Contents

Chapter 1

Intro

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休言万事转头空，未转头时皆梦。

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1.1 Intro

Definition 1.1.1.

1. Homework: 15%, weekly. Friday 8am, Crowdmark.
2. Office hours: Wed $12:30-2:30$.
3. Midtern: In class, on 2/10, 25%.
4. Final: 60%.
5. Textbook: Invitation to Algebraic Geometry by Karen Smith.

Remark 1.1.2. Algebraic geometry has connections to

1. complex analysis,
2. differential geometry,
3. number theory,
4. logic,
5. combinatorics

From time to time, we will use facts from Comm algebra facts without proof.

Example 1.1.3 (Pythogorean Triplets)**.** Let a, b, c be positive integers with the property $a^2 + b^2 = c^2$.

Question: Write down all Pythagorean triplets.

If (a, b, c) is a triplet, then (na, nb, nc) is a triplet where $n \in \mathbb{Z}_{\geq 0}$. So, it is enough to write them all down up to scalar by positive integers.

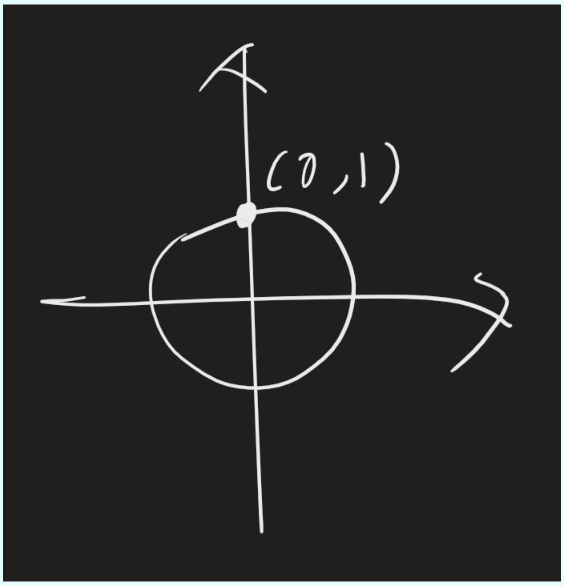
Therefore, we want to write down all positive integer solutions to

$$x^2 + y^2 = z^2 \Rightarrow (\frac{x}{z})^2 + (\frac{y}{z})^2 = 1$$

up to scalars. I.e., it is enough to solve for all $\mathbb{Q}$ solutions to

$$x^2 + y^2 = 1$$

We know what the set of real solutions to $x^2 + y^2 = 1$ looks like:



Next, pick a rational solution, say $(0, 1)$, and consider another solution (x, y). We join the two points by a line ℓ, and we have ℓ has rational slope if and only if $(x, y) \in \mathbb{Q}^2$.

Next, we solve for (x, y), say $x^2 + y^2 = 1$ and the line is $\ell : y - 1 = tx$, $t \in \mathbb{Q}\backslash\{0\}$. Solve for x, we get $x = -\frac{2t}{1+t^2}$ and solve for y we get $y = \frac{1-t^2}{1+t^2}$.

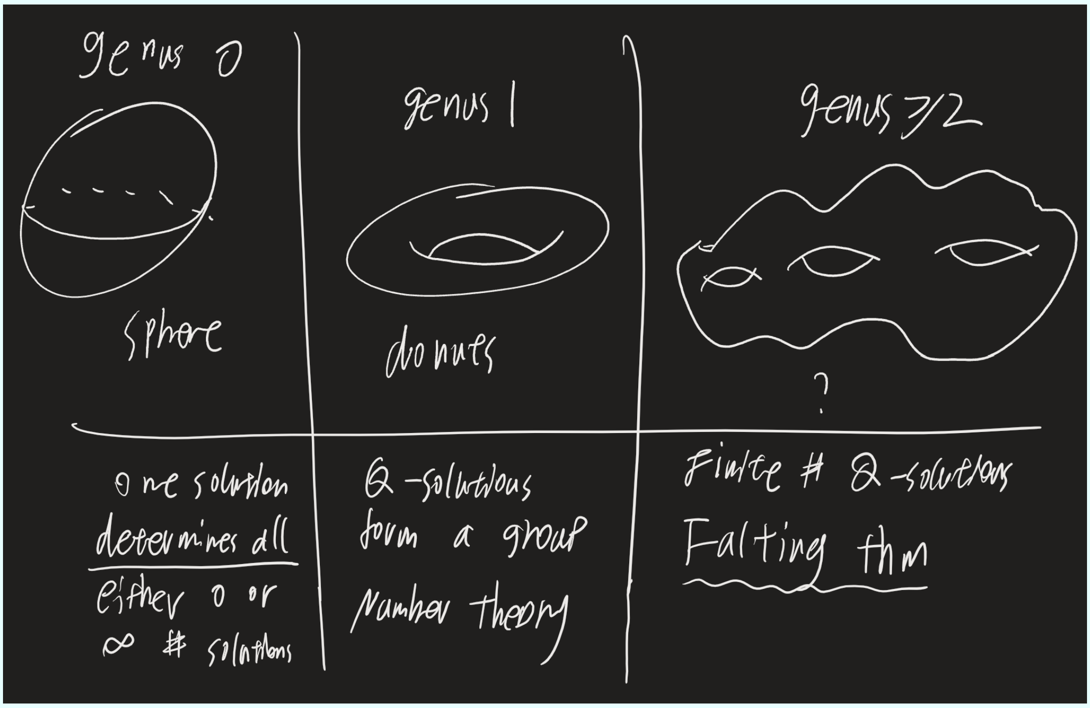
Therefore, up to scalar, all $\mathbb{Q}-$solutions to $x^2 + y^2 = z^2$ are given by

$$(-2t, 1 - t^2, 1 + t^2)$$

Remark 1.1.4 (Bigger Picture)**.** The bigger picture of the above example is that we have some rational equation $f(x, y, z) = 0$ and we want to find all $\mathbb{Q}$ solutions up to scalar.

In our case of $f = x^2 + y^2 - z^2$, one solution $(0, 1, 1)$ determines all others. This is not always the case. Therefore, the question is, when does one $\mathbb{Q}$ solution determines the others?

This is related to complex analysis and differential geometry. In our particular case, the $\mathbb{C}$ solutions to $f(x, y, z) = 0$, up to scalar, and after some module topolgical operations, has only three cases:



This trichotomy also turns out in differential geometry:

1. $g = 0$, then it has positive curvature,
2. $g = 1$, then it has zero curvature,
3. $g \geq 2$, then it has negative curvature.

Definition 1.1.5. Let K be a field, an ***affine algebraic variety*** is the common zero set of a set of polynomials $\{f_i\}_{i\in I}$, $f_i \in K[x_1, ..., x_n]$, i.e.

$$V(\{f_i\}_{i\in I}) := \{(a_1, ..., a_n) \in K^n : \forall i \in I, f_i(a_1, ..., a_n) = 0\}$$

Remark 1.1.6. An affine algebraic variety is sometiems called ***algebraic sets***.

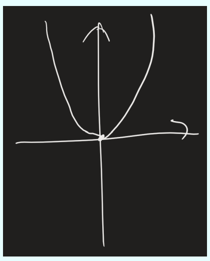
Example 1.1.7.

1. In K^n, $V(\{1\}) = 0$ and $V(\{0\}) = K^n$.
2. In K^2, $V(\{x, y\}) = \{(0, 0)\}$. In K^3, we have $V(\{x, y\}) = \{(0, 0, z) : z \in K\}$.
3. Every point in K^n is an affine algebraic variety. Indeed, $\{(a_1, ..., a_n)\} = V(x_1 - a_1, ..., x_n - a_n)$

Definition 1.1.8. Let $0 \neq f \in K[x, y]$, we say $V(f)$ is an ***affine plane curve***.

Example 1.1.9.

1. $V(y - x^2) \subseteq K^2$ has the following picture



Note we have a convention in algebraic geometry to draw the picture on $\mathbb{R}$.

2. $SL_n(K) = \{\text{n} \times \text{n matrix with determinant equal } 1\}$ is an affine algebraic variety. Indeed, the set of n by n matrices is equal K^{n^2} and hence $SL_n(K) \subseteq K^{n^2}$, where we note $det(A)$ is a polynomial.
3. Fix r. Then $\{\text{n} \times \text{n matrices of rank } < r\}$ is an affine algebraic variety. This is the vanishing of all $r \times r$ minors.

Definition 1.1.10. Given an ideal $I \subseteq k[x_1, ..., x_n]$, we define $V(I) = V(\{f : f \in I\})$.

Lemma 1.1.11. *If $f_i \in k[x_1, ..., x_n]$, then $V(\{f_i\}_{i \in I}) = V(I)$, where I is the ideal generated by $\{f_i : i \in I\}$.*

Proof. Let $a \in V(\{f_i\})$, we need to show $\forall h \in I, h(a) = 0$. By definition, $h = \sum f_i g_i$ where $g_i \in k[x_1, ..., x_n]$ and so $h(a) = \sum f_i(a) g_i(a) = 0$. Viz $V(\{f_i\}) \subseteq V(I)$.

On the other hand, let $a \in V(I)$, then trivially $f_i(a) = 0$ as $f_i \in I$. Viz $V(I) \subseteq V(\{f_i\})$. ♡

Corollary 1.1.11.1. *Let $X \subseteq k^n$ is an affine variety iff there exists $I \leq k[x_1, ..., x_n]$ such that $X = V(I)$, where $I \leq k[x_1, ..., x_n]$ means I is an ideal of $k[x_1, ..., x_n]$.*

Remark 1.1.12. If $I = \langle f_1, ..., f_m \rangle$, then $V(I) = V(\{f_1, ..., f_m\})$.

1.2 Topology on Affine varieties

Remark 1.2.1. As this course is called algebraic geometry, next we are going to introduce some geometries.

We will define the closed sets of k^n to be affine algebraic varieties.

Proposition 1.2.2. *The affine algebraic varieties form the closed sets of a topology of k^n.*

Proof. We have $\emptyset$ is closed since $\emptyset = V(1)$. We have k^n is closed sicne $k^n = V(0)$.

Next, we need to show $\bigcap_{j \in J} V(I_j)$ is closed. More specifically, we show $\bigcap_{j \in J} V(I_j) = V(\sum_{j \in J} I_j)$.

We first show $\bigcap_{j \in J} V(I_j) \subseteq V(\sum_{j \in J} I_j)$. Let $a \in \cap V(I_j)$ and let $f \in \sum I_j$ be arbitrary. Then $f = \sum f_j$ where $f_j \in I_j$, and so $f(a) = \sum f_j(a) = \sum 0 = 0$. Next, let $a \in V(\sum I_j)$, then let $f_j \in I_j$, we have $f_j(a) = 0$ as $f_j \in \sum I_j$. Thus $a \in \cap V(I_j)$. Therefore, we have double inclusion and so $\bigcap_{j \in J} V(I_j) = V(\sum_{j \in J} I_j)$.

Now we show $V(I) \cup V(J)$ is again closed. In particular, $V(I) \cup V(J) = V(IJ)$. First, we show $V(I) \cup V(J) \subseteq V(IJ)$. It is enough to show $V(I) \subseteq V(IJ)$. Let $a \in V(I)$ and $f \in IJ$. We have $f = \sum f_i g_i$ where $f_i \in I, g_i \in J$ and so $f(a) = \sum f_i(a) g_i(a) = \sum 0 \cdot g_i(a) = 0$. Next, we show $V(IJ) \subseteq V(I) \cup V(J)$. Let $a \in V(IJ)$

and assume $a \notin V(I)$. Let $j \in J$ be arbitrary. Since $a \notin V(I)$, there exists $g \in I$ such that $g(a) \neq 0$. Consider $fg \in IJ$, we have $f(a)g(a)$ must be zero but $g(a)$ is not, so $f(a) = 0$, i.e. $a \in V(J)$. Thus $V(IJ) = V(I) \cup V(J)$. ♡

Definition 1.2.3. The topology on k^n where closed sets are affine varieties are called ***Zariski topology***. In particular, we let $\mathbb{A}^n_k = k^n$ with the Zariski topology.

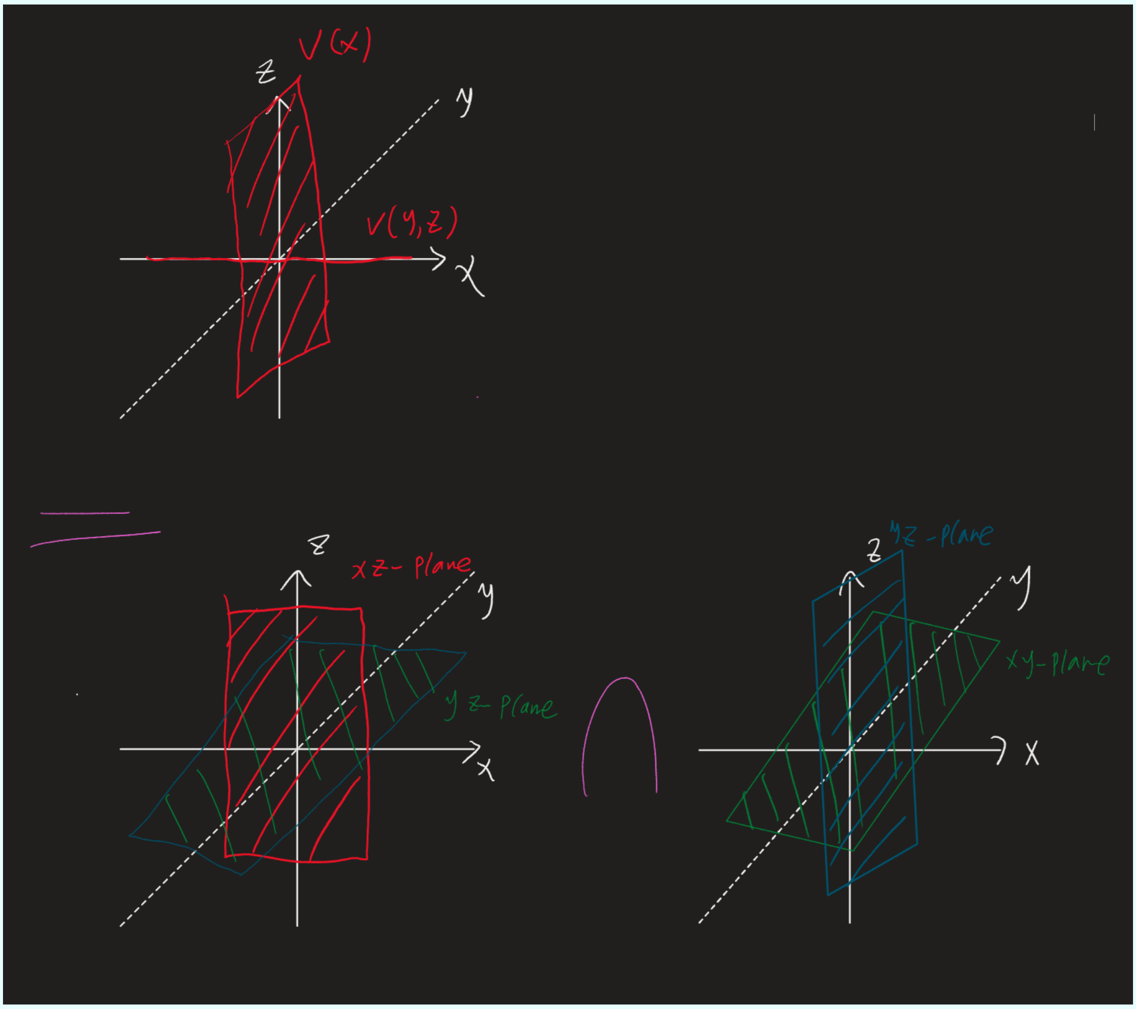
Example 1.2.4. Let k be an algebraic closed field, what is $\mathbb{A}^1_k$ looks like topologically?

Let us write down non-trivial closed sets $V(I)$ where $I \leq k[x]$ (note $k[x]$ is PID, i.e. $I = \langle f(x) \rangle$). Since k is algebraic closed, we have $f(x) = \lambda \prod_{i=1}^{d}(x - r_i)$ where $\lambda \neq 0$. Therefore, $V(I) = \{r_1, ..., r_d\}$.

So, the non-trivial closed sets of $\mathbb{A}^1_k$ are finite union of points. In particular, $\mathbb{A}^1_k$ is ***not*** Hausdorff.

Definition 1.2.5. If X is an affine algebraic variety, then $X \subseteq k^n$. We define the ***Zariski topology on*** X to be the subspace topology $X \subseteq \mathbb{A}^n$.

Example 1.2.6. From the proof of Proposition 1.2.2, we know $V(y,z) \cup V(x) = V((y,z)x) = V(xy, xz)$. Thus, in k^3, we have



Remark 1.2.7. We note affine algebraic varieties are cut out by by polynomials, so natural definition of morphism is polynomial maps.

Definition 1.2.8. A ***morphism*** $F : \mathbb{A}^n_k \to \mathbb{A}^m_k$ is a set map such that, $\exists F_1, ..., F_m \in k[x_1, ..., x_n]$ such that

$$F(a_1, ..., a_n) = (F_1(a_1, ..., a_n), ..., F_m(a_1, ..., a_n))$$

Example 1.2.9. We have $\mathbb{A}^1 \to \mathbb{A}^2$ given by $x \mapsto (x^3 + 7, x^2 + 2x)$ is a morphism.

Definition 1.2.10. A ***morphism between affine algebraic varieties*** $V \subseteq \mathbb{A}^n_k$ to $W \subseteq \mathbb{A}^m_k$ is a set map $F : V \to W$ such that there exists a morphism $G : \mathbb{A}^n_k \to \mathbb{A}^m_k$ such that $F = G|_V$.

Example 1.2.11. Consider $X := V(y - x^2) \subseteq \mathbb{A}^2$, then the map $F : X \to \mathbb{Z}^1$ given by $F(x, y) = x$ is a morphism because it is the restriction of the morphism $\mathbb{A}^2 \to \mathbb{A}$ given by $(x, y) \mapsto x$.

Definition 1.2.12. An ***isomorphism*** between affine algebraic varieties is a morphism $F : V \to W$ such that there exists morphism $G : W \to V$ with $G \circ F = Id_V$ and $F \circ G = Id_W$.

Example 1.2.13. The map $F : V(y - x^2) \to \mathbb{A}^1$ given by $(x, y) \mapsto x$ is an isomorphism because $G : \mathbb{A}^1 \to V(y - x^2)$ given by $x \mapsto (x, x^2)$ is the inverse of F.

Example 1.2.14. Warning: Bijective morphism are not necessarily isomorphisms!

Consider the example: $F : \mathbb{A}^1 \to V(y^3 - x^2)$ given by $t \mapsto (t^3, t^2)$. It will be left as an exercise that F is bijective but the inverse is not a morphism.

Example 1.2.15. Warning: If $F : V \to W$ is a morphism, the image of a subvariety might not be a subvariety.

Consider the example: Consider $F : \mathbb{A}^2 \to \mathbb{A}$ given by $(x, y) \mapsto x$. Let $X = V(xy - 1) \subseteq \mathbb{A}^2$. Then $F(X) = \mathbb{A}^1 \backslash \{0\}$ and so $F(X)$ is open, not closed, i.e. not a subvariety.

1.3 Dimension and Irreducibility

Definition 1.3.1. If X is a topological space, we say X is ***reducible*** if $X = Y \cup Z$ where $Y \subsetneq X, Z \subsetneq X$ are both closed.

Example 1.3.2. We have $X = V(xy) \subseteq \mathbb{A}^2$. Then $X = V(x) \cup V(y)$.

Remark 1.3.3. To think about reducible spaces, we should try to think it as reducible polynomials. For instance, the polynomial in the above example is indeed reducible.

Moreover, most non-algebraic geometry topological spaces are reducible. For instance, $\mathbb{R}$ as usual topology, we have $\mathbb{R} = (-\infty, 0] \cup [0, \infty)$.

Definition 1.3.4. A topological space is ***irreducible*** if it is not reducible.

Definition 1.3.5. Let X be an irreducible affine algebraic variety. The ***dimension*** of X is d if d is the max number for which there exists $X = X_d \supsetneq X_{d-1} \supsetneq X_{d-2} ... \supsetneq X_0 \supsetneq \emptyset$ with each X_i closed and irreducible.

Example 1.3.6.

1. If X be a point, then $dim(X) = 0$.
2. If $k = \overline{k}$, then $dim(\mathbb{A}^1_k) = 1$. This is because the clsoed subsets are $\mathbb{A}^1, \emptyset$ and finite unions of points. So, the longest chain is
$$\mathbb{A}^1 \supsetneq \text{ point } \supsetneq \emptyset$$
3. We see that $\mathbb{A}^n \geq n$ because we have a chain
$$\mathbb{A}^n \supsetneq V(x_1) \supsetneq V(x_1, x_2) \supsetneq ...$$
It takes some work to show $dim(\mathbb{A}^n) = n$.

Definition 1.3.7. Let X be a topological space, we say $Z \subseteq X$ is an ***irreducible component*** if Z is irreducible and whenver $Z \subseteq W$ with W irreducible, we have $Z = W$.

Definition 1.3.8. Let X be an affine algebraic variety in general, then $dim(X) = max\{dim(X_i)\}$ where X_i are irreducible components of X.

Example 1.3.9. Consider $X = V(xz, yz)$, then the irreducible components are $V(z)$ and $V(x, y)$. We have $dim(X) = 2$.

Also, if $Y \subseteq X$, then $dim(Y) \leq dim(X)$. Indeed, take any irreducible components of Y, they are closed, hence they are closed subsets of X intersect with Y. Therefore, the irreducible component in Y must have less than or equal dimension than the closed subset in X. This imply $dim(Y) \leq dim(X)$.

Remark 1.3.10. Note when we say the max in the definition, we are assuming there exists only finite many components. Eventually, we will show $dim(\mathbb{A}^n) = n$, so every algebraic variety is finite-dimensional because it is contained in $\mathbb{A}^n$.

Remark 1.3.11. If $X \subseteq \mathbb{A}^n_{\mathbb{R}}$ is an affine algebraic variety that happens to be a manifold in the usual Euclidean topology, then dimension in algebraic geometry matches manifold dimension. Similarly for $X \subseteq \mathbb{A}^n_{\mathbb{C}}$.

Chapter 2

Hilberts' Nullstellensatz

忽魂悸以魄动，恍惊起而长嗟。
惟觉时之枕席，失向来之烟霞。
世间行乐亦如此，古来万事东流水。

李白

2.1 Some More Algebra

Remark 2.1.1. This week and next we will try to form a dictionary between geometric objects and algebraic objects.

In this course, rings are always commutative with unit.

Definition 2.1.2. If A is a ring, we say $f : A \to B$ is an A-***algebra*** if B is a ring and f is a ring homomorphism.

Sometimes we just say (B, f) or just B is an A-algebra where f is understood.

Example 2.1.3.

1. Every ring B is a $\mathbb{Z}$-algebra in a unique way. Indeed, we have only one non-trivial ring homomorphism from $\mathbb{Z} \to B$, namely $1_{\mathbb{Z}} \mapsto 1_B$.
2. $A \to A[x_1, ..., x_n]$ is an algebra.
3. $A \to A[x, y]/(y^2 - x^3)$ is an algebra.
4. $\mathbb{R} \to \mathbb{C}$ via inclusion map is an algebra.
5. $\mathbb{C}[x] \to \mathbb{C}$ via $x \mapsto 5$ is an algebra (and the map is surjective).

Definition 2.1.4. A ***morphism*** of A-algebra

$$(A \xrightarrow{f} B) \longrightarrow (A \xrightarrow{g} C)$$

is given by a ring homomorphism $h : B \to C$ such that $h \circ f = g$. In particular, this means the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[d, "h"] \\ & C \end{tikzcd}$$

Example 2.1.5.

1. A morphism of $\mathbb{Z}$-algebra $B \to C$ is just a morphism of rings.
2. Consider the map $f : \mathbb{C}[x] \to \mathbb{C}[x]$ via $\sum a_j x^j \mapsto \sum \overline{a_j} x^j$ is a ring homomorphism but not $\mathbb{C}$-algebra morphism. Indeed, consider

$$\begin{tikzcd} \mathbb{C} \arrow[r, "Id"] \arrow[dr, "f"'] & \mathbb{C}[x] \arrow[d, "Id"] \\ & \mathbb{C}[x] \end{tikzcd}$$

   and choose $z = i$, then the diagram does not commute. However, this is a $\mathbb{R}$-algebra homomorphism.

Definition 2.1.6. We say an A-algebra $f : A \to B$ is ***generated by*** $\{b_i\}_{i \in I}$ with $b_i \in B$ if $\forall b \in B$, there exists $a_i \in A$ such that

$$b = \sum_{i \in I} f(a_i) b_i$$

and all but finitely many of $a_i = 0$.

Definition 2.1.7. Given $X \subseteq \mathbb{A}_k^n$ an affine algebra variety, let $I(X) \leq k[x_1, ..., x_n]$ to be

$$I(X) = \{f \in k[x_1, ..., x_n] : \forall a \in X, f(a) = 0\}$$

Remark 2.1.8. Note $I(X)$ is an ideal. Indeed, let $f, h \in I(X), g \in k[x_1, ..., x_n]$, then $\forall a \in X, fg(a) = f(a)g(a) = 0 \cdot g(a) = 0$ and $(f + h)(a) = 0 + 0$ so $I(X)$ is indeed an ideal.

Proposition 2.1.9. *If $X \subseteq \mathbb{A}_k^n$ is an affine algebra variety, then $V(I(X)) = X$.*

Proof. By definition, $X = V(J)$ where J is an ideal.

We first show $X \subseteq V(I(X))$. Let $a \in X$ be arbitrary, let $f \in I(X)$ be arbitrary. Then $f(a) = 0$ by the definition of $I(X)$.

Then, we show $V(I(X)) \subseteq X = V(J)$. It suffice to show $J \subseteq I(X)$ by homework question. Let $f \in J$, then f vanishes all of X, i.e. $f \in I(X)$. Thus $J \subseteq I(X)$ and so $V(I(X)) \subseteq V(J)$.

The proof follows. ♡

2.2 Hilbert's Nullstellensatz

Remark 2.2.1. The above proposition 2.1.9 also says $V(I(V(\mathfrak{J}))) = V(\mathfrak{J})$.

Now, what about given $I(V(\mathfrak{J}))$? Do we have $I(V(\mathfrak{J})) = \mathfrak{J}$? The answer is no, unfortunately.

Example 2.2.2. Consider $\mathfrak{J} = \langle x^2 \rangle \subseteq k[x]$. Then, we have $V(\mathfrak{J}) = \{0\} \subseteq \mathbb{A}^1$. Now, consider $I(V(\mathfrak{J})) = \{f(x) : f(0) = 0\} = \langle x \rangle$

Definition 2.2.3. If R is a ring and $J \leq R$ is an ideal, we define the ***radical*** of J, $\sqrt{J}$, to be

$$\sqrt{J} := \{r \in R : \exists n \in \mathbb{Z}_+, r^n \in J\}$$

Example 2.2.4. We have $\sqrt{\langle x^5 \rangle} = \langle x \rangle$.

Definition 2.2.5. We say J is a ***radical ideal*** if $J = \sqrt{J}$.

Theorem 2.2.6 (Hilbert's Nullstellensatz V1)**.** *If $\mathfrak{J} \leq k[x_1, ..., x_n]$ is an ideal, and $k = \overline{k}$ is algebraically closed, then*

$$I(V(\mathfrak{J})) = \sqrt{\mathfrak{J}}$$

Corollary 2.2.6.1. *There is an inclusion reversing bijection between affine algebraic varieties and radical ideals in $k[x_1, ..., x_n]$ with $X \mapsto I(X)$ and $\mathfrak{J} \mapsto V(\mathfrak{J})$. In particular, inclusion reversing means $X \subseteq Y$ iff $I(Y) \subseteq I(X)$.*

Remark 2.2.7. Consider $\mathbb{R}[x] \supseteq \mathfrak{J} = \langle x^2+1 \rangle$, then $V(\mathfrak{J}) = \emptyset$ but $I(V(\mathfrak{J})) = \mathbb{R}[x] \neq \sqrt{\mathfrak{J}}$. Thus algebraically closed is important here.

Remark 2.2.8. Note Nullstellensatz gives a bijection between points of $\mathbb{A}^n_k$ and maximal ideals of $k[x_1, ..., x_n]$ if $k = \overline{k}$.

Given $m_a = \langle x_1 - a_1, ..., x_n - a_n \rangle$. Let $a = (a_1, ..., a_n) \in \mathbb{A}^n_k$. We know m_a is maximal, and by first day of class, we know $V(m_a) = \{a\}$.

If we start with $a \in \mathbb{A}^n_k$, let's calculate $I(\{a\})$. We have $I(\{a\}) = I(V(\{m_a\})) = \sqrt{m_a} = m_a$ as m_a is maximal.

Remark 2.2.9. Note Nullstellensatz is a generalization of fundamental theorem of algebra.

Indeed, let k be algebraically closed. The fundamental theorem of algebra states the maximal ideals of $k[x]$ are exactly $\langle x - a \rangle$, where $a \in k$. In addition, Nullstellensatz says maximal ideals of $k[x_1, ..., x_n]$ are $\langle x_1 - a_1, ..., x_n - a_n \rangle$, where $a_i \in k$.

Definition 2.2.10. If B is an A-algebra, we say B is ***finitely generated*** if there exists $b_1, ..., b_n \in B$ such that $\forall b \in B, \exists p \in A[x_1, ..., x_n]$ such that $b = p(b_1, ..., b_n)$.

Remark 2.2.11. The above definition is equivalent to $A[x_1, ..., x_n] \to B$ via $x_i \mapsto b_i$ is surjective.

Theorem 2.2.12 (Nullstellensatz V2)**.** *If K/F is a field extension and suppose K is a finitely generated F-algebra, then K is a finite extension of F, i.e. K is a finite dimensional F vector space.*

Corollary 2.2.12.1. *If $k = \overline{k}$, then every maximal ideal of $k[x_1, ..., x_n]$ is of the form $\langle x_1 - c_1, ..., x_n - c_n \rangle$ where $c_i \in k$.*

Proof. Let $m \leq k[x_1, ..., x_n] := R$ be a maximal ideal. Thus $F = R/m$ is a field and we have the quotient map $\pi : k[x_1, ..., x_n] \to F$. Note π is surjective, we have F is a finitely generated k-algebra so Theorem 2.2.12 says F is finite extension of k, thus $F = k$ as k is algebraically closed. Thus we can view π as a map between R and k. Therefore, there exists $c_i \in k$ so $\pi(x_i) = c_i$, with $m = Ker(\pi) = \langle x_1 - c_1, ..., x_n - c_n \rangle$. ♡

Corollary 2.2.12.2. *If $k = \overline{k}$ and $I \subsetneq k[x_1, ..., x_n]$ is an ideal, then $V(I) \neq \emptyset$.*

Proof. We know $I \subseteq m$ for some maximal ideal m. Thus $m = \langle x_1 - c_1, ..., x_n - c_n \rangle$ where $c_i \in k$. Hence we have $V(m) \subseteq V(I)$ where $V(m) \neq \emptyset$. ♡

Remark 2.2.13. Note that K is a field in the above proof is necessary. Consider $k[x]$, which is a finitely generated k-algebra. However, clearly it is not a fintie extension (f.g. module) of k.

Normally, we do not have f.g. k-algebra imply f.g. k-module.

Theorem 2.2.14. *Theorem 2.2.12 imply Theorem 2.2.6. A small remark that Theorem 2.2.12 is sometimes called Zariski's lemma.*

Proof. We use $V2$ to show $V1$. In particular, we will show $I(V(\mathfrak{I})) = \sqrt{\mathfrak{I}}$.

