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山抹微云，天连衰草，画角声断谯门。暂停征棹，聊共引离尊。多少蓬莱旧事，空回首，烟霭纷纷。斜阳外，寒鸦万点，流水绕孤村。
销魂，当此际，香囊暗解，罗带轻分。谩赢得青楼，薄幸名存。此去何时见也，襟袖上，空惹啼痕。伤情处，高城望断，灯火已黄昏。

秦观

Special thanks to Sean and Catherine for their help with the note.

1 Introduction

So, before we go real, let's answer the question, "why stacks"?

To answer this question, we back up and look ad moduli spaces first. Frequently, when studying geometric objects (such as enumerating genus g curves, Abelian varieties, vector bundles, etc), it helps to look at the "space of all such objects".

When that space exists, it is called a moduli space.

Why is this helpful?

Example 1.1. How many plane conics (i.e. a degree 2 curve in $\mathbb{P}^2$) go through 5 general points?

We can prove there exists a unique conic. To do this, we consider the moduli space of conics in $\mathbb{P}^2$. A degree 2 curve $C \subseteq \mathbb{P}^2$ (with coordinate x, y, z) correspond bijectively to $C = V(ax^2 + by^2 + cz^2 + dxy + exz + fyz)$. Observe if we take (a, b, c, d, e, f) and scale it by $\lambda \neq 0$ in field k, then we see we still get the same conic C. In particular, this is the only relation we need to mod out, i.e. two conics C, C' are the same iff their coefficients differ by a scalar.

Thus moduli space of plane conics are given by $\{(a, b, c, d, e, f)\}/k^*$, which is exactly $\mathbb{P}^5_k$ with coordinates (a, b, c, d, e, f).

Now, for a fixed point $p = (x_0 : y_0 : z_0)$ in $\mathbb{P}^2$, a conic $C := V(f)$ with $f = ax^2+by^2+cz^2+dxy+exz+fyz$ contains p iff $f(x_0, y_0, z_0) = 0$. To translate this condition to a condition on a, b, c, d, e, f, we see this becomes linear constraint on a, b, c, d, e, f, i.e. $H_p := \{C \in \mathbb{P}^5 : p \in C\}$ is a hyperplane.

Thus, choosing 5 general points $p_1, ..., p_5$, we see $\{C \in \mathbb{P}^5 : p_1, ..., p_5 \in C\} = H_{p_1} \cap ... \cap H_{p_5}$. But since the points p_i are general, we have $H_{p_1} \cap ... \cap H_{p_5}$ is exactly one point, i.e. there is a unique conic that does the job.

The above toy example shows it is helpful to consider moduli spaces. This is all good and sound, but we have a problem.

Moduli spaces rarely exists, or, they are rarely schemes. They are typically stacks!

Thus, in order to use moduli spaces, we are forced to consider stacks. Well, there are ways to get around this, but we lose information along the way (e.g. GIT).

So, why aren't moduli spaces schemes?

Example 1.2. Let $\mathcal{M}$ be the moduli space of vector bundles, i.e. a map $X \to \mathcal{M}$ is equivalent to giving vector bundle $\mathscr{E}$ on X. Suppose it is a scheme. Then, lets consider two maps $\mathbb{P}^1 \to \mathcal{M}$, the first one given by $\mathbb{P}^1 \xrightarrow{\mathcal{O}} \mathcal{M}$ and the second one given by $\mathbb{P}^1 \xrightarrow{\mathcal{O}(1)} \mathcal{M}$.

Then, since $\mathbb{P}^1 = U_1 \cup U_2$ with $U_i = \mathbb{A}^1$, we get

$$\begin{tikzcd} U_1 \arrow[d, "\subseteq"] \arrow[dr, "\mathscr{O}(1)|_{U_1}"] & \\ \mathbb{P}^1 \arrow[r, "\mathscr{O}(1)"] & \mathcal{M} \end{tikzcd}$$

where $\mathscr{O}(1)|_{U_1} \cong \mathscr{O}|_{U_1}$. On the other hand, we also have

$$\begin{tikzcd} U_2 \arrow[d, "\subseteq"] \arrow[dr, "\mathscr{O}(1)|_{U_2}"] & \\ \mathbb{P}^1 \arrow[r, "\mathscr{O}(1)"] & \mathcal{M} \end{tikzcd}$$

where $\mathscr{O}(1)|_{U_2} \cong \mathscr{O}$ as well. However, if $f|_{U_1} = g|_{U_1}$ and $f|_{U_2} = g|_{U_2}$, and if $\mathcal{M}$ were a scheme, then $f = g$, i.e. in our case we get $\mathscr{O} \cong \mathscr{O}(1)$ on $\mathbb{P}^1$, which is a contradiction.

So, the intuition for stacks.

With a moduli space (e.g. plane conic), a map $X \to \mathcal{M}$ corresponds to some geometric object on X. If $\mathcal{M}$ is a scheme, then $\mathrm{Hom}_{(\mathrm{Sch})}(X, \mathcal{M})$ is a set. For stack $\mathcal{M}$, the set $\mathrm{Hom}(X, \mathcal{M})$ will be a category. For example, in the case of vector bundles, $\mathrm{Hom}(X, \mathcal{M})$ just become the category of vector bundles. In this case, if two maps f, g agree on open cover, they might not be the same map, hence the problem occured in the above example is throw out of the window.

To define stacks, there are two main points:

1. Pure category theory (which is roughly analogous to a sheaf, and we call this "categorical stacks"). The main input here is what's called a fibered category.
2. Geometry (which is roughly extra constraints on the "categorical stacks"). This additional geometry makes them algebraic stacks.

So, let's jump into the math. Before we do anything new, let's talk about sheaves.

Recall a presheaf on a topological space X is a choice of sets $\mathscr{F}(U)$ for all $U \subseteq X$ open, and a choice of restriction maps $\rho^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$ whenever $V \subseteq U$, with a compatibility condition: if $W \subseteq V \subseteq U$ then $\rho^V_W \rho^U_V = \rho^U_W$.

We want to reformulate this definition. Another way to saying this is: consider the category $\mathrm{Op}(X)$ whose objects are open subsets $U \subseteq X$, and arrows are inclusions, i.e. $V \to U$ iff $V \subseteq U$. Then a presheaf $\mathscr{F}$ is a functor $\mathrm{Op}(X)^{opp} \to (\mathrm{Sets})$.

Definition 1.3. If $\mathcal{C}$ is any category, a ***presheaf on $\mathcal{C}$*** is defined by

$$\mathrm{Pre}(\mathcal{C}) := \hat{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}^{opp}, (\mathrm{Sets}))$$

where $\mathrm{Fun}(\mathcal{C}, \mathcal{B})$ is the collection of functors.

Theorem 1.4 (Yoneda's Lemma). *Let $\mathcal{C}$ be a category. Then*

$$\mathcal{C} \to \operatorname{Pre}(\mathcal{C})$$

$$X \mapsto h_X := \operatorname{Hom}(-, X)$$

is an embedding, i.e. if $h_X \cong h_Y$ then $X \cong Y$, and

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) = \operatorname{Hom}_{\hat{\mathcal{C}}}(h_X, h_Y) = \{\text{natural trans } h_X \to h_Y\}$$

In fact, we have $\operatorname{Hom}(h_X, \mathscr{F}) = \mathscr{F}(X)$ if $\mathscr{F} \in \operatorname{Fun}(\mathcal{C}^{opp}, (Sets))$.

Before we give a sketch proof, we note this is useful because it allows us to put $\mathcal{C}$ and $\hat{\mathcal{C}}$ objects into a same diagram. For example, rather than writing $h_X \to \mathscr{F}$, we can write $X \mapsto \mathscr{F}$. This is handy, say, given $f \in \mathscr{F}(X)$, we know that this corresponds to $h_X \to \mathscr{F}$, and hence we can write $X(f)$, instead of $h_X(f)$. This is good because, say we have $Y \xrightarrow{F} X$, then let $f \in \mathscr{F}(X)$, we get $F^*(f) \in \mathscr{F}(Y)$. Then we can write a diagram

$$\begin{tikzcd} Y \arrow[r, "F"] \arrow[dr, "F^*(f)"'] & X \arrow[d, "f"] \\ & \mathscr{F} \end{tikzcd}$$

In practice, we might want $\mathscr{F}$ be to a moduli space. Then Yoneda says that $X \to \mathscr{F}$ is the same as $\mathscr{F}(X)$.

Now lets give a sketch proof.

Proof. Notice if $\mathscr{F}(X) = \operatorname{Hom}(h_X, \mathscr{F})$, then in particular, $\operatorname{Hom}(h_X, h_Y) = h_Y(X) = \operatorname{Hom}_{\mathcal{C}}(X,Y)$. Thus we just need to show $\mathscr{F}(X) = \operatorname{Hom}(h_X, \mathscr{F})$. It suffices to show how to go back and forth. Suppose we are given $h_X \xrightarrow{\eta} \mathscr{F}$, we get

$$\eta(X) : h_X(X) = \operatorname{Hom}(X,X) \to \mathscr{F}(X)$$

In particular we get $\operatorname{Id} \in \operatorname{Hom}(X,X)$ and it correspond to

$$(\eta(X))(\operatorname{Id}) \in \mathscr{F}(X)$$

Viz, given $h_x \to \mathscr{F}$, we get an element of $\mathscr{F}(X)$. Conversely, given an element $g \in \mathscr{F}(X)$, we get a map $\eta : h_X \to \mathscr{F}$ defined by, for all $Y \in \mathcal{C}$, $\eta(Y) : h_X(Y) \to \mathscr{F}(Y)$, that $f \in h_X(Y)$ is mapped to $\mathscr{F}(f)(g)$, where we note $f : Y \to X$ implies $\mathscr{F}(f) : \mathscr{F}(X) \to \mathscr{F}(Y)$. One can check $\eta : h_X \to \mathscr{F}$ is a natural transformation, and it is inverse of the first map. ♡

Definition 1.5. We say a presheaf $\mathscr{F} \in \hat{\mathcal{C}}$ is ***representable*** (i.e. a "moduli space") if $\mathscr{F} \cong h_X$ for some $X \in \mathcal{C}$.

Now we have a more general definition of presheaves, what about sheaves in this kind of generality?

To do this, let's look at vanilla definition of sheaves. We say presheaf $\mathscr{F}$ on a topological space X is a sheaf if, whenever $U \subseteq X$ is open, and $U = \bigcup_i U_i$ is an open cover, if we have $f_i \in \mathscr{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, then there exists unique $f \in \mathcal{F}(U)$ so $f|_{U_i} = f_i$.

This is nicely summarized by saying

$$\mathscr{F}(U) \xrightarrow{\rho} \prod_i \mathscr{F}(U_i) \overset{\rho_1}{\underset{\rho_2}{\rightrightarrows}} \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

is an equalizer diagram. Here $\rho(f) = (f|_{U_i}) \in \prod_i \mathscr{F}(U_i)$, and $\rho_1(f_i) = (f_i|_{U_i \cap U_j})$ and $\rho_2(f_i) = (f_j|_{U_i \cap U_j})$. Recall $A \xrightarrow{f} B \overset{h}{\underset{g}{\rightrightarrows}} C$ is an equalizer if $gf = hf$ and for all $A' \xrightarrow{f'} B$ such that $gf' = hf'$ there exists unique $\alpha : A' \to A$ such that $f\alpha = f'$. In other word, we have the following diagrams

$$\begin{tikzcd}
A' \arrow[dr, dashed, "\exists!\alpha"'] \arrow[drr, "f'"] & & & \\
& A \arrow[r, "f"'] & B \arrow[r, shift left, "h"] \arrow[r, shift right, "g"'] & C
\end{tikzcd}$$

In other word, in the sheaf axiom equalizer, if we get $(f_i) \in \mathscr{F}(U_i)$ such that $\rho_1(f_i) = \rho_2(f_i)$, then there exists unique $f \in \mathscr{F}(U)$ such that f maps to (f_i).

Then, Grothendieck's insigh is that, to define sheaves, we don't need toe full strength of topology (we don't need unions!). We just need intersections and a notion of when a collection forms a cover (e.g. $U_i \subseteq U$ and $U = \bigcup_i U$).

By doing this, we get what's called Grothendieck topology, which works for any category. Next time we will define Grothendieck topologies and define sheaves.

One can read (not required): equalizers and limits in categories.

2 Sites

Last time we talked a lot about motivations. Recall if $\mathcal{C}$ is a category, then $\mathrm{Pre}(\mathcal{C}) = \hat{\mathcal{C}}$ denotes the presheaves on $\mathcal{C}$, i.e. they are just functors $F : \mathcal{C}^{opp} \to (\mathrm{Sets})$.

This is a lot of abstraction, and the presheaves we grow up with are when $\mathcal{C} = \mathrm{Op}(X)$, the category of open sets in X. In particular, we sort of need a topology to define presheaves, but this is false, and Grothendieck realized we only need some weaker notion of abstract coverings.

Recall, $\mathscr{F}$ is a sheaf on a topological space X if whenever $U = \bigcup_i U_i$, we have an equalizer

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

In sets, this is actually pretty concrete, i.e. in the category (Sets), we have

$$A \longrightarrow B \underset{g}{\overset{f}{\rightrightarrows}} C$$

is an equalizer iff $A = \{b \in B : f(b) = g(b)\}$.

So, back to the Grothendieck's insight into topology. In particular, he observed we don't need the full union axiom for topologies, while intersection is still needed.

Definition 2.1. Let $\mathcal{C}$ be a category. A ***Grothendieck topology on $\mathcal{C}$*** is: for every $X \in \mathcal{C}$, a particular subset $\mathrm{Cov}(X) \subseteq \mathrm{PowerSet}(\{Y \to X : Y \in \mathcal{C}\})$, which are called the ***covering of*** X. This $\mathrm{Cov}(X)$ must satisfy:

1. if $V \xrightarrow{\sim} X$, then $\{V \xrightarrow{\sim} X\} \in \mathrm{Cov}(X)$
2. if $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $Y \to X$, then the fiber product $Y \times_X X_i$ exists in $\mathcal{C}$, and $\{Y \times_X X_i \to Y\}_{i \in I} \in \mathrm{Cov}(Y)$.
3. if $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $\{V_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(X_i)$, then

$$\{V_{ij} \to X\}_{i \in I, j \in J_i} \in \mathrm{Cov}(X)$$

We note, the three conditions correspond to the traditional topological spaces:

1. the first condition means X is a covering of X itself
2. the second condition means the pullback of covering is a covering
3. the third condition means we can refine coverings

We will see this example below in more details.

Definition 2.2. A category $\mathcal{C}$ with a choice of Grothendieck topology is called a ***site***.

Example 2.3. Let X be a topological space, consider the category $\mathrm{Op}(X)$. Then, we define

$$\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$$

if and only if $\bigcup_{i \in I} U_i = U$. Recall in $\mathrm{Op}(X)$ we have $U_i \to U$ iff $U_i \subseteq U$. This makes $\mathrm{Op}(X)$ an site.

Let's check the three axioms:

1. In our case, $V \xrightarrow{\sim} U$ if and only if $V = U$ (we have double inclusions). This is indeed true, because $U = \bigcup U$.
2. This is a little bit more intersting, but if we actually think about what fibered products are in $\mathrm{Op}(X)$, we would realize it is just intersections. Indeed, suppose we have

$$\begin{array}{ccc} & & V \\ & & \downarrow \\ U & \longrightarrow & X \end{array}$$

then the fibered product is indeed $U \cap V$ as the arrows above are all inclusions. Hence, we see this means, if we have $U = \bigcup_i U_i$ and $V \subseteq U$, then from basic topology we see

$$\{U_i \times_U V \to V\}_{i\in I} = \{U_i \cap V \subseteq V\}_{i\in I}$$

is indeed a cover of V.

3. In our case, this is just $U = \bigcup_i U_i$ and $U_i = \bigcup_j V_{ij}$ then $U = \bigcup_{i,j} V_{ij}$. Hence the third axiom is satisfied.

Definition 2.4. If X is a scheme, then $\mathrm{Op}(X)$ with the above Grothendieck topology is called a ***small Zariski site*** of X.

Of course there is also a big Zariski site. First, let (Sch) be the category of schemes, and (Sch/X) be the category of X-schemes.

Example 2.5 (Big Zariski Site)**.** Let $\mathcal{C} = (\mathrm{Sch/X})$, then we consider the Grothendieck topology obtained by define

$$\{Y_i \to Y\}_{i\in I} \in \mathrm{Cov}(Y \to X)$$

if and only if $Y_i \hookrightarrow Y$ is an open immersion, and $\bigcup_i Y_i = Y$. This is what's called ***big Zariski sites***. We can think of this as, consider all small Zariski sites of Y where $Y \in (\mathrm{Sch/X})$, then sandwich all those small Zariski sites together, we get the big Zariski site of X.

Example 2.6. In general, we can localizing a site. If $\mathcal{C}$ is a site, and $X \in \mathcal{C}$, consider $\mathcal{C}_X$ be the category with objects $Y \to X$ and morphisms being

$$\begin{tikzcd} Y' \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}$$

Then we define the localization on $\mathcal{C}_X$ to be,

$$\left\{ \begin{tikzcd} Y' \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd} \right\} \in \mathrm{Cov}(Y \to X)$$

if and only if $\{Y_i \to Y\}_{i\in I} \in \mathrm{Cov}(Y)$ in $\mathcal{C}$.

In particular, we have big Zariski site of X localized at $Y \to X$ is the big Zariski site of Y.

To make sure we don't lose track, we note the main motivation for this whole business of Grothendieck topology is to get new cohomology theories (e.g. etale cohomology). This is because sheaf cohomology has its problems, e.g. if X is a complex manifold (smooth projective $\mathbb{C}$ scheme), you would like a cohomology theory $H^*(X)$ that recovers topological cohomology.

The reason why Grothendieck want to do this is because he wants to prove the Weil conjectures.

Definition 2.7. A presheaf $\mathscr{F}$ on a site $\mathcal{C}$ is a ***sheaf***, if for all $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, the sequence

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathscr{F}(U_i \times_U U_j)$$

is an equalizer (sometimes we say this is "exact sequence").

If you don't like equalizers, the above can be reformulated concretely. This above definition says, if for all $f_i \in \mathscr{F}(U_i)$ such that the image of f_i in $\mathscr{F}(U_i \times_U U_j)$ equal image of f_j in $\mathscr{F}(U_i \times_U U_j)$ for all i, j, then there exists unique $f \in \mathscr{F}(U)$ such that f_i is equal the image of f in $\mathscr{F}(U_i)$ for all i.

Definition 2.8. A ***presheaf*** $\mathscr{F}$ on a site $\mathcal{C}$ is called ***separated*** if for all $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, $\mathscr{F}(U) \hookrightarrow \prod_i \mathscr{F}(U_i)$ is an injection.

Theorem 2.9. *If $\mathcal{C}$ is a site, then $\mathrm{sh}(\mathcal{C}) := (\text{Sheaves on } \mathcal{C}) \hookrightarrow \mathrm{Pre}(\mathcal{C})$ (by definition this is a full subcategory) has a left adjoint.*

If you are not swimming in the language of category, this can be translated concretely. The above statement says, for all $\mathscr{F} \in \mathrm{Pre}(\mathcal{C})$, there exists $\mathscr{F} \to \mathscr{F}^a$ with $\mathscr{F}^a \in \mathrm{sh}(\mathcal{C})$ such that $\forall \mathscr{G} \in \mathrm{sh}(\mathcal{C})$, we have diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr] & \mathscr{G} \\ & \mathscr{F}^a \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

We call $\mathscr{F}^a$ the ***sheafification*** of $\mathscr{F}$.

In Hartshorne, the idea is that $\mathscr{F}^a(U)$ is going to be a subset of $\prod_{p\in U} \mathscr{F}_p$ such that consists of "compatible germs". However, in sites, we cannot do this at all, because we don't even have a topology, i.e. for us $U \in \mathcal{C}$, it does not have points, so our proof is going to be complicated.

Proof. We will only give sketch. So, we have

$$\mathrm{sh}(\mathcal{C}) \subseteq (\text{Separated presheaves}) \subseteq \mathrm{Pre}(\mathcal{C})$$

and it suffices to show we have each inclusion has a left adjoint. We will only do the left adjoint between separated presheaf and presheaf.

Suppose $\mathscr{F} \in \mathrm{Pre}(\mathcal{C})$ is a presheaf. We want left adjoint $\mathscr{F} \to \mathscr{F}^s$. We define

$$\mathscr{F}^s(U) := \mathscr{F}(U)/\sim$$

where $a, b \in \mathscr{F}(U)$ we say $a \sim b$ if there exists $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$ such that a, b map to same thing in $\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i)$.

Why is $\mathscr{F}^s$ a presheaf? If $V \to U$ and $a, b \in \mathscr{F}(U)$ such that $a \sim b$, then there exists a covering $\{U_i \to U\} \in \mathrm{Cov}(U)$ such that in $\mathscr{F}(U) \to \prod_i \mathscr{F}(U_i)$ we have a, b maps to the same image. Then by axiom 2, we know $\{U_i \times_U V \to V\} \in \mathrm{Cov}(V)$ is a covering of V. Then we have a commutative diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r] \arrow[d, "\rho"] & \prod_i \mathscr{F}(U_i) \arrow[d] \\ \mathscr{F}(V) \arrow[r] & \prod_i \mathscr{F}(U_i \times_U V) \end{tikzcd}$$

since a, b maps to the same image in the above vertical arrow, it maps to the same image in $\prod_i \mathscr{F}(U_i \times_U V)$. However, since the diagram is commutative, this means $\rho(a)$ and $\rho(b)$ must map to the same image in the bottom vertical arrow.

Hence we see $\rho(a) \sim \rho(b)$ and so

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{F}^s(U) \arrow[d, dashed, "\exists!"] \\ \mathscr{F}(V) \arrow[r] & \mathscr{F}^s(V) \end{tikzcd}$$

which shows $\mathscr{F}^s$ is a presheaf. Also, it is clearly separated (by the definition of our equivalent relation). Next it remains to show this is an adjoint, i.e.

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr] & \mathscr{G} \\ & \mathscr{F}^s \arrow[u, "\exists!"'] \end{tikzcd}$$

To that end, just note for each $V \to U$, we get diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "g_U"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \mathscr{F}(V) \arrow[r, "g_V"] & \mathscr{G}(V) \end{tikzcd}$$

we just define the map $\mathscr{F}^s(U) \to \mathscr{G}(U)$ to be $[a] \mapsto g_U(a)$. To see this is well-defined, we just note by the above exact same argument, we get $[a] = [b]$ then $g_U(a) = g_U(b)$ and hence this map $\mathscr{F}^s(U) \to \mathscr{G}(U)$ is well-defined. The functoriality is easy to check.

Next, if $\mathscr{F}$ is separated, we want a sheafification $\mathscr{F}^a$. What we want is that we get the right equalizer. Thus, what we have to have is $\mathscr{F}^a(U)$ must be identified as the elements of $\prod_i \mathscr{F}^a(U_i)$ that gets identified in the product $\prod_{i,j} \mathscr{F}(U_i \times_U U_j)$. But this depends on the covering, hence we define an equivalent relation on all covering.

In particular, we define

$$\mathscr{F}^a(U) := \text{ set of all pairs } \{(\{U_i \to U\}_{i \in I}, \{a_i\}_{i \in I})\} \text{ module } \sim$$

This time, we say

$$(\{U_i \to U\}, \{a_i\}) \sim (\{V_j \to U\}, \{b_j\})$$

iff $a_i|_{U_i \times_U V_j} = b_j|_{U_i \times_U V_j}$ for all i, j.

Thus, we get $\mathscr{F}(U) \to \mathscr{F}^a(U)$ is given by

$$a \mapsto (\{U = U\}, \{a\})$$

Then, we leave it as an exercise to check $\mathscr{F}^a$ is a sheaf and the map is left adjoint. ♡

Remark 2.10. All of this works for sheaves/presheaves valued in groups, modules, rings, etc.

Let's stop talk about categories and do some geometry (of etale morphisms).

So, the idea is that etale morphisms are "covering spaces" (from algebraic topology). There are tons of characterizations of etale morphisms, and we will talk about five of them.

Definition 2.11. A morphism $f : Y \to X$ is ***quasi-finite*** if it is of finite type and for all $x \in X$, the fiber $Y_x := Y \times_X \operatorname{Spec} \kappa(x)$ is a finite set.

Definition 2.12. A morphism $f : X \to Y$ is ***locally quasi-finite*** if for all $y \in Y$, there exists open neighbourhood $y \in U$ such that $f(U) \subseteq V$ and $f|_U : U \to V$ is quasi-finite (equivalently, $f|_U : U \to X$ is quasi-finite).

Example 2.13. Consider $\coprod_{i=1}^{\infty} \operatorname{Spec} K \to \operatorname{Spec} K$ is locally quasi-finite but not quasi-finite. This is because if we take the inverse image of the point in $\operatorname{Spec} K$, you get infinitely many elements in the fiber, but locally this is finite.

Definition 2.14. A morphism $f : Y \to X$ is ***etale*** if f is smooth and locally quasi-finite (i.e. etale is smooth plus relative dimension 0).

Let's consider the key example that we will use through out the course.

Example 2.15. Let K be a field, then $Y \to \operatorname{Spec} K$ is etale iff $Y = \coprod_i \operatorname{Spec} L_i$ where L_i/K is finite separable extension of fields.

Proposition 2.16.

1. *Compostion of etale morphisms are etale, i.e. $f : Z \to Y, g : Y \to X$ are both etale, then $g \circ f$ is etale.*
2. *If $f : Y \to X$ is etale, then the base change is etale, i.e. if we have*

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r, "et"] & X \end{tikzcd}$$

then $Y \times_X Z \to Z$ is etale.

3. If

$$\begin{tikzcd} Z \arrow[r] \arrow[dr, "\text{etale}"'] & Y \arrow[d, "\text{etale}"] \\ & X \end{tikzcd}$$

then $Z \to Y$ is etale.

Example 2.17. We consider the ***small etale site*** of X to be the following category: objects being $Y \to X$ with arrows being etale, and morphisms are etale triangles

$$\begin{tikzcd} Y' \arrow[r, "\text{etale}"] \arrow[dr, "\text{etale}"'] & Y \arrow[d, "\text{etale}"] \\ & X \end{tikzcd}$$

Then, the Grothendieck topology would be, $\{Y_i \xrightarrow{et} Y\} \in \mathrm{Cov}(Y \xrightarrow{et} X)$ iff

$$\coprod_i Y_i \twoheadrightarrow Y$$

is surjective. Here the *et* means etale.

Example 2.18. If $\mathscr{F}$ is a sheaf on the small etale site of $\operatorname{Spec} K$, K a field. Then, suppose L/K is Galois extension with Galois group G. Then

$$\mathscr{F}(\operatorname{Spec} K) = \mathscr{F}(\operatorname{Spec} L \xrightarrow{et} \operatorname{Spec} K)^G$$

which is the G-invariant of $\mathscr{F}(\operatorname{Spec} L)$. Here that $\mathscr{F}(\operatorname{Spec} L \xrightarrow{et} \operatorname{Spec} K)^G$ is just functors on the objects.

3 Etale Maps and Topos

Last time we defined what a site is, what sheaves are on sites, and showed sheafification exists. Then we started talking about etale maps.

Today we continue the brief review of etale maps. Recall etale maps are just smooth map with relative dimension 0 (or smooth and locally quasi-finite).

Then, we covered an example of etale map, where K is a field, then Y/K is etale iff $Y = \prod_{i \in I} \operatorname{Spec} L_i$ with L_i/K finite separable extension of K.

Example 3.1. Suppose $f : \mathbb{A}^1 \to \mathbb{A}^1$ is given by $x \mapsto x^2$. We claim this is not etale. This is clearly quasi-finite, thus we need to show it is not smooth.

Consider the pullback F_0 where $\operatorname{Spec} k \to \mathbb{A}^1$ is the point 0,

$$\begin{tikzcd} F_0 \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

Let's compute what the fibered product is. We see

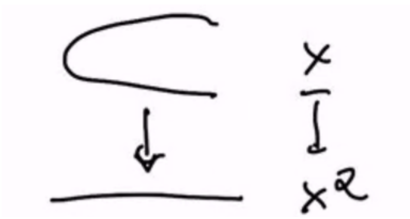
$$F_0 = \operatorname{Spec}(k[x] \otimes_{k[x]} k)$$

but this is equal (isomorphic) to $\operatorname{Spec}(k[x] \otimes_{k[x]} k[x]/(x))$ and hence

$$F_0 = \operatorname{Spec}(k[x]/(x^2))$$

But then F_0 is the dual number and hence it is not a field, which implies we get a map $F_0 \to \operatorname{Spec} k$ that's not etale. Hence the original map $f : \mathbb{A}^1 \to \mathbb{A}^1$ is not etale.

The problem of our map f is that it is ramified at the origin, i.e. we are sandwich the parabola to a single line, and hence the origin is squeezed twice:



Put it in the other perspective, we see $\frac{d}{dx}[x^2] = 2x$ and hence it vanishes at 0. In other word, the module of differential

$$\Omega^1_{\mathbb{A}^1/\mathbb{A}^1} = \frac{k[x]dx}{d(x^2)} = \frac{k[x]dx}{2xdx} \cong \frac{k[x]}{x}$$

Viz, this is a $k[x]$-module supported at 0, i.e. if we specialize x to be non-zero, the module is 0, but if $x \mapsto 0$, the module is non-zero.

The above example actually give an different way to define what etale maps are.

Definition 3.2. If $f : Y \to X$ is locally of finite presentation, then f is ***unramified*** if $\Omega^1_{Y/X} = 0$.

Proposition 3.3. *Let f be locally of finite presentation. Then $f : Y \to X$ is etale if and only if f is smooth and unramified iff f is smooth and $\Omega^1_{Y/X} = 0$.*

Note almost all maps in this course will be locally of finite presentation. Hence we will just drop this assumption.

Proposition 3.4. *The map $f : Y \to X$ is etale iff f locally of finite presentation and for all $y \in Y$, there exists open neighbourhood $y \in U$ and $f(y) \in V$, such that $f(U) \subseteq V$ and $f|_U : U \to V$ is "standard etale", i.e. $V = \operatorname{Spec} R$ and $U = \operatorname{Spec}(R[x]/f)_g$ for some g, where $f \in R[x]$ and $\frac{d}{dx} f$ is a unit in $(R[x]/f)_g$.*

Definition 3.5. We say f is ***formally smooth***, if

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] & Y \arrow[d, "f"] \\ \operatorname{Spec} A/I \arrow[u, hook] \arrow[r] \arrow[ur, dashed, "\exists"] & X \end{tikzcd}$$

where $I \subseteq A$ is an ideal with $I^2 = 0$. We say f is ***formally etale*** if it is formally smooth and the dotted arrow is unique, i.e. we have $\exists!$.

The way to think about this is that, we would have (where $A = k[x]/(x^2)$ and $I = (x)$)

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d, hook] & Y \arrow[d] \\ \operatorname{Spec} k[x]/(x^2) \arrow[r] & X \end{tikzcd}$$

and hence the dotted arrow is just choosing a tangent vector at y, i.e. formal smoothness is every $y \in Y$ can be extended to a tangent vector at y.

The following characterization is the usual definition of etale.

Proposition 3.6. *Let $f : Y \to X$, then f is etale (resp. smooth) iff it is formally etale (resp. formally smooth) and locally of finite presentation.*

Proposition 3.7. *Let $f : Y \to X$, then f is etale iff f is flat and unramified.*

Example 3.8. We define what's called ***small etale site*** of X to be as follows. The objects are $Y \xrightarrow{et} X$, and morphisms are etale triangles. THen the coverings are, $\{Y_i \to Y\}_{\in I} \in \operatorname{Cov}(Y \to X)$ where $\coprod Y_i \twoheadrightarrow Y$.

We also have the ***big etale site*** of X, where now $\mathcal{C} = (\mathrm{Sch})/X$, and the coverings are given by

$$\{Y_i \to Y\} \in \operatorname{Cov}(Y \to X)$$

if $Y_i \xrightarrow{et} Y$ and $\coprod Y_i \twoheadrightarrow Y$.

Example 3.9. Let $\mathscr{F}$ be a sheaf on small etale site of $\operatorname{Spec} K$, K a field. Let L/K be (finite) Galois with group G. Then, suppose we are given the singleton $\{\operatorname{Spec} L \to \operatorname{Spec} K\}$, then this is in $\operatorname{Cov}(\operatorname{Spec} K)$.

Now we know G acts on L over K, thus for all $g \in G$, we get

$$\begin{tikzcd} L \arrow[r, "g", "\cong"'] & L \\ & K \arrow[ul, hook] \arrow[u, hook] \end{tikzcd}$$

Thus we obtain a map

$$\begin{tikzcd} \operatorname{Spec} L \arrow[r, "g"] \arrow[dr] & \operatorname{Spec} L \arrow[d] \\ & \operatorname{Spec} K \end{tikzcd}$$

This is a self map of $\operatorname{Spec} L \to \operatorname{Spec} K$ in the etale site. So since $\mathscr{F}$ is a presheaf, we get $\mathscr{F}(L) \xrightarrow{g^*} \mathscr{F}(L)$. Thus G acts on $\mathscr{F}(L)$.

The fact $\mathscr{F}$ is a sheaf implies

$$\mathscr{F}(K) \longrightarrow \mathscr{F}(L) \rightrightarrows \mathscr{F}(L \otimes_K L)$$

is exact. To know what's going on, we compute $L \otimes_K L$. But first, note L/K is separable, we get $L = K(\alpha)$ for some $\alpha \in L$. The minimal polynomial of α is exactly $f(x) = \prod_{g \in G}(x - g(\alpha)) \in K[x]$. Hence we see

$$\begin{aligned} L \otimes_K L &= L \otimes_K K(\alpha) \\ &= L \otimes_K K(x)/f(x) \\ &= L[x]/f(x) \\ &\cong \bigoplus_{g \in G} L(x)/(x - g(\alpha)) \\ &\cong \bigoplus_{g \in G} L \end{aligned}$$

It turns out, the two maps from L to $L \otimes_K L \cong \bigoplus_{g \in G} L$ are given by, $\beta \mapsto (\beta)_{g \in G}$ and $\beta \mapsto (g(\beta))_{g \in G}$.

Then, since $\mathscr{F}$ is a sheaf, we get

$$\mathscr{F}(K) = \{\gamma \in \mathscr{F}(L) : \text{ under the two maps } \gamma \text{ has same image}\}$$

But $\gamma \in \mathscr{F}(L)$ has the same image under the two maps iff $(\gamma)_{g \in G} = (g(\gamma))_{g \in G}$. Hence

$$\mathscr{F}(K) = \{\gamma \in \mathscr{F}(L) : \forall g, g^*(\gamma) = \gamma\} = \mathscr{F}(L)^G$$

A side note about the above example: there is such a thing called etale fundamental group π_1^{et}, which gives you $\pi_1^{et}(K) = \operatorname{Gal}(\overline{K}/K)$ and

$$\pi_1^{et}(\text{smooth scheme } X \text{ over } \mathbb{C}) = \widehat{\pi}_1(X(\mathbb{C}))$$

where on the right hand side we get profinite completion of the fundamental group of $X(\mathbb{C})$.

Definition 3.10. A ***topos*** is a category $\mathcal{C}$ equivalent to the category of sheaves on a site.

Definition 3.11. A ***morphism of topoi*** $f : T \to T'$ is a triple $f := (f^*, f_*, \phi)$, where

$$f_* : T' \to T$$

$$f^* : T \to T'$$

and these are an adjoint pair with f^* equal the left adjoint of f_*, and $\phi : \mathrm{Hom}_T(f^*(-), -) \xrightarrow{\sim} \mathrm{Hom}_{T'}(-, f_*(-))$ is a choice of isomorphism that's natural in the two dashes such that f^* commutes with finite limits (note by definition of adjoint we have the two sets being isomorphic, and we just make a particular choice of ϕ).

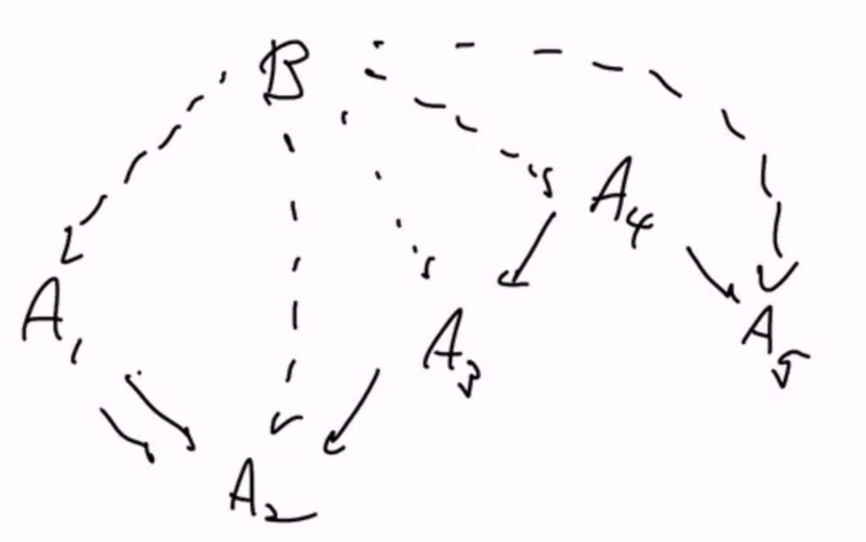
We will not give a full definition of what is the meaning of finite limit, but f^* commutes with finite limits is equivalent to saying that,

$$f^*(\mathscr{F}_1 \times \ldots \times \mathscr{F}_n) = f^*\mathscr{F}_1 \times \ldots \times f^*\mathscr{F}_n$$

and if $\mathscr{F} \to \mathscr{G} \rightrightarrows \mathscr{H}$ is equalizer, then $f^*\mathscr{F} \to f^*\mathscr{G} \rightrightarrows f^*\mathscr{H}$.

