-1}(\frac{\alpha}{a}, \infty)$ which is measurable. If $a < 0$ then $(af^{-1})(\alpha,\infty) = f^{-1}(-\infty, \frac{\alpha}{a})$. If $a = 0$ then the claim is trivial.

We now show $f + g$ is measurable. For $\alpha \in \mathbb{R}$ we see

$$\begin{aligned}
(f+g)^{-1}(\alpha, \infty) &= \{x \in A : f(x) + g(x) > \alpha\} \\
&= \{x \in A : f(x) > \alpha - g(x)\} \\
&= \{x \in A : \exists q \in \mathbb{Q}, f(x) > q > \alpha - g(x)\} \\
&= \bigcup_{q \in \mathbb{Q}} (\{x \in A : f(x) > q\} \cap \{x \in A : g(x) > \alpha - q\}) \\
&= \bigcup_{q \in \mathbb{Q}} (f^{-1}(\alpha, \infty) \cap g^{-1}(\alpha - q, \infty))
\end{aligned}$$

which is measurable and the proof follows.

By the Quiz 4, we see $|f|$ is measurable. Thus

$$(f^2)^{-1}(\alpha, \infty) = \{x \in A : f(x)^2 > \alpha\} = \begin{cases} A, & \alpha < 0 \\ (|f|)^{-1}(\sqrt{\alpha}, \infty), & \alpha \geq 0 \end{cases}$$

and since $|f|$ is measurable we see f^2 is measurable. Finally, observe we have

$$(f+g)^2 = f^2 + 2fg + g^2 \Rightarrow fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$$

is measurable as desired. ♡

Example 2.1.7. Let $\psi : [0,1] \to \mathbb{R}$ and $\psi(x) = x + \phi(x)$. Note there exists $A \subseteq [0,1]$ such that A is measurable but $\psi(A)$ is not measurable. Now extend $\psi : \mathbb{R} \to \mathbb{R}$ continuously to a strictly increasing surjective function such that ψ^{-1} is continuous. Now consider $\chi_A \circ \psi^{-1}$, where we see both functions are continuous. However, the composition $\chi_A \circ \psi^{-1}$ is not measurable as $(\chi_A \circ \psi^{-1})^{-1}(\frac{1}{2}, \frac{3}{2}) = \psi(A)$ which is not measurable.

Proposition 2.1.8. *Let $A \subseteq \mathbb{R}$ be measurable. If $g : A \to \mathbb{R}$ is measurable and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f \circ g$ is measurable.*

Proof. Let U be open, we see $(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))$ where $f^{-1}(U)$ is open and hence $g^{-1}(f^{-1}(U))$ is measurable as desired. ♡

Definition 2.1.9. Let $A \subseteq \mathbb{R}$, we say a property $P(x)$ is true ***almost everywhere*** if $m(\{x \in A : \neg P(x)\}) = 0$.

Proposition 2.1.10. *Let $f : A \to \mathbb{R}$ be measurable. If $g : A \to \mathbb{R}$ is a function and $f = g$ almost everywhere, then g is measurable.*

Proof. Note $B = \{x \in A : f(x) \neq g(x)\}$ has measure 0. Let $\alpha \in \mathbb{R}$ be arbitrary, we see

$$\begin{aligned} g^{-1}(\alpha, \infty) &= \{x \in A : g(x) > \alpha\} \\ &= \{x \in A \backslash B : g(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ &= \{x \in A \backslash B : f(x) > \alpha\} \cup \{x \in B : g(x) > \alpha\} \\ &= (f^{-1}(\alpha, \infty) \cap A \backslash B) \cup \{x \in B : g(x) > \alpha\} \end{aligned}$$

where we see $\{x \in B : g(x) > \alpha\}$ is measurable as it is a subset of measure zero set B. On the other hand we see $f^{-1}(\alpha, \infty) \cap A \backslash B$ is also measurable thus the proof follows. ♡

Proposition 2.1.11. *Let A be measurable, $B \subseteq A$ be measurable. A function $f : A \to \mathbb{R}$ is measurable iff $f|_B$ and $f|_{A \backslash B}$ are measurable.*

Proof. ($\Rightarrow$): Suppose $f : A \to \mathbb{R}$ is measurable. Let $\alpha \in \mathbb{R}$, then $(f|_B)^{-1}(\alpha, \infty) = \{x \in B : f(x) > \alpha\} = f^{-1}(\alpha, \infty) \cap B$ is measurable. The proof for $A \backslash B$ is identical.

($\Leftarrow$): Suppose $f|_B$ and $f|_{A \backslash B}$ are measurable. For $\alpha \in \mathbb{R}$,

$$\begin{aligned} f^{-1}(\alpha, \infty) &= \{x \in A : f(x) > \alpha\} \\ &= \{x \in B : f(x) > \alpha\} \cup \{x \in A \backslash B : f(x) > \alpha\} \\ &= (f|_B)^{-1}(\alpha, \infty) \cup (f|_{A \backslash B})^{-1}(\alpha, \infty) \end{aligned}$$

and hence the proof follows. ♡

Proposition 2.1.12. *Let (f_n) be a sequence of measurable functions and $f_n \to f$ converges pointwise almost everywhere. Then f is measurable.*

Proof. Let $B = \{x \in A : f_n(x) \not\to f(x)\}$ and by assumption we see $m(B) = 0$. For $\alpha \in \mathbb{R}$, we see $(f|_B)^{-1}(\alpha, \infty) = f^{-1}(\alpha, \infty) \cap B$ is measurable as it has measure 0.

Thus it suffices to show $f|_{A \backslash B}$ is measurable. By replacing f by $f|_{A \backslash B}$ we may assume $f_n \to f$ pointwise. Let $\alpha \in \mathbb{R}$, since $f_n \to f$ pointwise, we see for $x \in A$, $f(x) > \alpha$ iff

$$\exists n, N \in \mathbb{N}, \forall i > N, f_i(x) > \alpha + \frac{1}{n}$$

We then see that

$$f^{-1}(\alpha, \infty) = \bigcup_{n \in N} \bigcup_{N \in \mathbb{N}} \bigcap_{i=N}^{\infty} f_i^{-1}(\alpha + \frac{1}{n}, \infty)$$

is measurable and the proof follows. ♡

Definition 2.1.13. A function $\phi : A \to \mathbb{R}$ is called ***simple*** if ϕ is measurable and $\phi(A)$ is finite.

Remark 2.1.14 (Canonical Representation)**.** Let $\phi : A \to \mathbb{R}$ be simple, then $\phi(A) = \{c_1, ..., c_k\}$. We see $A_i := \phi^{-1}(c_i)$ is measurable and $A = \bigcup A_i$. Thus we see we get a canonical representation of a simple function

$$\phi = \sum_{i=1}^{k} \chi_{A_i}$$

Lemma 2.1.15. *Let $f : A \to \mathbb{R}$ be measurable and bounded. For all $\epsilon > 0$ there exists simple $\phi_\epsilon, \psi_\epsilon : A \to \mathbb{R}$ such that:*

1. $\phi_\epsilon \leq f \leq \psi_\epsilon$.
2. $0 \leq \psi_\epsilon - \phi_\epsilon < \epsilon$.

Proof. Let $f(A) \subseteq [a, b]$, since f is bounded. Let $\epsilon > 0$ and consider a partition of $[a, b]$ with each interval less than ϵ, i.e. say

$$a = y_0 < y_1 < ... < y_n = b$$

with $y_{i+1} - y_i < \epsilon$. Then let $I_k = [y_{k-1}, y_k)$ and $A_k = f^{-1}(I_k)$. Now define

$$\phi_\epsilon : A \to \mathbb{R}, \psi_\epsilon : A \to \mathbb{R}$$

be the following:

$$\phi_\epsilon = \sum_{k=1}^{n} y_{k-1}\chi_{A_k}, \quad \psi_\epsilon \sum_{k=1}^{n} y_k \chi_{A_k}$$

We see both of them are simple. Let $x \in A$ be arbitrary, since $f(x) \in [a, b]$, we see $\exists k \in \{1, ..., n\}$ such that $f(x) \in I_k$. Viz, $y_{k-1} \leq f(x) < y_k$ and hence $x \in A_k$. Moreover, $\phi_\epsilon(x) = y_{k-1}$ and $\psi_\epsilon(x) = y_k$. Thus we see $\phi_\epsilon \leq f \leq \psi_\epsilon$ and $0 \leq \psi_\epsilon - \phi_\epsilon < \epsilon$ as desired. ♡

Theorem 2.1.16 (Simple Approximation)**.** *A function $f : A \to \mathbb{R}$ is measurable if and only if there is a sequence (ϕ_n) of simple functions on A such that $\phi_n \to f$ converges pointwise and $\forall n, |\phi_n| \leq |f|$.*

Proof. ($\Leftarrow$): We are done as each ϕ_n are measurable and it converges pointwise.

($\Rightarrow$): Suppose $f : A \to \mathbb{R}$ be measurable.

Case 1: $f \geq 0$. For each $n \in \mathbb{N}$ define $A_n = \{x \in A : f(x) \leq n\}$, so that A_n is measurable and $f|_{A_n}$ is measurable and bounded. Thus we can use the lemma on $f|_{A_n}$ so that there exists simple functions $(\phi_n), (\psi_n)$ so that

$$0 \leq \phi_n \leq f \leq \psi_n$$

on A_n and $0 \leq \psi_n - \phi_n < \frac{1}{n}$. Fix $n \in \mathbb{N}$ and extend $\phi_n : A \to \mathbb{R}$ by setting $\phi_n(x) = n$ if $x \notin A_n$. Thus we see $0 \leq \phi_n \leq f$ and hence for each $n \in \mathbb{N}$, we have $\phi_n : A \to \mathbb{R}$ is simple.

We claim $\phi_n \to f$ converges pointwise. Let $x \in A$ and let $N \in \mathbb{N}$ such that $f(x) \leq N$, i.e. $x \in A_N$. Then for $n \geq N$ we have $x \in A_n$ and so $0 \leq f(x) - \phi_n(x) \leq \psi_n(x) - \phi_n(x) < \frac{1}{n}$ which implies the sequence $\phi_n \to f$ pointwise as desired.

Case 2: let $f : A \to \mathbb{R}$ be measurable. Now let $B = \{x \in A : f(x) \geq 0\}$ and $C = \{x \in A : f(x) < 0\}$. We see both B and C are measurable and we see both $g := f|_B = \chi_B \cdot f$ and $h := -f|_C = -\chi_C \cdot f$ are non-negative and measurable so we can apply case 1. Hence we get $\phi_n \to g$ and $\psi_n \to h$ pointwise. Then we see

$$\phi_n - \psi_n \to g - h = f$$

pointwise and the proof follows. Also note

$$|\phi_n - \psi_n| \leq |\phi_n| + |\psi_n| = \phi_n + \psi_n \leq g + h = |f|$$

♡

2.2 Littlewood's Principles

Remark 2.2.1. Up to certain finiteness conditions:

1. Measurable sets are "almost" fintie, disjoint union of bounded open intervals.
2. Measurable functions are almost continuous.
3. Pointwise limits of measurable functions are "almost" uniform limits.

Theorem 2.2.2 (Littlewood 1). *Let A be measurable, $m(A) < \infty$. For all $\epsilon > 0$ there exists finitely many open, bounded, disjoint intervals $I_1, ..., I_n$ such that*

$$m(A \Delta U) < \epsilon$$

where $U = I_1 \cup ... \cup I_n$.

