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Definition 0.0.1. Prof Kevin Purbhoo. Office: MC 5122.

Remark 0.0.2. Regrade must be hand in person.

Chapter 1

Enumerative Combinatorics

1.1 How many ways to place k balls in n bins

Example 1.1.1. How many ways to place k balls in n bins?

Definition 1.1.2. Note we have the permutation group on a set Λ to be $S_\Lambda := \{f : \Lambda \to \Lambda : f \text{ is bijective}\}$. We have the power set X to be $P(X)$, the collection of all subsets of X.

Remark 1.1.3. We note that there are 12 variations of this question.

Consider the following three restrictions:

1. None,
2. at most one ball per bin,
3. at least one ball per bin.

Consider the following four distinguishabilities:

(a) Balls and bins are distinguishable.
(b) Balls are indistinguishable.
(c) Bins are indistinguishable.
(d) Both are both indistinguishable.

Formally, each way to place balls into bins is a function $f : K \to N$ where $|K| = k$, i.e. the set of balls, and $|N| = n$, i.e. the set of bins.

Now, we note for the three restrictions, we have,

1. arbitrary functions from $K \to N$
2. injective,
3. surjective.

The distinguishabilities tells us that there exists equivalence relations on the set of functions from K to N. In particular,

(a) $f \sim g \iff f = g$
(b) $f \sim g \iff \exists \sigma \in S_K, f = g \circ \sigma$
(c) $f \sim g \iff \exists \beta \in S_N, f = \beta \circ g$
(d) $f \sim g \iff \exists \sigma \in S_K, \beta \in S_N, f = \beta \circ g \circ \sigma$

Example 1.1.4. In each of the 12 possibilities, find the number of equivalence classes of functions satisfying the restrictions.

Solution.

1.(a) We have n^k ways.

1.(b) The interpretations here is that we are looking for $k-$elements multisets from an $n-$element set, e.g. $n = 2$ and $k = 4$, then there are 5 k-multisets from $N = \{a, b\}$, they are

$$\{a, a, a, a\}, \{a, a, a, b\}, \{a, a, b, b\}, \{a, b, b, b\}, \{b, b, b, b\}$$

We define $\left(\!\binom{x}{k}\!\right) = \frac{x^{\bar{k}}}{k!}$ to be the multichoose. Then, we have the number we are looking for for $1.(b)$ is $\left(\!\binom{n}{k}\!\right)$ by Theorem 1.1.7.

2.(a) We have $n(n-1)(n-2)...(n-k+1) = \prod_{i=k}^{n}(n-i+1) = \frac{n!}{(n-k)!} := n^{\underline{k}}$. Note similarly, we have $x^{\bar{k}} = x(x+1)(x+2)...(x+k-1)$

2.(b) We define $\binom{n}{k}$ to be $\frac{x^{\underline{k}}}{k!}$. If $0 \leq k \leq n$, then $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$. We have $\binom{n}{k}$ ways to do $2.(b)$, consider Theorem 1.1.5

2.(c) 1 if $k \leq n$ and 0 otherwise

2.(d) 1 if $k \leq n$ and 0 otherwise

3.(a) Surjective functions, so the answer is $n!\left\{ {k \atop n} \right\}$.

3.(b) We will try to use generating function to show this one. The interpretation of this question is the composition of an integers, i.e. how many ways to write $k = c_1 + ... + c_n$ for k, n be given, and $c_1, ..., c_n$ are positive integers. This is similar to $1.(b)$.

3.(c) This is the problem of set partitions, i.e. how many ways to partition the set K into n disjoint non-empty subsets. For example, we have

$$\{\{1, 4\}, \{2, 5, 6\}, \{3\}, \{7\}\}$$

We will use the notation $\left\{ {k \atop n} \right\}$ = the number of partitions of $\{1, ..., k\}$ into n non-empty subsets. We note $1.(c)$ is related to this.

3.(d) This is the partition of an integer, i.e. how many ways to write $k = c_1 + ... + c_n$ where k, n are given and $c_1 \geq c_2 \geq ... \geq c_n$ are positive integers. We also note $1.(d)$ is related to this. ♠

Theorem 1.1.5. *The answer to* $2.(b)$ *is* $\binom{n}{k}$.

Proof. Let $X := \{f : K \to N : f \text{ is injective}\}$. For $f, g \in X$, let $f \sim g$ iff $\exists \alpha \in S_K$ such that $f = g \circ \alpha$. Let $Y = X/\sim$ be the set of equivalence classes of X and $\sim$.

We want to show $|Y| = \binom{n}{k}$. We see $|X| = n^{\underline{k}}$, and $|X| = \sum_{c \in Y} |c|$ since equivalence classes form disjoint sets. Suppose $c \in Y$, $g \in C$, then $c = \{f \in X : \exists \alpha \in S_K, f \in g \circ \alpha\} = \{g \circ \alpha : \alpha \in S_K\}$ Since g is injective, $g \circ \alpha = g \circ \beta \iff \alpha = \beta$ for $\alpha, \beta \in S_K$. Hence, $|C|$ is the number of bijections from K to K, which is $k!$.

Hence, we have $|X| = \sum_{c \in Y} |c| = \sum_{c \in Y} k! = k! \cdot |Y|$, thus $|Y| = \frac{|X|}{k!} = \frac{n^{\underline{k}}}{k!}$.

♡

Remark 1.1.6. We remark an alternate interpretation of the above Theorem 1.1.5. We have $\binom{n}{k}$ is the number of k-elements subsets of N where $|N| = n$. Try to show that if $f, g \in X$, then $f \sim g \iff Image(f) = Image(g)$.

Theorem 1.1.7. *The number of $k-$element multisets with elements from $\{1, ..., n\}$ is $\left(\!\binom{n}{k}\!\right)$.*

Proof. Let $X = \{\text{set of k-element multisets from } \{1, ..., n\}\}$. Let y be the set of $k-$element subsets of $\{1, 2, ..., n+k-1\}$. We know that $|y| = \binom{n+k-1}{k} = \left(\!\binom{n}{k}\!\right)$. We want to show $|X| = \left(\!\binom{n}{k}\!\right)$.

We define a bijection from X to Y. Let $f : X \to Y$. Let $A \in X$, we write $A = \{a_1, a_2, ..., a_k\}$ with $a_1 \leq ... \leq a_k$. Define $f(A) = \{a_1, a_2+1, a_3+2, ..., a_k+k-1\}$. We note this is well-defined, and we need to show it is bijection. We define f^{-1} to be $B = \{b_1, ..., b_k\}$, then let $f^{-1}(B) = \{b_1, b_2 - 1, ..., b_k - k + 1\}$, we have this is an two side inverse and we are done. ♡

Definition 1.1.8. Let $\mathbb{N} = \{0, 1, 2, 3, ...\}$ in this course. We have $[n] = \{1, ..., n\}$. We have

$$x^{\underline{n}} = x(x-1)...(x-n+1)$$
$$x^{\overline{n}} = (x+1)(x+2)...(x+n-1)$$
$$n! = n^{\underline{n}}$$
$$\binom{x}{n} = \frac{x^{\underline{n}}}{n!}, \left(\!\binom{x}{n}\!\right) = \frac{x^{\overline{n}}}{n!}$$

Example 1.1.9. We have $x^{\underline{0}} = x^{\overline{0}} = 0! = \binom{x}{0} = 1$.

We say an empty product is 1 and an empty sum is 0.

We have $[0] = \emptyset$ and $0^0 := 1$ in this course.

We have $n \in \mathbb{N}$ and $n < 0$, then $\frac{1}{n!} = 0$.

1.2 Intro to Generating Functions

Example 1.2.1 (Combinatorics Identities)**.** Prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Solution. Let $X = \{S \in P([n]) : |S| = k\}$. Let $Y = \{S \in P([n-1]) : |S| \in \{k, k-1\}\}$. It suffice to show $|X| = |Y|$.

For $A \in X$, let $f(A) = A \backslash \{n\}$, this is the desired bijection and the proof follows. ♠

Remark 1.2.2. Other than using double count (or define a bijection), we can also solve combinatoric questions algebraicly.

Definition 1.2.3. If $A(x) = \sum_{j=0}^{\infty} a_j x^j$ is a polynomial or power series, then we define $[x^k]A(x) := a_k$ to be the coefficient of x^k in $A(x)$ is a_k.

Example 1.2.4. Show that for $|x| < 1$ and $a \in \mathbb{R}$, we have $[x^k](1+x)^a = \binom{a-1}{k} + \binom{a-1}{k-1}$

Solution. Start with $(1+x)^{a-1} = \sum_{j \geq 0} \binom{a-1}{j} x^j$, we multiply both sides by $(1+x)$, then we have

$$\begin{aligned}
(1+x)^a &= (1+x) \sum_{j \geq 0} \binom{a-1}{j} x^j \\
&= \sum_{j \geq 0} \binom{a-1}{j} x^j (1+x) = \sum_{j \geq 0} \binom{a-1}{j} (x^j + x^{j+1}) \\
&= \sum_{j \geq 0} \binom{a-1}{j} x^j + \sum_{t \geq 0} \binom{a-1}{t} x^{t+1} \\
&= 1 + \sum_{j \geq 1} \binom{a-1}{j} x^j + \sum_{j \geq 1} \binom{a-1}{j-1} x^j, \quad \text{let } j = t+1 \\
&= 1 + \sum_{j \geq 1} \left(\binom{a-1}{j} + \binom{a-1}{j-1} \right) x^j \\
&= 1 + \sum_{j \geq 1} \binom{a}{j} x^j
\end{aligned}$$

The proof follows by comparing the coefficients of the polynomial, i.e.

$$1 + \sum_{j \geq 1} \left(\binom{a-1}{j} + \binom{a-1}{j-1} \right) x^j = 1 + \sum_{j \geq 1} \binom{a}{j} x^j$$

Finally, we remark that we need to check the formula is also correct for $k = 0$. ♠

Remark 1.2.5. We also have the negative binomial theorem,

$$(1-x)^{-a} = \sum_{k \geq 0} \left(\!\binom{a}{k}\!\right) x^k$$

The reader should try to show that this is equivalent to the usual statement of the binomial theorem.

Remark 1.2.6. When we want to exam the sum $\binom{a-1}{k-1} + \binom{a-1}{k}$, we are trying to understand the sum of two consecutive terms in the sequence $\{\binom{a-1}{i}\}_{i=0}^{\infty}$.

The technique to this is to ***form a power series where coefficients are the sequence of interest***. E.g. we would have

$$f(x) := \sum_{j\geq 0}\binom{a-1}{j}x^j$$

Then, if we want to learn about the consecutive sums, we exam

$$(1+x)f(x) = \binom{a-1}{0} + (\binom{a-1}{1} + \binom{a-1}{0})x + (\binom{a-1}{2} + \binom{a-1}{1})x^2 + ...$$

The binomial theorem let us to compare this with LHS.

Example 1.2.7. Consider the Fibonacci sequence $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$. Find an explicit formula for f_n.

Proof. Consider the series $F(x) := \sum_{n\geq 0} f_n x^n$. We have

$$\begin{aligned} F(x) &= f_0 + f_1x + f_2x^2 + f_3x^3 + ... \\ xF(x) &= \quad +f_0x + f_1x^2 + f_2x^3 + ... \\ x^2F(x) &= \quad + \quad + f_0x^2 + f_1x^3 + ... \end{aligned}$$

Thus, we have $F(x) - xF(x) - x^2F(x) = f_0 + (f_1 - f_0)x + 0 + 0 + ...$ Therefore, we have

$$F(x) - xF(x) - x^2F(x) = x \Rightarrow F(x) = \frac{x}{1-x-x^2}$$

Let $\phi_1 = \frac{1+\sqrt{5}}{2}$ and $\phi_2 = \frac{1-\sqrt{5}}{2}$. Next, we use method of partial fractinos, and we have

$$\begin{aligned} F(x) &= \frac{1}{\sqrt{5}}(\frac{1}{1-\phi_1 x}) - \frac{1}{\sqrt{5}}(\frac{1}{1-\phi_2 x}) \\ &= \frac{1}{\sqrt{5}}(1-\phi_1 x)^{-1} - \frac{1}{\sqrt{5}}(1-\phi_2)^{-1}, \quad \text{then, use geometric series} \\ &= \frac{1}{\sqrt{5}}\sum_{n\geq 0}\phi_1^n x^n - \frac{1}{\sqrt{5}}\sum_{n\geq 0}\phi_2^n x^n \\ &= \sum_{n\geq 0}(\frac{\phi_1^n - \phi_2^n}{\sqrt{5}})x^n \end{aligned}$$

Then, by compare coefficients, we have $f_n = \frac{\phi_1^n - \phi_2^n}{\sqrt{5}}$. ♡

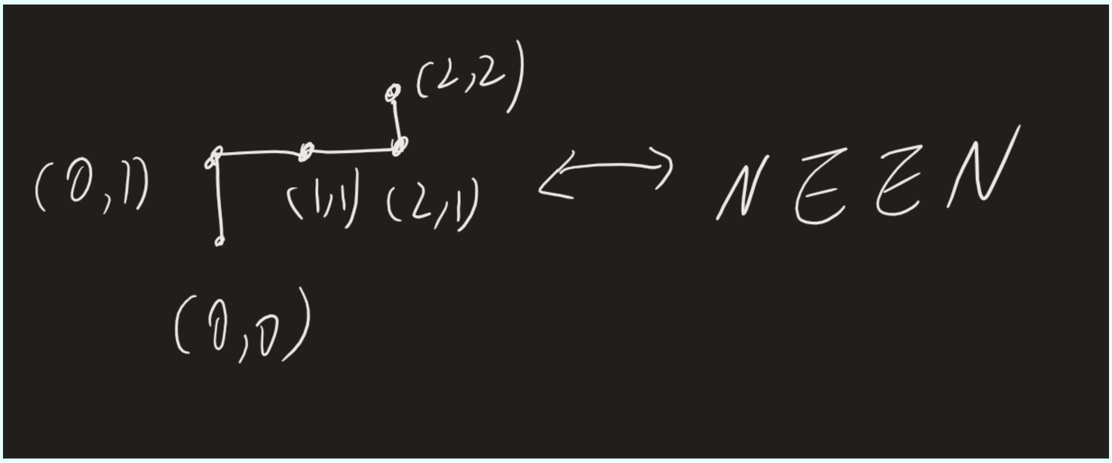
Remark 1.2.8. We note on partial fractions, to expand $\frac{P(x)}{Q(x)}$ where $Q(x)$ has repeated roots, there are two forms. The better one for this course is, for example,

$$\frac{1}{(1-2x)^2(1-3x)^3} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{(1-3x)} + \frac{D}{(1-3x)^2} + \frac{E}{(1-3x)^3}$$

1.3 Lattice Path

Definition 1.3.1. A ***latice path*** is a sequence $p_0, p_1, ..., p_n$ on points in $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ such that $p_{i+1} - p_i \in \{(1,0),(0,1)\}$ where $(1,0)$ is called east step and $(0,1)$ is called north step.

Remark 1.3.2. We often represent a path starting at p_0, where usually(but not always) $p_0 = (0,0)$, by a string of N's and S's, e.g.

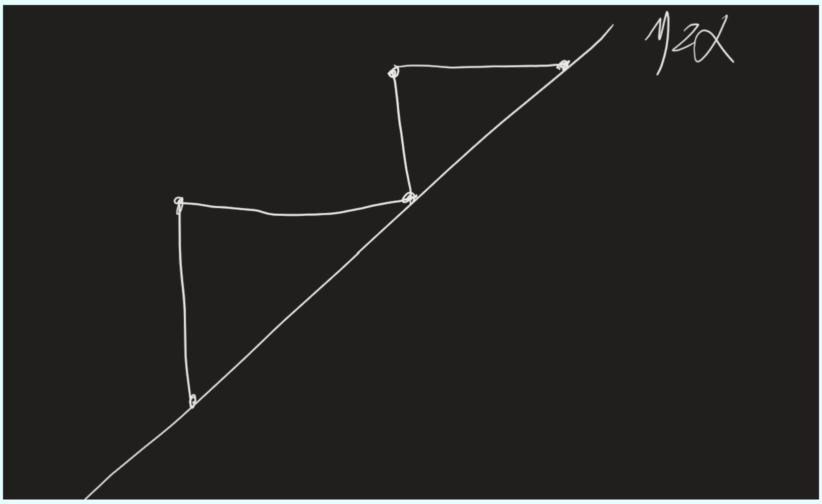


Example 1.3.3. How many latice paths from $(0,0)$ to (m,n)?

Solution. We have $\binom{n+m}{n}$. Indeed, each such path has $m+n$ steps, n are N and m are E. These may be in any order. If the path is represented by the string $S = S_1S_2,...,S_{n+m}$, let $\alpha(S) = \{i \in [m+n] : S_i = N\}$, then α is a bijection from set of lattice paths to number of n subsets in $[m+n]$. ♠

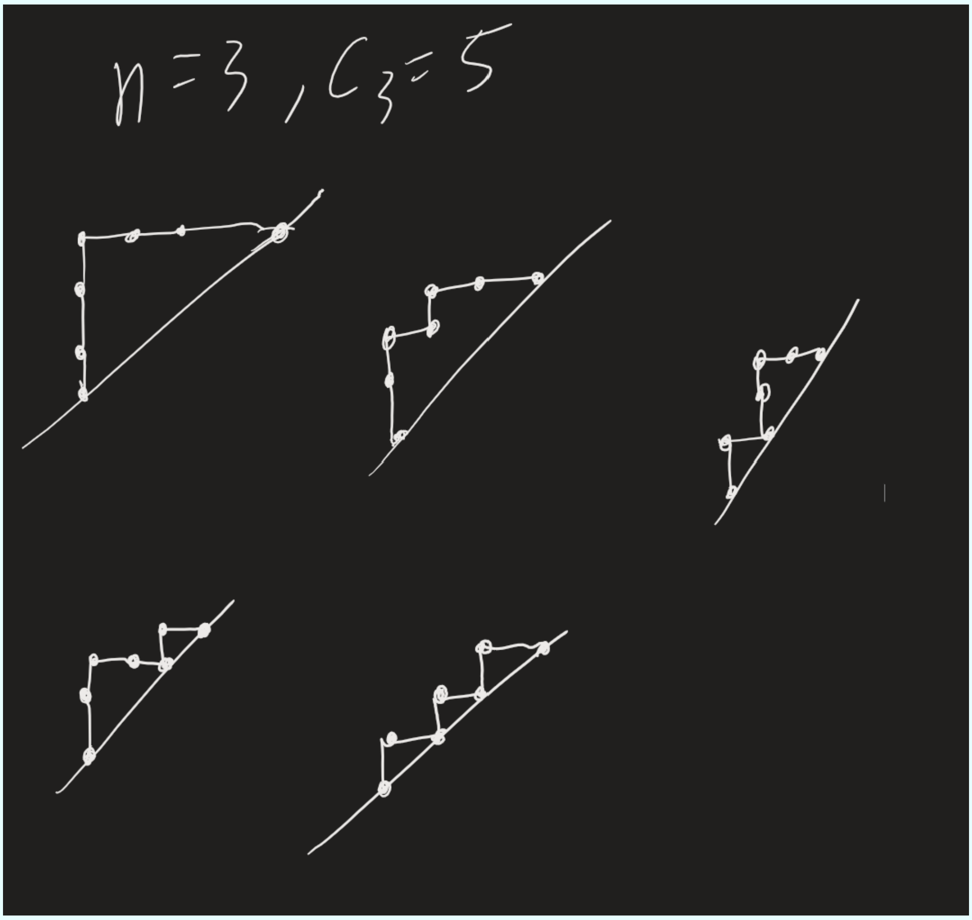
Definition 1.3.4. A ***Dyck path*** is a lattice path such that $p_0 = (0,0)$ and $p_n(n,n)$ and $p_i = (x_i, y_i)$ where $x_i \leq y_i$ for all i.

Remark 1.3.5. A Dyck path never goes below the line $y = x$. Consider the example



Definition 1.3.6. For $n \in \mathbb{N}$, let c_n, the ***Catalan numbers***, denote the number of Dyck paths of length $2n$.

Example 1.3.7. When $n = 3$, we have $c_3 = 5$. Indeed,



Theorem 1.3.8. *We have* $c_n = \frac{1}{n+1}\binom{2n}{n}$ *for all* $n \in \mathbb{N}$.

Proof. We do in three steps.

STEP 1 Find a recurrence relation for the sequence $\{c_n\}_{n\geq 0}$. In particular, we will show $c_0 = 1$ and $c_n = \sum_{k=0}^{n-1} c_k c_{n-k-1}$.

We note LHS is the number of Dyck path of length $2n$ and the RHS is the number of pairs of Dyck paths where combined length is $2n-2$.

Let $n \geq 1$ and let π be a Dyck path of length $2n$, the first step of π must be N and π ends on the line $y = x$. Thus, consider the first time that π returns to the line $y = x$. Suppose this is after $2k+2$ steps where $k \geq 0$. Then the $2k+2$th step must be E. We can write π as $N\pi_1 E\pi_2$ where π_1 is of length $2k$ and π_2 is of length $2(n-k-1)$. Note that π_1 is a Dyck path shifted by $(0,1)$ and π_2 is a Dyck path shifted by $(k+1, k+1)$. Thus, there are c_k possibilities of π_1 and $n-k-1$ possibilities of π_2. Summing over all possibilities k gives, we have $c_n = \sum_{k=0}^{n-1} c_k c_{n-k-1}$.

STEP 2 Turn this into a fact about series. To solve this recurrence, let $C(x) = \sum_{n\geq 0} c_n x^n$, we claim that

$$xC(x)^2 - C(x) + 1 = 0$$

We have

$$\begin{aligned}
xC(x)^2 &= x(\sum_{j\geq 0} c_j x^j)(\sum_{k\geq 0} c_k x^k) \\
&= \sum_{j\geq 0}\sum_{k\geq 0} c_j c_k x^{i+k+1} \\
&= \sum_{n\geq 1}(\sum_{k=0}^{n-1} c_k c_{n-k-1}) x^n \\
&= \sum_{n\geq 1} c_n x^n = C(x) - 1
\end{aligned}$$

STEP 3 ~~Profits~~. No, I mean extract answers.

By the quadratic formula, we have $C(x) = \frac{1\pm\sqrt{1-4x}}{2x} = \frac{1\pm(1-4x)^{1/2}}{2x}$. We note

$$(1-4x)^{1/2} = 1 + \sum_{k\geq 1}\binom{1/2}{k}(-4)^k x^k$$

We note

$$\begin{aligned}
\binom{1/2}{k}(-4)^k &= \frac{\frac{1}{2}(-\frac{1}{2})(\frac{-3}{2})\dots(\frac{-2k+3}{2})}{k!}(-1)^k 2^k 2^k \\
&= -\frac{1\cdot 3\cdot \dots \cdot (2k-3)}{k!}2^k \cdot \frac{(k-1)!}{(k-1)!} \\
&= -\frac{1\cdot 3\cdot \dots \cdot (2k-3)}{k!(k-1)!} 2\cdot 2\cdot 4\cdot \dots \cdot (2k-2) \\
&= \frac{-2}{k}\frac{(2k-2)!}{(k-1)!(k-1)!} = \frac{-2}{k}\binom{2k-2}{k-1}
\end{aligned}$$

Hence, we get

$$C(x) = \frac{1}{2x} \pm \left(\frac{1}{2x} - \frac{1}{x}\sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^k\right)$$

We remark that we cannot have $+$ in the $\pm$ as we know $C(x)$ has positive degrees. Therefore, we must have

$$C(x) = \frac{1}{2x} - \frac{1}{2x} + \frac{1}{x}\sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^k = \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n$$

In particular, this gives us $c_n = \frac{1}{n+1}\binom{2n}{n}$ as desired. ♡

Example 1.3.9. Let D_n be the set of Dyck paths of length $2n$. Let P_n be the set of lattice paths from $(0,0)$ to (n,n). We know that $|P_n| = \binom{2n}{n}$. The goal is to show that $(1+n)|D_n| = |P_n|$.

We will give a bijection $f : P_n \to D_n \times [n+1]$. Let $s_1, s_2, ..., s_{2n} \in P_n$, and let $s_0 = N$. Consider the following $2n+1$ paths

$$s_0 s_1 s_2 ... s_{2n-2} s_{2n-1}$$

$$s_1 s_2 s_3 ... s_{2n-1} s_{2n}$$

$$s_2 s_3 s_4 ... s_{2n} s_0$$

$$s_3 s_4 s_5 ... s_{2n-1} s_{2n} s_0 s_1$$

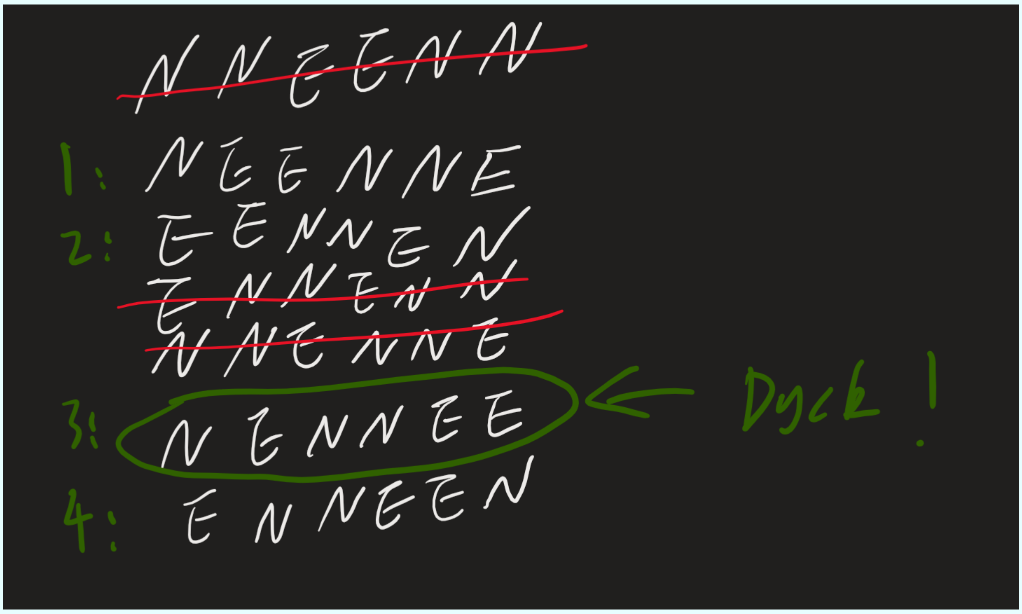
$$\vdots$$

$$s_{2n} s_0 s_1 ... s_{2n-2}$$

Next, we cross out all paths that do not end at (n,n), i.e. crossout all with n+1 N steps and n-1 E steps.

We claim that there are $n+1$ paths remaining and exactly one of these is in D_n.