The full definition of finite limit is just limits over finite diagrams (hence we don't take infinite diagrams).

For example, if we have the following diagram (the diagram is the ones with the A_i)



Then we say B is the limit if there exists unique dotted arrows making the above diagram commutes, and if C also has dotted arrows, then C factors through B.

So why is that we only neec to check products and equalizer?

Well, because we have another way to construct B: We isolate the source and target (so we isolate all objects that has arrows going out, and all objects that receives arrows), and make product between those, i.e. we have

$$A_1 \times A_3 \times A_4 \rightrightarrows A_2 \times A_5$$

then the limit B is just the equalizer of the above diagram, i.e.

$$B = \mathrm{Eq}\Big(A_1 \times A_3 \times A_4 \rightrightarrows A_2 \times A_5 \Big)$$

Definition 3.12. A ***continuous map*** of sites $f : \mathcal{C} \to \mathcal{C}'$ is a functor such that for all $X \in \mathcal{C}$, for all $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ we have $\{f(X_i) \to f(X)\}_{i\in I} \in \mathrm{Cov}(f(X))$, and f commutes with finite fiber product, if they exists.

Example 3.13. SUppose $f : X \to Y$ be a map between topological spaces. Then we get the corresponding map between categories to be $f^{-1} : \mathrm{Op}(Y) \to \mathrm{Op}(X)$. Then we see $U \subseteq Y$ maps to $f^{-1}(U)$ and hence it is continuous map between the two sites $\mathrm{Op}(Y)$ to $\mathrm{Op}(X)$.

Next time, we will show $f : \mathcal{C} \to \mathcal{C}'$ is continuous then there exists $f_* : T \to T'$.

In particular, we will have the following (but we will not prove it):

Proposition 3.14. *If $f : \mathcal{C}' \to \mathcal{C}$ is continuous map of sites, then $f_* : T' \to T$ has left adjoint $f^* : T \to T'$. If $\mathcal{C}'$ has finite limit and f commutes with finite limits, then f^* commutes with finite limits, i.e. (f^*, f_*) yields $T' \to T$ a map of topoi.*

4 Fppf Sites

First, let's recall last time we defined what's a continuous map between sites. In particular, we say $f : \mathcal{C}' \to \mathcal{C}$ is continuous if it takes covering to covering (in particular in the topological space example, continuous map f between topological spaces correspond to the functor f^{-1}). Let's consider another example of continuous maps.

Example 4.1. If $f : X \to Y$ is a map of schemes. Then we get a map from the small etale site of Y to the small etale site of X given by

$$(Z \xrightarrow{et} Y) \mapsto (Z \times_X Y \xrightarrow{et} X)$$

This is a continuous map of sites by properties of fibered product/pullback.

As a result (by the theorem below), we get a map of topoi $X_{et} \to Y_{et}$.

Example 4.2. The above example is not a single case. We can also use pullback to define continuous maps from small Zariski site on X to the small etale site of X, by the map $U \subseteq X \mapsto U \subseteq X$. In particular, we get a map of topoi $X_{et} \to X_{Zar}$

Last time we stated a proposition about how continuous maps induce map between topoi.

Let's now define pushforward on topoi. Given $f : \mathcal{C}' \to \mathcal{C}$ continuous map of sites. We get

$$f_* : T \to T'$$

on topoi by

$$(f_*\mathscr{F})(X') = \mathscr{F}(f(X'))$$

We note unlike pushforward for schemes, where we define $f_*\mathscr{F}(U) := \mathscr{F}(f^{-1}(U))$, we used $f(X')$. This is because f itself in our context should be thought as an "inverse" (think of the topological space example).

So why is $f_*\mathscr{F}$ a sheaf?

If $\{X'_i \to X'\}$ is a cover of X', then we want to check the following sequence is exact

$$(f_*\mathscr{F})(X') \longrightarrow \prod_{i\in I} f_*\mathscr{F}(X'_i) \rightrightarrows \prod_{i,j} f_*\mathscr{F}(X'_i \times_X X'_j)$$

However, by definition, the above sequence is the same as

$$\mathscr{F}(f(X')) \longrightarrow \prod_{i\in I} \mathscr{F}(f(X'_i)) \rightrightarrows \prod_{i,j} \mathscr{F}(f(X'_i \times_X X'_j))$$

but f commutes with finite fiber product, and hence

$$\mathscr{F}(f(X'_i \times_X X'_j)) = \mathscr{F}(f(X'_i) \times_{f(X)} f(X'_j))$$

However, now we see the whole sequence is exact because $\{f(X'_i) \to f(X')\}$ is a covering of $f(X')$.

Theorem 4.3. *If $f : \mathcal{C}' \to \mathcal{C}$ is continuous map of sites. Then $f_* : T \to T'$ has a left adjoint $f^* : T' \to T$. Moreover, if $\mathcal{C}'$ has finite limits and f commutes with finite limits, then f^* commutes with finite limits. In particular, (f^*, f_*) is a map of topoi $T \to T'$.*

We will not prove this, but will mention how to construct f^*.

We just need to construct a left adjoint of f_*. Note f_* also gives a map $\mathrm{Pre}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{C}')$, thus if we can construct a adjoint of $f_* : \mathrm{Pre}(\mathcal{C}) \to \mathrm{Pre}(\mathcal{C}')$, because if f^*_{Pre} is the left adjoint then take f^* be the sheafification $(f^*_{\mathrm{Pre}})^a$ (we composed two left adjoints, so the whole thing is left adjoint).

The construction of f^* is also similar to what we do in baby algebraic geometry.

In particular, we take

$$(f^*\mathscr{F})(U) = \varinjlim_{U'} \mathscr{F}(U')$$

where the colimit is taken over a diagram with objects (U', ρ) where $\rho : U \to f(U')$

Remark 4.4. If $X' \in \mathcal{C}'$, then we get representable functor $h_{X'} \in \operatorname{Pre}(\mathcal{C}')$. Thus we get $h^a_{X'} \in T'$ is a sheaf in the topos. Thus we see $f^*(h^a_{X'}) = h^a_{f(X')}$ if $f : \mathcal{C}' \to \mathcal{C}$ is a continuous map of sites.

Let's prove this, as this is just unravel definitions. Note

$$\operatorname{Hom}_T(f^*(h^a_{X'}), \mathscr{F}) = \operatorname{Hom}_{T'}(h^a_{X'}, f_*\mathscr{F}) = \operatorname{Hom}_{\operatorname{Pre}(\mathcal{C}')}(h_{X'}, f_*\mathscr{F})$$

where on the first equality we used f_* and f^* are adjoints, and the second equality is because sheafification is adjoint. Next, by Yoneda, we get

$$\operatorname{Hom}_{\operatorname{Pre}(\mathcal{C}')}(h_{X'}, f_*\mathscr{F}) = (f_*\mathscr{F})(X')$$

However, $(f_*\mathscr{F})(X') = \mathscr{F}(f(X'))$ and Yoneda again we get

$$\mathscr{F}(f(X')) = \operatorname{Hom}_{\operatorname{Pre}(\mathcal{C})}(h_{f(X')}, \mathscr{F})$$

Now use the same trick, we observe

$$\operatorname{Hom}_{\operatorname{Pre}(\mathcal{C})}(h_{f(X')}, \mathscr{F}) = \operatorname{Hom}_T(h^a_{f(X')}, \mathscr{F})$$

and hence we get $h^a_{f(X')} = f^*(h^a_{X'})$.

This is about enough category for today, but before geometry, we need to define one more site, the fppf site.

Definition 4.5. We say $A \to B$ is ***of finite presentation*** if B is a finitely generated A-algebra and

$$A^m \to A^n \to B \to 0$$

is exact with n, m finite, i.e. the kernel is finitely generated.

We also have another formulation. Note if we assume B is f.g. A-algebra, then we get surjection $\pi : A[x_1, ..., x_n] \twoheadrightarrow B$. Then we say $A \to B$ is of finite presentation if $\operatorname{Ker}(\pi)$ is f.g. ideal of $A[x_1, ..., x_n]$. In other word, we should think of finite presentation as being finitely generated A-algebra with finitely many relations (i.e. B is f.g. A-algebra means finite many generators, and B is of finite presentation means B has finite number of relations on the finite number of generators).

We can generalize this to schemes.

Definition 4.6. We say $f : X \to Y$ is ***locally of finite presentation*** if for all $y \in Y$, there exists open neighbourhood $U = \operatorname{Spec} A \subseteq Y$ of y and $f^{-1}(U)$ has an open affine cover $\bigcup V_i = \bigcup \operatorname{Spec} B_i$ such that $A \to B_i$ is of finite presentation.

Definition 4.7. $f : X \to Y$ is ***faithfully flat*** if f is flat and surjective.

Definition 4.8. $f : X \to Y$ is ***fppf*** if f is faithfully flat and locally of finite presentation.

Example 4.9. If X is a scheme, the ***fppf site*** has category (Sch/X) and coverings

$$\{Y_i \to Y\} \in \mathrm{Cov}(Y)$$

iff each $Y_i \to Y$ is flat and locally of finite presentation, and together we have surjection

$$\coprod_i Y_i \twoheadrightarrow Y$$

The fppf sites play a very very important role in algebraic geometry. One reason is that we have what's called faithfully flat descent.

The idea is that, let P be a property of morphisms of schemes. Then we get commutative diagram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, two heads, "fppf"] & X \end{tikzcd}$$

Then, frequently, f has P iff f' has P. This is what's called faithfully flat descent.

We are familiar with the fact that if $X = \bigcup U_i$ is open covering, then $f : Y \to X$ has property P iff $f|_{f^{-1}(U)}$ has property P for all i. This is in fact an example of faithfully flat descent (in the Zariski topology).

Remark 4.10. We note Zariski cover is a subset of Etale cover and Etale cover is a subset of fppf cover. Hence because faithfully flat descent on fppf covers, then it holds for Etale and Zariski as well.

Remark 4.11. We get maps on topoi:

$$X_{fppf} \to X_{et} \to X_{zar}$$

We will want to work up our way to faithfully flat descent, and the first thing we do is give tons of characterizations.

Definition 4.12. A module M is:

1. ***flat*** if $- \otimes_R M$ is exact functor.
2. ***faithfully flat*** if $N' \to N \to N''$ is exact iff $N' \otimes_R M \to N \otimes_R M \to N'' \otimes_R M$ is exact.

Proposition 4.13. *The following are equivalent:*

1. *M is flat and we have injection $\mathrm{Hom}_R(N, N') \hookrightarrow \mathrm{Hom}(N \otimes M, N' \otimes M)$ for all N, N'.*
2. *M is flat and for all N', we have injection $N' \hookrightarrow \mathrm{Hom}(M, N' \otimes M)$ by $y \mapsto (y \mapsto y \otimes m)$.*
3. *M is faithfully flat.*
4. *$N' \to N$ is injection iff $N' \otimes M \to N \otimes M$ is injection.*
5. *M is flat and $N \otimes M = 0$ then $N = 0$.*

6. M *is flat and for all maximal ideal* $\mathfrak{m} \subseteq R$, $M/\mathfrak{m}M \neq 0$.
7. M *is flat and for all prime ideal* $\mathfrak{p} \subseteq R$, $M/\mathfrak{p}M \neq 0$.

Proof. (1) $\Leftrightarrow$ (2): If $F \twoheadrightarrow N$, then $F \otimes M \twoheadrightarrow N \otimes M$. Thus we get

$$\begin{tikzcd} \mathrm{Hom}(N, N'), \arrow[r, "\phi_{N,N'}"] \arrow[d, hook] & \mathrm{Hom}(N \otimes M, N' \otimes M) \arrow[d, hook] \\ \mathrm{Hom}(F, N') \arrow[r, "\phi_{F,N'}"] & \mathrm{Hom}(F \otimes M, N' \otimes M) \end{tikzcd}$$

where we have injections because $F \otimes M \twoheadrightarrow N \otimes M$ is surjection. We always can take $F = R^I$ to be a free module, thus $\phi_{N,N'}$ is injection iff $\phi_{F,N'}$ is injection, i.e. (1) is equivalent to checking (1) when $N = R^I$ is free.

However, $\phi_{N,N'} : \mathrm{Hom}(R^I, N') \to \mathrm{Hom}(R^I \otimes M, R^I \otimes N')$ is just

$$\phi_{N,N'} = \prod_{i \in I} \mathrm{Hom}(R, N') \to \prod_{i \in I} \mathrm{Hom}(M, R^I \otimes N')$$

But if we think what is this map doing, we can assume $|I| = 1$ as all factors are preserved in products. Thus we have

$$N' \xrightarrow{\sim} \mathrm{Hom}(R, N') \to \mathrm{Hom}(M, M \otimes N')$$

and this is exactly the map

$$y \mapsto (1 \mapsto y) \mapsto (m \mapsto m \otimes y)$$

This concludes (1) holds iff (2) holds.

Let's go from (2) to (4). If M is flat, then $N \hookrightarrow N'$ implies $N \otimes M \hookrightarrow M \otimes N'$. So, we have to show the converse, i.e. if $N \to N'$ and $N \otimes M \hookrightarrow N' \otimes M$ then $N \hookrightarrow N'$.

We see

$$\begin{tikzcd} N \arrow[r, hook, "\text{by (2)}"] \arrow[d] & \mathrm{Hom}(M, N \otimes M) \arrow[d] \\ N' \arrow[r, hook, "\text{by (2)}"] & \mathrm{Hom}(M, N' \otimes M) \end{tikzcd}$$

but the right vertical arrow is injective since $N \otimes M \hookrightarrow N' \otimes M$. This forces $N \to N'$ to be injective as the diagram commutes.

(4) $\to$ (5): Suppose $N \otimes M = 0$, then tensor the map $N \to 0$ by M, we get $N \otimes M = 0 \to 0$ is injective. Thus $N \to 0$ is injective and hence $N = 0$ as desired.

(5) $\Rightarrow$ (7): Let $N = R/\mathfrak{p} \neq 0$ if $\mathfrak{p}$ is prime. Thus $N \otimes M \neq 0$ where $N \otimes M = M/\mathfrak{p}M$.

(7) $\Rightarrow$ (6): Maximal ideals are prime.

(5) $\Leftarrow$ (3): If $N \otimes M = 0$, then consider the sequence $(0 \to N \to 0)$ and tensor with M we get exact sequence $0 \to 0 \to 0$, hence the original sequence $0 \to N \to 0$ must be exact, hence $N = 0$ as desired.

(5) ⇒ (3): We have $N' \xrightarrow{\alpha} N \xrightarrow{\beta} N''$ and exact sequence

$$N' \otimes M \xrightarrow{\alpha'} N \otimes M \xrightarrow{\beta'} N'' \otimes M$$

Let H be the cohomology

$$H := \frac{\operatorname{Ker}(\beta)}{\operatorname{Im}(\alpha)}$$

We want $H = 0$. However, since M is flat, we have

$$H \otimes M = \frac{\operatorname{Ker}\beta'}{\operatorname{Im}\alpha'} = 0$$

and hence $H = 0$ as desired.

It remains to show (6) ⇒ (2). We do this by contradiction. Suppose there exists N with $N \to \operatorname{Hom}(M, N \otimes M)$ by $y \mapsto (m \mapsto y \otimes m)$ is not injective, i.e. $\exists x \in N$ such that $x \otimes m = 0$ for all $m \in M$. We will produce a maximal ideal with the quotient equal 0.

Let $L \subseteq N$ be the submodule generated by x, i.e. $L = Rx$. Then we let $\mathcal{L} = \operatorname{Ker}(R \to L)$ where the map $R \to L$ is given by $1 \mapsto x$. Thus we see $L = R/\mathcal{L}$. We will show we can replace N by L.

Note $L \hookrightarrow N$, and since M is flat, we see $L \otimes M \hookrightarrow N \otimes M$. We see $x \otimes m = 0$ when x is viewed as element of N. Hence $x \otimes m = 0$ when x is viewed as element of L. But x generates L, hence $L \otimes M = 0$. Thus we see $0 = L \otimes M = R/\mathcal{L} \otimes M = M/\mathcal{L}M$. If $\mathfrak{m} \supseteq \mathcal{L}$ is the maximal ideal, then $M/\mathfrak{m}M = 0$, which is a contradiction.

Next time we will show (6) ⇒ (2). We will use ♡

Corollary 4.13.1. *$\operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A$ is faithfully flat iff B is a faithfully flat A-module.*

Proof. By definition, f is flat iff B is flat A-module. So, f is flat and surjective iff B is flat and if we do base change at closed point $\operatorname{Spec} A/\mathfrak{p} \to \operatorname{Spec} A$, we get non-empty the fibered product is non-empty, i.e. we have

$$\begin{array}{ccc} \neq \varnothing & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \\ \operatorname{Spec} A/\mathfrak{m} & \longrightarrow & \operatorname{Spec} A \end{array}$$

But this is condition (6) and we are done. ♡

5 Fppf Sites

Last time we talked about tons of characterizations of flat modules. We get the following theorem from the proposition.

Theorem 5.1 (Chevalley). *If $f : X \to Y$ is locally of finite presentation, then $f(U)$ is constructible if U is constructible. In particular, $f(U)$ is constructible if U is open.*

Theorem 5.2. *If $f : X \to Y$ is flat and locally of finite presentation, then f is open, i.e. $f(U)$ is open if U is open.*

Proof. We are going to prove this, but we need to import a block box theorem, i.e. the Chevalley's theorem. The statement is recorded above. In particular, all we need from Chevalley's theorem is that $f(U)$ is constructible when U is open.

We also need another fact from topology: let $E \subseteq Y$ be a constructible set. Assume E is stable under generalization, i.e. if y' specializes to y ($y \in \overline{\{y'\}}$) and $y \in E$, then $y' \in E$. Then E is open, i.e. stable under generalization+constructible implies open.

As a corollary of the above facts, we see if $f : X \to Y$ locally of finite presentation and $U \subseteq X$ open and $f(U)$ is stable under generalization, then $f(U)$ is open. Now let's prove the theorem.

Let $U \subseteq X$ be open, we just need to show $f(U)$ is stable under generalization. We can reduce this to the local case and assume $f : \operatorname{Spec} B \to \operatorname{Spec} A$. We have $y \in f(U)$ and $y \in \overline{\{y'\}}$. Thus, say $f(x) = y$. We want to find x' specialize to x such that $f(x') = y'$.

Thus, we have

$$\begin{tikzcd} x \arrow[d] & x' \arrow[l, dotted] \arrow[d, dotted] \\ y & y' \arrow[l] \end{tikzcd}$$

where the dotted arrows are what we wanted. In particular, this means we get $\mathfrak{q}$ in B that's lies above $\mathfrak{p}$ that correspond to y, and we get $\mathfrak{p}' \subseteq \mathfrak{p}$ that correspond to y'. We want $\mathfrak{q} \supseteq \mathfrak{q}'$ such that $\mathfrak{q}'$ lies over $\mathfrak{p}'$. This is known as "going down".

In other word, after localizing, we get

$$\begin{tikzcd} B_{\mathfrak{q}} & \\ A_{\mathfrak{p}} \arrow[u, "flat"] & \mathfrak{p}' A_{\mathfrak{p}} \arrow[l, "\supseteq"'] \end{tikzcd}$$

and we want a prime of $B_{\mathfrak{q}}$ lying over $\mathfrak{p}' A_{\mathfrak{p}}$. This is the same as

$$\begin{tikzcd} \operatorname{Spec} B_{\mathfrak{q}} \arrow[d, "g"] \\ \operatorname{Spec} A_{\mathfrak{p}} \end{tikzcd}$$

and we want $\mathfrak{p}' \in \operatorname{Im}(g)$, i.e. we want g to be surjective.

However, g is flat and $\operatorname{Spec} A_{\mathfrak{p}}$ has only one closed point and g surjects on that closed point. Thus, by the equivalence of (6) and (7) of the proposition we proved, g is surjective on all points. ♡

Corollary 5.2.1. *If $f : X \to Y$ is fppf, and $U \subseteq Y$ is open and quasi-compact (e.g. affine). Then there exists open cover $f^{-1}(U) = \bigcup_j V_j$ with V_j are quasi-compact and $f(V_j) = U$.*

Proof. It is enough to show every $x \in f^{-1}(U)$ has open quasi-compact neighbourhood V with $f(V) = U$.

Choose affine neighbourhood W' of x with $W' \subseteq f^{-1}(U)$. Choose affine cover $\bigcup W_i = f^{-1}(U)$. We see f is open means $f(W_i)$ are open. Since f is surjective, we see $U = f(f^{-1}(U))$. Thus $U = \bigcup f(W_i)$. However, since U is quasi-compact, so we can assume the index set is finite. Let $V := W' \cup \bigcup_{i \in I} W_i$, we see this is a finite union of affine, so in particular it is quasi-compact. ♡

Definition 5.3. We say a property P of morphisms of schemes is ***local on the base (target) for the fppf (et, Zar, etc.) topology***, if for all Cartesian diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, two heads, "fppf"] & S \end{tikzcd}$$

if f' has P then f has P.

Theorem 5.4. *The following properties are local on the base for the fppf topology: surjective, locally of finite type, locally of finite presentation, of finite type, of finite presentation, universally closed, universally open, separated, proper, unramified, smooth, etale, flat, affine, isomorphism, open immersion, closed immersion, finite, locally quasi-finite, quasi-finite, quasi-compact, quasi-separated, universally injective,universally homeomorphism, and the list goes on.*

Proof. The whole proof is on Stacks Project, Tag 02YJ.

We will only prove some of these.

Universally Closed: we need to prove that in the following diagram

$$\begin{tikzcd} X_T \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r, "\forall"] & S \end{tikzcd}$$

that $X_T \to T$ is closed for all base change. To do this, we consider the following cube of Cartesian:

$$\begin{tikzcd}
X'_{T'} \arrow[rr] \arrow[dd, "g'"] \arrow[dr] & & X_T \arrow[dd] \arrow[dr] & \\
& X' \arrow[rr, "g"] \arrow[dd] & & X \arrow[dd, "f"] \\
T' \arrow[rr, "f'"] \arrow[dr] & & T \arrow[dr] & \\
& S' \arrow[rr] & & S
\end{tikzcd}$$

In the above, all squares are Cartesian. Thus f' is universally closed implies g' is closed. Hence, we get

$$\begin{tikzcd}
X'_{T'} \arrow[r, two heads, "\pi'"] \arrow[d, "g'"] & X_T \arrow[d, "g"] \\
T' \arrow[r, two heads, "\pi \quad fppf"] & T
\end{tikzcd}$$

and we want to show g' is closed. In particular, π is fppf implies π is open.

Thus, π surjective and open, we see if $W \subseteq T$, then W is closed iff $\pi^{-1}(W)$ is closed. Thus take $Z \subseteq X_T$ closed, we want $g(Z)$ to be closed. It is enough to show $\pi^{-1}(g(Z))$ is closed. Take

$$Z' \coloneqq (\pi')^{-1}(Z) = \{(t', z) : z \in Z, \pi(t') = g(z)\}$$

We see Z' is closed and

$$g'(Z') = \{t' : \exists z \in Z \text{ with } \pi(t') = g(z)\}$$

but we see this is just

$$g'(Z') = \pi^{-1}(g(Z))$$

Thus we see $g'(Z')$ is closed because g' is a closed map. This concludes our proof.

Separated: Recall $f : X \to S$ is separated means $\Delta_{X/S} : X \to X \times_S X$ is closed immersion. But Δ is always immersion, thus we just need to show $\Delta_{X/S}$ is closed.

Consider Cartesian diagram

$$\begin{tikzcd}
X_T \arrow[r] \arrow[d] & X \arrow[d] \\
T \arrow[r] & S
\end{tikzcd}$$

We get another diagram

$$\begin{tikzcd}
X_T \arrow[r] \arrow[d, "\Delta_{X_T/T}"] & X \arrow[d, "\Delta_{X/S}"] \\
X_T \times_T X_T \arrow[r] \arrow[d] & X \times_S X \arrow[d] \\
T \arrow[r] & S
\end{tikzcd}$$

Since separated maps base change to be separated, saying $\Delta_{X/S}$ is closed is the same as saying $\Delta_{X/S}$ being universally closed.

Thus, we get

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\Delta_{X'/S'}"] & X \arrow[d, "\Delta_{X/S}"] \\ X' \times_{S'} X' \arrow[r] \arrow[d] & X_S \times X \arrow[d] \\ S' \arrow[r, "fppf"] & S \end{tikzcd}$$

but the bottom arrow is fppf, and hence we see $\Delta_{X'/S'}$ is universally closed implies $\Delta_{X/S}$ is universally closed.

Locally of Finite type: we can reduce to the affine case

$$\begin{tikzcd} \operatorname{Spec} B' \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} A' \arrow[r, "fppf"] & \operatorname{Spec} A \end{tikzcd}$$

where we have B' is f.g. A-algebra (and the diagram is Cartesian). Let $y'_1, \ldots, y'_m \in B'$ be generators. We see $B' = A' \otimes_A B$ and hence $y'_i = \sum_j a'_{ij} \otimes x_{ij}$. Let $C \subseteq B$ be the A-algebra generated by the x_{ij}.

By flatness of $A \to A'$, we see

$$C \otimes_A A' \hookrightarrow B \otimes_A A' = B'$$

Thus we see $C \otimes_A A' = B'$. By faithful flatness, we see this forces $C = B$.

Quasi-compact: easy.

Proper: We have

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "fppf"] & S \end{tikzcd}$$

with f' proper. However, recall proper is the same as universally closed, separated, locally of finite type and quasi-compact. Thus we are done.

♡

Next, we will show if X is a scheme, then h_X is sheaf for the fppf topology.

We expand the definitions to see what this means. We see we need to work with fppf cover $\{Z_i \to Z\}_{i \in I}$, where $Z_i \to Z$ flat locally of finite presentation and $\coprod Z_i \to Z$ surjective. Then we need to show $h_X : (\mathrm{Sch}/\operatorname{Spec} \mathbb{Z})^{opp}_{fppf} \to (\mathrm{Sets})$ is a functor, which says it is presheaf. We also need to show exact sequence

$$h_X(Z) \longrightarrow \prod_i h_X(Z_i) \rightrightarrows \prod_{i,j} h_X(Z_i \times_Z Z_j)$$

Then, h_X sheaf for fppf topology (i.e. we get the above exact sequence) means we want

$$\begin{tikzcd}
Z_{ij} = Z_i \times_Z Z_j \arrow[r] \arrow[d] & Z_i \arrow[d] \arrow[ddr, bend left] & \\
Z_j \arrow[r] \arrow[drr, bend right] & Z \arrow[dr, dashed, "\exists!"] & \\
& & X
\end{tikzcd}$$

the unique dashed arrow from Z to X.

Remark 5.5. To see those two are the same, we expand what it means to have $h_X(Z) = \mathrm{Eq}(\ \prod_i h_X(Z_i) \rightrightarrows \prod_{i,j} h_X(Z_i \times_Z Z_j)\ )$. Note $h_X(Z) = \mathrm{Hom}(Z, X)$, $h_X(Z_i) = \mathrm{Hom}(Z_i, X)$, and $h_X(Z_{ij}) = \mathrm{Hom}(Z_{ij}, X)$. Since products commutes with what we are doing, we just need to check on a pair of indices i, j. Thus, we get $Z_i \to Z$ and $Z_j \to Z$ by assumption (from the covering), and $Z_{ij} \to Z_i$ and $Z_{ij} \to Z_j$. Then the two arrows $\prod_i h_X(Z_i) \rightrightarrows \prod_{ij} h_X(Z_{ij})$ correspond to the composition $Z_{ij} \to Z_i \to X$ and $Z_{ij} \to Z_j \to X$. Now we want to say that every such pair of arrows come from a unique arrow in $h_X(Z)$, i.e. we want unique $Z \to X$ so the above diagram commutes, as desired.

A special case of this is that, $Z = \bigcup Z_i$ open cover and $f : Z \to X$ is equivalent to $f_i : Z_i \to X$ such that $f_i|_{Z_{ij}} = f_j|_{Z_{ij}}$.

6 Fppf Sites

The big theorem we are working towards at this point is that, if X is a scheme, then h_X is a sheaf for the fppf site on all schemes, i.e. the fppf site on $\mathrm{Spec}\,\mathbb{Z}$. In particular, if $X \to Y$ then h_X is a sheaf on the fppf site on (Sch/Y).

At the end of last lecture, we talked about what it means for h_X be a sheaf for fppf topology.

Example 6.1. Let L/K be a Galois field extension with Galois group G. Then $\mathrm{Spec}\, L \to \mathrm{Spec}\, K$ is etale and hence fppf. We showed for any sheaf $\mathscr{F}$, we see $\mathscr{F}(K) = \mathscr{F}(L)^L$. In particular, taking $\mathscr{F} = h_X$, we see a morphism $\mathrm{Spec}\, K \to X$ is the same as a G-invariant morphism $\mathrm{Spec}\, L \to X$.

Since Zariski and etale covers are examples of fppf covers, the big theorem also says h_X is a sheaf for the (big) etale and Zariski topologies. We will start prove the theorem.

Proposition 6.2. *If $A \to B$ is faithful flat and M is A-module, then*

$$M \xrightarrow{f} M \otimes_A B \underset{p_2}{\overset{p_1}{\rightrightarrows}} M \otimes_A B \otimes_A B$$

is exact. Here the maps are given by $f : m \mapsto m \otimes 1$ and $p_1 : m \otimes b \mapsto m \otimes b \otimes 1$ and $p_2 : m \otimes b \mapsto m \otimes 1 \otimes b$.

Proof. Exactness is equivalent to exactness of $0 \to M \xrightarrow{f} M \otimes B \xrightarrow{p_1 - p_2} M \otimes B \otimes B$. Thus it is enough to show (since $A \to B$ is faithfully flat, the old sequence is exact iff we tensor with B) that

$$\begin{tikzcd} M \otimes_A B \arrow[r, "f' := f \otimes \mathrm{Id}"] & M \otimes B \otimes B \arrow[r, shift left, "p_1'"] \arrow[r, shift right, "p_2'"'] & M \otimes B \otimes B \otimes B \end{tikzcd}$$

where

$$p_1'(m \otimes b \otimes b') = m \otimes b \otimes 1 \otimes b'$$
$$p_2'(m \otimes b \otimes b') = m \otimes 1 \otimes b \otimes b'$$

and $f'(m \otimes b) = f(m) \otimes \mathrm{Id}(b) = m \otimes 1 \otimes b$. The point of doing this is that now we get a section, i.e. we have

$$\begin{tikzcd} M \otimes B & M \otimes B \otimes B \arrow[l, "\gamma"'] & M \otimes B \otimes B \otimes B \arrow[l, "\tau"'] \end{tikzcd}$$

where

$$\tau(m \otimes b \otimes b' \otimes b'') = m \otimes b \otimes b'b''$$
$$\gamma(m \otimes b \otimes b') = m \otimes bb'$$

In particular, we get $\tau p_1' = \mathrm{Id}_{M \otimes B \otimes B}$ and $\tau p_2' = f'\gamma$. Indeed,

$$\tau p_1'(m \otimes b \otimes b') = \tau(m \otimes b \otimes 1 \otimes b') = m \otimes b \otimes 1b' = m \otimes b \otimes b'$$

$$\tau p_2'(m \otimes b \otimes b') = \tau(m \otimes 1 \otimes b \otimes b') = m \otimes 1 \otimes bb'$$

where we note

$$f'\gamma(m \otimes b \otimes b') = f'(m \otimes bb') = m \otimes 1 \otimes bb'$$

One can also check $\gamma f' = \mathrm{Id}$. Hence, we indeed get a section which implies f' is injective as desired (which proves exactness on the left).

Next, we check exactness on the middle. Suppose $\alpha \in M \otimes B \otimes B$, then we have $p_1'(\alpha) = p_2'(\alpha)$. In particular, we get

$$\tau p_1'(\alpha) = \tau p_2'(\alpha) = f'(\gamma(\alpha)) = \alpha$$

This concludes the exactness on the middle as well. ♡

Corollary 6.2.1. *If U, V, X are affine schemes. If $V \twoheadrightarrow U$ is fppf cover, then we get exact sequence*

$$\begin{tikzcd} h_X(U) \arrow[r] & h_X(V) \arrow[r, shift left] \arrow[r, shift right] & h_X(V \times_U V) \end{tikzcd}$$

Proof. Say $U = \operatorname{Spec} A, V = \operatorname{Spec} B$ and $X = \operatorname{Spec} R$. Then t ake $M = A$ in previous proposition, we get

$$A \xrightarrow{\iota} B \rightrightarrows B \otimes_A B$$

is exact. In particular, ι is injective. We want that, when we take $\operatorname{Hom}(R, -)$ to the above sequence, we get exact sequence, i.e. we want to show the following sequence is exact

$$\operatorname{Hom}(R, A) \longrightarrow \operatorname{Hom}(R, B) \rightrightarrows \operatorname{Hom}(R, B \otimes_A B)$$

Exact on the left: Suppose we have $R \underset{\beta}{\overset{\alpha}{\rightrightarrows}} A \xrightarrow{\iota} B$ with $\iota\alpha = \iota\beta$ and ι injective. However, this implies $\alpha = \beta$ as ι is injective, as desired.

Exact on the middle: Say $f : R \to B$ such that for all $v \in R$, $f(r) \otimes 1 = 1 \otimes f(r)$. By previous proposition, we know

$$f(r) \in A \subseteq B$$

Hence f factors through A, which proves our claim. ♡

Lemma 6.3. *Let $\mathscr{F} : (Sch)^{opp} \to (Sets)$ be a big Zariski sheaf. Then $\mathscr{F}$ is a sheaf for fppf topology iff for all $V \twoheadrightarrow U$ fppf, we get exact sequence*

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V)$$

Proof. Clearly if $\mathscr{F}$ is sheaf for fppf topology, then we get the desired exact sequence. We just need to show the converse.

Let $\{U_i \to U\} \in \operatorname{Cov}(U)$ be fppf cover. Let $V = \coprod U_i$. Then we get a sequence

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j)$$

However, we can complete the above diagram to

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] & \prod_i \mathscr{F}(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{ij} \mathscr{F}(U_i \times_U U_j) \\
\mathscr{F}(U) \arrow[u, "="] \arrow[r] & \mathscr{F}(V) \arrow[u, "\sim"] \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}(V \times_U V) \arrow[u, "\sim"]
\end{tikzcd}$$

The vertical maps are isomorphisms because U_i form open (Zariski) cover of V, and $U_i \cap U_j = \varnothing$ in V. Thus, if we assume the top row is exact, then we can indeed conclude the bottom row is exact, which implies $\mathscr{F}$ is sheaf for fppf topology, as desired.

♡

Lemma 6.4. *Let $\mathscr{F} : (Sch)^{opp} \to (Sets)$ be a presheaf. Assume $\mathscr{F}$ is a sheaf for the big Zariski topology. Then $\mathscr{F}$ is fppf sheaf iff for all $V \twoheadrightarrow U$ fppf, V, U affine, we have*

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V)$$

is exact.

Proof. By previous lemma, it is enough to check sheaf axioms on $V \twoheadrightarrow U$ not necessarily affine but singleton covers.

Exactness on the left: We need to show $\mathscr{F}(U)$ injects into $\mathscr{F}(V)$. Let $U = \bigcup U_i$ be open affine cover and $f : V \twoheadrightarrow U$ be fppf. Then $f^{-1}(U_i) = \bigcup_j V_{ij}$ be an open affine cover. Hence $V = \bigcup_{ij} V_{ij}$. We get the following diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r] \arrow[d, hook] & \mathscr{F}(V) \arrow[d, hook] \\ \prod_i \mathscr{F}(U_i) \arrow[r] & \prod_{i,j} \mathscr{F}(V_{ij}) \end{tikzcd}$$

Note the vertical arrows are injective since $\mathscr{F}$ is a Zariski sheaf. Thus, to prove the exactness on the left, it is enough to show the bottom arrow is injective. Since f is flat locally of finite presentation, we see f is open. Thus $f(V_{ij})$ are open in U_i. Also, f is surjective as its fppf, we see

$$U_i = f(f^{-1}(U_i)) = \bigcup_j f(V_{ij})$$

where the each $f(V_{ij})$ are open. Since U_i is affine, it is quasi-compact, we see we can take finite subcover $U_i = \bigcup_{k=1}^{l} f(V_{ij_k})$. In particular, we see the map

$$\coprod_{k=1}^{l} V_{ij_k} \twoheadrightarrow U$$

is fppf because

$$\begin{tikzcd} V_{ij} \subseteq f^{-1}(U_i) \arrow[r, "fppf"] \arrow[d, "\subseteq"] & U_i \arrow[d, "\subseteq"] \\ V \arrow[r, "fppf"] & U \end{tikzcd}$$

where the inclusion $V_{ij} \subseteq f^{-1}(U_i)$ is flat. Thus we see since $\coprod_{k=1}^{l} V_{ij_k}$ and U_i are both affine, so by assumption, we get

$$\mathscr{F}(U_i) \hookrightarrow \prod_{k=1}^{l} \mathscr{F}(V_{ij_k}) \xrightarrow{\subseteq} \prod_j \mathscr{F}(V_{ij})$$

which shows exactness on the left.