Proof. Recall $M(A\Delta U) = m(A \backslash U) + m(U \backslash A)$. Let $\epsilon > 0$ be given, we may find an open set U such that $A \subseteq U$ and $m(U \backslash A) < \frac{\epsilon}{2}$. By basic topology, we see we can find bounded, open, disjoint intervals I_i such that

$$U = \bigcup_{i=1}^{\infty} I_i$$

Note $\sum_{i=1}^{\infty} \ell(I_i) = m(U)$ where we see $m(U) < \infty$. Thus we can find $N \in \mathbb{N}$ so $\sum_{i=N+1}^{\infty} \ell(I_i) < \frac{\epsilon}{2}$. Now let $V = I_1 \cup ... \cup I_{N+1}$ and we see

$$m(A \backslash V) \leq m(U \backslash V) \leq \sum_{i=N+1}^{\infty} \ell(I_i) < \frac{\epsilon}{2}$$

On the other hand, note

$$m(V \backslash A) \leq m(U \backslash A) < \frac{\epsilon}{2}$$

Thus the proof follows as we have

$$m(A \Delta V) < \epsilon$$

♡

Lemma 2.2.3. *Let A be measurable, $m(A) < \infty$, (f_n) is sequence of measurable functions, with $f_n : A \to \mathbb{R}$ such that $f_n \to f$ pointwise. For all $\alpha, \beta > 0$ there exists measurable subset $B \subseteq A$ and $N \in \mathbb{N}$ such that:*

1. *$|f_n(x) - f(x)| < \alpha$ for all $x \in B, n \geq N$.*
2. *$m(A \backslash B) < \beta$.*

Proof. Let $\alpha, \beta > 0$ be given. For $n \in \mathbb{N}$, define

$$A_n = \{x \in A : \forall k \geq n, |f_k(x) - f(x)| < \alpha\}$$

Note by definition

$$A_n = \bigcap_{k=n}^{\infty} |f_k - f|^{-1}(-\infty, \alpha)$$

and so all A_n are measurable. Since $f_n \to f$ pointwise, we see

$$A = \bigcup_{n=1}^{\infty} A_n$$

since $A_1 \subseteq A_2 \subseteq ...$ is assending, we see by continuity of measure we get

$$m(A) = \lim_{n \to \infty} m(A_n) < \infty$$

Hence we see we can find $N \in \mathbb{N}$ such that $\forall n \geq N$ we have

$$m(A) - m(A_n) < \beta$$

Pick $B = A_n$ and then the proof follows. ♡

Theorem 2.2.4 (Littlewood 3/Egorov's Theorem)**.** *Let A be measurable, $m(A) < \infty$, (f_n) be a sequence of measurable functions from A to $\mathbb{R}$ and $f_n \to f$ pointwise. For all $\epsilon > 0$ t here exists a closed set $C \subseteq A$ such that:*

1. *$f_n \to f$ uniformly on C.*
2. *$m(A \backslash C) < \epsilon$.*

Proof. Let $\epsilon > 0$ be given, by the Lemma 2.2.3, for all $n \in \mathbb{N}$ we can find measurable $A_n \subseteq A$ and $N(n) \in \mathbb{N}$ such that, for all $x \in A_n$ and $k \geq N(n)$ we have

$$|f_k(x) - f(x)| < \frac{1}{n}$$

and $m(A \backslash A_n) < \frac{\epsilon}{2^{n+1}}$. Now take

$$B = \bigcap_{n=1}^{\infty} A_n$$

and we see it is measurable. For $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$, we have $k \geq N(n)$ and $x \in B$ we have

$$|f_k(x) - f(x)| < \frac{1}{n} < \epsilon$$

which implies $f_n \to f$ uniformly on B as desired. Finally, note

$$m(A \backslash B) = m(\bigcup (A \backslash A_n)) \leq \sum m(A \backslash A_n) < \sum \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2}$$

Now by A1, we can find closed set C such that $C \subseteq B$ and $m(B \backslash C) < \frac{\epsilon}{2}$ as B is measurable. Then we get

$$m(A \backslash C) < \epsilon$$

and $f_n \to f$ uniformly on C as desired. ♡

Remark 2.2.5. Let $f_n \to \mathbb{R} \to \mathbb{R}$ and $f_n(x) = \frac{x}{n}$, then we see $f_n \to 0$ pointwise. Then $f_n \not\to 0$ uniformly on any measurable set $B \subseteq \mathbb{R}$ such that $m(\mathbb{R} \backslash B) < 1$.

Lemma 2.2.6. *Let $f : A \to \mathbb{R}$ be simple. For all $\epsilon > 0$ there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed $C \subseteq A$ such that:*

1. *$f = g$ on C.*
2. *$m(A \backslash C) < \epsilon$.*

Proof. Let $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ be the canonical representation. We see each A_i are measurable. By Assignment 1, we see we can find $C_i \subseteq A_i$ closed in $\mathbb{R}$ with $m(A_i \backslash C_i) < \frac{\epsilon}{n}$. By the construction, we see

$$A = \bigcup_{i=1}^{n} A_i$$

as a disjoint union, and thus we define disjoint union $C := \bigcup_{i=1}^{n} C_i$. For $x \in C_i$, we see $f(x) = a_i$ and hence by A1 we see f is continuous on C and we can extend $f|_c$ to a continuous map $g : \mathbb{R} \to \mathbb{R}$. Finally, note

$$m(A \backslash C) \leq \sum_{i=1}^{n} m(A_i \backslash C_i) < \epsilon$$

♡

Theorem 2.2.7 (Littlewood 2/Luzin's Theorem)**.** *Let $f : A \to \mathbb{R}$ be measurable. Then for all $\epsilon > 0$ there exists a continuous $g : \mathbb{R} \to \mathbb{R}$ and a closed set $C \subsetneq A$ such that:*

1. *$f = g$ on C*
2. *$m(A \backslash C) < \epsilon$.*

Proof. Let $\epsilon > 0$ be given.

Case 1: consider $m(A) < \infty$. Let $f : A \to \mathbb{R}$ be measurable. By the simple approximation theorem 2.1.16 we have a sequence f_n of simple functions such that $f_n \to f$ pointwise. By the above lemma 2.2.6 we can find continuous $g_n : \mathbb{R} \to \mathbb{R}$ and closed $c_n \subsetneq A$ such that $f_n = g_n$ on C_n and $m(A \backslash C_n) < \frac{\epsilon}{2^{n+1}}$.

By Egorov's theorem 2.2.4 we can find closed set $C_0 \subsetneq A$ such that $f_n \to f$ uniformly on C_0 and $m(A \backslash C_0) < \frac{\epsilon}{2}$. Now let

$$C := \bigcap_{i=0}^{\infty} C_i$$

and we have $g_n = f_n \to f$ uniformly on $C \subsetneq C_0$ and hence f is continuous on C. By Assignment 1 we may extend $f|_C$ to a continuous function $g : \mathbb{R} \to \mathbb{R}$. Moreover, note

$$m(A \backslash C) = m(A \backslash \bigcap_{i=0}^{\infty} C_i) \leq \sum_{i=0}^{\infty} m(A \backslash C_i) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which concludes the proof.

Case 2: consider $m(A) = \infty$. Then we set $A_n := \{a \in A : |a| \in [n-1, n)\}$ so that

$$A = \bigcup_{n=1}^{\infty} A_n$$

is a disjoint union. By case 1, we know for every n there exists $g_n : \mathbb{R} \to \mathbb{R}$ and closed set $C_n \subsetneq A_n$ such that $f = g_n$ on C_n and $m(A_n \backslash C_n) < \frac{\epsilon}{2^n}$. Now consider

$$A = \bigcup_{n=1}^{\infty} C_n$$

and one can show C is closed. Then we see

$$m(A \backslash C) \leq \sum m(A_n \backslash C_n) < \epsilon$$

Next, define $g : C \to \mathbb{R}$ and let $x \in C$ so that $x \in C_n$ for exactly one $n \in \mathbb{N}$, then we define $g(x) = g_n(x) = f(x)$. One can show g is continuous and hence by A_1 we can extend g continuous to all of $\mathbb{R}$. ♡

2.3 Integration

Definition 2.3.1. Let $\phi : A \to \mathbb{R}$ be simple with $m(A) < \infty$ and $\phi = \sum a_i \chi_{A_i}$. Then the ***(Lebesgue) integral of ϕ over*** A is

$$\int_A \phi = \sum_{i=1}^{n} a_i m(A_i)$$

Lemma 2.3.2. *Let $m(A) < \infty$, if $B_1, ..., B_n \subseteq A$ are measurable and disjoint, and $\phi : A \to \mathbb{R}$ is defined by $\phi = \sum b_i \chi_{B_i}$, then*

$$\int_A \phi = \sum_{i=1}^{n} b_i m(B_i)$$

Proof. For $n = 2$, if $b_1 \neq b_2$ then $\phi = b_i \chi_{B_i} + b_2 \chi_{B_2}$ is the canonical representation, thus by definition we get the desired conclusion. If $b_1 = b_2$ then $\phi = b_1 \chi_{B_1 \cup B_2}$ is the canonical representation. Thus

$$\int_A \phi = b_1 m(B_1 \cup B_2) = b_1 m(B_1) + b_1 m(B_2)$$

and the proof follows. ♡

Proposition 2.3.3. *Let $\phi, \psi : A \to \mathbb{R}$ be simple with $m(A) < \infty$. For all $\alpha, \beta \in \mathbb{R}$ we have*

$$\int_A (\alpha\phi + \beta\psi) = \alpha \int_A \phi + \beta \int_A \psi$$

Proof. Let $\phi(A) = \{a_1, ..., a_n\}$ be distinct and $\psi(A) = \{b_1, ..., b_m\}$ be distinct. Consider $C_{ij} = \{x \in A : \phi(x) = a_i, \psi(x) = b_j\}$, then we see they are measurable. In particular, note

$$\alpha\phi + \beta\psi = \sum_{i,j} (\alpha a_i + \beta b_j) \chi_{C_{ij}}$$

but C_{ij}'s are pairwise disjoint, hence by the above lemma we get

$$\begin{aligned}
\int_A \alpha\phi + \beta\psi &= \sum_{i,j} (\alpha a_i + \beta b_j) m(C_{ij}) \\
&= \sum_{i,j} (\alpha a_i) m(C_{ij}) + \sum_{i,j} (\beta b_j) m(C_{ij}) \\
&= \sum_i \alpha a_i (\sum_j m(C_{ij})) + \sum_j \beta b_j (\sum_i m(C_{ij})) \\
&= \alpha \sum_i a_i m(\phi^{-1}(a_i)) + \beta \sum_j b_j m(\psi^{-1}(b_j)) \\
&= \alpha \int_A \phi + \beta \int_A \psi
\end{aligned}$$

♡

Proposition 2.3.4. *Let $\phi, \psi : A \to \mathbb{R}$ be simple with $m(A) < \infty$. If $\phi \leq \psi$ then*

$$\int_A \phi \leq \int_A \psi$$

Proof. Note

$$\int_A \psi - \int_A \phi = \int_A (\psi - \phi) \geq 0$$

This concludes the proof. ♡

Definition 2.3.5. Let $f : A \to \mathbb{R}$ be bounded, measurable, $m(A) < \infty$. We define ***lower Lebesgue integral***

$$\underline{\int_A} f := \sup\{\int_A \phi : \phi \leq f \text{ is simple}\}$$

and ***upper Lebesgue integral***

$$\overline{\int_A} f := \inf\{\int_A \phi : \phi \geq f \text{ is simple}\}$$

Proposition 2.3.6. *Let $m(A) < \infty$, $f : A \to \mathbb{R}$ be bounded and measurable. Then*

$$\underline{\int_A} f = \overline{\int_A} f$$

Proof. For all $n \in \mathbb{N}$, we can find simple functions $\phi_n, \psi_n : A \to \mathbb{R}$ such that $\phi_n \leq f \leq \psi_n$ and $\psi_n - \phi_n \leq \frac{1}{n}$. Thus we see

$$0 \leq \overline{\int_A} f - \underline{\int_A} f \leq \int_A \psi_n - \int_A \phi_n \leq \frac{m(A)}{n}$$

for all $n \geq 1$ and so the proof follows as by taking $n \to 0$. ♡

Definition 2.3.7. If $f : A \to \mathbb{R}$ is bounded and measurable with $m(A) < \infty$, then we define the ***(Lebesgue) integral of f over A*** by

$$\int_A f := \underline{\int_A} f = \overline{\int_A} f$$

Proposition 2.3.8. *Let $f, g : A \to \mathbb{R}$ be bounded and measurable with $m(A) < \infty$. For $\alpha, \beta \in \mathbb{R}$, then*

$$\int_A (\alpha f + \beta g) = \alpha \int_A f + \beta \int_A g$$

Proof. The scalar multiplication is left as an exercise.