For example, if we have $NEENNE \in P_3$, then we have



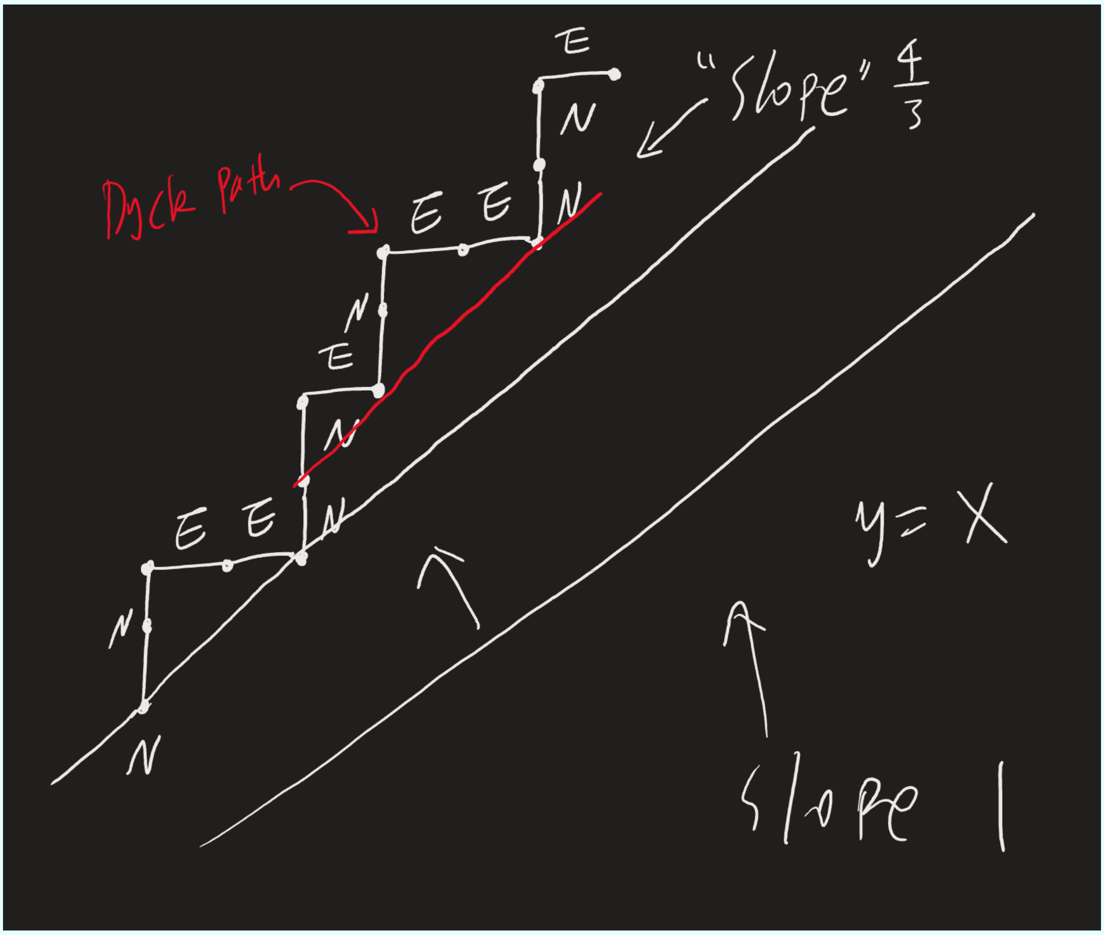
We define $f(NEENNE) = (NENNEE, 3)$ and more generally, we have

$$f(s_1 s_2 ... s_n) = (s_k s_{k+1} ... s_{k-2}, j)$$

where $s_k ... s_{k-2}$ is the unique path in D_n on the list and j indicates which element on the list it is (after crossing out all paths with too many N steps).

The question is, why does it work? Think about the infinite path

$$s_0 s_1 s_2 ... s_{2n} s_0 s_1 s_2 ... s_{2n} s_0 s_1 ...$$



For a complete proof, check Golden's note.

Example 1.3.10. We show a second proof of the Example 1.3.9. We have Theorem 1.3.11.

Theorem 1.3.11. *For $0 \le a \le b$. The number of lattice paths from $(0,0)$ to (a,b) that do not go below the line $y = x$ is*

$$\binom{a+b}{a} - \binom{a+b}{a-1}$$

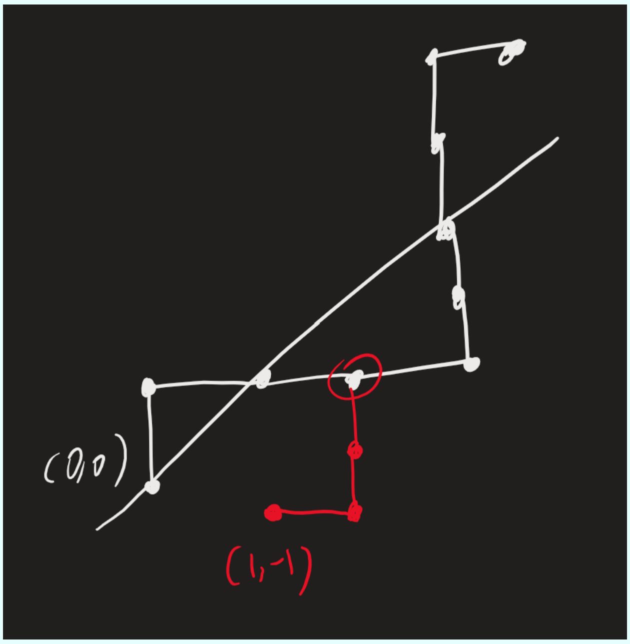
Proof. Let X be the set of lattice paths from $(0,0)$ to (a,b) that do go below $y = x$, we will show $|X| = \binom{a+b}{a-1}$.

Let Y be the set of all latice paths from $(1,-1)$ to (a,b), we know that $|Y| = \binom{a+b}{a-1}$. Note that all paths in y go below $y = x$. We give a bijection $f : X \to Y$. Let $s_1 s_2 ... s_{a+n} \in X$. Since there is at least one point below $y = x$, this path must have a point on the line $y = x - 1$. Suppose the first such point is after $2k - 1$ steps, define

$$f(s_1 s_2 ... s_{a+n}) = \overline{s_1 s_2} ... \overline{s_{2k-1}} s_{2k} ... s_{a+n}$$

where $\overline{s_i}$ means switch north and east. One should check this is a bijection.

Consider the following example



♡

1.4 The Formalism of Generating Functions

Definition 1.4.1. ***Formalism*** means one of the following:

1. the doctrine that formal structure rather than content is what should be represented
2. (philosophy) the philosophical theory that formal (logical or mathematical) statements have no meaning but that its symbols (regarded as physical entities) exhibit a form that has useful applications
3. the practice of scrupulous adherence to prescribed or external forms

Remark 1.4.2. The above is totally unrelated to the class.

Remark 1.4.3 (Ingredients of Generating Functions)**.** We would have

- A set Λ of "combinatorial objects"
- A function $\omega : \Lambda \to \mathbb{N}$ called the ***weight function***. For $\sigma \in \Lambda$, $\omega(\sigma)$ is called the weight of σ.
- General counting problems: for $n \in \mathbb{N}$, determine the number of elements of Λ that have weight n, i.e. $|\{\sigma \in \Lambda : \omega(\sigma) = n\}| := \#\{\sigma \in \Lambda : \omega(\sigma) = n\}$
- A weight function is ***good*** if answer is finite for all n.

Then, the ***basic strategy*** is shove almost every counting problem we encounter into this framework.

Example 1.4.4. Fix $m \in \mathbb{N}$, let $\Lambda_m = \{0,1\}^m$ be the set of binary strings of length m. For example, we have $m = 3$ and the numbers we have is

$$\{000, 001, 010, 011, 100, 101, 110, 111\}$$

Next, if $\sigma \in \Lambda_m$, define $\omega(\sigma)$ be the number of 1's in σ. Then, the counting problem becomes: determine the number of binary strings of length m with n 1s and the answer is $\binom{m}{n}$.

Example 1.4.5. Let C be the set of all Dyck paths of any length. Let's define the weight to be, for $\pi \in C$, we let $\omega(\pi)$ to be the number of N steps. Then, the general counting problem becomes: determine the number of Dyck paths of length $2n$. We know the answer to this is $\frac{1}{n+1}\binom{2n}{n}$

Definition 1.4.6. In general, we define the ***generating function*** for Λ with respect to ω to be the power series

$$\Phi_\Lambda(x) = \sum_{\sigma \in \Lambda} x^{\omega(\sigma)}$$

Example 1.4.7. In the above Example 1.4.4, we have

$$\Phi_{\Lambda_3}(x) = x^0 + x^1 + x^1 + x^2 + x^1 + x^2 + x^2 + x^3$$

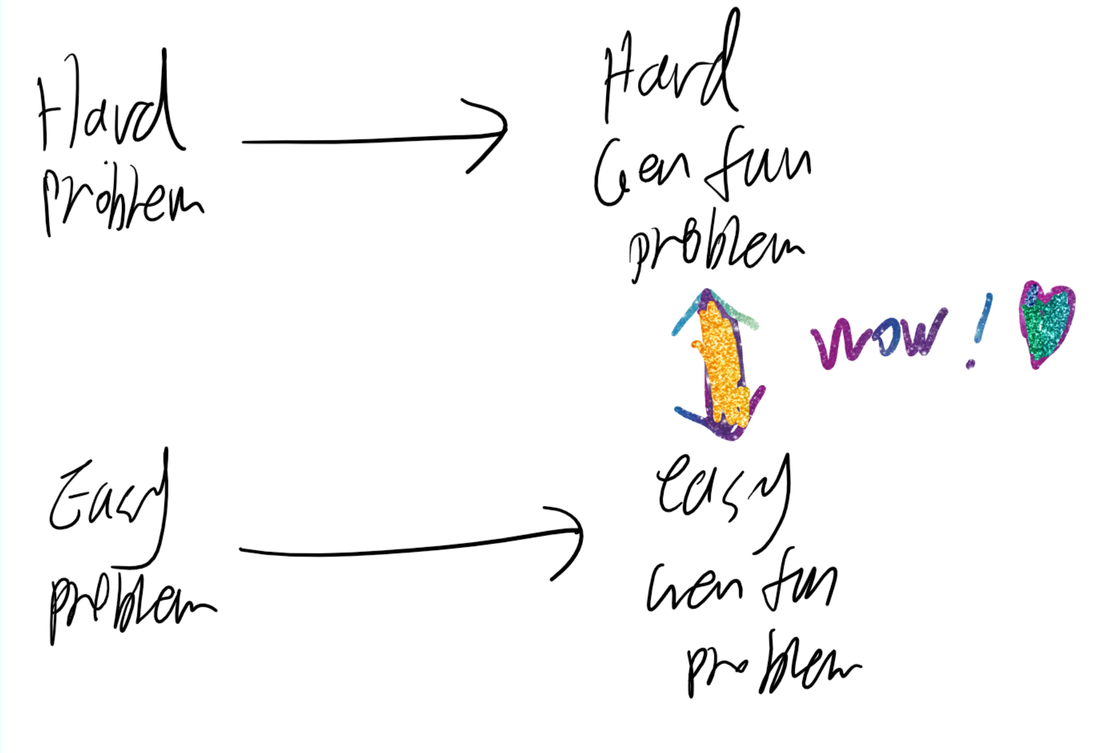
Proposition 1.4.8. *The answer to the counting problem is* $[x^n]\Phi_\Lambda(x)$*, i.e. we have* $[x^n]\Phi_\Lambda(x) = |\{\sigma \in \Lambda : \omega(\sigma) = n\}|$

Proof. We have

$$\begin{aligned}
[x^n]\Phi_\Lambda(x) &= [x^n]\sum_{\sigma\in\Lambda} x^{\omega(\sigma)} \\
&= [x^n]\sum_{m\geq 0}\sum_{\substack{\sigma\in\Lambda \\ \omega(\sigma)=m}} x^{\omega(\sigma)} \\
&= [x^n]\sum_{m\geq 0}\sum_{\substack{\sigma\in S^m \\ \omega(\sigma)=m}} x^m \\
&= [x^n]\sum_{m\geq 0}(\sum_{\substack{\sigma\in\Lambda \\ \omega(\sigma)=m}} 1)x^m \\
&= \sum_{\substack{\sigma\in\Lambda \\ \omega(\sigma)=n}} 1
\end{aligned}$$

♡

Remark 1.4.9. We have two kinds of problems, ***hard problems*** and ***easy problems***, where both of them can be translated into generating function problems.



Lemma 1.4.10 (The Sum Lemma)**.** *Let Λ be a set with weight function w. Let A be a subset of Λ. Suppose $\Lambda = A \cup B$. Then we have*

$$\Phi_{A\cup B}(x) = \Phi_A(x) + \Phi_B(x) - \Phi_{A\cap B}(x)$$

Proof. For a subset $A \subseteq \Lambda$, let $\chi_a : \Lambda \to \mathbb{Z}$ be the indicator function, i.e.

$$\chi_A(\sigma) = \begin{cases} 1, & \sigma \in A \\ 0, & \sigma \notin A \end{cases}$$

Note that

$$\Phi_A(x) = \sum_{\sigma\in A} x^{w(\sigma)} = \sum_{\sigma\in\Lambda} \chi_A(\sigma) x^{w(\sigma)}$$

Then, we have

$$\forall \sigma \in \Lambda, \chi_A(\sigma) + \chi_B(\sigma) - \chi_{A\cap B}(\sigma) = 1$$

Thus, we have

$$\begin{aligned} \Phi_A(x) + \Phi_B(x) - \Phi_{A\cap B}(x) &= \sum_\sigma \chi_a(\sigma)x^{w(\sigma)} + \sum_\sigma \chi_B(\sigma)x^{w(\sigma)} - \sum_\sigma \chi_{A\cap B}x^{w(\sigma)} \\ &= \sum(\chi_A(\sigma) + \chi_B(\sigma) - \chi_{A\cap B}(\sigma))x^{w(\sigma)} = \Phi_\Lambda(x) \end{aligned}$$

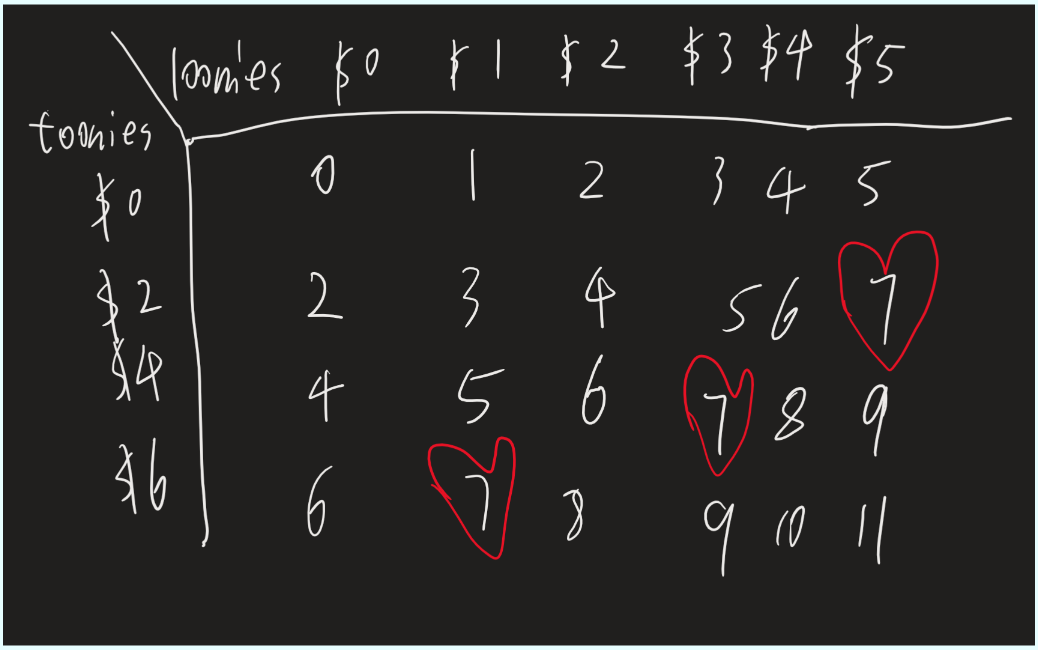
♡

Remark 1.4.11. Note this is used mostly in the case where A and B intersect trivially, i.e. $\emptyset$. In which case we would have $\Phi_{A\cap B}(x) = 0$.

If $\Lambda = \bigcup_i A_i$ where the union is fintie or countable such that pair-wise disjoint, then we have $\Phi_\Lambda(x) = \sum_i \Phi_{A_i}(x)$

Example 1.4.12. You have 5 loonies and 3 toonies and the question is how many ways to make 7 dollar.

Solution. We can try to make a table of all possible combinations of loonies and toonies.



toonies \ loonies	\$0	\$1	\$2	\$3	\$4	\$5
\$0	0	1	2	3	4	5
\$2	2	3	4	5	6	7
\$4	4	5	6	7	8	9
\$6	6	7	8	9	10	11

Hence the answer is 3 ♠

Example 1.4.13. Compute $[x^7](1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^2 + x^4 + x^6)$.

Solution. We can try to make a nicer table of all possibilities.

Hence the answer is 3 ♠

Remark 1.4.14. Note that the Example 1.4.12 and Example 1.4.13 are the same question in different terms.

In the two examples, the question of how to translate one to another is how to transform a word question into a generating function question.

Thus, let A be the set of configurations that I can make using only loonies. Thus, we have $A = \{\$0, ..., \$5\}$. Similarly, we can let $B = \{\$0, ..., \$6\}$ be the set of configurations of toonies. Then, $A \times B$ is the set of configurations I can make using both loonies and toonies.

Next, for each of the set, define the weight of a configuration to be the total number value, for example, $w(2, 4) = 6$, and $w(2) = 2$ and so on. Note this ***three*** weight functions satisfy the hypothesis of the product lemma below. Thus we have $\Phi_{A\times B} = \Phi_A(x)\Phi_B(x)$ and we want $[x^7]\Phi_{A\times B}(x)$, which is going to be our answer.

Lemma 1.4.15 (The Product Lemma)**.** *Suppose A, B are sets, and α and β be weight functions on A and B respectively. Let $\Lambda = A \times B$, let $w : \Lambda \to \mathbb{N}$ be a weight function on Λ. Let γ be a constant.*

If $w(a, b) = \alpha(a) + \beta(b) + \gamma$ for all $a \in A, b \in B$. Then, we have

$$\Phi_\Lambda(x) = x^\gamma \cdot \Phi_A(x) \cdot \Phi_B(x)$$

Proof. We have

$$\begin{aligned}\Phi_\Lambda(x) &= \sum_{(a,b)\in A\times B} x^{w(a,b)}\\ &= \sum_{a\in A}\sum_{b\in B} x^{w(a,b)} = \sum_{a\in A}\sum_{b\in B} x^{\alpha(a)+\beta(b)+\gamma}\\ &= x^\gamma \sum_{a\in A} x^{\alpha(a)} \sum_{b\in B} x^{\beta(b)}\\ &= x^\gamma \Phi_A(x)\Phi_B(x)\end{aligned}$$

♡

Lemma 1.4.16 (Product Lemma in General). *If $A_1, ..., A_k$ and $A_1 \times A_2 \times ... \times A_k$ be sets with weight functions $\alpha_1, ..., \alpha_k$ and w, respectively. In addition, suppose $w(a_1, ..., a_k) = \gamma + \alpha_1(a_1) + ... + \alpha_k(a_k)$ for all $(a_1, ..., a_k) \in A_1 \times ... \times A_k$. Then, we have*

$$\Phi_{A_1\times...\times A_k}(x) = x^\gamma \prod_{i=1}^{k} \Phi_{A_i}(x)$$

Remark 1.4.17. In product lemma, we do not talk about countable products of sets as that may result in uncountable sets.

Example 1.4.18. Let $k = 3$ and $n = 2$, how many ways to write out the k-tuples such that makes their sum equal n?

Solution. We have $0+0+2$, $0+2+0$, $2+0+0$, $1+1+0$, $1+0+1$ and $0+1+1$. ♠

Example 1.4.19 (Week Composition of an Integer). For $k, n \in \mathbb{N}$, determine the number of $k-$tuples $(c_1, ..., c_k) \in \mathbb{N}^k$ such that $\sum c_i = n$.

Solution. Consider $\mathbb{N}^k$ with weight function $w : \mathbb{N}^k \to \mathbb{N}$ such that $w(c_1, ..., c_k) = c_1 + ... + c_k$. The answer will be $[x^n]\Phi_{\mathbb{N}^k}(x)$. Observe that this is a Cartesian product and thus we may use product lemma.

Let $\alpha : \mathbb{N} \to \mathbb{N}$ be the weight function on $\mathbb{N}$ to be $\alpha(i) = i$. Notice that $w(c_1, ..., c_k) = \alpha(c_1) + ... + \alpha(c_k)$. Thus, the generating function is

$$\Phi_{\mathbb{N}^k}(x) = (\Phi_{\mathbb{N}}(x))^k$$

Hence, we only need to figure out what $\Phi_{\mathbb{N}}(x)$ is, but this is easy and can be computed directly! In particular, we have

$$\Phi_{\mathbb{N}}(x) = \sum_{i\in\mathbb{N}} x^{\alpha(i)} = \sum_{i=1}^{\infty} x^i = (1-x)^{-1}$$

Therefore, we have $\Phi_{\mathbb{N}^k}(x) = (1-x)^{-k}$. By the negative binomial theorem, we have $[x^n](1-x)^{-k} = \left(\!\binom{k}{n}\!\right)$ ♠

1.5 Composition

Definition 1.5.1. A ***composition*** of a integer n with k parts is a k-tuple

$$(c_1, ..., c_k) \in \mathbb{N}_{\geq 1}^k$$

where $\mathbb{N}_{\geq i} = \{i, i+1, i+2, ...\}$ such that $\sum_{i=1}^k c_i = n$. The numbers $c_1, ..., c_k$ are called the ***parts*** of the composition.

Remark 1.5.2. We will insist that there is a unique composition of 0, it has 0 parts.

Example 1.5.3. For $k, n \in \mathbb{N}$, determine the number of compositions of n with k parts.

Solution. Similar to Example 1.4.19, but use $\mathbb{N}_{\geq 1}$ instead of $\mathbb{N}$, i.e. $\Phi_{\mathbb{N}_{\geq 1}}(x) = x(x-1)^{-1}$ and so $\Phi_{\mathbb{N}_{\geq 1}^k}(x) = x^k(1-x)^k$ and the answer would be $[x^k]\Phi_{\mathbb{N}_{\geq 1}^k}(x)$, which is equal to $[x^{n-k}](1-x)^{-k} = \left(\!\binom{k}{n-k}\!\right)$. ♠

Example 1.5.4. Determine the number of compositions of n in which there are an even number of parts and all parts are odd. For instance, $n = 4$ then we have $(3, 1)$, $(1, 3)$, and $(1, 1, 1, 1)$.

Solution. Let S be the set of all compositions in which there are an even number of parts and all parts are odd. Define the weight w of a composition to be the sum of its parts.

Let $\mathbb{N}_o$ be the set of odd numbers, i.e. $\mathbb{N}_o = \{1, 3, 5, 7, ...\}$.

Then, since all parts are odd, we have

$$S = \bigcup_{k \geq 0} (\mathbb{N}_o)^{2k}$$

Using the weight function $\alpha(i) = i$ on $\mathbb{N}_o$, we have

$$\Phi_{\mathbb{N}_o}(x) = x + x^3 + x^5 + ... = \frac{x}{1 - x^2}$$

Moreover, the weight functions w, α satisfy the hypothesis of product lemma, i.e. we have $w(c_1, ..., c_{2k}) = \alpha(c_1) + ... + \alpha(c_{2k})$. Then, by the sum lemma, we have

$$\begin{aligned}
\Phi_S(x) &= \sum_{k \geq 0} \Phi_{\mathbb{N}_o^{2k}}(x) \\
&= \sum_{k \geq 0} (\Phi_{\mathbb{N}_o}(x))^{2k} \\
&= \sum_{k \geq 0} (x(1-x^2)^{-1})^{2k} = \frac{1}{1 - (x(1-x^2)^{-1})^2}
\end{aligned}$$

Hence, the answer to the problem is

$$[x^n](1 - x^2(1 - x^2)^{-2})^{-1}$$

From here, in principle, one could "simplify(?)" further via partial fraction to get closed form answer or find a recurrence. This is often messy, and not always better or viable.

We also note that $\Phi_S(x)$ is an even function, and thus answer is 0 for n odd. ♠

Remark 1.5.5 (Plugging in numbers)**.** If S is a finite set, then $\Phi_S(x)$ is a polynomial. Then, we would get

1. $\Phi_S(1) = |S|$
2. $\Phi'_S(1) = \sum_{\sigma \in S} w(\sigma)$
3. $\frac{\Phi'(1)}{\Phi(1)} = \frac{\sum_{\sigma \in S} w(\sigma)}{|S|}$
4. We can also calculate the variance formula, i.e. $\Phi''_S(1) + \Phi'_S(1) - (\Phi'_S(1))^2$

If S is an infinite set, $\Phi_S(x)$ is a power series. In some problems, we have $\Phi'_S(p)$, $0 \leq p < 1$, computes an expected value. In addition, the radius of convergence tells us something about asymptotics. In some cases, radius of convergence is 0.

1.6 Formal Power Series

Example 1.6.1. Consider the question: "to be or not to be"

Remark 1.6.2. We have analytic power series and formal power series. Analytic power series can take in values, and formal power series does not take values. We need to understand the difference between polynomial functions and polynomials and we can extend this to power series.

In particular, we have the formal power series is the ring of infinite polynomials, i.e. $R^{\mathbb{N}}$, the collections of functions from $\mathbb{N}$ to R where R is any ring.

Remark 1.6.3. ***The following definition is different from lecture, so DO REFER TO YOUR OWN NOTES***

Definition 1.6.4. Let R be a ring, then the ***ring of formal power series***, $R[[x]]$, is the set $R^{\mathbb{N}}$, the infinite sequences in R, with two operations, addition and multiplication.

The addition: let $(a_0, a_1, a_2, ...) + (b_0, b_1, b_2, ...) = (a_0 + b_0, ...)$.

We define the inverse of addition to be $((a_i)_{i=0}^{\infty})^{-1} = (-a_i)_{i=0}^{\infty}$.

We define the multiplication to be $(a_i)_{i=0}^{\infty} \cdot (b_i)_{i=0}^{\infty} = (\sum_{k=0}^{i} a_i b_{k-i})_{i=0}^{\infty}$ and for $c \in R$ we have $c(a_i)_{i=0}^{\infty} = (ca_i)_{i=0}^{\infty}$.

We define $\frac{(a_i)_{i=0}^{\infty}}{(b_i)_{i=0}^{\infty}} = (c_i)_{i=0}^{\infty}$ if $(b_i)_{i=0}^{\infty} \neq (0)_{i=0}^{\infty}$ and $(a_i)_{i=0}^{\infty} = (b_i)_{i=0}^{\infty} \cdot (c_i)_{i=0}^{\infty}$.

The above are standard ring stuff. We immediately identify $(a_i)_{i=0}^{\infty}$ to be $A(x) = \sum_{n\geq 0} a_i x^i$.

Next are not standard ring stuff. We have coefficient extraction, namely $[x^n]A(x) = a_n$.