Exactness on the middle: Suppose $V \twoheadrightarrow U$ is fppf with V, U not necessarily affine. We will do step by step.

Step 1: we show we may assume U affine. Let $U = \bigcup U_i$ be affine open cover. Let $V_i = f^{-1}(U_i)$. Then we get

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, hook, "a"] & \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook, "b"] & \mathscr{F}(V \times_U V) \arrow[d] \\
\prod_i \mathscr{F}(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \mathscr{F}(V_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \mathscr{F}(V_i \times_{U_i} V_i) \\
\prod_{i,j} \mathscr{F}(U_i \cap U_j) \arrow[r, hook, "c"] & \prod_{i,j} \mathscr{F}(V_i \cap V_j) &
\end{tikzcd}$$

We get a, b injective since $\mathscr{F}$ is big Zariski sheaf, the c is injective since $V_i \cap V_j \twoheadrightarrow U_i \cap U_j$ is fppf and we apply exactness on the left shown above. A diagram chase shows exactness in the middle as desired (if we can show the U affine case), i.e. we want $\mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V)$ to be exact.

The diagram chase is roughly as follows: start with the top middle bullet $\bullet_1$, we want to ask if there exists $\bullet_?$ in $\mathscr{F}(U)$ that maps to $\bullet_1$:

$$\begin{tikzcd}
\bullet_? \in \mathscr{F}(U) \arrow[r, "?"] & \bullet_1 \in \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] & \bullet_2 = \bullet_3 \in \mathscr{F}(V \times_U V)
\end{tikzcd}$$

There is not much we can do at this point, thus we send $\bullet_1$ to the bottom via b and get

$$\begin{tikzcd}
\bullet_? \in \mathscr{F}(U) \arrow[r, "?"] & \bullet_1 \in \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \bullet_2 = \bullet_3 \in \mathscr{F}(V \times_U V) \\
& \bullet_b &
\end{tikzcd}$$

We want to show $\bullet_b$ comes from $\prod \mathscr{F}(U_i)$. Thus we get

$$\begin{tikzcd}
\bullet_? \in \mathscr{F}(U) \arrow[r, "?"] \arrow[d, hook] & \bullet_1 \in \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \bullet_2 = \bullet_3 \in \mathscr{F}(V \times_U V) \arrow[d] \\
\bullet_a \arrow[r, "?"] & \bullet_b \arrow[r, shift left] \arrow[r, shift right] & \bullet_{2b} = \bullet_{3b}
\end{tikzcd}$$

However, note the middle row is exact by assumption, we indeed get

$$\begin{tikzcd}
\bullet_? \in \mathscr{F}(U) \arrow[r, "?"] \arrow[d, hook, "?"] & \bullet_1 \in \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \bullet_2 = \bullet_3 \in \mathscr{F}(V \times_U V) \arrow[d] \\
\bullet_a \arrow[r] & \bullet_b \arrow[r, shift left] \arrow[r, shift right] & \bullet_{2b} = \bullet_{3b}
\end{tikzcd}$$

Viz, we have $\bullet_b$ comes from $\bullet_a$ and it remains to show $\bullet_a$ comes from the injection ? arrow from $\mathscr{F}(U)$. To that end, we note the left vertical line is exact, hence to show $\bullet_a$ lives in the image of $\mathscr{F}(U)$, we just need to show $\bullet_a$ has the same image in $\prod_{i,j} \mathscr{F}(U_i \cap U_j)$. To show that, we map $\bullet_a$ forward via the two different maps, and get $\bullet_{a2}$ and $\bullet_{a3}$, i.e. we get

$$\begin{tikzcd}
\bullet_? \in \mathscr{F}(U) \arrow[r, "?"] \arrow[d, hook, "?"] & \bullet_1 \in \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \bullet_2 = \bullet_3 \in \mathscr{F}(V \times_U V) \arrow[d] \\
\bullet_a \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \bullet_b \arrow[r, shift left] \arrow[r, shift right] & \bullet_{2b} = \bullet_{3b} \\
\bullet_{a2}, \bullet_{a3} \arrow[r, hook] & {}
\end{tikzcd}$$

where at the bottom, we must have $\bullet_{a2}$ and $\bullet_{a3}$ map to the same element because the middle column $\mathscr{F}(V) \to \prod_i \mathscr{F}(V_i) \rightrightarrows \prod_{i,j} \mathscr{F}(V_i \cap V_j)$ is exact and the image of $\bullet_{a2}$ and $\bullet_{a3}$ must equal the image of $\bullet_1$. Hence, this forces $\bullet_{a2} = \bullet_{a3}$ which forces $\bullet_a$ to come from $\bullet_?$ and hence shows $\bullet_1$ indeed comes from $\bullet_?$ as desired.

After this point, we assume U is affine.

Step 2: we show we can assume V is quasi-compact. We showed last time there exists $V = \bigcup V_j$ open cover by quasi-compacts such that $V_j \twoheadrightarrow U$ fppf. Consider the restriction map (for each j)

$$\begin{tikzcd}
x \in \arrow[d, mapsto] & \mathrm{Eq}(\mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathscr{F}(V \times_U V)) \arrow[d] \\
x_j \in & \mathrm{Eq}(\mathscr{F}(V_j) \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}(V_j \times_U V_j))
\end{tikzcd}$$

Our goal is to show x comes from $\mathscr{F}(U)$, where we assume that this x comes from $\mathscr{F}(U)$ when V is quasi-compact.

Hence, assume quasi-compact case. Since $V_j \twoheadrightarrow U$ is fppf, we get the following sequence

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] & \mathscr{F}(V_j) \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}(V_j \times_U V_j)
\end{tikzcd}$$

is exact. Thus there exists unique $y_j \in \mathscr{F}(U)$ mapping to x_j. We claim y_j is independent of j. Indeed, consider the diagram

$$\begin{tikzcd}
V_i \times_U V_j \arrow[r, two heads] \arrow[d, two heads] & V_j \arrow[d, "\subseteq"] \arrow[ddr, bend left, two heads] & \\
V_i \arrow[r, "\subseteq"] \arrow[drr, bend right, two heads] & V \arrow[dr, two heads] & \\
& & U
\end{tikzcd}$$

where we used fppf maps $V_j \to U$ and $V_i \to U$ to get the fibered product. Now we apply $\mathscr{F}$ to the whole diagram. First, we get the following injections

$$\begin{tikzcd}
\mathscr{F}(V_i \times_U V_j) & \mathscr{F}(V_j) \arrow[l, hook'] & \\
\mathscr{F}(V_i) \arrow[u, hook] & \mathscr{F}(V) \arrow[l, hook'] \arrow[u, hook] & \\
 & & \mathscr{F}(U) \arrow[ul, hook'] \arrow[uul, hook'] \arrow[ull, hook']
\end{tikzcd}$$

Now let's chase elements:

$$\begin{tikzcd}
\mathscr{F}(V_i \times_U V_j) & x_j \in \mathscr{F}(V_j) \arrow[l, hook'] & \\
x_i \in \mathscr{F}(V_i) \arrow[u, hook] & x \in \mathscr{F}(V) \arrow[l, hook'] \arrow[u, hook] & \\
 & & \mathscr{F}(U) \arrow[ul, hook'] \arrow[uul, hook', bend right, "y_j \mapsto x_j"'] \arrow[ull, hook', bend left, "y_i \mapsto x_i"]
\end{tikzcd}$$

However, since x maps to x_i and x_j, we know x_i and x_j must map to the same thing in $\mathscr{F}(V_i \times_U V_j)$. However, $V_i \times_U V_j \to V_i \to U$ is fppf cover, thus the two arrows $\mathscr{F}(U) \rightrightarrows \mathscr{F}(V_i \times_U V_j)$ are injective. Thus, we must have $y_i = y_j$, hence we can denote this as $y = y_i = y_j$. Moreover we have $y \mapsto x$. This is exactly what we wanted, and hence this finishes step 2.

After this step, we assume U is affine awnd V is quasi-compact.

Step 3: finish the proof. We may assume $V \twoheadrightarrow U$ fppf with V quasi-compact and U affine. Let $V = \bigcup V_j$ be a finite affine cover. In particular, since the union is finite, we see $\coprod V_j$ is affine and hence $\coprod V_j \twoheadrightarrow U$ is fppf.

Thus we get

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, "="] & \mathscr{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & \mathscr{F}(V \times_U V) \arrow[d, hook] \\
\mathscr{F}(U) \arrow[r] & \mathscr{F}(\coprod_j V_j) \arrow[r, shift left] \arrow[r, shift right] & \mathscr{F}(\coprod_j V_j \times_U \coprod_j V_j)
\end{tikzcd}$$

The vertical arrows are injective since $\coprod V_j \twoheadrightarrow V$ is fppf cover. The bottom row is exact because $\coprod_j V_j \twoheadrightarrow U$ is fppf and both of them are affine and we are assuming the affine case holds. Hence the top row is exact. This concludes the proof. ♡

Corollary 6.4.1. *If X is affine, then h_X is fppf sheaf.*

Proof. We proved sheaf axiom for $V \twoheadrightarrow U$ where V, U are affine and the arrow is fppf. Also, it is easy to check h_X is big Zariski sheaf. Hence h_X is fppf sheaf. ♡

7 Fppf Sites and Fibered Caregory

Last time, we proved a big lemma says if $\mathscr{F}$ is sheaf for big Zariski topology, then $\mathscr{F}$ is sheaf for fppf topology iff for affine singleton covers we get the exact sequence.

The next step is to show we can move from affine X to any X.

Theorem 7.1. *Let X be a scheme, then h_X is a fppf sheaf.*

Proof. Let $X = \bigcup_i X_i$ be open affine cover. Let $V \twoheadrightarrow U$ be fppf with U, V affine. We just need to show

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V)$$

is exact (where of course $\mathscr{F} = h_X$).

Exact on the left: pick $f, g \in h_X(U)$ with f, g maps to the same element in $\mathscr{F}(V)$. That is, say we have

$$\begin{tikzcd} V \arrow[r, two heads, "t"] & U \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \end{tikzcd}$$

such that $ft = gt$. We want $f = g$. Set-theoretically, we know $f = g$ as t is surjection. Thus we just need to show this equality is scheme-theoretically. Since $X = \bigcup X_i$ and $f = g$ as set maps, we see $f^{-1}(X_i) = g^{-1}(X_i)$ as sets. Thus let's define $U_i := f^{-1}(X_i) = g^{-1}(X_i)$. Now take the diagram and restrict to U_i with $V_i := t^{-1}(U_i)$, we get

$$\begin{tikzcd} V_i \arrow[r, two heads, "t|_{V_i}", "fppf"'] & U_i \arrow[r, shift left, "f|_{U_i}"] \arrow[r, shift right, "g|_{U_i}"'] & X_i \end{tikzcd}$$

In particular, we get $f|_{U_i} \circ t|_{V_i} = g|_{U_i} \circ t|_{V_i}$. However, h_{X_i} is a sheaf as X_i is affine, thus we see $f|_{U_i} = g|_{U_i}$ scheme-theoretically. However, this holds for all U_i and hence we see $f = g$ scheme-theoretically globally.

Exact on the middle: say we have

$$\begin{tikzcd} V \times_U V \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & V \arrow[r, two heads, "t"] \arrow[dr, "f"] & U \arrow[d, dotted, "\exists h"] \\ & & X \end{tikzcd}$$

with $fp_1 = fp_2$. We want to show the dash arrow exists, i.e. we want to show there exists h. Let $|T|$ be the underlying topological space of any scheme T. Then we see we get the same diagram for topological spaces

$$\begin{tikzcd} {|V \times_U V|} \arrow[r, shift left, "|p_1|"] \arrow[r, shift right, "|p_2|"'] & {|V|} \arrow[r, "|t|"] \arrow[dr, "|f|"] & {|U|} \arrow[d, dotted, "\exists h"] \\ & & {|X|} \end{tikzcd}$$

However, in this case, $|h|$ exists because of the following claim.

Claim:

$$\begin{tikzcd} {|V \times_U V|} \arrow[r, shift left, "|p_1|"] \arrow[r, shift right, "|p_2|"'] & {|V|} \arrow[r, "|t|"] & {|U|} \end{tikzcd}$$

is a coequalizer of topological spaces.

Suppose this claim holds, then h exists topologically , and so we can talk about subschemes $U_i := h^{-1}(X_i) \subseteq U$, $V_i := f^{-1}(X_i) \subseteq V$. Then, we get

$$\begin{tikzcd} V_i \arrow[r, "fppf", "t|_{V_i}"'] \arrow[dr, "f|_{V_i}"'] & U_i \arrow[d, dotted, "\exists ! h_i"] \\ & X_i \end{tikzcd}$$

where the existence of h_i is by affine case. Moreover, we see $h_i|_{U_i \cap U_j} = h_j|_{U_i \cap U_j}$ because we can cover $X_i \cap X_j$ by affine open and h_i restrict to open affine agrees with h_j restricts to open affine by the uniqueness statement in the affine case. Since $h_i|_{U_i \cap U_j} = h_j|_{U_i \cap U_j}$, the h_i's glue to a map $h : U \to X$ scheme-theoretically. Thus, if we can prove the above claim, we are done. We are going to prove it in small steps.

Next, we claim that for fppf $t : V \twoheadrightarrow U$,:

1. there exists natural surjection $|V \times_U \times V| \twoheadrightarrow |V| \times_{|U|} |V|$
2. $R \subseteq |U|$ open iff $|t|^{-1}(R) \subseteq |V|$ is open.
3. $\begin{tikzcd} {|V \times_U V|} \arrow[r, shift left, "|p_1|"] \arrow[r, shift right, "|p_2|"'] & {|V|} \arrow[r, "|t|"] & {|U|} \end{tikzcd}$ is coequalizer in category of topological spaces.

(1): Take $x, x' \in |V|$ with the same image $\overline{x} \in |U|$. Then we see we get

$$\begin{tikzcd} \operatorname{Spec} \kappa(x) \arrow[drr, bend left] \arrow[ddr, bend right, two heads] & & \\ & \operatorname{Spec} \kappa(x') \arrow[r] \arrow[d, two heads] & V \arrow[d, "t"] \\ & \operatorname{Spec} \kappa(\overline{x}) \arrow[r] & U \end{tikzcd}$$

What we do next is to take fibered products of the two residue fields over $\operatorname{Spec} \kappa(\overline{x})$ and we get

$$\operatorname{Spec} \kappa(x) \times_{\operatorname{Spec}(\overline{x})} \operatorname{Spec} \kappa(x') \longrightarrow V \times_U V$$

Here the arrow above is the dashed arrow (by universal property) in the following

$$\begin{tikzcd} \operatorname{Spec} \kappa(x) \times_{\operatorname{Spec} \kappa(\overline{x})} \operatorname{Spec} \kappa(x') \arrow[r] \arrow[d] \arrow[drr, dashed] & \operatorname{Spec} \kappa(x) \arrow[r] & V \arrow[dd, "t"] \\ \operatorname{Spec} \kappa(x') \arrow[d] & & V \times_U V \arrow[u] \arrow[dll] \\ V \arrow[rr, "t"] & & U \end{tikzcd}$$

In particular, we see

$$\operatorname{Spec}\kappa(x) \times_{\operatorname{Spec}(\overline{x})} \operatorname{Spec}\kappa(x') = \operatorname{Spec}(\kappa(x) \otimes_{\kappa(\overline{x})} \kappa(x'))$$

and we just choose any point of $\operatorname{Spec}(\kappa(x) \otimes_{\kappa(\overline{x})} \kappa(x'))$ and its image inside $V \times_U V$ would be a point that correspond to (x, x') in $|V| \times_{|U|} |V|$. This yields a point of $|V \times_U V|$ mapping to $(x, x') \in |V| \times_{|U|} |V|$.

(2): $R \subseteq |U|$ is open then since t is continuous we get $|t|^{-1}(R)$ is open. Conversely, t is fppf implies it is surjective, thus $R = t(t^{-1}(R))$ and hence its open as $t^{-1}(R)$ is open and t is open map.

(3): we need to show the diagram is a coequalizer diagram. In this part, we will drop the bars, and just move to the category of topological spaces. What we want is that for any W topological space, we want

$$\begin{tikzcd} V \times_U V \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & V \arrow[r, "t"] \arrow[dr, "f"] & U \arrow[d, dotted, "\exists! h"] \\ & & W \end{tikzcd}$$

If h exists, then it is unique as t is surjective, i.e. $u \in |U|$, then choose $v \in |V|$ such that $t(v) = u$, then $h(u) = f(v)$.

Thus we just need to show if $v, v' \in |V|$ and $t(v) = t(v')$, then $f(v) = f(v')$. However, since $v, v' \in |V|$ with $t(v) = t(v')$, this means $(v, v') \in |V| \times_{|U|} |V|$. Earlier, we showed there is surjection

$$q : |V \times_U V| \twoheadrightarrow |V| \times_{|U|} |V|$$

Let $v'' \in |V \times_U V|$ so $q(v'') = (v, v')$. Then, we see we know $fp_1 = fp_2$ by assumption, thus we see $fp_1(v'') = fp_2(v'')$ but by definition $fp_1(v'') = f(v)$ and $fp_2(v'') = f(v')$ (this is universal property of fibered product on cat of top spaces etc and the fact our map $|V \times V| \to |V| \times |V|$ is natural).

At this point, we have defined h as set map, and we need to show h is continuous. Thus take $W' \subseteq W$ be open, then we see $h^{-1}(W') \subseteq U$ is open iff $t^{-1}h^{-1}(W')$ is open in $|V|$ by part (2). However,

$$t^{-1}h^{-1}(W') = f^{-1}(W')$$

where f is continuous, hence h is continuous as desired. ♡

The next topic is fibered category. At this point, we talked about descents. Then, descents plus fibered category gives categorical stacks, then plus geometry then we get algebraic stacks. In particular, categorical stacks are special kind of fibered categories which satisfy descent.

Frequently, given a diagram

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

we say take the fibered product and we get

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

(where we use $\square$ to indicate Cartesian diagram) However, $X \times_Y Z$ is only defined up to (canonical) isomorphism.

Thus, any two objects W_1, W_2 that claim to be the fibered product are isomorphic up to unique isomorphism. Usually it is good enough to make a choice between W_1 and W_2 and the choice does not make a difference.

However, since stacks are about automorphisms, we need to keep track of the choices we made, which is bad. Thus, we can speak of when

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is Cartesian, rather than saying W is "the" fibered product.

Definition 7.2. Let $\mathcal{C}$ be a category, then a ***category over*** $\mathcal{C}$ is a category $\mathcal{F}$ and a functor $p : \mathcal{F} \to \mathcal{C}$.

A morphism $\phi : U \to V$ in $\mathcal{F}$ is ***Cartesian*** if for the following diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, mapsto] \arrow[dr, phantom, "\square"] & V \arrow[d, mapsto] \\ p(U) \arrow[r] & p(V) \end{tikzcd}$$

and for all $\psi : W \to V$ and factorization $p(W) \xrightarrow{h} p(U) \xrightarrow{p(\phi)} p(V)$, there exists unique $\lambda : W \to U$ with

$$\begin{tikzcd} W \arrow[r, dotted, "\exists ! \lambda"] \arrow[rr, bend left, "\forall \psi"] \arrow[d, mapsto] & U \arrow[r, "\phi"] \arrow[d, mapsto] & V \arrow[d, mapsto] \\ p(W) \arrow[r, "h"] & p(U) \arrow[r] & p(V) \end{tikzcd}$$

such that $\phi\lambda = \psi$ and $p(\lambda) = h$.

The point of this is that, with this, the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, mapsto] & V \arrow[d, mapsto] \\ p(U) \arrow[r] & p(V) \end{tikzcd}$$

looks like a pullback.

Definition 7.3. Let $\mathcal{F}$ be category over $\mathcal{C}$ and $\phi : U \to V$ be Cartesian, then we say U is a ***pullback of*** V ***along*** $p(\phi)$.

If $U' \xrightarrow{\phi'} V$ and $U \xrightarrow{\phi} V$ are both pullbacks along the samething $p(\phi) = p(\phi')$, then we have

$$\begin{tikzcd}
U' \arrow[r, dotted, "\exists!\lambda"] \arrow[rr, bend left, "\phi'"] \arrow[d] & U \arrow[r, "\phi"] \arrow[d] & V \arrow[d] \\
p(U') \arrow[r, "\mathrm{Id}"] & p(U) \arrow[r] & p(V)
\end{tikzcd}$$

and hence $p(\lambda) = \mathrm{Id}$ which implies λ is an isomorphism (with $\phi' \circ \lambda = \phi$).

Remark 7.4. Now given $p : \mathcal{F} \to \mathcal{C}$, for any $U \in \mathcal{C}$, let $\mathcal{F}(U)$ be a category defined as follows: the objects are $X \in \mathcal{F}$ such that $p(X) = U$ and morphisms are $X \xrightarrow{\phi} Y$ such that $p(\phi) = \mathrm{Id}_U$.

The idea above is that we are supposed to think of $\mathcal{F}$ as a map from $\mathcal{C}$ to categories, where we input U and output a "category" $\mathcal{F}(U)$.

Definition 7.5. Let $p : \mathcal{F} \to \mathcal{C}$ be a category over $\mathcal{C}$, then we say $\mathcal{F}$ is a ***fibered category*** if "pullbacks exist", i.e. given diagram

$$\begin{tikzcd}
 & v \arrow[d] \\
U \arrow[r, "h"] & V := p(v)
\end{tikzcd}$$

then there exists Cartesian arrow $\phi : u \to v$ such that $p(\phi) = h$, i.e.

$$\begin{tikzcd}
u \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\square"] & v \arrow[d] \\
U \arrow[r, "h"] & V
\end{tikzcd}$$

is a pullback.

Example 7.6. Let $\mathcal{C} = (\mathrm{Sch})$ be the category of schemes, and let M_g be the category of genus g curves. In other words, objects of M_g are $C \xrightarrow{\pi} S$ where π is smooth and geometric fibers of π are genus g curves(recall geometric fiber means $X \times \mathrm{Spec}(\overline{\kappa(x)})$). The morphisms are diagrams

$$\begin{tikzcd}
C' \arrow[r] \arrow[d, "\pi'"] & C \arrow[d, "\pi"] \\
S' \arrow[r] & S
\end{tikzcd}$$

Then, our $p : M_g \to (\mathrm{Sch})$ is going to be $(C \xrightarrow{\pi} S) \mapsto S$. Since pullback exists because they exists in (Sch), we see M_g is a fibered category. Indeed, just take the fibered product in (Sch), say

$$\begin{tikzcd} & C \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

then we get

$$\begin{tikzcd} C \times_S S' =: C' \arrow[r] \arrow[d, "\pi'"] & C \arrow[d, "\pi"] \\ S' \arrow[r] & S \end{tikzcd}$$

where since π is smooth, then π' is smooth, and π' has the same geometric fibers as π.

Thus, we just constructed the moduli space of genus g curves.

Example 7.7. If $\mathcal{C}$ is any category, $X \in \mathcal{C}$, then consider $\mathcal{C}/X$ as the category with objects $Y \to X$ and morphisms

$$\begin{tikzcd} Y' \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}$$

Then $p : \mathcal{C}/X \to \mathcal{C}$ with $Y \mapsto Y$ gives $\mathcal{C}/X$ the structure of a fibered category. This is because every arrow in $\mathcal{C}/X$ is Cartesian. Indeed, take an arrow $\phi : Y' \to Y$ in $\mathcal{C}/X$, and let $\psi : Y'' \to Y$ be any arrow in $\mathcal{C}/X$ with a factorization $Y'' \xrightarrow{h} Y' \xrightarrow{\phi} Y$, then we see we indeed have dashed arrow in the following diagram

$$\begin{tikzcd} Y'' \arrow[r, dashed] \arrow[rr, bend left, "\psi"] \arrow[d, "\mathrm{Id}"] & Y' \arrow[r, "\phi"] \arrow[d, "\mathrm{Id}"] & Y \arrow[d, "\mathrm{Id}"] \\ Y'' \arrow[r, "h"] & Y' \arrow[r, "\phi"] & Y \end{tikzcd}$$

That is, we just take the dashed arrow be h, which is indeed unique and it exists.

Example 7.8. Let's consider the category (QCoh) over (Sch). Here objects of (QCoh) are pairs $(S, \mathscr{F})$ where $\mathscr{F}$ is quasi-coherent sheaf on S, and morphisms between $(S', \mathscr{F}') \to (S, \mathscr{F})$ are given by $f : S' \to S$ and $\epsilon : \mathscr{F}' \to f^*\mathscr{F}$.

In the above examples, as in Remark 7.4, we see $(\mathrm{QCoh})(S)$ is exactly the category of quasi-coherent sheaves on S, and $M_g(S)$ is exactly the category of genus g curves on S.

Indeed, $(\mathrm{Qcoh})(S)$ is by definition the category with objects being $(X, \mathscr{F}) \in (\mathrm{Qcoh})$ such that $p(X, \mathscr{F}) = S$ and morphisms being $f : (X, \mathscr{F}) \to (Y, \mathscr{G})$ such that $p(f) = \mathrm{Id}_S$. Well, $p(f, \epsilon) = \mathrm{Id}_S$ means we need to have $f : X \to Y$ is the identity, i.e. $X = Y = S$ and ϵ is just a morphism between quasi-coherent sheaves on S.

Similarly $M_g(S)$ is category of genus g curves because the projection forces any object to be live over S.

Definition 7.9. If $p_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$ and $p_{\mathcal{G}} : \mathcal{G} \to \mathcal{C}$ be two fibered categories, then a ***morphism of fibered categories*** is a functor $g : \mathcal{F} \to \mathcal{G}$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r, "g"] \arrow[dr, "p_{\mathcal{F}}"'] & \mathcal{G} \arrow[d, "p_{\mathcal{G}}"] \\ & \mathcal{C} \end{tikzcd}$$

and g sends Cartesian arrows to Cartesian arrows.

We note, for all $U \in \mathcal{C}$, we get $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$. Indeed, since $\mathcal{F}(U), \mathcal{G}(U)$ are categories, g_U is a functor between categories. It is just the same as g, i.e. $g_U(x) = g(x)$. We can check this is indeed a functor. Indeed, let $x_1 \to x_2 \in \mathcal{F}(U), y_1 \to y_2 \in \mathcal{G}(U)$ be two arrows, we need to show we get a diagram

$$\begin{tikzcd} x_1 \arrow[r, "\phi"] \arrow[d, "g_U"] & x_2 \arrow[d, "g_U"] \\ y_1 \arrow[r, "\psi"] & y_2 \end{tikzcd}$$

However, note we get the following diagram

$$\begin{tikzcd} x_1 \arrow[r, "\phi"] \arrow[d, "g_U"] \arrow[dd, bend right, "p_{\mathcal{F}}"'] & x_2 \arrow[d, "g_U"] \arrow[dd, bend left, "p_{\mathcal{G}}"] \\ y_1 \arrow[r, "\psi"] \arrow[d, "p_{\mathcal{G}}"] & y_2 \arrow[d, "p_{\mathcal{G}}"] \\ U \arrow[r, "\mathrm{Id}"] & U \end{tikzcd}$$

where the two triples of vertical arrows (i.e. $(g_U, p_{\mathcal{G}}, p_{\mathcal{F}})$) commutes, and the outer square and inner square both commutes, which forces the upper square to commute as desired.

Definition 7.10. If $g, g' : \mathcal{F} \to \mathcal{G}$ are two morphisms of fibered categories, then a ***base-preserving natural transformation*** $\alpha : g \to g'$ is a natural transformation of functors such that for all $U \in \mathcal{F}$, the map $\alpha_U : g(U) \to g'(U)$

satisfies $p_{\mathcal{G}}(\alpha_U) = \mathrm{Id}_{p_{\mathcal{F}}(U)}$, i.e. if we have the following diagram

$$\begin{tikzcd}
U \arrow[r, mapsto] \arrow[rr, bend left, mapsto] \arrow[d, mapsto, "p_{\mathcal{F}}"] & g(U) \arrow[r, "\alpha_U"'] \arrow[dl, mapsto, "p_{\mathcal{G}}"'] & g'(U) \arrow[dll, mapsto, "p_{\mathcal{G}}"'] \\
p_{\mathcal{F}}(U) & &
\end{tikzcd}$$

then we must have $p_{\mathcal{G}}(\alpha_U) = \mathrm{Id}$. In other word, we want α_U to be a morphism in $\mathcal{G}(p_{\mathcal{F}}(U))$.

We note this gives $\mathrm{HOM}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$ the category with objects being morphisms of fibered category $g : \mathcal{F} \to \mathcal{G}$ and morphisms being base-preserving natural transformations.

Now, suppose we have $g' : \mathcal{F} \to \mathcal{G}$, then we get $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for all $U \in \mathcal{C}$. This is kind of looks like a map of presheaves.

8 Fibered Category

Last time we defined a fibered category, which is a functor $p : \mathcal{F} \to \mathcal{C}$ such that pullbacks exist, i.e. for all z and for all $z \to y$, there exists unique $z \to x$ so the following commutes

$$\begin{tikzcd}
z \arrow[r, dotted, "\exists!"] \arrow[rr, bend left, "\forall"] \arrow[d] & x \arrow[r] \arrow[d] & y \arrow[d] \\
p(z) \arrow[r, "\forall"] & p(x) \arrow[r] & p(y)
\end{tikzcd}$$

The morphisms of fibered categories are given by functors such that g sends Cartesian arrows to Cartesian arrows.

Then, for $p : \mathcal{F} \to \mathcal{C}$ fibered category, for all $U \in \mathcal{C}$, we let $\mathcal{F}(U)$ be the category with objects being $x \in \mathcal{F}$ such that $p(x) = U$, and morphisms being $\phi : x \to y$ such that $p(\phi) = \mathrm{Id}_U$.

Thus, say $g : \mathcal{F} \to \mathcal{G}$ be maps of fibered categories over $\mathcal{C}$, then we get $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for all $U \in \mathcal{C}$. At the end of last class, we mentioned this implies fibered categories look like presheaves but instead of $\mathcal{F}(U)$ being sets, we have $\mathcal{F}(U)$ is category.

Example 8.1. Consider $\mathcal{M}_{g,n} \to (\mathrm{Sch})$ be the fibered category of genus g curves with n marked points. In this category, objects of $\mathcal{M}_{g,n}$ are $C \xrightarrow{\pi} S$ with π smooth and on geometric fibers of S, C is genus g curves. Next, we need to explain what marked points are. Those are given by sections of p, say $p_i : S \to C$, which are distinct points on the geometric fibers. Then the morphisms are diagrams

$$\begin{tikzcd} C' \arrow[r, "f"] \arrow[d, shift left] & C \arrow[d, shift left] \\ S' \arrow[r, "g"] \arrow[u, bend left, "p'_1,\dots,p'_n"] & S \arrow[u, bend right, "p_1,\dots,p_n"'] \end{tikzcd}$$

such that $\pi f = g\pi'$ and $fp'_i = p_i g$. Then, the projection $p : \mathcal{M}_{g,n} \to (\mathrm{Sch})$ is given by $(C \xrightarrow{\pi} S) \mapsto S$.

One should check that pullbacks exist.

In particular, $\mathcal{M}_{1,1}$ is the moduli space of genus 1 curves with one marked point, i.e. they are exactly elliptic curves.

Next, note we have the fibered category (QCoh) and we get $F_i : \mathcal{M}_{g,n} \to$ (QCoh) map of fibered category, for $1 \le i \le n$. The map is given by

$$F_i(C \xrightarrow{\pi} S) := (S, p_i^* \Omega^1_{C/S})$$

where we take the pullback of relative differential via p_i. We also need to define what F_i does on morphisms. Well, suppose we have morphism

$$\begin{tikzcd} C' \arrow[r, "f"] \arrow[d, shift left] & C \arrow[d, shift left] \\ S' \arrow[r, "g"] \arrow[u, bend left, "p'_1,\dots,p'_n"] & S \arrow[u, bend right, "p_1,\dots,p_n"'] \end{tikzcd}$$

We need a map between $(p'_i)^* \Omega^1_{C'/S'} \to g^* P_i^* \Omega^1_{C/S}$. Well, we do have a canonical morphism (and in fact its isomorphism), as we will show next. First, note $g^* p_i^* \Omega^1_{C/S}$ is equal to $(p'_i)^* f^* \Omega^1_{C/S}$ as the diagram commutes, and we also have $(p'_i)^* f^* \Omega^1_{C/S} = (p'_i)^* \Omega^1_{C'/S'}$. Hence we get the desired canonical (iso)morphism as desired.

Lemma 8.2. *Suppose $g : \mathcal{F} \to \mathcal{G}$ is a map of fibered category. Then g is fully faithful as a map of categories (not fibered category) if and only if $\forall U \in \mathcal{C}$, $g_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is fully faithful.*

This result should remind you of: if $\mathcal{F} \to \mathcal{G}$ is a map of presheaves then it is injective if and only if for all U, $\mathcal{F}(U) \to \mathcal{G}(U)$ is injective.

Proof. Recall fully faithful means we have a bijection between Hom sets (full means surjection between hom sets and faithful means injection between hom sets). Thus, let $x, y \in \mathcal{F}$, we get

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{F}}(x,y) \arrow[r, "g"] \arrow[dr, "p_{\mathcal{F}}"] & \mathrm{Hom}_{\mathcal{G}}(g(x), g(y)) \arrow[d, "p_{\mathcal{G}}"] \\ & \mathrm{Hom}_{\mathcal{C}}(p_{\mathcal{F}}(x), p_{\mathcal{F}}(y)) \end{tikzcd}$$

Then g is fully faithful if and only if for all $h : p_{\mathcal{F}}(x) \to p_{\mathcal{F}}(y)$ in $\mathrm{Hom}_{\mathcal{C}}(p_{\mathcal{F}}(x), p_{\mathcal{F}}(y))$, g induces a bijection

$$\{x \xrightarrow{\phi} y : p_{\mathcal{F}}(\phi) = h\} \xrightarrow{\sim} \{g(x) \xrightarrow{\psi} g(y) : p_{\mathcal{G}}(\psi) = h\}$$

This is sort of like we show bijection on each of the fibers.

Thus, we fix h downstairs

$$\begin{tikzcd}
 & y \arrow[d] \\
p_{\mathcal{F}}(x) \arrow[r, "h"] & p_{\mathcal{F}}(y)
\end{tikzcd}$$

and let y' be a fibered product/pullback

$$\begin{tikzcd}
y' \arrow[r] \arrow[d, "\tilde{h}"] \arrow[dr, phantom, "\square"] & y \arrow[d] \\
p_{\mathcal{F}}(x) \arrow[r, "h"] & p_{\mathcal{F}}(y)
\end{tikzcd}$$

Then we see for all $x \to y$, we get unique arrow $x \to y'$, i.e. we get

$$\begin{tikzcd}
x \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left=10] \arrow[ddr] & & \\
 & y' \arrow[r, "\tilde{h}"'] \arrow[d] \arrow[dr, phantom, "\square"] & y \arrow[d] \\
 & p_{\mathcal{F}}(x) \arrow[r, "h"] & p_{\mathcal{F}}(y)
\end{tikzcd}$$

so, we have a bijection

$$\{x \xrightarrow{\phi} y : p_{\mathcal{F}}(\phi) = h\} = \{x \xrightarrow{\phi'} y' : p_{\mathcal{F}}(\phi') = \mathrm{Id}\}$$

This is because when we actually using definition of pullback, what we get is a diagram

$$\begin{tikzcd}
x \arrow[r, dashed, "\exists!\phi'"] \arrow[rr, bend left, "\forall\phi"] \arrow[d] & y' \arrow[r, "\tilde{h}"'] \arrow[d] & y \arrow[d] \\
p_{\mathcal{F}}(x) \arrow[r, "\mathrm{Id}"'] & p_{\mathcal{F}}(x) \arrow[r, "h"'] & p_{\mathcal{F}}(y)
\end{tikzcd}$$

Now, since $g(\tilde{h})$ is Cartesian arrow as $\tilde{h}$ is Cartesian (by def of morphism of fibered cat), we see we get

$$\begin{tikzcd}
g(y') \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & g(y) \arrow[d] \\
p_{\mathcal{F}}(x) \arrow[r] & p_{\mathcal{F}}(y)
\end{tikzcd}$$

We also have the $g(x)$ floating around, and we get

$$\begin{tikzcd}
g(x) \arrow[drr, bend left] \arrow[dr, dotted, "\exists!"] \arrow[ddr] & & \\
& g(y') \arrow[r] \arrow[d, "g(\tilde{h})"] \arrow[dr, phantom, "\square"] & g(y) \arrow[d] \\
& p_{\mathcal{F}}(x) \arrow[r] & p_{\mathcal{F}}(y)
\end{tikzcd}$$

Hence we get bijection

$$\{g(x) \to g(y) \text{ lying over } h\} = \{g(x) \to g(y') \text{ lying over Id}\}$$

with the similar reasoning as above. Hence,

$$\{x \to y \text{ lying over } h\} \xrightarrow{g} \{g(x) \to g(y) \text{ lying over } h\}$$

is a bijection if and only if $g_{p_{\mathcal{F}}(x)} : \mathcal{F}(p_{\mathcal{F}}(x)) \to \mathcal{G}(p_{\mathcal{F}}(x))$ is fully faithful. This concludes the proof. ♡

Definition 8.3. We say $g : \mathcal{F} \to \mathcal{G}$ a map of fibered categories is an ***equivalence*** if there exists $h : \mathcal{G} \to \mathcal{F}$ map of fibered categories and exists a base preserving isomorphism $\alpha : g \circ h \xrightarrow{\sim} \mathrm{Id}_{\mathcal{G}}$ and $\beta : h \circ g \xrightarrow{\sim} \mathrm{Id}_{\mathcal{F}}$.