Let $\phi_1, \phi_2, \psi_1, \psi_2$ be simple with $\phi_1 \leq f \leq \psi_1$ and $\phi_2 \leq g \leq \psi_2$. Then we see

$$\int_A f + g \leq \int_A (\psi_1 + \psi_2) = \int_A \psi_1 + \int_A \psi_2$$

Hence

$$\int_A f + g \leq \inf\{\int_A \psi_1 + \int_A \psi_2 : f \leq \psi_1, g \leq \psi_2\}$$

but the inf above is just $\int_A f + \int_A g$. Hence $\int_A f + g \leq \int_A f + \int_A g$.

On the other hand, note $\int_A f + g \geq \int_A \phi_1 + \int_A \phi_2$. Thus we see $\int_A f + g \geq \int_A f + \int_A g$ as ϕ_1, ϕ_2 are arbitrary and the proof follows. ♡

Proposition 2.3.9. *Let $f, g : A \to \mathbb{R}$ be bounded and measurable with $m(A) < \infty$. If $f \leq g$ then*

$$\int_A f \leq \int_A g$$

Proof. Note $g - f \geq 0$ and so $\int_A (g - f) \geq \underline{\int_A} (g - f) \geq \int_A 0 = 0$ and the proof follows as $\int_A (g - f) = \int_A g - \int_A f$. ♡

Proposition 2.3.10. *Let $f : A \to \mathbb{R}$ be bounded measurable and $B \subseteq A$ be measurable with $m(A) < \infty$. Then*

$$\int_B f = \int_A f \cdot \chi_B$$

Proof. First assume $f = \chi_C$ where $C \subseteq A$ is measurable. Then $\int_A \chi_C \chi_B = \int_A \chi_{B \cap C} = m(B \cap C) = \int_B \chi_C|_B$.

Now suppose f is simple, say $f = \sum_{i=1}^n a_i \chi_{A_i}$. Then

$$\int_A f \chi_B = \sum a_i \int_A \chi_{A_i} \chi_B = \sum a_i \int_B \chi_{A_i} = \int_B f$$

Next, suppose $f : A \to \mathbb{R}$ is bounded, measurable. Then note if ψ is simple with $f \leq \psi$ then

$$\int_A f \chi_B \leq \int_A \psi \chi_B = \int_B \psi$$

Thus by taking inf over all ψ such that $f \leq \psi$ we get

$$\int_A f \chi_B \leq \overline{\int_B} f = \int_B f$$

Similarly taking $\phi \leq f$ we get $\int_B f = \underline{\int_B} f \leq \int_A f \chi_B$ and the proof follows. ♡

Proposition 2.3.11. *Let $f : A \to \mathbb{R}$ be bounded measurable, $m(A) < \infty$. If $B, C \subseteq A$ are measurable and disjoint, then*

$$\int_{B\cup C} f = \int_B f + \int_C f$$

Proof.

$$\begin{aligned}\int_{B\cup C} f &= \int_A f\chi_{B\cup C}\\ &= \int_A f(\chi_B + \chi_C)\\ &= \int_A f\chi_B + \int_A f\chi_C\\ &= \int_B f + \int_C f\end{aligned}$$

♡

Proposition 2.3.12. *Let $f : A \to \mathbb{R}$ be bounded, measurable and $m(A) < \infty$. Then*

$$|\int_A f| \leq \int_A |f|$$

Proof. Note $-|f| \leq f \leq |f|$ so

$$-\int_A |f| \leq \int_A f \leq \int_A |f|$$

and the proof follows. ♡

Proposition 2.3.13. *Let $f_n : A \to \mathbb{R}$ be a sequence of bounded measurable functions with $m(A) < \infty$. If $f_n \to f$ converges uniformly, then*

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof. Let $\epsilon > 0$ be given, let $N \in \mathbb{N}$ such that

$$|f_n - f| < \frac{\epsilon}{m(A) + 1}$$

for $n \geq N$. Then for all $n \geq N$ we have

$$|\int_A f_n - \int_A f| = |\int_A (f_n - f)| \leq \int_A |f_n - f| \leq m(A) \cdot \frac{\epsilon}{m(A)+1} < \epsilon$$

♡

Example 2.3.14. Suppose $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \begin{cases} 0, & 0 \le x < \frac{1}{n} \\ n, & \frac{1}{n} \le x \le \frac{2}{n} \\ 0, & x \ge \frac{2}{n} \end{cases}$ Then $f_n \to 0$ pointwise but $\int_{[0,1]} f_n = 1$ while $\int_{[0,1]} 0 = 0$. This shows the above proposition fails if $f_n \to f$ is not uniformly.

Theorem 2.3.15 (Bounded Convergence Theorem). *Let $f_n : A \to \mathbb{R}$ be a sequence of measurable functions, with $m(A) < \infty$ and $|f_n| \le M$. Suppose $f_n \to f$ converges pointwise, then*

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof. Let $\epsilon > 0$ be given, then by Egorov's theorem we get a measurable $B \subseteq A$ and $N \in \mathbb{N}$ such that $n \ge N$ then $f_n \to f$ on B uniformly and $m(A \backslash B) < \frac{\epsilon}{4M}$. Thus we get $|f_n - f| < \frac{\epsilon}{2(m(B)+1)}$ for $n \ge N$.

Then

$$|\int_A f_n - \int_A f| \le \int_A |f_n - f| = \int_B |f_n - f| + \int_{A\backslash B} |f_n - f|$$

where we see

$$\int_{A\backslash B} |f_n - f| \le \int_{A\backslash B} (|f_n| + |f|)$$

where by assumption $|f_n| \le M$ and since $f_n \to f$ pointwise we have $|f| \le M$ as well. In other word, we have

$$|\int_A f_n - \int_A f| \le \int_B |f_n - f| + 2M \cdot m(A\backslash B) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

♡

Definition 2.3.16. Let $f : A \to \mathbb{R}$, then we define $\operatorname{Supp} f := \{x \in A : f(x) \neq 0\}$.

Definition 2.3.17. Let $f : A \to \mathbb{R}$ be measurable:

1. we say f has ***finite support*** if $\operatorname{Supp} f = \{x \in A : f(x) \neq 0\}$ has finite measure.
2. we say f is a ***BF function*** if f is bounded with finite support.
3. if $f : A \to \mathbb{R}$ is BF then we define $\int_A f := \int_{A_0} f$.

Definition 2.3.18. Let $f : A \to \mathbb{R}$ be measurable, $f \ge 0$, then we define

$$\int_A f := \sup\{\int_A h : 0 \le h \le f, h \text{ is BF}\}$$

Proposition 2.3.19. *Let $f, g : A \to \mathbb{R}$ be measurable, $f, g \ge 0$, then:*

1. $\forall \alpha, \beta \in \mathbb{R}$,

$$\int_A (\alpha f + \beta g) := \alpha \int_A f + \beta \int_A g$$

2. *If $f \leq g$ then*

$$\int_A f \leq \int_A g$$

3. *If $B, C \subseteq A$ and $B \cap C = \emptyset$, then*

$$\int_{B \cup C} f = \int_B f + \int_C f$$

Proposition 2.3.20 (Chebychev's Inequality). *If $f : A \to \mathbb{R}$ is measurable, non-negative, then for all $\epsilon > 0$,*

$$m(\{x \in A : f(x) \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_A f$$

Proof. Let $\epsilon > 0$ be given and $A_\epsilon := \{x \in A : f(x) \geq \epsilon\}$.

First assume $m(A_\epsilon) < \infty$. Consider $\phi := \epsilon \chi_{A_\epsilon}$, then $\phi \leq f$. We observe ϕ is BF and $\int_A \phi = \epsilon m(A_\epsilon)$ by definition.

Next assume $m(A_\epsilon) = \infty$, then consider $B_n := A_\epsilon \cap [-n, n]$, and by continuity of measure, we get

$$\infty = m(A_\epsilon) = \lim_{n \to \infty} m(B_n)$$

However, for $n \in \mathbb{N}$ we define $\phi_n := \epsilon \chi_{B_n}$, then ϕ_n is BF and $\phi_n \leq f$. Therefore,

$$\infty = m(A_\epsilon) = \lim_{n \to \infty} m(B_n) = \lim_{n \to \infty} \frac{1}{\epsilon} \int_A \phi_n \leq \int_A f$$

♡

Proposition 2.3.21. *Let $f : A \to \mathbb{R}$, $f \geq 0$, then $\int_A f = 0$ if and only if $f = 0$ almost everywhere.*

Proof. ($\Rightarrow$): suppose $\int_A f = 0$. Then we see

$$m(\operatorname{Supp} f) \leq \sum m(\{x \in A : f(x) \geq \frac{1}{n}\}) \leq \sum n \int_A f = 0$$

This concludes $f = 0$ almost everywhere.

($\Leftarrow$): Suppose $\operatorname{Supp} f$ has measure 0, then

$$\int_A f = \int_B f + \int_{A \setminus B} f = \int_B f$$

However, note since $m(B) = 0$ we have $\int_B f = 0$, as one can show this is true. ♡

Lemma 2.3.22 (Fatou's Lemma). *Let $f_n : A \to \mathbb{R}$ be a sequence of measurable, non-negative functions. If $f_n \to f$ pointwise, then*

$$\int_A f \leq \liminf \int_A f_n$$

Proof. Let $0 \leq h \leq f$ be a BF function, let $A_0 := \operatorname{Supp} h$. It suffices to show

$$\int_A h \leq \liminf \int_A f_n$$

Since h is BF, we see $m(A_0) < \infty$. For each $n \in \mathbb{N}$, let $h_n := \min\{h, f_n\}$, we see h_n is measurable. Note h, f_n are non-negative, we have

$$0 \leq h_n \leq h \leq M$$

for some $M > 0$ because h is bounded for all $n \in \mathbb{N}$. Also, note for $x \in A_0$ and $n \in \mathbb{N}$, we have $h_n(x) = h(x)$ or $h_n(x) = f_n(x) \leq h(x)$, and so

$$0 \leq h(x) - h_n(x) = h(x) - f_n(x) \leq f(x) - f_n(x) \to 0$$

In other word, we have $h_n \to h$ pointwise on A_0. Then by BCT, we have

$$\lim_{n\to\infty} \int_{A_0} h_n = \int_{A_0} h \Rightarrow \lim_{n\to\infty} \int_A h_n = \int_A h$$

Since $h_n \leq f_n$ on A, we have

$$\int_A h = \lim_{n\to\infty} \int_A h_n = \liminf_{n\to\infty} \int_A h_n \leq \liminf_{n\to\infty} \int_A f_n$$

Thus the proof follows. ♡

Example 2.3.23. Let $A = (0, 1]$ and $f_n = n\chi_{(0,\frac{1}{n})}$. Then $f_n \to 0$ pointwise, with $\int_A 0 = 0$ and $\int_A f_n = nm(0, \frac{1}{n}) = 1$, i.e. we get strict inequality.