We have formal derivative rule, namely

$$\frac{d}{dx}A(x) = A'(x) = \sum_{n\geq 0} n a_n x^{n-1} = \sum_{n\geq 0}(n+1)a_{n+1}x^n$$

We want to show that $(A(x)B(x))' = A'(x)B(x) + A(x)B'(x)$ which I will not show.

We can also do substitution in $R[[x]]$. Let $A(x)$ and $B(x)$ be two formal power series, then $A(B(x))$ is well-defined in two cases,

1. If $A = \sum_{k=0}^{n} a_k x^k$, i.e. $a_k = 0$ for all $k > n$. In particular, in this case we have

$$A(B(x)) = \sum_{k=0}^{a} a_k B(x)^k$$

2. If $[x^0]B(x) = 0$, then we have

$$A(B(x)) = \sum_{k\geq 0} a_k B(x)^k$$

   This makes sense because if I want to know any particular coefficients of the right hand side, I can compute it. Indeed, say I want to know

$$[x^m]\sum_{k\geq 0} a_k B(x)^k$$

   Then, this is the same as compute

$$[x^m]a_0B(x)^0 + [x^m]a_1B(x)^1 + ... + [x^m]a_mB(x)^m + [x^m]a_{m+1}B(x)^{m+1} + ...$$

   In particular, we have $B(x) = a_1x_1 + a_2x^2 + ...$, $B(x)^2 = a_1^2x^2 + ...$ and $B(x)^{m+1}$ does not even have a x^m term! Thus, we get a finite sum, i.e.

$$[x^m]A(B(x)) = [x^m]\sum_{k=0}^{m} a_k B(x)^k$$

   Moreover, the fully correct definition of $A(B(x))$ in this case is

$$A(B(x)) := \sum_{m\geq 0}\left([x^m]\sum_{k=0}^{m} a_k B(x)^k\right)x^m$$

All other cases, $A(B(x))$ is not DEFINED. In particular, we cannot plug in any number other than 0, the additive identity.

Theorem 1.6.5. *A formal power series $A \in R[[X]]$ is invertible if and only if $[x^0]A(x) \in R^\times$ if and only if $A(0) \in R^\times$ where $R^\times$ is the set of invertible elements of the ring R.*

Proof. Suppose $A(x) = \sum a_n x^n$ with $a_0 \in R^\times$. Then, let $F(x) = a_0 - x$ and $G(x) = \sum a_0^{-n-1} x^n$, observe that $F(x)G(x) = 1$ and so we have $F(x)$ and $G(x)$ are inverses. Let $B(x) = a_0 - A(x)$ and note that $[x^0]B(x) = 0$. Thus, we have $F(B(x))G(B(x)) = 1$, however, $F(B(x)) = A(x)$ and so $G(B(x))$ is the inverse of $A(x)$.

Conversely, suppose $A(x)$ is invertible and then it has an inverse. Thus, there exists $Z(x) \in \mathbb{R}[[x]]$ such that $A(x)Z(x) = 1$. Therefore, we have $A(0)Z(0) = 1$ and so $A(0)$ is invertible. ♡

Remark 1.6.6. What about the binomial theorem? Aha, we can just define the binomial theorem to be a definition!

Definition 1.6.7. We define $(1+x)^a = \sum_{n\geq 0} \binom{a}{n} x^n$.

Example 1.6.8. Consider $\sqrt{4+x^2+X^3}$, we have

$$\begin{aligned}\sqrt{4+x^2+X^3} &= 4\sqrt{1+\frac{x^2+x^3}{4}} \\ &= 2(1+\frac{x^2+x^3}{4})^{1/2}\end{aligned}$$

Note $\frac{x^2+x^3}{4}$ can be substituted into $(!+x)$, we get

$$\sqrt{4+x^2+X^3} = 2\sum_{n\geq 0}\binom{1/2}{n}(\frac{x^2+x^3}{4})^n$$

Proposition 1.6.9. *Any identity which is true for analytic series is also a true identify for formal power series.*

Proof. Suppose $A(x) = B(x)$ as analytic power series. Then $\frac{d^n}{dx^n}A(x) = \frac{d^n}{dxn^2}B(x)$ and plug in 0 we get $[x^n]A(x) = [x^n]B(x)$ for all $n \in \mathbb{N}$. Thus $A(x) = B(x)$ as formal power series. ♡

1.7 Hey, Catalan Strikes Back!!

Definition 1.7.1. Let S_1 and S_2 be sets with weight functions w_1 and w_2, respectively. If $f : S_1 \to S_2$ is a bijection such that $w_2 \circ f = w_1$ then we say that f is a ***weight preserving bijection***.

Proposition 1.7.2. *If $f : S_1 \to S_2$ is weight preserving bijection, then $\Phi_{S_1}(x) = \Phi_{S_2}(x)$.*

Proof. Muda Muda Muda! ♡

Example 1.7.3. We will calculate the Catalan number again.

Let C be the set of all Dyck paths, define the weight of the path to be the number of north steps.

We have a bijection f from $C \times C$ to $C \backslash \{\emptyset\}$ where $\emptyset$ is the empty Dyck path given by following: $(\pi_1, \pi_2) = N\pi_1 E\pi_2$. Consider the weight function $w \circ f : C \times C \to \mathbb{N}$, note that $w \circ f(\pi_1, \pi_2) = 1 + w(\pi_1) + w(\pi_2)$.

Thus, by product lemma, we have

$$\Phi_{C\times C}(x) = x^1 \Phi_C(x)\Phi_C(x)$$

By construction, f is a weight preserving bijection. Thus, we have

$$\Phi_{C\times C}(x) = \Phi_{C\backslash\{\emptyset\}}(x) = \Phi_C(x) - 1$$

Therefore, we get

$$\Phi_C(X) - 1 = x\Phi_C(x)^2$$

Now we can solve for $\Phi_C(x)$.

1.8 Partitions of an Integer

Definition 1.8.1. A ***partition*** of n with k parts is a $k-$tuple $\lambda := (\lambda_1, ..., \lambda_k)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k > 0$ and $\sum \lambda_i = n$.

Definition 1.8.2. We say n is the ***size*** of λ and we write $|\lambda| = n$ and $\lambda \vdash n$. We say k is the ***length*** of λ and write $\ell(k) = k$. We say $\lambda_1, ..., \lambda_k$ are all parts.

Example 1.8.3. For $m, n \in \mathbb{N}$, determine the number of partitions of n, in which all parts are less than or equal to n.

For example, if $n = 5$ and $m = 3$, we have $(3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1)$, and $(1, 1, 1, 1, 1)$ is the complete list.

Solution. Let P_m be the set of all partitions in which all parts are less than or equal to m. We define the weight function to be $w : P_m \to \mathbb{N}$ to be $w(\lambda) = |\lambda|$. Let

$$S_m = \mathbb{N} \times 2\mathbb{N} \times 3\mathbb{N} \times ... \times m\mathbb{N}$$

where $k\mathbb{N} = \{ki : i \in \mathbb{N}\}$.

We define a weight function $f : S_m \to \mathbb{N}$ to be $f(c_1, ..., c_m) = c_1 + ... + c_m$. We claim

$$\Phi_{P_m}(x) = \Phi_{S_m}(x)$$

The idea is combine all parts of the same size into a single number. We give a bijection $g : P_m \to S_m$ where

$$c_i = \sum_{j:\lambda_j = i} \lambda_j = i \times |\{j | \lambda_j = i\}|$$

For example, say we have $\lambda = (5, 5, 3, 3, 3, 3, 2, 2, 1)$, then we have $g(\lambda) = (1, 2 + 2, 3 + 3 + 3 + 3, 0, 5 + 5, 0, 0) = (1, 4, 12, 0, 10, 0, 0)$. Clearly this is a bijection and so we only need to show it is weight preserving. However, this is obviously weight preserving as we are only adding up the same numbers in different ways. One should try to write up a formal proof of this, but not me, not this moment :)

Therefore, it suffices to compute $\Phi_{S_m}(x)$.

Let $\alpha_i : i\mathbb{N} \to \mathbb{N}$ be the standard weight function, $a_i(n) = n$. Since $f(c_1, ..., c_n) = \alpha_1(c_1) + ... + \alpha_n(c_n)$, we can use the product lemma. Hence, we have

$$\Phi_{S_m}(x) = \prod_{i=1}^{m} \Phi_{i\mathbb{N}}(x) = \prod_{i=1}^{m} \frac{1}{1 - x^i}$$

Hence, the answer is $[x^n] \prod_{i=1}^{m} \frac{1}{1-x^i}$.

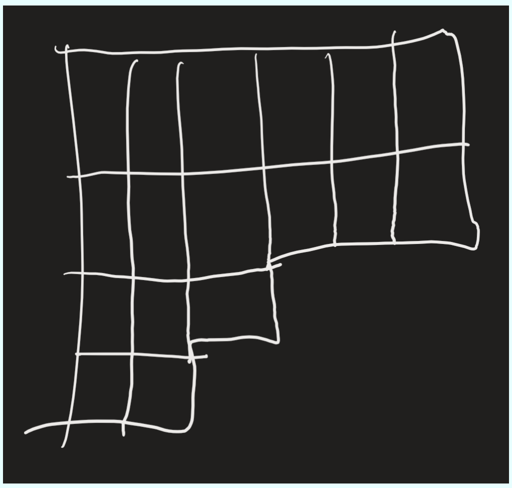
We will stop here and left as an exercise for the readers to simplify the answer. ♠

Example 1.8.4. For $n, m \in \mathbb{N}$, determine the number of partitions of n with length $\leq m$.

Solution. Let P'_m be the set of partitions with length less than or equal to m. Define weight function $w' : P'_m \to \mathbb{N}$ to be $w'(\lambda) = |\lambda|$.

We will show this is the same as Example 1.8.3. We see this from a picture.

We define the ***Ferres diagram*** of λ to be an array of boxes (left justified) with λ_i boxes in row i, for example, if $\lambda = (6, 6, 3, 2)$, then we have



In terms of Ferres diagrams, we have $g : P_m \to S_m$ in Example1.8.3 to be

$$g(\lambda_1, ..., \lambda_k) = (c_1, ..., c_m)$$

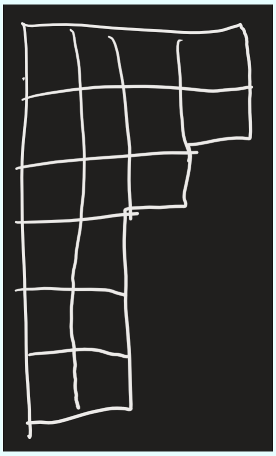
where c_i is the number of boxes in a row of length i.

This time, we use columns instead of rows, say $g' : P'_m \to S_m$ to be $g'(\lambda) = (c'_1, ..., c'_m)$ where c'_i is equal the number of boxes in a column of length i.

Hence, we know this problem has the same answer as Example 1.8.3.

Alternatively, there is a weight preserving bijection between P_m and P'_m. This is obtained by reflecting the Ferres diagram along a diagonal. The partition obtained from λ in this way is called the conjugate of λ, written as $\tilde{\lambda}$.

For example, recall $\lambda = (6, 6, 3, 2)$, then we have



which is equal to $(4, 4, 3, 2, 2, 2)$. ♠

Example 1.8.5. How many partitions of n with exactly k parts?

Solution. It is the number of partitions of n with $\leq k$ parts minus the number of partitions of n with $\leq k - 1$ parts. Hence, we have

$$[x^n]\Phi_{P_k}(x) - [x^n]\Phi_{P_{k-1}}(x)$$

♠

Example 1.8.6. We consider another way to solve the partition with exactly m parts.

Solution. To get a partition with exactly m parts, we start with any $\lambda \in P'_m$, we have at most m parts. Then, we add a column of length m to the Ferres diagram.

This gives a bijection between P'_m and and partitions with exactly m parts, i.e. $P_{\lambda(m)=m}$ which increases size by m. Thus, the generating function is

$$\Phi_{P_{\lambda(m)=m}}(x) = x^m \Phi_{P'_m}(x) = x^m \prod_{j=1}^{m} \frac{1}{1-x^j}$$

♠

Example 1.8.7. Determine the number of partitions of n.

Solution. Let $m \geq n$, then every partition of n has all parts less than or equal to m and length $\leq m$. Then, by previous example, we have the answer is

$$[x^n] \prod_{i=1}^{m} \frac{1}{1-x^i}$$

Since this is true for all $m \geq n$, we have the answer to be

$$[x^n] \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

♠

Definition 1.8.8. Let $A_1(x), A_2(x), ... \in \mathbb{R}[[x]]$, we say that $a_n = [x^n] \prod_{i=1}^{\infty} A_i(x)$ if there exists $N \in \mathbb{N}$, such that for all $m \geq N$, we have

$$a_n = [x^n] \prod_{i=1}^{m} A_i(x)$$

Remark 1.8.9. If there exists a series $A_i(x) \in \mathbb{R}[[x]]$ and $A(x) \in \mathbb{R}[[x]]$ such that for all n $[x^n]A(x) = [x^n] \prod_{i=1}^{\infty} A_i(x)$, then we say that $A(x) = \prod_{i=1}^{\infty} A_i(x)$.

Example 1.8.10. Compute $\prod_{i=1}^{\infty}(1 + x^{2^{i-1}})$

Solution. Consider

$$\prod_{i=1}^{m}(1 + x^{2^{i-1}}) = (1+x)(1+x^2)(1+x^4)(1+x^8)...(1+x^{2^{m-1}})$$

Then, we can show that

$$\prod_{i=1}^{m}(1 + x^{2^{i-1}}) = 1 + x + x^2 + ... + x^{2^m - 1}$$

In particular, when $N > \log_2(n)$, we have $[x^n] \prod_{i=1}^{m}(1 + x^{2^{i-1}}) = 1$ and so

$$[x^n] \prod_{i=1}^{\infty}(1 + x^{2^{i-1}}) = [x^n] \frac{1}{1-x}$$

Thus

$$\prod_{i=1}^{\infty}(1+x^{2^{i-1}}) = \frac{1}{1-x}$$

♠

Example 1.8.11. Show that the numer of partitions of n in which all parts are distinct is equal to the number of partitions of n in which all parts are odd.

For example, distinct parts partition is something like $\lambda = (\lambda_1, ..., \lambda_k), \lambda_1 > \lambda_2 > ... > \lambda_k$ and odd parts partition is like $(7, 7, 3, 3, 1)$.

Solution. We left this as an exercise to show that the distinct part generating function is $\prod_{i=1}^{\infty}(1+x^i)$. In addition, one should try to show that the odd part generating function is $\prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}}$

Then, we claim the two generating functions are equal.

Consider

$$\begin{aligned}
\prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}} &= \prod_{i=1}^{\infty} \frac{1}{1-x^{2i-1}} \cdot \prod_{i=1}^{\infty} \frac{1-x^{2i}}{1-x^{2i}} \\
&= \prod_{j=1}^{\infty} \frac{1}{1-x^{j}} \cdot \prod_{j=1}^{\infty}(1-x^{2j}) \\
&= \prod_{j=1}^{\infty} \frac{1-x^{2j}}{1-x^{j}} = \prod_{j=1}^{\infty}(1+x^{j})
\end{aligned}$$

We left this as an exercise to find a bijection between this two sets. ♠

1.9 Let's Talk About Strings

Definition 1.9.1 (Kidding)**.** To use strings, we just include $\#include\langle sstream\rangle$ in the header.

Remark 1.9.2. When we have more than one weight functions, what should we do?

Suppose S is a set of combinatorial objects and we have two weight functions, say w_1 and w_2. Then we can form the bivariate generating function

$$\Phi_S(x, y) = \sum_{\sigma \in S} x^{w_1(\sigma)} x^{w_2(\sigma)}$$

Then, we have $[x^m y^n]\Phi_S(x, y)$ answers the question "how many elements $\sigma \in S$ such that $w_1(\sigma) = m$ and $w_2(\sigma) = n$"

Remark 1.9.3. The sum and product lemmas both generalize to any number of weight functions.

Lemma 1.9.4 (Suum Lemma). *If $S = A \cup B$ and w_1, w_2 are weight functions, then*

$$\Phi_S(x, y) = \Phi_A(x, y) + \Phi_B(x, y) - \Phi_{A\cap B}(x, y)$$

Remark 1.9.5. Two u's in the above lemma to indicate we have two weight functions. Similarly, we would have ***suuuuuuuuum lemma*** to indicate we have 9 weight functions. (No, I'm just kidding)

Lemma 1.9.6 (Prooduct Lemma). *Suppose A, B and $A \times B$ are sets with weight functions and α_1, α_2 be weight functions for A, β_1, β_2 for B and w_1, w_2 for $A \times B$ and γ_1, γ_2 are constants. If*

$$\forall a \in A, b \in B, i \in \{1, 2\}, w_i(a, b) = \gamma_i + \alpha_i(a) + \beta_i(b)$$

then

$$\Phi_{A\times B}(x, y) = x^{\gamma_1} \cdot y^{\gamma_2} \Phi_A(x, y)\Phi_B(x, y)$$

Remark 1.9.7 (Kidding). The two o's is the same as above, and if we have 6 weight functions, then it is called ***proooooduct lemma***.

Example 1.9.8. [Composition Revisited] How many compositions of n with k parts?

Solution. Let S be the set of all compositions. Let $w_1 : S \to \mathbb{N}$ be $w_1(c_1, ..., c_k) = \sum c_i$ and $w_2 : S \to \mathbb{N}$ be $w_2(c_1, ..., c_k) = k$.

Hence, the answer is

$$[x^n y^k]\Phi_S(x, y)$$

Moreover, note $S = \bigcup_{k\geq 0}(\mathbb{N}_{\geq 1})^k$ and we need to put two weight functions on $\mathbb{N}_{\geq 1}$. Namely, let $\alpha_1 : \mathbb{N}_{\geq 1} \to \mathbb{N}$ be $\alpha(i) = i$ and we indeed have $w_1(c_1, ..., c_k) = \sum \alpha_1(c_i)$. Moreover, let $\alpha_2 : \mathbb{N}_{\geq 1} \to \mathbb{N}$ to be $\alpha_2(i) = 1$. Then we have $w_2(c_1, ..., c_k) = \sum \alpha_2(c_i)$.

To decide what α_i should be, we just need to make them so that the product lemma works.

Therefore,

$$\begin{aligned}
\Phi_S(x, y) &= \sum_{k\geq 0} \Phi_{(\mathbb{N}_{\geq 1})^k}(x, y) \\
&= \sum_{k\geq 0} (\Phi_{\mathbb{N}_{\geq 1}}(x, y))^k \\
&= \sum_{k\geq 0} (\frac{xy}{1 - x})^k \\
&= \frac{1}{1 - \frac{xy}{1-x}} = \frac{1 - x}{1 - x - xy}
\end{aligned}$$

Thus the number of compositions of n with k parts is

$$[x^n y^k]\frac{1-x}{1-x-xy}$$

♠

Remark 1.9.9 (A Word of Caution). Consider the example

$$F(x,y) = 1 + 3x + 5xy^2 - yx^2 = 1 + (3+5y^2)x - yx^2$$

What is $[x]F(x,y)$? It could be 3 or $3+5y^2$.

This is legitimately ambiguous. However, in this class, we always consider $[x]F(x,y)$ to be $(3+5y^2)$ and if we want 3, we write $[xy^0]F(x,y)$.

Remark 1.9.10. Note we have

1. $\Phi_S(1,1) = |S|$
2. $\Phi_S(x,1)$ is the generating function with respect to just w_1. Similarly we have $\Phi_S(1,y)$ is the same.
3. We have $[x^n]\Phi_S(x,y)$ is the generating function for $\{\sigma \in S : w_1(\sigma) = n\}$ with respect to w_2 and $[y^n]\Phi_S(x,y)$ means the similar thing.

Example 1.9.11. Determine the number of compositions of n.

Solution. Continue from the previous example 1.9.8, this is

$$[x^n]\Phi_S(x,1) = [x^n]\frac{1-x}{1-2x} = \begin{cases} 1, & \text{if } n = 0 \\ 2^{n-1}, & \text{otherwise} \end{cases}$$

♠

Definition 1.9.12. An ***alphabet*** is a set contains distinct elements.

Example 1.9.13. $\{0,1\}$ is an alphabet.

Definition 1.9.14. A ***string*** is a finite ordered list of symbols taking from an alphabet Σ. Moreover, we insist the existence of an ***empty string***, and is denoted by ϵ.

Example 1.9.15. Consider the binary strings, i.e. strings with alphabet from $\{0,1\}$. For instance, we have $a = 11010$ is a string and $b = 0001$ is also a string.

Definition 1.9.16. The ***length*** of a string is the length of a string.....

Example 1.9.17. The length of $a = 11010$ is 5 and $b = 0001$ is 4. We have ϵ has length 0.

Remark 1.9.18. We have a ***concatenation product*** on strings, if $a = 101$ and $b = 110$, then $ab = 101110$ and $ba = 110101$ and $a\epsilon = 101 = \epsilon a$.

This operation is associative, has identity elements, but no inverse, thus it is a ***monoid***. This monoid is not abelian and since we have no inverse elements, it is not a group[1].

[1] Recall semigroup is only with associative, monoid is semigroup with identity and group is monoid with inverse

Definition 1.9.19. If A, B are sets of strings, we define

$$AB = \{ab : a \in A, b \in B\}$$

We define

$$A^* = \{\epsilon\} \cup A \cup AA \cup AAA \cup ... = \bigcup_{i \geq 0} A^i$$

where $A^i = AA...A$, not the set product $A \times A \times ... \times A$.

Example 1.9.20. We have $\{0,1\}^*$ is the set of all binary strings.

Consider $A = \{0, 01\}$ and $B = \{1, 11\}$. Then, we have

$$A \times B = \{(0,1), (0,11), (01,1), (01,11)\}$$

and

$$AB = \{01, 011, 0111\}$$

Remark 1.9.21. Concatenation given a surjective map

$$A \times B \to AB$$

$$(a, b) \mapsto ab$$

If this map is a bijection, we say that AB is an unambiguous expression. The example above where $A = \{0, 01\}$ and $B = \{1, 11\}$ is ambiguous.

Definition 1.9.22. For unions, if $A \cap B$ is empty, we say that $A \cup B$ is an unambiguous expression.

More complicated expressions (e.g. A^*) is unambiguous if all constituents operations are.

Remark 1.9.23. For example, we have $\{0,1\}^*$ is unambiguous expression.

Example 1.9.24. Consider $a = \{0, 00\}$, $B = \{1, 11\}$ and $C = \{\epsilon, 0\}$.

1. We have ABC is unambiguous, ACB is ambiguous because AC is ambiguous.
2. We have $A \cup B$ is unambiguous and $A \cup B \cup C$ is ambiguous.
3. We have A^* is ambiguous because 00 can be in A as 00 or in AA as (0)(0).

Definition 1.9.25. A weight function is ***additive*** if $w(ab) = w(a) + w(b)$.

Example 1.9.26. The length of a string, the number of 1 in the string and the number of 0 in the string are three additive weight function.

Unless otherwise specified, the weight funciton on string will be length.

Theorem 1.9.27. *If $A, B \subseteq \{0,1\}^*$ and $w : \{0,1\}^* \to \mathbb{N}$ is an additive weight function. Then,*

1. *if $A \cup B$ is unambiguous, then $\Phi_{A \cup B}(x) = \Phi_A(x) + \Phi_B(x)$*
2. *If AB is unambiguous, then $\Phi_{AB}(x) = \Phi_A(x) \cdot \Phi_B(x)$*
3. *If A^* is unambiguous, then $\Phi_{A^*}(x) = (1 - \Phi_A(x))^{-1}$*

Proof. Number one is the sum lemma.

Number two. Since AB is unambiguous, we have a bijection $f\ A \times B \to AB$, define $w(a,b) = w(a) + w(b)$, since w is additive, this f is a weight preserving bijection. By product lemma, we are finished.

Number three. We have $\Phi_{A^*}(x) = \sum_{i\geq 0} \Phi_{A^i}(x) = \sum_{i \geq 0} (\Phi_A(x))^i = (1-\Phi_A(x))^{-1}$ ♡

Example 1.9.28. Consider $\Phi_{\{0,1\}^*}(x) = \frac{1}{1-\Phi_{\{0,1\}}(x)} = \frac{1}{1-2x}$. Note that

$$[x^n]\frac{1}{1-2x} = 2^n$$

Example 1.9.29. Consider

$$(\{1\}^*\{0\})^*\{1\}^*$$

This is saying, we want to write some 1's, then follow by one 0, then repeated this as many times as we want, then write some 1's following this. In particular, we have

$$(\{1\}^*\{0\})^*\{1\}^* = \{0,1\}^*$$

Note we have

$$\begin{aligned}
\Phi_{(\{1\}^*\{0\})^*\{1\}^*}(x) &= \Phi_{(\{1\}^*\{0\})^*}(x)\Phi_{\{1\}^*}(x) \\
&= (1-\Phi_{(\{1\}^*\{0\})}(x))^{-1}(1-\Phi_{\{1\}}(x))^{-1} \\
&= (1-\Phi_{\{1\}^*}(x)\Phi_{\{0\}}(x))^{-1}(1-\Phi_{\{1\}}(x))^{-1} \\
&= (1-(1-\Phi_{\{1\}}(x))^{-1}x)^{-1}(1-x)^{-1} \\
&= (1-(1-x)^{-1}x)^{-1}(1-x)^{-1} \\
&= \frac{1}{1-\frac{x}{1-x}}\frac{1}{1-x} = \frac{1}{1-2x}
\end{aligned}$$

Remark 1.9.30. Given any unambiguous expression for A, theorem gives a bunch of rules you can follow to compute $\Phi_A(x)$ if you follow those rules, and get the wrong answer for $\Phi_A(x)$, the expression you had is ambiguous. If you follow those rules and get the right answer for $\Phi_A(x)$, the expression you had is unambiguous.

Remark 1.9.31. All of the above works with multiple weight functions.

Example 1.9.32. Let A be the set of 01-strings that do not have 111 as a substring.

Solution. We have $A = \{0, 10, 110\}^*\{\epsilon, 1, 11\}$. This is an unambiguous expression for A and so

$$\Phi_A(x) = \frac{1}{1-(x+x^2+x^3)}(1+x+x^2)$$

as we note

$$\begin{aligned}
\Phi_A(x) &= \Phi_{\{0,10,110\}^*}(x)\Phi_{\{\epsilon,1,11\}}(x) \\
&= \frac{1}{1-\Phi_{\{0,10,110\}}(x)}\Phi_{\{\epsilon,1,11\}}(x)
\end{aligned}$$

♠

Theorem 1.9.33. *If $A_1, \dots, A_k$ are sets of strings, and $p(A_1, \dots, A_k)$ is an unambiguous expression and $B_1, \dots, B_k$ are subsets of $A_1, \dots, A_k$, then $p(B_1, \dots, B_k)$ is also unambiguous.*

Definition 1.9.34. An expression build from fintie sets of finite strings and finite many operations $\cup, \cdot, *$ is called a ***regular expression***.