Proposition 8.4. *For a map of fibered categories $g : \mathcal{F} \to \mathcal{G}$, g is equivalence iff $\forall U \in \mathcal{C}$, g_U is an equivalence (in category theory sense) iff $\forall U \in \mathcal{C}$, g_U is fully faithful and essentially surjective.*

Proof. We already showed g is fully faithful if and only if all g_U are. Thus we just need to show the claims about essentially surjective (recall essentially surjective for $g : \mathcal{F} \to \mathcal{G}$ means each object $y \in \mathcal{G}$ is isomorphic to an object of the form $g(x)$ where $x \in \mathcal{F}$).

($\Rightarrow$) : if g is equivalence, then we want g_U to be essentially surjective. Given $y \in \mathcal{G}(U)$, we have $gh(y) \xrightarrow{\sim} y$ and h is a morphism of fibered cats[1], so $h(y) \in \mathcal{F}(U)$.

($\Leftarrow$): now assume g_U is essentially surjective for all U. We need to construct an equivalence of fibered categories $h : \mathcal{G} \to \mathcal{F}$. Given $y \in \mathcal{G}(U)$, since g_U is equivalence, we know there exists $h(y) \in \mathcal{F}(U)$ such that $\alpha_y : y \xrightarrow{\sim} g_U(h(y))$. Given any $y \xrightarrow{\phi} y'$ in $\mathcal{G}(U)$, there exists unique $h(\phi) : h(y) \to h(y')$ in $\mathcal{F}(U)$ such that

$$\begin{tikzcd}
y \arrow[r, "\phi"] \arrow[d, "\alpha_y", "\sim"'] & y' \arrow[d, "\alpha_{y'}", "\sim"'] \\
g_U(h(y)) \arrow[r, "g_U(h(\phi))"] & g_U(h(y'))
\end{tikzcd}$$

[1]from time to time we will write cat to mean category

because g_U is fully faithful.

This gives functor $h : \mathcal{G} \to \mathcal{F}$ and also $\alpha : \mathrm{Id}_{\mathcal{G}} \xrightarrow{\sim} g \circ h$. We need h sends Cartesian arrows to Cartesian arrows. If $y \xrightarrow{\phi} y'$ is Cartesian in $\mathcal{G}$, then we get

$$\begin{tikzcd}
h(y) \arrow[r] \arrow[d] & h(y') \arrow[d] \\
p_{\mathcal{F}}(h(y)) \arrow[r] & p_{\mathcal{F}}(h(y'))
\end{tikzcd}$$

and suppose we are given arbitrary w with the following diagram

$$\begin{tikzcd}
w \arrow[rr, bend left] \arrow[d] & h(y) \arrow[r] \arrow[d] & h(y') \arrow[d] \\
p_{\mathcal{F}}(w) \arrow[r] & p_{\mathcal{F}}(h(y)) \arrow[r] & p_{\mathcal{F}}(h(y'))
\end{tikzcd}$$

where we want to show there exists unique arrow $w \to h(y)$. Since g is fully faithful, there exists dotted arrow $w \to h(y)$ if and only if it holds after we apply g. Thus we want to show there exists unique dotted arrow in the following:

$$\begin{tikzcd}
g(w) \arrow[rr, bend left] \arrow[r, dotted, "\exists!"] \arrow[d] & gh(y) \arrow[r] \arrow[d] & gh(y') \arrow[d] \\
p_{\mathcal{F}}(g(w)) \arrow[r] & p_{\mathcal{F}}(gh(y)) \arrow[r] & p_{\mathcal{F}}(gh(y'))
\end{tikzcd}$$

However, note we can complete the diagram with

$$\begin{tikzcd}
y \arrow[r, "\phi"] \arrow[d, "\alpha_y", "\sim"'] & y' \arrow[d, "\alpha_{y'}", "\sim"'] \\
gh(y) \arrow[r] & gh(y')
\end{tikzcd}$$

In other word, we get

$$\begin{tikzcd}
 & y \arrow[r] \arrow[d] & y' \arrow[d] \\
g(w) \arrow[rr, bend left=70] \arrow[r, dotted] \arrow[d] & gh(y) \arrow[r] \arrow[d] & gh(y') \arrow[d] \\
p_{\mathcal{F}}(g(w)) \arrow[r] & p_{\mathcal{F}}(gh(y)) \arrow[r] & p_{\mathcal{F}}(gh(y'))
\end{tikzcd}$$

but $y \to y'$ is Cartesian, hence we indeed have the dotted arrow as desired.

Lastly, we need $\beta : \mathrm{Id}_{\mathcal{F}} \xrightarrow{\sim} h \circ g$. If $x \in \mathcal{F}$, we want $x \xrightarrow[\beta_x]{\sim} h(g(x))$. By full faithfulness of g, we just need to show $y(x) \xrightarrow{\sim} g(h(g(x)))$. But we do have an isomorphism, $\alpha_{g(x)}$. Then, we left as an exercise that β_x is a natural transformation. ♡

Now we have seen fibered categories are analogous to presheaves (over (Sets)), we ask a natural question: what is analogue of sheaf? The answer is stacks.

Next, we ask if fibered categories are analogous to presheaves, is there a type of Yoneda lemma? Well, there is, and its called 2-Yoneda lemma.

Before we do this, let's recall if $X \in \mathcal{C}$, we have fibered category maps $\mathcal{C}/X \to \mathcal{C}$ with morphism $(Y \to X) \mapsto Y$. The analogy is that, $\mathcal{C}/X$ should correspond to h_X.

Theorem 8.5 (2-Yoneda Lemma)**.** *For any fibered category $\mathcal{F} \to \mathcal{C}$, and all $X \in \mathcal{C}$, we have a category $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F})$ with morphisms being base-preserving natural transformations. Then, we have a equivalence of categories*

$$\begin{aligned} \zeta : \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F}) &\to \mathcal{F}(X) \\ g &\mapsto g(X \xrightarrow{\mathrm{Id}_X} X) \end{aligned}$$

We will prove the 2-Yoneda lemma next time, and we first prove an corollary of this.

Corollary 8.5.1. *For $X, Y \in \mathcal{C}$, we have*

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{C}/Y) &\to \mathrm{Hom}_{\mathcal{C}}(X, Y) \\ f &\mapsto f(\mathrm{Id}_X) \end{aligned}$$

is an equivalence of categories.

Note in the above, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set viewed as a category with objects equal set elements, and morphisms being only identity maps.

Proof. Well, apply 2-Yoneda lemma to $\mathcal{F} = \mathcal{C}/Y$, we get

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{C}/Y) = (\mathcal{C}/Y)(X)$$

where the objects are $X \xrightarrow{\phi} Y$ and morphisms are

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[dr, "\phi"] & X \arrow[d, "\phi'"] \\ & Y \end{tikzcd}$$

living over Id_X, i.e. $\psi = \mathrm{Id}_X$. Thus, the only morphism we have in $(\mathcal{C}/Y)(X)$ are identity maps, i.e. $(\mathcal{C}/Y)(X)$ is exactly the category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. ♡

Thus, we will introduce some notations: we will frequently write $X \to \mathcal{F}$ in place of $\mathcal{F}(X)$ if $\mathcal{F}$ is fibered category. This is justified by 2-Yoneda because $\mathcal{C}/X \to \mathcal{F}$ is the same as $\mathcal{F}(X)$. So it is just a convenience to write X in place of $\mathcal{C}/X$. The corollary shows $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{C}/Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$, so $X \to Y$ in place of $\mathcal{C}/X \to \mathcal{C}/Y$ is unambiguous.

9 Fibered Category

Last time, we have the 2-Yoneda lemma

Theorem 9.1 (2-Yoneda Lemma). *For any fibered category $\mathcal{F} \to \mathcal{C}$, and all $X \in \mathcal{C}$, we have a category $\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F})$ with morphisms being base-preserving natural transformations. Then, we have a equivalence of categories*

$$\begin{aligned} \zeta : \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F}) &\to \mathcal{F}(X) \\ g &\mapsto g(X \xrightarrow{\mathrm{Id}_X} X) \end{aligned}$$

Just like the proof of Yoneda lemma, this is not involved, but also not a short proof.

Proof. We need to construct $\eta : \mathcal{F}(X) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{F})$ that maps $x \in \mathcal{F}(X)$ to a fibered category $(\eta_x : \mathcal{C}/X \to \mathcal{F})$.

First, we define η_x on objects: an object in $\mathcal{C}/X$ is a map $Y \xrightarrow{\phi} X$. In particular, we have diagram

$$\begin{tikzcd} & x \arrow[d, mapsto] & \mathcal{F} \arrow[d] \\ Y \arrow[r, "\phi"] & X & \mathcal{C} \end{tikzcd}$$

Well, the natural thing to do is just take a pullback $\phi^* x$, i.e. we make a choice and get the following diagram

$$\begin{tikzcd} \phi^* x \arrow[r] \arrow[d, mapsto] \arrow[dr, phantom, "\square"] & x \arrow[d, mapsto] \\ Y \arrow[r, "\phi"] & X \end{tikzcd}$$

and define $\eta_x(\phi) := \phi^* x \in \mathcal{F}(Y)$.

Next, we define η_x on morphisms. Suppose we are given morphism

$$\begin{tikzcd} Y' \arrow[r, "\xi"] \arrow[dr, "\phi'"] & Y \arrow[d, "\phi"] \\ & X \end{tikzcd}$$

in $\mathcal{C}$. We want to know what $\eta_x(\xi)$ is. Well, we get the following diagram

$$\begin{tikzcd}
(\phi')^*x \arrow[dd] \arrow[rrd, bend left] & & \\
& \phi^*x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d] \\
Y' \arrow[r] & Y \arrow[r] & X
\end{tikzcd}$$

but then we get a dotted arrow between $(\phi')^*x$ to ϕ^*x as the squares are Cartesian. Hence we have

$$\begin{tikzcd}
(\phi')^*x \arrow[dd] \arrow[rrd, bend left] \arrow[dr, dotted, "\exists!"] & & \\
& \phi^*x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d] \\
Y' \arrow[r] & Y \arrow[r] & X
\end{tikzcd}$$

This unique dotted arrow gives the desired map on morphisms (i.e. $\eta_x(\xi) : (\phi')^*x \to \phi^*x$).

Now we know η as a functor $\mathcal{F} \to \mathcal{C}$, why is η a morphism of fibered categories. We have to check two things: the first thing is that it respect fibers. However, we checked that already, i.e. $(Y \xrightarrow{\phi} X) \mapsto$ something in $\mathcal{F}(Y)$.

The second thing we need to show is that η takes Cartesian arrows to Cartesian. First, in $\mathcal{C}/X$ all arrows are Cartesian, so we need $\eta_x(\text{any arrow}) = \text{Cartesian}$. However, note by basic category theory, since the inner square and outer squares are both Cartesian, the dotted arrow must also be Cartesian.

So, we now know η on objects. What about morphisms?

Given $f : x' \to x$ in $\mathcal{F}(X)$, we want $\eta_f : \eta_{x'} \to \eta_x$, i.e. we need η_f to be base-preserving natural trans.

Given $\phi : Y \to X$ in $\mathcal{C}/X$ in $\mathcal{C}/X$, we need $\eta_f(\phi) : \eta_{x'}(\phi) = \phi^*x' \to \eta_x(\phi) = \phi^*x$. Lets draw out the diagrams

$$\begin{tikzcd}
(\phi)^*x' \arrow[rr] \arrow[ddr, mapsto] & & x' \arrow[d, "f"'] \arrow[dd, mapsto, bend left] \\
& \phi^*x \arrow[r] \arrow[d, mapsto] \arrow[dr, phantom, "\square"] & x \arrow[d, mapsto] \\
& Y \arrow[r, "\phi"] & X
\end{tikzcd}$$

but then we get a unique dotted arrow

$$\begin{tikzcd}
(\phi)^*x' \arrow[rr] \arrow[dr, dotted, "\exists!"] \arrow[ddr, mapsto] & & x' \arrow[d, "f"'] \arrow[dd, mapsto, bend left] \\
& \phi^*x \arrow[r] \arrow[d, mapsto] \arrow[dr, phantom, "\square"] & x \arrow[d, mapsto] \\
& Y \arrow[r, "\phi"] & X
\end{tikzcd}$$

this is our definition of $\eta_f(\phi)$. We still need to show this is base-preserving, i.e. it lives over the identity. Indeed, note $f : x' \to x$ lives over the identity (as f is a morphism in $\mathcal{F}(X)$), we see the pullback, i.e. the dotted arrow, also lives over identity. Hence η_f is base-preserving natural transformation as desired.

We have now defined η. Next we need to show this is an equivalence.

First, we show $\zeta\eta = \mathrm{Id}$. Note we have

$$\zeta\eta(x) = \zeta(\eta_x) = \eta_x(X \xrightarrow{\mathrm{Id}_X} X)$$

but note

$$\eta_x(\mathrm{Id}_x) : \mathcal{C}/X \to \mathcal{F}(X)$$

is given by the following square

$$\begin{tikzcd}
(\mathrm{Id}_X)^*x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d, mapsto] \\
X \arrow[r, "\mathrm{Id}_X"] & X
\end{tikzcd}$$

but $(\mathrm{Id}_X)^*x = x$ and hence $\zeta\eta(x) = x$ as desired.

Next, we show $\eta\zeta \cong \mathrm{Id}$. We see we get

$$\eta\zeta(f : \mathcal{C}/X \to \mathcal{F}) = \eta(f(\mathrm{Id}_X)) = \eta_{f(\mathrm{Id}_X)}$$

where we see

$$\eta_{f(\mathrm{Id}_X)} : \mathcal{C}/X \to \mathcal{F}$$

and by definition we get the following diagram

$$\begin{tikzcd}
& f(\mathrm{Id}_X) \arrow[d, mapsto] \\
Y \arrow[r, "\phi"] & X
\end{tikzcd}$$

and taking pullback we get

$$\begin{tikzcd}
\phi^*(f(\mathrm{Id}_X)) = \eta_{f(\mathrm{Id}_X)}(\phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & f(\mathrm{Id}_X) \arrow[d] \\
Y \arrow[r, "\phi"] & X
\end{tikzcd}$$

and we want to show $\eta_{f(\mathrm{Id}_X)}(\phi) = \phi^*(f(\mathrm{Id}_X)) \cong f(\phi)$ because then $\eta_{f(\mathrm{Id}_X)}(\phi) = f(\phi)$, i.e. $\eta\zeta(f) \cong f$.

To show this, it is enough to show the arrow $f(\phi) \to f(\mathrm{Id}_X)$ is Cartesian because if we have Cartesian diagrams

$$\begin{tikzcd} f(\phi) \arrow[drr] \arrow[ddr] \arrow[dr, phantom, "\square"] & & \\ & \phi^*(f(\mathrm{Id}_X)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & f(\mathrm{Id}_X) \arrow[d] \\ & Y \arrow[r] & X \end{tikzcd}$$

then since 2 pullbacks are canonically isomorphic we get the desired isomorphism.

Now, why is the arrow $f(\phi) \to f(\mathrm{Id}_X)$ Cartesian? We always have unique canonical map $\phi \to \mathrm{Id}_X$, as recall morphism of morpihsmes in our case is just try to fill the following diagram:

$$\begin{tikzcd} Y \arrow[dr, "\phi"] & X \arrow[d, "\mathrm{Id}_X"] \\ & X \end{tikzcd}$$

but there is only a unique way to do this, which is

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[dr, "\phi"] & X \arrow[d, "\mathrm{Id}_X"] \\ & X \end{tikzcd}$$

This gives an arrow $\phi \to \mathrm{Id}_X$ which is automatically Cartesian. Thus $f(\phi) \to f(\mathrm{Id}_X)$ is Cartesian because f preserves Cartesian arrows. This concludes the proof. ♡

Definition 9.2. A ***caregory fibered in sets over*** $\mathcal{C}$ is a fibered category $\mathcal{F} \to \mathcal{C}$ such that for all $U \in \mathcal{C}$, $\mathcal{F}(U)$ is a set, i.e. only maps are identity.

We note in category theory, a set means a category where the only maps are identity (this is definition).

Note for such $\mathcal{F}$, we have a well-defined pullback map. Indeed, we get diagram

$$\begin{tikzcd} y' \arrow[drr] \arrow[ddr] \arrow[dr, phantom, "\square"] & & \\ & y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & x \arrow[d] \\ & U \arrow[r] & V \end{tikzcd}$$

Then we get $y' \to y$ lying over Id_U, i.e. $y' \to y$ is morphism in $\mathcal{F}(U)$. But $\mathcal{F}(U)$ is a set, so it must be identity by definition, i.e. $y' = y$.

So, given $U \xrightarrow{f} V$, we get $f^* : \mathcal{F}(V) \to \mathcal{F}(U)$ compatible with composition, i.e. $\mathcal{F}$ naturally yields a presheaf $F_{\mathcal{F}}$ where $F_{\mathcal{F}}(U) \coloneqq \mathcal{F}(U)$.

The next task is to show categories fibered in sets are the same as presheaves.

Lemma 9.3. *If $\mathcal{F} \to \mathcal{C}$ is a category fibered in sets and $\mathcal{G} \to \mathcal{C}$ is any fibered category, then $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, \mathcal{F})$ is a set.*

We note again that in higher category theory, sets are, by definition, categories with only identity morphism.

Proof. For $f, g : \mathcal{G} \to \mathcal{F}$ morphisms of fibered category, and $\alpha : f \to g$ morphism, we want to show $\alpha = \mathrm{Id}$. For all $x \in \mathcal{G}(X)$, $\alpha_x : f(x) \to g(x)$ is in $\mathcal{F}(X)$, so $\alpha_x = \mathrm{Id}$. Also, given $\phi : y \to x$, we get diagram

$$\begin{tikzcd}
f(y) \arrow[r, "f(\phi)"] \arrow[d, "\alpha_y", "="'] & f(x) \arrow[d, "\alpha_x", "="'] \\
g(y) \arrow[r, "g(\phi)"] & g(x)
\end{tikzcd}$$

and hence $f(\phi) = g(\phi)$. Hence $f = g$ and α is Id_f. ♡

Corollary 9.3.1. *Categories fibered in sets over $\mathcal{C}$ is a (locally small) category, i.e. $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, \mathcal{F})$ is a set.*

Example 9.4. Given a presheaf $F : \mathcal{C}^{opp} \to (\mathrm{Sets})$, let $\mathcal{F} \coloneqq \mathcal{F}_F$ be the following category: objects are (U, u) with $U \in \mathcal{C}$, $u \in F(U)$, morphisms $(U', u') \to (U, u)$ are $g : U' \to U$ in $\mathcal{C}$ such that $g^* : F(U) \to F(U')$ so $u \mapsto u'$.

This is a fibered category because: we just let $p : \mathcal{F} \to \mathcal{C}$ be $(U, x) \mapsto U$, and we need to show pullback exists. Suppose we have

$$\begin{tikzcd}
g^*x \arrow[r] \arrow[d] & x \arrow[d] \\
U' \arrow[r, "g"] & U
\end{tikzcd}$$

we claim this is a pullback, i.e. all maps are Cartesian.

To check this, we have the following diagram

$$\begin{tikzcd}
U'' \arrow[r, "f"] \arrow[rr, bend right, "h"'] & U' \arrow[r, "g"] & U
\end{tikzcd}$$

In this diagram, we get $x \mapsto U, g^*x \mapsto U'$ and suppose we are given h^*x,

$$\begin{tikzcd}
h^*x \arrow[drr, bend left] & & \\
& g^*x \arrow[r] & x \\
U'' \arrow[r, "f"] \arrow[rr, bend right, "h"'] & U' \arrow[r, "g"] & U
\end{tikzcd}$$

We are trying to show there is unique map $h^*x \to g^*x$, i.e. we want

$$\begin{tikzcd}
h^*x \arrow[drr, bend left] \arrow[dr, dotted, "\exists!"] & & \\
& g^*x \arrow[r] & x \\
U'' \arrow[r, "f"] \arrow[rr, bend right, "h"'] & U' \arrow[r, "g"] & U
\end{tikzcd}$$

Why this map exists? By definition, $h^*x \to x$ means $(gf)^*x = f^*g^*x = f^*(g^*x)$ as we are working with presheaf. Hence we indeed get the desired arrow $h^*x \to g^*x$.

We should also check $\mathcal{F}(U)$ is a set, so that $\mathcal{F} \to \mathcal{C}$ is fibered in sets.

The objects of $\mathcal{F}(U)$ are (U, x) with $x \in F(U)$. The morphisms are $(U, y) \to (U, x)$ lying over identity Id_U, i.e.

$$\begin{tikzcd}
y \arrow[r] \arrow[d] & x \arrow[d] \\
U \arrow[r, "\mathrm{Id}_U"] & U
\end{tikzcd}$$

but $y = \mathrm{Id}_U^*(x)$ and hence $y = x$. Thus morphisms in $\mathcal{F}(U)$ are identity, i.e. $\mathcal{F}(U)$ is a set.

Proposition 9.5. *There is an equivalence of categories between*

$$\left\{\begin{matrix}\text{presheaves on} \\ \mathcal{C}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{categories fibered} \\ \text{in sets over } \mathcal{C}\end{matrix}\right\}$$

given by

$$F \mapsto \mathcal{F}_F$$
$$F_{\mathcal{F}} \leftarrow\!\shortmid \mathcal{F}$$

We defined the maps already, so one should check these are quasi-inverse maps

Throughout the course, we will identify $\mathcal{F}$ with $F_{\mathcal{F}}$.

Definition 9.6. A category is a ***groupoid*** if all morphisms are isomorphisms.

Example 9.7. If G is a group, then the category with one object and morphisms being G is a groupoids (i.e. if object is $\bullet$, then we get an arrow $\bullet \xrightarrow{g} \bullet$ for each $g \in G$). Thus groups are examples of groupoids.

Definition 9.8. A ***category fibered in groupoids*** is a fibered category $\mathcal{F} \to \mathcal{C}$ such that all $\mathcal{F}(U)$ are groupoids.

Proposition 9.9. *If $\mathcal{F}, \mathcal{F}'$ are categories fibered in groupoids over $\mathcal{C}$, then the category $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{F}')$ is a groupoid.*

Proof. Let $f, g : \mathcal{F} \to \mathcal{F}'$ with $\xi : f \to g$. We need to show ξ is isomorphism, i.e. forall $x \in \mathcal{F}$, we need to show $\xi_x : f(x) \xrightarrow{\sim} g(x)$. Let $X = p_{\mathcal{F}}(x)$, then since ξ is base-preserving natural trans, then ξ_x lies over Id_X, i.e. $\xi_x \in \mathcal{F}'(X) =$ groupoid, i.e. ξ_x is isomorphism. $\heartsuit$

We note in the proof, we only need $\mathcal{F}'$ being fibered in groupoids. Also, by the same argument, one can show that if $\mathcal{F} \to \mathcal{C}$ is category fibered in groupoids, then all arrows are Cartesian.

Next we consider a nice example of categories fibered in groupoids.

Definition 9.10. A ***groupoid in $\mathcal{C}$*** is $(X_0, X_1, s, t, \epsilon, i, m)$ such that $X_0, X \in \mathcal{C}$ and

$$\begin{tikzcd} X_1 \arrow[loop left, "i"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X_0 \arrow[l, bend left, "\epsilon"] \end{tikzcd}$$

and we get Cartesian square

$$\begin{tikzcd} X_1 & & \\ & X_1 \times_{s,X_0,t} X_1 \arrow[ul, "m"'] \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\square"] & X_1 \arrow[d, "t"] \\ & X_1 \arrow[r, "s"] & X_0 \end{tikzcd}$$

We note in the above, we used the notation $X_1 \times_{s,X_0,t} X_1$, which is just the fibered product, but we put emphasis on the two maps s and t that defines the fibered product.

So, here is the intuition: Here:

1. s: source
2. t: target
3. ϵ: identity
4. i: inverse

5. m: multiplication/composition

Here is how you supposed to think of this set of data.

X_0 is supposed to be like objects in a category, X_1 is supposed to be arrows in the category. Then what's going on with s and t in

$$X_1 \overset{s}{\underset{t}{\rightrightarrows}} X_0$$

is that, s takes an arrow and sends it to the source, while t takes an a rrow and sends it to the target.

$\epsilon : X_0 \to X_1$ takes objects to the identity arrow.

$i : X_1 \to X_1$ takes an arrow to its inverse (this supposed to exists because its a groupoid).

$m : X_1 \times_{s,X_0,t} X_1 \to X_1$ is like an arrow (α, β) such that the source of α is equal the target of β. In other word, (α, β) under m is supposed to be "$\alpha \circ \beta$".

The above is the intuition, and let's give the actual axioms about groupoids in $\mathcal{C}$.

Axioms:

1. $s \circ \epsilon = \mathrm{Id} = t \circ \epsilon$
2. $t \circ i = s$ and $s \circ i = t$.
3. $s \circ m = s \circ p_2$
4. $t \circ m = t \circ p_1$
5. (Associativity): the following two maps (we dropped the s, X_0, t in all the fibered products here)

$$X_1 \times X_1 \times X_1 \overset{m \times \mathrm{Id}}{\underset{\mathrm{Id} \times m}{\rightrightarrows}} X_1 \times X_1 \xrightarrow{m} X_1$$

are equal.
6. (Identity):

$$\begin{tikzcd}
 & X_1 \times_{s,X_0,\mathrm{Id}} X_0 \arrow[dr, "\epsilon \times \mathrm{Id}"] & & \\
X_1 \arrow[ur, "="] \arrow[dr, "="'] & & X_1 \times_{s,X_0,t} X_1 \arrow[r, "m"] & X_1 \\
 & X_0 \times_{\mathrm{Id},X_0,t} X_1 \arrow[ur, "\mathrm{Id} \times \epsilon"'] & &
\end{tikzcd}$$

This says $\alpha \circ \mathrm{Id} = \alpha$ and $\mathrm{Id} \circ \alpha = \alpha$.

7. (Inverse): we get diagrams

$$\begin{tikzcd} X_1 \arrow[r, "i\times \mathrm{Id}"] \arrow[d, "s"] & X_1 \times_{s,X_0,t} X_1 \arrow[d, "m"] \\ X_0 \arrow[r, "\epsilon"] & X_1 \end{tikzcd}$$

$$\begin{tikzcd} X_1 \arrow[r, "\mathrm{Id}\times i"] \arrow[d, "t"] & X_1 \times_{s,X_0,t} X_1 \arrow[d, "m"] \\ X_0 \arrow[r, "\epsilon"] & X_1 \end{tikzcd}$$

This says that $\alpha^{-1} \circ \alpha = \mathrm{Id}$ and $\alpha \circ \alpha^{-1} = \mathrm{Id}$.

10 Fibered Category

Last time we end up listing the axioms of groupoids in $\mathcal{C}$. The definition seems complicated, but the idea is not bad. Basically, X_0 should be objects, X_1 be morphisms, s takes morpihsms to its source, and t to the target, ϵ sends objects to identity map, i to inverse arrow, and m is composition of arrows.

We remark that, groupoids are sort of generalization of groups, where groups in category consists of only one object, and groupoids consists of more than one objects.

Today we are going to show we can get from groupoids in $\mathcal{C}$ to categories fibered in groupoids over $\mathcal{C}$.

Given $U \in \mathcal{C}$, let $\{X_0(U)/X_1(U)\}$ be a category defined as follows: objects are $X_0(U) = \mathrm{Hom}_{\mathcal{C}}(U, X_0)$, and morphisms for $u \to u'$ is an element $\alpha \in X_1(U)$ such that $s(\alpha) = u$ and $t(\alpha) = u'$ (here $u, u' : U \to X_0$, $\alpha : U \to X_1$ and hence $s(\alpha) = s \circ \alpha$ is an arrow $U \to X_0$, i.e. it make sense to ask $s(\alpha) = u$ and so on). This is a category, where composition of arrows are given by apply m, i.e. say we have $\eta : u'' \to u'$ and $\xi : u' \to u$, where $u'' \to u$ is given by $m(\xi, \eta)$.

Next, we define a fibered category $\mathcal{F} = \{X_0/X_1\}$ over $\mathcal{C}$ as follows: objects are (U, u), where $U \in \mathcal{C}$ and $u \in \{X_0(U)/X_1(U)\}$ (recall objects of this category is just $X_0(U)$). To get the morphisms, note given $f : V \to U$, we get the arrow (which is a functor)

$$\{X_0(U)/X_1(U)\} \xrightarrow{f^*} \{X_0(V)/X_1(V)\}$$

is well-defined (i.e. $f : U \to V$ induces $f^* : X_0(U) \to X_0(V)$ and $f^* : X_1(U) \to X_1(V)$ and hence $f^* : \{X_0(U)/X_1(U)\} \to \{X_0(V)/X_1(V)\}$). Then, morphisms $(V, v) \to (U, u)$ will be given by pairs $f : V \to U$ and $\alpha : v \xrightarrow{\sim} f^*u$ an isomorphism in $\{X_0(V)/X_1(V)\}$.

Then the projection $p : \mathcal{F} \to \mathcal{C}$ is going to be $p(U, u) = U$.

It remains to check $p : \mathcal{F} \to \mathcal{C}$ is a category fibered in groupoids.

The fiber $\mathcal{F}(U)$ is the category defined by: objects are, by definition, just $X_0(U)$ (as it is the same as the objects of $\{X_0(U)/X_1(U)\}$). The morphism for $(U, v) \to (U, u)$ is given by $U \xrightarrow{f} U$ and $\alpha : v \xrightarrow{\sim} f^*u$. However, since f must live over the identity, $f = \mathrm{Id}$ and hence $f^*u = u$. In other word, morphisms are just $X_1(U)$, i.e. $\mathcal{F}(U) = \{X_0(U)/X_1(U)\}$ is a groupoid, as desired.

Aside, if $\mathcal{F} \xrightarrow{p} \mathcal{C}$ is category fibered in groupoids, for $X \in \mathcal{C}$ we can define $p/X : \mathcal{F}/X \to \mathcal{C}/X$, which is a category fibered in groupoids where $\mathcal{F}/X$ behaves like objects of $\mathcal{F}$ over X. This notion is hardly been used, so if we need it in the future we will define it, but for now its just aside.

The next notion is rather important.

Definition 10.1. Given $\mathcal{F} \xrightarrow{p} \mathcal{C}$ a category fibered in groupoids, $x, x' \in \mathcal{F}(X)$. We define a preasheaf

$$\mathrm{Isom}(x, x') : (\mathcal{C}/X)^{opp} \to (\mathrm{Sets})$$

as follows.

Let $f : Y \to X$ in $\mathcal{C}$,

$$\mathrm{Isom}(x, x')(f) := \mathrm{Isom}_{\mathcal{F}(Y)}(f^*x, f^*x') = \mathrm{Hom}_{\mathcal{F}(Y)}(f^*x, f^*x')$$

because $\mathcal{F}(Y)$ is a groupoid (hence all arrows are isomorphisms). This depends on choices of pullbacks but we just fix one for all f.

Why is $\mathrm{Isom}(x, x')$ a presheaf?

Say we have have our arrows $Y \xrightarrow{f} X$, and $Z \xrightarrow{g} Y$. Then we get

$$\begin{tikzcd}
(gf)^*x' & g^*f^*x' & f^*x' & x' \\
(gf)^*x & g^*f^*x \arrow[u, "\sim"', "g^*\alpha"] & f^*x \arrow[u, "\sim"', "\alpha"] & x
\end{tikzcd}$$

where $(gf)^*$ and g^*f^* are two choices of pullback. However, note all pullbacks are isomorphic, we see that we get

$$\begin{tikzcd}
(gf)^*x' \arrow[r, "\gamma", "\sim"'] & g^*f^*x' & f^*x' & x' \\
(gf)^*x \arrow[r, "\beta", "\sim"'] & g^*f^*x \arrow[u, "\sim"', "g^*\alpha"] & f^*x \arrow[u, "\sim"', "\alpha"] & x
\end{tikzcd}$$

In particular, this means that β is canonical isomorphism and hence we get (canonical) arrow

$$\mathrm{Isom}(f^*x, f^*x') \to \mathrm{Isom}((gf)^*x, (gf)^*x')$$
$$\alpha \mapsto \gamma^{-1}\alpha\beta$$

which concludes $\mathrm{Isom}(x, x')$ is a presheaf (as it is compatible with composition of arrows).

Next, we define fibered products of groupoids. So, unlike normal fibered products in 1-category, now we are working with 2-categories, hence we also need to consider arrows between arrows.

We will start with a diagram of groupoids

$$\begin{tikzcd} & \mathcal{G}_1 \arrow[d, "f"] \\ \mathcal{G}_2 \arrow[r, "g"] & \mathcal{G} \end{tikzcd}$$

We are going to define $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ so that we get the following diagram:

$$\begin{tikzcd} \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathcal{G}_1 \arrow[d, "f"] \arrow[dl, Rightarrow, "\Sigma"'] \\ \mathcal{G}_2 \arrow[r, "g"] & \mathcal{G} \end{tikzcd}$$

where Σ is arrow between arrows.

Next, we will define some arbitrary category $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$, then we talk about universal properties that will convince us this is what fibered products should be for groupoids.

Definition 10.2. For groupoids $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}$ with $f : \mathcal{G}_1 \to \mathcal{G}$ and $g : \mathcal{G}_2 \to \mathcal{G}$, let $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ be the following category.

The objects are (x, y, σ) where $x \in \mathcal{G}_1, y \in \mathcal{G}_2$ and $f(x) \xrightarrow[\sigma]{\sim} g(y)$.

The morphisms for (x', y', σ') to (x, y, σ) will be a pair $(a : x' \to x, b : y' \to y)$ so that we get diagram

$$\begin{tikzcd} f(x') \arrow[r, "\sim", "f(a)"'] \arrow[d, "\sim"', "\sigma'"] & f(x) \arrow[d, "\sim"', "\sigma"] \\ g(y') \arrow[r, "g(b)", "\sim"'] & g(y) \end{tikzcd}$$

Next, we need to define p_1 and p_2. They are given by $p_1(x, y, \sigma) = x$ and $p_2(x, y, \sigma) = y$, and $\Sigma(x, y, \sigma) = \sigma$.

To add a few words on Σ, we note $\Sigma : f \circ p_1 \to g \circ p_2$, hence by definition this means we want that, inside $\mathcal{G}_1 \times \mathcal{G}_2$, for any $(x', y', \sigma') \to (x, y, \sigma)$, we get the following commutative square

$$\begin{tikzcd} f \circ p_1(x', y', \sigma') \arrow[r] \arrow[d, "\Sigma"] & f \circ p_1(x, y, \sigma) \arrow[d, "\Sigma"] \\ g \circ p_2(x', y', \sigma') \arrow[r] & g \circ p_2(x, y, \sigma) \end{tikzcd}$$

But if you expand the definition of p_i, Σ, this becomes exactly the square in definition of morphisms in $\mathcal{G}_1 \times \mathcal{G}_2$. Hence Σ is indeed natural trans as desired.

The universal property will be that, for all diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[drr, "\alpha"] \arrow[ddr, "\beta"'] & & \\
& & \mathcal{G}_1 \arrow[d, "f"] \\
& \mathcal{G}_2 \arrow[r, "g"'] & \mathcal{G} \arrow[ul, phantom, "\Swarrow\gamma"]
\end{tikzcd}$$

where $\mathcal{H}$ is a groupoid with $\mathcal{H} \xrightarrow{\alpha} \mathcal{G}_1$, $\mathcal{H} \xrightarrow{\beta} \mathcal{G}_2$ and isomorphism (which is natural transformation) $\gamma : f \circ \alpha \to g \circ \beta$, we get unique $(h, \lambda_1, \lambda_2)$ with diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[dr, "h"', "\exists!"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[u, phantom, "\Downarrow\lambda_1"] \arrow[l, phantom, "\Rightarrow\lambda_2"] & \mathcal{G}_1 \arrow[d, "f"] \\
& \mathcal{G}_2 \arrow[r, "g"'] & \mathcal{G} \arrow[ul, phantom, "\Swarrow\Sigma"]
\end{tikzcd}$$

where the two arrows λ_1, λ_2 are natural trans between arrows α (β, respectively) and the composing arrows of h and p_1 (h and p_2, respectively), so that the diagram

$$\begin{tikzcd}
f \circ \alpha \arrow[r, "f(\lambda_1)"] \arrow[d, "\gamma"] & f \circ p_1 \circ h \arrow[d, "\Sigma \circ h"] \\
g \circ \beta \arrow[r, "g(\lambda_2)"] & g \circ p_2 \circ h
\end{tikzcd}$$

commutes.

Well, why is this object exists? To answer this, we want to construct the unique $h : \mathcal{H} \to \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$. This will be the most "obvious" thing to do, which is $z \mapsto (\alpha(z), \beta(z), \gamma(z))$. We left the details of how h acts on morphisms, as it should be natural.