Theorem 2.3.24 (Monotone Convergence Theorem)**.** *Let $f_n : A \to \mathbb{R}$ be a sequence of measurable functions. If f_n is increasing, and $f_n \to f$ pointwise, then*

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof. By Fatou's lemma, we have

$$\int_A f \leq \liminf \int_A f_n \leq \limsup \int_A f_n$$

However, note since f_n is increasing, we get

$$\limsup \int_A f_n \leq \int_A f$$

In other word, we must have

$$\liminf \int_A f_n = \limsup \int_A f_n \Rightarrow \int_A f = \lim_{n\to\infty} \int_A f_n$$

♡

Remark 2.3.25. If $\phi : A \to \mathbb{R}$ is simple and $m(A) < \infty$, then

$$\int_A \phi < \infty$$

On the other hand, if $f : A \to \mathbb{R}$ is bounded and measurable and $m(A) < \infty$, then

$$\int_A f < \infty$$

Definition 2.3.26. If $f : A \to \mathbb{R}$ is measurable and $f \geq 0$, then we say f is ***integrable*** iff $\int_A f < \infty$.

Definition 2.3.27. Let $f : A \to \mathbb{R}$ be measurable, then we define $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$.

Remark 2.3.28. Clearly $f^+ + f^- = |f|$, $f^+ - f^- = f$ and f^+, f^- are measurable.

Proposition 2.3.29. *Let $f : A \to \mathbb{R}$ be measurable, then f^+, f^- are integrable iff $|f|$ is integrable.*

Proof. $\Rightarrow$: $|f| = f^+ + f^-$ and hence we see

$$\int_A |f| = \int_A f^+ + \int_A f^-$$

and since both terms on the right are finite, we get $\int_A |f|$ is finite and hence integrable.

$\Leftarrow$: $\int_A f^+ \leq \int_A |f| < \infty$ and $\int_A f^- \leq \int_A |f| < \infty$, thus both are integrable as desired. ♡

Definition 2.3.30. We say measurable function $f : A \to \mathbb{R}$ is ***integrable*** iff $|f|$ is integrable iff f^+ and f^- are integrable and define

$$\int_A f = \int_A f^+ - \int_A f^-$$

Proposition 2.3.31 (Comparison Test)**.** *Let $f : A \to \mathbb{R}$ be measurable, $g : A \to \mathbb{R}$ non-negative integrable. If $|f| \leq g$ then f is integrable and*

$$|\int_A f| \leq \int_A |f|$$

Proof. Note $\int_A |f| \leq \int_A g < \infty$ and hence $\int_A |f| < \infty$. Also observe

$$|\int_A f| = |\int_A f^+ - \int_A f^-| \leq \int_A f^+ + \int_A f^- = \int_A (f^+ + f^-) = \int_A |f|$$

This concludes the proof. ♡

Proposition 2.3.32. *Let $f, g : A \to \mathbb{R}$ be integrable, then:*

1. *$\forall \alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is integrable and $\int_A \alpha f + \beta g = \alpha \int_A f + \beta \int_A g$.*
2. *If $f \leq g$ then $\int_A f \leq \int_A g$.*
3. *If $B, C \subseteq A$ are measurable with $B \cap C = \emptyset$ then*

$$\int_{B \cup C} f = \int_B f + \int_C f$$

Theorem 2.3.33 (Lebesgue Dominated Convergence Theorem)**.** *Let $f_n : A \to \mathbb{R}$ be measurable with $f_n \to f$ pointwise. If there exists an integrable $g : A \to \mathbb{R}$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$, then f is integrable and $\lim_{n\to\infty} \int_A f_n = \int_A f$.*

Proof. Since $f_n \to f$ pointwise and $|f_n| \leq g$, we see $|f| \leq g$. Thus by comparison test, we get f is integrable.

Next, observe $g - f \geq 0$ and by Fatou, we see

$$\int_A g - \int_A f = \int_A g - f \leq \liminf_{n\to\infty} \int_A g - f_n = \int_A g - \limsup_{n\to\infty} \int_A f_n$$

Therefore, we see

$$\limsup \int_A f_n \leq \int_A f$$

With similar argument, we see

$$\int_A g + \int_A f = \int_A f + g \leq \liminf \int_A g + f_n = \int_A g + \liminf \int_A f_n$$

and hence

$$\int_A f \leq \liminf \int_A f_n$$

This completes the proof as we get $\int_A f = \liminf \int_A f_n = \limsup \int_A f_n$. ♡

2.4 Riemann Integration

Definition 2.4.1. A ***partition*** of $[a, b]$ is a finite set $P = \{x_0, ..., x_n\} \subseteq [a, b]$ such that $a = x_0 < x_1 < ... < x_n = b$.

Definition 2.4.2. Given partition P of $[a, b]$ and $f : [a, b] \to \mathbb{R}$, we define the ***lower Darboux sum***

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

with $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ and ***upper Darboux sum***

$$U(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

with $m_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$.

Definition 2.4.3. Let $f : [a, b] \to \mathbb{R}$ be bounded, we define the ***lower Riemann integral***

$$L\int_a^b f = \sup\{L(f, P) : P \text{ is a partition}\}$$

and ***upper Riemann integral***

$$U\int_a^b f = \inf\{U(f, P) : P \text{ is a partition}\}$$

and we say f is ***Riemann integrable*** iff

$$L\int_a^b f = U\int_a^b f$$

and in this case we write

$$R\int_a^b f = L\int_a^b f = U\int_a^b f$$

Definition 2.4.4. Let $I_1, ..., I_n$ be pairwise disjoint intervals such that $[a, b] = \bigcup_{i=1}^n I_i$. A ***step function*** is a function of the form

$$f = \sum_{i=1}^n a_i \chi_{I_i}$$

for some $a_i \in \mathbb{R}$.

Remark 2.4.5. Say $f : [a, b] \to \mathbb{R}$ be bounded, $a = x_0 < x_1 < ... < x_n = b$. Let $I_i = [x_{i-1}, x_i)$ for $i = 1, 2, ..., n - 1$ and $I_n = [x_{n-1}, x_n]$. Then

$$L(f, P) = \sum_{i=1}^n m_i \ell(I_i) = R\int_a^b \phi$$

where $\phi(x) = m_i$ on I_i. Similarly we get

$$U(f, P) = R\int_a^b \psi$$

where $\psi = M_i = \sup\{f(x) : x \in I_i\}$ on I_i and we see $f \leq \psi$.

Thus we see

$$U\int_a^b f = \sup\{L(f, P)\} = \sup\{R\int_a^b \phi : \phi \leq f \text{ is step function}\}$$

$$L\int_a^b f = \inf\{U(f, P)\} = \inf\{R\int_a^b \psi : f \leq \psi \text{ is step function}\}$$

Definition 2.4.6. Let $x \in [a, b]$ and $\delta > 0$, we define:

1. $m_\delta(x) = \inf\{f(x) : x \in (x - \delta, x + \delta) \cap [a, b]\}$
2. $M_\delta(x) = \sup\{f(x) : x \in (x - \delta, x + \delta) \cap [a, b]\}$

3. ***Lower boundary*** of f: $m(x) = \lim_{\delta\to 0} m_\delta(x)$.
4. ***Upper boundary*** of f: $M(x) = \lim_{\delta\to 0} M_\delta(x)$.
5. ***Oscillation*** of f:$\omega(x) = M(x) - m(x)$.

Remark 2.4.7. Now we note, say $f : [a, b] \to \mathbb{R}$, then we have the following are equivalent:

1. f is continuous at $x \in [a, b]$.
2. $M(x) = m(x)$.
3. $\omega(x) = 0$.

Lemma 2.4.8. *Let $f : [a, b] \to \mathbb{R}$ be bounded, then*

1. *m is measurable.*
2. *If $\phi : [a, b] \to \mathbb{R}$ is step function with $\phi \le f$ then $\phi(x) \le m(x)$ for all points of continuity of ϕ.*
3. *$L\int_a^b f = \int_{[a,b]} m$.*

Proof. See Appendix. ♡

Lemma 2.4.9. *Let $f : [a, b] \to \mathbb{R}$ be bounded, then*

1. *M is measurable.*
2. *If $\psi : [a, b] \to \mathbb{R}$ is step function with $f \le \psi$ then $\psi(x) \ge M(x)$ for all points of continuity of ϕ.*
3. *$U\int_a^b f = \int_{[a,b]} M$.*

Theorem 2.4.10 (Lebesgue)**.** *Let $f : [a, b] \to \mathbb{R}$ be bounded, then f is Riemann integrable iff f is continuous a.e. In that case,*

$$R\int_a^b f = \int_{[a,b]} f$$

Proof. Note $L\int_a^b f = \int_{[a,b]} m \le \int_{[a,b]} M = U\int_a^b f$. Thus f is Riemann integrable iff $\int_{[a,b]} m = \int_{[a,b]} = M$ iff $\int_{[a,b]}(M - m) = 0$ with $M - m \ge 0$ iff $M = m$ almost everywhere iff f is continuous a.e.

If f is continuous a.e., then we see f is measurable and hence $L\int_a^b f = \int_{[a,b]} m \le \int_{[a,b]} f \le \int_{[a,b]} M = U\int_a^b f$. Thus $R\int_a^b = \int_{[a,b]} f$ as desired. This concludes the proof. ♡

Example 2.4.11. Let $f : [0, 1] \to \mathbb{R}$ be $\chi_{\mathbb{Q}}$. Then f is discontinuous on $[0, 1]$ and hence f is not Riemann integrable. However, $f = 0$ a.e. and so $\int_{[0,1]} f = \int_{[0,1]} 0 = 0$.

Example 2.4.12. Let $\mathbb{Q} \cap [0, 1] = \{q_1, q_2, \ldots\}$. Let $f_n = \chi_{\{q_1,\ldots,q_n\}}$, then $f_n \to f$ pointwise where $f = \chi_{\mathbb{Q}}$. We see (f_n) is increasing and $f_n \le 1$ are Riemann integrable. However, we have

$$R\int_0^1 f_n \not\to R\int_0^1 f$$

In other word, we do not have MCT, BCT or DCT.

2.5 L^p Spaces

Remark 2.5.1. Recall for $1 \le p < \infty$, we see $(C([a,b]), \|\cdot\|_p)$ is a normed vector space where

$$\|f\|_p^p = \int_a^b |f|^p$$

Also recall for $p = \infty$ we can define the uniform norm

$$\|f\|_\infty = \sup\{|f(X)| : x \in [a,b]\}$$

and make $(C([a,b]), \|\cdot\|_\infty)$ is a Banach space.

Remark 2.5.2. Let $A \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$, then $\|f\|_P = \left(\int_A |f|^p\right)^{1/p}$ is not a norm on the vector space of integrable functions $f : A \to \mathbb{R}$ because $\int_A |f|^p = 0$ iff f is zero almost everywhere.