Remark 1.9.35 (Decomposition of $\{0,1\}^*$)**.** The 0-decomposition. We have

$$\{0,1\}^* = (\{1\}^*\{0\})^*\{1\}^* = \{1\}^*(\{0\}\{1\}^*)^*$$

The 1-decomposition. We have

$$\{0,1\}^* = (\{0\}^*\{1\})^*\{0\}^* = \{0\}^*(\{1\}\{0\}^*)^*$$

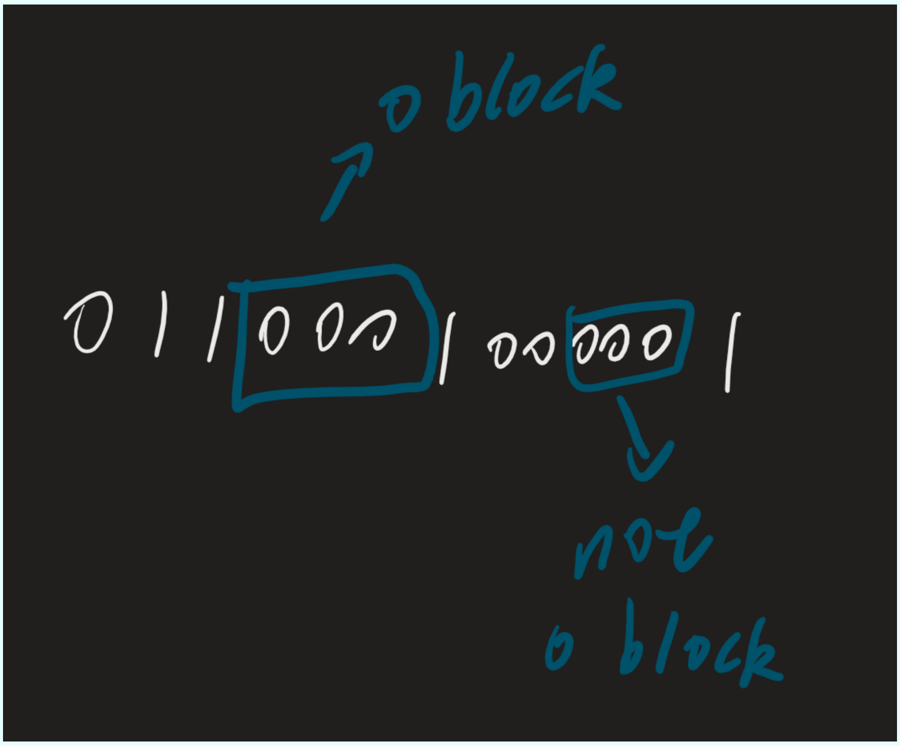
The block decomposition. We have

$$\begin{aligned}
\{0,1\}^* &= \{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^* \\
&= (\{\epsilon\} \cup \{0\}\{0\}^*)(\{1\}\{1\}^*\{0\}\{0\}^*)^*(\{\epsilon\} \cup \{1\}\{1\}^*) \\
&= \{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*
\end{aligned}$$

Definition 1.9.36. A 0-***block*** is a maximal non-empty substring of 0s.

Remark 1.9.37. We remark that maximal means cannot be extended, and maximum means theres nothing longer than this.

Example 1.9.38. We have



Example 1.9.39. Determine the number of binary strings in which all blocks have odd length.

Solution. Let S be the set of such strings, then we have

$$S = (\{\epsilon\} \cup \{1\}\{11\}^*)(\{0\}\{00\}^*\{1\}\{11\}^*)^*(\{\epsilon\} \cup \{0\}\{00\}^*)$$

The proof follows by etc. ♠

Example 1.9.40. Determine the number of 01-strings of length n such that every block of 0s is followed by a longer block of 1s.

Solution. Let S be the set of such strings. Then, we have

$$S = \{1\}^*(a\{1\}\{1\}^*)^*$$

where $a = \{01, 0011, 000111, 00001111, ...\} = \bigcup_{n\geq 1}\{0\}^n\{1\}^n$.

The proof follows and we remark the above expression is not a regular expression. In particular, we have

$$\Phi_S(x) = \frac{x^2}{1-x^2}$$

♠

Remark 1.9.41 (An Infinite Product Lemma)**.** Let $A_1, A_2, A_3, ...$ be sets of strings and assume $\epsilon \in A_i$ for all i. Define the ***infinite concatenation product***

$$A_1A_2A_3... := \bigcup_{m=1}^{\infty} A_1A_2...A_m$$

We say that this is ***unambiguous*** if $A_1A_2...A_m$ is unambiguous for each m.

We note this is different from before as here the union is never disjoint and in fact we have $A_1 \subseteq A_1A_2 \subseteq A_1A_2A_3...$

Thus, we should think $\bigcup_{m=1}^{\infty}$ as $\lim_{m\to\infty}$.

In particular, if $A = A_1A_2A_3A_4...$ is unambiguous, then we have

$$\Phi_A(x) = \prod_{i=1}^{\infty}\Phi_{A_i}(x)$$

We will provide a proof of this claim in the following.

Proof. Let $n \in \mathbb{N}$. Since there are only finite many elements of weight n in S.Then, they all must be in $A_1A_2...A_N$ for some N.

Thus, we have, $\forall m \geq N$, that

$$[x^n]\Phi_{A_1 \ldots A_m}(x) = [x^n]\prod_A(x)$$

where $[x^n]\Phi_{A_1 \ldots A_m}(x) = [x^n]\prod_{i=1}^m \Phi_{A_i}(x)$. By the definition of infinite products of formal power series, the result follows. ♡

1.10 Forbidden Substring Techniques

Example 1.10.1. We can consider the partition as an infinite concatenation product $P = \{1\}^*\{2\}^*\{3\}^*...$ in the alphabet $\{1, 2, 3, ...\}$.

Define the weight function w to be the sum of the numbers in the string. This is additive weight function, and P is an unambiguous expression. Therefore, we have

$$\Phi_P(x) = \prod_{i=1}^{\infty} \Phi_i^*(x) = \prod_{i=1}^{\infty}(1 - x^i)^{-1}$$

Example 1.10.2. Let S be the set of 01-strings that do not have 011 as a substring.

Determine the generating function $\Phi_S(x, y)$ where x means the number of 0 and y is the number of 1 in the string.

Solution. Find (using the power of your brain) an unambiguous regular expression. Then, we have $S = \{1\}^*\{0, 01\}^*$. Thus, we get

$$\Phi_S(x, y) = \frac{1}{1 - y} \cdot \frac{1}{1 - (x + xy)}$$

We provide method II. Notice that the set of all binary strings can be decomposed as

$$\{0, 1\}^* = S(\{011\}\epsilon)^*$$

Thus, we have

$$\Phi_{\{0,1\}^*}(x, y) = \Phi_{S(\{011\}\epsilon)^*}(x, y)$$

$$\frac{1}{1 - x - y} = \Phi_S(x, y) \cdot \frac{1}{1 - \Phi_{\{011\}}(x, y)\Phi_S(x, y)} = \Phi_S(x, y)\frac{1}{1 - xy^2\Phi_S(x, y)}$$

Then, solve for $\Phi_S(x, y)$ and the problem is solved. ♠

Remark 1.10.3. This works because 011 is ***non-overlapping***. For example, we have 010 is overlapping. I can write down strings like 01010, and which has two different decompositions (010)(01) and (01)(010). Hence this technique fails if 010 is the forbidden substring.

Example 1.10.4. We will provide a method III for the example 1.10.2 above.

Solution. We define an operation called substitution. Let Q be an alphabet, let $q \in Q$, let $R \subseteq Q^*$ be a set of strings.

Given $a \in Q^*$, we can write $a = a_0 q a_1 q a_2 q a_3 ... q a_m$ where $a_i \in (Q \backslash \{q\})^*$. Define $a[q \to R] = \{a_0\}R\{a_1\}R...R\{a_m\}$.

Consider the example $Q = \{0, 1\}$, $q = 0$ and $R = \{000, 010\}$ and $a = 0110$.

Then, we have $a[0 \to R] = \{00011000, 00011010, 01011000, 01011010\}$.

For $A \subseteq Q^*$, define $A[q \to R] = \bigcup_{a \in A} a[q \to R]$. We remark that the substitution is unambiguous if the concatenations and unions are. We need a theorem to proceed.

♠

Theorem 1.10.5. *Suppose $Q = \{0, 1, ..., k\}$. Let weight function $w_i : Q^* \to \mathbb{N}$ be the number of i's. Consider the generating series $\Phi_A(x_0, ..., x_k)$ and $\Phi_R(x_0, ..., x_k)$ and $\Phi_{A[k \to R]}(x_0, ..., x_k)$ with respect to $w_0, ..., w_k$. If $A[k \to R]$ is unambiguous, then*

$$\Phi_{A[k \to R]}(x_0, ..., x_k) = \Phi A(x_0, x_1, ..., \Phi_R(x_0, ..., x_k))$$

Proof. We have

$$\begin{aligned} \Phi A(x_0, x_1, ..., \Phi_R(x_0, ..., x_k)) &= \sum_{a \in A} x_0^{w_0(a)} x_1^{w_1(a)} ... x_{k-1}^{w_{k-1}(a)} \phi_R(x)^{w_k(a)} \\ &= \sum_{a \in A} \Phi_{a[k \in R]}(x_0, ..., x_k) = \Phi_{A[k \to R]}(x_0, ..., x_k) \end{aligned}$$

♡

Example 1.10.6. And we are bloody back to the example 1.10.2 AGAIN!

Solution. Consider the set A of 012-string that do not have 011 as a substring.

Notice $A[2 \to \{2, 011\}] = \{0, 1, 2\}^*$.

Then, we have

$$\Phi_{\{0,1,2\}^*}(x, y, z) = \Phi_A(x, y, \Phi_{\{2,011\}}(x, y, z))$$

Thus, we get

$$\frac{1}{1 - x - y - z} = \Phi_A(x, y, z + xy^2)$$

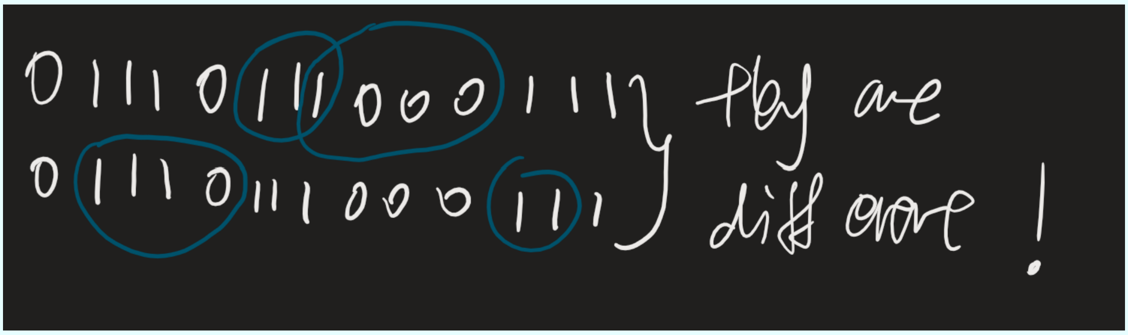
Note that (recall from above example) that we want $\Phi_S(x, y)$, which we have

$$\Phi_S(x, y) = \Phi_A(x, y, 0) = \frac{1}{1 - (x + y - xy^2)}$$

as $z + xy^2 = 0$ then $z = -xy^2$ and so $\frac{1}{1-(x+y+z)} = \frac{1}{1-(x+y-xy^2)}$ ♠

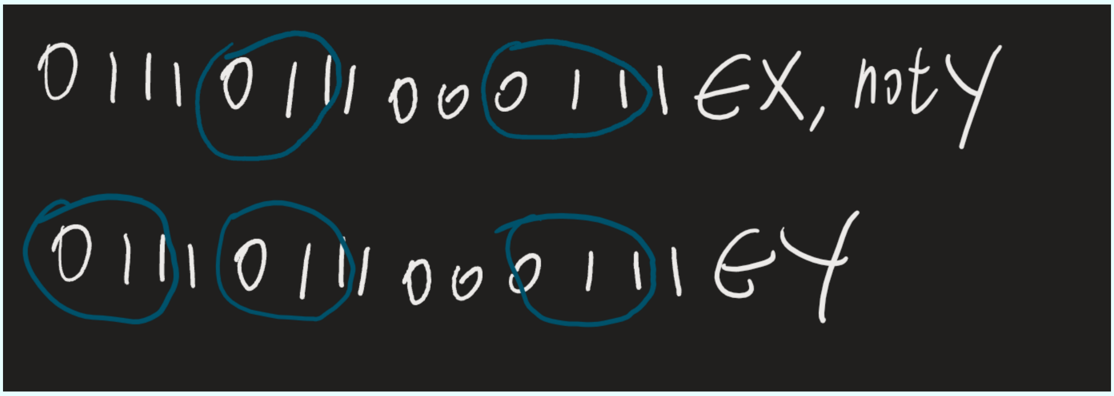
Example 1.10.7. We even have a Method 4 for the example 1.10.2. This is called ***marking technique***.

A marked 01-strings is a 01-string in which certain substring may be circled. For example,



Then, we consider the set X of 01-strings in which some occurrences of 011 may be marked. Let $Y \subseteq X$ be the subset in which every occurrences of 011 is marked.

For example,



The relation between this method and method 3 is that we can map circled 011 to 2, then we are back to method 3.

note: $X = \{0, 1, 011\}^*$

not circle

hence: $X = Y[\bigcirc \to \{\bigcirc, \ \}]$

(keep, or get rid of every circle!)

This GF with respect to # of 0s, # of 1s, and # of circles.

So,

$$\Phi_X(x,y,z) = \frac{1}{1 - \Phi_{\{0,1,011\}}(x,y,z)} = \frac{1}{1-(x+y+xy^2z)}$$

By sub thm,

$$\Phi_X(x,y,z) = \Phi_Y(x,y,\Phi_{\{\bigcirc, \ \}}(x,y,z)) = \Phi_Y(x,y,z+1)$$

So,

$$\Phi_Y(x,y,z+1) = \frac{1}{1-(x+y+xy^2z)}$$

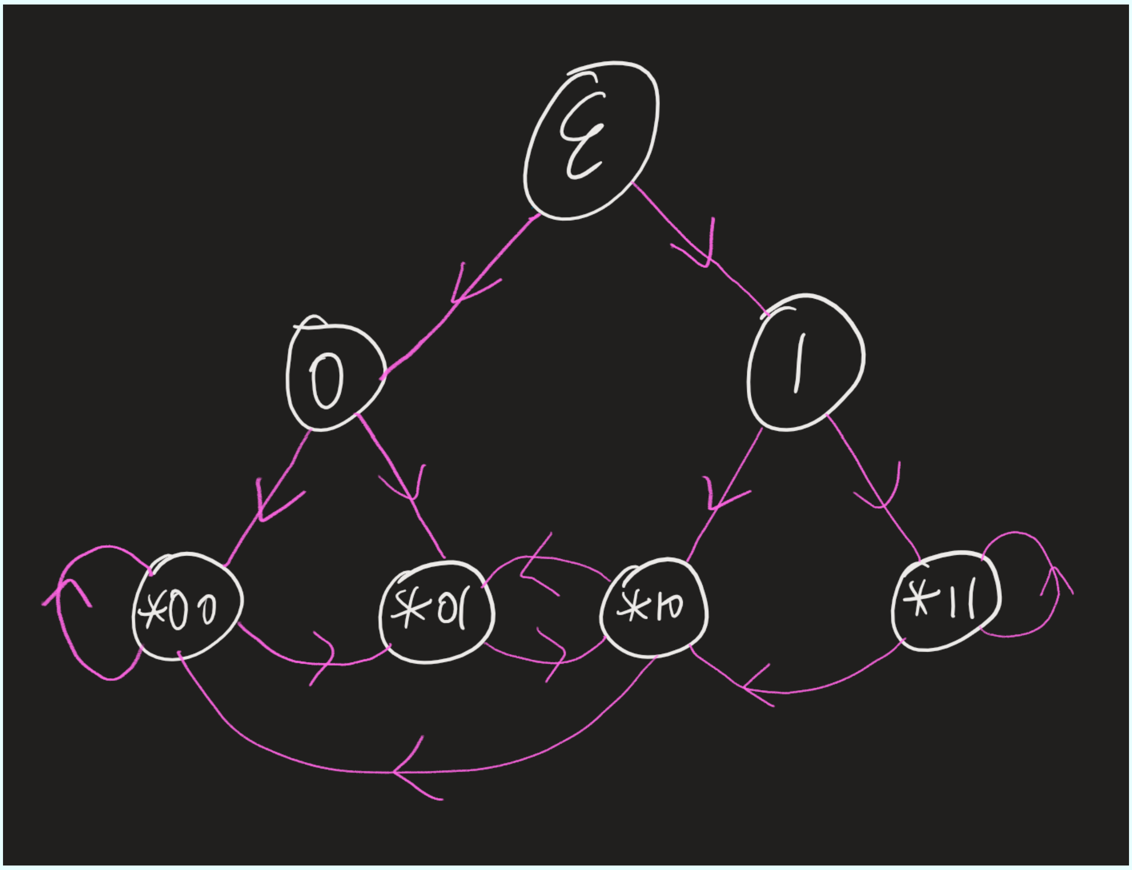
Note we want strings in Y with no circles, so the generating function is

$$\Phi_Y(x,y,0) = \frac{1}{1-(x+y-xy^2)}$$

1.11 METHOD FIVE!!!!

Example 1.11.1. We give a method five for example 1.10.2......

Solution. Consider the following graph.



In the above graph, the arrows means by adding one character to a string in the circle, then we get another one. In addition, we say $*xy$ to mean a string end with xy.

Therefore, we have the number of strings that do not have 011 as a substring is the same as ways of following arrows starting from ϵ.

In the next chapter of the course, one of the things we do is to study such walks in graphs. ♠

Chapter 2

Graph Theory

2.1 Intro

Definition 2.1.1. A ***graph*** (or a ***simple graph***)

$$G := \{V(G), E(G)\}$$

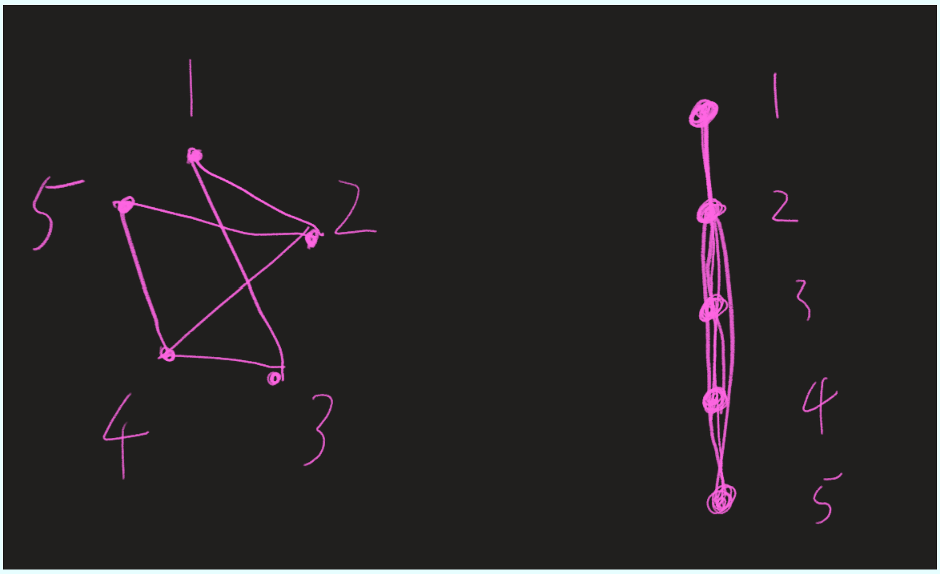
where $V(G)$ is just a set of elements and $E(G)$ consists of unordered pairs of distinct elements of $V(G)$.

The elements of $V(G)$ are called ***vertices*** (note the singular is vertex) and the elements of $E(G)$ are called ***edges***.

Example 2.1.2. Consider $V(G) = \{1, 2, 3, 4, 5\}$ and

$$E(G) = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}, \{2, 5\}\}$$

Then, we consider the drawings of G.



Remark 2.1.3. Variants of graphs (what we are ont primarily studying in this course):

1. Infintie graphs: $V(G)$ is an infinite set
2. Directed graphs: Edges are ordered pairs of vertices
3. Graphs with loops: two vertices in an edge can be the same
4. Graphs with multiple edges: $E(G)$ is an arbitrary set and we have a function on $E(G) \to \{$ unordered pairs of distinct vertices $\}$

Definition 2.1.4. If $e = \{x, y\}$ is an edge of G, we say the following things

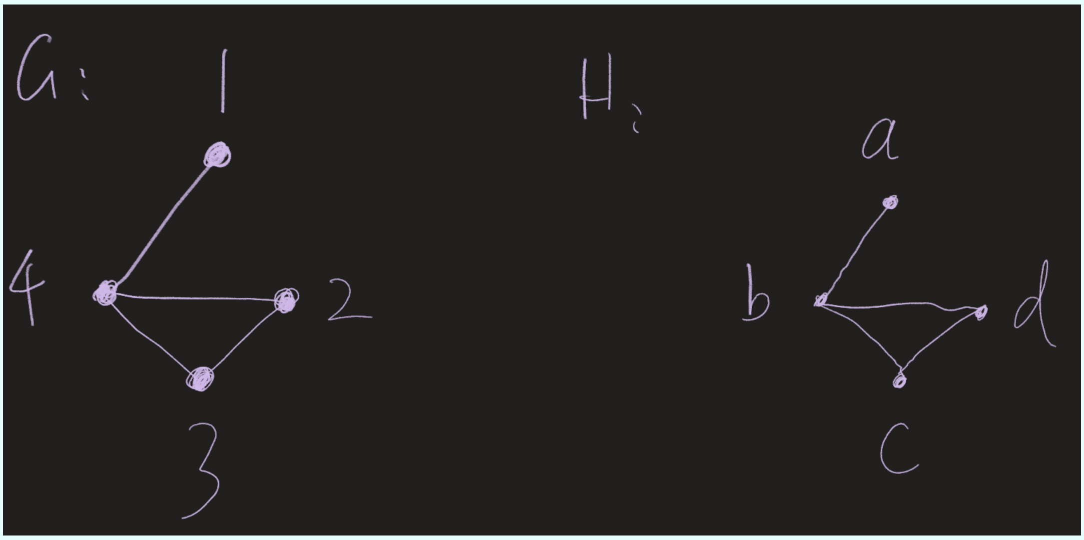
1. x is ***adjacent*** to y.
2. x and y are ***neighbours***.
3. x is ***incident*** with e or e is ***incident*** with x.
4. e ***joins*** x and y.

Definition 2.1.5 (Abuse of Notation)**.** We will write $e = xy$ instead of $e = \{x, y\}$.

Definition 2.1.6. If $v \in V(G)$, then $N(v)$ is the set of neighbours of v. The ***degree*** of v is $deg(v) = |N(v)|$.

Remark 2.1.7. We note $deg(v)$ is also the number of edges incident with v.

Example 2.1.8. Here are two graphs.



They are not the same graph as we have different vertices. We say they are ***isomorphic***.

Definition 2.1.9. Let G, H be two graphs, a bijection $f : V(G) \to V(H)$ is an ***isomorphism*** if f preserves adjacencies, i.e.

$$\{u, v\} \in E(G) \iff \{f(u), f(v)\} \in E(H)$$

We say that G and H are ***isomorphic*** if there exists an isomorphism between them.

Remark 2.1.10. Isomorphic graphs share all the same structure properties.

Example 2.1.11. An example of a structural property is the degree sequence: if we list the degree of vertices in increasing order, we have $1, 2, 2, 3$ for G in the example 2.1.8 above.

If we are only interested in structural properties, i.e. properties of the isomorphism type, we sometimes omit vertex label from the diagram.

Theorem 2.1.12 (Handshake Theorem)**.** *We have*

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|$$

Proof. Clear.

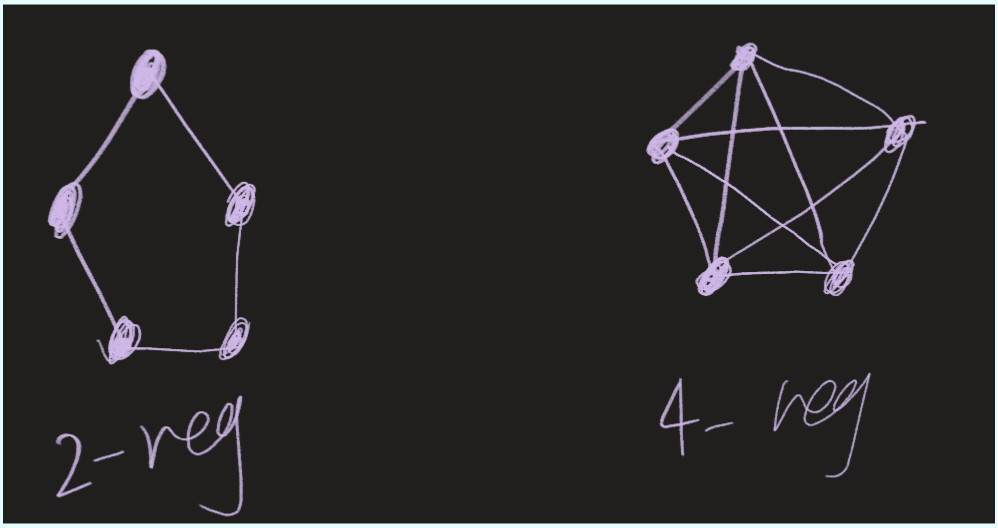
Consider the set $P = \{(v, e) : v \in V(G), e \in E(G), v \sim e\}$ where $v \sim e$ means v incident with e.

For each vertex $v \in V(G)$, there are $deg(v)$ edges e such that $(v, e) \in P$. Thus $|P| = \sum_{v \in V(G)} deg(v)$.