Now we want to make sure we get the natural transformations λ_1, λ_2. That is, we want a natural trans $\lambda_1 : \alpha \to p_1 \circ h$. This is the same as, for arbitrary z we want to get $\lambda_1(z) : \alpha(z) \to p_1(h(z)) = \alpha(z)$. Well, there is only one natural thing to do, which is take $\lambda_1(z)$ to be identity between $\alpha(z)$ and $\alpha(z)$. We do the same for λ_2.

Next we need to check commutativity of the following diagram

$$\begin{tikzcd}
f \circ \alpha \arrow[r, "f(\lambda_1)"] \arrow[d, "\gamma", "\sim"'] & f \circ p_1 \circ h \arrow[d, "\Sigma \circ h", "\sim"'] \\
g \circ \beta \arrow[r, "g(\lambda_2)"] & g \circ p_2 \circ h
\end{tikzcd}$$

where γ is given by definition:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathcal{G}_1 \arrow[d, "f"] \arrow[dl, Rightarrow, shorten=4ex, "\gamma"'] \\ \mathcal{G}_2 \arrow[r, "g"'] & \mathcal{G} \end{tikzcd}$$

Now for any z, we get

$$\begin{tikzcd} f(\alpha(z)) \arrow[r, "f(\lambda_1(z))"] \arrow[d, "\gamma(z)"] & f(\alpha(z)) \arrow[d, "\Sigma(h(z))"] \\ g(\beta(z)) \arrow[r, "g(\lambda_2(z))"] & g(\beta(z)) \end{tikzcd}$$

but then in particular $\Sigma(h(z)) = \Sigma(\alpha(z), \beta(z), \gamma(z)) = \gamma(z)$ by definition. Hence it is indeed commutative.

This concludes the definition of fibered products of groupoids, and we are heading to define fibered products of categories fibered in groupoids.

For this, let $\mathcal{F}_i \to \mathcal{C}$ be categories fibered in groupoids. Then we want to have a diagram

$$\begin{tikzcd} \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2 \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[d, "f"] \\ \mathcal{F}_2 \arrow[r, "g"] & \mathcal{F} \end{tikzcd}$$

so that for all $\mathcal{H} \to \mathcal{F}_1$ and $\mathcal{H} \to \mathcal{F}_2$ we get unique arrow $\mathcal{H} \to \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ with additional arrows between the arrows.

We want $\mathcal{G} = \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2$ to have the property that

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_1) \times_{\mathrm{Hom}_{\mathcal{C}}(\mathcal{H},\mathcal{F})} \mathrm{Hom}_{\mathcal{C}}(\mathcal{H}, \mathcal{F}_2)$$

However, on the RHS, they are just fibered products of groupoids, and by 2-Yoneda lemma, this determines $\mathcal{G}$ if the RHS exists.

Proposition 10.3 (Olsson, Prop 3.4.13). *The fibered product $\mathcal{G}$ exists.*

This concludes the topic about fibered products, and we are back to descents. After this, we will define what stacks are.

The idea of descents should be that, they are like sheaf axiom for fibered categories.

Example 10.4. Let X be a scheme and C be the category $\mathrm{Op}(X)$. Then consider $\mathcal{F} = (\mathrm{Vect}) \to \mathcal{C}$ where $\mathcal{F}(U)$ be the category of vector bundles on U. Then, if $U = \bigcup_i U_i$, a vector bundle on U is not equivalent to $\mathcal{E}_i$ on U_i with double intersections isomorphic (i.e. $\sigma_{ij} : \mathcal{E}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{E}_j|_{U_{ij}}$).

In this case, the naive sheaf axioms fit into the picture, i.e. we get

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij})$$

and this diagram is not exact.

We are missing the "cocycle" condition (to make the above diagram exact/equalizer). This means that, on $U_{ijk} = U_i \cap U_j \cap U_k$, we get diagram

$$\begin{tikzcd} \mathcal{E}_i|_{U_{ijk}} \arrow[r, "\sigma_{ij}"] \arrow[dr, "\sigma_{ik}"] & \mathcal{E}_j|_{U_{ijk}} \arrow[d, "\sigma_{jk}"] \\ & \mathcal{E}_k|_{U_{ijk}} \end{tikzcd}$$

In other word, the right "exact" diagram we need will be something like

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij}) \Rrightarrow \prod_{ijk} \mathcal{F}(U_{ijk})$$

Therefore, we want to formalize this triple arrow thing in fibered categories.

Let $p : \mathcal{F} \to \mathcal{C}$ be a fibered category. Given $f : X \to Y$ in $\mathcal{C}$, let $\mathcal{F}(X \xrightarrow{f} Y)$ be the category defined as follows (this is called category of descent data). The object should be (E, σ) with $E \in \mathcal{F}(X)$ and

$$X \times_Y X \times_Y X \Rrightarrow X \times_Y X \rightrightarrows X \xrightarrow{f} Y$$

where the triple arrows are p_{12}, p_{13}, p_{23} and the double arrows are p_1, p_2, and $\sigma : p_1^* E \to p_2^* E$ is an isomorphism in $\mathcal{F}(X \times_Y X)$ such that we get the following commutative diagram

$$\begin{tikzcd} p_{13}^* p_1^* E \arrow[r, "="] \arrow[d, "p_{13}^*\sigma"] & p_{12}^* p_1^* E \arrow[r, "p_{12}^*\sigma"] & p_{12}^* p_2^* E \arrow[d, "="] \\ p_{13}^* p_2^* E \arrow[r, "="] & p_{23}^* p_2^* E & p_{23}^* p_1^* E \arrow[l, "p_{23}^*\sigma"] \end{tikzcd}$$

where $=$ means canonical isomorphism. This is called the cocycle condition.

Remark 10.5. Here is just a brife recall of what all the above notations (i.e. $p_i^* E$, p_{ij}, etc) means.

First, recall that $p_i^* E$ are defined as the pullback of the following diagram

$$\begin{tikzcd} p_i^* E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \times_Y X \arrow[r, "p_i"] & X \end{tikzcd}$$

Similarly, $p_{jk}^* p_i E^*$ are defined as the pullback of the following diagram

$$\begin{tikzcd}
p_{jk}^* p_i^* E \arrow[r] \arrow[d] & p_i^* E \arrow[d] \\
X \times_Y X \times_Y X \arrow[r, "p_{jk}"] & X \times_Y X
\end{tikzcd}$$

Second, we note p_{ij} are projections come from the universal property of (fibered) products. In other word, note we would define $X_1 \times_S X_2 \times_S X_3$ as the unique object satisfies the following diagram

$$\begin{tikzcd}
& X_1 \times X_3 \arrow[dl] \arrow[dd, dashed] & & X_1 \times_Y X_2 \times_Y X_3 \arrow[ll] \arrow[dl] \arrow[dd] \\
X_1 \arrow[dd] & & X_1 \times X_2 \arrow[ll] \arrow[dd] & \\
& X_3 \arrow[dl, dashed] & & X_2 \times X_3 \arrow[ll, dashed] \arrow[dl] \\
Y & & X_2 \arrow[ll] &
\end{tikzcd}$$

Finally, a word on the isomorphisms $p_{jk}^* p_i^* E$. Continue with the above diagram (where now we let $X_1 = X_2 = X_3 = X$), we get the following

$$\begin{tikzcd}
& p_1^* E \arrow[dl] \arrow[dd, dashed] & & p_{13}^* p_1^* E \cong p_{12}^* p_1^* E \arrow[ll] \arrow[dl] \arrow[dd] \\
E \arrow[dd] & & p_1^* E \arrow[ll] \arrow[dd] & \\
& X_1 \times X_3 \arrow[dl, "p_1"] \arrow[dd, dashed] & & X_1 \times_Y X_2 \times_Y X_3 \arrow[ll, "p_{13}"] \arrow[dl, "p_{12}"] \arrow[dd] \\
X_1 \arrow[dd] & & X_1 \times X_2 \arrow[ll, "p_1"] \arrow[dd] & \\
& X_3 \arrow[dl, dashed] & & X_2 \times X_3 \arrow[ll, dashed] \arrow[dl] \\
Y & & X_2 \arrow[ll] &
\end{tikzcd}$$

where the two pullbacks along p_{13} and p_{12} must be the same object living over $X_1 \times X_2 \times X_3$, hence the canonical isomorphisms between $p_{13}^* p_1^* E \cong p_{12}^* p_1^* E$. The others are similar.

The idea is that, if we have $\mathcal{F}(Y) \xrightarrow{\epsilon} \mathcal{F}(X \xrightarrow{f} Y)$, then stack would be $\mathcal{F}$ fibered in groupoids where ϵ is an equivalence[1].

11 (Cat) Stack

Today we are finally going to define what a cat stack is:

It is a cat that stacks on your computer such that all pullbacks are still cat stacks on your computer!

Last time, we let $p : \mathcal{F} \to \mathcal{C}$ be a fibered category (not necessarily fibered in groupoids). Given $X \xrightarrow{f} Y$ in $\mathcal{C}$, we defined a category $\mathcal{F}(X \xrightarrow{f} Y)$ be the category of descent data.

The objects of this category is (E, σ) where $E \in \mathcal{F}(X)$ and $\sigma : p_1^* E \to p_2^* E$ is an isomorhism between $p_1^* E \xrightarrow{\sim} p_2^* E$. Here we have a diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and hence the two pullbacks of E should be isomorphic, and we just choose a particular σ. However, this σ cannot be arbitrary, as we need one additional condition called the cocycle condition.

The cocycle condition says that, all the ways to pullback E should all be commuting (we have three arrows p_{ij} from $X \times X \times X$ to $X \times X$, and two arrows from $X \times X$ to X, then we want $p_{23}^* \sigma \circ p_{12}^* \sigma = p_{13}^* \sigma$).

The point is that, we then get $\mathcal{F}(Y) \xrightarrow{\epsilon} \mathcal{F}(X \xrightarrow{f} Y)$ by

$$F \mapsto (f^* F, \sigma_{\text{can}})$$

[1]Here is some of my own understanding of this. Take it with a grind of salt. If we think those descent data are "triple intersection"/"cocycle" which are used in gluing, then this equivalence is the categorical way to say we are safe to glue things together (i.e. $\mathcal{F}(Y)$ are the same as glued things out of $X \to Y$).

This is because, $p_1^* f^* F$ and $p_2^* f^* F$ are pullbacks of F along $g : X \times_Y X \to Y$. For example, we get $p_1^* f^* F$ by the following diagram (where g is equal both fp_1 and fp_2 at the same time)

$$\begin{tikzcd}
p_1^* f^* F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & f^* F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d, "p"] \\
X \times_Y X \arrow[r, "p_1"] \arrow[rr, bend right, "g = fp_1 = fp_2"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Therefore we get a canonical map $\sigma_{\text{can}} : p_1^* f^* F \xrightarrow{\sim} p_2^* f^* F$.

In this case, we say f is an ***effective descent morphism*** if ϵ is equivalence of categories. In this case we also say $\mathcal{F}$ satisfies descent for f.

Before we say explain what this means, we recall the morphisms of $\mathcal{F}(X \xrightarrow{f} Y)$ is the following. From (E', σ') to (E, σ), a morphism is $\alpha : E' \to E$ in $\mathcal{F}(X)$ such that we get the following commuting diagram

$$\begin{tikzcd}
p_1^* E' \arrow[r, "p_1^* \alpha"] \arrow[d, "\sigma'"] & p_1^* E \arrow[d, "\sigma"] \\
p_2^* E' \arrow[r, "p_2^* \alpha"] & p_1^* E
\end{tikzcd}$$

More generally, we can define $\mathcal{F}(\{X_i \xrightarrow{f_i} Y\}_{i \in I})$ as $(\{E_i\}_{i \in I}, \{\sigma_{ij}\}_{i,j})$ such that $E_i \in \mathcal{F}(X_i)$ and $\sigma_{ij} : p_i^* E_i \xrightarrow{\sim} p_j^* E_j$ is an isomorhism, where

$$\begin{tikzcd}
X_{ij} := X_i \times_Y X_j \arrow[r, "p_j"] \arrow[d, "p_i"] \arrow[dr, phantom, "\square"] & X_j \arrow[d, "f_i"] \\
X_i \arrow[r, "f_i"] & Y
\end{tikzcd}$$

We also need the cocycle condition $\sigma_{jk} \circ \sigma_{ij} = \sigma_{ik}$.

Normally, we do not need to think about this more general case because of the following lemma.

> **Lemma 11.1** (Olsson, Lemma 4.2.7). *Assume coproducts exist in $\mathcal{C}$ and coproducts commute with fiber products when they exists. Assume for all sets of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$ the natural map $\mathcal{F}(\coprod X_i) \to \prod \mathcal{F}(X_i)$ is an equivalence. Then if $\{X_i \to Y\}$ are morphisms in $\mathcal{C}$ and $Q = \coprod_i X_i$, then $\mathcal{F}(Y) \to \mathcal{F}(\{X_i \to Y\})$ is equivalence (of categories) if and only if $\mathcal{F}(Y) \to \mathcal{F}(Q \to Y)$ is equivalence (of categories).*

Note this is coproduct, not co-fiber product. Co-fiber product may not exists in schemes. On the other hand, coproducts exists in schemes, as they are just disjoint union of schemes.

Thus, the point is that, we can always think $\mathcal{F}(\{X_i \to Y\})$ as $\mathcal{F}(\coprod X_i \to Y)$, which means we back to the first case.

Last time we mentioned informally what stacks are. This time we write down the formal definition.

Definition 11.2. A ***stack on a site*** $\mathcal{C}$ is a category fibered in groupoids $p : \mathcal{F} \to \mathcal{C}$, such that descent data is effective for covering maps, i.e. if $\{X_i \to Y\}_i \in \mathrm{Cov}(Y)$, then $\mathcal{F}(Y) \xrightarrow{\epsilon} \mathcal{F}(\{X_i \to Y\})$ is an equivalence between categories.

In short, stacks are just category fibered in groupoids where descent holds.

Remark 11.3. This is what we really call categorical stack, as we haven't done any actual geometry yet. Fibered categories are analogous to presheaves (they are presheaves when fibered in sets). So stacks are presheaves where sheaf axiom holds.

What this means is that, for example, take fppf topology on schemes and choose any sheaf $\mathcal{F}$. Then we say $\mathcal{F}$ is "geometric" if $\mathcal{F} = h_X$ for some scheme X. Those $\mathcal{F}$ are of course example of stacks.

Hence, in general, we want "representable stacks" (the technical term is algebraic stacks, or Artin stack) instead of arbitrary stacks.

Our next goal is to get a feeling about cat stacks, and then we try to find what would be a nice notion for algebraic stacks.

Well, I lied. Here is the punch line for what algebraic stacks are.

Remark 11.4 (Spoiler Alert)**.** The idea for algebraic stacks is that, if $\mathcal{F}$ is a stack over schemes. To import/involve geometry, we require that there exists X and arrow $X = h_X \to \mathcal{F}$, so that $h_X \twoheadrightarrow \mathcal{F}$ is a "smooth cover". This makes no sense, as we don't know what smooth covers are.

Thus, what should smooth cover $X \twoheadrightarrow \mathcal{F}$ be? Well, it should have the property that, for any scheme T, if we take fibered product

$$\begin{tikzcd}
X \times_{\mathcal{F}} T \arrow[r] \arrow[d, two heads, "g"', "sm"] \arrow[dr, phantom, "\square"] & X \arrow[d, two heads] \\
T \arrow[r] & \mathcal{F}
\end{tikzcd}$$

then the arrow $X \times_{\mathcal{F}} T \to T$ is a smooth cover. Well, this helps a little bit, as now over base becomes a scheme T. But, what is $X \times_{\mathcal{F}} T$? We don't know, hence we just insists that it should be nice, i.e. it should be a scheme by definition (this is actually not the full definition, i.e. the actual def is $X \times_{\mathcal{F}} T$ should be algebraic space).

In other word, algebraic stacks are stacks over scheme that we get a smooth cover $X \twoheadrightarrow \mathcal{F}$, where smooth cover means when we pullback along scheme T we always get $X \times_{\mathcal{F}} T$ be a scheme and $X \times_{\mathcal{F}} T \twoheadrightarrow T$ is a smooth cover of schemes.

Next, we consider an example of effective descent morphism.

Proposition 11.5. *If $f : X \to Y$ admits a section $s : Y \to X$ then descent data is effective for f.*

Note this holds for any category.

Proof. We have $\mathcal{F}(Y) \xrightarrow{\epsilon} \mathcal{F}(X \xrightarrow{f} Y)$ by $F \mapsto (f^*F, \sigma_{\text{can}})$. Thus we define $\eta : \mathcal{F}(X \to Y) \to \mathcal{F}(Y)$ by

$$(E, \sigma) \mapsto s^*E$$

where we recall $E \in \mathcal{F}(X)$ and $\sigma : p_1^*E \xrightarrow{\sim} p_2^*E$. Let's check this is what we wanted. Indeed, we see $\eta\epsilon(F) = \eta(f^*F, \sigma_{\text{can}}) = s^*f^*F \cong F$.

Now we need to go the other way, i.e. we start with (E, σ). We get the diagram

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then, we get

$$\begin{tikzcd} X \arrow[dr, dotted, "h"] \arrow[drr, bend left, "\mathrm{Id}"] \arrow[ddr, bend right, "sf"'] & & \\ & X \times_Y X \arrow[r, "p_1"'] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

This commutes because $f \circ \mathrm{Id} = f$ and $fsf = f$. Hence the dotted arrow h in the above exists.

Then we get $\sigma : p_1^*E \xrightarrow{\sim} p_2^*E$ and hence we get diagram

$$\begin{tikzcd} h^*p_1^*E \arrow[r, "\sim", "h^*\sigma"'] \arrow[d, "="] & h^*p_2^*E \arrow[d, "="] \\ E & f^*s^*E = \epsilon\eta(E, \sigma) \end{tikzcd}$$

Here $h^*p_2^* \cong (p_2h)^* \cong (sf)^* \cong f^*s^*$ hence $h^*p_2^*E \cong f^*s^*E$ and similarly $h^*p_1^*E \cong E = \mathrm{Id}^* E$. This gives an arrow $E \to f^*s^*(E, \sigma)$. Next, by pullback the cocycle diagram along the morphism $(\mathrm{Id}_{X \times_Y X}, sfp_2) : X \times_Y X \to (X \times_Y X) \times_Y X$, we get this map is compatible with σ. Hence this yields an isomorphism $(E, \sigma) \xrightarrow{\sim} \epsilon\eta(E, \sigma)$. ♡

This is a silly example, but it is actually very useful, as we can frequently reduce to the case where we have sections.

Now we consider descent for sheaves. Let $\mathcal{C}$ be a site where finite limits exist. Then take $f : X \to Y$ in $\mathcal{C}$, we get $\widetilde{\mathcal{C}/X} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \widetilde{\mathcal{C}/Y}$ maps between their topoi (here we use $\widetilde{\mathcal{C}/X}$ to denote topos). Here we get $(f^*\mathcal{F})(W \to X) = \mathcal{F}(W \to X \xrightarrow{f} Y)$. So, $f^*g^* = (gf)^*$ are equivalent.

Let $p : (\mathrm{Sh}) \to \mathcal{C}$ be the following category. The objects are (X, E) where $X \in \mathcal{C}$ and $E \in \widetilde{\mathcal{C}/X}$. The morphisms from (X, E) to (Y, F) are given by $X \xrightarrow{f} Y$ plus $E \to f^*F$. The projection is $p(X, E) = X$. We note this is not fibered in groupoids.

Theorem 11.6. *If $f : X \to Y$ is a covering in $\mathcal{C}$, then f is effective descent morphism for (Sh).*

Proof. Consider $(\mathrm{Sh})(X \to Y)$. This has objects (E, σ) where $E \in \widetilde{\mathcal{C}/X}$ and $\sigma : p_1^*E \xrightarrow{\sim} p_2^*E$ in $\mathcal{C}/\widetilde{(X \times_Y X)}$ that satisfies cocycle condition. For

$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[rr, bend right, "g"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

we want to construct an inverse functor $\eta : (\mathrm{Sh})(X \to Y) \to (\mathrm{Sh})(Y)$.

In this case, we get diagram

$$\begin{tikzcd} g_*p_2^*E \arrow[r, "g_*\sigma^{-1}", "\sim"'] \arrow[d, "="] & g_*p_1^*E \arrow[d, "="] \\ f_*(p_2)_*p_2^*E & f_*(p_1)_*p_1^*E \\ f_*E \arrow[u, "f_*(\text{adjunction})"] & f_*E \arrow[u] \end{tikzcd}$$

This is not a commutative diagram, thus we want to take equalizer. That is, we want to take $\mathrm{Eq}(f_*E \underset{\sigma^{-1}p_2^*}{\overset{p_1^*}{\rightrightarrows}} g_*p_1^*E) =: \eta((E, \sigma))$ as our definition. Here we abused notations. In particular, we write p_1^* to mean the arrow $f_*E \to f_*(p_1)_*p_1^*E$ given by apply f_* to the adjunction map $E \to (p_1)_*p_1^*E$ then take the reverse arrow of the arrow $g_*p_1^*E \to f_*(p_1)_*p_1^*E$. Similarly p_2^* is the arrow $f_*E \to g_*p_2^*E$ obtained from the above diagram. Also, that $\sigma^{-1}p_2^*$ is also abuse of notations, as what we really meant is $(g_*\sigma^{-1})p_2^*$ as in the above diagram.

Then, we claim $\mathrm{Id} \xrightarrow{\sim} \eta \circ \epsilon$. Indeed, if $F \in (\mathrm{Sh})(Y)$, then $\epsilon(F) = (f^*F, \sigma_{\mathrm{can}})$. Then $\eta\epsilon(F) = \eta(f^*F, \sigma_{\mathrm{can}}) = \mathrm{Eq}(f_*f^*F \underset{\sigma_{\mathrm{can}}^{-1}p_2^*}{\overset{p_1^*}{\rightrightarrows}} g_*p_1^*f^*F)$. Now note $p_1^*f^*F = g^*F$ and hence we just want to show that F is an equalizer of the arrow $f_*f^*F \rightrightarrows g_*g^*F$, then it will conclude $\eta\epsilon(F) = F$, where we note there is a natural map $F \to f_*f^*F$, i.e. we want to show $F \to f_*f^*F \rightrightarrows g_*g^*F$ is an equalizer diagram.

To do this, we prove it on Z-valued points, i.e. we take arbitrary $Z \to Y$, and we show it holds when we apply F to Z. Let

$$\begin{tikzcd} Z \arrow[r] & Y \\ X_Z \arrow[u, two heads, "\text{covering}"] \arrow[r] \arrow[ur, phantom, "\square"] & X \arrow[u, two heads, "f"'] \\ X_Z \times_Z X_Z \arrow[u, shift left] \arrow[u, shift right] \arrow[r] \arrow[ur, phantom, "\square"] & X \times_Y X \arrow[u, shift left] \arrow[u, shift right] \end{tikzcd}$$

Then, we see we get

$$F(Z) \longrightarrow F(X_Z) = f_* f^* F(Z \to Y) \rightrightarrows F(X_Z \times_Z X_Z) = g_* g^* F(Z \to Y)$$

That is exactly by def of sheaf an equalizer. Hence we see $\eta \circ \epsilon \cong \mathrm{Id}$ by Yoneda.

Next, we let (E, σ) be given and set $F := \eta(E, \sigma)$. Then $F \to f_* E$ by construction and so we get $f^* F \to f^* f_* E \to E$, i.e. we get canonical map $\rho(E, \sigma) : (f^* F, \sigma_{\text{can}}) = \epsilon\eta((E, \sigma)) \to (E, \sigma)$. We want to show $\rho(E, \sigma)$ is isomorphism.

To show $\rho(E, \sigma)$ is isomorphism, it is okay to do that locally on Y (left as exercise). To say do this locally, we mean that if

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, two heads, "g"] & Y \end{tikzcd}$$

then g^* on topoi is exact, so $g^* \circ \text{equalizer} = \text{equalizer} \circ g^*$. In other word, we get commutative diagram

$$\begin{tikzcd} (\mathrm{Sh})(X' \to Y') \arrow[d, "\eta'"] & (\mathrm{Sh})(X \to Y) \arrow[l, "(g')^*"] \arrow[d, "\eta"] \\ (\mathrm{Sh})(Y') & (\mathrm{Sh})(Y) \arrow[l, "g^*"] \end{tikzcd}$$

Then $(g')^* \rho(E, \sigma) = \rho((g')^* E, (g')^* \sigma)$. In particular, we see ρ' (this is the image of $\rho(E, \sigma)$ in $(\mathrm{Sh})(X' \to Y')$ in the above diagram) is isomorphism implies $\rho(E, \sigma)$ is isomorphism (this claim is left as exercise).

Now here is the trick: since we can take any cover g, we choose $g = f$. Now we get

$$\begin{tikzcd} X \times_y x \arrow[r] \arrow[d, two heads, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ X \arrow[u, bend left, "s'"] \arrow[r, two heads, "f"] & Y \end{tikzcd}$$

and s' is a section via the diagonal map, i.e. $x \mapsto (x, x) \in X \times_Y X$. Hence by the last theorem, we are done. ♡

12 (Cat) Stack

Last time, we defined fibered category of sheaves $(\mathrm{Sh}) \to \mathcal{C}$ over $\mathcal{C}$. We showed (Sh) is a "stack not fibered in groupoids", i.e. we have descent for covering in $\mathcal{C}$.

As a corollary of the theorem we proved above, we get

Proposition 12.1. *Let X, Y, S be schemes with diagram:*

$$\begin{tikzcd}
Y'' \arrow[r, shift left] \arrow[r, shift right] \arrow[dd, bend right] & Y' \arrow[r] \arrow[dd, bend left] & Y \arrow[dd, bend left] \\
X'' \arrow[u] \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X' \arrow[u, "f'"] \arrow[r] \arrow[d] & X \arrow[d] \\
S'' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & S' \arrow[r, two heads, "fppf", "g"'] & S \\
S' \times_S S' & &
\end{tikzcd}$$

where $X' = X \times_S S', X'' = X \times_S S''$, and similarly for Y', Y''. THe $f' : X' \to Y'$ over S' is such that $p_1^* f' = p_2^* f'$. Then there exists unique $f : X \to Y$ over S so that $g^* f = f'$.

Proof. We showed a big theorem: h_X, h_Y are fppf sheaves. Thus f' yields $h_{X'} \to h_{Y'}$. In particular, $p_1^* f' = p_2^* f'$ means this extends to $(h_{X'}, \sigma_{\mathrm{can}}) \to (h_{Y'}, \sigma_{\mathrm{can}})$ in $(\mathrm{Sh})(S' \twoheadrightarrow S)$. By big theorem last time, we see $(\mathrm{Sh})(S) \xrightarrow{\sim} (\mathrm{Sh})(S' \twoheadrightarrow S)$ and hence this arrow $(h_{X'}, \sigma_{\mathrm{can}}) \to (h_{Y'}, \sigma_{\mathrm{can}})$ correspond to an arrow $h_X \to h_Y$. Now Yoneda lemma tells us we have the desired arrow $f : X \to Y$. ♡

Next, we talk about variant of descent for sheaves. Let $\mathscr{O}$ be a sheaf of rings on a site $\mathcal{C}$. For all $X \in \mathcal{C}$, let $\mathscr{O}_X \in \widetilde{\mathcal{C}/X}$ be defined by $\mathscr{O}_X(Y \to X) := \mathscr{O}(Y)$. Then for all $f : X \to Y$, we get a map $(\widetilde{\mathcal{C}/X}, \mathscr{O}_X) \to (\widetilde{\mathcal{C}/Y}, \mathscr{O}_Y)$ map of "ringed topoi", i.e. map of topoi plus $\mathscr{O}_Y \to f_* \mathscr{O}_X$.

We will show this is almost a stack.

For $X \in \mathcal{C}$, let Mod_X be the category of $\mathscr{O}_X$-modules in $\widetilde{\mathcal{C}/X}$. Then for all $f : Y \to X$, we get $f^* : \mathrm{Mod}_X \to \mathrm{Mod}_Y$ by $(f^* M)(Z \to Y) := M(Z \to Y \to X)$.

Now we define fibered category $\mathrm{MOD} \xrightarrow{p} \mathcal{C}$ as follows: it has object (E, X) where $X \in \mathcal{C}$, $E \in \mathrm{Mod}_X$. The morphisms are $(F, Y) \to (E, X)$ is $f : Y \to X$ in $\mathcal{C}$ and $\epsilon : F \to f^* E$ in Mod_Y.

Theorem 12.2. *For all $f : Y \to X$ covers in $\mathcal{C}$, $\mathrm{Mod}_X \xrightarrow{\sim} \mathrm{MOD}(Y \to X)$.*

Now we have defined modules, the next topic is of course quasi-coherent sheaves.

Let S be a scheme, $\mathcal{C} = ((\mathrm{Sch})/S)_{fppf}$ be the fppf site associated with $(\mathrm{Sch})/S$.

Now $\mathscr{O}$ be the presheaf of rings on $\mathcal{C}$ defined as: $\mathscr{O}(T \to S) := \Gamma(\mathscr{O}_T) = \operatorname{Hom}_S(T, \mathbb{A}^1_S)$. This is just the global sections, i.e. it is the sheaf represented by $\mathbb{A}^1_S$. In other word, $\mathscr{O} = h_{\mathbb{A}^1_S}$ and hence we see this is a fppf sheaf as $h_{\mathbb{A}^1_S}$ is fppf sheaf.

Next, we want to figure out what's a reasonable notion of quasi-coherent for fppf topology.

For any scheme S, let $(\mathrm{Qcoh})(S)$ be the category of quasi-coherent $\mathscr{O}_S$-modules in Zariski topology, i.e. for S_{Zar}.

Given $\mathscr{F} \in (\mathrm{Qcoh})(S)$ we get $\mathscr{F}_{big}$, a presheaf of $\mathscr{O}$-modules on $\mathcal{C} = ((\mathrm{Sch})/S)_{fppf}$, defined as follows

$$\mathscr{F}_{big}(T \xrightarrow{f} S) := (f^* \mathscr{F})(T)$$

Note this depends on choice of pullback.

Lemma 12.3. *$\mathscr{F}_{big}$ is an fppf sheaf.*

Proof. Recall from awhile ago, to prove $\mathscr{F}_{big}$ is an fppf sheaf, we just need to check:

1. $\forall T \to S$, $\mathscr{F}_{big}|_{T_{Zar}}$ is a sheaf, and
2. sheaf condition on fppf arrow $\operatorname{Spec} B \twoheadrightarrow \operatorname{Spec} A$.

To see (1), we see it is enough to show $\mathscr{F}_{big}|_{T_{Zar}}$ is sheaf for small Zariski site. However, we see this is clearly a sheaf, because

$$\mathscr{F}_{big}|_{T_{Zar}} = f^* \mathscr{F} \in (\mathrm{Qcoh})(T)$$

by definition. So it is indeed a sheaf.

For (2), let $f : \operatorname{Spec} A \to S$ and $f^* \mathscr{F}$ be M an A-module. We need to check

$$\mathscr{F}_{big}(A) = M \longrightarrow \mathscr{F}_{big}(B) = M_B \rightrightarrows \mathscr{F}_{big}(B \otimes_A B) = M_{B \otimes_A B}$$

We showed this when showing schemes are fppf sheaves. ♡

So, $\mathscr{F} \in (\mathrm{Qcoh})(S)$ yields $\mathscr{F}_{big} \in (\widetilde{(\mathrm{Sch})/S})_{fppf}$. Conversely, given $\mathscr{H} \in (\widetilde{(\mathrm{Sch})/S})_{fppf}$ sheaf of $\mathscr{O}$-mods, and $T \xrightarrow{f} S$, we get $\mathscr{H}_T \in (\mathrm{Qcoh})(T)$ defined by

$$\mathscr{H}_T(U \subseteq T) = \mathscr{H}(U)$$

By construction, $\mathscr{F} \in (\mathrm{Qcoh})(S)$ is given by $(\mathscr{F}_{big})_S = \mathscr{F}$.

Proposition 12.4. *Let $\mathscr{F} \in (\mathrm{Qcoh})(S)$, $\mathscr{G}$ a fppf sheaf of $\mathscr{O}$-mods. Then*

$$\operatorname{Hom}_{\mathscr{O}}(\mathscr{F}_{big}, \mathscr{G}) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{G}_S)$$

given by $\alpha \mapsto \alpha|_{S_{Zar}}$.

Proof. Exercise: it is enough to check Zariski local on S, so we can assume S is affine. Then we get

$$\mathscr{F}_2 := \mathscr{O}_S^J \to \mathscr{F}_1 := \mathscr{O}_S^T \to \mathscr{F} \to 0$$

Then f^* is right exact, so $\mathscr{F}_{2,big} \to \mathscr{F}_{1,big} \to \mathscr{F}_{big} \to 0$. As a result, we get

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(\mathscr{F}_{big}, \mathscr{G}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathscr{F}_{1,big}, \mathscr{G}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathscr{F}_{2,big}, \mathscr{G}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(\mathscr{F}, \mathscr{G}_S) \arrow[r] & \mathrm{Hom}(\mathscr{F}_1, \mathscr{G}_S) \arrow[r] & \mathrm{Hom}(\mathscr{F}_2, \mathscr{G}_S) \arrow[r] & 0 \end{tikzcd}$$

So, we can assume $\mathscr{F} = \mathscr{O}_S^I$. Therefore, we may assume $\mathscr{F} = \mathscr{O}_S$, i.e. $|I| = 1$. Now, observe elements in $\mathrm{Hom}_{\mathscr{O}}(\mathscr{O}, \mathscr{G})$ means that we have compatible maps $\phi_T : \mathscr{O}(T \to S) \to \mathscr{G}(T \to S)$. In particular, this means for any f with diagram

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[dr, "f"] & S \arrow[d] \\ & S \end{tikzcd}$$

we get $f : (T \to S) \to (S \to S)$. Thus we get diagram

$$\begin{tikzcd} \mathscr{O}(S) \arrow[r, "\phi_S"] \arrow[d, "f^*"] & \mathscr{G}(S) \arrow[d, "f^*"] \\ \mathscr{O}(T) \arrow[r, "\phi_T"] & \mathscr{G}(T) \end{tikzcd}$$

so ϕ_T is determined by ϕ_S. Thus the map $\mathrm{Hom}_{\mathscr{O}}(\mathscr{O}, \mathscr{G}) \xrightarrow{\eta} \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{O}_S, \mathscr{G}_S)$ is isomorphism and hence ϕ_S iff $\xi \in \mathscr{G}(S)$ then $\phi_T(1) = f^*\xi$.

♡

Definition 12.5. A ***big quasi-coherent sheaf on*** S of $\mathscr{O}$-mods is $\mathscr{F}$ on S_{fppf} such that:

1. $\forall T \to S$, $\mathscr{F}_T := \mathscr{F}|_{T_{Zar}} \in (\mathrm{Qcoh})(T)$
2. $\forall T \xrightarrow{f} S$, $\mathscr{F}_T \to f^*\mathscr{F}_S$ is an isomorphism.

Proposition 12.6. *THere is an equivalent of categories*

$$(Qcoh)(S_{Zar}) \xrightarrow{\sim} (Qcoh)(S_{fppf})$$
$$\mathscr{F} \mapsto \mathscr{F}_{big}$$
$$\mathscr{G}_S \leftarrow\!\shortmid \mathscr{G}$$

We get fibered category $(\mathrm{Qcoh}) \to \mathcal{C}$.

Theorem 12.7. *If we have fppf $f : Y \twoheadrightarrow X$ then $(Qcoh)(X) \xrightarrow{\sim} (Qcoh)(Y \to X)$.*

Proof. We only show the local case and the full proof can be found in the book.

Martin reduces to the case where $f : Y \to X$ is qcqs (quasi-compact, quasi-separated). In this case, pushforward of quasi-coherent sheaf is quasi-coherent, i.e. $f(\text{qcoh}) = \text{qcoh}$.

Then, we note

$$(\text{Sh})(Y \xrightarrow{f} X) \underset{\eta}{\xrightarrow{\sim}} (\text{Sh})(X)$$

$$(\mathscr{F}, \sigma) \mapsto \text{Equalizer of } f'_* S$$

Since $f_*(\text{qcoh}) = \text{qcoh}$, and equalizer of qcoh is qcoh, we see η sends $(\text{Qcoh})(Y \to X)$ to $(\text{Qcoh})(X)$. ♡

Now we consider something that we already know, but in the new language.

We consider descent for closed subschemes:

Proposition 12.8. *Suppose we have fppf cover*

$$X \times_Y X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \underset{fppf}{\overset{f}{\twoheadrightarrow}} Y$$

Then the set of closed $W \subseteq Y$ is equivalent to the set of closed $Z \subseteq X$ such that $p_1^* Z = p_2^* Z$ given by the map $W \mapsto f^{-1}(W)$.

Proof. We note closed $W \subseteq Y$ is the same as quasi-coherent sheaf of ideals $\mathscr{I}_W \subseteq \mathscr{O}_Y$. Because f, p_1, p_2 are flat, pullback of ideal is an ideal. The result follows from descent of quasi-coherent sheaves applied to ideal sheaves. ♡

In a very similar manner, we get descent for affine maps.

Let $(\text{Aff}) \to (\text{Sch})$ be the fibered category with objects $(X' \xrightarrow{g} X, X)$ with g affine map.