Definition 2.5.3. Let $A \subseteq \mathbb{R}$ be measurable, we define

1. $M(A) := \{f : A \to \mathbb{R} \text{ measurable}\}$
2. On $M(A)$ we define equivalence relation $\sim$ given by $f \sim g$ iff $f = g$ almost everywhere.
3. Then we see $M(A)/\sim$ is a vector space.

Remark 2.5.4. If $f \sim g$ and f is integrable, then g is integrable and $\int_A f = \int_A g$.

Definition 2.5.5. Let $A \subseteq \mathbb{R}$, $1 \le p < \infty$, we define

$$L^p(A) := \{f \in M(A)/\sim : \int_A |f|^p < \infty\}$$

Remark 2.5.6. Suppose $f, g \in L^p(A)$, then $\int_A |f|^p, \int_A |g|^p$ are less than infinity. In particular,

1. $|f+g|^p \le (|f|+|g|)^p \le (2\max\{|f|,|g|\})^p \le 2^p(|f|^p + |g|^p)$. Thus $|f+g|^p < \infty$ and $f + g \in L^p(A)$ as well.
2. In other word, $L^p(A)$ is a subspace of $M(A)/\sim$.

Definition 2.5.7. Let $A \subseteq \mathbb{R}$ be measurable, we define

$$L^\infty(A) = \{f \in M(A)/\sim : f \text{ bounded a.e.}\}$$

Remark 2.5.8. Note $[f], [g] \in L^\infty(A)$ with $|f| \le M$ off $B \subseteq A$ with $m(B) = 0$ and similarly $|g| \le N$ off $C \subseteq A$ with $m(C) = 0$. Thus we see for $x \notin B \cup C$, we have

$$|f(x) + g(x)| \le M + N$$

which concludes $L^\infty(A)$ is a subspace of $M(A)/\sim$.

Proposition 2.5.9. *Let $A \subseteq \mathbb{R}$ be measurable, then*

$$\|[f]\|_\infty = \inf\{M \ge 0 : |f| \le M \ a.e.\}$$

is a norm on $L^\infty(A)$.

Proof. For all $n \in \mathbb{N}$, we have

$$|f| \leq \|f\|_\infty + \frac{1}{n}$$

off $m(A_n) = 0$, i.e. $|f| \leq \|f\|_\infty + \frac{1}{n}$ on $A \backslash A_n$. Then we see

$$B = \bigcup_{n \geq 1} A_n$$

has measure zero, i.e. $|f| \leq \|f\|_\infty$ off B.

Next, we see $\|f\|_\infty = 0$ implies $|f| \leq \|f\|_\infty$ a.e. and hence $|f| = 0$ a.e. and so $f = 0$ a.e. In other word, we get $f = 0$ a.e. in $L^\infty(A)$. This shows one definition of norm.

To show triangle inequality, we see $|f| \leq \|f\|_\infty$ off B and $|g| \leq \|g\|_\infty$ off C. Then $B \cup C$ still has measure zero and off $B \cup C$ we get

$$|f+g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty$$

However, then by definition of inf we can conclude

$$\|f+g\|_\infty = \|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$$

♡

Example 2.5.10. Let $p = 1$, $A \subseteq \mathbb{R}$ measurable and $f, g \in L^1(A)$ with $|f+g| \leq |f| + |g|$. Then we see

$$\int_A |f+g| \leq \int_A |f| + \int_A |g|$$

and hence

$$\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$$

which concludes the triangle inequality. Thus, we see $\|\cdot\|_1$ is also a norm.

Definition 2.5.11. For $1 < p < \infty$, we define $q = \frac{p}{p-1}$ to be the ***Holder conjugate*** of p. For $p = 1$ or ∞, we define the Holder conjugate q to be unique element in the set $\{1, \infty\} \backslash \{p\}$.

Remark 2.5.12. Note $q = \frac{p}{p-1}$ iff $p = \frac{q}{q-1}$. Also, we see if p is Holder conjugates of q, then

$$\frac{1}{p} + \frac{1}{q} = 1$$

Proposition 2.5.13 (Young's Inequality)**.** *Let $p, q \in (1, \infty)$ be Holder conjugates of each other. For all $a, b > 0$, we have*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. Let $f(x) = \frac{1}{p}x^p + \frac{1}{q} - x$ on $(0, \infty)$. Then $f'(x) = x^{p-1} - 1$. In particular, note $f(1) = \frac{1}{p} + \frac{1}{q} - 1 = 0$. Thus $f(x) \geq 0$ on $(0, \infty)$ and $x \leq \frac{1}{p}x^p + \frac{1}{q}$ for all $x > 0$. Now taking $x = \frac{a}{b^{q-1}}$, we get

$$\frac{a}{b^{q-1}} \leq \frac{1}{p} \cdot \frac{a^p}{b^{(q-1)p}} + \frac{1}{q}$$

This is exactly

$$\frac{a}{b^{q-1}} \leq \frac{1}{p}\frac{a^p}{b^q} + \frac{1}{q}$$

Thus we get

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

♡

Proposition 2.5.14 (Holder's Inequality). *Let $A \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, q being Holder conjugate of p. If $f \in L^p(A)$ and $q \in L^q(A)$. Then*

$$fg \in L^1(A)$$

and $\int_A |fg| \leq \|f\|_p \|g\|_q$.

Proof. For $p = 1$ we have $q = \infty$. Then $|fg| = |f| \cdot |g| \leq |f| \cdot \|g\|_\infty$ a.e. Thus $fg \in L^1(A)$ by comparison and so

$$\int_A |fg| \leq \int_A |f| \cdot \|g\|_\infty = \|g\|_\infty \|f\|_1$$

Now let $1 < p < \infty$, then $q = \frac{p}{p-1}$. Then we see

$$|fg| = |f| \cdot |g| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q}$$

We see $|fg|$ is integrable by comparison and so $fg \in L^1(A)$. Also,

$$\int_A |fg| \leq \frac{1}{p}\int_A |f|^p + \frac{1}{q}\int_A |g|^q = \frac{1}{p}\|f\|_p^p + \frac{1}{q}\|g\|_q^q$$

If $\|f\|_q = \|g\|_q = 1$, then we get

$$\int_A |fg| \leq \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \cdot \|g\|_q$$

Now consider $\frac{f}{\|f\|_p}$ and $\frac{g}{\|g\|_q}$, and by the above case, we get

$$\frac{1}{\|f\|_p \|g\|_q}\int_A |fg| \leq 1$$

Hence

$$\int_A |fg| \leq \|f\|_p \|g\|_q$$

and the proof follows. ♡

Lemma 2.5.15. *Let p, q be Holder conjugates, $f \in L^p(A)$. If $f \neq 0$, then*

$$f^* = \|f\|_p^{1-p} \operatorname{sgn}(f)|f|^{p-1}$$

is in $L^q(A)$ where $\operatorname{sgn}(x) = 1$ if $x \geq 0$ and -1 if $x < 0$ and

$$\int_A f f^* = \|f\|_p , \|f^*\|_q = 1$$

Proof. Let $p = 1$, then $q = \infty$. We note $f^* = \operatorname{sgn}(f)$ in our case, which is bounded a.e. In particular, we see

$$\int_A f f^* = \int_A |f| = \|f\|_1$$

and

$$\|f^*\|_\infty = 1$$

which concludes the proof.

Let $1 < p < \infty$ and q is the Holder conjugate. Note

$$\int_A f f^* = \|f\|_p^{1-p} \int_A |f|^p = \|f\|_p^{1-p} \|f\|_p^p = \|f\|_p$$

Now consider

$$\|f^*\|_q^q = \|f\|_p^{(1-p)q} \int_A |f|^{(p-1)q} = \|f\|_p^{-p} \int_A |f|^p = \|f\|_p^{-p} \|f\|_p^p = 1$$

♡

Theorem 2.5.16 (Minkowski's Inequality). *Let $A \subseteq \mathbb{R}$ be measurable, $1 \leq p < \infty$, if $f, g \in L^p(A)$, then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. If $p = 1$ then we are done. Now say $1 < p < \infty$, then we see

$$\|f + g\|_p = \int_A (f + g)(f + g)^* = \int_A f(f + g)^* + \int_A g(f + g)^*$$

Thus by Holder inequality we get

$$\int_A f(f + g)^* + \int_A g(f + g)^* \leq \|f\|_p \|(f + g)^*\|_q + \|g\|_p \|(f + g)^*\|_q = \|f\|_p + \|g\|_p$$

and the proof follows. ♡

Theorem 2.5.17 (Riesz-Fisher). *For every $A \subseteq \mathbb{R}$ and $1 \leq p \leq \infty$, $L^p(A)$ is a Banach space.*

Proof. The case $p = \infty$ is left as an exercise.

Next, let $1 \le p < \infty$. Let $(f_n) \subseteq L^p(A)$ be strongly Cauchy, we will show it converges and the proof follows. In particular, there exists $\epsilon_n \in \mathbb{R}$ so $\|f_{n+1} - f_n\|_p \le \epsilon_n^2$ and $\sum \epsilon_n < \infty$.

The idea of the proof is that, since $\mathbb{R}$ is complete, if $(f_n(x))$ is strongly-Cauchy then it converges. Thus, for each $n \in \mathbb{N}$, we define

$$A_n := \{x \in A : |f_{n+1}(x) - f_n(x) \ge \epsilon_n\} = \{x \in A : |f_{n+1}(x) - f_n(x)|^p \ge \epsilon_n^p\}$$

Then by Chebychev, we get

$$m(A_n) \le \frac{1}{\epsilon_n^p} \int_A |f_{n+1} - f_n|^p \le \frac{1}{\epsilon_n^p} \epsilon_n^{2p} = \epsilon_n^p$$

Thus we see

$$\sum m(A_n) \le \sum_{\epsilon_n^p} \le (\sum \epsilon_n)^p < \infty$$

Hence we get

$$m(\limsup A_n) = 0$$

by Assignment 1. Fix $x \notin \limsup A_n$, let $N = \max\{n : x \in A_n\}$, for $n > N$, we have

$$|f_{n+1}(x) - f_n(x)| < \epsilon_n^2$$

with $\sum \epsilon_n < \infty$. Then we see $(f_n(x))$ is Cauchy and hence we can define $f(x) := \lim_n f_n(x)$ and we have $f_n(x) \to f(x) \in \mathbb{R}$ and we get $f_n \to f$ pointwise a.e.

For $k \in \mathbb{N}$, we have

$$\|f_{n+k} + f_n\|_p \le \|f_{n+k} - f_{n+k-1}\|_p + \dots + \|f_{n+1} - f_n\|_p \le \epsilon_{n+k-1}^2 + \dots + \epsilon_n^2 \le \sum_{i=n}^{\infty} \epsilon_i^2$$

In other words, $|f_{n+k} - f_n|^p$ converges pointwise a.e. to $|f_n - f|^p$ as $k \to \infty$.

By Fatou, we have

$$\int_A |f_n - f|^p \le \liminf_{k\to\infty} \int_A |f_{n+k} - f_n|^p = \liminf_{k\to\infty} \|f_{n+k} - f_n\|_p^p \le \sum_{i=n}^{\infty} \epsilon_i^2 \to 0$$

♡

Definition 2.5.18. A matric space X is ***separable*** if it has a countable dense.subset.

Example 2.5.19. Say $p = \infty$, then say $\{f_n : n \in \mathbb{N}\}$ is dense in $L^\infty[0,1]$. For every $x \in [0,1]$, we may find $\left\|\chi_{[0,x]} - f_{\theta(x)}\right\|_\infty < \frac{1}{z} \in L^\infty$. For $x \ne y$, we have $\left\|\chi_{[0,x]} - \chi_{[0,y]}\right\| = 1$. Thus $\theta(x) \ne \theta(y)$, i.e. $\theta[0,1] \to \mathbb{N}$ is injective. This is a contradiction as it implies $[0,1]$ is countable.