First we show $\sqrt{\mathfrak{I}} \subseteq I(V(\mathfrak{I}))$. Let $f \in \sqrt{\mathfrak{I}}$, then $f^n \in \mathfrak{I}$ so $\forall a \in V(\mathfrak{I})$, we have $0 = f^n(a) = f(a)^n$. Thus $f(a) = 0$, i.e. $f \in I(V(\mathfrak{I}))$. Now we show $I(V(\mathfrak{I})) \subseteq \sqrt{\mathfrak{I}}$. We can assume $\mathfrak{I} \neq 0$ and so we choose $0 \neq f \in I(V(\mathfrak{I}))$. Moreover, say $\mathfrak{I} = \langle g_1, ..., g_s \rangle$.

Consider the ideal $\mathfrak{I}' \in k[x_1, ..., x_{n+1}]$ to be $\mathfrak{I}' = \langle g_1, ..., g_s, x_{n+1}f - 1 \rangle$. Let's calculate $V(\mathfrak{I}')$, and suppose $V(\mathfrak{I}') \neq \emptyset$ for a contradiction. Let $(a_1, ..., a_{n+1}) \in V(\mathfrak{I}')$, i.e. $g_i(a_1, ..., a_n) = 0$ and $a_{n+1}f(a_1, ..., a_n) = 1$. Therefore, $f(a_1, ..., a_n) \neq 0$ and $(a_1, ..., a_n) \in V(\mathfrak{I})$. However, $f \in I(V(\mathfrak{I}))$, and this is a contradiction. Thus $V(\mathfrak{I}') = \emptyset$. By corollary 2.2.12.2, we have $\mathfrak{I}' = k[x_1, ..., x_{n+1}]$ and so $1 \in \mathfrak{I}'$. Thus, there exists $h, h_1, ..., h_s \in k[x_1, ..., x_{n+1}]$ such that

$$1 = \sum_{i=1}^{s} h_i g_i + h \cdot (x_{n+1}f - 1)$$

Now, plug in $x_{n+1} = \frac{1}{f}$ and we have

$$1 = \sum_{i=1}^{s} h_i(x_1, ..., x_n, \frac{1}{f}) g_i$$

This is not a valid polynomial, so let's clear the denominators, i.e. multiply by f^m:

$$f^m = \sum_{i=1}^{s} h_i'(x_1, ..., x_n) g_i \in \mathfrak{I}$$

Hence $f \in \sqrt{\mathfrak{I}}$. ♡

Lemma 2.2.15. *Let k be a field, and $R \subseteq k$ be a subring. Suppose $\alpha \in k$ such that $k = R[\alpha]$, then there exists $0 \neq s \in R$ such that $R[\frac{1}{s}]$ is a field and α is algebraic over $R[\frac{1}{s}]$.*

Proof. Note k is a field so R is an integral domain. So let F be the field of fraction of R. Thus $k = F[\alpha]$ and $\alpha \in F[\alpha]$. Note k is a field imply $\frac{1}{\alpha} \in k = F[\alpha]$. Thus $\frac{1}{\alpha}$ is a polynomial[1] in α, i.e.[2]

$$\alpha^n + f_{n-1}\alpha^{n-1} + ... + f_0 = 0$$

with $f_i \in F$.

Thus, choose s to be the common denominator of f_i's, then $f_i \in R[\frac{1}{s}]$, so α is algebraic over $R[\frac{1}{s}]$. Thus we just need to show $R[\frac{1}{s}]$ is a field.

Given $0 \neq r' \in R[\frac{1}{s}] \subseteq k$, we show $\frac{1}{r'} \in R[\frac{1}{s}]$. Note $\frac{1}{r'} \in k = R[\alpha]$, so we have $\frac{1}{r'} = g(\alpha)$ for some $g(x) \in R[x]$ and $g(x) \in R[\frac{1}{s}][x]$. In particular, note $g(\alpha) \in R[\frac{1}{s}]$ where α is algebraic in $R[\frac{1}{s}]$, so it's product and sum is also algebraic in $R[\frac{1}{s}]$, i.e. $g(\alpha) = \frac{1}{r'}$ is algebraic over $R[\frac{1}{s}]$.

Hence, we have a monic polynomial over $R[\frac{1}{s}]$ such that vanishes $\frac{1}{r'}$. Therefore,

$$\left(\frac{1}{r'}\right)^m = a_{m-1}\left(\frac{1}{r'}\right)^{m-1} + ... + a_0$$

Multiply by $(r')^{m-1}$ to see $\frac{1}{r'} \in R[\frac{1}{s}]$. ♡

Lemma 2.2.16. *If F is a field and $R = F[x]$, then $R[\frac{1}{r}]$ is not a field for any $r \in R$.*

Proof. Exercise. ♡

Lemma 2.2.17. *If k/F is a field extension and there exists $\alpha \in k$, $u \in F[\alpha]$ such that $k = F[\alpha, \frac{1}{u}]$, then $k = F[\alpha]$ and α is algebraic over F.*

Proof. Suppose that α is transcendental over F, then $F[\alpha] \cong F[x]$. So, by Lemma 2.2.16, $F[\alpha, \frac{1}{u}]$ is not a field, a contradiction.

Since α is algebraic, $F(\alpha) = F[\alpha]$ and so $\frac{1}{u} \in F(\alpha)$, so $F[\alpha, \frac{1}{u}] = F[\alpha]$. Since $F[\alpha] = F[\alpha, \frac{1}{u}] = k$, we are done. ♡

[1] Note $F[\alpha]$ may be defined as the smallest subring containing α but it can also defined as all polynomials in terms of α. Indeed, $F[\alpha]$ contains α so it must contain elements of the form $\sum r_i\alpha^i$. However, polynomials in α is a subring of $F[\alpha]$ and so $F[\alpha]$ is equal the polynomials in α.

[2] We have this vanishing polynomial because, we have $\sum r_i\alpha^i = \frac{1}{\alpha} \Rightarrow \sum r_i\alpha^{i+1} - 1 = 0$ so we indeed have a polynomial in F that vanishes α

Lemma 2.2.18. *Let k be a field, R be a subring of k such that there exists $\alpha \in k$, $u \in R[x]$ such that $R[\alpha, \frac{1}{u}] = k$, then there exists $v \in R$ so $R[\frac{1}{v}]$ is a field and α is algebraic over $R[\frac{1}{v}]$.*

Proof. We know R is integral domain as it is a subring of a field. Thus $R \leq frac(R) := F$, the field of fraction of R. Thus by Lemma 2.2.17, α is algebraic over F and $k = F[\alpha]$. We can write $\frac{1}{u} = f_m\alpha^m + ... + f_0$ for some $f_i \in F$, take s to be the common denominator of f_i's. Then $\frac{1}{u} \in R[\frac{1}{s}]$ so that $k = R[\alpha, \frac{1}{s}] = R[\frac{1}{s}][\alpha]$. By Lemma 2.2.15, there is some $t \in R[\frac{1}{s}]$ such that $R[\frac{1}{s}][\frac{1}{t}]$ is a field and α is algebraic over this field $R[\frac{1}{s}][\frac{1}{t}]$. We know that $t = \frac{r}{s^m}$ for some $m \in \mathbb{Z}$ and $r \in R$, so $R[\frac{1}{s}][\frac{1}{t}] = R[\frac{1}{rs}]$. So taking $v = rs$ gives the result. ♡

Lemma 2.2.19. *Let k be a field and $A \subseteq k$ be a subring such that $A[\alpha_1, ..., \alpha_m] = k$, then there exists $v \in A$ such that $A[\frac{1}{v}]$ is a field and $[K : A[\frac{1}{v}]]$ is finite.*

Proof. Let $R = A[\alpha_1, ..., \alpha_{m-1}]$ and apply Lemma 2.2.18 with $\alpha = \alpha_m$ and $u = 1$. Then there exists $s \in R$ such that $R[\frac{1}{s}]$ is a field and α_m is algebraic over $R[\frac{1}{s}]$. If $m = 1$, the base case is finished.

Otherwise, repeat the procedure, apply Lemma 2.2.18 with $R = A[\alpha_1, ..., \alpha_{m-2}]$ with $\alpha = \alpha_{m-1}$ and $u = s$. Since adjoining finitely many algebraic elements gives a finite extension, the proof follows. ♡

Theorem 2.2.20. *Proof of Theorem 2.2.12.*

Proof. Since k is finitely generated, $\exists \alpha_1, ..., \alpha_m \in k$ such that $F[\alpha_1, ..., \alpha_m] = k$.

By apply Lemma 2.2.19 with $A = F$, there exists $v \in F$ such that $F[\frac{1}{v}]$ a field and $[k : F[\frac{1}{v}]]$ is finite. Since $v \in F$, we have $\frac{1}{v} \in F$ and $F = F[\frac{1}{v}]$, we have $[K : F]$ is finite. ♡

Definition 2.2.21. A topological space X is ***Noetherian*** if it satisfies the descending chain condition: every chain of closed subsets $X \supseteq X_1 \supseteq X_2 ...$ stabilizes at one point, i.e. there exists i such that $X_i = X_j$ for all $j \geq i$.

Theorem 2.2.22 (Hilbert's Basis Theorem)**.** *If R is a Noetherian ring, then $R[x]$ is Noetherian.*

Proof. We do this by showing every ideals in $R[x]$ is finitely generated. Then, say we have $\{J_i\}_{i \in I}$ be the a aescending chain of ideals, consider $\bigcup_{i \in I} J_i = \langle f_1, ..., f_n \rangle$ where f_i must be coming from some ideals J_{k_i} and if we just take the maximum of k_i, say M, we would have $\bigcup_{i \in I} J_i = J_M$ as $f_1, ..., f_n \in J_M$, i.e. it stabilizes at one point.

Now, say for the sake of contradiction say that $I \subseteq R[x]$ is a non-finitely generated ideal. Then, we can (recursively) get a sequence of polynomials $f_0, f_1, f_2, ...$ such that if $B_i := \langle f_0, ..., f_{i-1} \rangle$ then $f_i \in I \backslash B_i$ is of minimal degree. Then, we have

$\{deg(f_0), deg(f_1), ...\}$ must be a non-decreasing sequence by construction (if $f_n < f_{n-1}$, then f_{n-1} will not get picked as the $n-1$ term since both f_n and f_{n-1} are not in $I\backslash B_{n-2}$).

Now, let a_n be the leading coefficient of f_n and let B be the ideal generated by all of a_i, $i \geq 0$. Since R is Noetherian, we have

$$\langle a_0 \rangle \subseteq \langle a_0, a_1 \rangle \subseteq \langle a_0, a_1, a_3 \rangle ...$$

must stabilize at one point, i.e. $\exists N \in \mathbb{N}$ such that $B = \langle a_0, ..., a_{N-1} \rangle$. Hence, we get $a_N = \sum_{i=0}^{N-1} r_i a_i$ where $r_i \in R$. Then, let $q_i = deg(f_N) - deg(f_i)$ and consider

$$g(x) = \sum_{i=0}^{N-1} r_i x^{q_i} f_i(x) \in R[x]$$

Note $g(x)$ has the same leading coefficient as f_N and it has the same degree and $g(x) \in B_N$. However, note $f_N \notin B_N$ and so $f_N - g \in I\backslash B_N$ and it has less degree than f_N, a contradiction. Thus we must have I to be finitely generated. ♡

Corollary 2.2.22.1. *Let R be Noetherian, then $R[x_1, ..., x_n]$ is Noetherian. In particular, if k is a field, then $k[x_1, ..., x_n]$ is Noetherian.*

Proof. Using induction on n and the proof follows. To see the second assertion, note we only have two possible ideals of k and surely every aescending chain of ideals must stabilize, i.e. k is Noetherian. ♡

Proposition 2.2.23. *Affine algebraic variety is Noetherian space.*

Proof. Let X be an affine algebraic varieties of $\mathbb{A}^n$, we will show it is Noetherian. Let $\{X_i\}_{i\geq 1}$ be a descending chain of closed subsets in X, we need to show it stabilizes at one point. Note X_i are closed in X so we have $X_i = X \cap V(\mathfrak{I}_i)$ where $\mathfrak{I}_i \leq k[x_1, ..., x_n]$ is an ideal. For an abuse of notation, we say $X_i = V(\mathfrak{I}_i)$ as we are talking about the topological space X and it is redundant to say intersect with X every time. Also, let $X = V(\mathfrak{I})$.

Therefore, we get $V(\mathfrak{I}) \supseteq V(\mathfrak{I}_1) \supseteq V(\mathfrak{I}_2)...$ and thus we get $\mathfrak{I} \subseteq \mathfrak{I}_1 \subseteq \mathfrak{I}_2 \subseteq ...$ by assignment 1. It suffice to show that this aescending chain of ideals stabilizes at one point. However, note k is Noetherian ring and we have, by Hilbert's basis theorem, that $k[x_1, ..., x_n]$ is Noetherian ring, hence we indeed have this aescending chain to be stabilized at one point, i.e. $V(\mathfrak{I}) \supseteq V(\mathfrak{I}_1) \supseteq ...$ stabilizes at one point and thus X is Noetherian. ♡

Definition 2.2.24. A topological space is ***quasi-compact*** if every open cover admits a finite subcover. A Hausdorff quasi-compact space is called ***compact***.

Proposition 2.2.25. *Let X be Noetherian space, then every subset of X is quasi-compact.*

Proof. It suffice to show X is quasi-compact, as every subset of X is also Noetherian[1].

Let $X = \bigcup_{i\in I} U_i$ where each U_i is open. Then, we have $Z_i := X \backslash U_i = X \cap U_i^c$ is closed as it is the intersection of two closed subsets of X. We claim $\bigcap_{i\in I} Z_i = \bigcap_{i=1}^m Z_{k_i}$ where $k_i \in I$ for all $1 \le i \le m$.

For the sake of contradiction, say $\cap_{i\in I} Z_i$ is not a finite intersection of Z_i's, then for arbitrary chain of the form $Z_1 \supseteq Z_1 \cap Z_2 \supseteq Z_1 \cap Z_2 \cap Z_3 \supseteq \ldots \supseteq \cap_{i=1}^k Z_i$ with $Z_1, \ldots, Z_k \in \{Z_i : i \in I\}$, we can always find Z_{k+1} such that $\cap_{i=1}^k Z_i \supsetneq Z_{k+1}$, for if this is not the case then we have $\bigcap_{i\in I} Z_i = \bigcap_{i=1}^k Z_i$. However, if we can always find such Z_{k+1}, it will contradict the Noetherian property, i.e. we must have $\bigcap_{i\in I} Z_i = \bigcap_{i=1}^n Z_{k_i}$ for some $k_i \in I$.

Thus we get $X = \bigcup_{i\in I} U_i = (\bigcap_{i\in I} Z_i)^c = (\bigcap_{i=1}^n Z_{k_i})^c = \bigcup_{i=1}^n U_{k_i}$ as desired. ♡

Corollary 2.2.25.1. *Every affine variety is quasi-compact.*

Proof. Note $\mathbb{A}^n$ is Noetherian. ♡

Example 2.2.26. Observe that we also have every open subset of $\mathbb{A}^n$ to be quasi-compact as well. In particular, $\mathbb{Z}^n \backslash \{p\}$ is always quasi-compact.

2.3 Equivalence Between Algebra and Geometry

Remark 2.3.1. From now on, unless otherwise stated, k will be an algebraically closed field.

Let's try to define (reasonable) functions on affine varieties.

Consider $X \subseteq \mathbb{A}^n$ be an affine variety. The set of functions we are interested in is

$$k[X] := \{f|_X : f \in [x_1, \ldots, x_n]\}$$

We can add and multiply these functions, so $k[X]$ is a ring.

Definition 2.3.2. We define $k[X]$ to be the ***coordinate ring of*** X.

Remark 2.3.3. Observe the map $k[x_1, \ldots, x_n] \to k[X]$ by $f \mapsto f|_X$ is surjective and the kernel of the map is $I(X)$ and we have

$$k[X] \cong k[x_1, \ldots, x_n]/I(X)$$

If $X = V(\mathcal{I})$, then $k[X] \cong k[x_1, \ldots, x_n]/\sqrt{\mathcal{I}}$ by Nullstellensatz.

[1] Let Y be a subset of X with the induced topology. Say it is not Noetherian with non-stabilizing chain $Y \supseteq Y_1 \supseteq \ldots$. Then observe $Y_i = X_i \cap Y$ for some closed subsets $X_i \subseteq X$. Then, let $Z_i = \bigcap_{j=1}^i X_j$, we have $X \supseteq Z_1 \supseteq Z_2 \supseteq Z_3 \ldots$ which cannot be stabilize.

Proposition 2.3.4.

1. *$k[X]$ is finitely generated k-algebra as it is the quotient of the polynomial ring.*
2. *$k[X]$ has no non-trivial nilpotents, i.e. $p \in k[X]$ and $p^n = 0$ then $p = 0$. This is due to HW 2 and the fact that $\sqrt{\mathcal{I}}$ is radical.*
3. *The coordinate ring is **functorial**. This means, given $F : X \to Y$ a morphism of varieties, we get $F^* : k[Y] \to k[X]$ a morphism of k-algebra. Given $f \in k[Y]$, this is a function $f : Y \to k$, so we get a function $f \circ F : X \to k$ and so $F^*(f) = f \circ F$.*

Example 2.3.5.

1. Consider $X = V(xy - 1) \subsetneq \mathbb{A}^2$. We have $k[X] = k[x, y]/\langle xy - 1\rangle \cong k[x, x^{-1}]$ where the last part's isomorphism is given by $(x, y) \mapsto (x, x^{-1})$.
2. Consider $F : \mathbb{A}^3 \to \mathbb{A}^2$ via $(x, y, z) \mapsto (x^2y, x + 5z)$. Then, we have $F^* : k[A^2] \to k[A^3]$ is given by $x \mapsto x^2y$ and $y \mapsto x + 5z$. The reason is, $x \in k[\mathbb{A}^2]$ is the function reading off x-coordinate, so $F^*(x) = x \circ F$, which reads off x^2y.

Definition 2.3.6. Let $s \in R$, we say s is ***nilpotent*** if $\exists n \in \mathbb{N}$ such that $s^n = 0$.

Definition 2.3.7. A ring R is ***reduced*** if the only nilpotent element is 0.

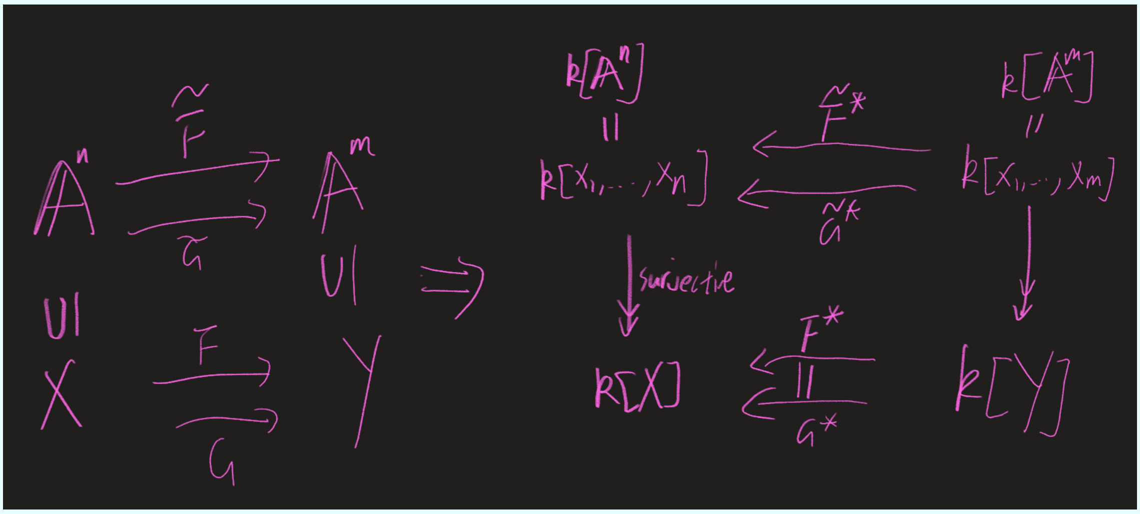
Theorem 2.3.8 (Equivalence of Algebra and Geometry)**.**

- *Every finitely generated reduced k-algebra R is isomorphic to $k[X]$ for some affine varieties X.*
- *Given a morphism of affine varieties $F : X \to Y$, then we get a k-algebra homomorphism from $k[Y]$ to $k[X]$, F^*. In particular, this map is bijective, i.e. we have an equivalence of categories between affine varieties of k to finitely generated reduced k-algebras.*

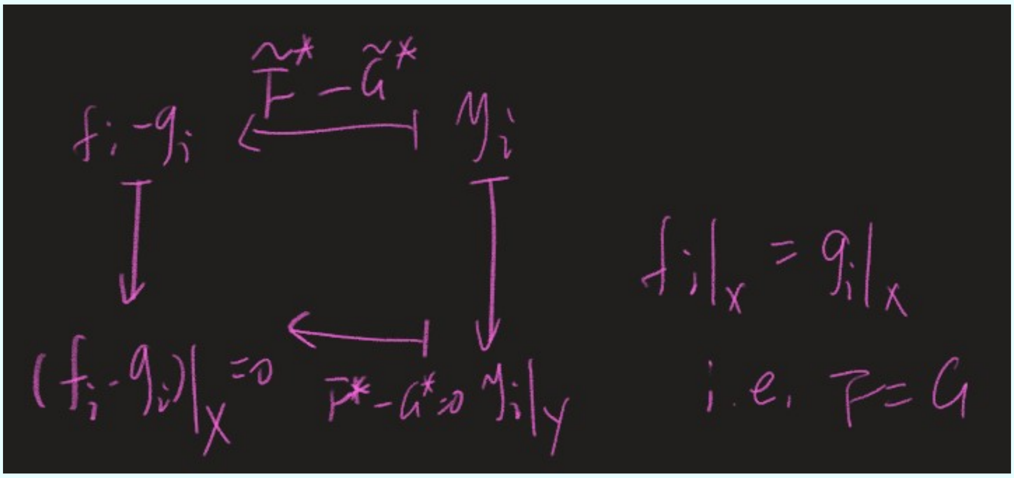
Proof. Let R be a finitely generated reduced k-algebra. Since R is finitely generated, we have $R \cong k[x_1, ..., x_n]/\mathcal{I}$ (as we can recall from 446). Since R is reduced, by HW2 we have $\mathcal{I}$ is radical.

Let $X = V(\mathcal{I})$, we have $k[X] \cong k[x_1, ..., x_n]/\sqrt{\mathcal{I}} \cong R$. In category language, we just proved "essential surjectivity".

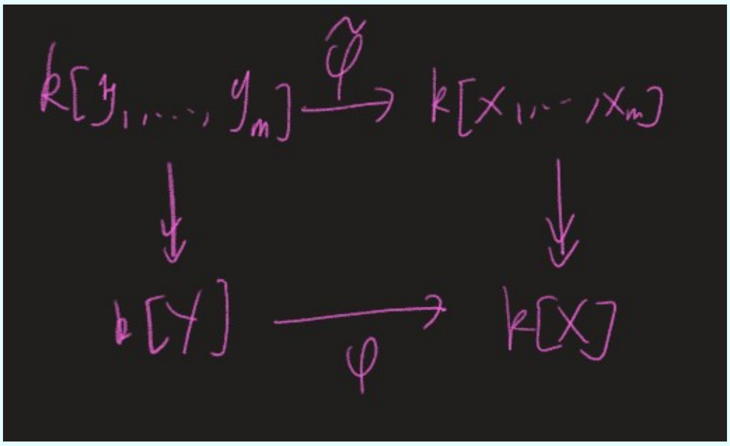
Now we prove "faithful" (which is from category language). Let $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$, $F : X \to Y$ and $G : X \to Y$ be two morphisms of affine varieties and $F^* = G^*$, then $F = G$. Let $X = V(\mathcal{I}), Y = V(\mathcal{J})$ where $\mathcal{I}, \mathcal{J}$ are radical ideals. In particular, note $F : X \to Y$ is the restriction of some $\tilde{F} : \mathbb{A}^n \to \mathbb{A}^m$ and similarly we have $\tilde{G}$. Thus, we have the diagram



Note $\tilde{F} = (f_1, ..., f_m)$ and $\tilde{G} = (g_1, ..., g_m)$ so that we have if we chase a particular element $y_i \in k[\mathbb{A}^m]$ in the following diagram, we get



Now we show "full". Start with $\phi : k[Y] \to k[X]$ be a k-algebra morphism. Choose $\tilde{\phi}$ arbitrarily as k-algebra map.



Then we define $\tilde{\phi}(y_i) =: f_i$. Let $\tilde{F} : \mathbb{A}^n \to \mathbb{A}^m$ to be $\tilde{F} = (f_1, ..., f_m)$, we need to show $\tilde{F}(X) \subseteq Y$, i.e. given $a \in X$ and $g \in \mathcal{J}$, we want $g(\tilde{F}(a)) = 0$. However, $g(\tilde{F})(a) = (g \circ \tilde{F})(a) = \tilde{\phi}(g)(a)$ where $\tilde{\phi}(\mathcal{J}) \subseteq \mathcal{I}$ and so $\tilde{\phi}(g) \in \mathcal{I}$, i.e. $\tilde{\phi}(g)(a) = 0$. ♡

Remark 2.3.9. What the above theorem says, is that given any finitely generated reduced k-algebra A, there exists an affine varieties X such that A is equal the space of functions on X.

Remark 2.3.10. Note our definition of affine varieties depends a priori on an embedding $X \subseteq \mathbb{A}^n$. However, $k[X]$ is just an abstract ring, so, the above theorem tells us that since $k[X]$ does not depend on an embedding, neither does X (for example, say we have $k[X]$ and $k[Y]$ are isomorphic, where X is in $\mathbb{A}^n$ and Y is in $\mathbb{A}^m$, we actually have X isomorphic to Y as well). Viz, this hints at the fact that you can develop affine varieties without dealing with subsets of $\mathbb{A}^n$.

Remark 2.3.11. We know from Nullstellensatz that points of $\mathbb{A}^n$ correspond to maximal ideals of $k[x_1, ..., x_n]$. We also know if $X = V(\mathfrak{I}) \subseteq \mathbb{A}^n$ where $\mathfrak{I} = \sqrt{\mathfrak{I}}$ then a point $p \in X$ if and only if $V(m) \subseteq V(\mathfrak{I})$ where m is the maximal ideal correspond to p. This happens iff $\mathfrak{I} \subseteq m$ iff $m/\mathfrak{I}$ is a maximal ideal of $k[x_1, ..., x_n]/\mathfrak{I} = k[X]$. Viz, points in X are exactly correspond to maximal ideals of $k[X]$.

So, X can be identified with $X = MaxSpec(k[X])$ where $MaxSpec(R)$ is defined as the set of maximal ideals of R and this definition of X does not depend on embedding of $\mathbb{A}^n$.

This tells us the inverse map in our equivalence (that takes finitely generated reduced k-algebra to affine varieties) is the map that sends R to $MaxSpec(R)$. The morphism is, given $f : R \to S$, we define a map that sents m to $f^{-1}(m)$ where m is maximal ideal in S.

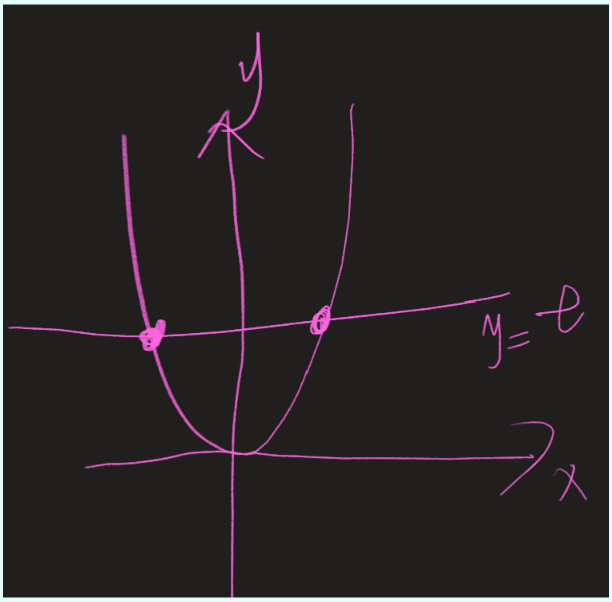
Now, what is the topology on $MaxSpec(R)$? It is the Zariski topology: we define the closed sets to be $V(\mathfrak{I}) := \{m : m \supseteq \mathfrak{I}\}$ where $\mathfrak{I}$ is an ideal of R and m is maximal, i.e. those closed sets are set of maximal ideals.

Remark 2.3.12. The next question is, why we look at max ideals? It turns out, we have $Spec(R)$, which is the collection of prime ideals of R and we can make a topology on $Spec(R)$ the same way we did to $MaxSpec(R)$, i.e. define closed sets to be $V(I) := \{p : p \supseteq I\}$ where I is an ideal of R and p is prime.

Moreover, this gives equivalence of categories between commutative rings to affine schemes where we send R to $Spec(R)$. In particular, if $k \neq \overline{k}$, we really want to look at $Spec(k[x_1, ..., x_n])$ instead of $MaxSpec(k[x_1, ..., x_n])$.

Example 2.3.13. Now we give an example where (affine) schemes arise naturally. For instance, it arises in the context of Intersection Theory.

Consider $X = V(y - x^2) \subseteq \mathbb{A}^2$ and $Y = V(y - t) \subseteq \mathbb{A}^2$ where $t \in k$. We want to look at the intersection.



The intersection has two points most of the time, unless $t = 0$. This is not good, we want to have two points in the intersection.

Note we have $k[X] = k[x,y]/\langle y - x^2\rangle$ and $k[Y] = k[x,y]/\langle y - t\rangle$ and we should have

$$k[X \cap Y] = k[x,y]/\langle y - x^2, y - t\rangle$$

Let's take a closer look at what this is. Note $y = x^2$ and $y = t$, so we have this is isomorphic to $k[x]/\langle x^2 - t\rangle$ and when $t \neq 0$, we have this is isomorphic to $k[x]/\langle x - \sqrt{t}\rangle \times k[x]/\langle x + \sqrt{t}\rangle$ with dimension as a k-vector space is 2. When $t = 0$, we have it is isomorphic to $k[x]/x^2$ and it's dimension as k-vector space is also 2. Viz, $k[x]/\langle x^2 - t\rangle$ always have dimension 2.

This says, if we want to the right answer to number of intersections, we need scheme intersection, not varieties intersection. In schemes, we get 1 points plus nilpotent.

Chapter 3

Projective Space

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3.1 Projective varieties

Definition 3.1.1. Let $k = \overline{k}$ be a field. We define $\mathbb{P}_k^n$ to be the 1 dimensional subspaces of k^{n+1}.

Example 3.1.2. $\mathbb{P}_k^1$ is lines in the plane, with each line is a point in $\mathbb{P}_k^1$.

Remark 3.1.3. We note a line $\ell \subseteq k^{n+1}$ is determined by $0 \neq (a_0, ..., a_n) \in \ell$ and the point is uniquely determined up to scalar. Therefore, another way to think $\mathbb{P}_k^n$ is, $\mathbb{P}_k^n = (k^{n+1} \backslash \{0\})/k^\times$ where $\lambda \in k^\times$ acts on $k^{n+1}\backslash\{0\}$ by $\lambda(a_0, ..., a_n) = (\lambda a_0, ..., \lambda a_n)$.

We use the notation $(a_0 : ... : a_n)$ to mean the equivalence class, i.e. $(a_0 : ... : a_n) = (b_0 : ... : b_n)$ iff there is a $\lambda \in k^\times$ such that $\forall i, a_i = \lambda b_i$.

Hence, we can define $\mathbb{P}_k^n = \{(a_0 : ... : a_n) : (a_0, ..., a_n) \neq 0\}$.

Example 3.1.4. Consider $0 \neq (x : y) \in \mathbb{P}^1$, say $y \neq 0$, then $(x : y) = (\frac{x}{y} : 1)$ and if $x \neq 0$, then $(x : y) = (1 : \frac{y}{x})$. So, $\mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1$, where we can think $\mathbb{P}^1$ as $\mathbb{A}^1$ glued to $\mathbb{A}^1$ where we glue $t \in \mathbb{A}^1$ to t^{-1} in other copy of $\mathbb{A}^1$.