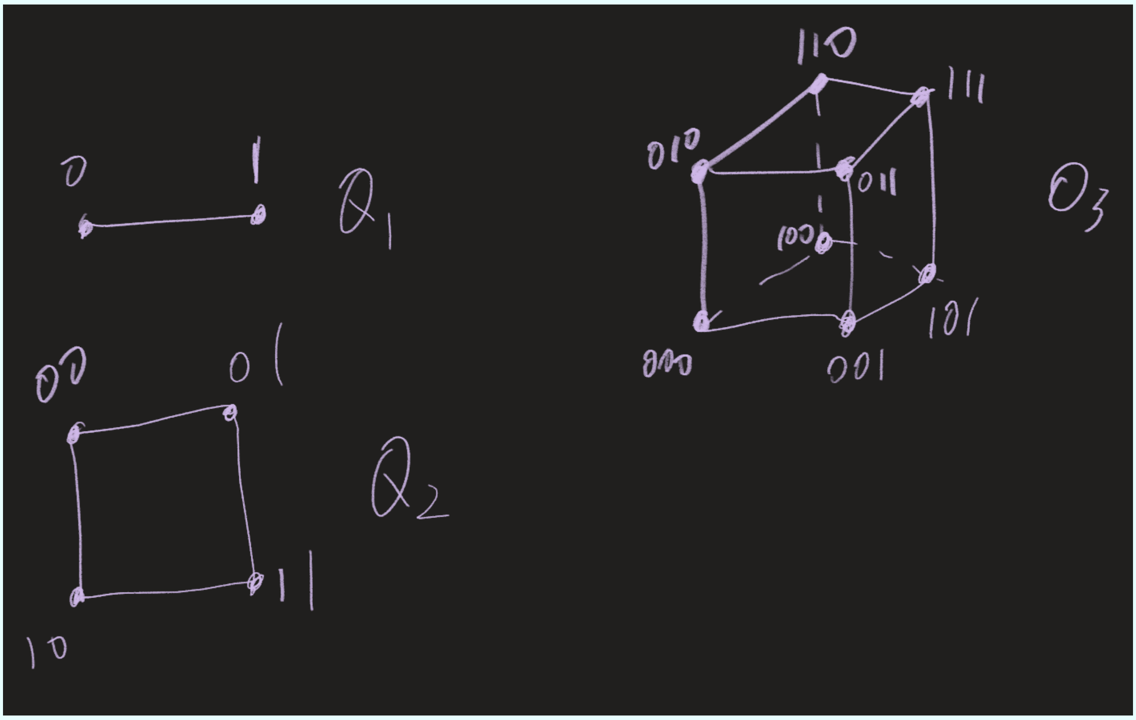
FOr each edge e, there are 2 vertices v such that $(v, e) \in P$. Thus $|P| = 2|E(G)|$. ♡

Definition 2.1.13. A graph in which all vertices has degree k is called k***-regular***.

Example 2.1.14. Consider



Example 2.1.15. Let Q_n be the graph with $V(Q_n) = \{01 - \text{string of length n}\} = \{0, 1\}^n$. Two vertices σ, σ' are adjacent in Q_n if they differ in exactly one position.



We have Q_n is called the ***n-cube***.

The question is, how many vertices/edges in Q_n? We have $|V(Q_n)| = 2^n$ and we note

$$2|E(Q_n)| = \sum deg(v) = n|V(Q_n)| = n2^n$$

Thus $|E(Q_n)| = n2^{n-1}$. We used that Q_n is n-regular.

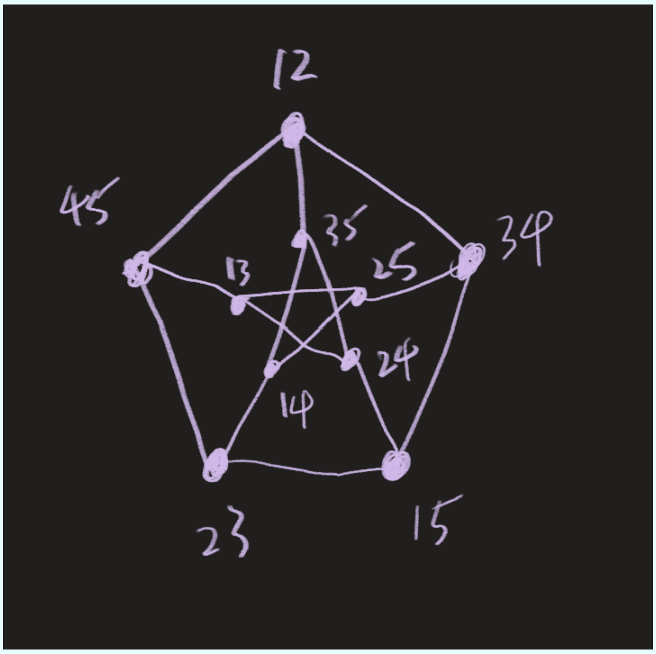
Example 2.1.16. Is there a 3-regular graph with 9 vertices?

No, because $\sum deg(v)$ would be 27, which is odd, so it cannot be $2|E(G)|$.

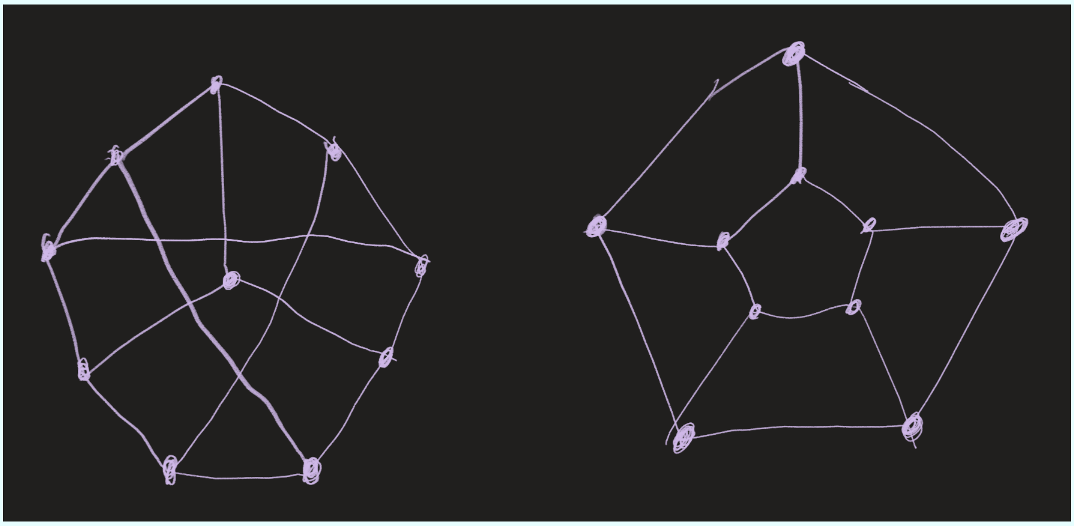
Example 2.1.17. Fix n, m, k, consider the graph wohse vertices are m-element subsets of $[n]$ and two vertices A, B are adjacent iff $|A \cap B| = k$.

We should try to compute the number of vertices and number of edges and draw the case when $n = 5, m = 2, k = 0$.

The answer is below.



Then, try to determine if the following two graphs is isomorphic to each other?

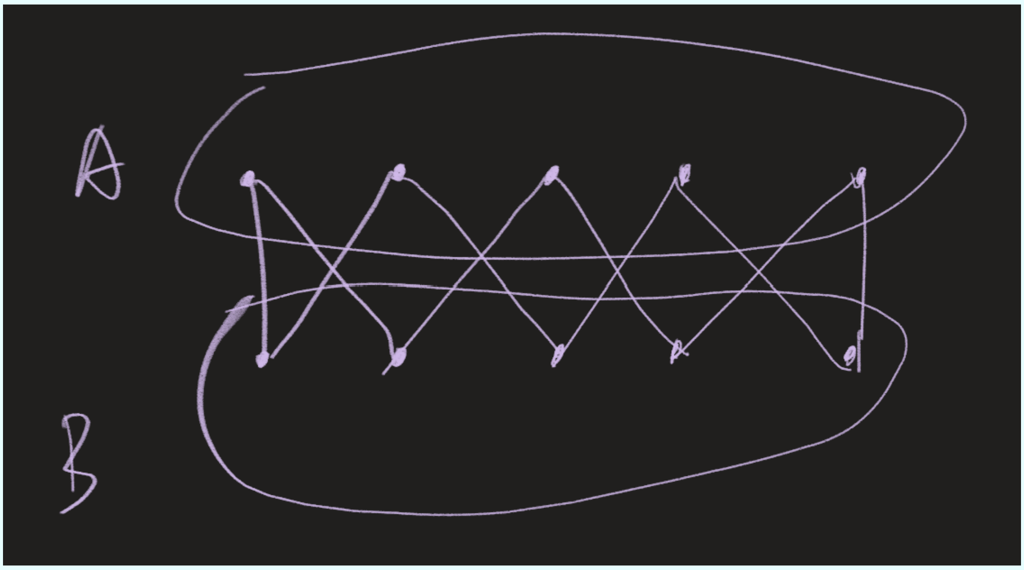


Remark 2.1.18. To prove two graphs are isomorphic, exhibit an isomorphism. To prove they are not isomorphic, find some structural property that distinguishes them.

Definition 2.1.19. A ***complete graph*** K_n has n vertices and all pairs of vertices are adjacent.

Definition 2.1.20. A graph is ***bipartite*** if there is a partition (A, B) of $V(G)$ such that every edges joins a vertax in A to a vertex in B. The pair (A, B) is called ***bipartition***.

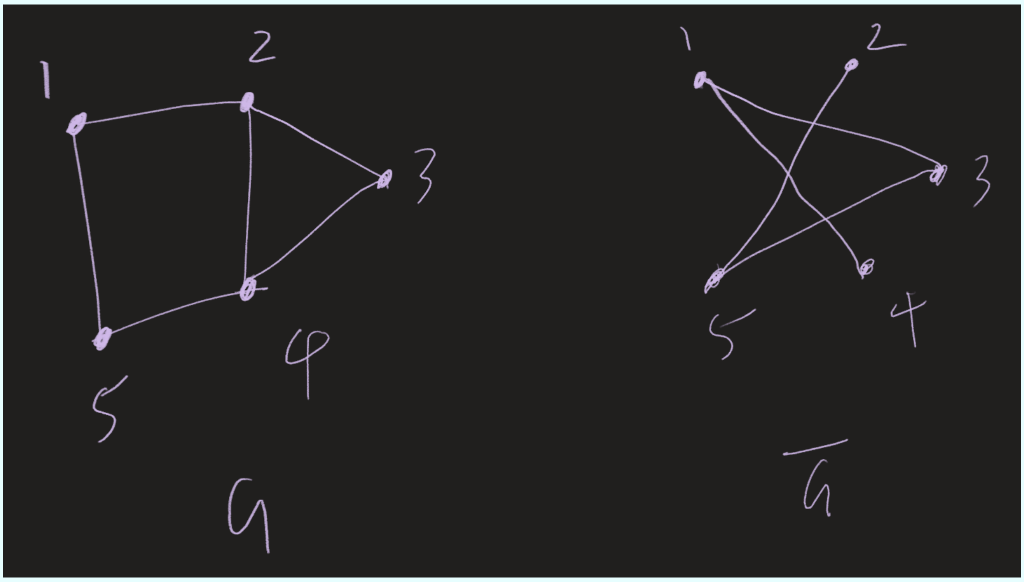
Example 2.1.21. Below is a bipartite graph.



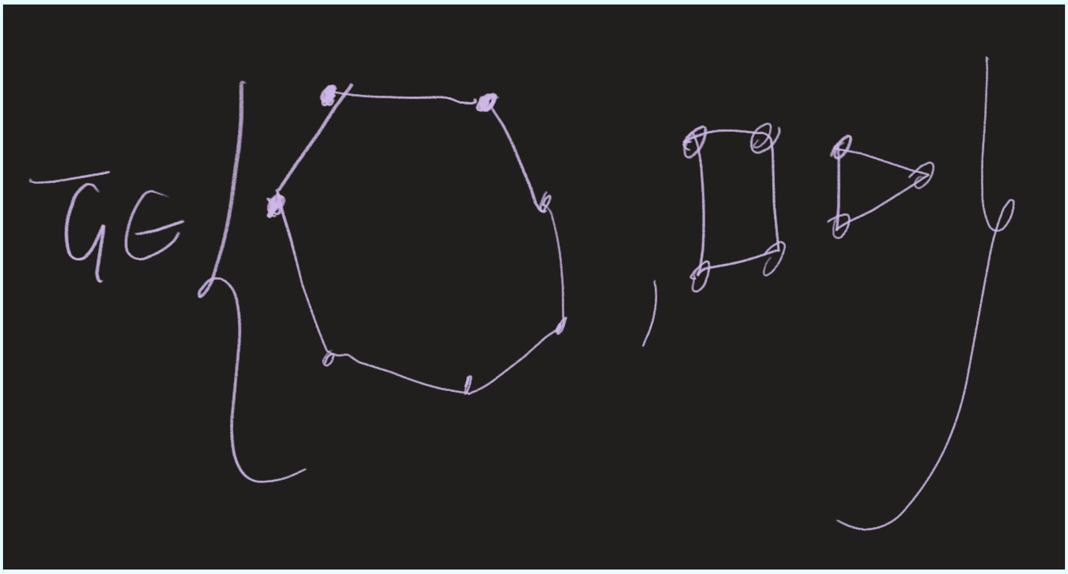
Remark 2.1.22. Note a bipartite graph is basically the same information as a relation between A and B.

Definition 2.1.23. If G is a graph, the complement of G is the graph $\overline{G}$ with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : u, v \in V(G), uv \notin E(G), u \neq v\}$

Example 2.1.24. Below is an example of a complement



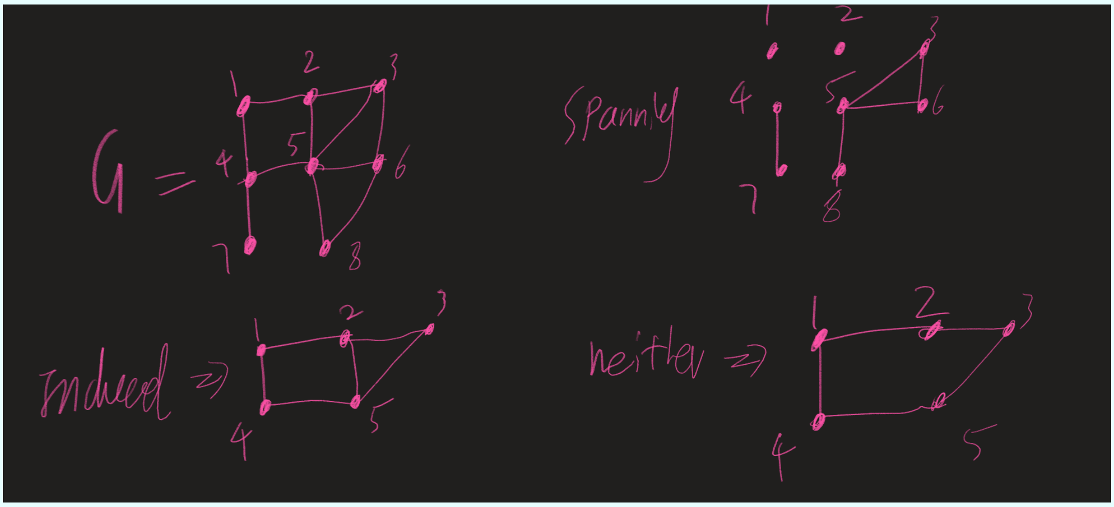
Next, try to draw all isomorphism types of 4-regular graphs with 7 vertiecs. Note the complement of such a graph must be 2-regular, and so we only have two possibilities.



Definition 2.1.25. A ***subgraph*** H of G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Let H be a subgraph of G, H is ***spanning*** if $V(G) = V(H)$ and H is ***induced*** if $E(H) = \{xy \in E(G) : v \in V(H) \land y \in V(H)\}$

Example 2.1.26. Below is an example.



2.2 Let's take a walk

Definition 2.2.1. A ***walk*** in a graph G is a sequence $v_0e_1v_1e_2v_2...e_nv_n$ in which $v_0, v_1, ..., v_n \in V(G)$ and e_i joins v_{i-1} and v_i.

In a simple graph, the e_i's are redundent info, so often we also write $v_0v_1...v_n$.

Remark 2.2.2. The walk $v_0e_1v_1...e_nv_n$ is said ***from*** v_0 to ***to*** v_n. We remark it is possible to have a walk of length 0, which has a single vertex.

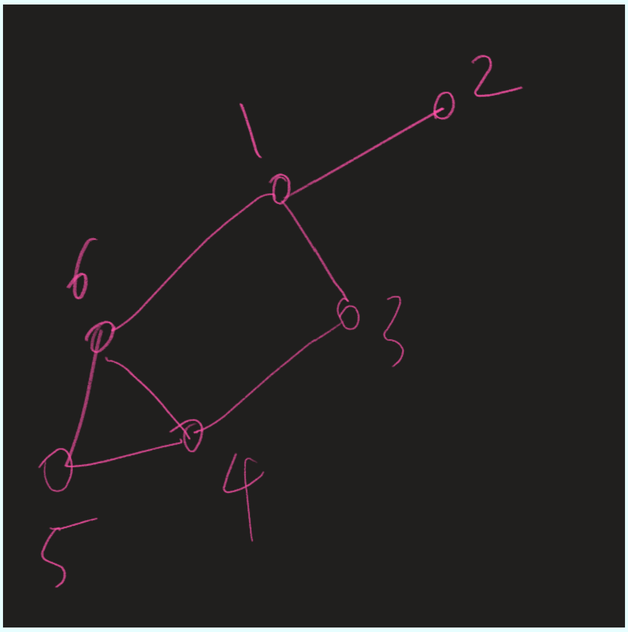
Definition 2.2.3. The ***reverse*** of the walk is $v_n e_n v_{n-1} e_{n-1} ... v_0$ from v_n to v_0, if our original walk is $v_0 e_1 v_1 ... v_n$.

Definition 2.2.4. A ***path*** is a walk with no vertices repeated.

A ***cycle*** (or ***cycle walk***) is a walk from a vertex to itself with no edges repeated and no vertices repeated (expect the first and last). In addition, we insist on at least one edges.

Remark 2.2.5. We also use the terms ***path*** and ***cycle*** to refer to the subgrpah defined by vertices and edges they are in.

Example 2.2.6. Consider the following graph



We have 12134645643 is a walk, 6543 is a path, and 134561 is a cycle.

Lemma 2.2.7. *If there is a walk from u to v, then any shortest walk from u to v is a path.*

Proof. Suppose $u = v_0 v_1 v_2 ... v_n = v$ is a shortest walk from u to v. Suppose this is not a path, so it must have a repeated vertex: say $v_i = v_j$ with $i < j$. But now $v_0 v_1 v_2 ... v_i v_{j+1} v_{j+2} ... v_n$ is a walk from u to v of length $n - (j - i) < n$. This contradicts the assumption that a is a shortest walk. ♡

Definition 2.2.8. Given a graph G, let P_G be the relation on $V(G)$ defined by $uP_G v$ if and only if there exists a path from u to v.

Theorem 2.2.9. *P_G is an equivalence relation.*

Proof. Reflexive: For $v \in V(G)$, v is a path (of length 0) from v to v.

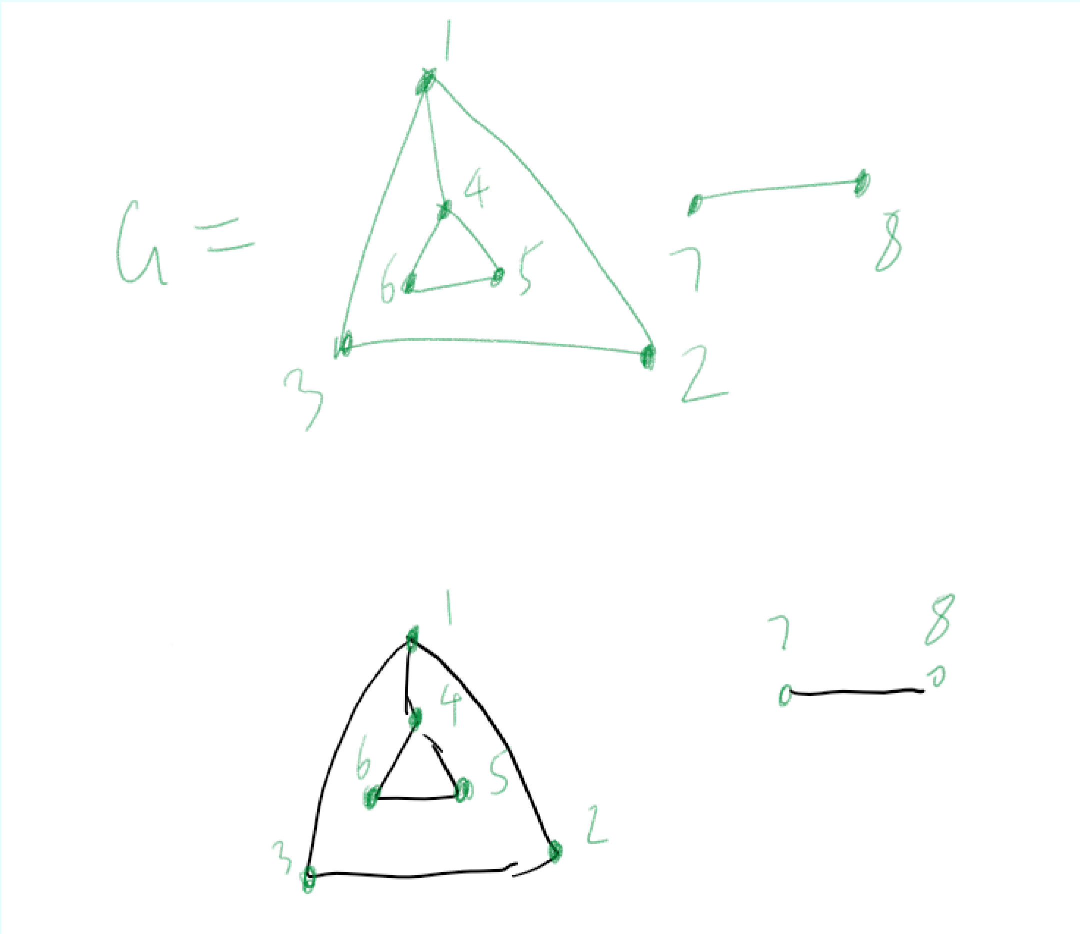
Symmetric: If $uP_G v$ then there is a path from u to v. The reverse is a path from v to u and so $vP_G u$.

Transitive: If uP_Gv and vP_Gw. Let $u = v_0...v_n = v$ be a path from u to v, and $v = w_0...w_m = w$ be a path from v to w. Then $v_0...v_nw_1...w_m$ is a walk from u to w, and by the above lemma, there exists a shortest path. Hence, uP_Gv. ♡

Definition 2.2.10. G is ***connected*** if P_G has exactly 1 equivalence class.

Definition 2.2.11. A ***component*** of G is a subgraph induced by an equivalence class of P_G.

Example 2.2.12. Below is a graph with two components.



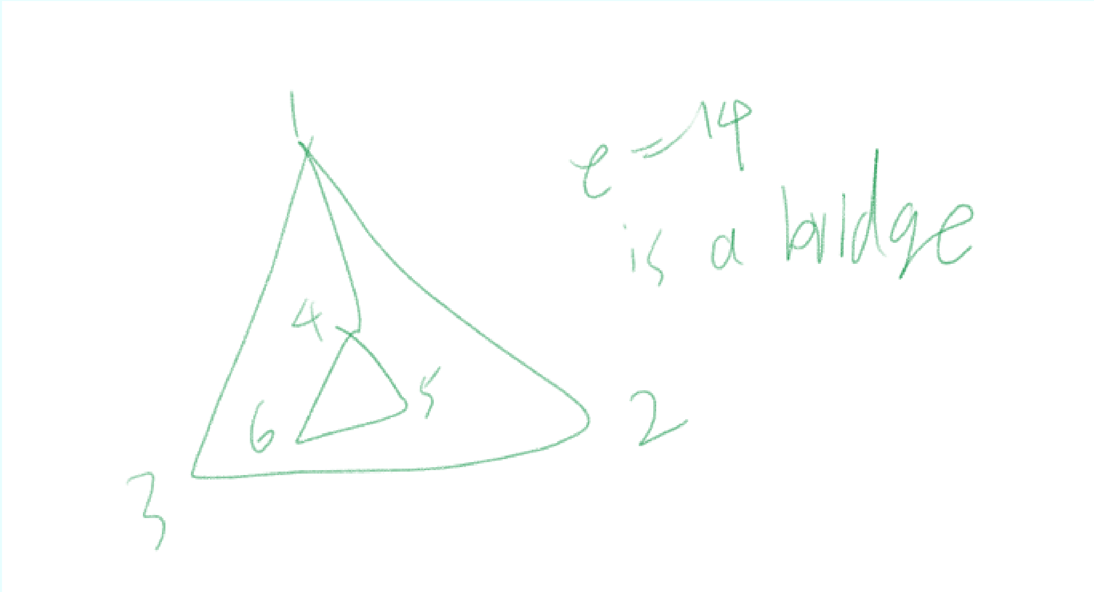
Remark 2.2.13. Let G be the graph, $V(G) = \emptyset$ and $E(G) = \emptyset$. This is called the empty graph. In this class, we will say that the empty graph is not connected.

2.3 Bridge

Definition 2.3.1. If e is an edge of a graph G, we define $G - e$ to be the spanning subgraph $E(G - e) = E(G)\backslash\{e\}$.

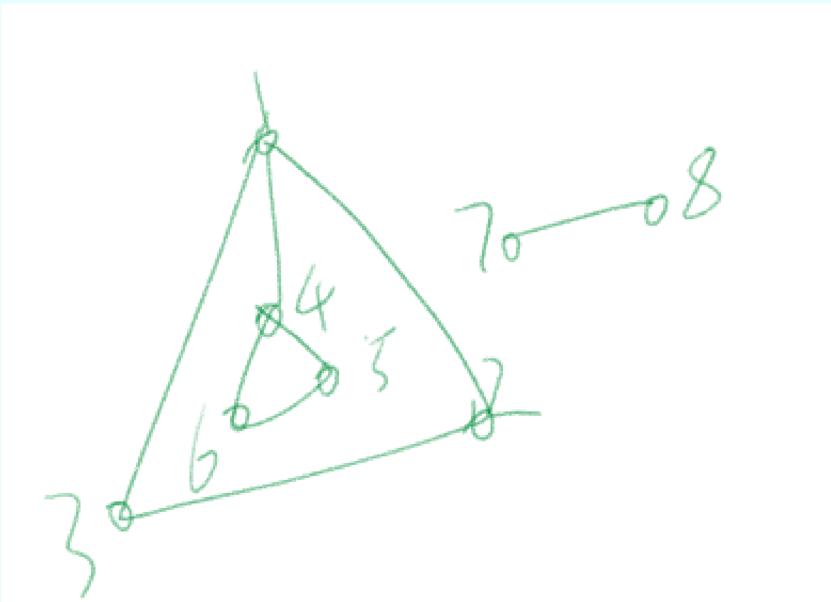
If G is a connected graph, we say that e is a ***bridge*** if $G - e$ is not connected.

Example 2.3.2. Below we have $e = 14$ is a bridge.



Remark 2.3.3. If G is not connected, $e \in E(G)$, then e is a bridge if and only if it is a bridge in its component.

Example 2.3.4. Below, we have two bridegs, $e_1 = 14$ and $e_2 = 78$.



Remark 2.3.5. An alternate definition of "component" is a maximal (non-empty) connected subgraph.

Theorem 2.3.6. *If G is connected and $e = xy$ is a bridge then $G - e$ has exactly 2 components, one containing x, the other containing y.*

Proof. Let $v \in V(G)$. Since G is connected. Let $v...x$ be a path from v to x in G. If e does not appear in this path then it is a path from v to x in $G - e$ so $vP_{G-e}x$.

Otherwise, the path must be of the form $v...yex$ and so $v...y$ is a path from v to y in $G-e$, so $vP_{G-e}y$.

This shows that every vertex is either in the component of x in $G-e$ or the component of y. Thus $G-e$ has at most 2 components. But by the definition of a brideg, $G-e$ is not connected, so it has at least 2 components and the result follows. ♡

Example 2.3.7. Prove that a 4-regular graph has no bridges.