Definition 2.5.20. We define $Simp(A)$ be the set of simple functions on measurable set A, $step[a,b]$ the set of step functions on $[a,b]$ and $step_{\mathbb{Q}}[a,b]$ be the set of step functions on $[a,b]$ with rational partition and rational function values. The last set is going to be countable.

Proposition 2.5.21. *Let $A \subseteq \mathbb{R}$ be measurable, $1 \le p < \infty$, then simp(A) is dense in $L^p(A)$.*

Proof. Let $f \in L^p(A)$, we have f is measurable. In particular, there exists ϕ_n simple, so that $\phi_n \to f$ pointwise and $|\phi_n| \le |f|$. Thus $|\phi_n|^p \le |f|^p$ and so by comparison, $|\phi_n|$ are integrable and so $(\phi_n) \subseteq L^p(A)$.

Note, $\|\phi_n - f\|_p^p = \int_A |\phi_n - f|^p$ and we have

$$|\phi_n - f|^p \le 2^p(|\phi_n|^p + |f|^p) \le 2^{p+1}|f|^p$$

which is integrable. Thus by Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_A |\phi_n - f|^p = \int_A 0 = 0$$

♡

Remark 2.5.22. Note the above proposition is true for $p = \infty$.

Proposition 2.5.23. *Let $1 \le p < \infty$, then Step$[a,b]$ is dense in $L^p[a,b]$.*

Proof. Note we can find simple function arbitrarily close to any function, thus we just need to approximate simple functions using step functions. However, simple functions are sum of characteristics functions, we just need to approximate characteristic functions.

Let $A \subseteq [a,b]$ be measurable and $\chi_A : [a,b] \to \mathbb{R}$. By Littlewood 1, we can find disjoint union of interval

$$\bigcup_{i=1}^{n} I_i =: U$$

such that $m(U\Delta A) < \epsilon^p$.

Then $\chi_U : [a,b] \to \mathbb{R}$ is a step function and

$$\|\chi_U - \chi_A\|_p^p = \int_A |\chi_U - \chi_A|^p = m(U\Delta A)$$

Then we get

$$\|\chi_U - \chi_A\|_p < \epsilon$$

and the proof follows. ♡

Corollary 2.5.23.1. *Let $1 \le p < \infty$, then Step$_{\mathbb{Q}}[a,b]$ is dense in $L^p[a,b]$.*

Proof. Clearly we can use density of rationals to approximate arbitrary step functions. ♡

Corollary 2.5.23.2. *$L^p[a,b]$ is separable.*

Proof. Immediate. ♡

Proposition 2.5.24. *Let $1 \leq p < \infty$, then $L^p(\mathbb{R})$ is separable.*

Proof. Let F_n be the set of functions in $L^p(\mathbb{R})$ such that $f|_{-n,n} \in Step_{\mathbb{Q}}[-n,n]$ and $f|_{\mathbb{R}\setminus[-n,n]} = 0$. Then let $F = \bigcup_{n=1}^{\infty} F_n$ be countable. Take $f \in L^p(\mathbb{R})$, fix $n \in \mathbb{N}$. THen $f|_{[-n,n]} \in L^p([-n,n])$. We show $f\chi_{[-n,n]} \to f$ in $L^p(\mathbb{R})$.

Note

$$\left\|f\chi_{[-n,n]} - f\right\|_p^p = \int_{\mathbb{R}} |f\chi_{[-n,n]} - f|^p = \int_{\mathbb{R}\setminus[-n,n]} |f|^p = \int_{\mathbb{R}} |f|^p \chi_{\mathbb{R}\setminus[-n,n]}$$

However, we also have

$$\left||f|^p \chi_{\mathbb{R}\setminus[-n,n]}\right| \leq |f|^p$$

Now by Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \left\|f\chi_{[-n,n]} - f\right\|_p^p = \lim_{n\to\infty} \int_{\mathbb{R}} |f\chi_{[-n,n]} - f|^p = \int_{\mathbb{R}} 0 = 0$$

Therefore, for each $n \in \mathbb{N}$, we can find $\phi_n \in F$ such that $\left\|f\chi_{[-n,n]} - \phi_n\right\|_p < \frac{1}{n}$ and so $\|f_n - f\|_p \to 0$. ♡

Theorem 2.5.25. *Let A be measurable, and $1 \leq p < \infty$, then $L^p(A)$ is separable.*

Proof. Let F as before, $\{f|_A : f \in F\}$ is a countable dense subset of $L^p(A)$. Thus the proof follows. ♡

2.6 Hilbert Spaces

Definition 2.6.1. Let V be a vector space over $\mathbb{F}$, an ***inner product*** on V is a map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{F}$ such that:

1. For all $v \in V$, $\langle v, v\rangle \in [0,\infty) \subseteq \mathbb{R}$.
2. For all $v, w \in V$, $\langle v, w\rangle = \overline{\langle w, v\rangle}$.
3. FOr all $\alpha \in \mathbb{F}$, $u, v, w \in V$, we have

$$\langle \alpha u + v, w\rangle = \alpha\langle u, w\rangle + \langle v, w\rangle$$

We call $(V, \langle\cdot,\cdot\rangle)$ is an ***inner product space***.

Proposition 2.6.2. *Let V be IPS, then $\|v\| = \sqrt{\langle v, v\rangle}$ is a norm on V, which is called induced norm.*

Example 2.6.3. Let $A \subseteq \mathbb{R}$, let $V = L^2(A)$. Then

$$\langle f, g\rangle = \int_A fg$$

is an IPS. In this case, the norm is given by

$$\sqrt{\langle f, f\rangle} = \left(\int_A |f|^2\right)^{1/2} = \|f\|_2$$

Example 2.6.4. Let $A \subseteq \mathbb{R}$ be measurable, let $V = L^2(A, \mathbb{C})$. Then define

$$\langle f, g \rangle = \int_A f\overline{g}$$

then we have $\sqrt{\langle f, f \rangle} = \|f\|_2$.

Proposition 2.6.5 (Parallelogram Law). *Let V be IPS, for all $u, v \in V$, we have*

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

Proof. Just expand and proof follows. Indeed,

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v \rangle + \langle u-v, u-v \rangle \\
&= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle + \langle u, u \rangle \\
&\quad - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle \\
&= 2(\langle u, u \rangle + \langle v, v \rangle)
\end{aligned}$$

and the proof follows. ♡

Example 2.6.6. Let $1 \leq p < \infty$, let $V = L^p[0, 2]$. Let $f = \chi_{[0,1]}$ and $g = \chi_{[1,2]}$. Then we have $\|f\|_p^2 = 1$ and $\|g\|_p^2 = 1$. On the other hand, we have $\|f+g\|_p^2 = 2^{2/p}$ and $\|f-g\|_p^2 = 2^{2/p}$. In other word, we see by the Parallelogram law, we have $2^{2/p} + 2^{2/p} = 2(1+1)$ iff $p = 2$. In other word, $\|\cdot\|_p$ is induced by an inner product iff $p = 2$.

Definition 2.6.7. A ***Hilbert space*** is a complete IPS, i.e. a Banach space whose norm is induced by an inner product.

Definition 2.6.8. Let V be IPS, we say $v, w \in V$ are ***orthogonal*** if $\langle v, w \rangle = 0$.

Example 2.6.9. Let $f, g \in L^2([-\pi, \pi], \mathbb{C})$, with $n \neq m$. Let $f(x) = e^{inx}$ and $g(x) = e^{imx}$. Then we have

$$\begin{aligned}
\langle f, g \rangle &= \int_{[-\pi,\pi]} f\overline{g} \\
&= \int_{-\pi}^{\pi} e^{ix(n-m)} dx \\
&= \int_{-\pi}^{\pi} \cos((n-m)x) dx + i \int_{-\pi}^{\pi} \sin((n-m)x) dx \\
&= \frac{1}{n-m} \sin((n-m)x)|_{-\pi}^{\pi} + \frac{-1}{n-m} \cos((n-m)x)|_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

Definition 2.6.10. Let V be IPS, we say $A \subseteq V$ is ***orthonormal*** if the elements of A are pair-wise orthogonal and $\|v\| = 1$ for all $v \in A$.

Theorem 2.6.11 (Pythagorean Theorem). *Let V be IPS, if $v_1, ..., v_n$ be pairwise orthogonal then*

$$\left\|\sum v_i\right\|^2 = \sum \|v_i\|^2$$

Corollary 2.6.11.1. *Let V be IPS, $\{v_1, ..., v_n\}$ be orthonormal, then*

$$\left\|\sum a_i v_i\right\|^2 = \sum |a_i|^2$$

Example 2.6.12. Consider $L^2([-\pi, \pi], \mathbb{C})$. Let $A = \{\frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z}\}$. Then we see

$$\begin{aligned} \frac{1}{2\pi} \left\|e^{inx}\right\|_2^2 &= \frac{1}{2\pi} \int_{[-\pi,\pi]} e^{inx} e^{-inx} dx \\ &= \frac{1}{2\pi} \int_{[-\pi,\pi]} 1 = 1 \end{aligned}$$

This shows A is orthonormal.

Definition 2.6.13. Let V be IPS. An ***orthonormal basis*** is a maximal (w.r.t. $\subseteq$) orthonormal subset of V.

Remark 2.6.14. An IPS always has an orthonormal basis. In addition, we note if H is Hilbert space, if $W \subseteq H$ is closed subspace then there exists a subspace $W^\perp \subseteq H$ such that $H = W \oplus W^\perp$ and $\langle w, z \rangle = 0$ for all $w \in W$ and $z \in W^\perp$.

Theorem 2.6.15. *Let H be Hilbert, then H has a countable orthonormal basis (ONB) if and only if H is separable.*

Proof. Suppose H has a countable ONB, say B. Let $W = \mathrm{Span}(B)$ and we claim $\overline{W} = H$, which will prove our claim. Suppose $\overline{W} \neq H$, since $H = \overline{W} \oplus \overline{W}^\perp$, we may find non-zero $x \in \overline{W}^\perp$. In particular, we may assume $\|x\| = 1$ since $x \neq 0$. However, this means $B \cup \{x\}$ is also orthonormal. This is a contradiction to the maximality of B. Since $\mathrm{Span}(B)$ is dense in H, we see $\mathrm{Span}_{\mathbb{Q}}(B)$ is dense where $\mathrm{Span}_{\mathbb{Q}}(B)$ is countable, i.e. H is separable.

Now suppose H does not have an orthonormal basis which is countable. Let B be ONB for H with B uncountable. For $u \neq v \in B$, consider

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 = 2$$

This means for any distinct elements in B, they have distance $\sqrt{2}$. This means H is not separable. Indeed, suppose $X \subseteq H$ such that $\overline{X} = H$. For every $u \in B$, there exists $x_u \in X$ such that $\|u - x_u\| < \frac{\sqrt{2}}{2}$. However, for $u \neq v \in B$, we have $x_u \neq x_v$ by triangle inequality. Thus we see $\phi : B \to X$ given by $\phi(u) = x_u$ and hence X is uncountable as well. In other word, H is not separable as desired. ♡

Example 2.6.16. We see $A = \{\frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z}\} \subseteq L^2([-\pi, \pi], \mathbb{C})$ is countable and orthonormal, but is it maximal?