Put in another word, consider the line $y = 1$, then each point (i.e. lines in k^2) in $\mathbb{P}^2$

with $y \neq 0$ could be mapped to the line $y = 1$ by taking the intersection with the line $y = 1$. This gives us a copy of $\mathbb{A}^1$. The another copy of $\mathbb{A}^1$ is all the points with $y = 0$.

Remark 3.1.5. We can cover $\mathbb{P}^n$ by $n+1$ sets by requiring the ith coordinate to be non-zero. The ith coordinate does not equal zero is the set

$$U_i := \{(\frac{a_0}{a_i} : ... : 1 : ... : \frac{a_n}{a_i}) : a_0, ..., a_n \in k, a_i \neq 0\}$$

which in turn is just k^n. Viz, we may cover $\mathbb{P}^n$ by $n+1$ copies of k^n(and it will be $\mathbb{A}^n$ when we make it into a topological space).

Now, what is the complement of U_i? This is the sets

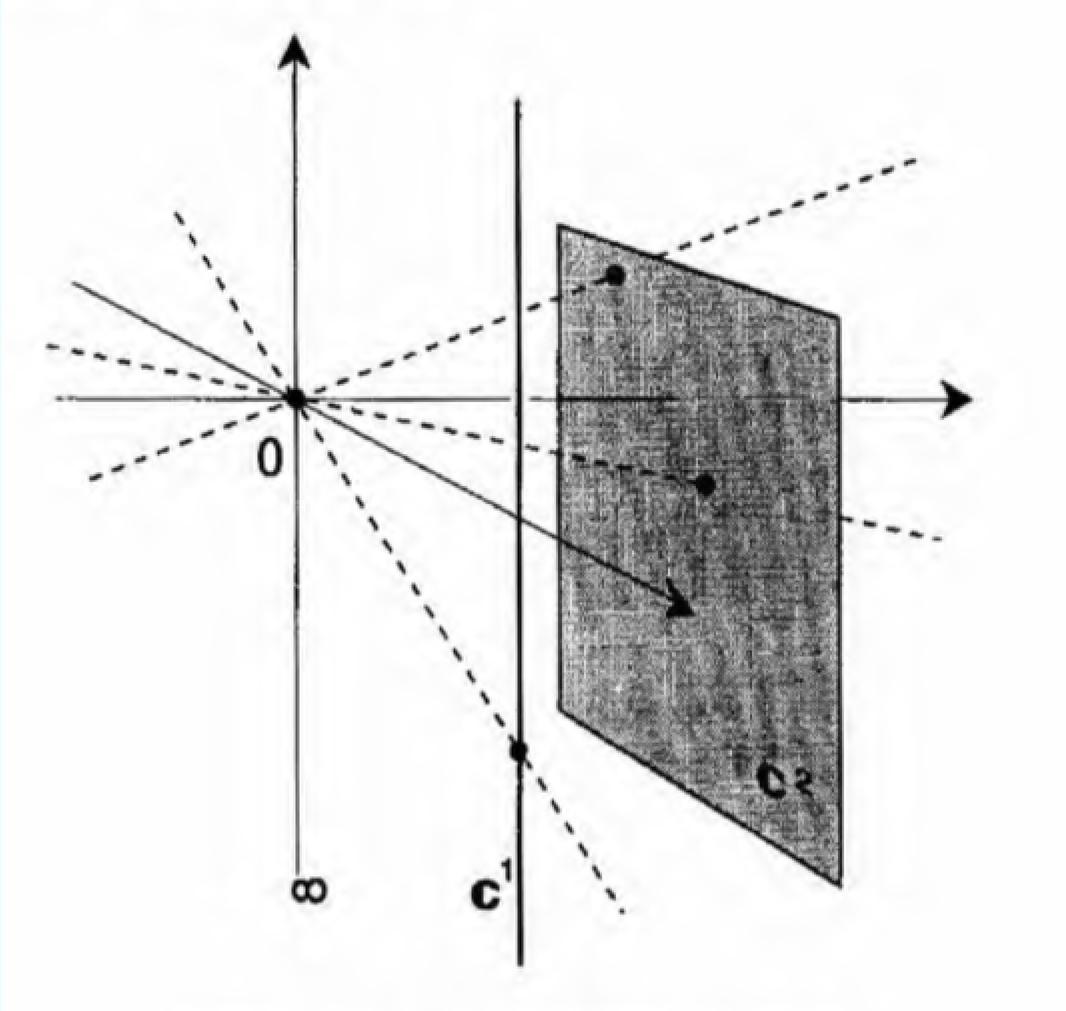
$$\{a_0 : ... : 0 : ... : a_n\} = \mathbb{P}^{n-1}$$

Therefore, we also have

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$$

We think of $\mathbb{P}^n$ as $\mathbb{A}^n$ plus a copy of $\mathbb{P}^{n-1}$ "at infinity". We will illustrate the reason behind point at infinity below in Example 3.1.6.

Example 3.1.6. Consider $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$. To see it is a union of the two subspaces, fix a plane away from $(0, 0, 0)$ in k^3, say $x_0 = 1$. Then, every point $(x_0 : x_1 : x_3)$ in $\mathbb{P}^2$ can be identified as a point projected to the plane $x_0 = 1$ if it's first coordinate is not 0. This gives us the copy of $\mathbb{A}^2$. Also, if $x_0 = 0$, then all the points in the form $(0 : x_1 : x_2)$ would give us a copy of $\mathbb{P}^1$.



In particular, those points in $\mathbb{P}^1$ are considered as points at infinity because we cannot project them to the plane and those lines "goes to infinity" without landing on our plane $x_0 = 1$. We finally remark that if we change our "perspective", i.e. the plane $x_0 = 1$ in this case, then the points of infinity will change as well. Say now we want to project to $x_1 = 1$, then the points at infinity will be different.

Remark 3.1.7. Observe in the above example, the choice of plane is arbitrary as long as it is not a subspace of k^3, e.g. we can use the plane $x_2 = 1$ or $x_1 + x_3 = 1$ and project the points in $\mathbb{P}^2$ that does not parallel to the given plane and form our copy of $\mathbb{A}^2$.

Since the choice of plane is arbitrary (we should see this idea extend naturally to k^n), we obtain a useful cover of $\mathbb{P}^n$ by $n + 1$ copies of $\mathbb{A}^n$. Namely, we have

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$$

where $U_i := \{(x_0 : \ldots : x_n) : x_i \neq 0\} = \{(x_0 : \ldots : x_n) : x_i = 1\}$.

Example 3.1.8. Consider $X := \{(x : y : z) : x = 5y\} \subseteq \mathbb{P}^2$ where $\mathbb{A}^2 \subseteq \mathbb{P}^2$ correspond to $z \neq 0$. When $z \neq 0$, we have $X \cap \mathbb{A}^2$ is the line $x = 5y$. However, at

∞, i.e. $z = 0$, we have $X \cap \{(x : y : 0)\} = \{(x : y : 0) : x = 5y\}$, which is equal a single point in $\mathbb{P}^2$, i.e. $(5 : 1 : 0)$ as it cannot be both 0.

3.2 Topology On Projective Space

Definition 3.2.1. Let $f \in k[x_0, ..., x_n]$, we say f is ***homogeneous of degree*** d if[1] $\forall \lambda \in k^\times$, we have $f(\lambda x_0, ..., \lambda x_n) = \lambda^d f(x_0, ..., x_n)$.

Example 3.2.2. We have $x_0^2 - x_1 x_2$ is homogeneous of degree 2.

Remark 3.2.3. We note, if f is homogeneous, we cannot ask what is the value of $f(a_0 : ... : a_n)$ as we note $(a_0 : ... : a_n)$ is an equivalence class of values. However, it is well-defined to ask whether $f(a_0 : ... : a_n) = 0$. Viz, we do have $V(f) = \{(a_0 : ... : a_n) : f(a_0 : ... : a_n) = 0\} \subseteq \mathbb{P}^n$.

Definition 3.2.4. An ideal $I \subseteq k[x_0, ..., x_n]$ is called ***homogeneous*** if it admits homogeneous generators.

Example 3.2.5. Exercise: If I, J are homogeneous ideals, then $IJ, I + J, I \cap J$ and $\sqrt{I}$ are all homogeneous.

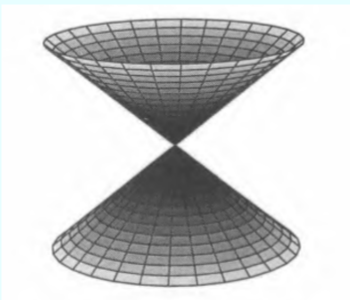
Also, I is homogeneous if and only if $\forall f \in I$, we have each $f_i \in I$, where $f = f_0 + ... + f_n$ and f_i is homogeneous of degree i.

Definition 3.2.6. A ***projective variety*** is a subset of the form $V(\mathfrak{I}) \subseteq \mathbb{P}^n_k$ where $V(\mathfrak{I}) := \{(a_0 : ... : a_n) : f(a_0 : ... : a_n) = 0, \forall f \in \mathfrak{I}$ such that is homogeneous$\}$ and $\mathfrak{I} \leq k[x_0, ..., x_n]$ is a homogeneous ideal.

Remark 3.2.7. Now we can topologize $\mathbb{P}^n$ by the Zariski topology, i.e. the closed sets are projective varieties.

Example 3.2.8. We have $V(x_0^2 - x_1 x_2, x_1^3 + x_2 x_3^2)$ is a projective variety.

Also, consider $V(x^2 + y^2 - z^2)$, which is also a projective variety. It looks like the following (observe each point below should be a line passing $(0, 0, 0)$)



Definition 3.2.9. Given $X \subseteq \mathbb{P}^n_k$ be a projective variety, let

$$I(X) := \langle \{f \in k[x_0, ..., x_n] : f \text{ is homogeneous}, f(\alpha) = 0, \forall \alpha \in X\} \rangle$$

[1] Note d should be positive

be the ideal generated by homogeneous polynomials that vanishes at all points in X.

Remark 3.2.10. One should check $I(X)$ is homogeneous.

Theorem 3.2.11 (Homogeneous Nullstellensatz)**.** *We have a bijective correspondence between projective varieties and homogeneous radical ideals in $k[x_0, ..., x_n]$ via $X \mapsto I(X)$ and $\mathfrak{I} \mapsto V(\mathfrak{I})$.*

Remark 3.2.12. This is a inclusion reversing bijection except for $\langle x_0, ..., x_n\rangle \leq k[x_0, ..., x_n]$. What goes wrong with $\langle x_0, ..., x_n\rangle$? We have $V(x_0, ..., x_n) = \emptyset$ and $I(\emptyset) = k[x_0, ..., x_n]$.

3.3 Projective Closure

Remark 3.3.1. Now, we want to "compactify" affine varieties. How?

Definition 3.3.2. Consider $X \subseteq \mathbb{A}^n$ be affine varieties, embed $\mathbb{A}^n \subseteq \mathbb{P}^n$ by

$$(x_1, ..., x_n) \mapsto (1 : x_1 : ... : x_n)$$

Take the closure[1] $\overline{X} \subseteq \mathbb{P}^n$, we call this the ***projective closure of*** X

Remark 3.3.3. You should think of $\overline{X} \subseteq \mathbb{P}^n$ as a compactification of X.

Example 3.3.4. Consider $X = V(y - x^2) \subseteq \mathbb{A}^2 \subseteq \mathbb{P}^2$. We have $X = \{(x : y : 1) : y = x^2\} = \{(\frac{x}{z} : \frac{y}{z} : 1) : \frac{x}{z} = (\frac{y}{z})^2, z \neq 0\}$. Observe $\frac{y}{z} = (\frac{x}{z})^2$ if and only if $yz = x^2$. We should convince ourself that $\overline{X} = V(yz - x^2)$. At infinity, i.e. when $z = 0$, we get $x^2 = 0$, i.e. $x = 0$ and $(0 : 1 : 0)$, i.e. $\overline{X} = X \cup \{(0 : 1 : 0)\}$. Note this process $y - x^2 \to yz - x^2$ is called ***homogenization***.

Definition 3.3.5. If $f \in k[x_1, ..., x_n]$ is of degree d, write $f = f_0 + f_2 + ... + f_d$ where f_i is homogeneous of degree i. Then the ***homogenization*** of f in $k[x_0, ..., x_n]$ is defined as

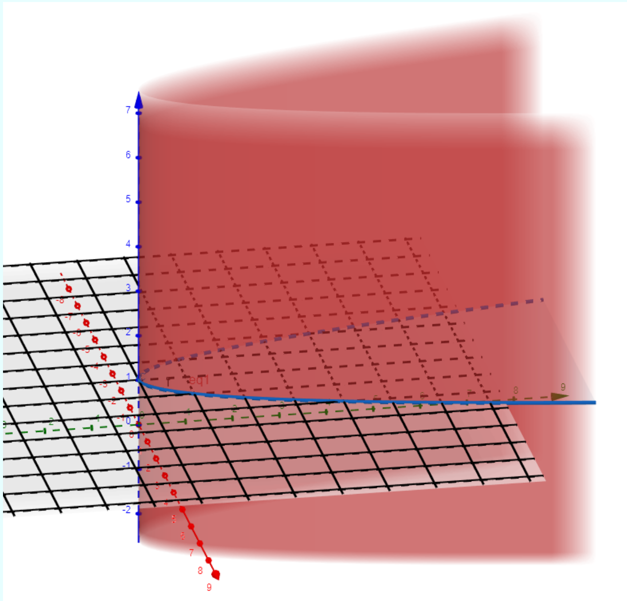
$$\tilde{f} = x_0^d f_0 + x_0^{d-1} f_1 + ... + f_d$$

Example 3.3.6. Let $f = x_1^3 + x_2 x_3^4$, then f has degree 5 and we have $\tilde{f} = x_0^2 x_1^3 + x_2 x^4$ and now $\tilde{f}$ is homogeneous of degree 5.

Next, we go back to the example $V(y - x^2)$ and present a illustration of the projective closure. Recall the homogenization of $y - x^2$ in $k[x, y, z]$ is $yz - x^2$.

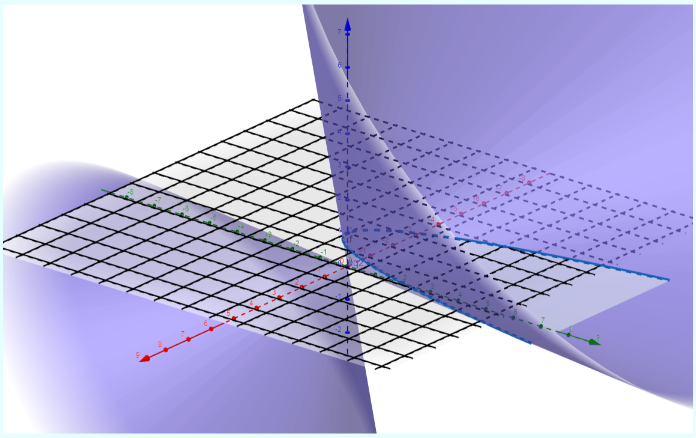
Think of $\mathbb{A}^2$ as the open set $U_z := \{(x : y : 1) : x, y \in k\}$, we have $V(f)$ in this copy of $\mathbb{A}^2$ looks like the intersection of the parabola $y = x^2$ in $\mathbb{C}^3$ and the plane $z = 1$, i.e. the blue line below:

[1] Note this means the smallest closed set containing X in $\mathbb{P}^n$



Then, image this in $\mathbb{P}^2$: the points in $V(f)$ as considered in $\mathbb{P}^2$ are lines that join a point on the blue line and the origin. Also, we missed one line here[1], which is the y-axis $x = z = 0$ and we need to add that in as well.

Hence, the closure of $V(f)$ is equal to $V(yz - x^2)$ and it is the following cone (in k^3):



We remark that this contains our original parabola $y - x^2$ in the open set U_z, plus one point at infinity $(0 : 1 : 0)$.

Remark 3.3.7. We see from the above example of $V(y - x^2)$, an guess of what the projective closure should be is: If $X = V(\mathfrak{I})$ with $\mathfrak{I} = \sqrt{\mathfrak{I}}$, where $\mathfrak{I} = \langle g_1, ..., g_s \rangle$. Then we should have $\overline{X} = V(\tilde{g}_1, ..., \tilde{g}_s)$.

This is wrong!!!!

[1]this line correspond to the point on the copy of $\mathbb{A}^2$ as the point where two branches of the parabola meets

Example 3.3.8 (Twisted cubic)**.** Let $X = V(y - x^2, z - xy) \subseteq \mathbb{A}^3$, this is the twisted cube. Then, after homogenization, we get $Y = V(yw - x^2, zw - xy)$. Now, consider $Y \cap \mathbb{A}^3 = Y \cap \{(w : x : y : z) \in \mathbb{P}^3 : w = 1\} = V(y - x^2, z - xy) = X$.

However, what happens "at infinity"? i.e. when $w = 0$. If $w = 0$, we have $yw = x^2, w = 0 \Rightarrow x = 0$ and so the second equation $zw = yx$ is automatically satisfied. Hence, we have $Y \cap \{(w : x : y : z) : w = 0\}$ is equal $\{(0 : y : z : 0) \in \mathbb{P}^3 : (y, z) \neq 0\}$. Hence, we have $Y = X \cup \mathbb{P}^1$.

This is bad because X has dimension 1 and $\mathbb{P}^1$ also has dimension 1. So Y cannot be the closure of X, because when you compactify X, you should only pick points at infinity, not curves.

Remark 3.3.9. Instead, to get the right answer, we homogenize every elements of $\mathfrak{I}$, not just the generators.

Definition 3.3.10. If $\mathfrak{I} \leq k[x_1, ..., x_n]$, then the ***homogenization*** is

$$\tilde{\mathfrak{I}} \leq k[x_0, ..., x_n]$$

given by $\tilde{\mathfrak{I}} := \langle \tilde{f} : f \in \mathfrak{I} \rangle$.

Remark 3.3.11. If $\mathfrak{I} = \langle g_1, ..., g_s \rangle$ when can we say $\tilde{\mathfrak{I}} = \langle \tilde{g}_1, ..., \tilde{g}_s \rangle$? This happens if $g_1, ..., g_s$ form a Groebner basis of $\mathfrak{I}$.

Theorem 3.3.12. *If $V(\mathfrak{I}) =: X \subseteq \mathbb{A}^n$, $\mathfrak{I} = \sqrt{\mathfrak{I}}$, then $\overline{X} = V(\tilde{\mathfrak{I}}) \subseteq \mathbb{P}^n$.*

Proof. We first show $\overline{X} \subseteq V(\tilde{\mathfrak{I}})$ and note $V(\tilde{\mathfrak{I}})$ is closed, so it is enough to just show $X \subseteq V(\tilde{\mathfrak{I}})$.

To do this, we need to show, given $f \in \mathfrak{I}$, we have $\tilde{f}|_X = 0$ and this is the same as showing $(\tilde{f}|_{\mathbb{A}_n})|_X$. However, $\tilde{f}|_{\mathbb{A}_n}$ is just set $x_0 = 1$, and note we have $\tilde{f}(1, x_1, ..., x_n) = f(x_1, ..., x_n)$ and so we have $(\tilde{f}|_{\mathbb{A}_n})|_X = f|_X = 0$ as $f \in \mathfrak{I}$.

Next, we show $V(\tilde{\mathfrak{I}}) \subseteq \overline{X}$. By Homogeneous Nullstellensatz, it is enough to show $I(\overline{X}) \subseteq \tilde{\mathfrak{I}}$.

Let $F \in k[x_0, ..., x_n]$ be a generator of $I(\overline{X})$, which is homogeneous of degree d. Note $F|_{\overline{X}} = 0$ and hence $F|_X = 0$ where $F|_X = (F|_{\mathbb{A}^n})|_X$. Note $F|_{\mathbb{A}^n} = F(1, x_1, ..., x_n) := f(x_1, ..., x_n)$, so $f|_X = 0$ and so $f \in I(X) = \mathfrak{I}$. Therefore, $\tilde{f} \in \tilde{\mathfrak{I}}$ and observe $\tilde{f}|_{\mathbb{A}^n} = f = F|_{\mathbb{A}^n}$. Then, it is left as an exercise to show that we get $F = \tilde{f} \cdot x_0^m$ for some $m \geq 0$. Hence, $\tilde{f} \in \tilde{\mathfrak{I}}$ and so $F \in \tilde{\mathfrak{I}}$. ♡

3.4 Morphisms of Projective varieties

Example 3.4.1. Consider $f : \mathbb{P}^1 \to \mathbb{P}^2$ given by $(x : y) \mapsto (x^2 : xy : y^2)$. This is well-defined as $f(ax : ay) = (a^2x^2 : a^2xy : a^2y^2) = f(x, y)$. It is also well-defined in a different sense, i.e. $f(x : y) \neq (0 : 0 : 0)$ for any $(x : y) \neq 0$. Indeed, $x^2 = 0 = y^0$ then $x = y = 0$.

Example 3.4.2. Consider $g : \mathbb{P}^1 \to \mathbb{P}^1$ given by $g(x : y) = (x^2 : xy)$. We have $g(0 : 1) = (0 : 0)$, so g is defined everywhere except at $(0 : 1)$. This is not a morphism from $\mathbb{P}^1 \to \mathbb{P}^1$ but it is a morphism from $\mathbb{P}\backslash\{(0 : 1)\} \to \mathbb{P}^1$. We note $\mathbb{P}\backslash\{(0 : 1)\}$ is a copy of $\mathbb{A}^1$.

Definition 3.4.3. Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be two projective varieties. Then $f : X \to Y$ is a ***morphism*** if, for all $x = (x_0 : ... : x_n) \in X$, there exists open subset $U \subseteq \mathbb{P}^n$ such that $x \in U$ and homogeneous polynomials $f_0, ..., f_m \in k[x_0, ..., x_n]$ of the same degree, such that $F|_U(x) = (f_0(x), ..., f_m(x))$.

Example 3.4.4. Let $X = V(xz - y^2) \subseteq \mathbb{P}^2$. Define $F : X \to \mathbb{P}^1$ by $F(x : y : z) = (x : y)$ if $x \neq 0$ (this is our first open set) and $F(x : y : z) = (y : z)$ if $z \neq 0$ (this is the second open set).

We need to make sure the two open sets covers all points. Say $U_1 = \{(x : y : z) : x \neq 0\}$ and $U_2 = \{(x : y : z) : z \neq 0\}$. Let $(x : y : z)$ be arbitrary point, we have either $x \neq 0$ or $x = 0$. If $x = 0$ then y must be zero as they are in X, so z is not zero. Hence F indeed defines on all points of X. Then to check well-defined (the value agree on intersection of U_1 and U_2), we suppose $x \neq 0$ and $z \neq 0$, Then $xz \neq 0$, so $y \neq 0$. Therefore, $(x : y) = (xz : yz) = (y^2 : y^2) = (y : z)$.

Definition 3.4.5. A morphism $F : X \to Y$ of projective varieties is an ***isomorphism*** if there exists a morphism $G : Y \to X$ of projective varieties such that $F \circ G$ and $G \circ F$ are both equal the identity. We write $X \cong Y$ and $G = F^{-1}$ in this case.

Example 3.4.6. Exercise: Let $X = V(xz - y^2)$ then $X \cong \mathbb{P}^1$ because the morphism $G : \mathbb{P}^1 \to X$ given by $(u : v) \mapsto (u^2, uv, v^2)$ is an inverse of F in example 3.4.4.

Remark 3.4.7. Warning: For affine varieties, the equivalence between algebra and geometry shows that if $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ then $X \cong Y$ if and only if $k[X] \cong [Y]$.

However, this is ***false*** for projective varieties. Indeed, we have $X = V(xz - y^2)$ and $\mathbb{P}^1$ are isomorphic, but $k[x, y, z]/\langle xz - y^2\rangle$ and $k[u, v]$ are not isomorphic. One way to see this is that x, y, z are linearly independent in $k[x, y, z]/\langle xz - y^2\rangle$ while u, v are linear independent in degree 1.

Definition 3.4.8. A ***linear change of coordinates on*** $\mathbb{P}^n$ is an isomorphism $L : \mathbb{P}^n \to \mathbb{P}^n$ such that there exists degree 1 homogeneous functions $\ell_0, ..., \ell_1$ for which $L(x) = (\ell_0(x) : ... : \ell_n(x))$.

Remark 3.4.9. L is globally given by this (not locally as in the definition of projective morphism).

Definition 3.4.10. We say projective varieties X and Y in $\mathbb{P}^n$ are ***projectively equivalent*** if there is a linear change of coordinates L such that $L(X) = Y$.

Example 3.4.11. $V(x)$ and $V(y)$ in $\mathbb{P}^2$ are projectively equivalent under the linear change of coordinates $L(x : y : z) = (y : x : z)$.

Example 3.4.12. The reason we care about this is that it allow us to present our varieties nicely. For instance, any pair of distinct points in $\mathbb{P}^1$ are linearly equivalent to $\{(1 : 0), (0 : 1)\}$, because the points correspond to distinct lines in k^2, so they form a basis, we then do linear change of coordinates to the $x-$ and $y-$ axes to get $\{(1 : 0), (0 : 1)\}$.

Similarly, any n points in $\mathbb{P}^{n-1}$ will be projectively equivalent to $\{(1 : ... : 0), ..., (0 : ... : 1)\}$.

3.5 Automorphisms of Projective Space

Remark 3.5.1. Observe if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, then the map $f_A : \mathbb{P}^1 \to \mathbb{P}^1$ given by $(x : y) \mapsto (ax + by : cx + dy)$ is an isomorphism with inverse $f_A^-1 = f_{A^{-1}}$.

Also. notice if $A = \lambda B$ for $\lambda \in k$, then $f_A = f_B$.

This means there is a map from $PGL_2(k)$ to $Aut(\mathbb{P}^1)$ where $PGL_2(k) \cong GL_2(k)/k^\times$.

We observe there is a connection with complex analysis. If we identify $\mathbb{P}^1_{\mathbb{C}}$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ by $(z : 1) \mapsto z \in \mathbb{C}$. In particular, consider $f_A(z : 1) = (az + b : cz + d) = (\frac{az+b}{cz+d} : 1)$, i.e. f_A is the Mobius transformation $z \mapsto \frac{az+b}{cz+d}$.

Theorem 3.5.2. *The automorphisms of $\mathbb{P}^n$ are given by $PGL_{n+1}(k)$.*

Proof. Later on, we are going to proof this using line bundles. However, feel free to assume this before that. ♡

3.6 Quasi-Projective varieties

Remark 3.6.1. The goal here is to unify affine varieties and projective varieties as instances of a common framework.

Definition 3.6.2. If Y is a topological space, then $X \subseteq Y$ is ***locally closed*** if there exists open $U \subseteq Y$ and closed $Z \subseteq Y$ such that $X = U \cap Z$.

Definition 3.6.3. A ***quasi-projective variety*** is a locally closed subset of some $\mathbb{P}^n$.

Example 3.6.4. Every projective variety is quasi-projective (because it is already closed).

Every affine variety is quasi-projective. Consider $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$, then $\overline{X}$ is closed and $\mathbb{A}^n$ is open[1], hence $X = \overline{X} \cap \mathbb{A}^n$.

Remark 3.6.5. Any open/closed subset of a quasi-projective variety is also quasi-projective.

Definition 3.6.6. A map $F : V \to W$ of quasi-projective variety $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ is a ***morphism***, if $\forall p \in V$, there exists an open nbhd U of p in V and homogeneous polynomials $f_0, f_1, ..., f_m \in k[x_0, ..., x_n]$ such that $\forall q \in U$, $F(q) = (f_0(q) : ... : f_m(q))$

[1] $\mathbb{A}^n$ can be considered as the complement of $V(x_0)$, hence open

Remark 3.6.7. A morphism F of quasi-projective varieties is locally defined by polynomials. Also, the above definition depends on the embedding of $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$.

Later, we will give an intrinsic definition of morphism. To do so, we need to extend what we call "affine".

Definition 3.6.8. A quasi-projective variety V is ***affine*** if it is isomorphic to some closed subset of $\mathbb{A}^n$ for some n.

Example 3.6.9. Consider $\mathbb{C}^\times = \mathbb{A}^1\backslash\{0\}$. This set is an open subset of $\mathbb{A}^1$, so it is not affine in the previous sense. However, since $\mathbb{A}^1\backslash\{0\} \cong V(xy-1) \subseteq \mathbb{A}^2$ with the isomorphism givne by $(x, y) \mapsto x$. Hence, it is an affine (quasi-projective) variety.

Remark 3.6.10. Now, we are going to show that any quasi-projective variety is ***locally affine***, i.e. for any point in that variety we can find an open nbhd that is affine.

We first observe that projective varieties are locally affine. Note that $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ where $U_i \cong \mathbb{A}^n$, then if $X \subseteq \mathbb{P}^n$ is projective, then $X = \bigcup_{i=0}^n (X \cap U_i)$, so they are indeed locally affine.

Lemma 3.6.11. *Let X be an affine (quasi-projective) variety and $f \in k[X]$ (the coordinate ring of X where we view X as in $\mathbb{A}^n$). Then $X\backslash V(f)$ is affine (quasi-projective variety).*

Proof. By definition, $X \cong V(\mathcal{I}) \subseteq \mathbb{A}^n$ for some radical ideal[1] $\mathcal{I} \leq k[x_1, ..., x_n]$. Then $f \in k[x_1, ..., x_n]/\mathcal{I}$ and we let $\mathcal{J} = \mathcal{I} + \langle x_{n+1}f - 1\rangle \leq k[x_1, ..., x_{n+1}]$.

We claim $V(\mathcal{J})$ is isomorphic to $X\backslash V(f)$ via the projection mapping

$$\pi : V(\mathcal{J}) \subseteq \mathbb{A}^{n+1} \to X\backslash V(f) \subseteq \mathbb{A}^n$$
$$(x_1, ..., x_{n+1}) \mapsto (x_1, ..., x_n)$$

Indeed, take $(x_1, ..., x_n) \in X\backslash V(f)$, i.e. $f(x_1, ..., x_n) \neq 0$ and $g(x_1, ..., x_n) = 0$ for all $f \neq g \in \mathcal{I}$. Then observe that $(x_1, ..., x_n, \frac{1}{f(x_1,...,x_n)}) \in V(\mathcal{J})$ and so $\pi(x_1, ..., x_n, \frac{1}{f(x_1,...,x_n)}) = (x_1, ..., x_n)$, i.e. we have surjective.

Next, say $a = (a_1, ..., a_{n+1}), b = (b_1, ..., b_{n+1}) \in V(\mathcal{J})$ such that $\pi(a) = \pi(b)$, then $a_i = b_i$ for $1 \leq i \leq n$. We note this would force $f(a_1, ..., a_n) = f(b_1, ..., b_n)$ Moreover, since $a, b \in V(\mathcal{J})$ we have $a_{n+1}f(a_1, ..., a_n) - 1 = 0 = b_{n+1}f(b_1, ..., b_n) - 1$ which imply $a_{n+1}f(a_1, ..., a_n) = b_{n+1}f(b_1, ..., b_n) = 1$. This forces $b_{n+1} = a_{n+1}$ as k is an integral domain, i.e. $a = b$ and so π is injective.

Now it suffice to find a quasi-projective morphism inverse from $X\backslash V(f)$ to $V(\mathcal{J})$ as we have shown π is bijective. Define a map $\tau : X\backslash V(f) \to V(\mathcal{J})$ to be $\tau = (f_1, ..., f_{n+1})$ where $f_1 = x_1, ..., f_n = x_n$ are coordinate map and $f_{n+1} = x_{n+1}$. This is because in $V(\mathcal{J})$ we have $x_{n+1} = \frac{1}{f}$ as this is how we defined $V(\mathcal{J})$, i.e. we have

[1] Note we can show $V(\mathcal{I}) = V(\sqrt{\mathcal{I}})$. Indeed, we have $V(\mathcal{I}) = V(I(V(\mathcal{I}))) = V(\sqrt{I})$

this relation $x_{n+1}f - 1 = 0$ in $V(\mathcal{J})$. Hence this map τ is polynomial and τ and π are inverse of each other.