Solution. Suppose to the contrary that G is a 4-regular graph and $e = xy$ is a bridge. Consider the component G_x of x in $G-e$. Every vertex in G_x has degree 4 except x, which has degree 3 (note $y \notin V(G_x)$). Therefore, $\sum_{v \in V(G_x)} deg(v)$ is odd, which is a contradiction to the handshake theorem.

More generally, this proof can be used on any graph with all vertices have even degree. ♠

Theorem 2.3.8. *Let G be a graph, $e \in E(G)$. Then e is a bridge if and only if e is not in any cycle.*

Proof. See the official note or exercise. ♡

Theorem 2.3.9. *Let G be a graph, the following are equivalent:*

1. *There exists vertices u, v such that there are two different paths from u to v.*
2. *There is a cycle.*

Proof. $2 \to 1$ is easy. We show $1 \to 2$.

Let $P_1 = [u = u_0 u_1 u_2 ... u_m = v]$ and $P_2 = [u = v_0 v_1 v_2 ... v_n = v]$ be two different paths from u to v. If $P_1 \neq P_2$, then there is an edge that appears in one path but not the other. With out lose of generality, suppose $e = u_{i-1}u_i$ is such an edge of P_1.

Then

$$u_i u_{i+1} ... u_m v_{n-1} ... v_0 u_1 ... u_{i-1}$$

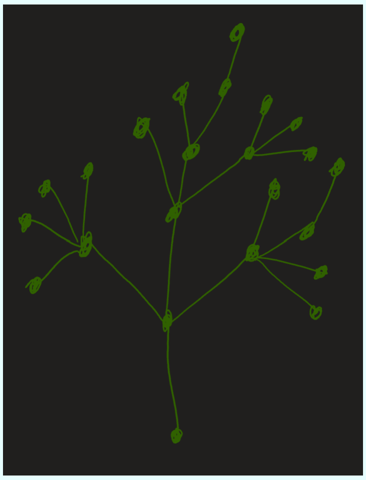
is a walk from u_i to u_{i-1} in $G-e$ and so e is not a bridge. Hence, G has a cycle by the above theorem. ♡

2.4 Tree

Definition 2.4.1. A ***tree*** is a graph with no cycles. A ***forest*** is a graph in which every component is a tree.

Remark 2.4.2. Equivalently, a forest is a graph with no cycles.

Example 2.4.3. The following is a tree



Theorem 2.4.4. *Let T be a connected graph, with p vertices and q edges. Then the following are equivalent:*

1. *T is a tree (T has no cycle)*
2. *Every edge of T is a bridge*
3. *There is exactly one path joining each pair of vertices*
4. *$q = p - 1$*

Proof. We have $1 \Leftrightarrow 2 \Leftrightarrow 3$ from previous theorems.

We show $1 \to 4$. We proceed by induction on p. If $p = 1$, then we are done. Suppose $p > 1$ and the result is true for graphs with fewer vertices (less than p).

Let T be a tree with p vertices. Let $e \in E(T)$ be any edge, then e is a bridge by (2). So $T - e$ has two components T_1 and T_2. These have no cycles and are connected, so T_1 and T_2 are trees. Then, we have $|E(T_1)| = |V(T_1)| - 1$ and $|E(T_2)| = |V(T_2)| - 1$ by induction. Therefore, we have

$$q = |E(T)| = |E(T_1)| + |E(T_2)| + 1 = p - 1$$

The proof follows.

We then show 4 imply 2. Suppose to the contrary that $q = p - 1$ and T has an edge that is not a bridge. Then $T - e$ is a connected graph with fewer than $p - 1$ edges. If we keep deleting non-bridges, we will eventually have to stop, at which point, we get a connected graph in which every edge is a bridge with p vertices and fewer than $p - 1$ edges.

This would be a tree that does not satisfy 4, but we just showed 1 imply 4, so a contradiction. ♡

Remark 2.4.5. If G is not connected, it is possible to have a graph with $q = p - 1$ such that it is not a tree.

Definition 2.4.6. A subgraph T of G is called a spanning tree if it is a spanning subgraph and it is a tree.

Theorem 2.4.7. *A graph is connected if and only if it has a spanning tree.*

Proof. $\Leftarrow$: If G has a spanning tree T, then since T is connected, there is a path joining any pair of vertices in T. Such a path is also in G, and so G is connected.

$\Rightarrow$: Suppose to the contrary that there exists a graph G such that G is connected and G has no spanning trees.

Consider a minimal such graph, i.e. a graph M with this property and as few edges as possible. Suppose M has no cycle, then M is a tree and it has itself as a spanning tree, which is a contradiction.

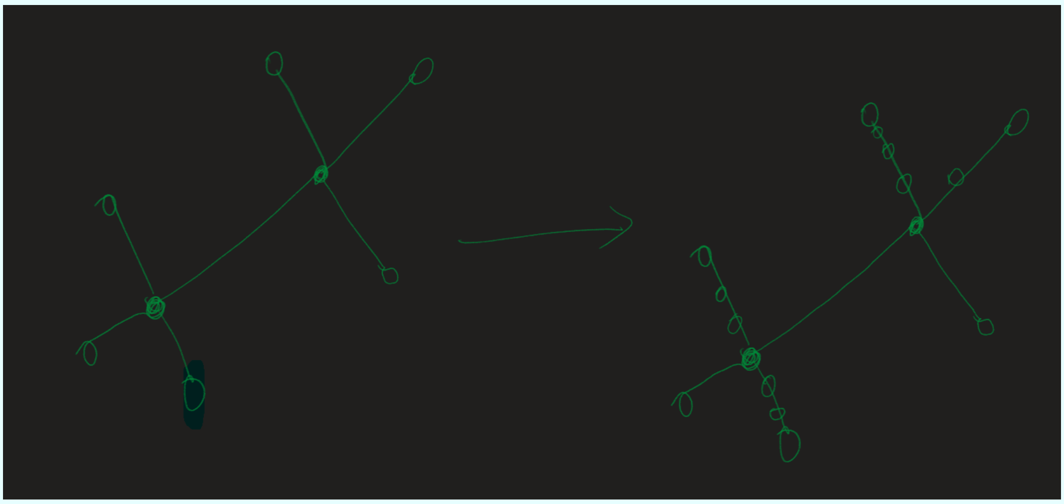
If M has at least one cycle, then M has an edge e that is not a bridge. Thus $M - e$ is connected and has fewer edges than M and so $M - e$ has a spanning tree T, but T is also a spanning tree of M and we are done. $\heartsuit$

Remark 2.4.8. Suppose T is a tree with p vertices, $p \geq 2$. Let n_i be the number of vertices of degree i, say $i = 0, 1, 2, 3, \ldots$. Then, we have
eq1. $n_0 = 0$
eq2. $\sum_{i\geq 1} n_i = p$
eq3. $\sum_{i\geq 1} in_i = 2(p-1)$
Part 1 says it due to no isolated vertices, part 2 is due to the total number of vertices and 3 is due to Handshake.

Then, consider $2 \cdot eq2 - eq3$, we have

$$(2n_1 + 2_n2 + 2n_3 + \ldots) - (n_1 + 2n_2 + 3n_3 + \ldots) = 2p - (2p - 2)$$

Thus, we get
eq4. $n_1 = 2 + \sum_{i\geq 3}(i-2)n_i$
Note n_2 does not appear in this formula, this is because we can do the following, i.e. replace each edge by a path:



This is called edge subdivision and it changes the number of n_2 but not any other n_i.

Definition 2.4.9. A vertex of degree 1 in a tree is called ***leaf***.

Theorem 2.4.10. *Every tree with at least 2 vertiecs has at least two leaves.*

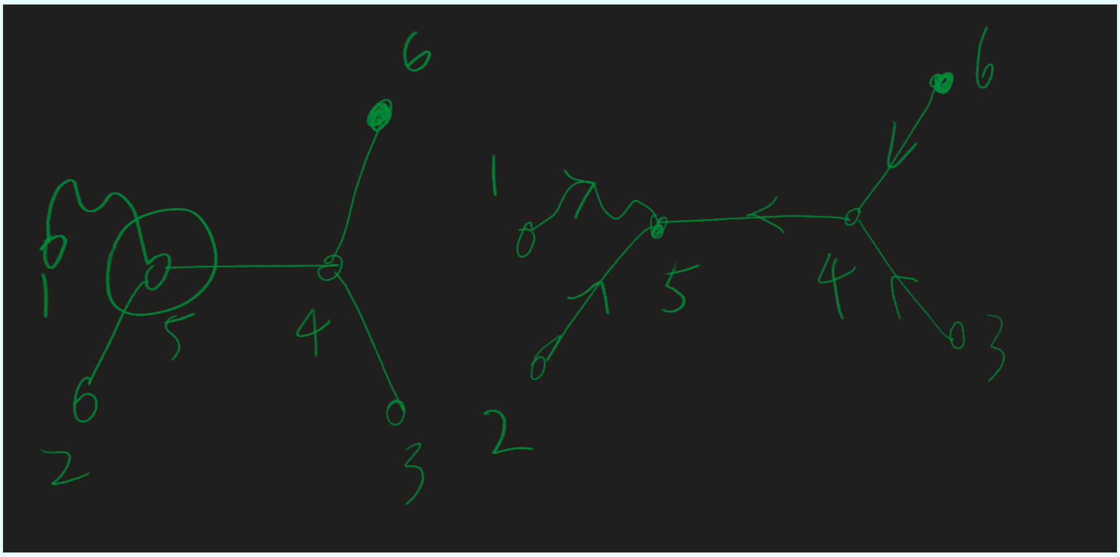
Example 2.4.11 (Exercise). If F is a forest with p vertices and q edges and c components, prove that $q = p - c$.

2.5 Enumeration of Trees

Definition 2.5.1. A ***rooted tree*** is a pair (T, r) where T is a tree, $r \in V(T)$ is a vertex called ***the root***.

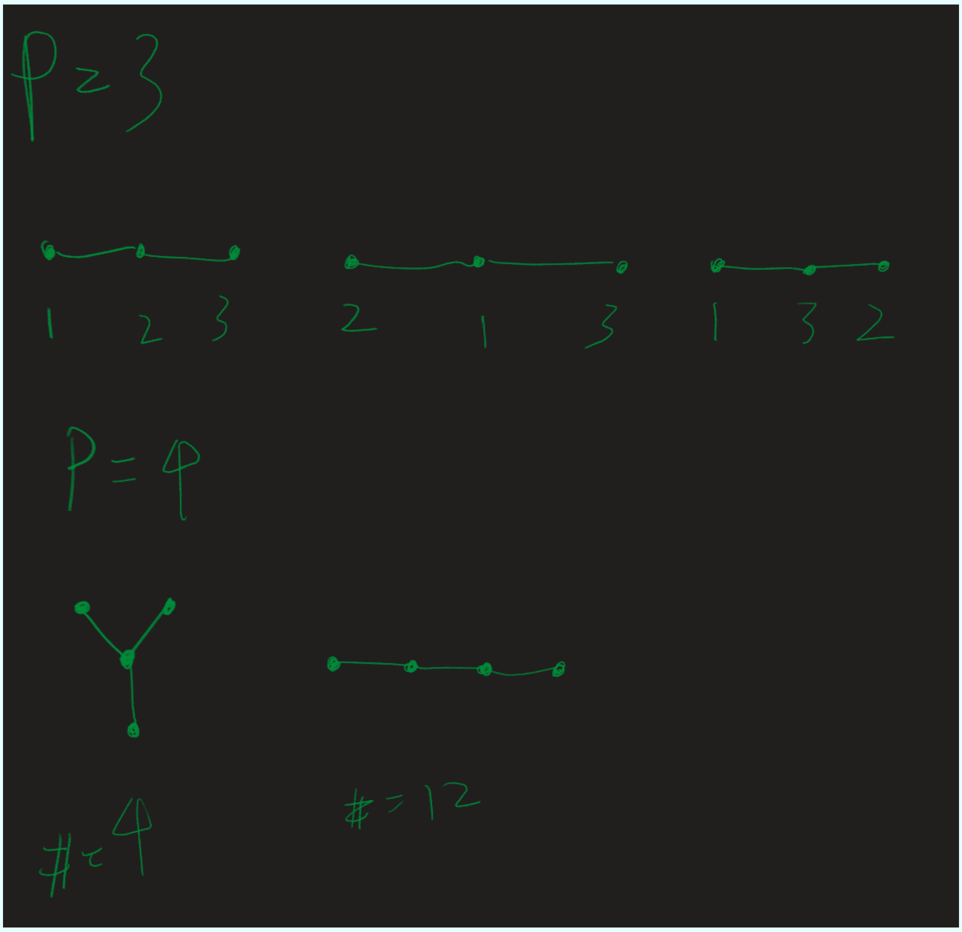
Remark 2.5.2. We can represent a rooted tree either by drawing the tree and circleing the root or by putting arrows on the edges so that the unique path any vertex to the root follows the arrows.

Example 2.5.3. Below is example of the two ways to indicate rooted graphs:



Theorem 2.5.4 (Cayley's). *There are p^{p-2} trees with vertex set $[p]$.*

Example 2.5.5. Below are examples when $p = 3$ and $p = 4$,



Proof. We will count sequences $(e_1, ..., e_{p-1})$ of directed edges such that these edges together form a rooted tree on vertex set $[p]$.

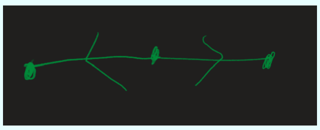
We give two ways to count this.

Method One: Let T_p be the set of all trees with vertex set $[p]$. To produce such a sequence, I can

1. Pick a tree $T \in T_p$
2. Pick a vertex $r \in [p]$ to be the root
3. Direct all edges towards the root
4. List the edges in same order

In particular, we can do the first step in $|T_p|$ ways, the second step p ways, thrid step in 1 way, and $(p-1)!$ ways for step 4. Thus, the number of sequences is $|T_p| \cdot p \cdot (p-1)! = |T_p| \cdot p!$.

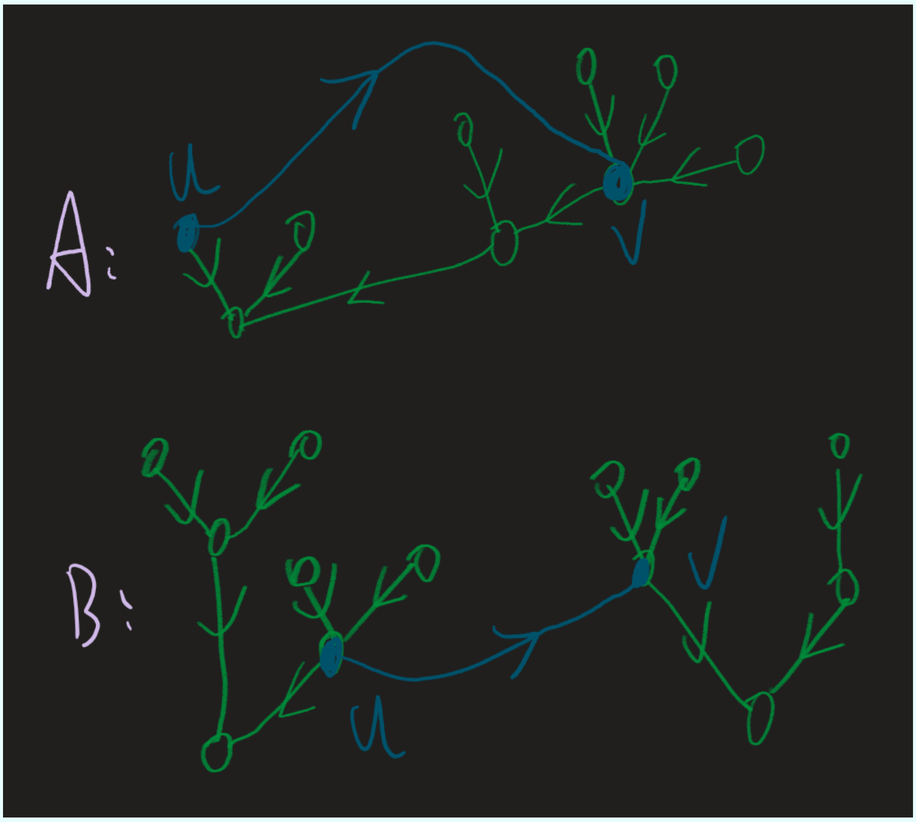
Method Two: Start with vertex set $[p]$ and no edges, we add directed edges one at a time try to avoiding the following case:



Thus, we do the following

1. Pick a vertex $v \in [p]$
2. Pick another vertex u such that adding $u \to v$ creates a graph in which every component is a rooted tree

Repeat until we have $p-1$ edges added. Thus, there are p ways to do step 1. To exam step 2, we consider the following:



Both case A and B will cause problem, in particular, from A we note u and v must be in different components as if not then we get a cycle. From B, we note u must be the root of its component, as if not, then we get a note that goes both ways, i.e. the case we want to avoid.

THerefore, the number of choice for step 2 is equal to the number of components minus 1, which is equal

$$p - 1 - \#\text{ of edges so far}$$

The number of choices for ith iteration is $p(p-i)$ and so the number of sequences is

$$\prod_{i=1}^{p-1}(p(p-i)) = p^{p-1}(p-1)!$$

Thus, by the double counting argument, we get

$$|T_p| \cdot p! = p^{p-1}(p-1)! \Rightarrow |T_p| = p^{p-2}$$

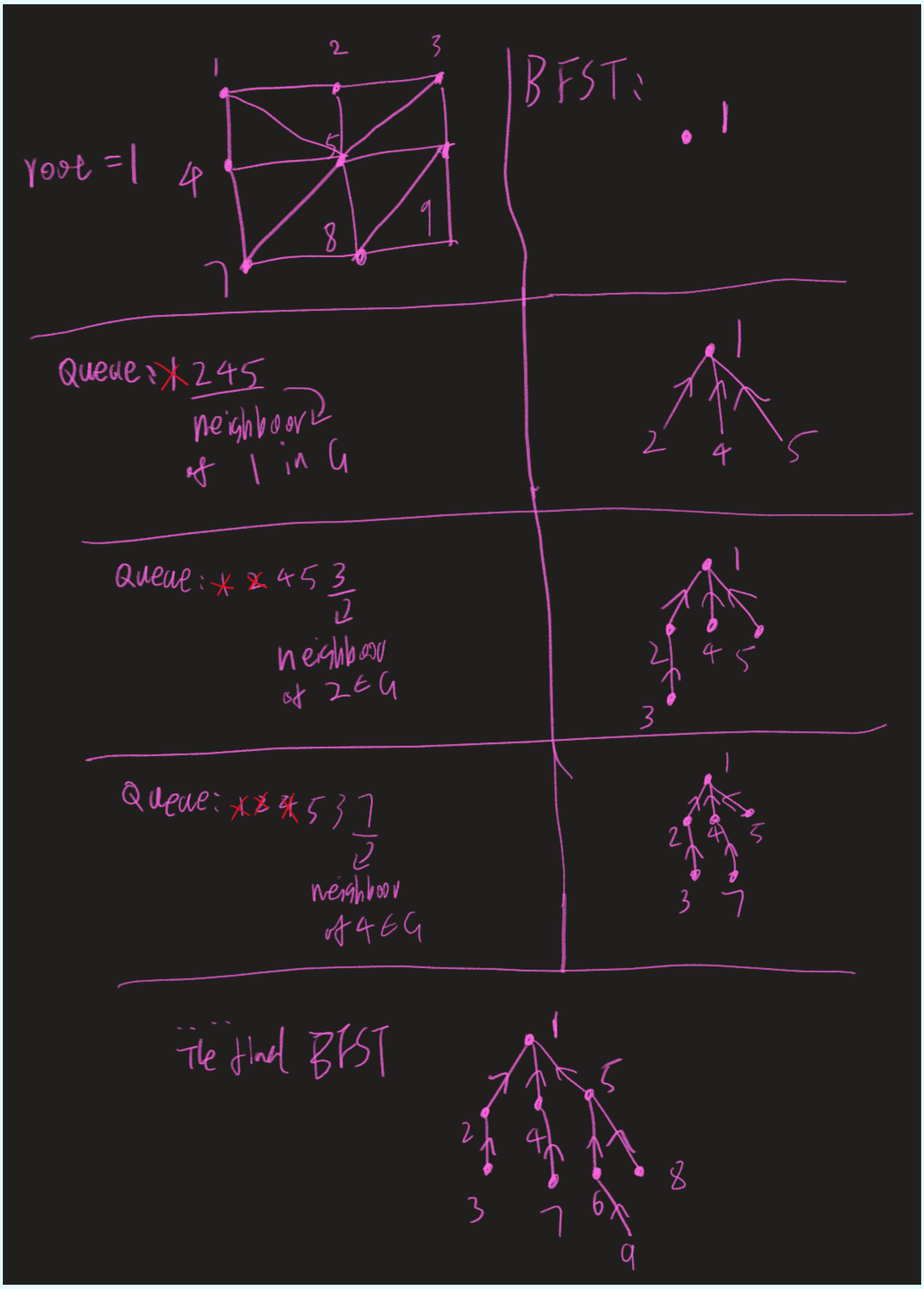
♡

2.6 Breadth First Search Trees(BFSTs)

Definition 2.6.1. A BFST of a graph G is a rooted tree output by the BFST algorithm.

Remark 2.6.2. In particular, for a fixed graph G, we can have more than one BFSTs. The input for the algorithm is a graph G and a vertex $r \in V(G)$. The output is a BFST, rooted at r.

Example 2.6.3. We do an example of BFST.



Definition 2.6.4. Let G be a graph and B be a BFST of G. Let $x \in V(G)$. The ***parent*** of x, denoted by $pr(x)$ is the vertex to which x is joined in the BFST algorithm. We write $pr(r) = \emptyset$ to say that the root has no parent.

The ***level*** of x is defined revursively as $level(r) = 0$ and $level(x) = level(pr(x)) + 1$ if $x \neq r$.

Theorem 2.6.5.

1. *A BFST of G is a tree.*
2. *Let H be a BFST, then H is a spanning tree if and only if G is connected.*

Proof. To show 1, we do by induction. We show that each time, we add a new vertex/edge we get a connected graph T with $|E(T)| = |V(T)| - 1$.

To show 2, $\Rightarrow$ is clear. Conversely, suppose G is connected, let T be a BFST rooted at r. Let $v = v_0 \ldots v_k = r$ be a path from v to r.

If $v \notin V(T)$ then consider the first vertex, say v_i, in this path such that $v_i \notin V(T)$ then $v_{i-1} \in V(T)$. When the algorithm reaches v_{i-1}, it adds all neighbours of v_{i-1} (unless they are already in the tree). Thus v_i gets added to T, this is a contradiction. Thus the proof follows. ♡

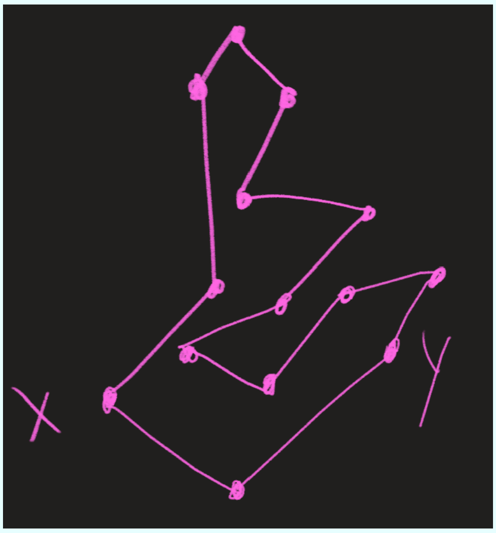
Theorem 2.6.6 (Fundamental Property of BFSTs)**.** *Let G be a connected graph and T is a BFST of G. Let $e = xy \in E(G)$. Then*

$$|level(x) - level(y)| \leq 1$$

Proof. Try to figure out by yourself. Related to the fact that BFST algorithm adds vertices to the queue in order of level. ♡

Definition 2.6.7. Let G be a connected graph. Let $x, y \in V(G)$, the ***distance*** from x to y is the length of a shortest path from x to y. We write $dist(x, y)$ to denote the distance.

Example 2.6.8. For example, $dist(x, y) = 2$ below



Theorem 2.6.9. *Let G to be a connected graph, if T is a BFST rooted at x then $dist(x,y) = level(y)$.*

Proof. Let $y = v_0 v_1 v_2 ... v_k = x$ be a path from x to y with length k. Then

$$\begin{aligned} k = \sum_{i=1}^{k} 1 &= \sum_{i=1}^{k} |level(N_i) - level(N_{i-1})| \\ &\geq |\sum_{i=1}^{k} level(N_i) - level(N_{i-1})| \\ &= |level(N_k) - level(N_0)| \\ &= |level(y) - level(x)| = |level(y)| \end{aligned}$$

So the length of every path from x to y is at least $level(y)$ but the path

$$y, pr(y), pr(pr(y)), ..., x$$

has length $|level(y)|$ so $dist(x,y) = |level(y)|$. ♡

Remark 2.6.10. To use this theorem one of x or y must be the root.

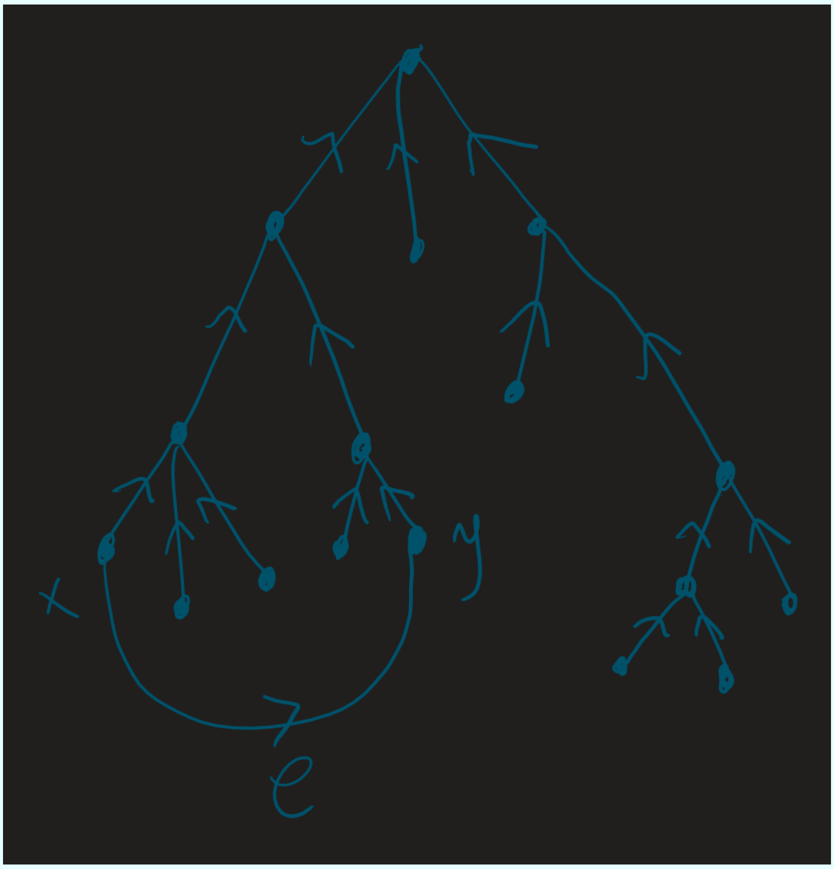
Definition 2.6.11. The ***girth*** of a graph G is the length of a shortest cycle. A graph with no cycles (i.e. trees) do not have a girth.