Remark 2.6.17. Let H be IPS, let $\{v_1, ..., v_n\}$ be orthonormal, if $v = \sum a_i v_i$, then $\lambda_i = \langle v, v_i \rangle$. We call $\langle v, v_i \rangle$ the ***Fourier coefficient*** of v w.r.t. $\{v_1, ..., v_n\}$.

Now let H be a separable Hilbert space and $\{v_1, v_2, \dots\}$ be countable orthonormal basis. Then for any $v \in H$, we define the ***Fourier series*** of v related to

$\{v_1, v_2, ...\}$ as

$$\sum_{i=1}^{\infty} \langle v, v_i \rangle v_i$$

In this case, we write $v \sim \sum_{i\geq 1} \langle v, v_i \rangle v_i$. Does this series converges? Does it converges to v?

Theorem 2.6.18 (Best Approximation). *Let H be Hilbert space, $\{v_1, ..., v_n\}$ be orthonormal. For $v \in H$, then $\|v - \sum \lambda_i v_i\|$ is minimized when $\lambda_i = \langle v, v_i \rangle$. Moreover,*

$$\left\| v - \sum \langle v, v_i \rangle v_i \right\|^2 = \|v\|^2 - \sum |\langle v, v_i \rangle|^2$$

Proof. Note $W = \mathrm{Span}\{v_1, ..., v_n\}$ is closed and $H = W \oplus W^{\perp}$. Let $x \in W$ and $v = w + z$ with $w \in W$, $z \in W^{\perp}$. Then we have

$$\begin{aligned} \|v - x\|^2 = \|w + z - x\|^2 = \|w - x + z\|^2 &= \|w - x\|^2 + \|z\|^2 \\ &\geq \|z\|^2 = \|v - w\|^2 \end{aligned}$$

This means $\|v - x\| \geq \|v - w\|$.

Thus we see $v = \sum \lambda_i v_i + z$ with $z \in W^{\perp}$. Then $\langle v, v_i \rangle = \lambda_i + \langle z, v_i \rangle = \lambda_i$. Thus $\|v\|^2 = \|\sum \langle v, v_i \rangle v_i\|^2 + \|z\|^2 = \sum |\langle v, v_i \rangle|^2 + \|z\|^2$. Thus

$$\left\| v - \sum |\langle v, v_i \rangle| v_i \right\|^2 = \|z\|^2 = \|v\|^2 - \sum |\langle v, v_i \rangle|^2$$

♡

Theorem 2.6.19 (Bessel's Inequality). *Let H be Hilbert space, $\{v_1, ..., v_n\}$ be orthonormal. If $v \in H$, then*

$$\sum_{i=1}^{n} |\langle v, v_i \rangle|^2 \leq \|v\|^2$$

Proof. Just note

$$\|v\|^2 - \sum_{i=1}^{n} |\langle v, v_i \rangle|^2 = \left\| v - \sum \langle v, v_i \rangle v_i \right\|^2 \geq 0$$

♡

Theorem 2.6.20 (Parseval's Identity). *Let H be Hilbert space, $\{v_1, v_2, v_3, \ldots\}$ be orthonormal. For $v \in H$, we have*

$$\sum_{i=1}^{\infty} |\langle v, v_i \rangle|^2 = \|v\|^2$$

if and only if

$$\lim_{n\to\infty} \left\| v - \sum_{i=1}^{n} \langle v, v_i \rangle v_i \right\| = 0$$

Proof. This is just best approximation theorem. ♡

Theorem 2.6.21 (Orthonormal Basis Test)**.** *Let H be separable Hilbert space, $S = \{v_1, v_2, \ldots\}$ orthonormal. Then the following are equivalent:*

1. *$\{v_1, v_2, \ldots\}$ is a basis.*
2. *$\overline{\operatorname{Span} v_1, v_2, \ldots} = H$.*
3. *$\lim_{n\to\infty} \|v - \sum_{i=1}^{n} \langle v, v_i\rangle v_i\| = 0$ for every $v \in H$.*

Proof. (1) $\Rightarrow$ (2) is done.

(2) $\Rightarrow$ (1): If S is not maximal, then we can find $u \in H$ with $\|u\| = 1$ and $\langle u, v_i\rangle = 0$ for all $i \in \mathbb{N}$. Since $C = \{x \in H : \langle x, u\rangle = 0\}$ is closed, we see $u \notin \overline{\operatorname{Span}(S)}$.

(2) $\Rightarrow$ (3): Let $v \in H$ and $\epsilon > 0$ be g iven. Let $\sum_{i=1}^{N} d_i v_i \in \operatorname{Span}(S)$ such that

$$\left\| v - \sum_{i=1}^{N} d_i v_i \right\| < \epsilon$$

Thus $\left\| v - \sum_{i=1}^{N} \langle v, v_i\rangle v_i \right\| < \epsilon$ as this is the best approximation. Hence for all $n \geq N$ we have

$$\left\| v - \sum_{i=1}^{n} \langle v, v_i\rangle v_i \right\| \leq \left\| v - \sum_{i=1}^{N} \langle v, v_i\rangle v_i \right\| + \left\| \sum_{i=N+1}^{n} \langle v, v_i\rangle v_i \right\| < \epsilon + \sqrt{\sum_{i=N+1}^{n} |\langle v, v_i\rangle|^2}$$

where we see $\sqrt{\sum_{i=N+1}^{n} |\langle v, v_i\rangle|^2} \to 0$ as $N \to \infty$.

(3) $\Rightarrow$ (2): similar. ♡

2.7 Fourier Series

Definition 2.7.1. Let $T = [-\pi, \pi)$ be the ***torus*** or the ***circle***. Then we define $L^p(T) = L^p(T, \mathbb{C})$ for $1 \leq p < \infty$. We define norm

$$\|f\|_p = \left(\frac{1}{2n} \int_T |f|^p \right)^{1/p}$$

and then $L^p(T)$ is a separable Banach space.

Remark 2.7.2. As a group under addition modulo 2π, then $T \cong \mathbb{R}/\mathbb{Z} \cong \{z \in \mathbb{C} : |z| = 1\}$. In this way, T is a locally compact abelian group and there is a one-to-one correspondence between $f : T \to \mathbb{C}$ and 2π-periodic functions $f : \mathbb{R} \to \mathbb{C}$.

Definition 2.7.3. Let $f \in L^1(T)$,

1. For $n \in \mathbb{Z}$, we define the nth ***Fourier coefficient*** of f by $\langle f, e^{inx}\rangle = \frac{1}{2n} \int_T f(x) e^{-inx} dx$.

2. We define the ***Fourier series*** of f by

$$f \sim \sum_{n\in\mathbb{Z}} a_n e^{inx}$$

where $a_n = \langle f, e^{inx}\rangle$.
3. We let $S_N(f,x) = \sum_{-N}^{N} a_n e^{inx}$ denote the Nth partial sum of the above Fourier series.

Proposition 2.7.4. *Consider the **trigonometric polynomial** $f \in L^1(T)$ given by*

$$f(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

for some $a_i \in \mathbb{C}$. For each $-N \le n \le N$, we have $\langle f, e^{inx}\rangle = a_n$.

Proof. We note

$$\frac{1}{2\pi}\int_T e^{imx}e^{-inx}dx = \delta_{m,n}$$

and the proof follows. ♡

Remark 2.7.5. Suppose $f \in L^1(T)$ is ***real-valued*** and $f \sim \sum_{n\in\mathbb{Z}} a_n e^{inx}$. Then

$$\begin{aligned}
S_N(f,x) &= \sum_{n=-N}^{N} a_n e^{inx} \\
&= a_0 + \sum_{n=1}^{N}(a_n e^{inx} + a_{-n}e^{-inx}) \\
&= a_0 + \sum_{n=1}^{N}((a_n + a_{-n})\cos(nx) + i(a_n - a_{-n})\sin(nx)) \\
&= a_0 + \sum_{n=1}^{N} b_n \cos(nx) + c_n \sin(nx), \quad b_n := a_n + a_{-n}, c_n := a_n - a_{-n}
\end{aligned}$$

Now, we see

$$a_0 = \frac{1}{2\pi}\int_T f(x)e^{-i0x}dx = \frac{1}{2\pi}\int_T f(x)dx$$

For b_n and c_n, we see

$$b_n = a_n + a_{-n} = \frac{1}{2\pi}\int_T f(x)(e^{-inx} + e^{inx})dx = \frac{1}{\pi}\int_T f(x)\cos(nx)dx$$

$$c_n = i(a_n - a_{-n}) = \frac{i}{2n}\int_T f(x)(e^{-inx} - e^{inx})dx = \frac{1}{\pi}\int_T f(x)\sin(nx)dx$$

Proposition 2.7.6. *Let $f, g \in L^1(T)$,*

1. *$\langle f+g, e^{inx}\rangle = \langle f, e^{inx}\rangle + \langle g, e^{inx}\rangle$*
2. *For $\alpha \in \mathbb{C}$, $\langle \alpha f, e^{inx}\rangle = \alpha\langle f, e^{inx}\rangle$.*

3. *If $\overline{f} : T \to \mathbb{C}$ is defined by $\overline{f}(x) = \overline{f(x)}$, then $\overline{f} \in L^1(T)$ with $\langle \overline{f}, e^{inx} \rangle = \overline{\langle f, e^{-inx} \rangle}$.*

Proof. We see (1) and (2) are immediate since integral is linear. For (3), we note $|f| = |\overline{f}|$ and so $\overline{f} \in L^1(T)$. On the other hand, we see

$$\begin{aligned}
\langle \overline{f}, e^{inx} \rangle &= \frac{1}{2\pi} \int_T \overline{f}(x) e^{-inx} dx = \frac{1}{2\pi} \overline{f(x) e^{inx}} dx \\
&= \frac{1}{2\pi} \int_T \mathrm{Re}(\overline{f(x)e^{inx}}) dx + \frac{i}{2\pi} \int_T \mathrm{Im}(\overline{f(x)e^{inx}}) dx \\
&= \frac{1}{2\pi} \int_T \mathrm{Re}(f(x)e^{inx}) dx - \frac{i}{2\pi} \int_T \mathrm{Im}(f(x)e^{inx}) dx \\
&= \overline{\frac{1}{2\pi} \int_T f(x) e^{inx} dx} \\
&= \overline{\langle f, e^{inx} \rangle}
\end{aligned}$$

♡

Proposition 2.7.7. *Let $f \in L^1(T)$, $\alpha \in \mathbb{R}$. By a previous remark, we may view $f : \mathbb{R} \to \mathbb{C}$ as a 2π-periodic function which is integrable over T. For $\alpha \in \mathbb{R}$, $f_\alpha : \mathbb{R} \to \mathbb{C}$ given by $f_\alpha(x) = f(x - \alpha)$ is integrable over T and $\langle f_\alpha, e^{inx} \rangle = \langle f, e^{inx} \rangle e^{-inx}$.*

Proof. Assignment. ♡

Proposition 2.7.8. *Let $f \in L^1(T)$, for all $n \in \mathbb{Z}$, we have $|\langle f, e^{inx} \rangle| \leq \|f\|_1$.*

Proof. We see

$$|\langle f, e^{inx} \rangle| = |\frac{1}{2\pi} \int_T f(x) e^{-inx} dx| \leq \frac{1}{2\pi} \int_T |f(x) e^{-inx}| dx = \frac{1}{2\pi} \int_T |f(x)| dx = \|f\|$$

♡

Corollary 2.7.8.1. *Say $f_k \to f$ in $L^1(T)$, then for all $n \in \mathbb{Z}$, we have $\langle f_k, e^{inx} \rangle \to \langle f, e^{inx} \rangle$.*

Proof. We see

$$|\langle f_k, e^{inx} \rangle - \langle f, e^{inx} \rangle| = |\langle f_k - f, e^{inx} \rangle| \leq \|f_k - f\|_1 \to 0$$

♡

Remark 2.7.9. Let $Trig(T)$ denote the set of trigonometric polynomials on T, by A3, we have $\overline{Trig(T)} = L^1(T)$.