Therefore, we indeed have they are isomorphic, i.e. $X \backslash V(f)$ is affine. ♡

Example 3.6.12. Is $\mathbb{A}^2 \backslash \{(0,0)\}$ affine or not?

Remark 3.6.13. Let X be affine quasi-projective variety, then we know $X \backslash V(f)$ is affine for $f \in k[X]$. In particular, we have

$$\begin{aligned} k[X \backslash V(f)] &\cong k[V(\mathcal{J})] \\ &\cong k[x_1, ..., x_{n+1}]/\mathcal{J} \\ &\cong k[x_1, ..., x_{n+1}]/(I + \langle x_{n+1}f - 1\rangle) \\ &\cong k[X][x_{n+1}]/\langle x_{n+1}f - 1\rangle \\ &\cong k[X][\frac{1}{f}] \end{aligned}$$

Note $k[X][\frac{1}{f}]$ is just the localization of the ring $k[X]$ at $\frac{1}{f}$.

Theorem 3.6.14. *Any quasi-projective variety is **locally affine**.*

Proof. Let X be a quasi-projective variety in $\mathbb{P}^n$. Then X has a finite open cover $X \cap U_i$ where $\bigcup_{i=1}^n U_i = \mathbb{P}^n$ where $U_i \cong \mathbb{A}^n$.

Now, $X \cap U_i$ is locally closed in U_i. Note U_i is open, so U_i^c is closed, i.e. $U_i^c = V(\mathcal{J})$ for some radical ideal $\mathcal{J}$. So, $X \cap U_i = X \backslash U_i^c = V(\mathcal{I}) \backslash V(\mathcal{J}) \subseteq U_i \cong \mathbb{A}^n$ for some radical ideals $\mathcal{I}$ and $\mathcal{J}$.

WLOG, we may assume that $\mathcal{J} = \langle g_{i1}, ..., g_{is_i}\rangle$ where $g_{i1}, ..., g_{is_i} \in k[x_1, ..., \hat{x_j}, ..., x_n]$. Then $X \cap U_i = V(\mathcal{I}) \backslash \bigcap_{j=1}^{s_i} V(g_{ij}) = \bigcup_{j=1}^{s_i} V(\mathcal{I}) \backslash V(g_{ij})$. Then, Lemma 3.6.11 asserts that $V(\mathcal{I}) \backslash V(g_{ij})$ is affine.

Lastly, it suffice to check that each $V(\mathcal{I}) \backslash V(g_{ij})$ is open in X. Because, if we have shown this, then $X = \bigcup_{i=0}^n (X \cap U_i) = \bigcup_{i=0}^n \bigcup_{j=1}^{s_i} (V(\mathcal{I}) \backslash V(g_{ij}))$.

We may assume that $V(I)$ is the closure of $U_i \cap X$ in $U_i \cong \mathbb{A}^n$. Now, $V(\mathcal{I}) \backslash V(g_{ij}) = (X \cap U_i) \cap (V(\mathcal{I}) \backslash V(g_{ij})) = X \cap (U_i \backslash V(g_{ij}))$ is open in X. ♡

Corollary 3.6.14.1. *A basis of open sets for quasi-projective varieties is affine varieties.*

Proof. If X is quasi-projective and $U \subseteq X$ is open. Then U is quasi-projective and so by the above theorem it has an open affine cover. ♡

Remark 3.6.15 (Midterm)**.** It will cover through projective varieties (not including quasi-projective).

1. 1 computational question
2. 3 pages of short answer proof or give examples.

Remark 3.6.16. Recall: Previously, affine meant $X = V(I) \subseteq \mathbb{A}^n$. Now it means X is a quasi-projective variety and X is isomorphic to an affine variety.

3.7 Regular Functions

Remark 3.7.1. In the theorem, we showed if X is affine, $U \subseteq X$ is open, then there exists $f \in k[X]$ such that $X \backslash V(f) \subseteq U$ and $X \backslash V(f)$ is affine. In addition, we computed $k[X \backslash V(f)] = k[X][\frac{1}{f}]$, which is the localization of $k[X]$ at $f \in k[X]$.

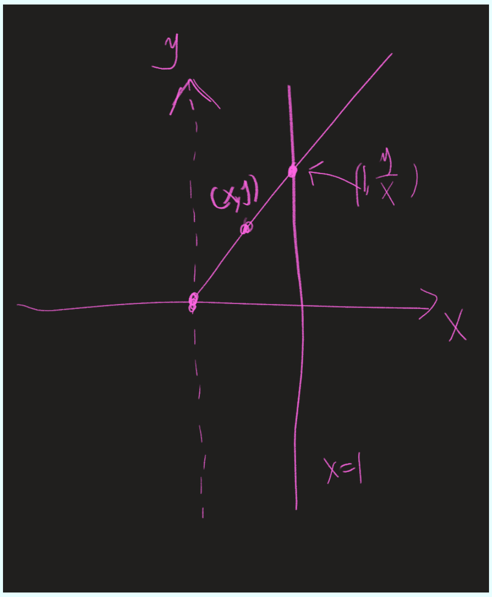
This is telling us that locally on X we get new functions of the form $\frac{g}{f}$ where $f, g \in k[X]$. Regular functions are maps $X \to k$ locally of the form $\frac{g}{f}$.

Definition 3.7.2. Let X be affine, $U \subseteq X$ open. We say a function $F : U \to k$ is called ***regular at*** $p \in U$ if there exists an open nbhd of p $W \subseteq U$ and $f, g \in k[X]$ such that $\forall q \in W, g(q) \neq 0$ and $F|_W = \frac{f}{g}|_W$.

We say F is ***regular on*** U if F is regular at all $p \in U$.

We define $O_X(U)$ to be the set of all regular functions on U.

Example 3.7.3. Consider $X := \mathbb{A}^2 \backslash V(x)$, X is affine. Consider $F : X \to k$ to be $(x, y) \mapsto \frac{y}{x}$. This is a regular function as $\frac{y}{x} \in k[x, y][\frac{1}{x}] = k[X]$. Geometrically, this can be viewed as



This is a projection away from the point $(0, 0)$ onto the line $x = 1$.

More generally, you can project away from any point p to get a regular function.

Remark 3.7.4. Note $k[X] \subseteq O_X(X)$.

Theorem 3.7.5. *If X is affine, then $k[X] = O_X(X)$.*

Proof. Given $g \in Q_X(X)$, by definition, we know for all $p \in X$, there exists open set U_p and $h_p, k_p \in k[X]$ such that $g_{U_p} = \frac{h_p}{k_p}|_{U_p}$ with $k_p(q) \neq 0$ for all $q \in U_p$.

We can shrink U_p if necessary, to assume U_p is affine of the form $X \backslash V(F_p)$ where $F_p \in k[X]$.

Note $\{U_p : p \in X\}$ is an open cover of X, and since X is quasi-compact[1], we can choose a finite subcover $U_{p_1}, ..., U_{p_t}$. For ease of notation, say $U_{p_i} = U_i$, $k_i = k_{p_i}$ and $h_i = h_{p_i}$ for $1 \leq i \leq t$.

Note k_i does not vanish on U_i. We have $X = \bigcup_{i=1}^t U_i$ and note $U_i \subseteq X \backslash V(k_i)$. Hence we get $X = \bigcup_{i=1}^t (X \backslash V(k_i))$, therefore, $\emptyset = \bigcap_i V(k_i) = V(k_1, ..., k_t)$.

By Nullstellensatz, we have $\sqrt{\langle k_1, ..., k_t \rangle} = k[X]$. Hence, $1 \in \sqrt{\langle k_1, ..., k_t \rangle}$ and so $1^p \in \langle k_1, ..., k_t \rangle$, i.e. $1 \in \langle k_1, ..., k_t \rangle$.

Therefore, there exists $l_i \in k[X]$ such that $1 = \sum_{i=1}^t l_i k_i$. (Note this is the algebraic analogue of partition of unity).

Hence, we get

$$g = 1 \cdot g = \sum_{j=1}^t l_j k_j g$$

On U_i, we have $g|_{U_i} = \sum_j l_j k_j \frac{h_i}{k_i}$, let $f = \sum_j h_j l_j \in k[X]$. We claim $f = g$, and if so, then we are done as $f \in k[X]$.

Note

$$g|_{U_i \cap U_j} = (g|_{U_i})|_{U_i \cap U_j} = (g|_{U_j})|_{U_i \cap U_j}$$

where

$$(g|_{U_j})|_{U_i \cap U_j} = (\frac{h_j}{k_j})|_{U_i \cap U_j}$$

and

$$(g|_{U_i})|_{U_i \cap U_j} = (\frac{h_i}{k_i})|_{U_i \cap U_j}$$

Hence we have

$$h_j k_i|_{U_i \cap U_j} = h_i k_j|_{U_i \cap U_j}$$

Exercise: Try to show this imply if X is irreducible then $h_j k_i = h_i k_j$ as functions globally.

---

[1] By HW question from A2

Now, if we assume the above fact, we have

$$
\begin{aligned}
g|_{U_i} &= \sum_j l_j k_j \frac{h_i}{k_i} \\
&= \sum_j l_j (k_j h_i) \frac{1}{k_i} \\
&= \sum_j l_j (k_i h_j) \frac{1}{k_i} \\
&= \sum_j l_j h_j = f|_{U_i}
\end{aligned}
$$

So, g and f agree as functions on an open cover, so they agree globally as functions and the proof follows (for irreducible X). ♡

Definition 3.7.6. If X is quasi-projective, $U \subseteq X$ and $F : U \to k$. Then F is ***regular at*** p if there exists an open affine $W \subseteq U$ that contains p such that $F|_W$ is regular at p, i.e. $F|_W \in Q_W(W) = k[W]$.

We say F is ***regular on*** U if F is regular at all p. Again, we denote $O_X(U)$ be the set of regular functions on U.

Remark 3.7.7. Think $F : X \to k$ is regular if locally it is in $k[W]$ for $W \subseteq X$ affine.

Remark 3.7.8. Properties of O_X:

1. We have $O_X(U)$ is a ring for all $U \subseteq X$. In fact, this is a k-algebra. Indeed, we can add and multiply functions on $U \to k$. Also, this is indeed a k-vector space, hence a k-algebra.
2. We have a restriction map: $U \subseteq V \subseteq X$, the restriction map $\rho_U^V : O_X(V) \to O_X(U)$ is given by $F \mapsto F_U$.
3. If $f, g \in O_X(U)$ and $\bigcup_{i \in I} U_i = U$ and $f|_{U_i} = g|_{U_i}$ for all $i \in I$, then $f = g$.
4. If $U = \bigcup_{i \in I} U_i$ and $f_i \in O_X(U_i)$ and $\forall i, j \in I, f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. Then there exists $f \in O_X(U)$ such that $f|_{U_i} = f_i$ for $i \in I$.

Remark 3.7.9. Note we can abstractify O_X. Given a topological space, define a functor as follows:

$$\mathcal{F} : \text{open sets}(X) \to \text{sets}$$

i.e. given $U \subseteq X$ be open, $\mathcal{F}(U)$ is equal some set. Given $U \subseteq V$, part of the data of $\mathcal{F}$ is restriction map $F(V) \to F(U)$.

If $\mathcal{F}$ satisfies properties 3 and 4, then it is called a ***sheaf***.

Remark 3.7.10. If $f : X \to Y$ is a morphism of quasi-projective varieties. If $F \in O_Y(U)$ then $F \circ f$ is regular on $f^{-1}(U)$, i.e. $F \circ f \in O_X(f^{-1}(U))$. Viz, we get a pullback map from $O_Y(U) \to O_X(f^{-1}(U))$, denoted by f^*.

Example 3.7.11. On HW, we will show $O_{\mathbb{A}^2}(\mathbb{A}^2 \backslash 0) = k[\mathbb{A}^2]$. Then we will use this to show $\mathbb{A}^2 \backslash 0$ is not affine.

Chapter 4

Classical Constructions

活水还须活火烹，自临钓石取深清。

大瓢贮月归春瓮，小杓分江入夜瓶。

茶雨已翻煎处脚，松风忽作泻时声。

枯肠未易禁三碗，坐听荒城长短更

苏轼

4.1 Veronese Map

Definition 4.1.1. The degree d ***Veronese map*** $v_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ is given by

$$v_d(x_0 : ... : x_n) = (x_0^d : x_0^{d-1}x_1 : x_0^{d-2}x_2 : ... : x_n^d)$$

Note we may want to fix an ordering on the degree d monomials.

Remark 4.1.2. Veronese map gives a non-trivial embedding of varieties to higher dimension projective space.

Example 4.1.3. Consider $v_3 : \mathbb{P}^1 \to \mathbb{P}^3$ with

$$(x, y) \mapsto (x^3 : x^2y : xy^2 : y^3) =: (X : Y : Z : W)$$

Then, we have $v_3(\mathbb{P}^1)$ is cut out by equations $XZ = Y^2, YW = Z^2, XW = YZ$. This is the twisted cubic.

Proposition 4.1.4. *v_d is an embedding, i.e. it defines an isomorphism onto its image.*

Proof. Let $X = v_d(\mathbb{P}^n) \subseteq \mathbb{P}^m$ where $m = \binom{n+d}{d} - 1$. Let the coordinates on $\mathbb{P}^m$ be Z_I where $I = (i_0, ..., i_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ with $\sum_{j=1}^{n+1} i_j = d$, i.e. the coordinates are indexed by degree d monomials. For example, $I = (i_0, ..., i_n)$ means we are talking about the place with the monomial $x_0^{i_0} x_1^{i_1} ... x_n^{i_n}$.

First we show v_d is well-defined because if $v_d(x_0 : ... : x_n) = (0 : ... : 0)$ then $x_i^d = 0$ for all i and hence $x_i = 0$.

Let $U_i \subseteq X$ be the open set where the x_i^d coordinate is not zero, i.e. $z_{(0,...,0,d,0,...,0)}$ with d at ith place.

Consider the map $\phi_i : U_i \to \mathbb{P}^n$ given by

$$\phi_i(z) = (z_{(1,...,d-1,...,0)} : ... : z_{(0,...,d-1,...,1)})$$

where $d-1$ occur at ith place. That is, we send z to the $(n+1)$-tuple of its coordinates indexed by $x_0 x_i^{d-1}, ..., x_n x_i^{d-1}$.

Note

$$\phi_i(v_d(x_0 : ... : x_n)) = (x_0 x_i^{d-1} : ... : x_n x_i^{d-1}) = (x_0 : ... : x_n)$$

Observe this inverse map only works if $v_d(x_0 : ... : x_n) \in U_i$. However, note $\bigcup_i U_i$ forms a cover of $v_d(\mathbb{P}^n)$.

Hence, we only left to show those ϕ_i's agree on the intersection. On $U_i \cap U_j$, we have x_i^d and x_j^d both non-zero. Hence, we have

$$(x_0 x_i^{d-1} : ... : x_n x_i^{d-1}) = (x_0 x_j^{d-1} : ... : x_n x_j^{d-1})$$

via scalling by $(\frac{x_j}{x_i})^{d-1}$. ♡

Remark 4.1.5. In general, $v_d(\mathbb{P}^n) \subseteq \mathbb{P}^{\binom{n+d}{d}-1}$ is cut out by the equations

$$z_I z_J = z_K z_L$$

where $I + J = K + L$ where $I, J, K, L \in \mathbb{Z}_{\geq 0}^{n+1}$.

One reason to care about Veronese is that v_d takes degree d equations on $\mathbb{P}^n$ and turns them into degree 1 equations on $\mathbb{P}^{\binom{n+d}{d}-1}$.

4.2 A Half-Day Away From Algebraic

Remark 4.2.1. Today[1] we gonna do some Enumerative Geometry, yeah, without Algebraic :)

[1] half today actually

Example 4.2.2. Question: if $V(f) = X \subseteq \mathbb{P}^3$ with f be degree 3. How many lines lie on X?

Answer: 27.

This is a hard question and we are not going to show it.

Theorem 4.2.3. *Given 5 points in $\mathbb{P}^2$, there exists a conic (degree 2 curve) C such that C contains the 5 points. Moreover, if no 4 of the points are collinear, then C is unique.*

Proof. A conic, by definition is of the form $C_{(a,b,c,d,e,f)} = V(ax^2 + by^2 + cz^2 + dxy + exz + fyz)$ with $a, b, c, d, e, f \in k$. In particular, the set of conics is the same as all possible values of (a, b, c, d, e, f). Observe that some choices of (a, b, c, d, e, f) yield same conic. In addition, note $C_{(a,b,c,d,e,f)} = C_{(a',...,f')}$ if and only if $(a, b, c, d, e, f) = \lambda(a', ..., f')$.

Hence, the set of conics can be identified with $\mathbb{P}^5$, i.e. the points in $\mathbb{P}^5$ can be identified with geometric objects (conic) in $\mathbb{P}^2$. We say $\mathbb{P}^5$ is the ***moduli space*** of conics.

If $p = (\alpha : \beta : \gamma) \in \mathbb{P}^2$, then a conic $C_{(a,b,c,d,e,f)}$ contains p if and only if $g(\alpha, \beta, \gamma) = 0$ where $g(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz$. Note this $g(\alpha, \beta, \gamma)$ is a linear equation in a, b, c, d, e, f.

Let $H_p = \{(a : b : c : d : e : f) : g(\alpha, \beta, \gamma) = 0\}$, which in turn is the set of conics contains p. In particular, observe $H_p \cong \mathbb{P}^4$.

Now, given 5 points, $p_1, p_2, ..., p_5$. A conic C contains $p_1, ..., p_5$ if and only if $C \in H_{p_1} \cap ... \cap H_{p_5}$, which is not empty(as we are solving five linear equations). Hence we get the existence.

The conic is unique if and only if the intersection of the hyperplanes is a single point if and only if some four of these are collinear. ♡

4.3 Segre Embedding

Remark 4.3.1. Recall we showed $\mathbb{A} \times \mathbb{A} \ncong \mathbb{A}^2$. So, one question arises is that how do we define $X \times Y$, where X, Y are quasi-projective?

Example 4.3.2. Consider $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ via $(a : b) \times (c : d) \mapsto (ac : ad : bc : bd)$. This map is indeed a morphism for sure.

We first observe this is an embedding. WLOG, say $ac \neq 0 \Rightarrow a \neq 0$ then we have $(ac : ad) = (c : d)$ as $a \neq 0$.

The image of this map is cut out by the equation $xw = yz$. We know topology on $\mathbb{P}^3$ so we define topology on $\mathbb{P}^1 \times \mathbb{P}^1$ via this embedding.

In particular, this gives a map from $\mathbb{A}^1 \times \mathbb{A}^1$ to $\mathbb{A}^3$ with $(a:1) \times (c:1) \mapsto (ac:a:c:1)$. This induces the topology on $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ (where $\mathbb{A}^1 \times \mathbb{A}^1$ is the same as $\mathbb{A}^2$ in the set sense).

Definition 4.3.3. The ***Segre embedding*** $s : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$ is given by

$$(x_0 : ... : x_n) \times (y_0 : ... : y_m) \mapsto (x_0y_0 : x_0y_1 : ... : x_ny_m)$$

Remark 4.3.4. Let Z_{ij} be the x_iy_j coordinate of $\mathbb{P}^{(n+1)(m+1)-1}$. Then $s((x_0 : ... : x_n) \times (y_0 : ... : y_m)) = \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_0 & ... & y_m \end{bmatrix} = \begin{bmatrix} x_0y_0 & ... & x_0y_m \\ x_1y_0 & ... & x_1y_m \\ \vdots & \ddots & \vdots \\ x_ny_0 & ... & x_ny_m \end{bmatrix} = [z_{ij}]$.

Definition 4.3.5. Let X, Y be quasi-projective varieties. We have $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, then $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ where $\mathbb{P}^n \times \mathbb{P}^m$ can be embeded to some $\mathbb{P}^N$ via Segre embedding. This will be the topology on $X \times Y$.

Theorem 4.3.6. *s is an embedding and the image is cut out by the vanishing of all 2 by 2 minors of $[z_{ij}]$.*

Proof. Let X be the image of s. Then $X \subseteq V(2 \times 2 \text{ minors})$, i.e. $s((x_0 : ... : x_n) \times (y_0 : ... : y_m))$ is a rank 1 matrix.

Conversely, given a $[z_{ij}]$ where all 2 by 2 minors vanish, it is rank 1. So $[z_{ij}] = \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_0 & ... & y_m \end{bmatrix}$ for some x_i and y_j. Not all x_i vanishes, because that would force all $z_{ij} = 0$.

Therefore, we have $X = V(2 \times 2 \text{ minors})$.

Thus, the question is, why is s an embedding? Let $[z_{ij}] = \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_0 & ... & y_m \end{bmatrix}$, we know it has rank 1, so all columns are scalar multiplies, i.e. not all columns are 0. Say the jth column is non-zero, then $\begin{bmatrix} x_0y_j \\ \vdots \\ x_ny_j \end{bmatrix} = \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix}$ up to scalar, i.e. as a point in $\mathbb{P}^n$.

So, we can recover $(x_0 : ... : x_n)$. Similarly we can recover $(y_0 : ... : y_m)$. ♡

4.4 Grassmannian

Definition 4.4.1. We define $Gr(d, n)$ to be the set of d-dimensional subspaces of k^n. This is called the ***Grassmannian***.

Example 4.4.2. We have $Gr(1, n+1) = \mathbb{P}^n$. Observe Grassmannian is the generalization of projective spaces.

Right now, $Gr(d, n)$ is a set but we will give it the structure of a variety.

Theorem 4.4.3. *$Gr(d, n)$ can be embedded in $\mathbb{P}^{\binom{n}{d}-1}$.*

Proof. Let $V \in Gr(d, n)$ where $V \subseteq k^n$. Choose basis $v_1, ..., v_d \in k^n$ of V. Let

$$A = [v_1, ..., v_d]$$

be a $n \times d$ matrix with ith column equal v_i. A is a full rank matrix.

Conversely, given a full rank $n \times d$ matrix A, then we get $V \subseteq k^n$ where V is the column space of A.

Given two such full rank $n \times d$ matrices, when do they yield the same V? This happens iff $B = AP$ where $P \in GL_d(k)$.

This shows

$$Gr(d, n) = \frac{\{\text{full rank n by d matrices}\}}{GL_d(k)}$$

Given $1 \leq i_1 < i_2 < ... < i_d \leq n$, let $I = (i_1, ..., i_d) \in \mathcal{I}$ where $\mathcal{I}$ is the collection of all I with a specific order (note $\mathcal{I}$ is finite and $|\mathcal{I}| = \binom{n}{d} - 1$). Let $\Delta_I(A)$ be the d by d minor of A with rows $i_1, ..., i_d$. We have a map

$$\iota : Gr(d, n) \mapsto \mathbb{P}^{\binom{n}{d}-1}$$

given by

$$\iota(A) = (\Delta_{I_i}(A) : 1 \leq i \leq \binom{n}{d} - 1)$$

Note not all $\Delta_{I_i}(A) = 0$ because A has full rank. Also, ι is well-defined because if A and B represent the same $V \subseteq k^n$, then $B = AP$ and so $\iota(B) = (\Delta_{I_i}(A) \cdot det(P) : 1 \leq i \leq \binom{n}{d} - 1)$

We will show injectivity later. ♡

Definition 4.4.4. If V is a finite dimensional vector space over k. Define $\mathbb{P}(V)$ to be the 1 dimensional subspaces of V. Note $\mathbb{P}(V) \cong \mathbb{P}^{dim(V)-1}$.

Definition 4.4.5. Let V be vector space, we define the ***exterior algebra*** $\Lambda^d(V)$ to be

$$\Lambda^d(V) = \bigotimes_{i=1}^{d} V/A$$

where A is a subspace of $\bigotimes_{i=1}^{d} V$ where $v_1 \otimes ... \otimes v_d \in A$ iff $v_1 \otimes ... \otimes v_d$ change sign when we swap two distinct component of $v_1 \otimes ... \otimes v_d$.

Remark 4.4.6. Why do we need exterior algebra?

Let $A = [v_1, ..., v_n]$ be n by n matrix. Let $e_1, ..., e_n$ be the standard basis, then we have

$$v_1 \wedge ... \wedge v_n = (det(A))e_1 \wedge ... \wedge e_n$$

ALso, note $dim(\Lambda^d(V)) = \binom{n}{d}$ with $\{e_I : I \in \mathcal{L}\}$ forms a basis, where $\mathcal{L} = \{(i_1, ..., i_k) : 1 \leq i_1 < i_2 < ... < i_k \leq n\}$ and e_I is defined to be $e_{i_1} \wedge ... \wedge e_{i_k}$ with $I = (i_1, ..., i_d)$.

Observe then we have

$$v_1 \wedge ... \wedge v_n = \sum_{I \in \mathcal{I}} \Delta_I(A) e_I$$

Definition 4.4.7. We define $Gr(d, W)$ to be d-dimensional subspaces of W.

Remark 4.4.8. Then the map $\iota : Gr(d, n) \to \mathbb{P}^{\binom{n}{d}-1}$ becomes a map between $Gr(d, W)$ and $\mathbb{P}(\Lambda^d W)$ given by $\iota(V) = \Lambda^d(V)$ where V is a d-dimensional subspace and $\Lambda^d(V)$ is a 1-dimensional subspace.

Remark 4.4.9. Now we can show ι is injective. Choose a basis $v_1, ..., v_d \in W$ for V. Let $\omega = v_1 \wedge ... \wedge v_d$ in $\Lambda^d(V)$. Let $\phi : W \to \Lambda^{d+1}W$ given by $w \mapsto w \wedge \omega$, then we have $V \subseteq Ker(\phi)$. This is because $v \in V$ then $v \wedge v_1 \wedge ... \wedge v_d \in \Lambda^{d+1}(V)$ where $\Lambda^{d+1}(V)$ has dimension $\binom{d}{d+1} = 0$.

It will be as an exercise to show $V = Ker(\phi)$. So, if we just understand $\omega \in \Lambda^d(W)$, then we can recover $V \subseteq W$ as the kernel of ϕ. However, $\Lambda^d(V) = span(\omega) \subseteq \Lambda^d(W)$, i.e. just knowing[1] $\Lambda^d(V) \subseteq \Lambda^d(W)$, we recover $V \subseteq W$.

Definition 4.4.10. The map $\iota : Gr(d, n) \to \mathbb{P}^{\binom{n}{d}-1}$ is called ***Plucker embedding***.

Remark 4.4.11. Observe $G_r(d, n)$ is irreducible. Via ι, $G_r(d, n) \subseteq \mathbb{P}^{\binom{n}{d}-1}$ is closed, so $Gr(d, n)$ is a projective variety.

Remark 4.4.12.

1. We have $dim(\mathbb{A}^n) = n$.
2. Also, X is irreducible, $\emptyset \neq U \subseteq X$ be open, then $dim(U) = dim(X)$.
3. Let $X \subseteq \mathbb{P}^n$ be irreducible projective variety. Let $H \subseteq \mathbb{P}^n$ be a hyperplane, i.e. H is the vanishing loci of linear equations. Then $dim(X) - 1 \leq dim(X \cap H) \leq dim(X)$. In fact, $dim(X \cap H) = dim(X) - 1$ unless $X \subseteq H$.

Remark 4.4.13. $\mathbb{P}^{\binom{n}{d}-1}$ is covered by $\{U_i\}_{1 \leq i \leq \binom{n}{d}}$, the standard affine patches. What is $\iota(Gr(d, n)) \cap U_i$?

This is the subset of a $n \times d$ matrix where we set one of the $d \times d$ minors to be non-zero. Permuting coordinates, we can assume $\Delta_{\{1,2,...,d\}}(A) \neq 0$, i.e. we have $A = \begin{bmatrix} B \\ C \end{bmatrix}$ where B is $d \times d$ and $det(B) \neq 0$. Hence, up to $GL_d(k)$-action, the vector space represented by A is $AB^{-1} = \begin{bmatrix} I \\ D \end{bmatrix}$ where I is the identity matrix and D is

[1] Note this means we found a left inverse

arbitrary.

Hence, the standard affine patch intersect with $Gr(d,n)$ is equal

$$\left\{ \begin{bmatrix} I \\ C \end{bmatrix} : C \text{ is } (n-d) \times d \right\} \cong \mathbb{A}^{(n-d)d}$$

So $Gr(d,n)$ is covered by $\binom{n}{d}$ opens which each is isomorphic to $\binom{n}{d}$. In particular, the dimension of $Gr(d,n)$ is the $(n-d)d$.

4.5 Degree of a Projective Variety

Definition 4.5.1. If $X \subseteq \mathbb{P}^n$ be irreducible projective variety. Then the ***degree of*** X is

$$deg(X) := \max\{|X \cap L| : L \subseteq \mathbb{P}^n \text{ linear }, dim(X) + dim(L) = n, |X \cap L| < \infty\}$$

where $L \subseteq \mathbb{P}^n$ is linear means $L = V(l_1, ..., l_r)$ where l_i are degree 1 homogeneous equations.

Example 4.5.2. Consider $X = V(yz - x^2) \subseteq \mathbb{P}^2$. We remark $dim(X) = 1$ and so we are looking at $L \subseteq \mathbb{P}^2$ with $dim(L) = 1$.

In $\mathbb{A}^2$ with $z = 1$, picture of X is $y = x^2$ and take $L : y = 2x + 3$, we have $X \cap L$ is two points. Now, at infinity, $z = 0$ and we have $x^2 = 0 = z$ and so $y = 1$, i.e. $(0 : 1 : 0)$ is in X at infinity. At ∞, the equation for L is the projective closure is $y = 2x + 3z$ where $(0 : 1 : 0) \notin L$. So, for this particular line L we have $|X \cap L| = 2$.

What if L is horizontal, i.e. $L : y = -1$. We have $|X \cap L| = 2$ over $\mathbb{C}$. If L is vertical, we have $l : x = cz$. In the affine patch $\mathbb{A}^2$ we have one point in the intersection. At infinity, we have $z = 0$ and $(0 : 1 : 0) \in X \cap L$, hence $|X \cap L| = 2$.

However, if we choose a tangent line, say $L : y = 0$, then $|X \cap L| = 1$. However, if we work with schemes, this is one point with multiplicity 2, i.e. we get $|X \cap L| = 2$.

To summerize, $deg(X) = 2$. Moreover, $|X \cap L| = 2$ unless L is tangent to X.

Note lines in $\mathbb{P}^2$ is $Gr(2,3)$. If L is not tangent, then $|X \cap L| = 2$. Consider $U \subseteq Gr(2,3)$ where U is the lines that is not tangent to X. We showed that $L \in U$ then $|X \cap L| = 2$, i.e. a general line has $|X \cap L| = 2$.

Specifically, the term "general plane" means there exists open $\emptyset \neq U \subseteq Gr(m,n)$ such that some property holds.

Theorem 4.5.3. *If F is irreducible homogeneous polynomial of degree d. Then $V(F) \subseteq \mathbb{P}^n$ then $deg(V(F)) = d$.*

Proof. We have $dim(V(F)) = n - 1$. Say $dim(L) = 1$ where $L \subseteq \mathbb{P}^n$ is linear. Then $L \cap V(F) = V(F|_L)$. Choose $L \cong \mathbb{P}^1$, choose a point $p \in L \backslash V(F|_L)$ where $L \backslash p \cong \mathbb{A}^1$. Note $V(F|_L) = V(F|_{L\backslash p})$.

Note $F|_{L\setminus p} \in k[t]$, this is a degree d polynomial. If L is general, then $F|_{L\setminus p}$ has distinct roots, i.e. $|V(F|_{L\setminus p})| = d$. If L is not general then $F|_{L\setminus p}$ can have fewer than d roots, i.e. it does not matter. ♡

Proposition 4.5.4. *Degree is a projective invariant, i.e. if $\sigma : \mathbb{P}^n \to \mathbb{P}^n$ is an automorphisms and $X \subseteq \mathbb{P}^n$, then $deg(X) = deg(\sigma(X))$.*

Proof. We know $Aut(\mathbb{P}^n) = PGL_{n+1}(k)$ and in particular, σ is given by a linear change of coordinates, i.e. multiplication by an invertible matrix.