Theorem 2.6.12. *Let G be a graph, for each $r \in V(G)$, let T_r be a BFST rooted at r. Let*

$$m_r = \min_{xy \in E(G) \setminus E(T_r)} \{level_{T_r}(x) + level_{T_r}(y) + 1\}$$

Then the girth of G is $min_{r \in V(G)} m_r$, i.e. $girth(G) = \min_{r \in V(G)} m_r$.

Proof. We have $T_r + e$ is the graph with $V(T_r + e) = V(T_r)$ and $E(T_r + e) = E(T_r) \cup \{e\}, e \notin E(T_r)$. E.g.



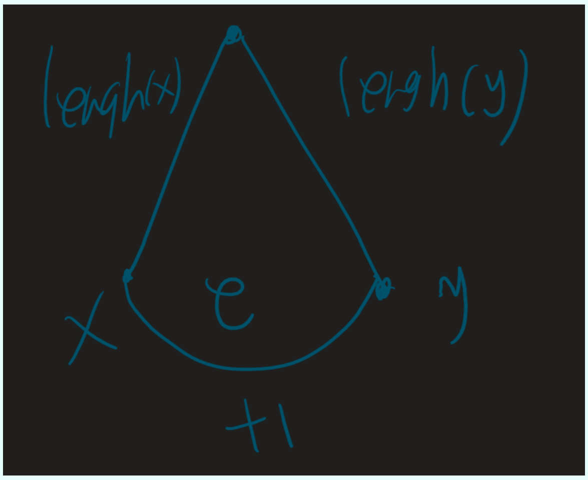
Length of the cycle is $\leq level(x) + level(y) + 1$. Get equality if the cycle includes the root. ♡

Theorem 2.6.13. *Let G be a connected graph, T be a BFST, then the following are equivalent:*

1. *G is bipartite*
2. *G has no cycles of odd length*
3. *For every $xy \in E(G)$, $|level(x) - level(y)| = 1$*

Proof. For $1 \Rightarrow 2$, we show $\neg 2 \Rightarrow \neg 1$.

For $2 \Rightarrow 3$, suppose to the contrary that there exists $e = xy \in E(G)$ with $level(x) = level(y)$. Then T has a cycle with odd length.



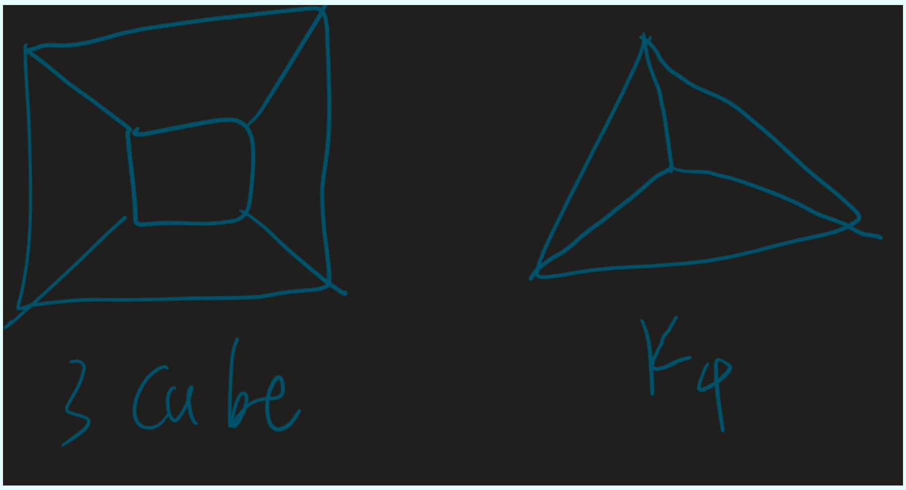
To show 3 $\Rightarrow$ 1. Consider the set $A = \{N \in V(G) : level(N) \equiv 1 \pmod 2\}$, $B = \{N \in V(G) : level(N) \equiv 0 \pmod 2\}$. This is a bipartition. ♡

2.7 Planar

Definition 2.7.1. A drawing of a graph (in the plane) with no edges crossing is called a ***planar embedding***.

Definition 2.7.2. A graph G is ***planar*** if it has at least one planar embedding.

Example 2.7.3. 3 cube and K_4 are both planar.



Definition 2.7.4. We define $K_{m,n}$ to be the complete bipartite graph of type (m, n).

$K_{m,n}$ is a bipartite graph with bipartition (A, B) where $|A| = m$ and $|B| = n$, and every vertex in A is joined to every vertex in B.

Remark 2.7.5. We note $|V(K_{m,n})| = m + n$ and $|E(k_{m,n})| = mn$. In addition, $K_{m,w}$ is planar.

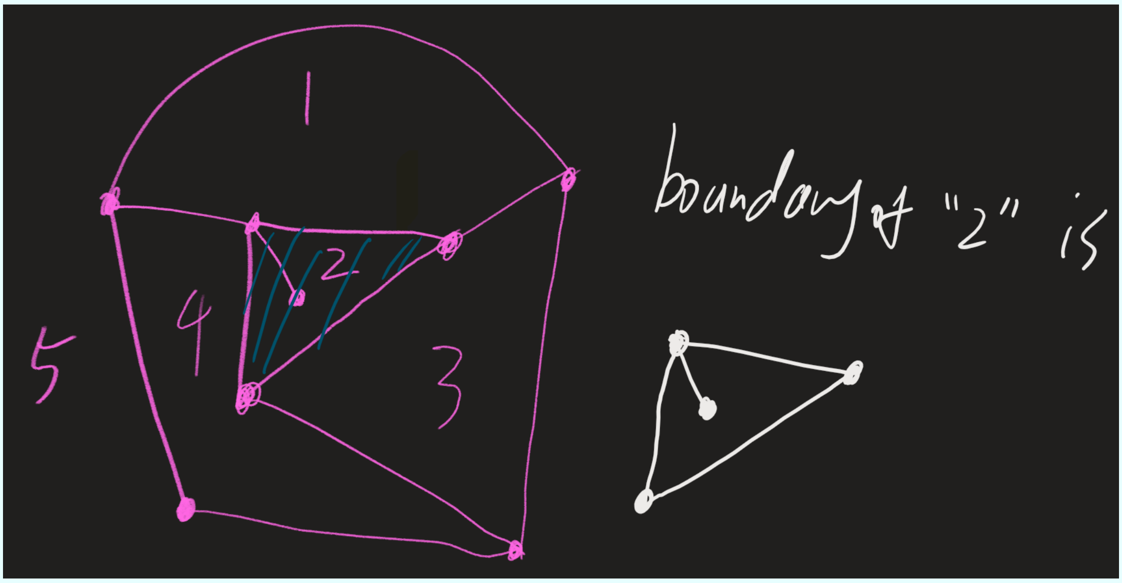
Example 2.7.6. We have K_5 is not planar.

Definition 2.7.7. A planar embedding divides the plane into regions called ***faces***.

Two faces are ***adjacent*** if they are ***incident*** with a common edge.

The ***boundary*** of a face is the subgraph of all vertices and edges incident with the face.

Example 2.7.8.



Definition 2.7.9. The degree of a face is

$$(\# \text{ of non-bridges}) + 2(\# \text{ of bridges})$$

Remark 2.7.10. Why brideg are special?

Let e be an edge. Starting on one side of e, trace along the boundary of the face.

If we get to the 'opposite side'. Then

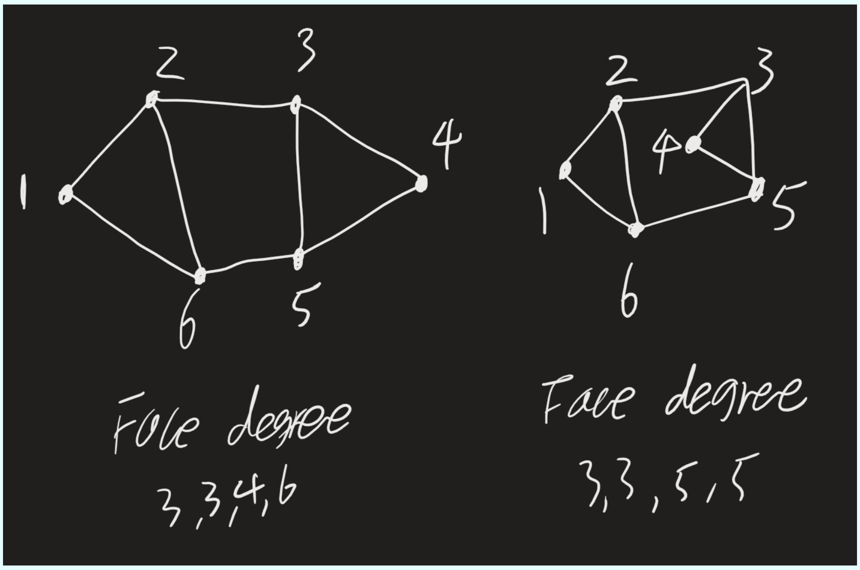
1. e is a bridge,
2. the two faces on both sides of e are the same.

If not, then

1. e is contained in a cycle, so e is not a bridge,
2. e is incident with two different faces.

Therefore, bridges are incident with one face and non-bridges are incident with two faces. We need to use Jordan curve theorem to prove this fact and we will skip this part.

Remark 2.7.11. Note a planar graph can have many "inequivalent" drawings.



Therefore, ***GRAPHS DO NOT HAVE FACES!***. We can only talk about faces in the context of a ***specific*** planar embedding (drawing).

Theorem 2.7.12 (Handshake For Faces)**.** *For a planar embedding P, let $F(P)$ be the set of faces of P. If P has q edges, then*

$$\sum_{f \in F(P)} deg(f) = 2q$$

Proof. Let

$$A = \{(f, e) : f \in F(P), e \in E(P), f \text{ is incident with } e, e \text{ not a bridge}\}$$

$$B = \{(f, e) : f \in F(P), e \in E(P), f \text{ is incident with } e, e \text{ a bridge}\}$$

Consider $|A| + 2|B|$, counting by edges, the remark above imply $|A| + 2|B| = 2q$.

Counting by faces, since the definition of degree counts bridges with a factor of 2, we have the desired result. ♡

Theorem 2.7.13 (Euler's Formula)**.** *For any planar embedding with p vertices, q edges, s faces and c components, we have*

$$p - q + s = c + 1$$

Proof. Let p be fixed, proceed by induction on q.

Base case: $q = 0$ then $s = 1$ and $c = p$. The proof follows.

Suppose $q > 0$ and the result holds for all planar embeddings with p vertices and $q-1$ edges.

Let p be a planar embedding with p vertices and q edges. Suppose p has s faces and c components.

Since $q > 0$, let e be an edge. Consider the graph $P - e$.

Case One: If e is not a bridge, then $P - e$ has p vertices, $q-1$ edges, $s-1$ faces (since the two faces incident with e in P get merged together) and c components. By the induction hypothesis applied to $P - e$, we have

$$p - (q-1) + (s-1) = c + 1 \Rightarrow p - q + s = c + 1$$

Case Two: If e is a bridge then $P - e$ has p vertices, $q-1$ edges, s faces (since no merging, the two faces incident with e are already the same) and $c+1$ components. By induction hypothesis, we have

$$p - (q-1) + s = (c+1) + c \Rightarrow p - q + s = c + 1$$

♡

Theorem 2.7.14. *Let P be a planar embedding that has a cycle. Then*

1. *The boundary of every face contains a cycle*
2. *The degree of every face is at least $girth(P)$*

Proof. Let f be a face of P. Recall that in a forest, the number of vertices, edges and components satisfy $q = p - c$. Since P has a cycle $q \geq p - c + 1$, by Euler's Formula, we have

$$s = q - p + c + 1 \geq 2$$

SInce there are at least 2 faces, f is not the whole plane. Let Q be the boundary of f, this is a drawing in the plane.

Then f is a face of Q too and still not the whoe plane.

SO we can find another face f' adjacent to f. By definition f and f' are incident with a common edge p. So $e \in E(Q)$ and e is not a bridge. Thus Q has a cycle.

For assertion 2, for any face f, we have $deg(f)$ is greater than or equal to the number of edges in the boundary of f. This is $\geq$ the length of any cycle in boundary of f. That is $\geq girth(p)$. ♡

Remark 2.7.15. We have four formulas that can be applied to any planar embeddings.

1. Handshake for vertices: $\sum_{v \in V(P)} deg(v) = 2q$
2. Handshake for faces: $\sum_{f \in F(P)} deg(f) = 2q$
3. Euler's Formula: $p - q + s = c + 1$
4. Min degree of faces: $deg(f) \geq girth(P)$

Example 2.7.16. K_5 is not planar.

Solution. Suppose that p is a planar embedding of K_5, we have $p = 5, q = 10, c = 1$. Thus, by Euler's formula, we have the number of faces is $s = q - p + c + 1 = 7$. Since the girth of K_5 is 5, the degree of every face of P is at least 3. Therefore, $20 = 2q = \sum_{f \in F(P)} deg(f) \geq 3 \cdot 7 = 21$. Contradiction. ♠

Example 2.7.17 (Exercise). Use a similar argument to show $K_{3,3}$, Petersen graph, Q_4 are non-planar.

Theorem 2.7.18. *Suppose P is a planar embedding with $p \geq 3$ vertiecs, q edges and every face has degree $\geq d^*$. Then $q \leq \frac{d^*(p-2)}{d^*-2}$*

Proof. We have

$$2q = \sum_{f \in F(P)} deg(f) \geq d^*$$
$$\Rightarrow s \leq \frac{2q}{d^*}$$

Then, P has at least one component, and so $c \geq 1$. Then, by Euler's, we have

$$p - q = c + 1 - s$$

Therefore, we have

$$p - q \geq 1 + 1 - \frac{2q}{d^*} \Rightarrow q \leq \frac{d^*(p-2)}{d^*-2}$$

♡

Corollary 2.7.18.1. *Let G be a graph with $p \geq 3$ and q edges. If G is planar then $q \leq 3(p-2)$.*

Proof. If G has a cycle, we can take $d^* = 3$ in the above theorem because every face of a planar embedding has degree $\geq girth(G) \geq 3$.

If G does not, then it is a forest, and $q \leq p - 1 \leq 3(p-2)$. ♡

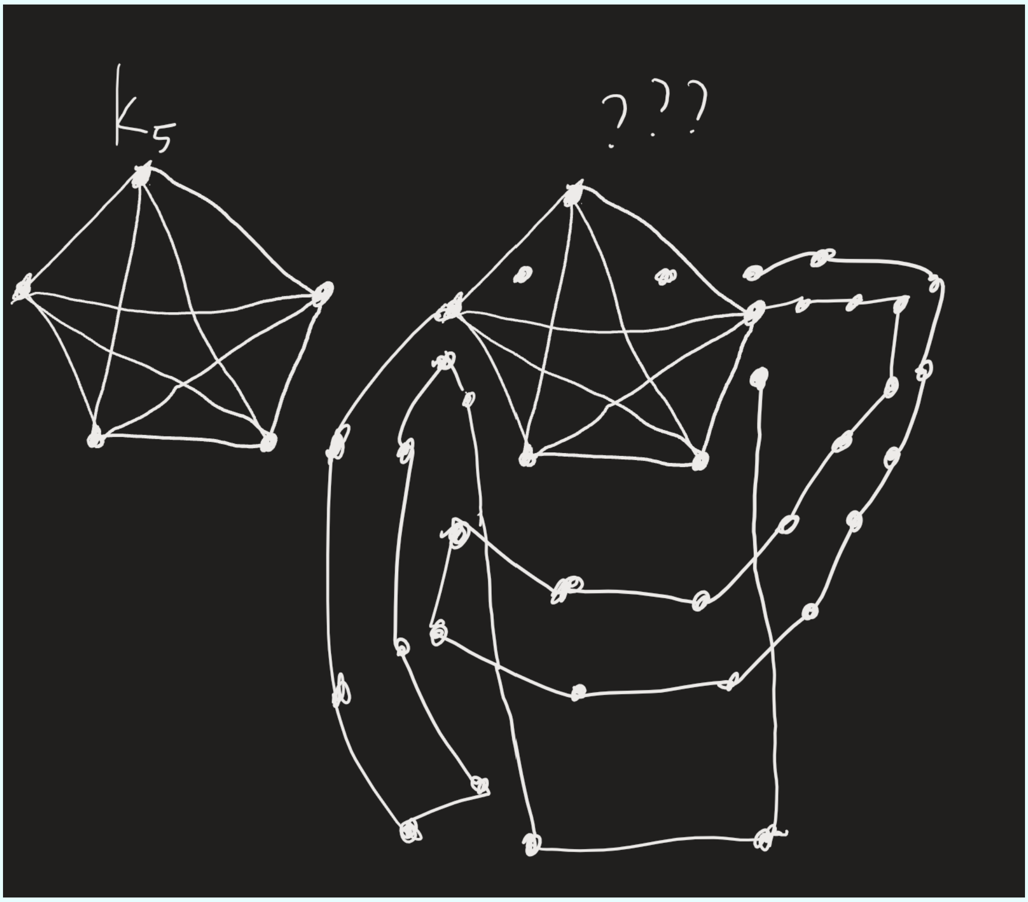
Corollary 2.7.18.2. *Let G be a graph with p vertices, q edges, $girth(G) = k$. If G is planar, we have*

$$q \leq \frac{k(p-2)}{k-2}$$

Proof. We can take $d^* = k$ in the theorem. ♡

2.8 Muda!

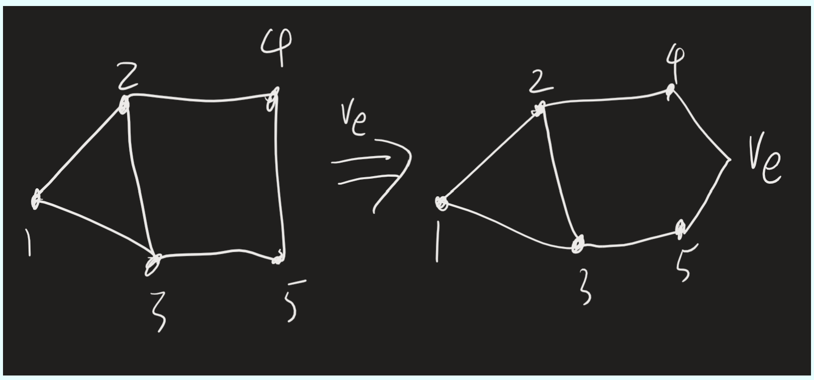
Example 2.8.1. Note the four formulas we had is not a if and only if statement. For example, K_5 and ??? are both not planar, but ??? would pass all the four formulas while K_5 would fail.



Definition 2.8.2. Let G be a graph, $e = xy \in E(G)$. We define a new graph G' with $V(G') = V(G) \cup \{v_e\}$ and $E(G') = (E(G) \backslash e) \cup \{xv_e, yv_e\}$.

A graph obtained by applying this process repeatedly, starting with G is called an ***edge subdivision*** of G.

Example 2.8.3. We have



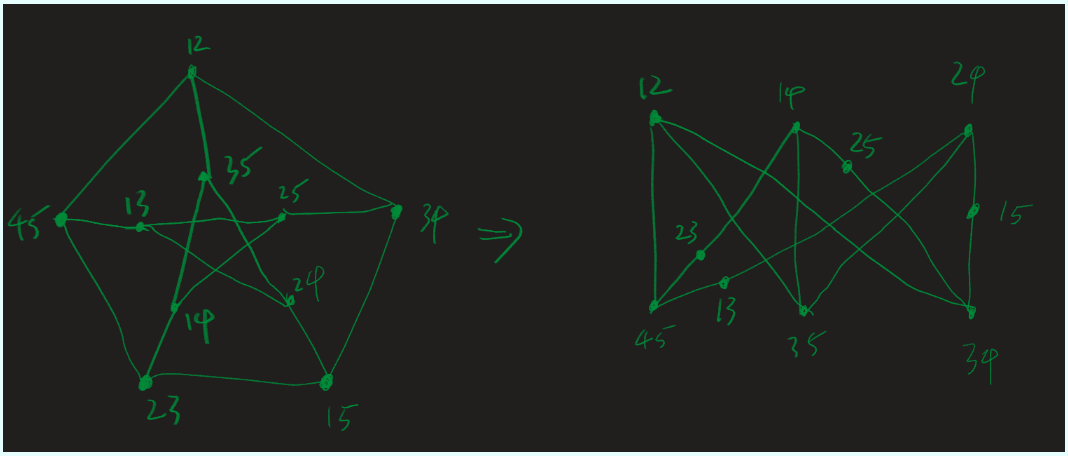
Theorem 2.8.4 (Kuratowski's Theorem). *Let G be a graph. Then G is planar if and only if both of the following holds*

1. *G has no subgraph isomorphic to an edge subdivision of K_5,*
2. *G has no subgraph isomorphic to an edge subdivision of $K_{3,3}$.*

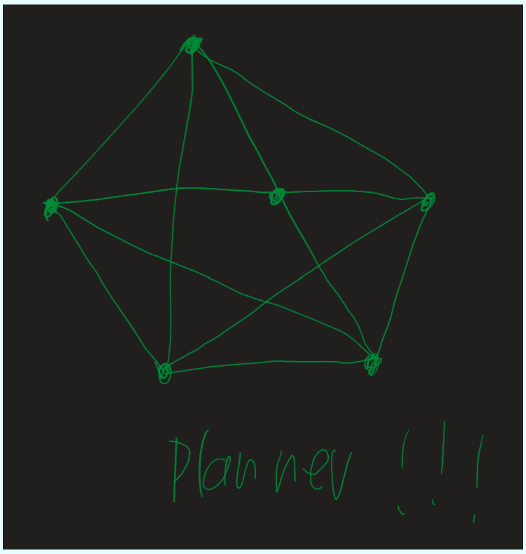
Proof. ($\Rightarrow$): Is true becauses a planar graph cannot have a non-planar subgraph. If 1 and 2 was false, then it would have a non-planar subgraph.

($\Leftarrow$): Hard, but also mainly of theoretical interest. This is covered in $CO342$. ♡

Example 2.8.5. Consider the Petersen graph, we have a subgraph that is isomorphic to an edge subdivision of $K_{3,3}$



Caution: You want to avoid the following



As this looks like an edge subdivision of K_5 but isn't.

Definition 2.8.6. Let G be a graph, $e = xy \in E(G)$. Define a new graph

$$V(G/e) = V(G) \cup \{v_e\} \backslash \{x, y\}$$

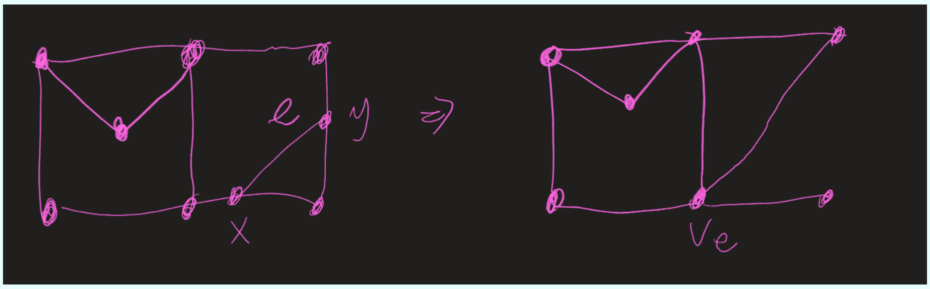
On the other hand, $E(G/e)$ consists of edges

1. st where $st \in E(G)$ and $s, t \notin \{x, y\}$
2. sv_e where either $sx \in E(G)$ or $st \in E(G)$

This is called the ***contracted graph of*** G and the process is called ***contracting*** e from G.

A graph obtained from G by contracting and deleting edges is called a ***minor*** of G.

Example 2.8.7. This of this as edge e shrinking down to zero length and get rid of multiple edges that arises.



Theorem 2.8.8 (Wagner's Theorem)**.** *A graph is planar if and only if the following two condition holds*

1. *K_5 is not a minor,*
2. *$K_{3,3}$ is not a minor.*

Proof. ($\Rightarrow$) Easy. If G is planar, then deleting an edge and contracting an edge can both be performed an planar embeddings without creating crossings.

So any minor of a planar graph must be planar.

($\Leftarrow$) Similar to Kuratowski. ♡

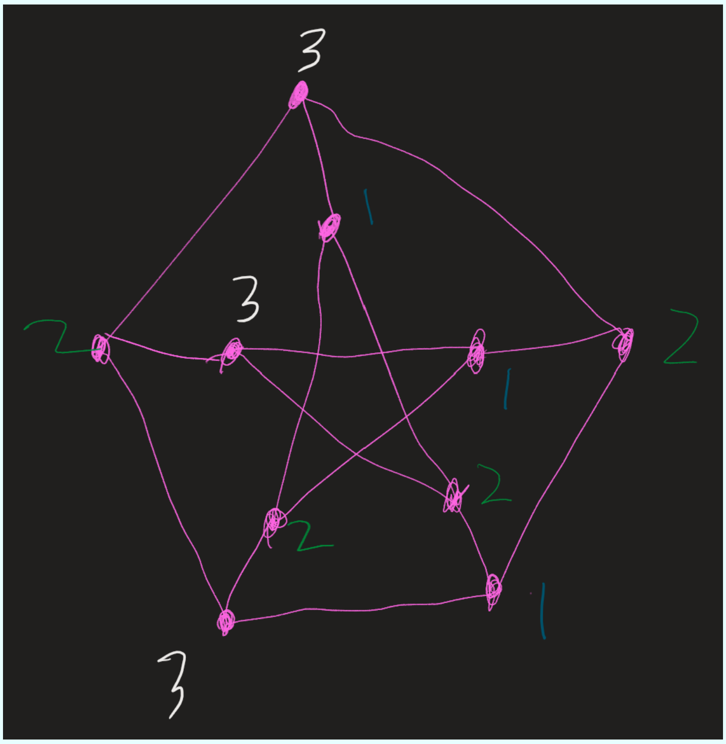
2.9 Graph Colouring

Definition 2.9.1. A ***k-colouring*** of a graph G is an assignment of k (or fewer) colours to the vertices, such that adjacent vertices are assigned different colours.

Formally, a k colouring is a function $f : V(G) \to [k]$ such that $f(x) \neq f(y)$ for all $xy \in E(G)$.

Definition 2.9.2. If a k colouring exists, we say that G is k-colourable.

Example 2.9.3. Consider the Petersen graph.



We observe that the k colouring is a generalization of bipartite graphs and G is bipartite if and only if G is 2 colourable.

Remark 2.9.4. There is no (known) efficient algorithm for determining if a graph is 3 colourable (or for the minimal number of colours required to colour a graph).

The most famous theorem in graph theory is the four colour theorem.