Proposition 12.9. *If we have fppf cover $S' \twoheadrightarrow S$, then $(Aff)_S \xrightarrow{\sim} (Aff)(S' \to S)$.*

Proof. Say $X \to S$ is affine, this is the same as $X = \operatorname{Spec} \mathscr{A}$ where $\mathscr{A}$ is qcoh sheaf of $\mathscr{O}_S$-algebra. Then we have descent for quasi-coherent sheaves, and we want to make sure it is still $\mathscr{O}_S$-algebra.

That is, how do we know if $\mathscr{A}' = f^* \mathscr{A}$ and $\mathscr{A}'$ a $\mathscr{O}_{S'}$-algebra, then $\mathscr{A}$ is $\mathscr{O}_S$-algebra?

Being an algebra means we have $m : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ with commutativity diagrams. But this is a map and hence $m' : \mathscr{A}' \times \mathscr{A}' \to \mathscr{A}'$ descents to get m and diagrams can be checked locally. ♡

13 G-Bundles

In this section, we are going to start with G-bundles/torsors.

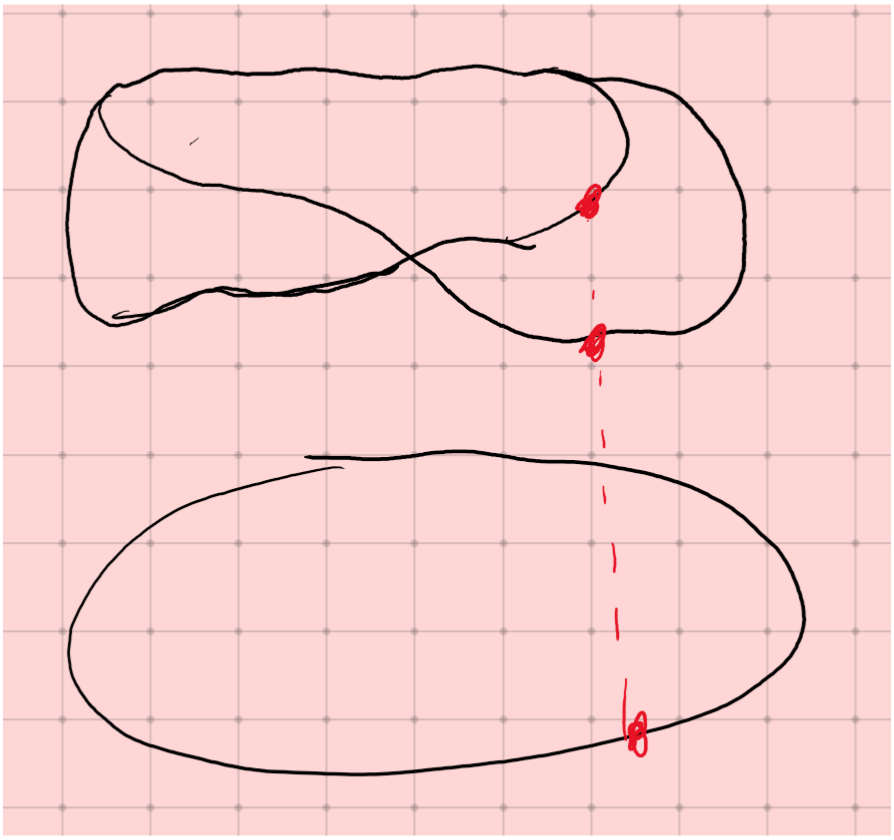
Thus, let X be a scheme and $\mathcal{C}((\mathrm{Sch})/X)_{fppf}$. We are going to start with a group scheme over X, say G, which is flat and locally of finite presentation over X.

Definition 13.1. A ***(principle)*** G***-bundle*** is a scheme $\pi : P \to X$ over X, where π is fppf, with an G-action $\rho : G \times P \to P$, such that the map $G \times_X P \xrightarrow{\sim} P \times_X P$ given by $(g,p) \mapsto (gp,p)$, is isomorphism, where $gp = \rho(g,p)$.

This $G \times_X P \xrightarrow{\sim} P \times_X P$ condition is equivalent to saying if $Y \to X$ and $P(Y) \neq \varnothing$, then the group action of $G(Y)$ acts on $P(Y)$ is simple and transitive, i.e. $G(Y)$ acts on $P(Y)$ has no stabilizers and for all $p, p' \in P(Y)$ there exists $g \in G(Y)$ so $gp = p'$, i.e. for all $p, p' \in P(Y)$ there exists unique $g \in G(Y)$ so $p' = gp$.

So, before we give examples, we talk about the idea of G-bundle. In particular, we can think of P as a group without a choice of identity. Here, if we give two elements, we only care about the difference, not which particular g we are working with(so it is similar to the idea of potential functions in physics, i.e. we only care about diff of potentials, not the initial value).

Example 13.2. Consider $\mathbb{C} \supseteq S^1 \to S^1$ given by $z \mapsto z^2$. Here is a picture:



Then there is a $\mathbb{Z}/2\mathbb{Z} = S^1 \coprod S^1$ action: we swap the 2 strands, i.e. $z \mapsto -z$. Locally, P is $S^1 \coprod S^1 = \mathbb{Z}/2$ but not globally.

Definition 13.3. A ***map of G-bundles*** $P \to P'$ is a map of S-schemes $f : P \to P'$ such that

$$\begin{tikzcd} G \times P \arrow[r, "\rho"] \arrow[d, "\mathrm{Id} \times f"] & P \arrow[d, "f"] \\ G \times P' \arrow[r, "\rho'"] & P' \end{tikzcd}$$

Definition 13.4. Let $\mathcal{C}$ be a site and μ a sheaf of groups. A ***μ-torsor*** is a sheaf $\mathcal{P}$ with $\mu \times \mathcal{P} \to \mathcal{P}$ such that:

1. for all $X \in \mathcal{C}$, there exists $\{X_i \to X\} \in \mathrm{Cov}(X)$ such that $\mathcal{P}(X_i) \neq \varnothing$ for all i.
2. we have $\mu \times \mathcal{P} \xrightarrow{\sim} \mathcal{P} \times \mathcal{P}$ where the map is given by $(g, p) \mapsto (gp, p)$.

Definition 13.5. A μ-torsor is ***trivial*** if $\mathcal{P} \cong \mu$ as μ-torsor.

Proposition 13.6. *If μ is representable on $\mathcal{C} = ((Sch)/X)_{fppf}$ by a group scheme G, then*

$$\{Principle\ G\text{-}bundle\} \xrightarrow{\epsilon} \{\mu\text{-}torsor\}$$
$$P \mapsto h_P$$

is fully faithful. And if $G \to X$ is affine then ϵ is equivalence.

Proof. Yoneda says $P \mapsto h_P$ is fully faithful. Why h_P is μ-torsor? In other word, in def of torsor, (2) holds by definition, but why we have (1)?

Consider

$$\begin{tikzcd} & P \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

We want fppf cover $Y' \twoheadrightarrow Y$ such that $P(Y') \neq \varnothing$. However, note we have

$$\begin{tikzcd} Y' = P_Y \arrow[r, "p \in P(Y')"] \arrow[d, two heads, "fppf"] \arrow[dr, phantom, "\square"] & P \arrow[d, two heads, "fppf"] \\ Y \arrow[r] & X \end{tikzcd}$$

Hence we are done.

Now assume $G \to X$ is affine and $\mathcal{P}/X$ is μ-torsor, we want $\mathcal{P} = h_P$ where P is a G-bundle.

Note by assumption, we can find fppf cover $\{X_i \twoheadrightarrow X\}$ such that $\mathcal{P}(X_i) \neq \varnothing$. But then $\mathcal{P}|_{X_i} \cong \mu_{X_i}$ since \mu$(X_i)$ acts on $\mathcal{P}(X_i)$ simple and transitive where $\mu_{X_i} = h_{G \times_X X_i}$, i.e. $\mathcal{P}$ is locally representable.

Since $\mathcal{P}$ is a sheaf over X, $\mathcal{P}|_{X_i}$ have canonical descent data. On the other hand, $\mathcal{P}|_{X_i} \cong G \times_X X_i$ is scheme affine over X_i. So, this yields descent data for the $G \times_X X_i$. But last result we showed last class is descent data is effective for affine morphisms. Therefore, we get a scheme $P \to X$ that also defines a sheaf h_P which must agree with $\mathcal{P}$ because they are sheaves with same descent data.

Last thing we need to check is that why $P \to X$ is a G-bundle.

We need to check there exists action $G \times P \to P$ such that the graph of the action $G \times_X P \xrightarrow{\sim} P \times_X P$ is isomorphism.

Why $G \times_X P \to P$ exists? Well, we have $G \times P \to P$ by Yoneda because we have $\mu \times \mathcal{P} \to \mathcal{P}$ action such that $\mu \times \mathcal{P} \xrightarrow{\sim} \mathcal{P} \times \mathcal{P}$ so by Yoneda we get $G \times_X P \to P$ with $G \times P \xrightarrow{P} \times P$. ♡

Remark 13.7. If $G \to X$ is smooth, then $P \to X$ is also smooth. This is by fppf descent, i.e. we get

$$\begin{tikzcd} G \times_X P \arrow[r, "\cong"] \arrow[dr, "smooth"'] & P \times_X P \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & P \arrow[d, two heads, "fppf"] \\ & P \arrow[r, two heads, "fppf"] & X \end{tikzcd}$$

and hence the arrow $P \to X$ is smooth as well. Note here smooth does nothing and it can be changed to basically any property.

Proposition 13.8. *Let X be a scheme, $\mathscr{F}$ be a sheaf on X, and $\mu = \mathrm{Aut}(\mathscr{F})$. Then there is an equivalence of categories*

$$\mu - \text{torsors on } X \xrightarrow{\sim} \begin{matrix} \text{sheaves locally} \\ \text{isomorphic to } \mathscr{F} \end{matrix}$$

given by map

$$\mathscr{P} \mapsto \mathscr{H}_{\mathscr{P}} := \mathrm{Hom}_\mu(\mathscr{P}, \mathscr{F})$$
$$\mathscr{P}_{\mathscr{H}} := \mathrm{Isom}(\mathscr{F}, \mathscr{H}) \leftarrow\!\shortmid \mathscr{H}$$

where Hom_μ means μ-equivariant maps.

Proof. We first show $\mathscr{P}_{\mathscr{H}}$ is μ-torsor. We deinfe a map $\mu \times \mathscr{P}_{\mathscr{H}} \to \mathscr{P}_{\mathscr{H}}$ by $\mu(Y) \times \mathscr{P}_{\mathscr{H}}(Y) \to \mathscr{P}_{\mathscr{H}}(Y)$ by $(f, \lambda) \mapsto f \circ \lambda$, i.e. we get a diagram

$$\begin{tikzcd} \mathscr{F}_Y \arrow[r, "f", "\sim"'] \arrow[dr, dashed, "f \circ \lambda"'] & \mathscr{F}_Y \arrow[d, "\lambda"', "\sim"] \\ & \mathscr{H}_Y \end{tikzcd}$$

Why is this simple and transitive? If $\mathscr{P}_{\mathscr{H}}(Y) \neq \varnothing$, then for $\lambda, \lambda' \in \mathscr{P}_{\mathscr{H}}(Y)$ we get

$$\begin{tikzcd} \mathscr{F}_Y \arrow[r, "f", "\sim"'] \arrow[dr, "p"'] & \mathscr{H}_Y \arrow[d, "(\lambda')^{-1}"', "\sim"] \\ & \mathscr{F}_Y \end{tikzcd}$$

i.e. $\lambda' = p \circ \lambda$ for a unique p.

Why $\mathscr{H}_{\mathscr{P}}$ is a sheaf? Given

$$Y'' = Y' \times_Y Y' \overset{p_1}{\underset{p_2}{\rightrightarrows}} Y' \xrightarrow[f]{fppf} Y$$

Then for $\phi' \in \mathscr{P}_{\mathscr{H}}(Y')$ with $\phi' : \mathscr{F}_{Y'} \to \mathscr{H}_{Y'}$ μ-equivariant map such that $p_1^*\phi' = p_2^*\phi'$, since sheaves satisfy descent so we get a unique $\phi : \mathscr{F}_Y \to \mathscr{H}_Y$ such that $f^*\phi = \phi'$.

Why ϕ is μ-equivariant? Viz, we want commutative diagram

$$\begin{tikzcd} \mu \times \mathscr{F}_Y \arrow[r] \arrow[d, "\mathrm{Id} \times \phi"] & \mathscr{F}_Y \arrow[d, "\phi"] \\ \mu \times \mathscr{H}_Y \arrow[r] & \mathscr{H}_Y \end{tikzcd}$$

However, diagram commutes fppf locally over Y', hence it commutes over Y.

Why $\mathscr{H}_{\mathscr{P}} \cong \mathscr{F}$ locally?

Locally we get μ-equivariant isomorphism $\mathscr{P} \cong \mu$. So we claim $\mathscr{H}_{\mathscr{P}}$ is locally isomorphic to $\mathrm{Hom}_\mu(\mu, \mathscr{F})$. Well, suppose we have $\mu \xrightarrow{\alpha} \mathscr{F}$. Then we see $\alpha(\zeta) = \alpha(\zeta \cdot 1) = \zeta \cdot \alpha(1)$ and hence $\mathrm{Hom}_\mu(\mu, \mathscr{F}) \cong \mathscr{F}$ given by $\alpha \mapsto \alpha(1)$. This concludes the proof.

♡

Remark 13.9. If $\mathscr{F}$ is an $\mathscr{O}_X$-module, and $\mu = \mathrm{Aut}_{\mathscr{O}_X}(\mathscr{F})$, then μ-torsor are equivalent to $\mathscr{O}_X$-modules $\mathscr{H}$ which are locally isomorphic to $\mathscr{F}$ as $\mathscr{O}_X$-modules.

Example 13.10. Take $\mathscr{F} = \mathscr{O}_X$ as $\mathscr{O}_X$-modules. Then we are shown that line bundles are isomorphic to μ-torsors, where $\mu = \mathrm{Aut}_{\mathscr{O}_X}(\mathscr{O}_X) = \mathscr{O}_X^*$, where $\mathscr{O}_X^*$ is representable by $\mathbb{G}_m = \mathrm{Spec}\,\mathbb{Z}[x, x^{-1}]$. Hence, we see the line bundles are the same as $\mathbb{G}_m$-bundles.

Clearly we don't have to restrict to $\mathscr{O}_X$. In other word, we can take $\mathscr{F} = \mathscr{O}_X^n$, then $\mu = \mathrm{GL}_n(\mathscr{O}_X)$, which is representable by scheme $\mathrm{GL}_n = \mathrm{Spec}\,\mathbb{Z}[x_{ij}, x_{ij}^{-1} : 1 \le i, j \le n]$. We can also define this as $M_n := \mathrm{Spec}\,\mathbb{Z}[x_{ij}, 1 \le i, j \le n]$ and then $\deg(x_{ij})$ is a polynomial and we look at $\mathrm{GL}_n := M_n \backslash V(\det)$, which is affine. In this case, we just get rank n vector bundles are equivalent to GL_n-bundles.

Example 13.11. We can also talk about Brauer-Sever varieties, which are locally isomorphic to $\mathbb{P}^n$. Those are PGL_n-torsors, and it is related to Azumaya algebras and the Brauer groups, where the Brauer groups are deeply related to class field theory in number theory.

14 Algebraic Space

Recall that stack is a category fibered in groupoids where descent holds for all covering maps.

Proposition 14.1. *If $\mathcal{F} \to \mathcal{C}$ is a stack, then for all $X \in \mathcal{C}$ and $x, y \in \mathcal{F}(X)$, we have* $\mathrm{Isom}(x, y)$ *is a sheaf on $\mathcal{C}/X$.*

Proof. If $Y' \xrightarrow{f} Y$ is a covering, then consider

$$Y'' = Y' \times_Y Y' \overset{p_2}{\underset{p_1}{\rightrightarrows}} Y' \xrightarrow{f} Y$$

in $\mathcal{C}/X$, i.e. we also have an arrow $Y \to X$ by default. Then we get $x, y \in X$ and hence we get a bunch of pullbacks

$$\begin{tikzcd} Y'' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & Y' \arrow[r, "f"] & Y \arrow[r, "g"] & X \end{tikzcd}$$

$$\begin{tikzcd} p_1^* f^* g^* x \arrow[d, shift right, "p_1^*\phi"'] \arrow[d, shift left, "p_2^*\phi"] & f^* g^* x \arrow[d, "\phi"] & g^* x & x \\ p_1^* f^* g^* y & f^* g^* y & g^* y & y \end{tikzcd}$$

Note here the double arrows of pullback is actually a square

$$\begin{tikzcd} p_1^* f^* g^* x \arrow[r, "\sim", "\mathrm{can}"'] \arrow[d, "p_1^*\phi"] & p_2^* f^* g^* x \arrow[d, "p_2^*\phi"] \\ p_1^* f^* g^* y \arrow[r, "\sim", "\mathrm{can}"'] & p_2^* f^* g^* y \end{tikzcd}$$

Suppose $p_1^*\phi = p_2^*\phi$. We want ϕ to descent to $g^*x \xrightarrow{\sim} g^*y$. By definition, ϕ defines an isomorphism $(f^*g^*x, \sigma_{\mathrm{can}}) \xrightarrow{\sim} (f^*g^*y, \sigma_{\mathrm{can}})$ in $\mathcal{F}(Y' \to Y)$. However, we know $\mathcal{F}(Y) \underset{\epsilon}{\xrightarrow{\sim}} \mathcal{F}(Y' \to Y)$ as $\mathcal{F}$ is a stack, thus we are done as desired. ♡

Proposition 14.2. *Let*

$$\begin{tikzcd} & \mathcal{F}_1 \arrow[d] \\ \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \end{tikzcd}$$

be maps of stacks over $\mathcal{C}$. Let $\mathcal{F} := \mathcal{F}_1 \times_{\mathcal{F}_3} \mathcal{F}_2$ be the fibered product of category fibered in groupoids. Then $\mathcal{F}$ is a stack.

Proof. By definition, $(\mathcal{F}_1 \times_{\mathcal{F}_3} \mathcal{F}_2)(X) = \mathcal{F}_1(X) \times_{\mathcal{F}_3(X)} \mathcal{F}_2(X)$. Similarly, an easy check shows

$$\mathcal{F}(\{X_i \to X\}) = \mathcal{F}_1(\{X_i \to X\}) \times_{\mathcal{F}_3(\{X_i \to X\})} \mathcal{F}_2(\{X_i \to X\})$$

Each $\mathcal{F}_i(X) \xrightarrow{\sim} \mathcal{F}_i(\{X_i \to X\})$ and hence $\mathcal{F}$ is a stack. ♡

Just like for sheaves we have sheafification, for stacks we have "stack-fication".

Theorem 14.3 (Theorem 4.6.5 in Martin). *Let $\mathcal{F}$ be category fibered in groupoids over $\mathcal{C}$. Then there exists stack $\mathcal{F}^a/\mathcal{C}$ and $\mathcal{F} \to \mathcal{F}^a$ such that for all stacks $\mathcal{H}/\mathcal{C}$, we have* $\mathrm{HOM}_{\mathcal{C}}(\mathcal{F}^a, \mathcal{H}) \xrightarrow{\sim} \mathrm{HOM}_{\mathcal{C}}(\mathcal{F}, \mathcal{H})$.

Next, we are going to define algebraic spaces, but first, we give some ideas.

Idea: what is a scheme? A scheme is affine schemes glued in Zariski topology. Then algebraic space is affine schemes glued in etale topology.

Of course, now we are just begging for the question of what if fppf topology. It turns out, it is not so easy to answer this question. The answer is that a theorem of Artin, where he showed they are the same as algebraic spaces.

Let S be a scheme, $\mathcal{C} = ((\mathrm{Sch})/S)_{et}$.

Definition 14.4. A morphism of sheaves $\mathcal{F} \to \mathcal{H}$ is ***representable by schemes*** if for all $T \to \mathcal{H}$ with $T = h_T$ scheme, the fibered product (as cat fibered in sets) $\mathcal{F} \times_{\mathcal{H}} T$ is a scheme.

So, when we say $\mathcal{F} \to \mathcal{H}$ is representable by schemes, we mean for all T we get the following diagram

$$\begin{tikzcd} \mathcal{F} \times_{\mathcal{H}} T \in (\mathrm{Sch}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{F} \arrow[d] \\ T \arrow[r] & \mathcal{H} \end{tikzcd}$$

Definition 14.5. Let P be a property of morphisms of schemes. If for all diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

we have:

1. f has P implies f' has P, then we say P is ***stable under base change***.
2. if g is a Zar (et, sm, fppf) covering, then f has P iff f' has P, then we say P is ***local on the base for the Zar (et, sm, fppf) topology***.

Finally, we say P is ***local on the source for the Zar (et, sm, fppf) topology***, if for all diagrams

$$\begin{tikzcd} X' \arrow[r, "\pi"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

with π a Zar (et, sm, fppf) covering, f has P iff g has P.

Definition 14.6. If $\mathcal{F} \xrightarrow{f} \mathcal{H}$ is representable by schemes and P is a property which is stable under base change and local on the base, then we say f ***has*** P iff for all $T \to \mathcal{H}$ with T schemes, $\mathcal{F} \times_{\mathcal{H}} T \to T$ has P.

We remark that if $\mathcal{F} = h_X$ and $\mathcal{H} = h_Y$ then $\mathcal{F} \to \mathcal{H}$ has P iff $X \to Y$ has P because we can consider the identity $Y \to \mathcal{H}$ map and pull it back using this.

Lemma 14.7. *Let $\mathcal{F}$ be a presheaf on $(Sch)/S$. Then $\Delta : \mathcal{F} \to \mathcal{F}_S \mathcal{F}$ is representable by schemes if and only if for all $T \to \mathcal{F}$ is representable by schemes for all schemes T.*

Proof. Consider

$$\begin{tikzcd} T \times_{\mathcal{F}} T' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "f"] \\ T' \arrow[r, "f'"] & \mathcal{F} \end{tikzcd}$$

Then we get

$$\begin{tikzcd} \mathcal{H} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \times T' \arrow[d, "{(f,f')}"] \\ \mathcal{F} \arrow[r, "\Delta"] & \mathcal{F} \times \mathcal{F} \end{tikzcd}$$

Here $\mathcal{H}$ is (x, t, t') with $(x, x) = \Delta(x) = (f(t), f'(t))$. So, $\mathcal{H} \cong T' \times_{\mathcal{F}} T$. Δ representable by schemes, so $\mathcal{H}$ is scheme, so $T' \times_{\mathcal{F}} T$ is scheme, as desired. ♡

Definition 14.8. An algebraic space over S is a sheaf $\mathcal{F}$ for the big etale topology on $(\mathrm{Sch})/S$ such that:

1. $\Delta : \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable by scheme.
2. there exists scheme U and etale covering $\pi : U \twoheadrightarrow F$

We note (2) makes sense because (1) implies π is representable by schemes and etale surjections are property that are stable under base change and local on the base.

Remark 14.9. Schemes are algebraic spaces because we see (1) is true as $\mathcal{F} = h_X$ is a scheme, and for (2) we choose $\pi = \mathrm{Id}_X$. This is because h_X is sheaf for fppf topology, so also it is also sheaf for etale topology.

Definition 14.10. A morphism $\mathfrak{X} \to \mathfrak{Y}$ of stacks is representable if for all diagrams with Y algebraic space

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ Y \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

we get X is an algebraic space.

Proposition 14.11. *A morphism of stacks $\mathfrak{X} \to \mathfrak{Y}$ is representable iff for all diagrams with Y a scheme*

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ Y \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

we get X is algebraic space.

Proof. If Y is algebraic space then we know X is a stack, which proves the forward direction. We show the converse. Hence, suppose we are given $Y \to \mathfrak{Y}$ where Y is algebraic space, we want to show $\mathfrak{Z}$ is an algebraic space, where the stack $\mathfrak{Z}$ is defined by

$$\begin{tikzcd} \mathfrak{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ Y \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

(1): we want to show $\mathfrak{Z}$ is a sheaf. Since $\mathfrak{Z}$ is a stack, so we just need to show $\mathfrak{Z}$ is fibered in sets. Now consider this diagram

$$\begin{tikzcd} T' \arrow[r, two heads, "g"] \arrow[d, shift right, "x'"'] \arrow[d, shift left, "y'"] \arrow[dr, phantom, "\square"] & T \arrow[d, shift right, "x"'] \arrow[d, shift left, "y"] & \\ \mathfrak{Z}' \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ Y' \arrow[r, two heads, "et"] & Y \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

where we know Y' is a scheme, Y an algebraic space, and T a scheme. We have $x \xrightarrow[\phi]{\sim} y$ in $\mathfrak{Z}(T)$ and we want ϕ to be the identity map. However, $\mathfrak{Z}'$ is algebraic space by hypothesis, T' is a scheme by definition of Y being algebraic space. We get $g^*\phi : x' \xrightarrow{\sim} y'$ and $\mathfrak{Z}'$ is algebraic space, so $g^*\phi = \mathrm{Id}$. g is etale covering and $\mathfrak{Z}$ is a stack, and $g^*\phi = \mathrm{Id}$, hence $\phi = \mathrm{Id}$.

(2): we want to show $\mathfrak{Z}$ has etale covering by scheme. We get sch $\twoheadrightarrow \mathfrak{Z}' \twoheadrightarrow \mathfrak{Z}$ where $\mathfrak{Z}'$ is algebraic space and all arrows are etale, hence we are done.

(3): It remains to show $\Delta_{\mathfrak{Z}}$ is repable by schemes. We do a diagram again:

$$\begin{tikzcd}
\mathfrak{Z}_T \arrow[r] \arrow[d, "\Delta_{\mathfrak{Z}_T}"'] \arrow[dr, phantom, "\square"] & \mathfrak{Z} \arrow[r] \arrow[d, "\Delta_{\mathfrak{Z}}"] & \mathfrak{X} \arrow[dd] \\
\mathfrak{Z}_T \times_T \mathfrak{Z}_T \arrow[r] \arrow[d] & \mathfrak{Z} \times_Y \mathfrak{Z} \arrow[d] \arrow[r, phantom, "\square"] & \\
T \arrow[r] \arrow[u, bend left, "s"] \arrow[ur, dashed, "g" description, "\square"'] & Y \arrow[r] & \mathfrak{Y}
\end{tikzcd}$$

where T is a scheme. This gives

$$\mathfrak{Z} \times_{\Delta, \mathfrak{Z} \times_Y \mathfrak{Z}, g} T = \mathfrak{Z}_T \times_{\Delta, \mathfrak{Z}_T \times_T \mathfrak{Z}_T, s} T =: W$$

where W is a scheme since $\mathfrak{Z}_T$ is an algebraic space. This concludes the proof. ♡

15 Algebraic Stacks

Today we are going to define algebraic stacks.

Definition 15.1. Let P be property of morphisms which is local on the source for the etale topology, local on the target for the sm topology, and stable under base change. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a representable map of stacks over S. We say f ***has*** P if for all schemes T and diagram

$$\begin{tikzcd}
\mathfrak{X}_T \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\
T \arrow[r] & \mathfrak{Y}
\end{tikzcd}$$

we have g has property P, where we know $\mathfrak{X}_T$ is algebraic space.

Recall that, if $g : X \to T$ where T is scheme and X is algebraic space, then we say g has P if there exists etale covering $\tilde{X}$ with

$$\tilde{X} \xrightarrow[\pi]{et} X \xrightarrow{g} T$$

such that $g\pi$ has P.

Example 15.2. P could be etale, smooth, relative dim d, affine, finite, closed, immersion, open immersion, surjection, and so on. We note this list is smaller than the list for fppf descent as we require local on the source.

Definition 15.3. A stack $\mathfrak{X}/S$ is an ***algebraic/Artin stack*** if:

1. $\Delta_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ is representable,
2. There exists scheme U so that $U \xrightarrow{sm} \mathfrak{X}$

We note $\Delta_{\mathfrak{X}}$ is representable implies for all algebraic spaces X, the maps $X \to \mathfrak{X}$ are representable. So, $U \to \mathfrak{X}$ being smooth surjection is well-defined.

Definition 15.4. A morphism of stacks over S, say $\mathfrak{X} \to \mathfrak{Y}$, is defined as an element of $\mathrm{Hom}_S(\mathfrak{X}, \mathfrak{Y})$, i.e. they are morphisms of fibered categories over S.

Lemma 15.5. *Let $\mathfrak{X}$ be stack over S. Then $\Delta_{\mathfrak{X}}$ is representable iff for all S-schemes U, V, $U \in \mathfrak{X}(U)$ and $V \in \mathfrak{X}(V)$ with*

$$\begin{tikzcd}
U \times V \arrow[r, "p_U"] \arrow[dr, "p_V"'] & U \arrow[d] \\
& V
\end{tikzcd}$$

we have $\mathrm{Isom}(p_U^* U, p_V^* V)$ is an algebraic space. This is also equivalent to: for all $U, V \in \mathfrak{X}(U)$, $\mathrm{Isom}(U, V)$ is an algebraic space.

Proof. Equivalence of last 2 conditions: for all $U, V \in \mathfrak{X}(U)$,

$$\begin{tikzcd}
\mathrm{Isom}(U,U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Isom}(p_U^* U, p_U^* V) \arrow[d] \\
U \arrow[r, "\Delta"] & U \times U
\end{tikzcd}$$

so we see $\mathrm{Isom}(p_U^* U, p_U^* V)$ is algebraic space imply $\mathrm{Isom}(U, V)$ is algebraic space. Conversely, $\mathrm{Isom}(p_U^* U, p_V^* V)$ is the special case with $U' = U \times V$, $U' = p_U^* U$, $V' = p_V^* V$.

Next, consider diagram

$$\begin{tikzcd}
\mathfrak{Y} \arrow[r] \arrow[d, "(u,v)"] \arrow[dr, phantom, "\square"] & U \times V \arrow[d] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

Just like the proof for algebraic space, we see we get

$$\begin{tikzcd}
\mathfrak{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d, "v"] \\
U \arrow[r, "u"] & \mathfrak{X}
\end{tikzcd}$$

So, $\mathfrak{Y}(T)$ gives

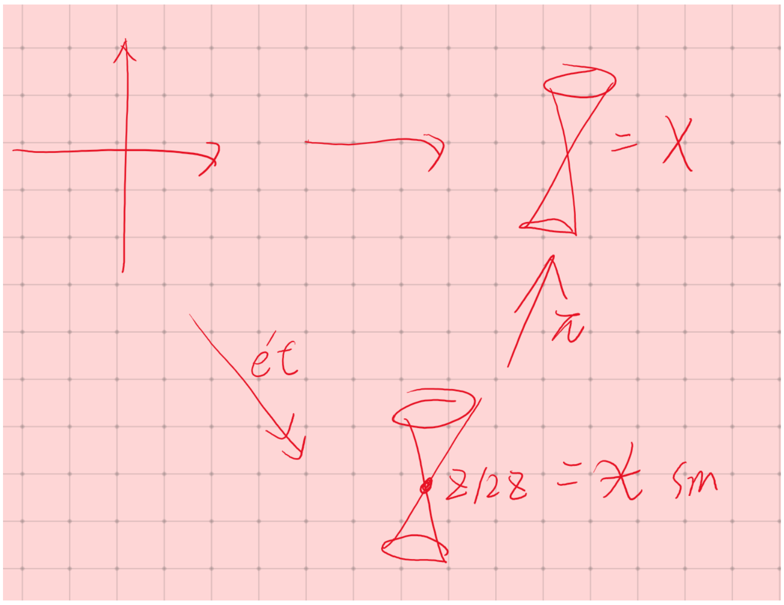
$$\begin{tikzcd}
T \arrow[rrd, bend left, "g"] \arrow[rdd, bend right, "f"'] \arrow[dr, phantom, "\xi"] & & \\
& \mathfrak{X} & V \arrow[l, "v"] \arrow[d, "\pi_1"] \\
& U \arrow[u, "u"] \arrow[r, "\pi_2"] & S
\end{tikzcd}$$

where $\xi : f(u) \xrightarrow{\sim} g(v)$. Then $\pi_1 f = \pi_g$ implies $T \xrightarrow{h} U \times_S V$ and hence $h^* p_U^* u = f^*(u) \xrightarrow[\xi]{\sim} g^* v = h^* p_V^* v$. Thus we see $\xi \in \operatorname{Isom}(p_U^* u, p_V^* v)(T)$ which concludes the proof. ♡

The next goal is to define one of the most important example of stacks, namely quotient stacks.

Example 15.6. Consider $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{A}^2$, i.e. $\mathbb{Z}/2\mathbb{Z} \circlearrowleft \mathbb{A}^2$ via $(x,y) \mapsto (-x,-y)$. In particular, we see $\mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z}) := \operatorname{Spec} k[x,y]^{\mathbb{Z}/2\mathbb{Z}}$ where the ring $k[x,y]^{\mathbb{Z}/2\mathbb{Z}} = \{f(x,y) : f(x,y) = f(-x,-y)\} = k[x^2, xy, y^2]$ is the ring of invariant of the $\mathbb{Z}/2\mathbb{Z}$ action. We also ntoe $k[x^2,xy,y^2] = k[a,b,c]/(ac-b^2)$ which is given by $\mathbb{A}^2 \to \mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z})$. Thus we get $\mathbb{A}^2 \to \mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z})$ is sort of like the xy-plane map to the cone defined by $ac-b^2$. But this is bad, because $V(ac-b^2)$ is singular.

That's why we want stakcs, where we replace the origin by the point $\mathbb{Z}/2\mathbb{Z}$, i.e. we get a "quotient stack" $[\mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z})] =: \mathfrak{X}$, and we get the following diagram



Here the map π is whats called a coarse space map, and it is proper. In particular, π is isomorphism over X^{sm}. This is sort of like blow-up, but we didn't introduce any divisors, hence it is not a blow-up. This π is a "stacky resolution", and it is an example of Vistoli's canonical stacks.

In particular, we get a diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[dr, "\text{can stack}"'] & \tilde{X} \arrow[d, "\text{minimal res}"] \\ & X \end{tikzcd}$$

> where $\tilde{X}$ is the minimal resolution of X, and $\mathfrak{X}$ is the canonical stack we discussed above, and X is any surface with mild singularity.
>
> In physics, we get McKay correspondence that comparing $\mathfrak{X}$ and $\tilde{X}$. We see a lot of interesting math about comparing the two, and it also relates to deriving categories.

Before we jump to definition, we give one or two words about the idea. Say we have group scheme $G \circlearrowleft X$ over S. Then we get $X \to [X/G]$ and what we want is to have the arrow $X \to [X/G]$ to be a G-torsor.

We don't realy know what $X \to [X/G]$ is G-torsor means, thus we want to pullback and get

$$\begin{tikzcd} G \circlearrowleft P \arrow[r, "\text{G-equiv}"] \arrow[d, "\text{G-torsor}"'] \arrow[dr, phantom, "\square"] & G \circlearrowleft X \arrow[d, "\text{G-torsor}"] \\ T \arrow[r] & {[X/G]} \end{tikzcd}$$

This is going to be our definition.

> **Definition 15.7.** Continue the above set-up, for any scheme T the category $[X/G](T)$ is defined as follows. The objects are
>
> $$\begin{tikzcd} P \arrow[r, "\text{G-equiv}", "\pi"'] \arrow[d, "\text{G-tors}"'] & X_T \arrow[dl] \\ T & \end{tikzcd}$$
>
> The morphisms are $(T', P', \pi') \to (T, P, \pi)$ is given by
>
> $$\begin{tikzcd} P' \arrow[r, "\cong", "f^\flat"'] \arrow[dr] & P_{T'} \arrow[r] \arrow[d] & P \arrow[d] \\ & T' \arrow[r, "f"] & T \end{tikzcd}$$
>
> where

We let $\mathfrak{X} = [X/G]$. Why is $\mathfrak{X}$ a stack? We know G-torsors are sheaves, so we have descent for sheaves. Then descent as sheaf with G-action we see G-action is $G \times P \to P$ map of sheaves, so those descent as well. We see $G \circlearrowleft P$ is torsor if $G \times p \xrightarrow{\sim} P \times P$ given by $(g, p) \mapsto (p, gp)$ and we can check this isomorphism locally.

Why is $\Delta_{\mathfrak{X}}$ representable? Let (P_1, π_1), (P_2, π_2) be over T. Let

$$I = \mathrm{Isom}((P_1, \pi_1), (P_2, \pi_2))$$

We want to show I is algebraic space.

First, we claim (its an exercise!) that if $\mathcal{F} \to W$ where W is scheme and $\mathcal{F}$ a sheaf, then we can check $\mathcal{F}$ is algebraic space etale locally on W.

Thus, to check I is algebraic space, we can make etale base change on T so that (P_i, π_i) are trivial torsors. Thus now we have

$$\begin{tikzcd} P_1 = G \arrow[r, "\xi", "\sim"'] \arrow[dr, "\pi_1"] & P_2 = G \arrow[d, "\pi_2"] \\ & X \end{tikzcd}$$

where $\xi \in I$. However, note if we have $\xi(1) = g$, then for any h we get $\xi(h) = \xi(h \cdot 1) = h \cdot \xi(1) = hg$ and hence ξ is right multiplication by g. Thus we see $\pi_1(1) = \pi_2(g)$. Thus we see I has a very simple description:

$$\begin{tikzcd} I \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G_T \arrow[d, "(1,g)"] \\ X_T \arrow[r, "\Delta"] & X_T \times_T X_T \end{tikzcd}$$

In particular, since $G_T, X_T, X_T \times_T X_T$ are all schemes, hence I is a scheme, as desired.