Hence, $L \subseteq \mathbb{P}^n$ is linear if and only if $\sigma(L)$ is linear. Thus, if $|X \cap L| < \infty$ then $|X \cap L| = |\sigma(X \cap L)| = |\sigma(X) \cap \sigma(L)|$. Thus the proof follows. ♡

Remark 4.5.5. Note degree ***does*** depends on the choice of embedding, i.e. depends on the ambient environment. Viz, if $X \cong Y$ and $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, we do not necessarily have $deg(X) = deg(Y)$.

Example 4.5.6. Consider $\mathbb{P}^1$ is isomorphic to $v_d(\mathbb{P}^1) \subseteq \mathbb{P}^d$. Or, we can embed $\mathbb{P}^1$ to $\mathbb{P}^d$ via the map ϕ given by $(x : y) \mapsto (x : y : 0 : ... : 0)$.

However, we have $deg(v_d(\mathbb{P}^1)) = d$ while $deg(\phi(\mathbb{P}^1)) = 1$.

Example 4.5.7. We know that $deg(V(F)) = deg(F)$. What about the degree of $V(F_1, ..., F_s)$? One guess is that $deg(V(F_1, ..., F_s)) = \prod_{i=1}^{s} deg(V(F_i))$. However, this is not true.

Consider $v_3(\mathbb{P}^1) \subseteq \mathbb{P}^3$, the twisted cubic. It had degree 3 but $v_3(\mathbb{P}^1) = V(xw - y^2, xw^2 - z^3)$ and the product would be 6, it is not equal 3.

The essential problem is that if $X = v_3(\mathbb{P}^1)$ then $I(X) \neq \langle xw - y^2, xw^2 - z^3\rangle$, i.e. $X = V(\mathcal{I})$ where $\mathcal{I}$ is a 2-generated ideal but $I(X) = \sqrt{\mathcal{I}}$ is not 2-generated.

Definition 4.5.8. If $X \subseteq \mathbb{P}^n$ then the ***codimension of*** X is $codim(X) = n - dim(X)$.

Definition 4.5.9. Let $X \subseteq \mathbb{P}^n$ be a projective variety, then X is a ***set-theoretic complete intersection*** if $X = V(\mathcal{I})$ where $\mathcal{I} = \langle F_1, ..., F_c\rangle$ where F_i are homogeneous and $c = codim(X)$.

Example 4.5.10. The twisted cubic has dimension 1 and codimension 2.

Next, $v_3(\mathbb{P}^1) = V(\mathcal{I})$ where $\mathcal{I}$ is 2-generated. Hence twisted cubic is a set-theoretic complete intersection.

Definition 4.5.11. Let $X \subseteq \mathbb{P}^n$ be projective variety. Then X is called ***complete intersection*** if $I(X)$ is generated by $codim(X)$ elements.

Remark 4.5.12. The difference between set-theoretic complete intersection and complete intersection is that for complete intersections, the radical of the ideal is generated by $codim(X)$ many elements, while for set-theoretic case, it is the ideal itself is generated by $codim(X)$ many elements.

Example 4.5.13.

1. Twisted cubic is not a complete intersection.
2. (Open Problem) If $C \subseteq \mathbb{P}^3$ is an irreducible curve (dimension 1), then is C a set-theoretic complete intersection?

Theorem 4.5.14. *If X is a complete intersection with $I(X) = \langle F_1, ..., F_c \rangle$ with $c = codim(X)$, then $deg(X) = \prod_{i=1}^{c} deg(F_i)$.*

Theorem 4.5.15 (Bezout's Theorem)**.** *If $C, C' \subseteq \mathbb{P}^2$ be curves and if they do not share any irreducible components, then $C \cap C'$ is $deg(C) \cdot deg(C')$ many points when counted with multiplicity.*

4.6 Hilbert Functions

Remark 4.6.1. Let $X \subseteq \mathbb{P}^n$ be projective variety. Consider $R = \frac{k[x_0,...,x_n]}{I(X)} = \bigoplus_{d \geq 0} R_d$ where R_d is the set of degree d homogeneous polynomials mod out by degree d elements of $I(X)$.

Definition 4.6.2. The ***Hilbert function*** of $X \subseteq \mathbb{P}^n$ is $h_X : \{0, 1, 2, ...\} \to \mathbb{Z}$ with $h_X(d) = dim_k(R_d)$.

Example 4.6.3. Let $X = V(xy - z^2) \subseteq \mathbb{P}^2$. Then $R = k[x, y, z]/\langle xy - z^2 \rangle$.

Next, note $R_0 = k$ and $h_X(0) = 1$. Note $R_1 = kx \oplus ky \oplus kz$ with $h_x(1) = 3$. Then, $R_2 = (kx^2 \oplus ky^2 \oplus kz^2 \oplus kxy \oplus kxz \oplus kyz)/k(xy - z^2)$ and so $h_x(2) = 6 - 1 = 5$.

Next, R_3 is equal the vector space generated by $x^3, y^3, ..., xyz$ mod out by vector space generated by $x(yz - z^2), y(xy - z^2), z(xy - z^2)$. Thus, $h_x(3)$ is equal the number of degree 3 monomial minus 3, which is equal 7. In this case, we should see $h_X(d) = 2d + 1$.

Theorem 4.6.4. *There exists a unique polynomial p_X such that $p_X(d) = h_X(d)$ for all d sufficiently large. Moreover, $deg(p_X) = dim(X)$ and the leading coefficient of p_X is given by $\frac{deg(X)}{(dim(X))!}$.*

Definition 4.6.5. We call p_x the ***Hilbert polynomial***.

Theorem 4.6.6 (Riemann-Roch Theorem)**.** *Every coefficient of p_X is interpreted in terms of Chern classes.*

Remark 4.6.7. Note h_X and p_X depend on the embedding $X \subseteq \mathbb{P}^n$, e.g. $\mathbb{P}^1 \subseteq \mathbb{P}^1$, then the leading coefficient of p_X will be 1. On the other hand, if we embed $\mathbb{P}$ to $\mathbb{P}^2$ using Veronese, then leading coefficient of p_X will be 2.

Remark 4.6.8. Suppose you want to study all subvarieties $X \subseteq \mathbb{P}^n$. Question: Is this a variety?

There is a way of making this a scheme.If you perturb the equations of X a little, i.e. "deform X", then you get a new variety with the same Hilbert polynomial. Hartshorne's thesis give us that connected components (of this) are parameterized by Hilbert polynomials.

The (connected) Hilbert scheme is fix Hilbert polynomial p then look at all subvarieties X of $\mathbb{P}^n$ with $p_X = p$. Note, if you want this to be compact, you need to look at subschemes, rather then subvarieties.

Chapter 5

Smoothness and Tangent Space

今人不见古时月，今月曾经照古人。

古人今人若流水，共看明月皆如此。

唯愿当歌对酒时，月光长照金樽里。

李白

5.1 Tangent Space

Definition 5.1.1. Let $p \in X \subseteq \mathbb{A}^n$ where $X = V(\mathcal{I})$ is affine variety with radical ideal $\mathcal{I} = \langle f_1, ..., f_r \rangle$. Translate to assume $p = 0 \in \mathbb{A}^n$. Given a line $\ell \subseteq \mathbb{A}^n$ through 0, when is ℓ tangent to X at 0?

Note $\ell = \{ta : t \in k\}$ where $a \in \mathbb{A}^n$ is fixed. Then, $X \cap \ell$ is cut out by $f_i(ta_1, ..., ta_n) = 0$ for all i where $f_i(ta_1, ..., ta_n)$ is now a polynomial in t. Since we are working with algebraically closed field, we have $f_i(at)$ can be written as a product of linear terms. Say the ***multiplicity*** of f_i at 0 is the maximal m_i such that t^{m_i} divides $f_i(at)$. Then, we define the ***multiplicity*** of ℓ to X at 0 is $\min\{m_1, ..., m_r\}$.

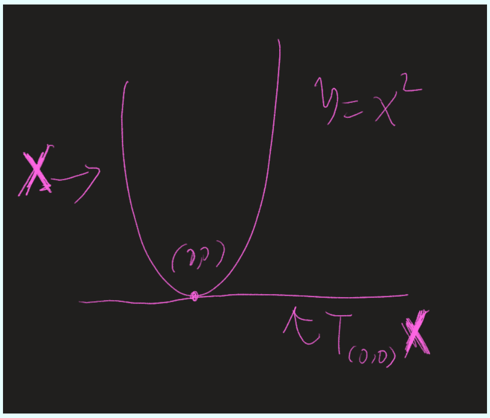
We say ℓ is ***tangent to*** X ***at*** 0 if multiplicity is greater than or equal to 2.

Example 5.1.2. Consider $X = V(y - x^2) \subseteq \mathbb{A}^2$. Then $\ell = \{(ta, tb) : t \in k\}$. Let $f = y - x^2$, we have $f(ta, tb) = tb - (ta)^2 = -t^2a^2 + tb$. If $b = 0$ then multiplicity is 1 and if $b = 0$ then multiplicity is 2.

So ℓ is tangent iff $b = 0$.

Definition 5.1.3. The ***tangent space*** of X at p is T_pX, which is the union of all the lines tangent to X at p.

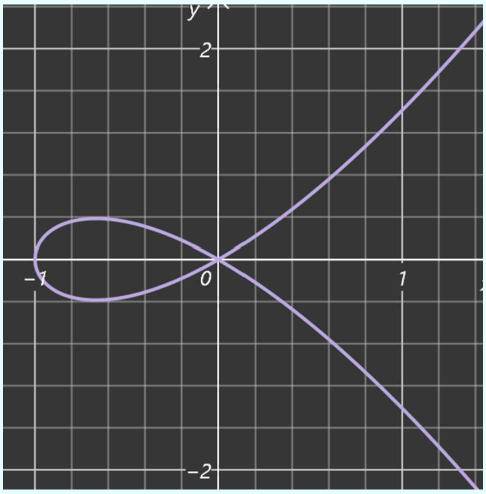
Example 5.1.4. We have



Remark 5.1.5. We will soon see that $T_pX \subseteq \mathbb{A}^n$ is a linear space, i.e. if you translate so that p is at 0, then T_pX is a sub-vector space of $\mathbb{A}^n$.

Also, we would have T_pX is independent of the choice of f_i.

Example 5.1.6. Consider the nodal cubic $X = V(y^2 - x^3 - x^2)$ with the graph

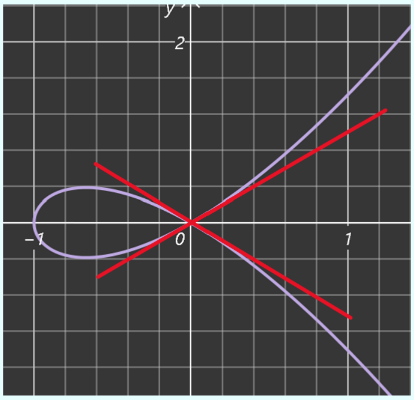


Let $\ell = \{(ta, tb) : t \in k\}$. When is ℓ tangent to $(0,0)$? We have

$$\begin{aligned}(tb)^2 - (ta)^3 - (ta)^2 &= t^2b^2 - t^3a^3 - t^2a^2 \\ &= t^2(b^2 - ta^3 - a^2)\end{aligned}$$

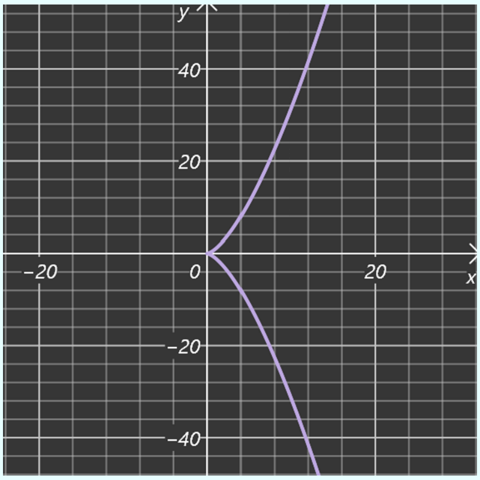
Viz, we always have a root at 0 of multiplicity greater than or equal to 2 and so every ℓ is tangent. Hence, $T_{(0,0)}X = \mathbb{A}^2$.

It is weird that it isn't just two tangent lines



Here we need to introduce a related concept, called the ***tangent cone*** at the origin of a plane curve defined by $F(x, y) = 0$. This is the variety defined by the lowest-degree terms of the polynomial $F(x, y)$. In our example, the lowest degree terms are $y^2 - x^2$ and so we get $V(y^2 - x^2) = V(y - x) \cup V(x + y)$ where we quickly observe the two red tangent lines in the above graph is defined[1] by $x - y = 0$ and $x + y = 0$.

Example 5.1.7. Consider the cuspidal cubic $V(y^2 - x^3)$ with graph



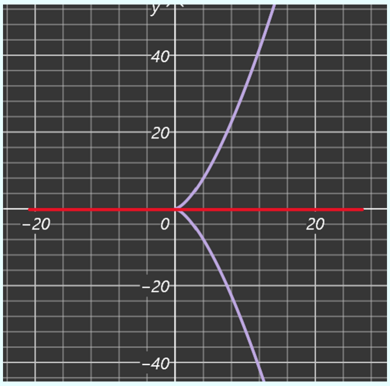
A easy calculation shows that

$$t^2a^2 - t^3b^3 = t^2(a^2 - tb^3)$$

and hence $T_{(0,0)}X = \mathbb{A}^2$ as well.

The tangent cone for this case is the vanish loci of y^2, i.e. $V(y^2)$. Thus, we get the tangent cone is the x-axis:

[1]Forgive my bad drawing, it is supposed to be the two lines, LOL



In particular, we note it should actually be interpreted as "the x-axis counted twice" as it is the vanish of y^2 rather than y.

Remark 5.1.8. If we believe T_pX is a linear space, i.e. vector space up to translattion, we can talk aboud $dim(T_pX)$ as k-vector spaces.

Also, if X is irreducible, then $dim(T_pX) \geq dim(X)$.

Definition 5.1.9. We say p is a ***smooth point of*** X if $dim(T_pX) = dim(X)$. Otherwise we say p is a ***singular point of*** X.

Definition 5.1.10. We say X is ***smooth*** if $\forall x \in X$, x is smooth.

Remark 5.1.11. 0 is a singular point of cuspidal and nodal cubics.

Also, if X is reducible, then $T_pX \geq dim(W)$ for all irreducible components W containing p.

Definition 5.1.12. Given $f \in k[x_1, ..., x_n]$ and $p \in \mathbb{A}^n$, the ***differential of*** f ***at*** p is the linear part in the "Taylor expansion" of f at p. Viz, we write $df|_p(x-p) := \sum_{j=1}^n \frac{\partial f}{\partial x_j}(p)(x_j - p_j)$.

Example 5.1.13. If $f = x^5 \in \overline{\mathbb{F}_5}[x]$ then $\frac{\partial f}{\partial x} = 5x^4 = 0$.

Theorem 5.1.14. *Let $p \in X = V(\mathcal{I}) \subseteq \mathbb{A}^n$ with $\mathcal{I} = \langle f_1, ..., f_r \rangle$. Then $T_pX = V(df_1|_p, ..., df_r|_p)$. Moreover, T_pX is linear space and independent of choice of generators of I.*

Proof. We can assume by translation that $p = 0$. Thus, $0 \in X$ means $f_i(0) = 0$ for all i. So, we can write $f_i = df_i|_p + g_i$ where $df_i|_p$ is the linear part and g_i is the higher order terms.

Let $\ell = \{(ta_1, ..., ta_n) : t \in k\}$, then

$$\begin{aligned} f_i(ta_1, ..., ta_n) &= df_i|_p(ta_1, ..., ta_n) + g_i(ta_1, ..., ta_n) \\ &= tdf_i|_p(a_1, ..., a_n) + \underbrace{g_i(ta_1, ..., ta_n)}_{\text{divisible by } t^2} \end{aligned}$$

Hence, ℓ is tangent iff for all i, we have $t^2 \mid f_i(ta_1, ..., ta_n)$ iff $\forall i, df_i|_p(a_1, ..., a_n) = 0$. Viz, $T_pX = V(df_1|_p, ..., df_r|_p)$ and note it is cut out by linear equations, so we have it is a linear space.

Lastly, let's show T_pX is independent of choice of generators of $\mathcal{I}$. If $\mathcal{I} = \langle g_1, ..., g_q \rangle$, then $g_j = \sum_{i=1}^r h_{ij} f_i$ and so we have

$$dg_j|_p = \sum_{i=1}^{r} (dh_{ij}|_p f_i(p) + h_{ij}(p) df_i|_p)$$

However, $f_i(p) = 0$ because $X = V(f_1, ..., f_r)$ and $p \in X$. So, we have

$$dg_i|_p = \sum h_{ij} df_i|_p$$

Thus, $\langle dg_1|_p, ..., dg_q|_p \rangle \subseteq \langle df_1|_p, ..., df_r|_p \rangle$. By symmetry, the two ideals are equal. $\heartsuit$

Example 5.1.15. Consider $V(f)$ where $f = y^2 - x^3 - x^2$, the nodal cubic. THen, we have $T_{(0,0)}X = V(df|_0)$ by above theorem. In particular, $df|_0 = -(3x^2+2x)|_0(x-0) + (2y)|_0(y-0) = 0$ and so $T_{(0,0)}X = V(0) = \mathbb{A}^2$.

Remark 5.1.16. If X is quasi-projective and $p \in X$, choose an open affine U containing p and we have $U \subseteq \mathbb{A}^n$, so we can define T_pU. However, if choose different open affine, you get different T_pU's for different choices of U.

It turns out, $dim(T_pU)$ is independent of choice of U. This allows us to define smoothness and singularities of quasi-projective varieties.

Remark 5.1.17. Note in commutative algebra, we would define tangent space to be, if $p \in X$ is affine, then p correspond to an maximal ideal $\mathfrak{m} \subseteq k[X]$, then the tangent space is $(\mathfrak{m}/\mathfrak{m}^2)^*$.

Remark 5.1.18. If X is reducible with irreducible components $X_1, ..., X_n$ then all points of $X_i \cap X_j$ with $i \neq j$ are singular. Thus, the notion of smoothness immediately reduces to the case of irreducibility.

Example 5.1.19.

1. $\mathbb{P}^n$ is smooth because locally it is $\mathbb{A}^n$, which is smooth.
2. We have $\mathbb{P}^n \cong v_d(\mathbb{P}^n) \subseteq \mathbb{P}^N$ and so $v_d(\mathbb{P}^n)$ is smooth.
3. Moreover, $\mathbb{P}^n \times \mathbb{P}^m$ is locally $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$ and so it is smooth.

5.2 Jacobi Criterion

Definition 5.2.1. We define the ***smooth locus*** of X to be $X^{sm} = \{p \in X : p \text{ is smooth}\}$. Similarly we define the ***singular locus***, X^{sing}, to be the set of points singular in X.

Remark 5.2.2. Let X be quasi-projective variety then the smooth locus X^{sm} is non-empty and open in X.

If $X = V(\mathcal{I}) \subseteq \mathbb{A}^n$ with X irreducible and $\mathcal{I} = \langle f_1, ..., f_r \rangle$ then the singular locus of X is $X \cap V(\mathcal{J})$ where $\mathcal{J}$ is generated by the $c \times c$ minors ($c = n - dim(X) =$

$codim(X)$) of the matrix

$$Jac := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n} \end{bmatrix}$$

Example 5.2.3. Let $X = V(f) \subseteq \mathbb{A}^2$ and $f = y^2 - x^3$. Then $c = 2 - 1 = 1$. Thus, we have the Jacobian matrix equal

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} -3x^2 \\ 2y \end{bmatrix}$$

Then, we are looking at the vanishing of 1×1 minors, and this is the same as saying $-3x^2 = 2y = 0$ and so $x = y = 0$. Thus the singular locus is $\{(0,0)\}$.

Example 5.2.4. Assume $char(k) \nmid n$. Let $X = V(f) \subseteq \mathbb{A}^2$ where $f = y^n - p(x)$ where $p(x) \in k[X]$ and $n \geq 2$. Then, $\mathcal{J} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle p'(x), y^{n-1} \rangle$. Thus, $X^{sing} = \{(a,0) : p'(a) = p(a) = 0\}$. This is the same as saying a is a root of $p(x)$ with multiplicity at least 2.

Hence, we can apply this to nodal cubic, which is $y^2 - (x^3 + x^2)$.

Theorem 5.2.5 (Jacobi Criterion)**.** *This will be the proof of Remark 5.2.2*

Proof. To show X^{sm} is open, it is enough to show the affine statement $X^{sing} = X \cap V(\mathcal{J})$. Then, let $p \in X \subseteq \mathbb{A}^n$ with $X = V(f_1, ..., f_r)$, then $T_P X = V(df_i|_p(x-p))$ and we observe

$$\begin{bmatrix} df_1|_p(x-p) \\ \vdots \\ df_r|_p(x-p) \end{bmatrix} = Jac(p) \cdot \begin{bmatrix} x_1 - p_1 \\ \vdots \\ x_n - p_n \end{bmatrix}$$

where $Jac(p)$ just means Jac evaluate at p.

Hence, $T_p X = Ker(Jac(p))$ and p is singular iff $dim(T_p X) > dim(X)$ iff

$$dim(k^n)/Ker(Jac(p)) < n - dim(X)$$

iff $dim(Im(Jac(p))) < n - dim(X)$ iff $rank(Jac(p)) < n - dim(X)$ iff p is in the vanish of $\mathcal{J}$.

Lastly, we need to show X^{sm} is not empty. Let $X = V(\mathcal{I}) \subseteq \mathbb{A}^n$ such that $\mathcal{I} = \sqrt{\mathcal{I}} = \langle f_1, ..., f_r \rangle$. We will only handle the case $r = 1$ and later in the course, we will say how to do $r > 1$.

Then $X = V(f) \subseteq \mathbb{A}^n$ and so $X^{sing} = V(\langle f \rangle + \langle \{\frac{\partial f}{\partial x_i} : 1 \leq i \leq n\} \rangle)$ and so if $X^{sing} = X$ then we would have $\frac{\partial f}{\partial x_i} \in \sqrt{\mathcal{I}} = \mathcal{I}$ and so $f \mid \frac{\partial f}{\partial x_i}$. If $char(k) = 0$ then by degree consideration this is impossible.

If $char(k) = p$, this is possible but we can check $f = g^p$ for some $g \in k[x_1, ..., x_n]$ but then f is not irreducible, which is a contradiction.

To handle $r > 1$, show there exists open subset $U \subseteq X$ such that $U = V(f)$, so we reduce to case $X = V(f)$. To show the existence of such U, we use technique called generic projection. ♡

Remark 5.2.6 (Resolution of Singularities)**.** Question: Given a singular X, can we "modify" X to make it smooth but do this by only modifying X^{sing}? Hironka showed this is possible in $char(k) = 0$: given singular X, there exists a $\pi : X' \to X$ where X' is relatively compact so that π is an isomorphism over X^{sm}.

Example 5.2.7 (Exercise)**.** If X is an irreducible projective variety of codimension c and $I(X) = \langle f_1, ..., f_r \rangle$, then $X^{sing} = X \cap V(\text{c by c minors of Jacobi})$. In particular, X is smooth if and only if $V(f_1, ..., f_r) \subseteq \mathbb{A}^{n+1}$ has only singular points at 0, i.e. $X^{sing} = \{(0 : ... : 0)\}$.

Example 5.2.8. Assume $char(k) = 0$. Consider $X = V(x^2+y^2-z^2) \subseteq \mathbb{P}^2$, consider $Y := V(x^2 + y^2 - z^2) \subseteq \mathbb{A}^3$, which is a cone over X. Then

$$Jac = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ -2z \end{bmatrix}$$

Thus, we are looking at 1 by 1 minors, i.e. $Y^{sing} = V(f, 2x, 2y, -2z) = 0$ and so X is smooth.

5.3 Family of Varieties

Definition 5.3.1. Let $X \subseteq \mathbb{A}^n$ be smooth of dimension d. We define the ***tangent bundle*** to be $TX := \{(p, v) : v \in T_pX\} \subseteq X \times \mathbb{A}^n$.

Remark 5.3.2. Note we have the following diagram commutes

$$\begin{tikzcd} TX \arrow[r, "\iota"] \arrow[d, "\pi"] & X \times \mathbb{A}^n \arrow[dl] \\ X & \end{tikzcd}$$

where π is projection and ι is the inclusion.

Note the fiber of $p \in X$ in the map $\pi : TX \to X$ is just T_pX, i.e. $\pi^{-1}(p) = T_pX$. Since X is smooth, all fibers are vector spaces of dimension d. This is why it's called a bundle.

Example 5.3.3. Consider $V(xy - t) \subseteq \mathbb{A}^2$ where t is a constant. Then, we have

$$\begin{tikzcd} X = V(xy-t) \arrow[r, "\subseteq"] \arrow[dr, "\pi"'] & \mathbb{A}^3_{x,y,t} \arrow[d, "\pi"] \\ & \mathbb{A}_t \end{tikzcd}$$

where π and π' are projections.

Now, to study when t is constant, we have $X_t = \pi^{-1}(t)$, e.g. $X_5 = \pi^{-1}(5) = V(xy-5) \subseteq \mathbb{A}^2_{x,y}$.

However, $X_0 = V(xy) \subseteq \mathbb{A}^2$. This is called the degeneration of X_0, i.e. it degenerated from a smooth variety $X_{t\neq 0}$ to X_0, which is singular.

Example 5.3.4. Consider $X = V(tx-ty) \subseteq \mathbb{A}^3_{x,y,t}$ with projection $\pi : \mathbb{A}^3_{x,y,z} \to \mathbb{A}_t$. Then $X_{t\neq 0} = V(tx-ty) = V(x-y) \subseteq \mathbb{A}^2_{x,y}$ and $X_0 = V(0x-0y) = \mathbb{A}^2_{x,y}$. Viz, when t is not zero, we get a dimension 1 variety, while if $t=0$, we jumped to a variety with dimension 2.

This is not a nice property for a "continuous family of varieties" that we may want.

Remark 5.3.5. What goes wrong with this example above?

Note $X = V(tx-ty) \to \mathbb{A}^1$ correspond to a map $k[t] \to \frac{k[x,y,z]}{\langle tx-ty\rangle} =: A$. The problem is that in A, $x-y \neq 0$ but $t(x-y)=0$, i.e. $x-y$ is a non-trivial t-torsion element. Hence, we only want to allow families $\pi : X \to \mathbb{A}^1$ such that $k[X]$ has no non-trivial t-torsion.

More generally, we only want families $\pi : X \to Y$ such that $k[Y] \to k[X]$ makes $k[X]$ into a flat $k[Y]$-module.

5.4 Bertini's Theorem

Remark 5.4.1. A Bertini's theorem is a result along the following line:

Let $X \subseteq \mathbb{P}^n$ be projective variety. Say X has some property P (e.g. P=smooth). Then $X \cap H$ also has property P if $H \subseteq \mathbb{P}^n$ is a **general** hyperplane.

Definition 5.4.2. Note $\{H \subseteq \mathbb{P}^n : H \text{ is hyperplane}\}$ is again a projective space $\mathbb{P}^n$. Thus, H gets identified with the point $(a_0 : ... : a_n) \in \mathbb{P}^n$, and is called the ***dual projective space*** $(\mathbb{P}^n)^*$.

Remark 5.4.3. Consider $\mathbb{P}^n \times (\mathbb{P}^n)^*$, we have an incidence subvariety $I := \{((x,H) : x \in H)\} = \{a_0x_0 + ... + a_nx_n = 0\}$.

Theorem 5.4.4. *Let $X \subseteq \mathbb{P}^n$ be smooth projective variety, then there exists open* [1] *$\emptyset \neq U \subseteq (\mathbb{P}^n)^*$ such that for all hyperplane $[H] \in U$, H does not contain an irreducible component of X and $H \cap X$ is smooth.*

Proof. Consider the bad locus $B \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^*$ given by $(x,H) \in B$ if and only if $x \in H$ and either $X \subseteq H$ or x is a singular point in $H \cap X$. Suppose that $X = V(f_1, ..., f_r) \subseteq \mathbb{P}^n$. Since X is smooth, the Jacobian $(\frac{\partial f_i}{\partial x_j}) := \begin{bmatrix} \frac{\partial f_1}{\partial x_0} & \cdots & \frac{\partial f_r}{\partial x_0} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_r}{\partial x_n} \end{bmatrix}$ has rank $n-d$ at every point $x \in X$.

[1] recall all open sets are dense in $\mathbb{P}^n$

Note $B \subseteq I = \{(x, H) : x \in H\}$ so $B \subseteq \{(x, H) : x \in X, a_0x_0 + ... + a_nx_0 = 0\}$. Now, if $X \cap H$ is singular at x then the Jacobian

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_0} & \cdots & \frac{\partial f_r}{\partial x_0} & a_0 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_r}{\partial x_n} & a_n \end{bmatrix}$$

has rank $n - d$.

If $X \subseteq H$, then by replacing $\mathbb{P}^n$ by $H \cong \mathbb{P}^{n-1}$ then rank of the Jacobian is $(n-1)-d$. Hence, our bad locus B is actually given by the information on the rank of the Jacobian (with the additional column $(a_0, ..., a_n)^T$). Since it's rank is equal $n - d$, we see that all $(n - d + 1)$-minors have vanishing determinants.

Exercise: we need d linear equations to capture the vanishing of these $n - d + 1$-minors. Hence the fiber of $\pi_1 : B \to X$ given by $\pi_1(x, H) = x$ over $x \in X$, say B_x, has dimension $n - d - 1$. Thus $dim(B) \leq n - (d + 1) + dim(X) = n - 1$.

Now, consider $\pi_2 : B \to (\mathbb{P}^n)^*$. It is easy to see that $dim(\pi_2(B)) \leq dim(B) \leq n - 1$. Take $U = (\mathbb{P}^n)^* \backslash \pi_2(B)$.

Exercise: show this U works. ♡

Remark 5.4.5.

1. A "general" hyperplane H cuts X smoothly.
2. A variant of Bertini's theorem for hypersurfaces: a general hypersurfaces[1] S cuts X smoothly.

[1] Moduli space of hypersurfaces of deg d in $\mathbb{P}^n$ is given by $\mathbb{P}^N$ where $N = \binom{n+d}{d} - 1$

Chapter 6

Additional Lecture Material

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滟滟随波千万里，何处春江无月明！

江流宛转绕芳甸，月照花林皆似霰。

空里流霜不觉飞，汀上白沙看不见。

江天一色无纤尘，皎皎空中孤月轮。

张若虚

6.1 Discrete Valuation Ring(DVR)

Definition 6.1.1. Given a field k, a ***discrete valuation*** on k is a surjective function $v : k \to \mathbb{Z} \cup \{\infty\}$ such that

1. $v(ab) = v(a) + v(b)$
2. $v(a + b) \geq \min\{v(a), v(b)\}$
3. $v(0) = \infty$ (a convention)

Remark 6.1.2.
1. Note $v(1) = 0$ using property 1.
2. Easy to check the p-adic valuation is a discrete valuation on $\mathbb{Q}$.

Definition 6.1.3. Given a discrete valuation on k, define the ***discrete valuation ring*** to be

$$R := \{a \in k : v(a) \geq 0\}$$

Remark 6.1.4. Observe there is a canonical ideal in R given by $\mathfrak{m} := \{a \in R : v(a) > 0\}$. It is easy to see it is an ideal.

Proposition 6.1.5. *Given DVR R. $\mathfrak{m} := \{a \in R : v(a) > 0\}$ is maximal and it is the unique maximal ideal.*

Proof. Pick an arbitrary $a \in R\backslash\mathfrak{m}$, i.e. $v(a) = 0$. Then property (1) of valuation give us $v(a^{-1}) = 0$. Hence $a^{-1} \in R$ and so a is invertible. This shows $\mathfrak{m}$ is maximal. ♡

Remark 6.1.6. Note this imply that the units of DVR R are precisely $\{a \in R : v(a) = 0\}$.