Theorem 2.9.5 (Four Colour Theorem)**.** *Every planar graph is* 4 *colourable.*

Proof. We won't prove it! It is hard and it is the first theorem proved by computer. ♡

Theorem 2.9.6 (Six Colour Theorem)**.** *Every planar graph is* 6 *colourable.*

Proof. To prove this kind of question, we normally prove a lemma of the following sort.

Lemma: Every planar graph has vertex of degree at most 5.

Proof: Suppose to the contrary there exists a planar embedding P with p vertices and q edges and every vertex has degree ≥ 6.

Note $2q = \sum_{v \in V(P)} deg(v) \geq 6p$ and so $q \geq 3p$. However, we already have $q \leq 3(p-2)$ and so it is a contradiction. The proof follows.

Then, we back to the theorem. We proceed by induction on the number of vertices of G. If $p = 1$, we are done.

Assume $p \geq 2$ and the result holds for planar graph with $p-1$ vertices.

Let G be planar with p vertices and by the lemma G has a vertex u such that $deg(v) \leq 5$. Let $G-v$ be the subgraph induced by $V(G)\backslash\{v\}$. Then $G-v$ is planar and has $p-1$ vertices. Thus it is 6 colourable. We can extend this colouring to a 6 colouring of G by assigning v a colour differently from any of its neighbours. The proof follows. ♡

Theorem 2.9.7 (Five Colour Theorem)**.** *Every planar graph is* 5 *colourable.*

Proof. As before, proceed by induction and let v be a vertex of degree ≤ 5. If $deg(v) \leq 4$, we can use the same argument as in the proof of the 6 colour theorem.

If $deg(v) = 5$, v has two neighbours x, y such that $xy \notin E(G)$. Indeed, otherwise all neighbours of v are adjacent, so the subgraph induced by $N(v)$ is K_5, and so G is not planar.

Let G' be the graph obtained by contracting vx and vy. By inductive hypothesis, G' is 5 colourable. Let's call the new vertex v'. Use this to get a 5 colouring of $G-v$ in which every vertex other than x, y get the colour from the colouring of G' and x, y are coloured with two colour of v'.

There are at most 4 colours used by the neighbours of v, so we can colour v with a different colour to get a 5 colouring of G. ♡

2.10 History of 4 Colour Theorem

Definition 2.10.1. Conjectured around 1852.

First proof is due to Appel and Hacken/Haken around 1976.

Efficient Algorithm is around 1996, by Robertson, Sanders, Seymour, and Thomas.

Remark 2.10.2 (Idea)**.** In the 6 colour theorem, we first showed cannot avoid a vertex of degree 1 to 5. Those are unavoidable configuration. The second step is that we used an induction algorithm, tells you what to do if you encounter each of these cases.

In the 5 colour theorem, we need two algorithm for the configurations, i.e. one algorithm works for vertex with degree $1, 2, 3, 4$ and one algorithm works for vertex with degree 5.

Therefore, we would thought for 4 colour theorem, we only need to come up with some new algorithms for vertex with degree 5, one algorithms for degree 4 vertices and one for degree $1, 2, 3$.

However, that is difficult. Therefore, the proof they did is that we expand the list of unavoidable configurations using technique called discharging.

For more, see $https://projecteuclid.org/download/pdf_1/euclid.ijm/1256049898$ and $https://pdfs.semanticscholar.org/3d50/72364f230e1ab5a0b3725d84d15689615077.pdf$

2.11 How many k colour do we have

Theorem 2.11.1 (Chromatic Polynomial)**.** *Let G be a graph with p vertices, there exists a polynomial $\chi_G(t)$ with integer coefficients and degree less than or equal to p such that $\chi_G(k)$ equal the number of k colours of G. This holds for all k. This $\chi_G(t)$ is called chromatic polynomial.*

Proof. We proceed by induction on the number of edges of G. If G has no edges, then G has k^p colouring. So $\chi_G(t) = t^p$ has the desired property.

Now, suppose G has at least one edge and the result is true for all graph with fewer edges. Let $e = xy \in E(G)$ be an edge, then every k colouring of G is a k colouring of $G - e$.

So, a k colouring of $G - e$ is a k colouring of G if and only if x and y are different colours. Also, a k colouring of $G - e$ in which x, y are same colour correspond to a k colouring of G/e.

Therefore,

$$\begin{aligned}\#\text{ of k-colourings of } G &= \#(\text{ of k-colourings of } G - e) - (\#\text{ of k-colourings of } G/e)\\ &= \chi_{G-e}(k) - \chi_{G/e}(k)\end{aligned}$$

By inducetive hypothesis, note both $G - e$ and G/e has less edges, and so

$$\chi_G(t) = \chi_{G-e}(t) - \chi_{G/e}(t)$$

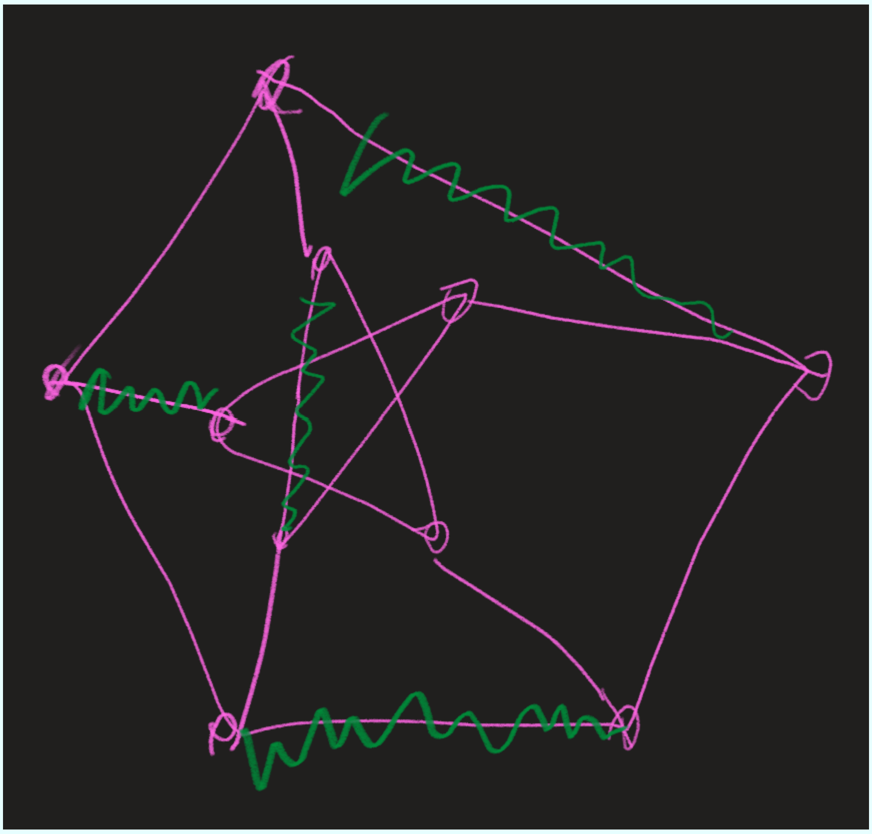
would be the desired polynomial and the inducetion follows.

♡

2.12 Matching

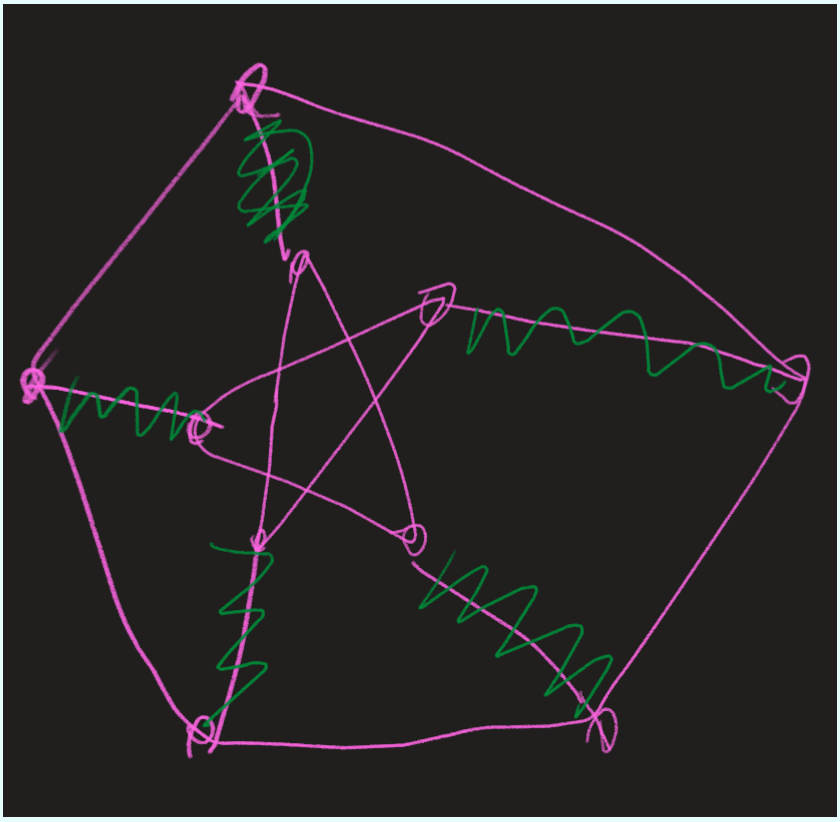
Definition 2.12.1. A ***matching*** M in a graph G is a subset of $E(G)$ such that no two edges in M are incident with a common vertex.

Example 2.12.2. Below is a matching



Note empty set is a matching for any graph.

Below is a perfect mathcing



Definition 2.12.3. A vertex $v \in V(G)$ is ***saturated*** by M, if v is incident with some edge in M.

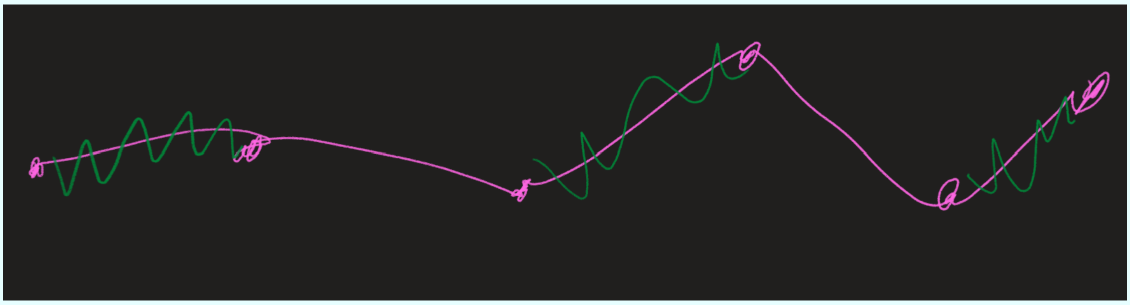
Definition 2.12.4. A ***maximum matching*** is a matching with the largest possible number of edges (among all matchings in G)

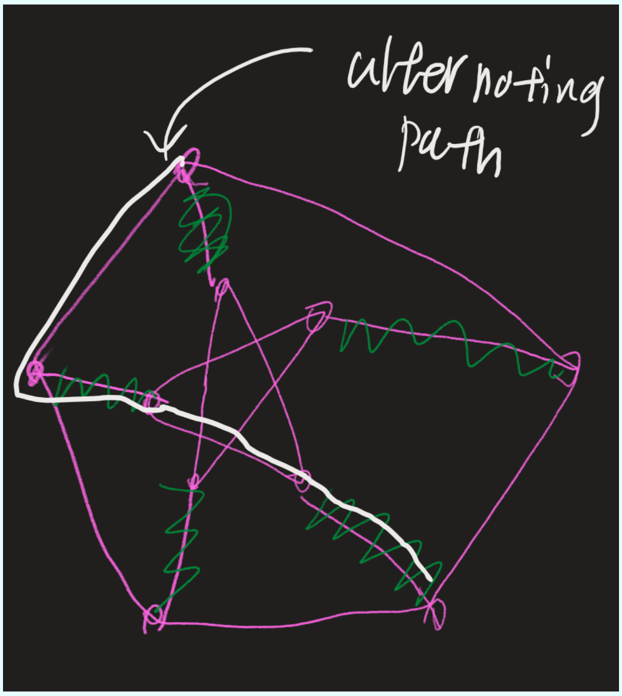
Definition 2.12.5. A ***perfect matching*** is a mathcing in which all vertices are saturated.

Remark 2.12.6 (Problem)**.** How to find maximum mathcings?

Definition 2.12.7. Given a graph G and a mathcing M, an ***alternating path*** is a path in which the edges are alternately in M and not in M.

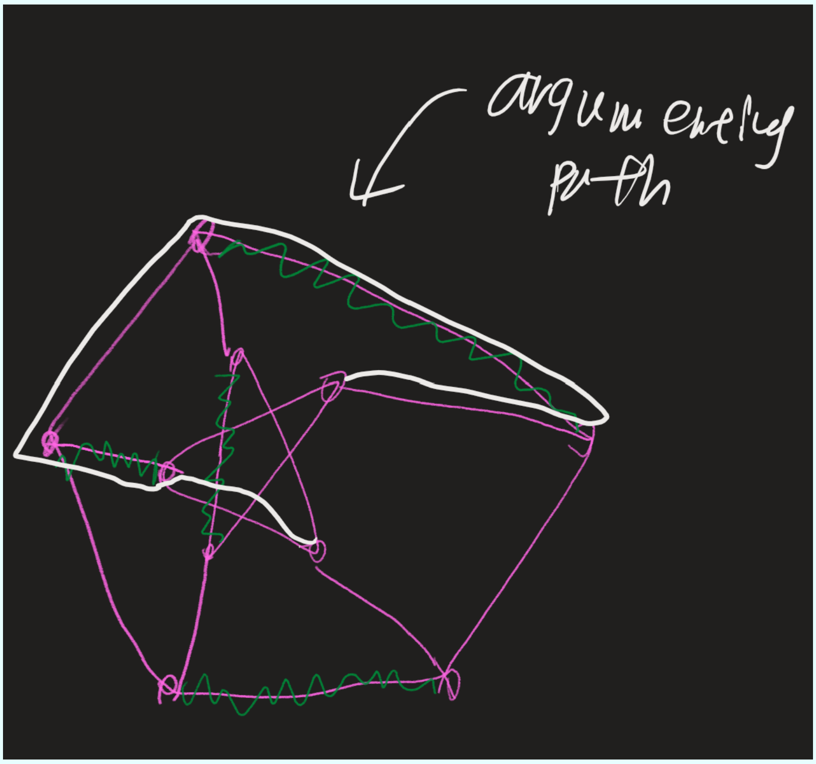
Example 2.12.8. Below are two alternating path





Definition 2.12.9. An ***argumenting path*** is an alternating path of length at least 1 from an unsaturated vertex to another unsaturated vertex.

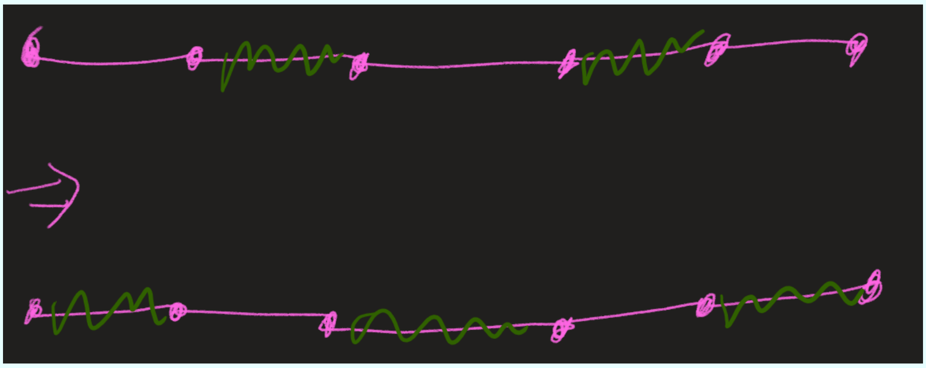
Example 2.12.10. Below is a argumenting path



Remark 2.12.11. Given an augmenting path P, replace M with

$$(M \backslash E(P)) \cup (E(P) \backslash M)$$

This creates a mathcing with 1 more edge.



Theorem 2.12.12. *If an augmenting path exists, then M is not a maximum matching.*

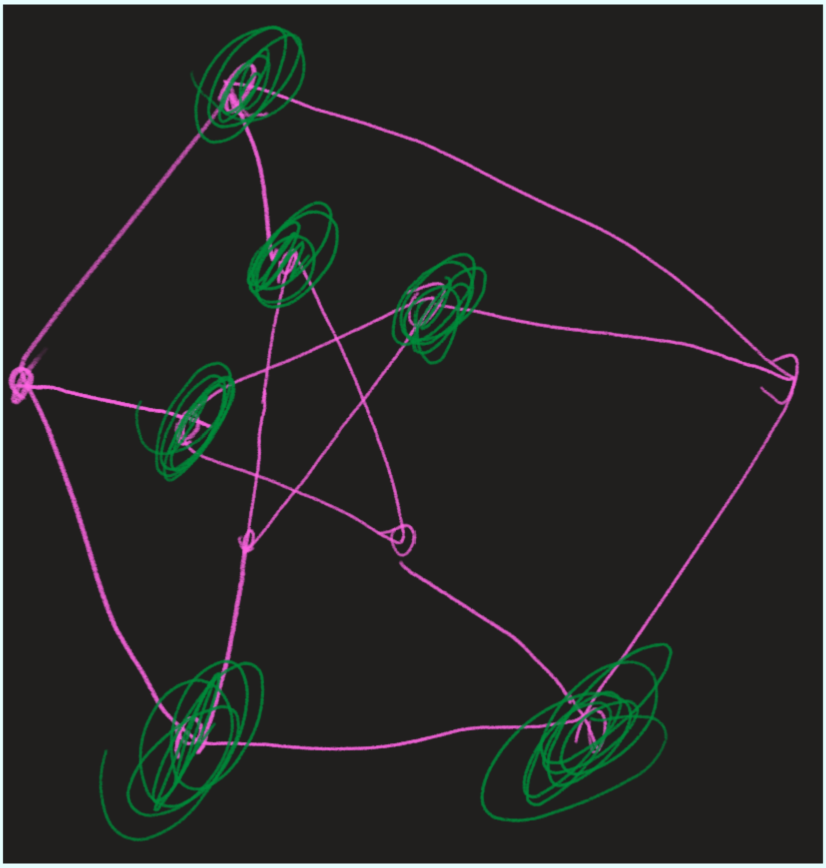
Remark 2.12.13. The converse is also true.

In this course, we will look for mathcing for bipartite case. This is because

1. it is slightly easier
2. more important for applications
3. stronger results

Definition 2.12.14. A ***cover*** C in a graph G is a subset of the vertices such that every edge of G is incident with at least one vertex in C.

Example 2.12.15. Below is a cover



Theorem 2.12.16. *If M is a matching and C is a cover in a graph. Then,*

$$|M| \leq |C|$$

Proof. Let $M = \{e_1, ..., e_k\}$, $e_i = u_i v_i$. For each i, either $u_i \in C$ or $v_i \in C$. WLOG, suppose $u_i = e$ then $\{u_1, ..., u_k\} \subseteq C$. Since M is a matching, $u_i \neq u_j$ for $i \neq j$, so

$$|C| \geq k = |M|$$

♡

Corollary 2.12.16.1. *If we have a mathcing M and a cover C such that $|C| = |M|$, then M is a maximum matching and C is a minimum cover.*

Theorem 2.12.17 (Konig's Theorem)**.** *In a bipartite graph, there exists a matching M and a cover C such that $|C| = |M|$.*

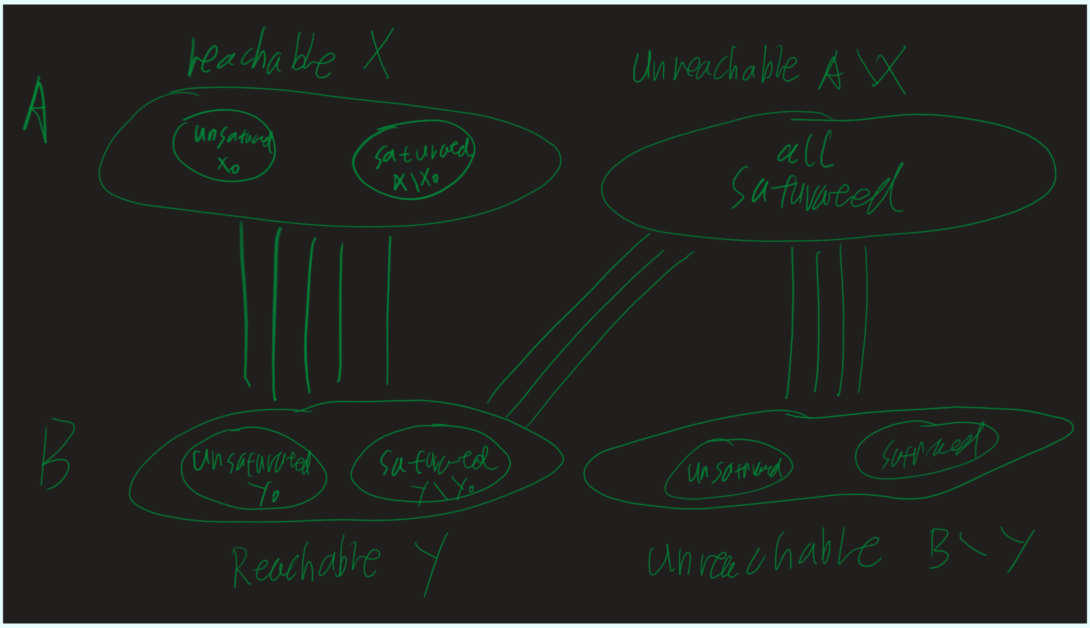
Proof. Here is the idea. Start with any mathcing,

- try to find an augmenting path
- in the process, build a cover C
- Find: either an augmenting path exists, or $|C| = |M|$

First step: search for augmenting paths.

Let M be a mathcing in G. (A, B) be a bipartition of G. Note that an augmenting path has one end in A and one end in B. Let X_0 be the set of unsaturated vertices in A and X be the set of "reachable" vertices in A. Let Y be the set of "reachable" vertices in B. Let Y_0 be the set of unsaturated vertices in Y. A vertex is reachable if there exists an alternating path from a vertex in X_0 to v. Note every vertex in X_0 is reachable. Note an augmenting path exists if and only if $Y_0 \neq \emptyset$, which is more or less by definition.

We have seven sets, consider the diagram below



Lemma 1: If $u \in X$, and $uv \in E(G)$, then $v \in Y$.

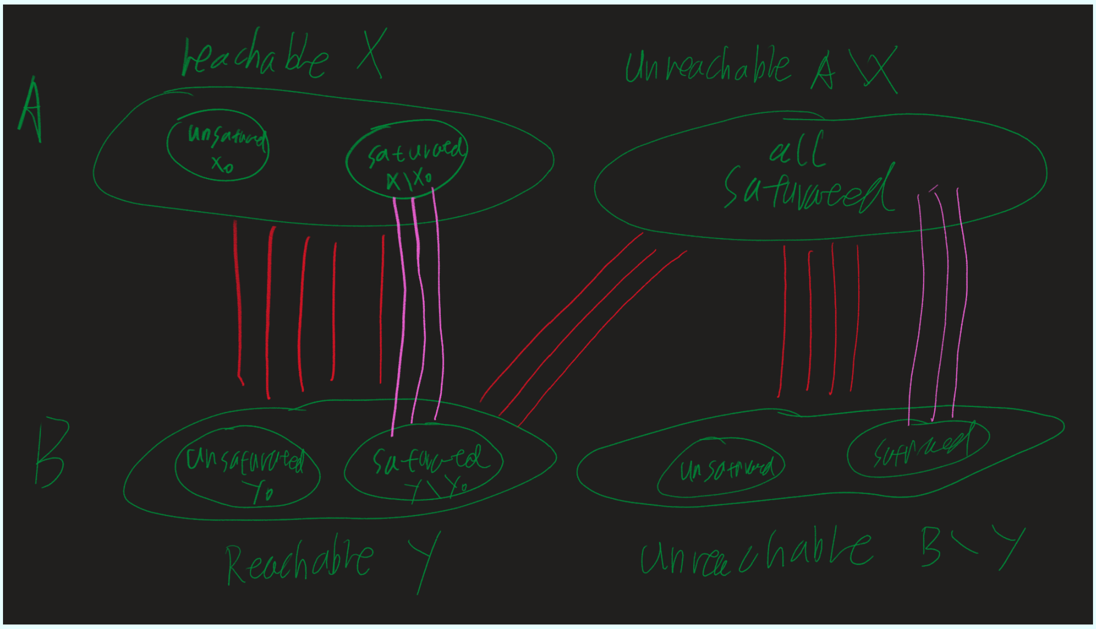
Indeed, since u is reachable, there exists an alternating path starting at some vertex $x \in X_0$, ending at u. Call this path $P(u)$.

- If $v \in P(u)$ then v is reachable
- If v is not in this path and $uv \notin M$, then $P(u)v$, which is obtained by extend the path to add a step from u to v, is an alternating path from x to v, and so v is reachable and so $v \in Y$.
- If v is not in this path and $uv \in M$, this is impossible, because the last edge in $P(u)$ must belong to M, and is incident with u. Since $uv \in M$, and the last edge of $P(u)$ is in M and both are incident to u, they must be the same edge. So uv is an edge in $P(u)$ and so v is in $P(u)$. A contradiction.

This shows the lemma.

Lemma 2: If $v \in Y$, and $e = uv \in M$, then $u \in X$. The proof is similar to Lemma 1, and is left as an exercise.

We have the picture below:



where red line is matching edge and purple line is all edges.

Second step: building a cover.

Lemma 3: $C = Y \cup (A \backslash X)$ is a cover and $|C| = |M| + |Y_0|$.

Indeed, by Lemma 1, every edge has at least one vertex in either Y or $(A \backslash X)$. Every edge in M is incident with either a vertex in $A \backslash X$ or a vertex in $Y \backslash Y_0$ but by Lemma 2, not both. Therefore, we have

$$|M| = |A \backslash X| + |Y \backslash Y_0| = |C| - |Y_0|$$

This is the end of the lemma 3.

Now we turn to the proof. Suppose M is a max. mathcing. Then there is no augmenting path. So $Y_0 = \emptyset$ and so $|C| = |M|$.

♡

Remark 2.12.18. For non-bipartite graphs, the above theorem may or may not be true.

Remark 2.12.19 (Algorithm for Max Mathcing)**.** The idea is

- construct X, Y, etc in a systematic way,
- Modify BFST algorithm.

In particular, the modification follows as

- Instead of starting with a single root, every vertex in X_0 is a root.
- If v is the active vertex
  - If $v \in X$, add only edges not in M,
  - If $v \in Y$, add only edges in M

In particular, X is the vertices of even level, Y is the vertices of odd level, X_0 is the vertices of level 0 and Y_0 is the vertices of odd level with no children.

If Y_0 is not empty, then the path from a vertex in Y_0 to its root is augmenting.

If Y_0 is empty, then we have a max matching and min cover, with $C = Y \cup (A \backslash X)$

Remark 2.12.20. We should practice BFST algorithm.