Why is $\mathfrak{X}$ has smooth cover by scheme?

We see we get $X \xrightarrow[sm]{q} [X/G] = \mathfrak{X}$ where q is defined as

$$\begin{tikzcd} U := G \times X \arrow[r, "\text{action}"] \arrow[d, "\text{G-tors}"] & X \\ X \end{tikzcd}$$

Why is q a smooth surjection? Well, we get

$$\begin{tikzcd} I := \mathrm{Isom}((U,\sigma),(P,\pi)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "q(u,\sigma)"] \\ T \arrow[r, "(P,\pi)"] & \mathfrak{X} \end{tikzcd}$$

where $G = U \to P$ is given by $1 \mapsto p$. So, we get map $I \xrightarrow{\sim} P$ given by $(U \xrightarrow{f} P) \mapsto f(1)$.

The point is that we get

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, two heads, "sm"] \arrow[dr, phantom, "\square"] & X \arrow[d, "q"] \\ T \arrow[r, "(P,\pi)"] & \mathfrak{X} \end{tikzcd}$$

and since $P \to T$ is smooth, q is smooth surjection as desired. Thus, we showed, every torsor is the pullback of the torsor $X \to [X/G]$!

> **Example 15.8.** Let $X = S$ and $G \circlearrowright S$ be the trivial action. Then we let $BG := [X/G]$ and hence we get
>
> $$\begin{tikzcd} P \arrow[r, "\text{G-equiv}"] \arrow[d, "\text{G-tors}"] & S \arrow[d] \\ T \arrow[r] & BG \end{tikzcd}$$

However, note the action is trivial, thus G-equivariant $P \to S$ is just any arrow $P \to S$. Hence we see $(BG)(T)$ is just G-torsors on T.

Example 15.9. Let $G = \mathbb{G}_m = \mathrm{GL}_1$, then $B\mathbb{G}_m$ is just line bundles, and $B\,\mathrm{GL}_n$ is vector bundles.

Definition 15.10. We say $\mathfrak{X}$ is quotient stack if $\mathfrak{X} \cong [X/G]$ for some X, G.

16 Algebraic Stacks

Before we do the math, we say something about final presentations:

1. It should be about 20 to 30 min, record a talk
2. Run the topic by prof first

There will be a stack workshop in Montreal, it is in-person and online at the same time. It is during May 30 to June 3.

Last time we talked about quotient stacks $\mathfrak{X} = [X/G]$ where G is a group scheme over S and $G \circlearrowright X$ with X a S-scheme.

We also showed that $X \to [X/G]$ is universal, in the sense that if we have $T \to [X/G]$ then we get the following diagram

$$\begin{tikzcd} P \arrow[r, "\text{G-equiv}"] \arrow[d, "\text{G-tors}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "sm"', "\text{G-tors}"] \\ T \arrow[r] & {[X/G]} \end{tikzcd}$$

We also talked about examples. In particular, $\mathbb{P}^n = [\mathbb{A}^{n+1}\backslash 0/\mathbb{G}_m]$ and more generally, if $G \circlearrowright X$ is a free action, i.e. all stablizers are trivial, then $[X/G]$ is an algebraic space that is exactly X/G. For example, if $X = \operatorname{Spec} A$ then $[X/G] = X/G = \operatorname{Spec}(A^G)$.

Proposition 16.1. *If $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ are Artin stacks with*

$$\begin{tikzcd} & \mathfrak{Y} \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{Z} \end{tikzcd}$$

then $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is an Artin stack.

Next, we are going to define a very useful stack, called the inertia stack.

Definition 16.2. For $\mathfrak{X}$, we define the ***inertia stack*** $I_{\mathfrak{X}} = I\mathfrak{X}$ to be the pullback

$$\begin{tikzcd} I_{\mathfrak{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "\Delta"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

The point of this is that, suppose we have

$$\begin{tikzcd} & I_{\mathfrak{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "\Delta"] \\ T \arrow[r, "x"] & \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

then a lift of the arrow x is equivalent to giving $x' \in \mathfrak{X}(T)$ and $\xi : (x,x) \xrightarrow{\sim} (x',x')$, i.e. $\xi_1, \xi_2 : x \xrightarrow{\sim} x'$. In particular, we get that for $\sigma \in \mathrm{Aut}(x)$, we have $\sigma = \xi_1^{-1} \circ \xi_2 : x \xrightarrow{\sim} x$.

A different point of $I\mathfrak{X}$ is $(x,x) \xrightarrow[(\mathrm{Id},\sigma)]{\sim} (x,x)$. This new point is isomorphic to the old point:

$$\begin{tikzcd} (x,x) \arrow[r, "(\xi_1,\xi_2)"] \arrow[d, "(\mathrm{Id},\mathrm{Id})"'] & (x',x') \arrow[d, "(x_1^{-1},\xi_1^{-1})"] \\ (x,x) \arrow[r, "(\mathrm{Id},\sigma)"'] & (x,x) \end{tikzcd}$$

So, we see

$$\begin{tikzcd} I\mathfrak{X} \arrow[d] & \\ \mathfrak{X} & T \arrow[l, "x"] \arrow[ul, dashed, "\sigma \in \mathrm{Aut}(x)"'] \end{tikzcd}$$

In other word, we get

$$\begin{tikzcd} \mathrm{Aut}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d] \\ I\mathfrak{X} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

and $I\mathfrak{X} \to X$ is a relative group algebraic space, but it is normally not flat!

The next topic is properties for stacks and morphisms.

Definition 16.3. Let P be a property that is local for smooth topology. Then we say $\mathfrak{X}$ has P if there exists smooth cover $X \twoheadrightarrow \mathfrak{X}$ with X scheme, such that X has P.

Example 16.4. P could be local Noetherian, regular, of finite type over S, of finite presentation over S.

Lemma 16.5. *If P is local for smooth topology, and $\mathfrak{X}$ has P, and for Y a scheme we have $Y \xrightarrow{sm} \mathfrak{X}$ then Y has P.*

Proof. Consider

$$\begin{tikzcd} Z \arrow[r, "sm"] \arrow[d, "sm"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "sm"] \\ Y \arrow[r, "sm"] & \mathfrak{X} \end{tikzcd}$$

where we assume X has P. But X has P implies Z has P and hence Y has P as desired. ♡

Remark 16.6. The proof shows that if $Y \to \mathfrak{X}$ is a morphism, then smooth locally on Y $Y \to \mathfrak{X}$ factors through $X \to \mathfrak{X}$.

Definition 16.7. If $f : \mathfrak{X} \to \mathfrak{Y}$ is morphism of Artin stacks, then a ***chart for*** f is a diagram

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[rr, bend left, "g"] \arrow[dr] & \mathfrak{Z} \arrow[r] \arrow[d, two heads] & Y \arrow[d, two heads, "sm"] \\ & \mathfrak{X} \arrow[r, "f"'] & \mathfrak{Y} \end{tikzcd}$$

If P is property of morphisms stable under base change, local on source and target for smooth topology, then we say f ***has*** P if g has P. In this case we also say that this ***chart*** g ***has*** P.

Example 16.8. P could be smooth, locally of finite presentation, surjective, etc.

Example 16.9. If we are given quotient stack $[X/G]$ over S, and

$$\begin{tikzcd} X \arrow[r, two heads, "sm"] \arrow[rr, bend right, "g"'] & \mathfrak{X} \arrow[r, "f"] & S \end{tikzcd}$$

Then $\mathfrak{X}/S$ is smooth iff X/S is smooth. For example, we see $[\mathbb{A}^2/(\mathbb{Z}/2)]$ is smooth because $\mathbb{A}^2$ is smooth. On the other hand, $\mathbb{A}^2/(\mathbb{Z}/2)$ is singular as its equal $\operatorname{Spec} k[x,y]^{\mathbb{Z}/2}$.

Proposition 16.10. *The morphism $f : \mathfrak{X} \to \mathfrak{Y}$ has P iff every chart has P.*

Proof. We start with a chart

$$\begin{tikzcd} X \arrow[r, two heads] & \mathfrak{Z} \arrow[r] \arrow[d] & Y \arrow[d] \\ & \mathfrak{X} \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

Then we get another chart

$$
\begin{tikzcd}
X \arrow[rr] & & \mathfrak{Z} \arrow[rr] \arrow[dd] & & Y \arrow[dd] & \\
& X' \arrow[rr] & & \mathfrak{Z}' \arrow[rr] \arrow[dl] & & Y' \arrow[dl] \\
& & \mathfrak{X} \arrow[rr] & & \mathfrak{Y} &
\end{tikzcd}
$$

Now we want that: $X \to Y$ has P iff $X' \to Y'$ has P.

Now take pullbacks of the squares of the two sides, we get

$$
\begin{tikzcd}
& X'' \arrow[rr] & & \mathfrak{Z}'' \arrow[dl, two heads] \arrow[dd] & & Y'' \arrow[dl, two heads] \arrow[dd] \\
X \arrow[rr] & & \mathfrak{Z} \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\square"] & & Y \arrow[dd] \arrow[dr, phantom, "\square"] & \\
& X' \arrow[rr] & & \mathfrak{Z}' \arrow[rr] \arrow[dl] & & Y' \arrow[dl] \\
& & \mathfrak{X} \arrow[rr] & & \mathfrak{Y} &
\end{tikzcd}
$$

and we also get natural arrows from $\mathfrak{Z}'' \to Y''$. Viz we get

$$
\begin{tikzcd}
& X'' \arrow[rr] & & \mathfrak{Z}'' \arrow[rr] \arrow[dl, two heads] \arrow[dd] & & Y'' \arrow[dl, two heads] \arrow[dd] \\
X \arrow[rr] & & \mathfrak{Z} \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\square"] & & Y \arrow[dd] \arrow[dr, phantom, "\square"] & \\
& X' \arrow[rr] & & \mathfrak{Z}' \arrow[rr] \arrow[dl] & & Y' \arrow[dl] \\
& & \mathfrak{X} \arrow[rr] & & \mathfrak{Y} &
\end{tikzcd}
$$

Thus it is good enough to show $X \to Y$ has P iff $X'' \to Y''$ has P, i.e. it is good enough to handle the case when $Y' \to \mathfrak{Y}$ factors as $Y' \twoheadrightarrow Y \twoheadrightarrow \mathfrak{Y}$.

First, consider the case $X' = X \times_{\mathfrak{Z}} \mathfrak{Z}'$ and we get diagram

$$
\begin{tikzcd}
X' \arrow[r] \arrow[d, two heads] \arrow[rr, bend left, "g'"] \arrow[dr, phantom, "\square"] & \mathfrak{Z}' \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Y' \arrow[d, two heads] \\
X \arrow[r] \arrow[rr, bend left, "g"] & \mathfrak{Z} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & Y \arrow[d, two heads] \\
& \mathfrak{X} \arrow[r] & \mathfrak{Y}
\end{tikzcd}
$$

where g' is pullback of g under $Y' \twoheadrightarrow Y$. We see P is smooth local on the target so g has P iff g' has P.

Now, we just need to compare two charts with the same Y. To see this, we note we have the following diagram

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d, two heads, "\pi"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, two heads] \arrow[dr, "g'"] & \\
X \arrow[r, two heads] \arrow[rr, bend right, "g"] & \mathfrak{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\
& \mathfrak{X} \arrow[r] & \mathfrak{Y}
\end{tikzcd}$$

and we see g' has P iff $g'\pi' = g\pi$ has P iff g has P. This concludes the proof. ♡

Next, we consider separatedness.

Proposition 16.11. *Consider the diagram*

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "f"] \arrow[rr, bend right, "h"'] & \mathfrak{Y} \arrow[r, "g"] & \mathfrak{Z}
\end{tikzcd}$$

of stacks with g representable. Then h is representable iff f is representable.

Proof. If f is representable, h is representable by definition. If h is representable, given $Z \xrightarrow{\alpha} \mathfrak{Y}$ we want $Z \times_{\mathfrak{Y}} \mathfrak{X}$ is an algebraic space. Thus we get the following diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X \arrow[l] \arrow[d] \\
\mathfrak{Y} \arrow[d, "g"'] & Y \arrow[l] \arrow[d] \\
Z & Z \arrow[l] \arrow[ul, "\alpha"]
\end{tikzcd}$$

where the bottom square is also Cartesian. We note X, Y are algebraic spaces because h, g are representable, respectively. Then, note we get a section $\beta : Z \to Y$ defined by α and hence we obtain the diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X \arrow[l] \arrow[d] \\
\mathfrak{Y} \arrow[d, "g"'] & Y \arrow[l] \arrow[d] \\
Z & Z \arrow[l] \arrow[ul, "\alpha"] \arrow[u, bend right, "\beta"']
\end{tikzcd}$$

Now we see $\mathfrak{X} \times_{\mathfrak{Y},\alpha} Z = X \times_{Y,\beta} Z$ is an algebraic space as X, Y, Z are all algebraic spaces. ♡

Proposition 16.12. *Let $\mathfrak{X}, \mathfrak{Y}$ be Artin stacks over S and $f : \mathfrak{X} \to \mathfrak{Y}$, then $\Delta_{\mathfrak{X}/\mathfrak{Y}}$ is representable.*

Proof. We have the following diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\Delta_{\mathfrak{X}/\mathfrak{Y}}"'] \arrow[rr, bend left, "\Delta_{\mathfrak{X}/S}", "repable"'] \arrow[dr, "f"'] & \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \times_S \mathfrak{X} \arrow[d, "f \times f"] \\
& \mathfrak{Y} \arrow[r, "repable", "\Delta_{\mathfrak{Y}/S}"'] & \mathfrak{Y} \times_S \mathfrak{Y}
\end{tikzcd}$$

THen $\Delta_{\mathfrak{Y}}$ repable implies g is repable. Hence $\Delta_{\mathfrak{X}}$ repable implies, by the proposition above, that $\Delta_{\mathfrak{X}/\mathfrak{Y}}$ is representable as desired. ♡

Definition 16.13. We say $f : \mathfrak{X} \to \mathfrak{Y}$ is ***separated*** if $\Delta_{\mathfrak{X}/\mathfrak{Y}}$ is proper.

Remark 16.14. For stacks, Δ keeps track of Isom or Aut which is a group so that is is rarely a closed immersion.

Remark 16.15. FOr X, Y schemes, $\Delta_{X/Y}$ is always an immersion, so they are separated and of finite type. Thus $\Delta_{X/Y}$ is proper iff $\Delta_{X/Y}$ universally closed but base changes of immersion is immersion and hence Δ is universally closed iff its closed.

Thus, $X \to Y$ separated in the usual definition ($\Delta_{X/Y}$ closed) iff its separated as stacks.

Example 16.16. Let $\mathfrak{X} = [X/G]$, then we see we get

$$\begin{tikzcd}
X \times_{\mathfrak{X}} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \times X \arrow[d, two heads] \\
\mathfrak{X} \arrow[r, "\Delta_{\mathfrak{X}}"] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

where $X \times_{\mathfrak{X}} X$ is the Aut of universal torsor. Hence we see

$$\begin{tikzcd}
X \times_{\mathfrak{X}} X \cong G \times X \arrow[r] \arrow[d, shift left] \arrow[dr, phantom, "\square"] & X \arrow[d, two heads] \\
X \arrow[u, shift left] \arrow[r, two heads] & \mathfrak{X}
\end{tikzcd}$$

where $G \times X \cong X \times_{\mathfrak{X}} X$ because we has a section $X \to X \times_{\mathfrak{X}} X$. Thus, we see the above diagram's arrows are given by

$$\begin{tikzcd}
(g,x) \arrow[rr, mapsto] \arrow[dd, mapsto] & & gx \\
& G \times X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, two heads] \\
x & X \arrow[r, two heads] & \mathfrak{X}
\end{tikzcd}$$

So, we see

$$(g,x) \longmapsto (x, gx)$$

$$\begin{tikzcd}
G \times X \arrow[r, "\Gamma"] \arrow[d] \arrow[dr, phantom, "\square"] & X \times X \arrow[d, two heads] \\
\mathfrak{X} \arrow[r, "\Delta"'] & \mathfrak{X} \times \mathfrak{X}
\end{tikzcd}$$

where Γ is the graph of the action. Therefore, we see $\mathfrak{X}$ is separated iff Γ is proper. By definition of $G \circlearrowleft X$ is called proper if Γ is proper.

Example 16.17. If X is separated and G is proper (e.g. G is finite) then

$$\begin{tikzcd}
& G \times_S X \arrow[dl] \arrow[dr, "proper"] \arrow[r] & X \times_S X \arrow[r] \arrow[d, "sep"] \arrow[dr, phantom, "\square"] & X \arrow[d, "sep"] \\
G \arrow[dr, "proper"'] \arrow[rr, phantom, "\square"] & & X \arrow[r] \arrow[dl] & S \\
& S & &
\end{tikzcd}$$

and hence Γ is proper, i.e. if X is separated, G proper (e.g. finite), then $[X/G]$ is separated.

Example 16.18. If A is Abelian variety, then BA is separated.

17 Deligne-Mumford Stacks

Last time we talked about separatedness. In particular, if X is separated over S, G proper over S, then $[X/G]$ is separated over S.

The next topic is to work towards the first big theorem in stacks.

Definition 17.1. If $\mathfrak{X}/S$ is Artin stack, then we say it is ***Deligne-Mumford*** (DM) if there exists scheme X with $X \xrightarrow{etale} \mathfrak{X}$.

In general, its not easy to check when a stack is DM. To check this, we recall the notion of formally etale/smooth. That is, we say $X \xrightarrow{f} Y$ is formally etale/smooth if for all diagrams

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} A/I \arrow[r] & Y
\end{tikzcd}$$

with $I^2 = 0$ in A, there exists unique arrow/exists arrow from $\operatorname{Spec} A/I$ to Y.

So, etale means exists unique arrow, smooth means exists arrow, and now we define formally unramified, means unique arrow.

Definition 17.2. We say $X \xrightarrow{f} Y$ is ***formally unramified*** if for all diagrams

$$\begin{tikzcd}
\operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, shift left, "\alpha"] \arrow[ur, shift right, "\beta"'] & Y
\end{tikzcd}$$

with $I^2 = 0$ then $\alpha = \beta$.

Theorem 17.3. *If $\mathfrak{X}$ is Artin stack over S, then $\mathfrak{X}$ is DM iff $\Delta_{\mathfrak{X}/S}$ is formally unramified.*

We first state some corollaries, before we prove this.

Corollary 17.3.1. *If $\mathfrak{X}$ is Artin stack over S, then $\mathfrak{X}$ is algebraic space over S iff for all $x \in \mathfrak{X}(U)$, $\operatorname{Aut}(x)$ is trivial, i.e. algebraic spaces are Artin stacks with no stablizers.*

Proof. First, if $\mathfrak{X}$ is algebraic space, then $\mathfrak{X}$ is sheaf, i.e. fibered in sets, so no automorphisms.

Conversely, if

$$\begin{tikzcd}
\operatorname{Isom}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "{(x,y)}"] \\
\mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times_S \mathfrak{X}
\end{tikzcd}$$

If $\operatorname{Isom}(x,y) = \varnothing$, then it is a scheme and $\operatorname{Isom}(x,y) \to U$ is formally unramified. If $\operatorname{Isom}(x,y) \neq \varnothing$, then it is an $\operatorname{Aut}(x)$-torsor. That is, if we have two isomorphisms $x \underset{\alpha}{\overset{\beta}{\rightrightarrows}} y$ then α, β are related by unique automorphism $\beta^{-1} \circ \alpha$. By assumption, $\operatorname{Aut}(x) \to U$ is the trivial group scheme, i.e. $U \xrightarrow{\mathrm{Id}} U$. So it is affine group scheme, and so $\operatorname{Isom}(x,y)$ is a scheme. Also, $\operatorname{Isom}(x,y) \to U$ is a monomorphism, so it is formally unramified. Therefore Δ is formally unramified and representable by schemes. Now by the big theorem above, we see there exists a etale cover.

Lastly, $\mathfrak{X}$ is a sheaf because the automorphisms are trivial and hence $\mathfrak{X}$ is algebraic space as desired. ♡

Remark 17.4. Suppose $\Delta_{\mathfrak{X}}$ is locally of finite presentation. Then the theorem says $\mathfrak{X}$ is DM iff for all $k = \overline{k}$ and $\operatorname{Spec} k \xrightarrow{x} \mathfrak{X}$, the automorphism group $\operatorname{Aut}(x) \to \operatorname{Spec} k$ is a finite group.

Why is this? We know $\Delta_{\mathfrak{X}}$ is formally unramified iff $\operatorname{Isom}(x,y) \to U$ is formally unramified for all $x, y \in \mathfrak{X}(U)$. However, since $\Delta_{\mathfrak{X}}$ is locally of finite presentation, $\operatorname{Isom}(x,y) \to U$ is locally of finite presentation. Hence, we can check formally unramified for $\operatorname{Isom}(x,y) \to U$ on geometric points, i.e. for all $k = \overline{k}$ with $\operatorname{Spec} k \xrightarrow{f} U$ we can check $\operatorname{Isom}(f^*x, f^*y) \to \operatorname{Spec} k$ formally unramified.

If $\operatorname{Isom}(f^*x, f^*y) = \varnothing$ there is nothing to do, else (i.e. its non-empty) it is isomorphic to $\operatorname{Aut}(x)$. But then $\operatorname{Aut}(x) \to \operatorname{Spec} k$ is locally of finite presentation, so its formally unramified if and only if etale if and only if $\operatorname{Aut}(x)$ is finite if and only if $\operatorname{Aut}(x)$ is a group.

Example 17.5. From the above remark, we see $B\mathbb{G}_m$ is not DM as $\mathbb{G}_m$ is not finite. More generally, BG is not DM if G is a positive dimension group scheme.

Example 17.6. A long time ago we talked about moduli space of genus g curves M_g, where $M_g(T)$ are given by $C \xrightarrow{\pi} T$ with π smooth proper and on geometric fibers it is genus g curve.

When Deligne and Mumford defined DM stacks, they showed that for $g \geq 2$, M_g is DM and defined a compactification $\overline{M}_g$. We will go through the idea of how to show M_g is DM for $g \geq 2$.

We start with a curve C of genus $g \geq 2$ over a ACF $k = \overline{k}$. We want $\operatorname{Aut}(C) \to \operatorname{Spec} k$ to be formally unramified (i.e. we want $\operatorname{Aut}(C)$ is finite group). So we want to show for all diagrams

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A' \arrow[r] \arrow[ur, shift left, "\alpha"] \arrow[ur, shift right, "\beta"'] & Y \end{tikzcd}$$

with $A = A'/I$, we have $\alpha = \beta$. Now, we consider the diagram

$$\begin{tikzcd} C_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_{A'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} A' \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

where we get two arrows from $C_{A'}$ to itself (α, β) and one arrow from C_A to itself (γ), where α, β both reduce to be γ over A. Now by deformation theory, "$\alpha - \beta$" is a class in $H^0(T_{C_A/A})$, where $T_{C_A/A}$ is tangent bundle. However, the degree of tangent bundles are given by $2 - 2g$. Hence we see for $g \geq 2$, the degree become negative, i.e. $H^0(T_{C_A/A}) = 0$ and hence $\alpha = \beta$, as desired.

What about $g = 0, 1$?

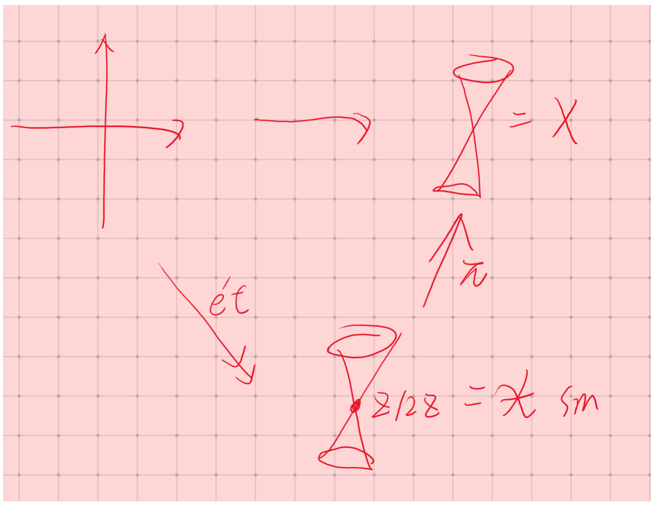
If $g = 0$, then M_0 is just $\mathbb{P}$ over ACF $k = \overline{k}$, and over T it is not. It is a Brauer-Severi variety. These are $\mathrm{Aut}(\mathbb{P}^1)$-torsors and so $M_0 = B(\mathrm{PGL}_2)$. We know $\dim \mathrm{PGL}_2 = 3 > 0$, so M_0 is Artin, not DM. In particular, the dimension of M_0 is dimension of a point subtract dimension of PGL_2, i.e. $\dim M_0 = -3$.

Next, we consider $M_{0,3}$, the moduli space of genus 0 curves with 3 marked points. Let C be a curve with $g = 0$ and three marked points, then its isomorphic to $\mathbb{P}^1$ with three additional points. Thus $M_{0,3}$ =points.

In terms of deformation theory, we get $\alpha - \beta$ lives in $H^0(T(-3 \quad \mathrm{pt}\, s))$, where $T(-3 \quad \mathrm{pt}\, s)$ is twisted down by three points, thus the degree is $2 - 2g - 3 < 0$.

Similarly, M_1 is Artin, but $M_{1,1}$ is genus 1 curves with one marked point, which is just the moduli space of elliptic curves. In particular, $\dim M_{1,1} = 1$, which is exactly the j-invariant of elliptic curves. We get a map $M_{1,1} \to \mathbb{A}^1$ which sends elliptic curve E to isomorphism class of E (i.e. sends it to the j-invariant).

We give a rough picture of what this looks like:



generically we have $\mathbb{Z}/2$-stablizers since E has automorphisms equal multiply by -1. However, for $j = 0$ and $j = 1728$, we have more automorphisms: $\mathbb{Z}/4$ and $\mathbb{Z}/6$.

We don't have time for the proof of the big theorme, but we go through the idea first.

If $X \to Y$ is a map of schemes, its smooth iff we can find Zariski cover $U \twoheadrightarrow X$

with

$$\begin{tikzcd}
U \arrow[rr, "F"] \arrow[dr, twoheadrightarrow] & & \mathbb{A}^n_Y \arrow[dd] \\
& X \arrow[dr, "f"'] & \\
& & Y
\end{tikzcd}$$

where $n = \dim X - \dim Y$ and F comes from the following: $\Omega^1_{X/Y}$ is locally free, we look locally on U where $U^1_{X/Y}|_U$ is free. We choose basis $df_1, ..., df_n$ which yields map to $\mathbb{A}^n_Y$ coming from $(f_1, ..., f_n)$.

For us, we have smooth $X \twoheadrightarrow \mathfrak{X}$, we would like to have something like "$\Omega^1_{X/\mathfrak{X}}$". We look etale locally where $\Omega^1_{X/\mathfrak{X}}$ is free to get

$$\begin{tikzcd}
X \arrow[r, "et"] \arrow[dr, "sm"] & \mathbb{A}^n_{\mathfrak{X}} \arrow[d] \\
& \mathfrak{X}
\end{tikzcd}$$

Formally unramified will allow us to "slice" $\mathbb{A}^n_{\mathfrak{X}} \to \mathfrak{X}$ to get $W \subseteq \mathbb{A}^n_{\mathfrak{X}}$ with etale arrow $W \to X$

$$\begin{tikzcd}
\mathbb{A}^n_{\mathfrak{X}} \arrow[r] & \mathfrak{X} \\
W \arrow[u, "\subseteq"] \arrow[ur, "et"] &
\end{tikzcd}$$

Before we end, we talk about how to define $\Omega^1_{X/\mathfrak{X}}$.

We don't realy know what to do, hence the first thing is to descent. Thus, consider the following diagram

$$\begin{tikzcd}
Z \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[r, twoheadrightarrow, "\pi"] \arrow[d, dashed, "\pi'"] \arrow[dr, phantom, "\square"] & X \arrow[d, twoheadrightarrow, "sm"] \\
Y \arrow[r, shift left] \arrow[r, shift right] & X \arrow[r, twoheadrightarrow, "sm"'] & \mathfrak{X}
\end{tikzcd}$$

We have $\Omega^1_{Y/X} = \Omega^1_{\pi'}$ and we get canonical isomorphism $p^*\Omega^1_{Y/X} \cong \Omega^1_{Z/Y} \cong q^*\Omega^1_{Y/X}$. Thus it satisfies the cocycle condition. Thus by descent of coherent sheaves, we get $\Omega^1_{X/\mathfrak{X}}$ such that $\pi^*\Omega^1_{X/\mathfrak{X}} \cong \Omega^1_{Y/X}$ where Y/X is via the map $\pi' : Y \to X$.

Moreover, $\Omega^1_{X/\mathfrak{X}}$ is locally free sheaf on X because $\pi^*\Omega^1_{X/\mathfrak{X}} \cong \Omega^1_{Y/X}$ is.

In addition, we get $\Omega^1_{X/S} \to \Omega^1_{X/\mathfrak{X}}$. To show this, use descent:

$$\pi^*\Omega^1_{X/S} \longrightarrow \Omega^1_{Y/S} \longrightarrow \Omega^1_{Y/X} = \Omega^1_{\pi'} = \pi^*\Omega^1_{X/\mathfrak{X}}$$

Thus we get $\Omega^1_{X/S} \to \Omega^1_{X/\mathfrak{X}}$.

We note that, for Artin stacks, this is usually not surjective. But it is surjective for DM stacks.

18 DM Stacks

Last time we defined DM stacks to be Artin stack $\mathfrak{X}$ such that it has etale cover by a scheme.

Theorem 18.1. *$\mathfrak{X}$ is DM iff $\Delta_{\mathfrak{X}/S}$ is formally unramified.*

Last time, for smooth $X \twoheadrightarrow \mathfrak{X}$ we defined $\Omega^1_{X/\mathfrak{X}}$ coherent locally free. We also mentioned the idea of the proof, which is to look at where $\Omega_{X/\mathfrak{X}}$ is free, then we get

$$\begin{tikzcd} X' \arrow[d, two heads, "et"] & \mathbb{A}^r_{\mathfrak{X}} \arrow[d, two heads, "g"', "et"] \\ X \arrow[r, two heads, "sm"] & \mathfrak{X} \end{tikzcd}$$

Then using $\Delta_{\mathfrak{X}/S}$ is formally unramified, we will "slice" g until it becomes relative dim 0, i.e. etale.

Now we start the proof.

Proof. We start with the easy direction. Assume $\mathfrak{X}$ is DM. Choose etale cover $X \twoheadrightarrow \mathfrak{X}$. Consider the following diagram

$$\begin{tikzcd} X \times_{\mathfrak{X}} X \arrow[r, "b"] \arrow[d, two heads, "et"] \arrow[dr, phantom, "\square"] & X \times_S X \arrow[d, two heads, "et"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times_S \mathfrak{X} \end{tikzcd}$$

The goal is to show b is formally unramified. Let $X \times_S X \to X$ be the projection, then we get the following diagram

$$\begin{tikzcd} X \times_{\mathfrak{X}} X \arrow[r, "\pi \circ b"] \arrow[d, "\pi \circ b"] & X \arrow[d, two heads] \\ X \arrow[r, two heads, "et"] & \mathfrak{X} \end{tikzcd}$$

However, note $\pi \circ b$ is etale, hence b is unramified, and hence Δ is unramified as desired.

Conversely, suppose Δ is formally unramified. Let $k = \overline{k}$ be a ACF, $y \in \mathfrak{X}(k)$. Choose $p : X \xrightarrow{sm} \mathfrak{X}$ with X affine, such that

$$\begin{tikzcd} \varnothing \neq X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k \arrow[d, "y"] \\ X \arrow[r, "p"] & \mathfrak{X} \end{tikzcd}$$

However, since k is ACF, we get a section x':

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k \arrow[l, bend right, "x'"'] \arrow[d, "y"] \\ X \arrow[r, "p"'] & \mathfrak{X} \end{tikzcd}$$

Let k_0 be the residue field of image of x' in S and let k_0^{sep} be the separable closure of k_0.

Our goal is to show etale locally on S and X, we will find $W \subseteq X$ closed such that $W \xrightarrow{et} \mathfrak{X}$ with $W_y \neq \varnothing$. Then we are done as $\coprod_y W_y \twoheadrightarrow \mathfrak{X}$ is etale.

Now we get the diagram

$$\begin{tikzcd} Z \arrow[r, "\pi"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & \mathfrak{X} \end{tikzcd}$$

which is the same as the following diagram

$$\begin{tikzcd} Z \arrow[r, "(\pi,\pi')"] \arrow[d] \arrow[dr, phantom, "\square"] & X \times_S X \arrow[d] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

Now let $\Omega'_{\pi'} := \Omega^1_{Z/X} = \pi^*\Omega^1_{X/\mathfrak{X}}$. Last time we also constructed $\Omega^1_{X/S} \to \Omega^1_{X/\mathfrak{X}}$. This is usually not surjective, but we will show it is the case for DM stacks. To show it is surjective, by descent, it is enough to show surjective after applying π^*. We see Δ is formally unramified implies (π, π') is formally unramified, and hence

$$(\pi,\pi')^*\Omega_{X\times_S X/S} = \pi^*\Omega^1_{X/S} \oplus (\pi')^*\Omega_{X/S} \twoheadrightarrow \Omega_{Z/S}$$

This gives

$$\begin{tikzcd} \pi^*\Omega^1_{X/S} \arrow[r, hook] \arrow[ddr, "\pi^*\phi"'] & \pi^*\Omega^1_{X/S} \oplus (\pi')^*\Omega^1_{X/S} \arrow[d] \arrow[dd, bend left, "0"] \\ & \Omega^1_{Z/S} \arrow[d] \\ & \Omega^1_{\pi'} = \pi^*\Omega^1_{X/\mathfrak{X}} \end{tikzcd}$$

Since we have that 0 map, it gives us that $\pi^*\phi$ is surjective, and hence ϕ is surjective.

We have

$$\begin{tikzcd} \mathscr{O}_X, \arrow[r, "d"] \arrow[dr] & \Omega^1_{X/S} \arrow[d, two heads] \\ & \Omega^1_{X/\mathfrak{X}} \end{tikzcd}$$

where we call the arrow $\mathscr{O}_X \to \Omega^1_{X/\mathfrak{X}}$ d as well. Locally, $d : \mathscr{O}_X \to \Omega^1_{X/S}$ has image generates, so the same is true for $d : \mathscr{O}_X \to \Omega^1_{X/\mathfrak{X}}$.

Since $\Omega^1_{X/\mathfrak{X}}$ is locally free, so we need to look etale locally where $\Omega^1_{X/\mathfrak{X}}$ is free and there exists $f_1, ..., f_r \in \Gamma(\mathscr{O}_U)$ such that $df_1, ..., df_r$ is a basis for $\Omega^1_{X/\mathfrak{X}}|_U$.

Shrinking X, we may assume $U = X$. Let $f_i \in \Gamma(\mathscr{O}_X)$ that df_i generates $\Omega^1_{X/\mathfrak{X}}$, so we get $F : X \xrightarrow{(p, f_1, ..., f_r)} \mathfrak{X} \times_S \mathbb{A}^r_S$. Then,

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "sm"'] & \mathfrak{X} \times_S \mathbb{A}^r_S = \mathbb{A}^r_{\mathfrak{X}} \arrow[dl, "sm"] \\ \mathfrak{X} & \end{tikzcd}$$

and $\dim_{x'}(X) = \dim_{x'}(\mathbb{A}^r_{\mathfrak{X}})$ because $\Omega'_{X/\mathfrak{X}}$ is free for rank r.

F is smooth map, representable and relative dim 0 in neighbourhood of x', so it is etale in neighbourhood of x'. Shrink to assume F is etale. Then $F_y : X_y \xrightarrow{et} \mathbb{A}^r_k$. Then F_y is etale implies the image is open. Let $f \in k[t_1, ..., t_r]$ such that $\varnothing \neq D(f) \subseteq F_y(X_y)$. Over k_0^{sep}, there exists $a_1, ..., a_r \in k_0^{sep}$ such that $f(a_1, ..., a_r) \neq 0$.