Example 6.1.7. Let k be algebraically closed and $K = k(x)$. Let $f(x) \in k[x]$, then we have $f(x) = (x-a)^l \prod_{i=1}^m (x - r_i)^{l_i}$. Thus we define $ord_a(f)$, the ***order of vanishing at*** a is defined to be $ord_a(f) = l$. Then, consider $\frac{f}{g} \in K$, we define $ord_a(\frac{f}{g}) = ord_a(f) - ord_a(g)$.

Then, for every fixed $a \in k$ we get a discrete valuation v given by $v(f) = ord_a(f)$ for all $f \in K$.

For example, say $v = ord_5$ and $f(x) = (x-5)^3(x-1)$ and $g(x) = (x-5)^2(x-2)$. Then we have $f + g = (x-5)^2((x-5)(x-1) + (x-2))$ so that $v(f+g) \geq 2 = \min\{f, g\}$.

Hence, we have associated to each element $a \in k$ a discrete valuation $v = ord_a$ on $k(x)$. It turns out that ord_a's are all the discrete valuations except one more: $v : k(x)^* \to \mathbb{Z}$ given by $f \mapsto -deg(f)$.

Remark 6.1.8. If we assume the above two types of discrete valuations are all of the valuations, then we have a canonical bijection

$$\begin{aligned}\{\text{discrete valuations on } k(x)\} &\leftrightarrow \mathbb{P}^1 \\ ord_a(f) &\mapsto (a:1) \equiv a \in k \\ -deg(f) &\mapsto (1:0) \equiv \infty\end{aligned}$$

Given any irreducible variety X, its ***function field*** $k(X)$ is equal to the following: choose an open affine $U \subseteq X$, then $k[U]$ is an integral domain, so let $k(U) := Frac(k[U])$. It turns out this $k(U)$ is independent of choice of U and we just let $k(X) = k(U)$.

Example 6.1.9. Consider $k(\mathbb{P}^n)$ and $U = \mathbb{A}^n \subseteq \mathbb{P}^n$. Then $Frac(k[\mathbb{A}^n])$ is

$$Frac(k[x_1, ..., x_n]) = k(x_1, ..., x_n)$$

Theorem 6.1.10. *There exists a canonical bijection between*

$$\{\textit{discrete valuations of } k(C)\} \leftrightarrow C$$

if C is a curve.

Lemma 6.1.11. *Let R be a DVR, then the maximal ideal $\mathfrak{m} := \{a \in R : v(a) > 0\}$ is principal.*

Proof. Note $v : k^\times \to \mathbb{Z}$ is surjective, so let $\pi \in K$ be so that $v(\pi) = 1$. We see $\pi \in \mathfrak{m}$ and we will show $\mathfrak{m} = \langle \pi \rangle$.

Let $a \in \mathfrak{m}$, we have $v(a) = n \geq 1$ and so $v(\frac{a}{\pi^n}) = 0$, i.e. $\frac{a}{\pi^n} \in R^\times$, i.e. $a = \frac{a}{\pi^n}\pi^n$. Hence we have $\mathfrak{m} = \langle \pi \rangle$. ♡

Remark 6.1.12. Every DVR is a PID.

Remark 6.1.13. Given C a curve, given a point $p \in C$. Consider the discrete valuation $v : k(C) \to \mathbb{Z}$ given by $v = ord_p$. Given $f \in \mathcal{O}_C(U)$, view f as living in the local ring $\mathcal{O}_{C,p}$. Then $\{\frac{g}{h} : h(p) \neq 0\} =: \mathcal{O}_{C,p} \subseteq Frac(k[U]) = k(C)$.

Theorem 6.1.14. *Let C be a curve, $p \in C$, then p is smooth if and only if $\mathcal{O}_{C,p}$ is a DVR.*

Remark 6.1.15. Consider $\mathcal{O}_{C,p} \supseteq \mathfrak{m}_p = \{\frac{g}{h} : h(p) \neq 0, g(p) = 0\}$, where $\mathcal{O}_{C,p}$ is functions where you can evaluate at p and $\mathfrak{m}_p$ is the functions that vanishes at p. Then, given $f = \frac{g}{h} \in \mathcal{O}_{C,p}$, the definition of order of vanishing at p is, $ord_p(f) = n$ if and only if $f \in \mathfrak{m}_p^n \backslash \mathfrak{m}_p^{n+1}$.

Example 6.1.16. Consider $p = (1:1:1) \in C = V(xy - z^2) \subseteq \mathbb{P}^2$. Look at open affine nbhd $U = \mathbb{A}^2 = \{(a, b, 1) : a, b \in k\} \subseteq \mathbb{P}^2$. Then, $p = (1, 1) \in U = V(xy - 1)$, so $\mathcal{O}_{U,p} = k[U]_\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal correspond to p, i.e. $\mathfrak{m} = \langle x-1, y-1 \rangle$.

Hence, $\mathcal{O}_{U,p} = \{\frac{f}{g} : f, g \in k[x, y]/\langle xy - 1\rangle, g(1, 1) \neq 0\}$. Then, we have $\mathfrak{m}_p = \{\frac{f}{g} : f, g \in k[x, y]/\langle xy - 1\rangle, f(1, 1) = 0, g(1, 1) \neq 0\} \subseteq \mathcal{O}_{U,p}$. Observe $x - 1$ has order of vanishing at $(1, 1)$ is 1, and so we have $\mathfrak{m}_p = \langle x - 1\rangle = \{\frac{(x-1)h(x,y)}{g(x,y)} : h, g \in k[x, y]/\langle xy - 1\rangle\}$

6.2 Divisor

Remark 6.2.1. Given a curve C, what is the smallest n needed to have an embedding $C \subseteq \mathbb{P}^n$?

For example, given $C \subseteq \mathbb{P}^{10}$, that means we used 11 variables and at least 10 equations to cut out C. Can we do this with fewer variables and fewer equations?

Suppose we have $C \subseteq \mathbb{P}^n$, then given a hypersurface $H = V(f)$ and consider $H \cap C$. We would expect $|H \cap C|$ is finite. Then, $|H \cap C| = \{p_1, ..., p_r\} \subseteq C$ with multiplicities. Represent this as $\sum_{i=1}^r n_i p_i$, where n_i is the multiplicity.

Definition 6.2.2. A ***divisor on*** C is a formal sum $\sum_{i=1}^r n_i p_i$, where $n_i \in \mathbb{Z}$ and $p_i \in C$.

Remark 6.2.3. Note the divisors are just formal sums, some of them may be as we described in Remark 6.2.1, i.e. it is obtained by an intersection of hypersurface and C, and some divisors may not be. Hence, the divisors of C is a very huge set.

Remark 6.2.4. Let $Div(C) = \{\text{divisor on } C\}$, then this is a group under formal addition. In particular, $Div(C) = \bigoplus_{p \in C} \mathbb{Z}$.

From an embedding $C \subseteq \mathbb{P}^n$, we get a divisor on C for every hypersurface $H \subseteq \mathbb{P}^n$.

Now, we hope that, by understanding $Div(C)$, perhaps we can reconstruct all possible embeddings of $C \subseteq \mathbb{P}^n$.

Remark 6.2.5. Given an embedding $C \subseteq \mathbb{P}^n$ and $H, H' \subseteq \mathbb{P}^n$ be two hypersurfaces. We would like to find condition so that the divisors of $H \cap C$ and $H' \cap C$ are "equivalent".

Definition 6.2.6. Given $f \in k(C)^\times$, we define a divisor $div(f) = \sum_{p \in C} v_p(f)p$. Divisors of this form are called ***principal divisors***.

Remark 6.2.7. First, those principal divisors are indeed divisors as we are going to show the sum if finite.

We know there exists open affine $U \subseteq C$ such that $f|_U = \frac{g}{h}$ where $g, h \in k[U]$. If $p \in U$, then $v_p(f) = v_p(g) - v_p(h)$ where g, h have only finitely many roots. So if p is not one of these finitely many points, then $v_p(g) = v_p(h) = 0$. Since $\emptyset \neq U \subseteq C$ is open, we have $C \backslash U$ have only finitely many points as C has dimension 1 (recall closed sets in $\mathbb{A}$ is collection of finite points and of the form $\mathbb{A} \backslash U$ where U is open). So, the divisor $div(f)$ can only have non-zero coefficients in $C \backslash U$ and the finite number of zeros of g, h.

Example 6.2.8. Let $C = V(xy - z^2) \subseteq \mathbb{P}^2$, choose open affine $U \subseteq C$. Suppose $U = \{(a : b : 1) : a, b \in k\} \cap C$, then $k[U] = k[x, y]/\langle xy - 1\rangle$ and we have $k(C) = Frac(k[U]) = \{\frac{f}{g} : f, g \in k[x, y]/\langle xy - 1\rangle\}$.

Consider $\phi \in k(C)$ to be $\phi = \frac{x^2}{(y+1)^5}$ and let us compute $div(\phi)$.

First, ϕ vanishes at $x = 0$ but that is not on our curve, i.e. not a point on U as U is cut out by $U = V(xy - 1) \subseteq \mathbb{A}^2$. However, we does see ϕ has a "pole" (meaning denominator vanishes) at $y = -1$ and it does yeild a point $(-1, -1)$ on our curve. This is a pole of order 5. So far we know that $div(\phi)|_U = -5(-1, -1)$.

Now, we need to figure out contributions to $div(\phi)$ at infinity. So to do that, need to change coordinates from $U = \{(a : b : 1) : a, b \in k\}$ to another pacth, say $U' = \{(1 : a : b) : a, b \in k\}$. To do that, homogenize ϕ: $\phi = \frac{x^2 z^3}{(y+z)^5}$. Now restrict to $x = 1$ patch, we have $V(y - z^2) \subseteq \mathbb{A}^2$ and ϕ on this pacth is $\frac{z^3}{(y+z)^5}$. However, note $y = z^2$, so we have $\phi = \frac{z^3}{(y+z)^5} = \frac{z^3}{(z^2+z)^5} = \frac{1}{z^2(z+1)^5}$. On this pacth ϕ has 2 poles, at $z = 0$ and $z = -1$. If $z = -1$ then $y = z^2 = 1$ on $x = 1$ patch. Hence, we get this point $(1 : 1 : -1) = (-1 : -1 : 1)$. This is a pole of order 5 and we saw this already in the U pacth, and it is good that this is consistent. The new information is a pole of order 2 at $z = 0$ and in this case, $y = 0$ and hence we get $(1 : 0 : 0)$.

Lastly, in $y = 1$ patch, we have $\phi = \frac{x^2 z^3}{(1+z)^5}$ and since we are in $V(x - z^2)$, we have $\phi = \frac{z^7}{(1+z)^5}$. Hence we get a new information at $(0 : 1 : 0)$ with $z = 0$ and vanishing of order 7.

Hence, overall, we have

$$div(\phi) = 7(0 : 1 : 0) - 2(1 : 0 : 0) - 5(-1 : -1 : 1)$$

Definition 6.2.9. Let $D \in Div(C)$ with $D = \sum_{i=1}^{r} n_i p_i$, we say the ***degree***, $deg(D)$, to be the sum of n_i's.

Proposition 6.2.10. *IF D is a principal divisor, then $deg(D) = 0$.*

Proof. Choose any embedding $C \subseteq \mathbb{P}^n$. Let $f \in k(C)$, represent $f|_U = \frac{g}{h}$ on U where $U \subseteq C \cap \mathbb{A}^n$ is open affine and $g, h \in k[U]$.

Let g', h' be the degree d homogenization of g and h. Consider $H_1 = V(g')$ and $H_2 = V(h')$, which are in $\mathbb{P}^n$ and are degree d hypersurfaces. In particular, $div(f) = zeros(g') - zeros(h')$ on C. Hence, $deg(div(f))$ is equal the number of zeros of g' minus number of zeros of h' counted with multiplicity. However, this is equal $|H_1 \cap C| - |H_2 \cap C|$, which is equal $d - d$ when we count multiplicity. ♡

Definition 6.2.11. The ***support of*** $D = \sum n_p p_i$ is $\{p \in C : n + p \neq 0\} = Supp(D)$.

Remark 6.2.12. Let $C \subseteq \mathbb{P}^n$ be a curve. Suppose $H = V(f)$ and $H' = V(f')$ and $deg(f) = deg(f')$. Then $\frac{f}{f'}|_C \in k(C)$ and we have $div(\frac{f}{f'}) + H' \cap C = H \cap C$.

Hence, as motivated in Remark 6.2.5, if we want to define $H \cap C$ and $H' \cap C$ are equivalent, we want to mod out by the set of principal divisors.

Lemma 6.2.13. *$Princ(C) := \{$principal divisors of $C\}$ is a subgroup of $Div(C)$.*

Proof. We have $div(f) + div(g) = \sum v_p(f)p + \sum v_p(g)p = \sum(v_p(f) + v_p(g))p = \sum v_p(fg)p$. Hence the proof follows. ♡

Definition 6.2.14. We say divisors D and D' are ***linearly equivalent*** if $\exists f \in k(C)^\times$ such that $D' = div(F) + D$. We write $D \sim D'$.

Definition 6.2.15. The group of divisors mod by linear equivalents defined above, i.e. $Div(C)/Princ(C)$, is called the ***class group*** of C and denoted by $Cl(C)$.

Remark 6.2.16. This is related to the class group from number theory. We have $Spec(\mathbb{Z})$ is a "curve", $Spec(\mathcal{O}_k)$ is curve if $\mathcal{O}_k$ a ring of integers in number field K.

It turns out, $Cl(Spec(\mathcal{O}_k))$ is just equal the usual class group of K.

Then, $spec(\mathbb{Z})$ be a curve, $k(Spec(\mathbb{Z})) = Frac(\mathbb{Z}) = \mathbb{Q}$. Every closed points of $Spec(\mathbb{Z})$ should correspond to a discrete valuation on $\mathbb{Q}$. Closed points of $Spec(\mathbb{Z})$ are primes p corresponding valuation is the p-adic valuation.

Remark 6.2.17. Consider $deg : Div(C) \to \mathbb{Z}$ given by $deg(Princ(0)) = 0$. So we get induced function $deg : Cl(C) \to \mathbb{Z}$. The degree function is surjective: choose any point $p \in C$, let $D = n \cdot p$, then $deg(D) = n$.

Let $Cl^0(C) := Ker(deg)$, this is called the Jacobian of C.

Proposition 6.2.18. *$Cl^0(\mathbb{P}^1) = 0$, i.e. $D \in Div(\mathbb{P}^1)$, then $D \sim 0$ if and only if $D = div(f)$ iff $deg(D) = 0$. Thus $Cl(\mathbb{P}^1) \cong \mathbb{Z}$.*

Proof. We just need to show if $deg(D) = 0$, then $D \sim 0$. Note $D = \sum n_p p = D' - D''$ where $D' = \sum_{n_p>0} n_p p$ and $D'' = -\sum_{n_p<0} n_p p$. Then, $0 = deg(D)$ imply $deg(D') = deg(D'')$.

To ease notation, let $D' = \sum a_p p$ and $D'' = \sum b_p p$ with $a_p, b_p > 0$. Changing coordinates, we can assume ∞ is not in the support of D, so we can identify $p \in \mathbb{P}^1$ with numbers.

Let $g = \prod(x - py)^{a_p}$ and $h = \prod(x - py)^{b_p}$ then we have $deg(g) = deg(h)$. So $\frac{g}{h} \in k(\mathbb{P}^1)$ and so $div(\frac{g}{h}) = deg(g) - deg(h) = D' - D'' = D$. ♡

Remark 6.2.19. We have $CL^0(\mathbb{P}) = 0$ is analogous to the fact that $Cl(\mathbb{Q}) = 0$.

In addition, in the above proof, the reason why it works is that in $\mathbb{P}^1$ we can think of points p in $\mathbb{P}^1$ as being numbers, but we cannot do that on more general curves.

Remark 6.2.20 (Fact)**.** We have $Cl^0(C) = 0$ iff $C \cong \mathbb{P}^1$ iff $\exists p, q \in C$, $p \neq q$ and $p \sim q$.

6.3 Elliptic Curves

Definition 6.3.1. An ***elliptic curve*** E is a smooth plane cubic, i.e. $E = V(f) \in \mathbb{P}^2$ with $deg(f) = 3$.

Theorem 6.3.2. *If E is an elliptic curve and $p_0 \in E$, then $E \to Cl^0(E)$ given by $p \mapsto p - p_0$ is a bijection.*

Proof. First we show injectivity. We need the following fact: E is not isomorphic to $\mathbb{P}^1$. ♡

Remark 6.3.3. As a result of this theorem, since $Cl^0(E)$ is a group and since there exists a bijection between this and E, we get a group structureon E for each point $p_0 \in E$.

Also, note the group structure is depends on the p_0 we have chosen as the identity is different.

Remark 6.3.4. Throughout this course we normally assumed k is algebraically closed field. However, one could also work over a number field K.

Suppose $E = V(f)$ where $f \in K[x, y, z]$, then we could ask the question what are the K-points of E, i.e. solutions (a, b, c) of $f(x, y, z) = 0$ with $a, b, c \in K$. We detnoe the set of K-points by $E(K)$ and again we have $E(K)$ is a group.

Theorem 6.3.5 (Mordell-Weil)**.** *Let K be number field and E a elliptic curve, then*

$$E(K) \cong \mathbb{Z}^r \times G$$

where G is a finite abelian group.

Remark 6.3.6. The quantity r in the above theorem is called the rank of E.

6.4 Linear Systems

Definition 6.4.1. Let C be a curve and $D_1 = \sum n_p p$ and $D_2 = \sum m_p p$ are divisors on C. Then we write $D_1 \geq D_2$ if $n_p \geq m_p$ for all p. We say D is ***effective*** if $D \geq 0$.

Definition 6.4.2. If C is a curve and $D \in Div(C)$, then we define

$$\mathcal{L}(D) := \{f \in k(C)^* : div(f) \geq -D\} \cup \{0\}$$

Example 6.4.3. Let $C = \mathbb{P}^1$ and $D = p - 2q$. Then $\mathcal{L}(D)$ consists of all rational functions f with order of vanishing at least 2 at p and allowed to have a pole of order at most 1 at p. For example, if $p = (5:1)$ and $q = (7:1)$, then $\frac{(x-7y)^3}{x-5y}$ and $(x-7y)^2$ are both in $\mathcal{L}(D)$.

Example 6.4.4. Let $C = \mathbb{P}^1$ and $D = 3\infty$. Then $\mathcal{L}(D)$ consists of all rational functions f that are allowed to have a pole of order at most 3 at ∞. Thus $f = \frac{g(x,y)}{y^3}$ with $deg(g) = 3$ and so $\mathcal{L}(D)$ is a fintie-dimensional vector space with basis $\frac{x^i y^{3-i}}{y^3}$ for $0 \leq i \leq 3$.

Lemma 6.4.5. *If $D \in Div(C)$, then $\mathcal{L}(D)$ is a k-vector space under addition.*

Proof. If $f, g \in \mathcal{L}(D)$ then $div(f+g) = \sum v_p(f+g)p \geq \sum \min(v_p(f), v_p(g))p \geq -D$. ♡

Lemma 6.4.6. *If $D \sim D'$, then $\mathcal{L}(D') \cong \mathcal{L}(D)$ as k-vector spaces, Specifically, if $D \neq D'$ and $D' = div(f) + D$, then the isomorphism $\mathcal{L}(D') \to \mathcal{L}(D)$ is given by $g \mapsto gf$.*

Proof. Note $div(g) + D' \geq 0$ iff $0 \leq div(g) + div(f) + D = div(gf) + D$, so $g \mapsto gf$ does send $\mathcal{L}(D')$ to $\mathcal{L}(D)$. We also see the map is k-linear and since $f \neq 0$, the map has inverse given by $h \mapsto hf^{-1}$. ♡

Remark 6.4.7. If $f \in k(C)^*$ has no zero or poles, then $f \in k^*$. We note this is the analogue of Louiville's theorem in complex analysis.

Proposition 6.4.8.

1. *If $deg(D) < 0$ then $\mathcal{L}(D) = 0$.*
2. *If $deg(D) \geq 0$ then $dim_k(\mathcal{L}(D)) \leq deg(D) + 1$.*

Definition 6.4.9. If D is a divisor, its complete linear system is $|D| := \{D' : D' \geq 0, D' \sim D\}$.

Proposition 6.4.10. *The projectivization of the vector space $\mathcal{L}(D)$ is identified with $|D|$: via the map*

$$\begin{aligned} \mathcal{L}(D) &\to |D| \\ f &\mapsto div(f) + D \end{aligned}$$

we obtain a bijection $\mathbb{P}(\mathcal{L}(D)) \cong |D|$.

6.5 Maps Coming From Divisors

Definition 6.5.1. Given $D \in Div(C)$, choose a basis $f_0, ..., f_n$ of $\mathcal{L}(D)$, then we define

$$\begin{aligned} \phi_D : C &\to \mathbb{P}^n \\ p &\mapsto (f_0(0) : ... : f_n(p)) \end{aligned}$$

Definition 6.5.2. We define $\ell(D) := dim_k(\mathcal{L}(D))$.

Definition 6.5.3. Let $p \in C$, we say p is a ***basepoint*** of a divisor if $f(p) = 0$ for all $f \in \mathcal{L}(D)$.

Proposition 6.5.4.

1. *For all $p \in C$, we have*

$$\ell(D) - 1 \leq \ell(D - p) \leq \ell(D)$$

2. *We have p is a basepoint if and only if $\ell(D) - 1 = \ell(D - p)$ if and only if $\mathcal{L}(D - p) = \mathcal{L}(D)$.*

Proposition 6.5.5.

1. *ϕ_D is a morphism if and only if $\ell(D - p) = \ell(D) - 1$ for all $p \in C$.*
2. *ϕ_D is a closed embeddin if and only if $\ell(D - p) = \ell(D) - 1$ for all $p \in C$ and $\ell(D - p - q) = \ell(D) - 2$ for all $p, q \in C$.*

6.6 Differentials

Definition 6.6.1. For any k-algebra R, the ***differential*** $\Omega^1_{R/k}$ or Ω^1_R is defined to be the set of finite formal sums $\sum_i a_i db_i$ modulo the relations:

1. $d(a + b) = da + db$
2. $d(ab) = adb + bda$
3. $da = 0$ for all $a \in k$.

Remark 6.6.2. Notice Ω^1_R is an R-module.

Example 6.6.3. We have $\Omega^1_{k[x]/k} \cong k[x]$ and $\Omega^1_{k[x_1,...,x_n]/k} \cong \bigoplus_{i=1}^n k[x_1, ..., x_n]$.

Chapter 7

Appendix 1: Category

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烛影摇风，一枕伤春绪。归不去。凤楼何处。芳草迷归路。

苏轼

Remark 7.0.1. This appendix uses "The Rising Sea: Foundations Of Algebraic Geometry" by Ravi Vakil and is my personal notes on reading this text for enrichment.

Remark 7.0.2. The intuition for schemes can be built on the intuition of affine varieties and it involves[1] three different simultaneous generalizations

1. We allow nilpotents in the coordinate ring, which is basically analysis (looking at near-solutions of equations instead of eaxt ones).
2. We glue affine schemes together, which is what we do in differential geometry.
3. Instead of working over algebraically closed fields, we work with more general rings, which is basically number theory (solving equations over number field, ring of integers, etc.)

Remark 7.0.3. From here now on, all rings are commutative unless says otherwise. All rings contain a unit 1. Maps of rings must send 1 to 1. We do not assume $0 \neq 1$, meaning we could have the zero ring $R = \{0\}$. We accept axoim of choice.

We may write[2] $(A, \mathfrak{m})$ for localization of A at maximal ideal $\mathfrak{m}$. Or, we will write $A_{\mathfrak{m}}$ instead. Also, $(A, \mathfrak{m}, k)$ would denote the localization of A at $\mathfrak{m}$ with residue field $k = A/\mathfrak{m}$.

We also assume the two following theorems.

[1] as Allen Knutson and Terry Tao points out that

[2] as in the book

Theorem 7.0.4 (FTFGMPID, Invariant Factor Form, Existence). *Let A be a PID, M be a finitely generated A-module, then*

$$M \cong A^r \oplus \bigoplus_{i=1}^{m} A/\langle a_i \rangle$$

for some $r \geq 0$, some non-zero, non-units $a_1 \mid a_2 \mid \ldots \mid a_m$.

Theorem 7.0.5 (FTFGMPID, Elementray Divisor Form, Existence). *Keep the same notation as Theorem 7.0.4, we also get*

$$M \cong A^r \oplus A/\langle p_1^{n_1} \rangle \oplus \ldots \oplus A/\langle p_t^{n_t} \rangle$$

where $p_1, \ldots, p_t$ are (not necessary distinct) primes in A and $n_1, \ldots, n_t > 0$ are positive integers.

7.1 Category

Definition 7.1.1. A ***category***, $\mathscr{C}$, consists of a collection of ***objects***, denoted by $obj(\mathscr{C})$ or just $\mathscr{C}$ itself, and for each pair of objects, a set of ***morphisms*** between them.

For two objects $A, B \in \mathscr{C}$, for the pair (A, B) the set of morphisms is denoted by $Mor(A, B)$. For $f \in Mor(A, B)$ we say A is the ***source*** of f and B is the ***target***.

We insists that morphism compose as expected, i.e. there is a composition

$$\begin{aligned} Mor(B, C) \times Mor(A, B) &\to Mor(A, C) \\ (g, f) &\mapsto g \circ f \end{aligned}$$

We also insists the existence of an ***identity morphism***, i.e. for each object $A \in \mathscr{C}$, there is always a morphism $Id_A : A \to A$ so that [1] for any morphism $f : A \to B$ and $g : B \to C$, we have $Id_B \circ f = f$ and $g \circ Id_B = g$.

Remark 7.1.2. Technically, this is the definition of a ***locally small category***. Also, we may call morphisms just maps.

Definition 7.1.3. A ***isomorphism*** between two objects of a category is a morphism $f : A \to B$ such that there exists a morphism $g : B \to A$, where $f \circ g$ and $g \circ f$ are the identity of B and A, respectively.

We also have ***automorphism***, which is an isomorphism of the object with itself.

Example 7.1.4.

1. A concrete example of a category would be the category of sets, denoted by $Sets$. The objects are sets and the morphisms are functions between sets.

[1] in another word, when you compose (both left and right) a morphism with the identity, you get the same morphism.

2. Another example would be the category Vec_k, where objects are vector spaces over a given field k and morphisms are linear transformations.
3. A category in which each morphism is an isomorphism is called ***groupoid***. One could show that a groupoid with one object is an equivalent definition of a group.
4. The category Ab consists of abelian groups as objects with group homomorphisms as the morphisms. Also, we have the category Grp consists of groups as object with group homomorphism as the morphisms.
5. Let A be a ring, then the A-modules form a with morphisms to be A-module homomorphisms, denoted by Mod_A. This category has an additional structure and it will be the prototypical example of an abelian category. In particular, take $A = k$ a field, we get Vec_k and take $A = \mathbb{Z}$ we get Ab.
6. There is a category $Rings$, where objects are rings and morphisms are ring homomorphisms.
7. The topological spaces, along with continuous functions as morphisms, form a category Top.

Example 7.1.5. Let $A \in \mathscr{C}$, show $Aut(A) := \{f \in Mor(A, A) : \text{f is invertible}\}$ forms a group using composition of maps. In addition, show that two isomorphic objects have isomorphic automorphism groups.

Solution. We have an identity element Id_A. Next, every element has inverse by definition of $Aut(A)$. Also, it is clearly associative. Finally, observe that for $f, g \in Aut(A)$ we have $f \circ g \in Aut(A)$ because $g^{-1} \circ f^{-1}$ give us the inverse morphism.

Let $f : A \to B$ be an isomorphism. Then we observe that $f^* : Aut(B) \to Aut(A)$ given by $f^*(g) = f^{-1}g \circ f$ would be an group isomorphism between $Aut(B)$ and $Aut(A)$. Indeed, it is needless to say this is well-defined, invertible and is an homomorphism. ♠

Example 7.1.6. We remark that in Example 7.1.4, all of the categories are ***concrete categories***. This need not be the case.

A ***partially ordered set*** (or ***poset***) is a set S along with a binary relation $\geq$ satisfying, for $x, y, z \in S$:

1. $x \geq x$
2. $x \geq y \wedge y \geq z \Rightarrow x \geq z$
3. $x \geq y \wedge y \geq z \Rightarrow x = y$

A poset $(S, \geq)$ is a category with objects to be elements of S and a single morphism from x to y if and only if $x \geq y$ (and no morphism otherwise).

Example 7.1.7. We have the category of subsets of a set and the category of open subsets of a topological space.

If X is a set, then the power set $P(X)$, which is the collection of all subsets of X, forms a poset with $\subseteq$. This is a category in the poset sense. Similarly, if X is a topological space then inclusion gives a poset and it is a category.

Definition 7.1.8. A ***subcategory*** $\mathscr{A}$ of a category $\mathscr{B}$ is a category such that has its objects to be some objects of $\mathscr{B}$. Then, the morphisms of $\mathscr{A}$ are morphisms from $\mathscr{B}$ so that it includes all the identity morphisms and are all closed under composition of each other.

7.2 Functor

Definition 7.2.1. A ***covariant functor*** F from a category $\mathscr{A}$ to $\mathscr{B}$, denoted $F : \mathscr{A} \to \mathscr{B}$, is the following date. It is a map of objects, $F : obj(A) \to obj(B)$ and for each $A_1, A_2 \in \mathscr{A}$ and morphism $m : A_1 \to A_2$, we have a morphism $F(m) : F(A_1) \to F(A_2)$ in $\mathscr{B}$.

We insists that F preserves identity morphisms and that F preserves composition, i.e. $F(m_1 \circ m_2) = F(m_1) \circ F(m_2)$.

Remark 7.2.2. Note composition of covariant functors make sense. Viz, for $F : \mathscr{A} \to \mathscr{B}, G : \mathscr{B} \to \mathscr{C}$ be two functors, we would have $G \circ F : \mathscr{A} \to \mathscr{C}$ is also a functor.

Example 7.2.3.

1. The ***identity functor***, denoted $id : \mathscr{A} \to \mathscr{A}$, is defined to be map same element to same element and same morphism to same morphism.
2. Consider the functor from Vec_k to $Sets$, that associates to each vector space its underlying set. The functor sends a lienar transformation to its underlying map of sets. This is an example of a ***forgetful functor*** where some additional structure is gorgotten. Another example of this would be $Mod_A \to Ab$ by forgetting the additional A-structure and keep the abelian group part.
3. Suppose A is an object of $\mathscr{C}$. Then there is a functor $h^A : \mathscr{C} \to Sets$ sending $B \in \mathscr{C}$ to $Mor(A, B)$ and sending $f : B_1 \to B_2$ to $Mor(A, B_1) \to Mor(A, B_2)$ given by
$$(g : A \to B_1) \mapsto (f \circ g : A \to B_2)$$
We remark that we may also call this functor the $Hom(A, -)$ functor because we are varying B while make A fixed.
4. Given two categories $\mathscr{A}, \mathscr{B}$ and $B \in \mathscr{B}$, we can define a ***constant functor*** $\Delta_{\mathscr{B}}$ to $\mathscr{B}$, given by $\Delta_{\mathscr{B}}(A) = B$ and $\Delta_D(f) = Id_B$.
5. Given the category of commutative rings $CRing$, we have family of functors $GL_n(-) : Cring \to Grp$, which send each commutative ring R to the group $GL_n(R)$ with each ring homomorphism f send to the group homomorphism defined by apply f to each entry.
6. Given the category $CRing$, we have a functor $(-)^{\times} : CRing \to Grp$ which send R to $R^{\times} := \{a \in R : \exists b \in R, ab = 1\}$ and send each ring homomorphism to the restriction of f to $R^{\times}$.

Definition 7.2.4. A covariant functor $F : \mathscr{A} \to \mathscr{B}$ is ***faithful*** if for all $A, A' \in \mathscr{A}$ the map
$$Mor_{\mathscr{A}}(A, A') \to Mor_{\mathscr{B}}(F(A), F(A'))$$
$$f : A \to A' \mapsto F(f) : F(A) \to F(A')$$

is injective and ***full*** if this map is surjective.