Theorem 2.7.10 (Riemman-Lebesgue Lemma)**.** *If $f \in L^1(T)$, then*

$$\lim_{|n| \to \infty} \langle f, e^{inx} \rangle = 0$$

Proof. Let $\epsilon > 0$ be given and let $P \in Trig(T)$ such that $\|f-p\|_1 < \epsilon$. Say $p(x) = \sum_{k=-N}^{N} a_k e^{ikx}$. For $n > N$ or $n < -N$, we have $\langle p, e^{inx}\rangle = a_n = 0$ by definition of p. For $|n| > N$, we see

$$|\langle f, e^{inx}\rangle| = |\langle f-p, e^{inx}\rangle| \leq \|f-p\|_1 < \epsilon$$

Thus the proof follows. ♡

Definition 2.7.11. Let $f, g \in L^1(T)$, then the ***convolution*** of f and g is the function $f * g : T \to \mathbb{C}$ given by

$$(f*g)(x) := \frac{1}{2\pi}\int_T f(t)g(x-t)dt = \frac{1}{2\pi}\int_T f(t)g_t(x)dt$$

Remark 2.7.12 (Facts)**.**

1. Given $f, g \in L^1(T)$, then $f * g \in L^1(T)$ as well.
2. $\|f*g\|_1 \leq \|f\|_1 \|g\|_1$.
3. This make $L^1(T)$ a Banach algebra.

Definition 2.7.13. Let $C(T)$ be the set of continuous functions from T to $\mathbb{C}$. Then we define a ***summability kernel*** to eb a sequence $(k_n) \subseteq C(T)$ such that:

1. $\frac{1}{2\pi}\int_T k_n = 1$.
2. $\exists M, \forall n, \|k_n\|_1 \leq M$.
3. For all $0 < \delta < \pi$,
$$\lim_{n\to\infty}\left(\int_{-\pi}^{\delta}|k_n| + \int_{\delta}^{\pi}|k_n|\right) = 0$$

Proposition 2.7.14. *Let $(B, \|\cdot\|_B)$ be a $\mathbb{C}$-Banach space. Let $\phi : T \to \mathbb{C}$ be continuous. Let $(k_n) \subseteq C(T)$ be a summability kernel. Then*

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_T k_n(t)\phi(t)dt = \phi(0)$$

in the B-norm.

Proof. Appendix. ♡

Remark 2.7.15. By A3, $\phi : T \to L^1(T)$ given by $\phi(t) = f_t = f(x-t)$ is continuous.

Theorem 2.7.16. *Let $f \in L^1(T)$, let (k_n) be summability kernel in $L^1(T)$, then*

$$f = \lim_{n\to\infty} k_n * f$$

Proof. We see

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_T k_n(t)\phi(t)dt = \phi(0)$$

where $\phi : T \to L^1$ is given by $\phi(t) = f_t$. Thus we see

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_T k_n(t)f(x-t)dt = f(x)$$

but $\frac{1}{2\pi}\int_T k_n(t)f(x-t)dt = k_n * f$. Thus the proof follows. ♡

Remark 2.7.17. For $f \in L^1(T)$, for $n \in \mathbb{Z}$, consider $\phi_n(x) = e^{inx} \in L^1(T)$. Then

$$\begin{aligned}(\phi_n * f)(x) &= \frac{1}{2\pi}\int_T \phi_n(t) f_t(x)dt \\ &= \frac{1}{2\pi}\int_T e^{int} f(x-t)dt \\ &= \frac{1}{2\pi} e^{inx} \int_T e^{-in(x-t)} f(x-t)dt \\ &= \frac{1}{2\pi} e^{inx} \int_T e^{-in(-t)} f(t)dt \\ &= \frac{1}{2\pi} e^{inx} \int_T e^{-int} f(t)dt \\ &= e^{inx}\langle f, e^{inx}\rangle\end{aligned}$$

In particular, if we consider $p(x) = \sum_{k=-n}^{n} a_k e^{ikx}$, then we get

$$\begin{aligned}(p * f)(x) &= \frac{1}{2\pi}\int_T p(t) f(x-t)dt \\ &= \sum_{k=-n}^{n} \frac{a_n}{2\pi}\int_T e^{ikt} f(x-t)dt \\ &= \sum_{-n\le k\le n} a_n(\phi_k * f)(x) \\ &= \sum_{k=-n}^{n} a_n e^{ikx}\langle f, e^{ikx}\rangle\end{aligned}$$

Definition 2.7.18. Define the ***Dirichlet kernel*** of order n to be

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx}$$

Remark 2.7.19. We observe if $f \in L^1(T)$, we get

$$(D_n * f)(x) = \sum_{k=-n}^{n} e^{ikx}\langle f, e^{ikx}\rangle = S_n(f, x)$$

is equal the nth partial sum of the Fourier series of f. However, we note D_n is ***not*** a summability kernel.

Remark 2.7.20. To fix the Dirichlet kernel, we recall for $x_n \in \mathbb{C}$ with $x_n \to x$, then

$$\frac{\sum_{i=1}^{n} x_i}{n} \to x$$

Thus, we get the following definition.

Definition 2.7.21. Define the ***Fejer kernel*** of order n to be

$$\frac{D_0(x) + D_1(x) + ... + D_n(x)}{n+1}$$

Remark 2.7.22. We see

$$F_0(x) = D_0(x) = 1$$
$$F_1(x) = \frac{e^{-ix} + 2e^{i0x} + e^{ix}}{2}$$
$$F_2(x) = \frac{e^{-i2x} + 2e^{-ix} + 3e^{i0x} + 2e^{ix} + e^{i2x}}{3}$$
$$F_n(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikx}$$

Remark 2.7.23. The sequence (F_n) is a summability kernel. In particular, consider

$$\begin{aligned} F_n * f &= \frac{1}{n+1} \sum_{k=0}^{n} D_k * f \\ &= \frac{1}{n+1} \sum_{k=0}^{n} S_k(f) \\ &= \frac{S_0(f) + S_1(f) + ... + S_n(f)}{n+1} \end{aligned}$$

Definition 2.7.24. We define the nth ***Cesaro mean***

$$\sigma_n(f) := F_n * f = \frac{S_0(f) + ... + S_n(f)}{n+1}$$

Theorem 2.7.25. *Let $f \in L^1(T)$, (F_n) be the Fejer kernel, then*

$$\lim_{n\to\infty} F_n * f = \lim_{n\to\infty} \sigma_n(f) = f$$

in $L^1(T)$.

Remark 2.7.26. If $(S_n(f))$ converges in $L^1(T)$, then $S_n(f) \to f$ in $L^1(T)$.

Theorem 2.7.27 (Fejer's Theorem)**.** *For $f \in L^1(T)$ and $t \in T$, consider*

$$\omega_f(t) := \frac{1}{2} \lim_{x\to 0} (f(t+x) + f(t-x))$$

provided the limit exists. T hen

$$\sigma_n(f,t) \to \omega_f(t)$$

In particular, if f is continuous at t, then

$$\sigma_n(f,t) \to f(t)$$

Remark 2.7.28. In practice,

1. Fix $x \in T$.
2. Prove $(S_n(f,x))$ converges.
3. Then $S_n(f,x) \to \omega_f(x)$.
4. If f is continuous at x, then $S_n(f,x) \to f(x)$, i.e. $S(f,x) = f(x)$.

Example 2.7.29. Let $f \in L^1(T)$ be $|x|$. Then we see

$$S_n(f,x) = a_0 + \sum_{k=1}^{n}(b_k \cos(kx) + c_k \sin(kx))$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| dx = \frac{\pi}{2}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(kx) dx = \frac{2(-1)^k - 2}{k^2\pi}$$

$$c_k = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin(kx) dx = 0$$

Hence we see

$$S_n(f,x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{\frac{n+1}{2}}\left(\frac{(-1)^k - 1}{k^2}\cos(kx)\right) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^{\frac{n+1}{2}}\frac{(-2)}{(2k-1)^2}\cos((2k-1)x)$$

We note $(S_n(f,x))$ converges by comparison text with $\sum \frac{1}{(2k-1)^2}$. Since f is continuous, we see

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos((2k-1)x)}{(2k-1)^2}$$

Taking $x = 0$, we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

In other word, we get

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

On the other hand, we see

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=1}^{\infty}\frac{1}{(2k)^2} + \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} + \frac{\pi^2}{8}$$

In other word, we get

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

Definition 2.7.30. A ***homogeneous Banach space*** is a Banach space $(B, \|\cdot\|_B)$ such that

1. B is a subspace of $L^1(T)$.
2. $\|\cdot\|_1 \le \|\cdot\|_B$.
3. $\forall f \in B, \forall \alpha \in T$, we have $f_\alpha \in B$ and $\|f_\alpha\|_B = \|f\|_B$.
4. $\forall f \in B, \forall t_0 \in T, \lim_{t\to t_0}\|f_t - f_{t_0}\|_B = 0$.

Theorem 2.7.31. *Let B be a homogeneous Banach space, (k_n) a summability kernel. Then for all $f \in B$,*

$$\lim_{n\to\infty} \|k_n * f - f\|_B = 0$$

Proof. We see $\frac{1}{2\pi}\int_T k_n(t) f_t dt = k_n * f$. Also, note by definition we get

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_T k_n(t)\phi(t)dt = \phi(0)$$

for all continuous $\phi : T \to B$. However, note $\phi : T \to B$ given by $\phi(t) = f_t$ is continuous, thus we get

$$\|k_n * f - f\|_B \to 0$$

♡

Remark 2.7.32. Let B be homogeneous Banach space, taking $k_n = F_n$, we have $\|\sigma_n(f) - f\|_B \to 0$ for all $f \in B$. Moreover, take $B = L^p(T)$, we see

$$\|\sigma_n(f) - f\|_P \to 0$$

$$\overline{Trig(T)} = L^P(T)$$

Remark 2.7.33. In $L^2(T)$, we know $\overline{Trig(T)} = L^2(T)$. Thus $\overline{\text{Span}\{e^{inx} : n \in \mathbb{Z}\}} = L^2(T)$ and hence $\{\frac{1}{\sqrt{2\pi}}e^{inx} : n \in \mathbb{Z}\}$ is orthonormal basis.

Now let the above basis be $v_1, v_2, \ldots$, then for all $f \in L^2(T)$, we see

$$\lim_{n\to\infty} \sum_{i=1}^{n} \langle f, v_i\rangle v_i = f$$

If $v = \frac{1}{\sqrt{2\pi}}e^{ikx}$, then

$$\langle f, v\rangle v = \left(\int_T f(x)\frac{1}{\sqrt{2\pi}}e^{-ikx}dx\right)\frac{1}{\sqrt{2\pi}}e^{ikx} = \frac{1}{\sqrt{2\pi}}(2\pi\langle f, e^{ikx}\rangle)\frac{1}{\sqrt{2\pi}}e^{ikx} = \langle f, e^{ikx}e^{ikx}\rangle$$

In particular, for all $f \in L^2(T)$, we get

$$\|S_n(f) - f\|_2 \to 0$$