A covariant functor that is full and faithful is, of course, called ***fully faithful***.

Definition 7.2.5. A subcategory $\mathscr{A}$ of $\mathscr{B}$ is ***full subcategory*** if the inclusion map $\iota : \mathscr{A} \to \mathscr{B}$ is full. We note this inclusion map is always faithful.

Example 7.2.6. The forgetful functor $Vec_k \to Sets$ is faithful but not full.

Definition 7.2.7. A ***contravariant functor*** is defined in a similar way as covariant functors, where we switch the direction of all the arrows. Viz, a contravariant functor $F : \mathscr{A} \to \mathscr{B}$ is the following data. It is a map of objects $F : obj(A) \to obj(B)$ and for each $A_1, A_2 \in \mathscr{A}$ and morphism $m : A_1 \to A_2$, we have a morphism $F(m) : F(A_2) \to F(A_1)$ in $\mathscr{B}$. In particular, for contravariant functors our composition should be $F(m_1 \circ m_2) = F(m_2) \circ F(m_1)$.

Definition 7.2.8. Let $\mathscr{C}$ be a category, then the ***opposite category of*** $\mathscr{C}$ is $\mathscr{C}^{opp}$ or $\mathscr{C}^{op}$, where objects are the same as $\mathscr{C}$ and for each pair of objects $A, B \in \mathscr{C}^{op}$, we have $Mor(A, B)$ in $\mathscr{C}^{op}$ is equal the set $Mor(B, A)$ in $\mathscr{C}$.

Remark 7.2.9. Note sometimes people describe a contravariant functor $\mathscr{A} \to \mathscr{B}$ as a covariant functor $\mathscr{A}^{opp} \to \mathscr{B}$.

We also remark that we can define fullness and faithfulness for contravariant functors.

Example 7.2.10.

1. Consider Vec_k, then taking dual gives a contravariant functor $(\cdot)^* : Vec_k \to Vec_k$. Indeed, for each linear map $f : V \to W$ we have $f^* : W^* \to V^*$ and $(f \circ g)^* = g^* \circ f^*$.
2. There is a contravariant functor $Top \to Rings$ by taking a topological space X to the ring of real-valued continuous functions on X. A morphism of topological spaces $X \to Y$ induces the pullback map from functions on Y to functions on X.
3. Suppose A is an object of a category $\mathscr{C}$. Then there is a contravariant functor $h_A : \mathscr{C} \to Sets$ sending $B \in \mathscr{B}$ to $Mor(B, A)$, and sending the morphism $f : B_1 \to B_2$ to the morphism
$$(g : B_2 \to A) \mapsto (g \circ f : B_1 \to A)$$
We may also say h_A as $Hom(-, A)$ functor.
This example and Example 7.2.3.3 are both special cases of the ***functor of points***.

Definition 7.2.11. Suppose F and G are two covariant functors from $\mathscr{A}$ to $\mathscr{B}$. A ***natural transformation of covariant functors*** $F \to G$ is a family of morphisms of $\mathscr{B}$, $m_A : F(A) \to G(A)$, indexed by $A \in \mathscr{A}$ such that for each $f : A \to A'$ in $\mathscr{A}$, the following diagram commutes

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "m_A"] & F(A') \arrow[d, "m_{A'}"] \\ G(A) \arrow[r, "G(f)"] & G(A') \end{tikzcd}$$

Definition 7.2.12. A ***natural isomorphism*** of functors is a natural transformation such that each m_A is an isomorphism.

Definition 7.2.13. Given a category $\mathscr{C}$. A ***representable functor*** is a functor from $\mathscr{C}$ to $Sets$ such that is naturally isomorphic to the functor $Hom(A, -)$ for some object $A \in \mathscr{C}$. In particular, when we say the corresponding representable functor of $A \in \mathscr{A}$, we mean the functor $Hom(A, -)$.

Example 7.2.14. Let $A, B \in \mathscr{C}$ be fixed with a morphism $f : A \to B$. Consider two functors $Hom(-, A)$ and $Hom(B, -)$, which are both functor from $\mathscr{C}$ to $Sets$, we will create a natural transformation $\mathscr{C}(f, -)$ from $Hom(B, -)$ to $Hom(-, A)$ given as follows.

For $C \in \mathscr{C}$ and $g \in Mor(B, C)$, we have $\mathscr{C}(f, -)_C$, which should be a morphism from $Hom(-, A)(C) = Mor(C, A)$ to $Hom(B, -)(C) = Mor(B, C)$, is given by $g \mapsto g \circ f$, i.e. $\mathscr{C}(f, -)_C(g) = g \circ f$. This $\mathscr{C}(f, -)_C$ is indeed an morphism from $Mor(B, C)$ to $Mor(C, A)$.

We can check this is a natural transformation.

Example 7.2.15. Recall the two functors from $CRing$ to Grp in Example 7.2.3, $GL_n(-)$ and $(-)^{\times}$. We have a natural transformation between them, namely Det_R, which send $M \in GL_n(R)$ to $Det(M)$, the determinant of M.

Definition 7.2.16. Let $\mathscr{A}, \mathscr{B}$ be two categories, we write $Fun(\mathscr{A}, \mathscr{B})$ as the category of covariant functors from $\mathscr{A}$ to $\mathscr{B}$ with morphisms be natural transformations of covariant functors. Equivalently we write $Fun^*(\mathscr{A}, \mathscr{B})$ be the category of contravariant functors form $\mathscr{A}$ to $\mathscr{B}$.

Definition 7.2.17. ***Equivalence of categories*** is an equivalence relation on categories given by $\sim$ where $\mathscr{A} \sim \mathscr{B}$ iff there exists functors $F : \mathscr{A} \to \mathscr{B}$ and $F' : \mathscr{B} \to \mathscr{A}$ such that $F \circ F'$ is naturally isomorphic to the identity functor $id_{\mathscr{B}}$ and $F' \circ F$ is naturally isomorphic to the identity functor $id_{\mathscr{A}}$.

Remark 7.2.18. The right notion of two categories are "essentially the same" is not an isomorphism between two categories, i.e. a functor giving bijection of objects and morphisms but equivalence of categories. Inded, think about a vector space V, the double dual V^{**} is not the same as V, even we say they are canonically isomorphic to V.

Theorem 7.2.19 (Yoneda Lemma)**.** *Consider*[1] *a functor $F : \mathscr{A} \to Sets$ from a category $\mathscr{A}$ to $Sets$, an object $A \in \mathscr{A}$, and the corresponding $Hom(A, -)$ functor. Then the following is a bijective correspondence:*

$$\theta_{F,A} : Nat(Hom(A, -), F) \to F(A)$$

$$\theta_{F,A}(\alpha) = \alpha_A(Id_A)$$

where $Nat(Hom(A, -), F)$ is the set of all natural transformations from $Hom(A, -)$ to F and $F(A)$ is the set given by F apply to the object A.

[1] This part of the note is due to Thomas Rud and Jean-Claude Ton in the note "Yoneda Lemma"

Proof. Consider a given element in $a \in F(A)$. We define, for every object $B \in \mathscr{A}$ a mapping

$$\tau(a)_B : Hom(A, B) \to F(B)$$

given by $\tau(a)_B(f) = F(f)(a)$. Hence, this class of mappings defines a natural transformation

$$\tau(a) : Hom(A, -) \to F$$

We claim for every morphism $g : B \to C$ in $\mathscr{A}$, the following relation holds

$$F(g) \circ \tau(a)_B = \tau(a)_C \circ Hom(A, -)(g)$$

i.e. the diagram commutes

$$\begin{tikzcd}
Hom(A,B) \arrow[r, "\tau(a)_B"] \arrow[d, "Hom(A,-)(g)"] & F(B) \arrow[d, "F(g)"] \\
Hom(A,C) \arrow[r, "\tau(a)_C"] & F(C)
\end{tikzcd}$$

Indeed, for all $f \in Hom(A, B)$, we have

$$\begin{aligned}
F(g) \circ \tau(a)_B(f) &= F(g)(F(f)(a)) \\
&= (F(g) \circ F(f))(a) \\
&= F(g \circ f)(a) \\
&= \tau(a)_C \circ (Hom(A, -)(g))(f)
\end{aligned}$$

Hence, to finish the proof, we only need to show $\theta_{F,A}$ and τ are the inverse of each other.

Let $a \in F(A)$, we have $\theta_{F,A}(\tau(a)) = \tau(a)_A(Id_A) = F(Id_A)(a) = Id_{F(A)}(a)$ and so $\theta_{F,A} \circ \tau = Id_{F(A)}$ as desired.

Conversely, let $\alpha : Hom(A, -) \to F$ be an natural transformation and suppose $f : A \to B$ is in $Mor(A, B)$, then

$$\begin{aligned}
\tau(\theta_{F,A}(\alpha))_B(f) &= \tau(\alpha_A(Id_A))_B(f) \\
&= F(f)(\alpha_A(Id_A)) \\
&= \alpha_B(Hom(A, -)(f)(Id_A)), \quad \text{due to} \alpha \text{ is natural} \\
&= \alpha_B(f \circ Id_A) \\
&= \alpha_B(f)
\end{aligned}$$

Thus $\tau(\theta_{F,A}(\alpha))$ and α coincide since they have the same components. ♡

Remark 7.2.20. Intuitively, the Yoneda's lemma states that for an object $A \in \mathscr{C}$ the functor $Hom(A, -)$ determines A up to unique isomorphism. In other word, this lemma says that for locally small categories, we can study a object of a category by study the category of sets, namely, the set of morphisms.

Theorem 7.2.21 (Contravariant Yoneda)**.** *Consider a contravariant functor $F : \mathscr{A} \to Sets$, an object $A \in \mathscr{A}$, and the corresponding contravariant representable functor $Hom(-, A) : \mathscr{A}^{op} \to Sets$. Then the following is a bijective correspondence:*

$$\theta_{F,A} : Nat(Hom(-, A), F) \to F(A)$$

$$\theta_{F,A}(\alpha) = \alpha_A(Id_A^{op})$$

between the natural transformation from $Hom(-, A)$ to F and the elements of $F(A)$.

Definition 7.2.22. Let $\mathscr{C}$ be a category, we define the ***Yoneda embedding functors*** as follows

$$\begin{aligned} Y^* : \mathscr{C}^{op} &\to Fun(\mathscr{C}, Sets) \\ Y^*(A) &= Hom(A, -) \\ Y^*(f) &= Hom(A, -)(f) \end{aligned} \qquad \text{and} \qquad \begin{aligned} Y_* : \mathscr{C} &\to Fun(\mathscr{C}, Sets) \\ Y_*(A) &= Hom(-, A) \\ Y_*(f) &= Hom(-, A)(f) \end{aligned}$$

Remark 7.2.23. We can check $Y^*(Id_B) = Id_{Y^*(B)}$ and also, if $f \in Mor(A, B), g \in Mor(B, C)$, we have $Y^*(f \circ g) = Y^*(f) \circ Y^*(g)$. Thus Y^* is contravariant and similarly Y_* is covariant.

Proposition 7.2.24 (Yoneda Embedding)**.** *The Yoneda embedding functors are fully faithful.*

7.3 Universal Property

Remark 7.3.1. Given some category that we come up with, we often will have ways to produce new objects from old. One such way to define new objects is by consider a universal property. We first show that if it exists, then it is essentially unique up to unique isomorphism, then we go about construct an example of such an object to show existence.

Example 7.3.2. The product of two categories is defined by a universal property. Indeed, we can define the product of $A, B \in \mathscr{C}$ to be the object $A \times B$ with two morphisms $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$, if exists, such that for any object $Y \in \mathscr{C}$ with $\alpha : Y \to A$ and $\beta : Y \to B$, we have the follow diagram

$$\begin{tikzcd} & Y \arrow[dl, "\alpha"'] \arrow[d, dotted, "\exists!"] \arrow[dr, "\beta"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

The product of multiply objects $\prod_{i \in I} P_i$ is defined similarly, i.e. for any object $Y \in \mathscr{C}$ we want a morphism $(f_i)_{i \in I}$ so that for all $i \in I$ we have

$$\begin{tikzcd} Y \arrow[d, dotted, "\exists!(f_i)_{i\in I}"] \arrow[dr, "\alpha_i"] & \\ \prod_{i \in I} P_i \arrow[r, "\pi_i"] & P_i \end{tikzcd}$$

Definition 7.3.3. An object of a category $\mathscr{C}$ is an ***initial object*** if it has precisely one map to every object. It is a ***final object*** if it has precisely one map from every object. It is a ***zero object*** if it is both an initial and final object.

Example 7.3.4. Any two initial objects are uniquely isomorphic and any two final objects are uniquely isomorphic.

Solution. We show the first assertion. Consider $A, B \in \mathscr{C}$ to be two initial objects. Then, we have only one morphism $f : A \to B$ and $g : B \to A$. In particular, observe that $f \circ g : B \to B$ and hence $f \circ g$ is a morphism from B to B and we only have one such morphism as B is initial. Thus $f \circ g = Id_B$ is the identity morphism. Similar proof goes for $g \circ f$. ♠

Example 7.3.5.

1. In the category *Sets*, the initial object is $\emptyset$ the empty set and the final object is any singleton set. Indeed, there is only one map from empty set to anything, which is the map that takes nothing to nothing. Clearly there is only one map between any object to a singleton, which is the map that sends everything to the same element.
2. In the category *Rings*, the initial object is $\mathbb{Z}$ the integer ring and the final object is the zero ring. $\mathbb{Z}$ is initial should be clear by consider the morphism $1 \mapsto 1$ and the final object is the zero ring as this is the only one singleton ring exists.
3. In the category Top, the empty space is initial and the one point space is final.
4. In the category of groups, the trivial group is both initial and final.
5. In the category of an particular poset, minimal elements are initial and maximal elements are final. In particular, this shows that we may or may not have initial and final objects as we consider $[0, 1)$, which lack final objects and $(0, 1]$ lack initial objects.

Example 7.3.6. Recall the construction of localization of ring and modules using multiplictively closed sets. Now, let's try to define localization of modules by universal property. Suppose M is an A-module, we define the A-module map $\phi : M \to S^{-1}M$ as being initial among A-module maps $M \to N$ such that elements of S are invertible in N, i.e. $s \times \cdot : N \to N$ is an isomorphism for all $s \in S$. More precisely, any such map $\alpha : M \to N$ factors uniquely through ϕ:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "\alpha"'] & S^{-1}M \arrow[d, "\exists!"] \\ & N \end{tikzcd}$$

This is saying, $M \to S^{-1}M$ is universal (initial) among A-module maps from M to modules that are actually $S^{-1}A$-modules.

Example 7.3.7. Suppose we have morphism $\alpha : X \to Z$ and $\beta : Y \to Z$ in any category $\mathscr{C}$. Then the ***fibered product*** is an object $X \times_Z Y$ along with morphisms $pr_X : X \times_Z Y \to X$ and $pr_Y : X \times_Z Y \to Y$ where the two compositions $\alpha \circ pr_X$

and $\beta \circ pr_Y$ agree, such that given any object W with maps to X and Y (whose composition to Z agree), these maps factor through some unique $W \to X \times_Z Y$:

$$\begin{tikzcd}
W \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"] & Y \arrow[d, "\beta"] \\
& X \arrow[r, "\alpha"] & Z
\end{tikzcd}$$

By the usual universal property argument, if it exists, it is unique up to unique isomorphism.

The diagram

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "pr_Y"] \arrow[d, "pr_X"] & Y \arrow[d, "\beta"] \\
X \arrow[r, "\alpha"] & Z
\end{tikzcd}$$

is called a ***fibered/pullback/Cartesian diagram/square***.

Example 7.3.8. In the category *Sets*, we could show that

$$X \times_Z Y = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}$$

More precisely, show that the right side, equipped with its evident maps to X and Y, satisfies the universal property of the fibered product.

Indeed, let pr_Y be the projection to Y and pr_X be the projection to X. Then, for any object W with $\omega_1 : W \to X$ and $\omega_2 : W \to Y$ we see there exists unique $\phi : W \to X \times_Z Y$ given by $\phi(w) = (\omega_1(w), \omega_2(w))$ so that our diagram commutes.

Example 7.3.9. Let X be a topological space, then in the category of open sets of X, the fibered product always exists for U, V open. Indeed, say we have $\alpha : U \to Y$ and $\beta : V \to Y$ and this means U, V are open and $U, V \subseteq Y \subseteq X$. Then, the fibered product should be $U \cap V$ as we observe that if $W \subseteq U$ and $W \subseteq V$ then $W \subseteq U \cap V$. Since by definition we only have one morphism if $W \subseteq U \cap V$, this morphism is indeed unique.

Example 7.3.10. Consider the category of commutative rings, denoted by $CRing$. Then let $A, B, C \in obj(CRing)$ with $\alpha : A \to C$ and $\beta : B \to C$. We have $A \times_C B$ is exactly the same as in the *Sets* case, i.e. $A \times_C B = \{(a, b) \in A \times B : \alpha(a) = \beta(b)\}$ with pr_A and pr_B be the projection.

Example 7.3.11. If Z is the final object in a category $\mathscr{C}$, then $X \times_Z Y = X \times Y$ for any $X, Y \in \mathscr{C}$.

Example 7.3.12. Note towers of Cartesian diagrams are still Cartesian diagrams. Viz, if we have two Cartesian diagrams

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & V \arrow[d] \\
W \arrow[r] & X
\end{tikzcd}, \quad
\begin{tikzcd}
W \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

Then, we get a new Cartesian diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

We see this is the outside rectangle of tower the diagrams

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

Example 7.3.13. Given morphisms $X_1 \to Y, X_2 \to Y$ and $Y \to Z$, then there exists a natural morphism $X_1 \times_Y X_2 \to X_1 \times_Z X_2$, assuming both fibered product exits.

Indeed, observe we have the following diagrams by definition

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X_1 \times_Z X_2 \arrow[r] \arrow[d] & X_1 \arrow[d, "\beta"] \\ & X_2 \arrow[r, "\alpha"] & Z \end{tikzcd}$$

Example 7.3.14. It can be shown that, given $X_1 \to Y, X_2 \to Y$ and $Y \to Z$, we have the following diagram is a Cartesian square

$$\begin{tikzcd} X_1 \times_Y X_2 \arrow[r] \arrow[d] & X_1 \times_Z X_2 \arrow[d] \\ Y \arrow[r] & Y \times_Z Y \end{tikzcd}$$

This is called the magic diagram.

Indeed, consider the following diagram

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[rrr] \arrow[dddr, phantom, ""] \arrow[dr, red] \arrow[ddd, blue] & & & X_2 \arrow[ddd] & & \\
& X_1 \times_Z X_2 \arrow[r] \arrow[d, red] & X_2 \arrow[d] \arrow[ur, leftrightarrow, "="] & & & \\
& X_1 \arrow[r] \arrow[dl, red, leftrightarrow, "="'] & Z & & & \\
X_1 \arrow[rrr, blue, shift left] \arrow[rrr, red, shift right] & & & Y \arrow[ul] \arrow[dr, red, shift left] \arrow[dr, blue, shift right] \arrow[drr] \arrow[ddr] & & \\
& & & & Y \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& & & & Y \arrow[r] & Z
\end{tikzcd}$$

where follow the red we get $X_1 \times_Y X_2 \to X_1 \times_Z X_2 \to Y \times_Z Y$ and follow blue we get $X_1 \times_Y X_2 \to Y \to Y \times_Z Y$.

Definition 7.3.15. We define the ***coproduct*** in a category by reversing all the arrows in the definition of product. Similarly we define ***fibered coproduct*** by reversing all arrows in the definition of fibered product.

Example 7.3.16. In the category $Sets$, the coproduct C of two disjoint set $A, B \in \mathscr{C}$ is the union with maps being inclusion maps $\pi_1 : A \to C = A \cup B$ and $\pi_2 : B \to C$, i.e. for any object $Y \in Sets$

$$\begin{tikzcd}
& Y & \\
A \arrow[ur] \arrow[r, "\pi_1"] & C \arrow[u, dotted, "\exists!"] & B \arrow[ul] \arrow[l, "\pi_2"']
\end{tikzcd}$$

Definition 7.3.17. A morphism $\pi : X \to Y$ is a ***monomorphism*** if any two morphism $\mu_1 : Z \to X$ and $\mu_2 : Z \to X$ such that $\pi \circ \mu_1 = \pi \circ \mu_2$ imply $\mu_1 = \mu_2$.

Remark 7.3.18. Intuitively, it is the categorical version of an injective map, and indeed this notion generalizes the familiar notion of injective maps of sets.

Definition 7.3.19. A morphism $\pi : X \to Y$ is a ***epimorphism*** if any two morphism $\mu_1 : X \to Z, \mu_2 : X \to Z$ such that $\mu_1 \circ \pi = \mu_2 \circ \pi$ imply $\mu_1 = \mu_2$.

7.4 Limits and Colimits

Definition 7.4.1. A category is a ***small category*** if the objects and the morphisms are sets. Suppose $\mathscr{S}$ is any small category and $\mathscr{C}$ is any category. Then a functor $F : \mathscr{S} \to \mathscr{C}$ is said to be a ***diagram indexed by*** $\mathscr{S}$ and $\mathscr{S}$ is said to be an ***index category***.

Remark 7.4.2. Note we normally use partially ordered sets as our index category. However, from time to time we may use other small category as well. Intuitively, functors from an index category $\mathscr{S}$ to $\mathscr{C}$ captures the objects and morphisms follows the same pattern as $\mathscr{S}$.

Example 7.4.3. Consider the small category $\mathscr{S}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

and $\mathscr{C}$ be any category. Then, a functor $F : \mathscr{S} \to \mathscr{C}$ is precisely the data of a commuting square in $\mathscr{C}$.

Definition 7.4.4. The ***limit of the diagram*** is an object $\varprojlim_{,i\in\mathscr{S}} F(i)$, or $\varprojlim_{i\in\mathscr{S}} F(i)$, with morphisms $f_j : \varprojlim_{i\in\mathscr{S}} F(i) \to F(j)$ for each $j \in \mathscr{S}$, such that if $m : j \to k$ is a morphism in $\mathscr{S}$ then

$$\begin{tikzcd} \varprojlim_{i\in\mathscr{S}} F(i) \arrow[d, "f_j"] \arrow[dr, "f_k"] & \\ F(j) \arrow[r, "F(m)"] & F(k) \end{tikzcd}$$

commutes, and this object and maps to each A_i are universal (final) with respect to this property[1]. More precisely, given any other object W with maps $g_i : W \to A_i$ such that commutes with $F(m)$(i.e. if $m : j \to k$ then $g_k = F(m) \circ g_j$), then there is a unique map $g : W \to \varprojlim_{i\in\mathscr{S}} F(i)$ such that $g_i = f_i \circ g$ for all i.

Definition 7.4.5. Let $\mathscr{S}$ be small category and $\mathscr{C}$ be any category. Given a functor $F : \mathscr{S} \to \mathscr{C}$, a ***cone*** of F is an object $N \in \mathscr{C}$ with morphisms $f_i : N \to F(i)$ such that for every morphism $m : i \to j$ in $\mathscr{S}$, we have $F(m) \circ f_i = f_j$, i.e. the triangle commutes:

$$\begin{tikzcd} N \arrow[d, "f_i"] \arrow[dr, "f_j"] & \\ F(i) \arrow[r, "F(m)"] & F_j \end{tikzcd}$$

Remark 7.4.6. Now we can give another definition of limit that explains the universal (final) part in our above definition.

A limit of $F : \mathscr{S} \to \mathscr{C}$ is a cone $L := \varprojlim_{i\in\mathscr{S}} F(i)$ with morphisms $f_i : L \to F(i)$ for each $i \in \mathscr{S}$ such that: for every cone N with morphisms $g_i : N \to F(i)$, there exists a unique morphism $\mu : N \to L$ such that $f_i = \mu \circ g_i$ for every $i \in \mathscr{S}$. Viz, for any cone N, we have the following commutes with fixed μ while $i, j \in \mathscr{S}$ and $m : i \to j$

---

[1]What this means is that, we could have many objects such that exists this universal property of commuting with $F(m)$, but the limit is the final one

are arbitrary

$$\begin{tikzcd}
& N \arrow[ddl, "g(i)"'] \arrow[ddr, "g(j)"] \arrow[d, dashed, "\exists!\mu"'] & \\
& L \arrow[dl, "f(i)"] \arrow[dr, "f(j)"'] & \\
F(i) \arrow[rr, "F(m)"'] & & F(j)
\end{tikzcd}$$

Example 7.4.7. If $\mathscr{S}_1$ is the partially ordered set

$$\begin{tikzcd}
& i \arrow[d, "m_i"] \\
j \arrow[r, "m_j"] & k
\end{tikzcd}$$

then the limit of functor $F : \mathscr{S}_1 \to \mathscr{C}$ which sends $i \to X \in \mathscr{C}$, $j \to Y \in \mathscr{C}$ and $k \to Z \in \mathscr{C}$ would be the fibered product $X \times_Z Y$.

Indeed, note we have morphisms $F(m_1) : X \to Z$ and $F(m_2) : Y \to Z$, then we consider the limit L of functor F to be the object in $\mathscr{C}$ with two morphisms $f_i : L \to F(i)$, $f_j : L \to F(j)$ and $f_k : L \to F(k)$ such that, for all objects $D \in \mathscr{C}$ with morphisms $g_1 : D \to F(i) = X$ and $g_2 : D \to F(j) = Y$, we have unique μ that makes the following diagram commutes

$$\begin{tikzcd}
D \arrow[dr, dashed, "g_2"] \arrow[drr, bend left, "g_1"] \arrow[ddr, "\mu"'] & & \\
& L \arrow[r, "f_i"] \arrow[d, "f_j"] \arrow[dr, dotted, "f_k"] & X \arrow[d, "F(m_1)"] \\
& Y \arrow[r, "F(m_2)"] & Z
\end{tikzcd}$$

Observe that we may omit f_k because it is effectively the same as $F(m_1) \circ f_i$ as in the definition we insists that part commutes. We should immediately see this is the same as the definition of the fibered product.

We remark that the fibered product is also called ***pullback*** because[1] we obtained this by a diagram that pulls back the objects (note k is pulling back i and j if we look at the direction of arrows).

Example 7.4.8. If $\mathscr{S}$ is a set with only morphisms being identity morphisms, then the limit of a functor $F : \mathscr{S} \to \mathscr{C}$ where $\mathscr{C}$ is arbitrary category is called the ***product*** of $F(i)$ and is denoted by $\prod_{i \in \mathscr{S}} F(i)$.

Example 7.4.9. Let p be a prime number, the p-adic integers, $\mathbb{Z}_p$, is informally defined as $\mathbb{Z}_p = \{\sum_{i \geq 0} a_i p^i : 0 \leq a_i < p\}$. This can be defined as a lmit in the category of rings. In particular, this is defined as the object with the following

[1] I just made that up, or perhaphs that is why it also called pullback

diagram

$$\begin{tikzcd}
 & \mathbb{Z}_p \arrow[d] \arrow[dr] \arrow[drr] & & \\
\dots \arrow[r] & \mathbb{Z}/p^3\mathbb{Z} \arrow[r] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r] & \mathbb{Z}/p\mathbb{Z}
\end{tikzcd}$$

Remark 7.4.10. Note limits do not always exist for any index category $\mathscr{S}$. However, you can often check that limits exist if the objects of your category can be interpreted as sets with additional structure and arbitrary product exists (respecting the set-like structure).

Example 7.4.11. In the category *Sets*,

$$\left\{(a_i)_{i\in\mathscr{S}} \in \prod_i F(i) : \forall j \in \mathscr{S}, \forall m \in Mor(j,k), F(m)(a_j) = a_k\right\}$$

with the projection maps to each $F(i)$ forms the limit $\lim\limits_{\overleftarrow{\mathscr{S}}} F(i)$

Definition 7.4.12. The ***colimit***, $\lim\limits_{\overrightarrow{\mathscr{S}}} F(i)$ is defined as the definition of limit except all the arrows are reversed. In particular, this is the object such that for all object $M \in \mathscr{C}$ with arrows coming from all $F(i)$ and $F(j)$, we have the following diagram commutes (where the μ is fixed for all i and j)

$$\begin{tikzcd}
 & M & \\
 & \lim\limits_{\overrightarrow{\mathscr{S}}} F(i) \arrow[u, dashed, "\exists!\mu"] & \\
F(i) \arrow[uur, "g(i)"] \arrow[ur, "f(i)"'] \arrow[rr] & & F(j) \arrow[uul, "g(j)"'] \arrow[ul, "f(j)"]
\end{tikzcd}$$

Remark 7.4.13. To remember the definition of limit and colimit, we recall the analogy with kernel and cokernel. Note we normally have kernels "map to" other objects while cokernels are "mapped to" by others. This is similar in the case of limit and colimit as limit is the object that maps to all the objects in the big commutative diagram while colimit is the object has a map from all the objects.

Remark 7.4.14. Similarly, we also have coproduct, which is the colimit of the category $\mathscr{S}$ with arbitrary many dots and no arrows.

Example 7.4.15.

1. Consider the poset category $P(X)$ where X is a set and morphisms being arrow if and only if we have a inclusion. Then, consider a collection of subsets $\{U_i : i \in I\}$ of X and the index category I with no arrows other than the identity morphisms and indexed diagram $F(i) = U_i$. Then we should see the colimit of this functor is $\bigcup_{i\in I} U_i$.
2. We have $\mathbb{Q} = \lim\limits_{\rightarrow} \frac{1}{n}\mathbb{Z}$. One way to think about this is that colimit of sets is just union of sets and if we take union over all $\frac{1}{n}\mathbb{Z}$ where n range over $n \geq 1$, this is the rational numbers.

Definition 7.4.16. A non-empty poset $(S, \geq)$ is ***filtered*** (or a ***filtered set***) if for each $x, y \in S$, there is a z such that $x \geq z$ and $y \geq z$. More generally, a nonempty category $\mathscr{F}$ is ***filtered*** if

1. for each $x, y \in \mathscr{F}$, there is a $z \in \mathscr{F}$ and arrows $x \to z$ and $y \to z$, and
2. for every two arrows $u : x \to y$ and $v : x \to y$, there is an arrow $w : y \to z$ such that $w \circ u = w \circ v$.

7.5 Adjoints

Remark 7.5.1. The idea of adjoint is similar to the notion of universal property. In particular, as universal property determines an object in a category (assuming such object exists), adjoints determine a functor (if such functor exists).

Definition 7.5.2. Two covariant functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ are ***adjoint*** if there is a natural bijection for every $A \in \mathscr{A}$, $B \in \mathscr{B}$

$$\tau_{AB} : Mop_{\mathscr{B}}(F(A), B) \to Mor_{\mathscr{A}}(A, G(B))$$

where natural means the following diagram commutes for all $f : A \to A'$ (and similarly for all $g : B \to B'$)

$$\begin{tikzcd} Mor_{\mathscr{B}}(F(A'), B) \arrow[r, "(Ff)^*"] \arrow[d, "\tau_{A'B}"] & Mor_{\mathscr{B}}(F(A), B) \arrow[d, "\tau_{AB}"] \\ Mor_{\mathscr{A}}(A', G(B)) \arrow[r, "f^*"] & Mor_{\mathscr{A}}(A, G(B)) \end{tikzcd}$$

where f^* is the map induced by $f : A \to A'$, i.e. $a : A' \to G(B)$ then $f^*(a) = a \circ f$ and $(Ff)^*$ is the map induced by Ff.

Remark 7.5.3. The map τ_{AB} has the following property: for each $A \in \mathscr{A}$ there is a map $\eta_A : A \to G(F(A))$ so that for any $g : F(A) \to B$, the corresponding $\tau_{AB}(g) : A \to G(B)$ is given by the composition

$$A \xrightarrow{\eta_A} G(F(A)) \xrightarrow{G(g)} G(B)$$

Similarly there is a map $\epsilon_B : F(G(B)) \to B$ for each B so that for any $f : A \to G(B)$, the corresponding $\tau_{AB}^{-1} : F(A) \to B$ is given by the composition

$$F(A) \xrightarrow{F(f)} F(G(B)) \xrightarrow{\epsilon_B} B$$

Example 7.5.4. Let N be a A-module. Then we have $(-) \otimes_A N$ and $Hom_A(N, -)$ are adjoint functors. For example, suppose X, Y and Z are A-modules, then there is a bijection between $Hom_A(Y \otimes_A X, Z)$ and $Hom_A(Y, Hom_A(X, Z))$

Example 7.5.5. The Frobenius reciprocity in representation theory may be understood in terms of adjoints. Suppose V is a finite-dimensional representation of fintie group G and W is a representation of subgroup $H \leq G$. Then the induced representation to G, Ind_H^G, and restriction of representation to H, Res_H^G, are adjoint pair of functors.

Example 7.5.6 (Groupification)**.** This example will demonstrate how to get an abelian group from an abelian semigroups[1]. Morphisms of abelian semigroups are maps of sets preserving the binary operation.

One example is the non-negative integers $\mathbb{Z}_{\geq 0} = \{0, 1, 2, ...\}$ under addition, which lacks inverses. Another example is $\mathbb{Z}_{\geq 1}$ under multiplication.

From abelian semigroups $(\mathbb{Z}_{\geq 0}, +)$ we can create abelian group $\mathbb{Z}$ under addition and from $(\mathbb{Z}_{\geq 1}, \times)$ we can create $\mathbb{Q}_{\geq 1}$ under multiplication.

A ***groupification*** of a semigroup S is a map of abelian semigroups $\pi : S \to G$ such that G is also an abelian group, and any map of abelian semigroups from S to an abelian group G' factors uniquely through G:

$$\begin{tikzcd} S \arrow[r, "\pi"] \arrow[dr] & G \arrow[d, "\exists!"] \\ & G' \end{tikzcd}$$

Remark 7.5.7. The following table is a table of most of the adjoints that will come up for us.

Situation	Category $\mathscr{A}$	Category $\mathscr{B}$	Left adjoint $F : \mathscr{A} \to \mathscr{B}$	Right adjoint $G : \mathscr{B} \to \mathscr{A}$
A modules	Mod_A	Mod_B	$(-) \otimes_A N$	$Hom_A(N, -)$
Ring maps $B \to A$	Mod_B	Mod_A	$(-) \otimes_B A$, i.e. extension of scalars	$M \mapsto M_B$, i.e. restriction of scalars
(pre)sheaves on a topological space X	Presheaves on X	Sheaves on X	Sheafification	Forgetful
(semi)groups	Semigroups	Groups	Groupification	Forgetful
Sheaves, $\pi : X \to Y$	Sheaves on Y	Sheaves on X	π^{-1}	π_*
Sheaves of abelian groups or $\mathcal{O}$ modules, open embeddings $\pi : U \to Y$	Sheaves on U	Sheaves on Y	$\pi_!$	π_*
Quasicoherent sheaves, $\pi : X \to Y$	$QCoh_Y$	$QCoh_X$	π^*	π_*
Ring maps $B \to A$	Mod_A	Mod_B	$M \mapsto M_B$ by restriction	$N \mapsto Hom_B(A, N)$
Quasicoherent sheaves, affine $\pi : X \to Y$	$QCoh_X$	$QCoh_Y$	π_*	$\pi^!_{sh}$

7.6 Abelian Category

Remark 7.6.1. We will first do some examples and motivate the definition of abelian categories. In particular, two central examples of an abelian category are

[1]Recall a semigroup is just group but lack the inverse and identity axiom

the category Ab and the category Mod_A of A-modules where the first is a special case of the second (when $A = \mathbb{Z}$).

However, before we give definition of abelian category, we will define additive category.

Definition 7.6.2. A category $\mathscr{C}$ is ***additive*** if it satisfies the following properties:

1. For each $A, B \in \mathscr{C}$, $Mor(A, B)$ is an abelian group with an addition, such that composition of morphisms distributes over addition[1].
2. $\mathscr{C}$ has a zero object, denoted by 0. (This is an object that is simultaneously an initial object and a final object).
3. It has products of two objects (a product $A \times B$ for any pair of objects), and hence by induction, product of any finite number of objects.

Remark 7.6.3. In an additive category, the morphisms are often called ***homomorphisms***, and Mor is denoted by Hom.

Definition 7.6.4. A functor between additive categories preserving the additive structure of Hom's is called an ***additive functor***.

Remark 7.6.5. It can be show that additive functors send zero object to zero object. In addition, the name 0-object is that the 0-morphism in the abelian group $Hom(A, B)$ is the composition $A \to 0 \to B$.

Definition 7.6.6. Let $\mathscr{C}$ be a category with a 0-object (and thus a 0-morphism). A ***kernel*** of a morphism $f : B \to C$ is an object A and a map $i : A \to B$ such that $f \circ i = 0_{AC}$, and that is universal with respect to this property, i.e. for all objects Z with a morphism $j : Z \to B$ with the property $f \circ j = 0_{ZC}$, there exists a unique morphism $\mu : Z \to A$ such that the following commutes

$$\begin{tikzcd}
 & B \arrow[dr, "f"] & \\
 & A \arrow[u, "i"] \arrow[r, "0_{AC}"] & C \\
Z \arrow[uur, bend left, "j"] \arrow[ur, dashed, "\exists!\mu"] \arrow[urr, bend right, "0_{ZC}"'] & &
\end{tikzcd}$$

The kernel is denoted by $Ker(f)$. Similarly we define the ***cokernel*** with all the arrows reversed.

Remark 7.6.7. Note the kernel is the limit of the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
B \arrow[r, "f"] & C
\end{tikzcd}$$

and similarly cokernel is the colimit.

[1]This means $Mor(A, B)$ is a abelian group, i.e. if $f, g \in Mor(A, B)$ then $f + g \in Mor(A, B)$. In addition, we have $h : B \to C$ a morphism, then $h \circ (f + g) = h \circ f + h \circ g$ where we note $h \circ f, h \circ g \in Mor(A, C)$ so addition makes sense (and this addition is different from $f + g$'s addition)

Definition 7.6.8. If $i : A \to B$ is a monomorphism, then we say that A is a ***subobject*** of B, where the map i is implicit. There is also the notion of ***quotient object***, defined dually to subobject.

Definition 7.6.9. An ***abelian category*** is an additive category satisfying three additional properties:

1. Every map has a kernel and cokernel.
2. Every monomorphism is the kernel of its cokernel.
3. Every epimorphism is the cokernel of its kernel.

Definition 7.6.10. The ***image*** of a morphism $f : A \to B$ is defined as $Im(f) = Ker(Coker(f))$.

Remark 7.6.11. The morphism $f : A \to B$ factor uniquely through $Im(f) \to B$ whenever $Im(f)$ exists, and $A \to Im(f)$ is an epimorphism and a cokernel of $Ker(f) \to A$ in every abelian category.

Remark 7.6.12. The cokernel of a monomorphism is called the ***quotient***. The quotient of a monomorphism $A \to B$ is often denoted B/A (with the map from B implicit).

7.7 Complexes, Exactness and Homology

Remark 7.7.1. In this section, we will assume we are working with abelian categories.

Definition 7.7.2. We say a sequence

$$\dots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \dots$$

is a ***complex at*** B if $g \circ f = 0$, and is ***exact at*** B if $Ker(g) = Im(f)$. A sequence is ***complex/exact*** if it is complex/exact at every point.

Example 7.7.3.

1. A ***short exact sequence*** is an exact sequence with five terms and first and last terms are both zero.
2. The sequence $0 \to A \to 0$ is exact if and only if $A = 0$.
3. The sequence $0 \longrightarrow A \xrightarrow{f} B$ is exact if and only if f is monomorphism.
4. The sequence $A \xrightarrow{f} B \xrightarrow{f} 0$ is exact if and only if f is epimorphism.
5. The sequence $0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$ is exact iff f is isomorphism.

Definition 7.7.4. Suppose we have a sequence

$$\dots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \dots$$

and it is a complex at B, then it's ***homology at*** B, often denoted by H, is $Ker(g)/Im(f)$, i.e. H is the cokernel of some monomorphism from $Im(f)$ to $Ker(g)$.

Remark 7.7.5. In this terminology, we have a sequence is exact at B iff its homology at B is zero.

Definition 7.7.6. Given a sequence

$$\dots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \dots$$

where the objects of the category are sets with additional structure. We say the elements of $Ker(g)$ are ***cycles*** and elements of $Im(f)$ are ***boundaries***. In this sense, homology is cycles mod out by boundaries.

Definition 7.7.7. For a sequence $A_{i+1} \longrightarrow A_i \longrightarrow A_{i-1}$, we write H_i as the homology at A_i. On the other hand, given a sequence $A_{i-1} \longrightarrow A_i \longrightarrow A_{i+1}$ with increasing index, then we say the homology at A_i is the ***cohomology*** and denote it to be H^i.

Remark 7.7.8. Note for any long exact sequences

$$\dots \longrightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} \dots$$

we can factor them into short exact sequences of the following form for each A_i

$$0 \longrightarrow Ker(f_i) \longrightarrow A^i \longrightarrow Ker(f_{i+1}) \longrightarrow 0$$

Definition 7.7.9. If $F : \mathscr{A} \to \mathscr{B}$ is a covariant additive functor between abelian categories, we say F is ***right-exact*** if for any exact sequences

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in A we have

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

is also exact.

We say F is ***left-exact*** if for exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

we have

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is exact.

Definition 7.7.10. For a contravariant functor F, we say it is ***left-exact*** if

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact imply

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is exact.

We say F is ***right-exact*** if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

is exact imply

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

is exact.

Definition 7.7.11. A functor is ***exact*** if it is both right and left exact.

Chapter 8

Appendix 2: Sheaves

多情多感仍多病，多景楼中。樽酒相逢，乐事回头一笑空。
停杯且听琵琶语细，捻轻拢。醉脸春融，斜照江天一抹红。

李白

8.1 Intro

Remark 8.1.1. We will first give a motivation example then formally define what a sheaf is.

Example 8.1.2. Consider differentiable functions on the topological space $X = \mathbb{R}^n$. The sheaf of differentiable functions on X is the data of all differentiable functions on all open subsets on X. We will see how to manage these data and observe some key properties.

1. On each open set $U \subseteq X$, we have a ring of differentiable functions and we denote this ring of functions to be $\mathcal{O}(U)$.
2. Given a differentiable function on an open set, we can restrict it to a smaller open set and obtain a differentiable function there. In other words, if $U \subseteq V$ then we have a restriction map $res_{V \to U} : \mathcal{O}(V) \to \mathcal{O}(U)$.
3. Consider $U \subseteq V \subseteq W$ be three open sets and $f \in \mathcal{O}(W)$. Then restrict f to V then U is the same as restrict f to U directly. In other word, we have the

following diagram commutes

$$\begin{tikzcd} \mathcal{O}(W) \arrow[r, "res_{W\to V}"] \arrow[dr, "res_{W\to U}"'] & \mathcal{O}(V) \arrow[d, "res_{V\to U}"] \\ & \mathcal{O}(U) \end{tikzcd}$$

4. Given two differentiable functions $f, g \in \mathcal{O}(U)$ and an open cover of U by collection of open subsets $\{U_i\}_{i\in I}$, i.e. $U = \bigcup_{i\in I} U_i$. Then if f, g agree on each U_i, we must have $f = g$ on U. In other word, $\forall i \in I, res_{U\to U_i}(f) = res_{U\to U_i}(g)$ imply $f = g$. This means we can identify functions on an open set by looking at them on a covering by smaller open sets.
5. Suppose we have open set U and an open cover $\{U_i\}_{i\in I}$. Consider a family of functions $f_i \in \mathcal{O}(U_i)$ with the property $f_i = f_j$ on $U_i \cap U_j$ for all i, j. Then we can define a differentiable function $f \in \mathcal{O}(U)$ by gluing all the f_i's together. In other word, given $f_i \in \mathcal{O}(U_i)$ such that $res_{U_i\to(U_i\cap U_j)}(f_i) = res_{U_i\to(U_i\cap U_j)}(f_j)$ for all i, j then there exists $f \in \mathcal{O}(U)$ such that $res_{U\to U_i}(f) = f_i$ for all i.

Remark 8.1.3. Observe that the above five properties holds for continuous functions, smooth functions and even just all functions.

Definition 8.1.4. Let X be a topolgical space. Given a point $p \in X$, consider the set of 2-tuples $C = \{(f, U) : \text{U is open}, p \in U, f \in \mathcal{O}(U)\}$. Then a ***germ at*** p is an equivalence class in the set

$$C/\sim$$

where $\sim$ is the equivalence relation defined by $(f, U) \sim (g, V)$ if and only if there exists open neighborhood W of p such that $W \subseteq U \cap V$ and $f|_W = g|_W$.

Definition 8.1.5. We call the set of germs at p the ***stalk at*** p and it is denoted by $\mathcal{O}_p$.

Remark 8.1.6. Observe $\mathcal{O}_p$ forms a ring: you can add and composite two germs by their representatives. Also, $\mathcal{O}_p$ is a local ring with one maximal ideal $\mathfrak{m}_p := \{\text{all germs vanishing at } p\}$.

8.2 Definition of Sheaves

Definition 8.2.1. A ***presheaf*** $\mathscr{F}$ on a topological space X is the following data:

1. To each open set $U \subseteq X$, we have a set $\mathscr{F}(U)$. The elements of $\mathscr{F}(U)$ are called ***section of*** $\mathscr{F}$ ***over*** U. In particular, if U is ommited, it is implicitly taken $U = X$ and this section is called ***global section***.
2. For each inclusion of open sets $U \subseteq V$, we have a restriction $res_{V,U} : \mathscr{F}(V) \to \mathscr{F}(U)$.
3. The map $res_{U,U}$ is the identity on $\mathscr{F}(U)$.

4. If $U \subseteq V \subseteq W$ then the following diagram commutes

$$\begin{tikzcd}
\mathscr{F}(W) \arrow[r, "res_{W,V}"] \arrow[dr, "res_{W,U}"'] & \mathscr{F}(V) \arrow[d, "res_{V,U}"] \\
& \mathscr{F}(U)
\end{tikzcd}$$

Remark 8.2.2. It is not hard to see that a presheaf $\mathscr{F}$ is just a contravariant functor $\mathscr{F} : X \to \mathscr{C}$ with $\mathscr{C}$ a category (in the above definition we assumed our $\mathscr{C}$ is $Sets$ category) and X the category of poset on the open sets of X, i.e. objects of X are open sets of X and morphism $U \to V$ exists if and only if $U \subseteq V$. We will use the two definitions interchangeably.

Definition 8.2.3. The ***stalk*** of a ***presheaf*** $\mathscr{F}$ at a point p is the set of germs of $\mathscr{F}$ at p, denoted by $\mathscr{F}_p$. In other word, the stalk is the collection of equivalence classes defined by $\mathscr{F}_p := C/\sim$ where $C = \{(f, U) : U \text{ open}, p \in U, f \in \mathscr{F}(U)\}$ and $(f, U) \sim (g, V)$ iff there exists open neighborhood $W \subseteq U \cap V$ of p such that $res_{U,W}(f) = res_{V,W}(g)$.

Remark 8.2.4. It is not so hard to see that a stalk $\mathscr{F}_p$ is just a colimit of all $\mathscr{F}(U)$ over all open sets U containing p, i.e. $\mathscr{F}_p = \varinjlim \mathscr{F}(U)$. Therefore, we can define stalks for sheaves of sets, groups, rings and other things for which colimit exists for directed sets.

Definition 8.2.5. If $p \in U$ and $f \in \mathscr{F}(U)$ then the image of f in $\mathscr{F}_p$ is called the ***germ of f at p***.

Definition 8.2.6. A sheaf $\mathscr{F} : X \to \mathscr{C}$ is a presheaf with two additional properties:

1. (Identity Axiom): If $\{U_i\}_{i\in I}$ is an open cover of U and $f, g \in \mathscr{F}(U)$, and $res_{U,U_i}(f) = res_{U,U_i}(g)$ for all $i \in I$, then $f = g$. We note a presheaf with this property is called a ***separated presheaf***.
2. (Gluability Axiom): If $\{U_i\}_{i\in I}$ is an open cover of U, and there exists $f_i \in U_i$ for all i such that $res_{U_i,U_i\cap U_j}(f_i) = res_{U_i,U_i\cap U_j}(f_j)$ for all $i, j \in I$, then there exists $f \in \mathscr{F}(U)$ such that $res_{U,U_i}(f) = f_i$ for all i.

Remark 8.2.7. Note the first additional property means there is at most one way to glue sections and the second property means there is at least one way to glue sections.

Definition 8.2.8. The ***stalk of a sheaf*** at a point p is just the stalk as a presheaf and similarly for ***germs***.

Example 8.2.9. Observe that bounded functions on $\mathbb{R}$ is a presheaf but not a sheaf. Indeed, it is a presheaf trivially. However, consider the function $f_i(x) = x^2$ defined on $(0, i)$, which is bounded for all $i \in \mathbb{Z}_{\geq 1}$. On the other hand, we see $(0, \infty) = \bigcup_{i\geq 1}(0, i)$, i.e. $\{(0, i)\}_{i\geq 0}$ forms an open cover of $(0, \infty)$ while there does not exists bounded function f such that the restriction on each $(0, i)$ is x^2 as $f(x) = x^2$ is unbounded on $(0, \infty)$.

Definition 8.2.10. Consider $\mathscr{F} : X \to \mathscr{C}$ a sheaf and let U be an open subset of X. Then the ***restriction of $\mathscr{F}$ to U***, denoted by $\mathscr{F}|_U$, is the collection $\mathscr{F}|U(V) = \mathscr{F}(V)$ for all open subsets $V \subseteq U$. Clearly this is a sheaf.

Example 8.2.11. Suppose X is a topological space, with $p \in X$ fixed and S a set. Let $i_p : p \to X$ be the inclusion map. Then, define $i_{p,*}S$ to be the ***skyscrapper sheaf***

$$i_{p,*}S(U) := \begin{cases} S, \text{if } p \in U \\ \{e\}, \text{if } p \notin U \end{cases}$$

where e is a a fixed element for all U and $\{e\}$ is a one-element set. In particular, if S is a group or ring, we always assume e is the identity element.

Let us check this is a sheaf. First, for each open set U, we indeed have a set $\mathscr{F}(U)$. Second, if $U \subseteq V$, then we have the restriction map to be: if $p \in U \subseteq V$ then $res_{V,U}$ is the identity map between S and S and if $p \notin U$ then $res_{V,U}$ is the identity map $\{e\} \to \{e\}$. Third, the restriction $res_{U,U}$ is indeed the identity on $i_{p,*}S(U)$. Finally, if $U \subseteq V \subseteq W$ we should see we indeed have the diagram commutes as in the definition of presheaves and clearly identity axiom and gluability axiom.

Example 8.2.12. Let X be a topological space and S a set. Then the ***constant presheaf*** is defined to be $\underline{S}_{pre}(U) = S$ for all open sets U and restriction being the identity. Note this is not a sheaf in general by consider S with more than one element and X being the two point space with the discrete topology.

Example 8.2.13. Let X be a topolgical space and S a set. Let $\mathscr{F}(U)$ be the map to S that are locally constant, i.e. for any point p in U there is an open neighborhood of p where the function is constant. Then this is the ***constant sheaf*** and is denoted to be $\underline{S}$.

Another description of this is endow S with the discrete topology and let $\mathscr{F}(U)$ be the continuous map from U to S.

Example 8.2.14. Let X be the topological space with two points p and q with discrete topology. X has four open sets, $\emptyset, \{p\}, \{q\}$ and $\{p, q\}$. Then we have five proper inducion relations, i.e. $\emptyset \subsetneq \{p\}, \{q\}, \{p, q\}$ and $\{p\}, \{q\} \subsetneq \{p, q\}$.

Then consider the constant presheaf $\mathscr{Z} : X \to Sets$, which sends each of the open sets to the same set $\mathbb{Z}$. This is not a sheaf because it fails the identity axiom: $\emptyset$ is covered by the empty family of sets and thus any two sections of $\mathscr{Z}$ over the empty set are equal when restricted to any set in the empty family, which there is none, so by the identity axiom it is vacuously true that any two section of $\mathscr{Z}(\emptyset)$ is equal, i.e. any two elements in $\mathbb{Z}$ is equal. This is a contradiction so it cannot be a sheaf.

Now consider another presheaf $\mathscr{F} : X \to Sets$ given by the following: let $\mathscr{F}(\emptyset) = 0$ where 0 is a one-element set. On all non-empty sets S, we have $\mathscr{F}(S) = \mathbb{Z}$. For each inclusion of open sets, $\mathscr{F}$ returns either the unique map to 0, if the smaller set is empty, or the identity map on $\mathbb{Z}$ if the smaller set is not. This presheaf satisfies the identity axiom but fails the gluing axiom. Consider $\{p, q\}$, which is covered by $\{p\}$ and $\{q\}$ where those sets have empty intersection. A section on $\{p\}$ or $\{q\}$ is an element of $\mathbb{Z}$, i.e. a number. Choose a section m over $\{p\}$ and n over $\{q\}$ with the assumption $m \neq n$. Note m and n restrict to the same element 0 over the intersection of $\{p\}$ and $\{q\}$, the gluability axiom requires the existence of a unique section s on $\mathscr{F}(\{p, q\})$ such that equal m when restricted to $\{p\}$ and n on

$\{q\}$. However, the restriction map is the identity between them, hence we have $s = m = n$, a contradiction.

Finally, consider the presheaf $\mathscr{H} : X \to Sets$ given by$\mathscr{H}(\{p,q\})$ to the set $\mathbb{Z} \oplus \mathbb{Z}$ with two projections π_1 and π_2. Then we define $\mathscr{H}(\{p\}) = Im(\pi_1)$ and $\mathscr{H}(\{q\}) = Im(\pi_2)$ and $\mathscr{H}(\emptyset) = 0$. This is then a sheaf of X to $Rings$.

Example 8.2.15. Let X and Y be two topological spaces. Then the continuous maps to Y forms a sheaf of sets on X. More precisely, to each open set U of X, we have $\mathscr{F}(U)$ to be the set of continuous maps from U to Y.

Definition 8.2.16. Let $\pi : X \to Y$ be a continuous map and $\mathscr{F}$ a presheaf on X. Then we define $\pi_*\mathscr{F}$, the ***pushforward of $\mathscr{F}$ by π***, to be the presheaf of Y such that $\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V))$ where V is open subsets of Y.

Example 8.2.17. The skyscrapper sheaf is the pushforward of the constant sheaf $\underline{S}$ on a one point space p under the inclusion morphism $i_p : \{p\} \to X$.

Remark 8.2.18. Note if $\pi : X \to Y$ is continuous and $\mathscr{F}$ a sheaf of sets on X. Then the pushforward induces a natural morphism of stalks from q to $\pi(q)$.

Definition 8.2.19. A ***ringed space*** is a topolgical space X with a sheaf $\mathscr{F} : X \to Rings$. In particular, we will denote the sheaf $\mathscr{F}$ by $\mathcal{O}_X$ to indicate X with $\mathcal{O}_X$ is a ringed space. This sheaf $\mathcal{O}_X$ will be called ***structure sheaf*** of the ringed space.

Definition 8.2.20. Let X be a ringed space and U a open subset of X. Then we denote $\mathcal{O}_U$ be the restriction of sheaf $\mathcal{O}_X$ to U. Moreover, the stalk of $\mathcal{O}_X$ at a point $p \in X$ is denoted by $\mathcal{O}_{X,p}$.

Definition 8.2.21. Let X be a ringed space with $\mathcal{O}_X$. A ***$\mathcal{O}_X$ module*** is a sheaf $\mathscr{F} : X \to Ab$ from X to abelian groups with the following property: for each U, $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module and for each $U \subseteq V$ we have the diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_X(V) \times \mathscr{F}(V) \arrow[r, "\mathcal{O}_X(V)-\text{module action}"] \arrow[d, "res_{V,U} \times res_{V,U}"] & \mathscr{F}(V) \arrow[d, "res_{V,U}"] \\
\mathcal{O}_X(U) \times \mathscr{F}(U) \arrow[r, "\mathcal{O}_X(U)-\text{module action}"] & \mathscr{F}(U)
\end{tikzcd}$$

8.3 Morphisms of Presheaves and Sheaves

Definition 8.3.1. Let $\mathscr{F}, \mathscr{S}$ be two presheaves from X to another category $\mathscr{C}$. A ***morphism of presheaves $\mathscr{F}$ and $\mathscr{S}$*** on X, say $\phi : \mathscr{F} \to \mathscr{S}$, is the following: for all open subsets U, we have $\phi(U) : \mathscr{F}(U) \to \mathscr{S}(U)$ is a map makes the following commutes

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "res_{V,U}"] & \mathscr{S}(V) \arrow[d, "res_{V,U}"] \\
\mathscr{F}(U) \arrow[r, "\phi(U)"] & \mathscr{S}(U)
\end{tikzcd}$$

where the left $res_{V,U}$ is for $\mathscr{F}$ and the right $res_{V,U}$ is from $\mathscr{S}$.

Definition 8.3.2. A ***morphism of sheaves*** $\mathscr{F}$ ***and*** $\mathscr{S}$ are defined to be a morphism of between $\mathscr{F}$ and $\mathscr{S}$ as presheaves.

Example 8.3.3. The forgetful map between sheaf of differentiable functions to sheaf of continuous functions is a morphism, i.e. we just forget the function is differentiable and keep the fact they are continuous.

Remark 8.3.4. Note if we consider a presheaf on X is a contravariant functor, then a morphism between presheaves is just a natural transformation of functors.

Definition 8.3.5. We define $Sets_X$, Ab_X and etc. to be the category of sheaves of sets, abelian groups, etc. on a topological space X. Let $Mod_{\mathcal{O}_X}$ be the category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$. Let $Sets_X^{pre}$, etc. denote the category of presheaves of sets, etc. on X.

Remark 8.3.6. If $\phi : \mathscr{F} \to \mathscr{S}$ is a morphism of presheaves on X and $p \in X$, then we get an induced morphism of stalks $\phi_p : \mathscr{F}_p \to \mathscr{S}_p$, i.e. $\phi_p(f, U) = (\phi(U)(f), U)$.

We can see this is a morphism by realize $\mathscr{F}_p$ and $\mathscr{S}_p$ are colimits and use universal property to show such morphism exists.

However, we may also check it the hard way, namely, we are going to show ϕ_p is well-defined, i.e. if $(f, U) \sim (g, V)$ then we have $\phi_p(f, U) \sim \phi_p(g, V)$. Observe there exists $W \subseteq U \cap V$ so that $res_{U,W}(f) = res_{V,W}(g)$. Then, by definition of morphism of presheaves, we see $\phi(U)(f)$ and $\phi(V)(g)$ agrees on W again.

In particular, what this means is that taking stalk at a fixed point p is a functor. Viz, consider the category $Sets_X^{pre}$, then we get a functor $\mathscr{F} : Sets_X^{pre} \to Sets$ with $\mathscr{F}(F) = F_p$ where $F \in Sets_X^{pre}$ and for morphisms $\phi : F \to G$ we have $\mathscr{F}(\phi) = \phi_p$.

Remark 8.3.7. Let $\pi : X \to Y$ be a continuous map of topological spaces. Then the pushforward gives a functor $\pi_* : Sets_X \to Sets_Y$ where $Sets$ here can be replaced by other categories. Note here π_* is actually a functor between categories of functors.

Let $\mathscr{F}, \mathscr{G} : X \to Sets$ be two objects of $Sets_X$ and let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism. We need to show π_* makes the following diagram commutes

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\pi_*"] \arrow[d, "\phi"] & \pi_*(\mathscr{F}) \arrow[d, "\pi_*(\phi)"] \\
\mathscr{G} \arrow[r, "\pi_*"] & \pi_*(\mathscr{G})
\end{tikzcd}$$

Note if we define $\pi_*(\phi)$ to be $\pi_*(\phi)(V)$ to be $\phi(\pi^{-1}(V))$, then it will make the diagram commute as we note $\pi_*(\mathscr{F})(V)$ is just $\mathscr{F}(\pi^{-1}(V))$ and so on.

Definition 8.3.8. Suppose $\mathscr{F}$ and $\mathscr{S}$ are two sheaves of sets on X. Then we define $Hom(\mathscr{F}, \mathscr{S})$, called ***sheaf Hom***, to be the sheaf of X given by

$$\forall U \text{ open in } X, Hom(\mathscr{F}, \mathscr{S})(U) = Mor(\mathscr{F}|_U, \mathscr{S}|_U)$$

where $\mathscr{F}|_U$ is the restriction of sheaves and $Mor(\mathscr{F}|_U, \mathscr{S}|_U)$ is the collection of all the morphisms from $\mathscr{F}|_U$ to $\mathscr{S}|_U$.

Remark 8.3.9. Note the sheaf Hom is indeed a sheaf (a functor from X to $Sets$) and we can check this. Let $\mathscr{F}, \mathscr{S}$ be two sheaves from X to $\mathscr{C}$ where $\mathscr{C}$ is a category. Let $\mathscr{H}$ be the sheaf Hom of $\mathscr{F}$ to $\mathscr{S}$.

Then $\mathscr{H}$ is a presheaf. We note for open set $U \subseteq X$, a section of $\mathscr{H}(U)$ is a morphism from $\mathscr{F}_U$ to $\mathscr{S}_U$. In particular, let $U \subseteq V$, we define the restriction for $\mathscr{H}$ to be, $res_{V,U}(\sigma)(W) = \sigma(W)$ for all open subset W of U. Note in the above, $res_{V,U}(\sigma)$ should be a morphism in $Mor(\mathscr{F}|_U, \mathscr{S}|_U)$ and thus we should know how it works on open subsets of U. We add a final remark that $\sigma(W)$ is a map from $\mathscr{F}|_U(W)$ to $\mathscr{S}|_U(W)$, rather than a set or something as we recall the definition of morphisms.

Now we show the identity axiom. Let U be an open subset of X and $U = \bigcup_{i \in I} U_i$ be an open cover. We will show, if $\sigma \in Mor(\mathscr{F}|_U, \mathscr{S}|_U)$ with $\sigma_i := res_{U,U_i}(\sigma) = 0$ for all $i \in I$, then $\sigma = 0$. This is effectively the same as the identity axiom if we subtract f and g in the definition of identity axiom.

Let $g \in \mathscr{F}(U)$ be a fixed section. Note since $\sigma_i(U_i)$ is the zero morphism from $\mathscr{F}|_{U_i}$ to $\mathscr{S}|_{U_i}$, we have

$$\sigma_i(U_i)(res_{U,U_i}(g)) = 0$$

In particular, because $\mathscr{S}$ is already a sheaf, we have identity axiom at our disposal and we get (because $\sigma_i(U_i)(res_{U,U_i}(g)) = 0 \in \mathscr{S}(U_i)$ for all $i \in I$, where we see $\sigma_i(U_i)(res_{U,U_i}(g))$ is just $\sigma(U)(g)$ restricted to U_i. Now we apply the identity axiom to conclude the following)

$$\sigma(U)(g) = 0$$

Because g was arbitrary, we have $\sigma(U)$ maps every section of $\mathscr{F}(U)$ to zero and hence σ is the zero morphism as desired.

Now we show the gluability axiom. For this one, say U is an open subset of X with open cover $\bigcup_{i \in I} U_i$. Suppose we have $\{\phi_i : i \in I, \phi_i \in Mor(\mathscr{F}|_{U_i}, \mathscr{S}|_{U_i})\}$, we want to make sure we have a global ϕ such that $\phi|_{U_i} = \phi_i$. However, this follows just like the identity axiom follows from identity axioms of $\mathscr{F}$ and $\mathscr{S}$.

Hence, we indeed have $\mathscr{H}$ is a sheaf as desired.

Remark 8.3.10. Note $Hom(-, \mathscr{S})$ is a contravariant functor and $Hom(\mathscr{F}, -)$ is a covariant functor. In addition, Hom does not commute with taking stalks, i.e. $Hom(\mathscr{F}, \mathscr{S})_p$ is not isomorphic to $Hom(\mathscr{F}_p, \mathscr{S}_p)$.

Chapter 9

Appendix 3: Schemes