In particular, we see

$$\begin{array}{ccc} (a_1, ..., a_r) \in & & \mathbb{A}_{k_0^{sep}} \\ \Big\downarrow & & \Big\downarrow \\ \text{closed point } Q & & \mathbb{A}^r_{k_0} \end{array}$$

Now, we import a fact, which is that, we can take an etale neighbourhood $S' \to S$ of x' such that there exists closed E with $E \subseteq \mathbb{A}^r_{S'}$ and diagram

$$\begin{tikzcd} E \arrow[r, "\subseteq"] \arrow[dr, "et"'] & \mathbb{A}^r_{S'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^r_S \arrow[d] & \operatorname{Spec} k_0^{sep} \arrow[l, "Q"'] \arrow[d] \\ & S' \arrow[r, "et"'] & S & \operatorname{Spec} k_0 \arrow[l, "x'"] \end{tikzcd}$$

then there exists unique arrow $\operatorname{Spec} k \to S'$ with $Q \in E$, i.e. we get the following diagram

$$\begin{tikzcd} E \arrow[r, "\subseteq"] \arrow[dr, "et"'] & \mathbb{A}^r_{S'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^r_S \arrow[d] & \operatorname{Spec} k_0^{sep} \arrow[l, "Q"'] \arrow[d] \\ & S' \arrow[r, "et"'] & S & \operatorname{Spec} k_0 \arrow[l, "x'"] \arrow[ll, dashed, bend left, "\exists"] \end{tikzcd}$$

Therefore, we see

$$\begin{tikzcd} W \arrow[r, "\subseteq", "closed"'] \arrow[d, "et"'] \arrow[dr, phantom, "\square"] & X \times_S \mathbb{A}^r_{S'} \arrow[d, "F \times \mathrm{Id}"] \\ \mathfrak{X} \times_S E \arrow[r, "\subseteq"'] & \mathfrak{X} \times_{S'} \mathbb{A}^r_{S'} \end{tikzcd}$$

Therefore, $W_y \neq \varnothing$ by construction and hence

$$\begin{tikzcd} W \arrow[r, "et"] & E \times_{S'} \mathfrak{X} \arrow[r, "et"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ & E \arrow[r, "et"'] & S' \end{tikzcd}$$

Now $W \to \mathfrak{X}$ is our desired map. ♡

Corollary 18.1.1. *If G is a finite group, then $\mathfrak{X} = [X/G]$ DM.*

Proof. We just need Aut groups are finite over $k = \overline{k}$. Over $k = \overline{k}$, G-torsors are trivial, so our point is

$$\begin{tikzcd} G \arrow[r, "G\text{-equiv}", "\alpha"'] \arrow[d] & X \\ \operatorname{Spec} k & \end{tikzcd}$$

and Aut group is

$$\begin{tikzcd} G \arrow[dr, "\gamma"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right] & & \\ & G \arrow[r, "\alpha"] \arrow[d] & X \\ & \operatorname{Spec} k & \end{tikzcd}$$

So $\gamma(1) = g$. Then $\gamma(h) = h \circ \gamma(1) = hg$ and hence we get

$$\begin{tikzcd} 1 \arrow[r, mapsto] \arrow[d, mapsto] & \alpha(1) \arrow[d, mapsto, "="] \\ \gamma(1) \arrow[r, mapsto] & \gamma(1) \cdot \alpha(1) \end{tikzcd}$$

In other word, Aut is stablizer of $\alpha(1)$. Hence we get

$$\begin{tikzcd} \mathrm{Aut} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k \arrow[d] \\ G \arrow[r] & X \end{tikzcd}$$

$$g \longmapsto g \cdot \alpha(1)$$

and so Aut is finite as desired. ♡

At the start of the course, we talked about 5 general points determine conic. We did this through geometry on moduli space. In particular, moduli space of singular conics is exactly $\mathbb{P}^5$ (if we drop singular, then this is not true!).

Even if we only interested in smooth curves, i.e. M_g. To do intersection theory, we need a compactification $\overline{M_g}$. So, we need a notion of properness. To do this, we need quasi-coherent sheaves on $\mathfrak{X}$.

Theorem 18.2 (Eisenbud-Harris). *It is impossible to write down a general $g \geq 24$ curve.*

This uses intersection theory on $\overline{M_g}$. Note here we are asking for general $g \geq 24$ curve. To see what this means, we consider the example of elliptic curve. For that, we know the short form for elliptic curve is given by $y^2 = x^3 + ax + b$. Thus it is the same as a dominant rational map $\mathbb{A}^2_{a,b} \dashrightarrow M_{1,1}$, i.e. $\mathbb{A}^2_{a,b} \backslash (\text{discriminant} = 0) \to M_{1,1}$. They showed M_g is of general type.

There is also another similar problem, where we look at $\mathscr{A}_g$, the moduli space of dim g abelian varities. Then we know $\mathscr{A}_{\geq 7}$ is of general type, $\mathscr{A}_{\leq 5}$ is , and $\mathscr{A}_6$ is unknown.

Now we jump to quasi-coherent/coherent sheaves on stack.

Definition 18.3. We define ***Lisse-etale site*** $\text{Lis-et}(\mathfrak{X})$ on $\mathfrak{X}$ as follows (the topos is denoted by $\mathfrak{X}_{\text{Lis-et}}$). The site $\text{Lis-et}(\mathfrak{X})$ has objects $T \xrightarrow[sm]{t} \mathfrak{X}$ with T being scheme, and morphisms

$$\begin{tikzcd} T' \arrow[r, "f"] \arrow[dr, "t'"'] & T \arrow[d, "t"] \arrow[dl, phantom, "\overset{f^\flat}{\Longrightarrow}"] \\ & \mathfrak{X} \end{tikzcd}$$

The coverings are given by family of diagrams of the form

$$\begin{tikzcd} T_i \arrow[r, "f_i"] \arrow[dr, "t_i"'] & T \arrow[d, "t"] \arrow[dl, phantom, "\Longrightarrow"] \\ & \mathfrak{X} \end{tikzcd}$$

such that we have etale surjection $\coprod T_i \twoheadrightarrow T$.

Then, we define $\mathscr{O}_{\mathfrak{X}} \in \mathfrak{X}_{\text{Lis-et}}$ as $\mathscr{O}_{\mathfrak{X}}(T \xrightarrow{t} \mathfrak{X}) := \Gamma(\mathscr{O}_T)$.

Now let $\mathcal{C}$ be the following category. The objects are: for all $(T,t) \in \text{Lis-et}(\mathfrak{X})$ a choice of etale sheaf of sets $\mathcal{F}_{(T,t)} \in T_{et}$ and for all $(f, f^\flat) : (T', t') \to (T, t)$ a choice

$$f^{-1}\mathcal{F}_{(T,t)} \xrightarrow{\rho_{(f,f^\flat)}} \mathcal{F}_{T',t'}$$

such that:

1. if $f : T' \to T$ is etale then $\rho_{(f,f^\flat)}$ is isomorphism.
2. For diagram

$$\begin{tikzcd} (T'', t'') \arrow[r, "(g,g^\flat)"] \arrow[dr] & (T', t') \arrow[d, "(f,f^\flat)"] \\ & (T, t) \end{tikzcd}$$

we have

$$\begin{tikzcd}
g^{-1}f^{-1}\mathcal{F}_{(T,t)} \arrow[r, "g^{-1}\rho_{(f,f^\flat)}"] \arrow[d, "\mathrm{can}"'] & g^{-1}\mathcal{F}_{(T',t')} \arrow[d, "\rho_{(g,g^\flat)}"] \\
(fg)^{-1}\mathcal{F}_{(T,t)} \arrow[r, "\rho_{(f,f^\flat)\circ(g,g^\flat)}"'] & \mathcal{F}_{(T'',t'')}
\end{tikzcd}$$

A morphism between $(\{\mathcal{F}_{(T,t)}\},\{\rho_{(f,f^\flat)}\}) \to (\{\mathcal{G}_{(T,t)}\},\{\lambda_{(f,f^\flat)}\})$ in $\mathcal{C}$ is a collection of morphisms $\gamma_{(T,t)} : \mathcal{F}_{(T,t)} \to \mathcal{G}_{(T,t)}$ so the following diagram commutes

$$\begin{tikzcd}
f^{-1}\mathcal{F}_{(T,t)} \arrow[r, "f^{-1}\gamma_{(T,t)}"] \arrow[d, "\rho_{(f,f^\flat)}"] & f^{-1}\mathcal{G}_{(T,t)} \arrow[d, "\lambda_{(f,f^\flat)}"] \\
\mathcal{F}_{(T',t')} \arrow[r, "\gamma_{(T',t')}"] & \mathcal{G}_{(T',t')}
\end{tikzcd}$$

We have $\mathcal{C} \to \mathfrak{X}_{\text{Lis-et}}$ given by $(\{\mathcal{F}_{(T,t)}\},\{\rho_{(f,f^\flat)}\})$ maps to the sheaf $\mathcal{F}$ given by $\mathcal{F}(T \xrightarrow{t} \mathfrak{X}) = \mathcal{F}_{(T,t)}(T)$.

This is a presheaf where transition maps of $\mathcal{F}$ come from $\rho_{(f,f^\flat)}$.

It is a sheaf because etale covers in Lis-et$(\mathfrak{X})$ are already coverings in T_{et}.

Conversely, $\mathfrak{X}_{\text{Lis-et}} \to \mathcal{C}$ given by $\mathcal{F}$ maps to the object $\mathcal{F}_{(T,t)} := \mathcal{F}|_{T_{et}}$.

Therefore, we get $\mathcal{C} \cong \mathfrak{X}_{\text{Lis-et}}$.

Definition 18.4. Let Λ be a sheaf of rings on Lis-et$(\mathfrak{X})$. Let $\mathcal{F}$ be a sheaf of Λ-module. Then:

1. We say $\mathcal{F}$ is ***Cartesian*** if for all $(T',t') \to (T,t)$, we get
$$f^*\mathcal{F}_{(T,t)} := f^{-1}\mathcal{F}_{(T,t)} \otimes_{f^{-1}\Lambda_{(T,t)}} \Lambda_{(T',t')} \xrightarrow{\sim} \mathcal{F}_{(T',t')}$$
2. We say $\mathcal{F}$ is ***quasi-coherent*** if it is Cartesian $\mathscr{O}_{\mathfrak{X}}$-module and for all (T,t), we have $\mathcal{F}_{(T,t)} \in (\text{Qcoh})(T_{et})$ is quasi-coherent.
3. We say $\mathcal{F}$ is ***coherent*** if $\mathcal{X}$ is locally Noetherian (note this implies for anyu $(T,t) \in \text{Lis-et}(\mathfrak{X})$ we get T locally Noetherian), $\mathcal{F}$ is quasi-coherent and for all (T,t), we have $\mathcal{F}_{(T,t)} \in (\text{Coh})(T_{et})$.

19 Quasi-coherent Sheaves

Last time we defined the Lis-et site. We also gave an alternative description, namely if $\mathcal{F} \in \mathfrak{X}_{\text{Lis-et}}$ is a sheaf of $\mathscr{O}_{\mathfrak{X}}$-modules, then for all $t : T \xrightarrow{sm} \mathfrak{X}$ we let $\mathcal{F}_{(T,t)} := \mathcal{F}|_{T_{et}}$.

Given any

$$\begin{tikzcd} T' \arrow[r, "f"] \arrow[dr, "t'"'] & T \arrow[d, "t"] \arrow[dl, phantom, "\Rightarrow"] \\ & \mathfrak{X} \end{tikzcd}$$

we have a module-theoretic pullback $f^*\mathcal{F}_{(T,t)} := f^{-1}\mathcal{F}_{(T,t)} \otimes_{f^{-1}\mathscr{O}_T} \mathscr{O}_{T'}$. In particular we get natural map $f^*\mathcal{F}_{(T,t)} \to \mathcal{F}_{(T',t')}$.

We also defined Cartesian $\mathcal{F}$, which is that, if for all $T' \to T$ we get $f^*\mathcal{F}_{(T,t)} \xrightarrow{\sim} \mathcal{F}_{(T',t')}$. We say $\mathcal{F}$ is quasi-coherent if $\mathcal{F}$ is Cartesian and all $\mathcal{F}_{(T,t)}$ are quasi-coherent. If $\mathfrak{X}$ is locally Noetherian, then $\mathcal{F}$ is coherent if all $\mathcal{F}_{(T,t)}$ are coherent.

This is a lot of data to keep track of, but as you would expect, to check this, we only need to do it on an open cover.

Proposition 19.1. *Let $X \twoheadrightarrow \mathfrak{X}$ be smooth covering and $\mathcal{F}$ Cartesian. Then $\mathcal{F}$ is quasi-coherent iff $\mathcal{F}_{(X,x)}$ is quasi-coherent. If $\mathfrak{X}$ is locally Noetherian, then $\mathcal{F}$ is coherent iff $\mathcal{F}_{(X,x)}$ is coherent.*

Proof. For all $Y \xrightarrow{y} \mathfrak{X}$, take pullback

$$\begin{tikzcd} Z \arrow[r, "p"] \arrow[d, two heads, "sm"', "\pi"] \arrow[dr, phantom, "\square"] & X \arrow[d, two heads, "sm"', "x"] \\ Y \arrow[r, "y"] & \mathfrak{X} \end{tikzcd}$$

We see $\pi^*\mathcal{F}_{(Y,y)} \cong \mathcal{F}_{(Z,z\pi)}$ because $\mathcal{F}$ is Cartesian. But $\mathcal{F}_{(Z,y\pi)} \cong p^*\mathcal{F}_{(X,x)}$ where $\mathcal{F}_{(X,x)}$ is quasi-coherent, hence $\mathcal{F}_{(Z,z)}$ is quasi-coherent. Therefore $\mathcal{F}_{(Y,y)}$ is by descent quasi-coherent.

The argument for coherent is similar. ♡

If $\mathfrak{X}$ is DM, then we have an etale site $\mathrm{Et}(\mathfrak{X})$ defined as follows (it always exists, but in general we don't have a good theory).

The objects are $T \xrightarrow[et]{t} \mathfrak{X}$, and morphisms are

$$\begin{tikzcd} T' \arrow[r, "et"] \arrow[dr, "et"'] & T \arrow[d, "et"] \arrow[dl, phantom, "\Leftarrow"] \\ & \mathfrak{X} \end{tikzcd}$$

The coverings are just etale coverings.

You can define Cartesian, quasi-coherent, and coherent in the same way as $\mathfrak{X}_{\text{Lis-et}}$ for $\mathfrak{X}_{et}$.

From now on, if we are talking about $\mathfrak{X}_{et}$, we always assume $\mathfrak{X}$ is DM.

In particular, the notion of (quasi)-coherent on $\mathfrak{X}_{et}$ agrees with $\mathfrak{X}_{\text{Lis-et}}$. In particular, we get the restriction map $r : \mathrm{Et}(\mathfrak{X}) \to \text{Lis-et}(\mathfrak{X})$ given by $(T \xrightarrow[et]{f}) \mapsto (T \xrightarrow[sm]{f} \mathfrak{X})$. Thus we get $r_* : (\mathrm{Qcoh})(\mathfrak{X}_{\text{Lis-et}}) \to (\mathrm{Qcoh})(\mathfrak{X}_{et})$.

Proposition 19.2. *r_* is an equivalence. It is also equivalence for coherent sheaves if $\mathfrak{X}$ is locally Noetherian.*

Proof. Start with $\mathcal{F} \in (\mathrm{Qcoh})(\mathfrak{X}_{et})$, we will extend to $\tilde{\mathcal{M}} \in (\mathrm{Qcoh})(\mathfrak{X}_{\text{Lis-et}})$ in a unique way.

To do this, we choose a smooth cover $X \xrightarrow{et} \mathfrak{X}$. Suppose we are given $Y \xrightarrow{sm} \mathfrak{X}$, we want to define $\tilde{\mathcal{M}}|_{Y_{et}}$. Well, we pullback the arrow $Y \xrightarrow{sm} \mathfrak{X}$, namely

$$\begin{tikzcd} Z \arrow[r, "sm", "f"'] \arrow[d, two heads, "et"] \arrow[dr, phantom, "\square"] & X \arrow[d, two heads, "et"] \\ Y \arrow[r, "sm"] & \mathfrak{X} \end{tikzcd}$$

By descent, it is enough to define $\tilde{\mathcal{M}} \in Z_{et}$ with descent data. Because we want $\tilde{\mathcal{M}}$ to be Cartesian, so we need $\tilde{\mathcal{M}}_Z = f^*\tilde{\mathcal{M}}_X = f^*\mathcal{M}_X$. We get a diagram

$$\begin{tikzcd} Z' \arrow[r, "g"] \arrow[d, shift right, "\pi_2"'] \arrow[d, shift left, "\pi_1"] & X' \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\ Z \arrow[r, "f"'] \arrow[d, "\pi"'] & X \arrow[d, "p"] \\ Y \arrow[r] & \mathfrak{X} \end{tikzcd}$$

Since $\mathcal{M}$ is Cartesian, so $p_1^*\mathcal{M}_X \cong p_2^*\mathcal{M}_X$. We want $\tilde{\mathcal{M}}$ to be Cartesian, so

$$\pi_1^*\tilde{\mathcal{M}}_Z \cong g^*p_1^*\mathcal{M}_X \cong g^*p_2^*\mathcal{M}_X \cong \pi_2^*\tilde{\mathcal{M}}_Z$$

Thus $\tilde{\mathcal{M}}_Z$ has descent data and it descends to $\tilde{\mathcal{M}}_Y$, as desired. ♡

The next topic is differentials on stacks.

W e have shown how to define $\Omega^1_{X/\mathfrak{X}}$ if we have smooth cover $X \twoheadrightarrow \mathfrak{X}$.

If $\mathfrak{X}$ is DM, you can even define $\Omega^1_{\mathfrak{X}} \in (\mathrm{Coh})(\mathfrak{X}_{et})$. Well, to do this, consider

$$X' = X \times_{\mathfrak{X}} X \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X \xrightarrow[\pi]{} \mathfrak{X}$$

We then descend Ω^1_X to get $\Omega^1_{\mathfrak{X}}$. Here π_1, π_2 are etale, so $\pi_1^*\Omega^1_X \cong \Omega^1_{X'} \cong \pi_2^*\Omega^1_X$ are canonical isomorphisms, thus we indeed have descent data. Therefore, we get $\Omega^1_{\mathfrak{X}}$ such that $\pi^*\Omega^1_{\mathfrak{X}} = \Omega^1_X$.

For Artin stack, this does not work because all you get $\pi_1 : X' \xrightarrow{sm} X$ and you have a map

$$0 \longrightarrow \pi_1^*\Omega^1_X \longrightarrow \Omega^1_{X'} \longrightarrow \Omega^1_{X'/X} \longrightarrow 0$$

where $\Omega^1_{X'/X} \neq 0$ unless it is etale. Therefore we cannot descend Ω^1_X because we have no isomorphism $\pi_1^*\Omega^1_X \to \Omega^1_{X'}$.

There are some remediations.

For exmaple, there is something called pseudo-differentials, but in that theory, we get $\Omega^1_{\mathfrak{X}/\mathfrak{X}} \neq 0$, which is weird.

Another solution is the cotangent complex $L_{\mathfrak{X}}$. This is a 2-term complex such that $\pi^* L_{\mathfrak{X}} \cong [\Omega^1_X \to \Omega^1_{X/\mathfrak{X}}]$, where $\pi : X \twoheadrightarrow \mathfrak{X}$. If $\mathfrak{X} = [X/G]$, then we can say more. In particular, in this case we get $\Omega^1_{X/\mathfrak{X}} \cong \mathscr{O}_X \otimes \mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual of the Lie algebra of G, i.e. $g = \Omega^1_{G,e}$ where $e \in G$ is the identity point.

Example 19.3. We have seen (in the proofs above) that to define

$$\mathcal{F} \in (\mathrm{Qcoh})(\mathfrak{X}_{et})$$

it is good enough to define $\mathcal{M} \in (\mathrm{Qcoh})(X)$ plus descent data where $\pi : X \twoheadrightarrow \mathfrak{X}$ is etale covering.

Let's look at $\mathfrak{X} = [X/G]$. In this case, we know $X \times_{\mathfrak{X}} X \cong G \times X$, and hence

$$X \times_{\mathfrak{X}} X \cong G \times X \underset{p}{\overset{\sigma}{\rightrightarrows}} X \xrightarrow{\pi} \mathfrak{X}$$

where σ is the action, and p is the projection, i.e. $(g,x) \mapsto gx$ and $(g,x) \mapsto x$. We want $\mathcal{M}$ quasi-coherent on X and $\sigma^*\mathcal{M} \cong p^*\mathcal{M}$ plus the cocycle condition. But that last part (i.e. $\sigma^*\mathcal{M} \cong p^*\mathcal{M}$ and cocycle condition) is just saying we have G acts on $\mathcal{M}$. Therefore, $(\mathrm{Qcoh})(\mathfrak{X}_{et})$ is just quasi-coherent sheaves on $\mathfrak{X}_{et}$ with G-linearization, i.e. G-action.

Example 19.4. For example, if $\mathfrak{X} = BG$, and we take $\operatorname{Spec} k \twoheadrightarrow \mathfrak{X}$, then $(\mathrm{Qcoh})(BG)$ is isomorphic to vector space plus G-action, which is G representations.

Example 19.5. If $G \circlearrowright X$ acts freely, i.e. X/G is algebraic space (e.g. if $X = \operatorname{Spec} A$ then $X/G = \operatorname{Spec}(A^G)$). Then $(\mathrm{Qcoh})(X/G) = (\mathrm{Qcoh})(X)$ plus G-linearization

Example 19.6. Suppose $X = \operatorname{Spec} L$ where L/K is Galois with Galois group G. Then $[X/G] = X/G = \operatorname{Spec} K$. Therefore, we see K-vector spaces is the same thing as L-vector spaces with G-action.

In general, given $\mathcal{F} \in (\mathrm{Qcoh})([X/G])$, it corresponds to $\mathcal{M} \in (\mathrm{Qcoh})(X)$ with G-action. If X is affine, then the cohomology $H^i(\mathcal{F})$ is equal the group cohomology $H^i(G, \Gamma(\mathcal{M}))$. If X is not affine, there is a spectral sequence relating $H^i(\mathcal{F})$ with $H^p(G, H^q(\mathcal{M}))$.

In general, if $f : \mathfrak{X} \to \mathfrak{Y}$ is quasi-compact and quasi-separated, then $f_*\mathcal{F}$ quasi-coherent if $\mathcal{F}$ is quasi-coherent. In particular, we get $f_*\mathcal{F}(Y \xrightarrow{sm} \mathfrak{Y}) = \mathcal{F}(Y \times_{\mathfrak{Y}} \mathfrak{X} \xrightarrow{sm} \mathfrak{X})$. There is a left adjoint f^*, but we are not going to define it.

Definition 19.7. We say $f : \mathfrak{X} \to \mathfrak{Y}$ is ***affine*** if it is representable and affine.

Given $\mathscr{A}$ a sheaf of $\mathscr{O}_{\mathfrak{X}}$-algebra, we can define $\operatorname{Spec}(\mathscr{A}) \to \mathfrak{X}$ as follows: the T-points are given as follows:

$$\begin{tikzcd} & \operatorname{Spec}\mathscr{A} \arrow[d] \\ T \arrow[ur] \arrow[r] & \mathfrak{X} \end{tikzcd}$$

is a choice $\rho : t^*\mathscr{A} \to \mathscr{O}_T$ morphism of $\mathscr{O}_T$-algebras. By construction, we see we get

$$\begin{tikzcd} \operatorname{Spec}(t^*\mathscr{A}) \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \operatorname{Spec}\mathscr{A} \arrow[d, "p"] \\ T \arrow[r, "t"] & \mathfrak{X} \end{tikzcd}$$

So π is an affine map of schemes, and so p is representable and affine.

Moreover, $\operatorname{Spec}\mathscr{A}$ is a stack because $\mathscr{O}_X$-algebra satisfy descent. Hence $\operatorname{Spec}\mathscr{A}$ is an Artin stack.

Like for schemes, we have the bijective correspondence

$$\left\{\begin{matrix} \text{affine} \\ \text{maps to } \mathfrak{X} \end{matrix}\right\} \leftrightarrow \{\mathscr{O}_X\text{-algebras}\}$$

given by $\mathscr{A} \mapsto (\operatorname{Spec}\mathscr{A} \to \mathfrak{X})$ and $(\mathfrak{Y} \xrightarrow{f} \mathfrak{X}) \mapsto f_*\mathscr{O}_{\mathfrak{Y}}$.

The next topic is closed substacks.

Definition 19.8. We say $\mathfrak{Z} \to \mathfrak{X}$ is ***closed/open immersion*** if it is representable and it is closed/open immersion.

Definition 19.9. Given $\mathfrak{X} \xrightarrow{f} Y$ with Y a scheme, the ***image*** of f is given by: choose $X \xrightarrow{\pi} \mathfrak{X} \xrightarrow{f} Y$, then $\operatorname{Im}(f) := \operatorname{Im}(f \circ \pi)$.

We say f is ***closed*** if for all $\mathfrak{Z} \subseteq \mathfrak{X}$ closed substacks, $\operatorname{Im}(\mathfrak{Z} \subseteq \mathfrak{X} \to Y)$ is closed.

Definition 19.10. We say $g : \mathfrak{X} \to \mathfrak{Y}$ is ***universally closed*** if for all $Y \to \mathfrak{Y}$ with Y scheme and diagram

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Y}} Y \arrow[r] \arrow[d, "g_Y"] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "g"] \\ Y \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

we have g_Y is closed.

Definition 19.11. We say $f : \mathfrak{X} \to \mathfrak{Y}$ is ***proper*** if f is separated, of finite type, and universally closed.

Example 19.12. Let A be Abelian variety, then $BA \to \operatorname{Spec} k$ is proper (it is separated because A is proper).

Example 19.13. If G is finite group, then $BG \to \operatorname{Spec} k$ is proper.

20 Valuative Criteria

Theorem 20.1. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be finite type map of locally Noetherian Artin stacks.*

1. *Then f is separated iff for all DVR R with $K = \operatorname{Frac}(R)$ and diagram*

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, dashed, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, shift left, "y"] \arrow[ur, shift right, "x"'] & \mathfrak{Y} \end{tikzcd}$$

   we have $x \cong y$.
2. *Then f is proper iff f is separated and for all DVR R with $K = \operatorname{Frac}(R)$, there exists finite extension K'/K and normalization of R in K' such that we have the dashed arrow in the following diagram*

$$\begin{tikzcd} \operatorname{Spec} K' \arrow[r, "\exists"] \arrow[d] & \operatorname{Spec} K \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, dashed, "f"] \\ \operatorname{Spec} R' \arrow[r, "\exists"'] \arrow[urr, dashed, "\exists"] & \operatorname{Spec} R \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

So, how do you think about this? To get some intuition, lets take $\mathfrak{Y}$ to be $\operatorname{Spec} k$. Then $f : \mathfrak{X} \to \operatorname{Spec} k$ is a moduli space, and the theorem says, f is proper if and only if, if we are given $x \in \mathfrak{X}(K)$, e.g. a curve over K, then we can extend it to $\mathfrak{X}(R)$, e.g. a curve over R, if we allow for finite extensions of K.

Example 20.2. Let's consider an exmaple where the "weak form" of valuative criteria does not hold, i.e. if we don't allow finite extension.

We know BG is proper if G is finite. Let's take $B(\mathbb{Z}/2)$ with $\operatorname{Char} k = 0$. Then we have

$$\operatorname{Spec} K \longrightarrow B(\mathbb{Z}/2)$$

with $R = k[[t]]$ and $K = \operatorname{Frac}(R) = k((t))$. Then we have a $\mathbb{Z}/2$-torsor $\operatorname{Spec} k((\sqrt{t})) \to \operatorname{Spec} K$ with the action being $\sqrt{t} \mapsto -\sqrt{t}$.

Then, we can't extend this to a $\mathbb{Z}/2$-torsor over R. We will not give a proof, but let's check the obvious choice does not work. The obvious choice would be extend to $\operatorname{Spec} k[[\sqrt{t}]] \to \operatorname{Spec} k[[t]]$, where we still have $\mathbb{Z}/2$-action extends to

$\sqrt{t} \mapsto -\sqrt{t}$.

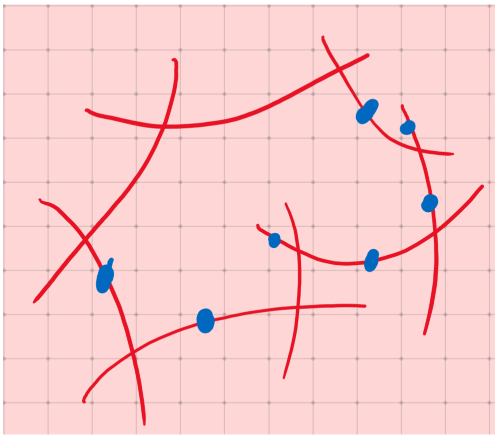
However, we have a problem: this is not a torsor because it is ramified over $t = 0$. We see $k[[\sqrt{t}]] = k[[t]][x]/(x^2 - t)$ and hence if $t = 0$ we get $k[[t]][x]/x^2$ which is not reduced and so $\operatorname{Spec} k[[\sqrt{t}]] \to \operatorname{Spec} k[[t]]$ is not etale. Thus it is not a torsor.

However, if we take finite extension $K' = k((\sqrt{t}))$ over K (this is degree 2 extension), then the torsor over K' is trivial, i.e. $\mathbb{Z}/2 \times \operatorname{Spec} K' \to \operatorname{Spec} K'$, in other word, we get

$$\begin{tikzcd}
\mathbb{Z}/2 \times \operatorname{Spec} R' \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{Z}/2 \times \operatorname{Spec} K' \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K' \arrow[d] \\
\operatorname{Spec} R' & \operatorname{Spec} K' \arrow[l] \arrow[r] & \operatorname{Spec} K
\end{tikzcd}$$

and now it extends.

Example 20.3. Deligne Mumford introduced a compactification $\overline{M}_{g,n}$ of $M_{g,n}$, i.e. $M_{g,n} \subseteq \overline{M}_{g,n}$ dense open substack and $\overline{M}_{g,n}$ is proper. We have $\overline{M}_{g,n}$ is like $M_{g,n}$ but the geometric fibers are "stable curves", i.e. nodel curves which look like:



where each line is a curve, and intersections are nodes, and dots are marked points with the property that every genus 1 component has ≥ 1 special points and every genus 0 component has ≥ 3 special points, where we say a point is special if it is a node or a marked point.

This ensures automorphism group is finite. As a result, $\overline{M}_{g,n}$ is DM.

Next, how do we show $\overline{M}_{g,n}$ is proper? We use valuative criteria, but a stronger version of it. That is,

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & M_{g,n} \arrow[d, "\subseteq"] \\ \operatorname{Spec} R \arrow[r, dashed, "\exists!"'] & \overline{M}_{g,n} \end{tikzcd}$$

where we can get the dashed arrow after finite extension. Viz, we start with genus g curves and n marked points over K, we want to extend to stable curve over R, after finite extension.

Let's consider the basic case: consider $\overline{M}_{1,1}$, i.e. we have elliptic curves over $\mathbb{Q}$. A theorem of Tate says it is impossible to extend to an elliptic curve over $\mathbb{Z}$, we always have a node (i.e. semi-stable reduction) or worth (e.g. cusp, which is called additive reduction).

Tate's algorithm tells you how to make a finite extension so that you end up with semi-stable reduction.

In general, extending from K-points of $M_{g,n}$ to R-points of $\overline{M}_{g,n}$ is called semi-stable reduction theorem.

David Smyth in 2010 constructed "alternative compactification" of $M_{g,n}$. It is a different stack where you replace the $g = 1$ constrain with cusps.

So, why we even care about this?

This arises naturally when you run minimal model program (MMP) on coarse space of $\overline{M}_{g,n}$.

This should be enough motivation, and let's give a sketch proof of the theorem.

Proof. For separatedness: Suppose we have

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, dashed, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, shift left, "x"] \arrow[ur, shift right, "x'"'] & \mathfrak{Y} \end{tikzcd}$$

then note we can base change to $\mathfrak{X}_R \coloneqq \mathfrak{X} \times_{\mathfrak{Y}} R$ and get

$$\begin{tikzcd} \mathfrak{X}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}_R \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} R \arrow[u, bend left, "x"] \arrow[u, bend right, "x'"'] \end{tikzcd}$$

But then this means we get

$$\begin{tikzcd}
\mathrm{Isom}(x_K, x'_K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K \arrow[d, dashed, "x_K"] & \operatorname{Spec} R \arrow[d, dashed, "x"] & \mathrm{Isom}(x, x') \arrow[l] \arrow[d] \arrow[dl, phantom, "\square"] \\
\operatorname{Spec} K \arrow[r, "x'_K"] & \mathfrak{X}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}_R \arrow[d] & \operatorname{Spec} R \arrow[l, "x'"'] \\
& \operatorname{Spec} K \arrow[r] & \operatorname{Spec} R \arrow[u, bend left, "x"] \arrow[u, bend right, "x'"']
\end{tikzcd}$$

where $x_K \cong x'_K$ means we have K-points of $\mathrm{Isom}(x, x')$. This extending to R-point just means $\mathrm{Isom}(x, x')$ is proper for all x, x'. Equivalent to $\Delta_{\mathfrak{X}}$ is proper but by definition this means $\mathfrak{X}$ is separated.

Now we prove the valuative criteria for properness: If f is proper, then it satisfies the valuative criteria. Again, pulling back to $\operatorname{Spec} R$, we have

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[d, "x"] \arrow[dd, bend right, "\mathrm{Id}"'] & \\
\mathfrak{X}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}_R \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

where $\operatorname{Spec} K \xrightarrow{x} \mathfrak{X}_K$ is a point and $\mathfrak{X}_K \to \operatorname{Spec} K$ is proper and hence x is a closed immersion. Therefore, we can take $\mathfrak{Z}$ to be the closure of x in $\mathfrak{X}_R$ and we get

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, "x"] \arrow[dd, bend right, "\mathrm{Id}"'] & \mathfrak{Z} \arrow[d, "\subseteq"] \\
\mathfrak{X}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}_R \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

but since we have no extra irreducible components of $\mathfrak{Z}$ over closed point of $\operatorname{Spec} R$ because $\mathfrak{Z}$ is the closure of x, we see the arrow $\mathfrak{Z} \to \operatorname{Spec} R$ is flat, i.e.

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d, "x"] \arrow[dd, bend right, "\mathrm{Id}"'] & \mathfrak{Z} \arrow[d, "\subseteq"] \arrow[dd, bend left, "flat"] \\
\mathfrak{X}_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X}_R \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

Now choose a smooth cover $Z \twoheadrightarrow \mathfrak{Z}$ and we note

$$\begin{tikzcd}
Z \arrow[r, two heads, "sm"] \arrow[dr, "{flat,surj}"'] & \mathfrak{Z} \arrow[d] \\
& \operatorname{Spec} R
\end{tikzcd}$$

and hence

$$
\begin{tikzcd}
 & Z \arrow[d] \\
\operatorname{Spec} R' \arrow[r] \arrow[ur, dashed, "\exists"] & \operatorname{Spec} R
\end{tikzcd}
$$

with K'/K finite extension.

Conversely, assume valuative criteria holds. Then consider

$$
\begin{tikzcd}
\mathfrak{X}_Y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "f"] \\
Y \arrow[r, two heads, "sm"] & \mathfrak{Y}
\end{tikzcd}
$$

by descend, we just need to show g is proper, so we may as well assume $\mathfrak{Y} = Y$.

Now note by Chow's lemma, which says there eixsts proper morphism $p : P \twoheadrightarrow \mathfrak{X}$, because $\mathfrak{X}$ is separated. Thus we get

$$
\begin{tikzcd}
P \arrow[r, two heads, "proper", "p"'] \arrow[dr, "h"] & \mathfrak{X} \arrow[d, "f"] \\
 & Y
\end{tikzcd}
$$

and it is enough to show h is proper.

However, note p, h are representable, so proper iff we have valuative criteria, i.e. we have valuative criteria for f, p and hence we have valuative criteria for h (to see this, note we get

$$
\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[dd] & P \arrow[d, "p"] \\
 & \mathfrak{X} \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r] \arrow[uur, dashed] \arrow[ur, dashed] & Y
\end{tikzcd}
$$

where the $\operatorname{Spec} R \to \mathfrak{X}$ arrow is due to val crit for f and the $\operatorname{Spec} R \to P$ arrow is due to val crit for p) and hence h is proper. ♡

The next topic is coarse moduli space (and some newer stuff).

Definition 20.4. Let $\mathfrak{X}$ be Artin stack, X be algebraic space. Then $\pi : \mathfrak{X} \to X$ is ***coarse (moduli) space*** if:

1. for all $\mathfrak{X} \to Y$ with Y algebraic space, we get

$$
\begin{tikzcd}
\mathfrak{X} \arrow[r, "\pi"] \arrow[dr, "\forall"] & X \arrow[d, dashed, "\exists!"] \\
 & Y
\end{tikzcd}
$$

2. π induces a bijection between $\mathfrak{X}(k)/\mathrm{iso} \to X(k)$ for all $k = \overline{k}$.

Example 20.5. Let G be a group scheme over k. Then $BG \xrightarrow{k} \operatorname{Spec} k$ with π the structure map is a coarse space map. To see this, note we have

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[d, two heads] \arrow[dr, "\mathrm{Id}"] \arrow[dd, bend right=60, "y"'] & \\
BG \arrow[d] & \operatorname{Spec} k \arrow[dl, dashed, "y"', "\exists!"] \\
Y &
\end{tikzcd}$$

where we recall $BG = [\operatorname{Spec} k/G]$. On the other hand, if $\Omega = \overline{\Omega}$ is ACF, then $BG(\Omega)/\mathrm{iso}$ only has one point, namely the trivial torsor, thus we only have one map $\operatorname{Spec}\Omega \to \operatorname{Spec} k$.

Example 20.6. If G is a finite group with $G \circlearrowleft \operatorname{Spec} A$, then the invariant map $\mathfrak{X} = [\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is coarse space map.

Example 20.7. The coarse space of $M_{1,1}$ is $\mathbb{A}^1$, where the coarse space map is sending E to its j-invariant, i.e. $E \mapsto j(E)$.

Example 20.8. Let's consider a non-example. Say we have $\mathbb{G}_m \circlearrowleft \mathbb{A}^1$ with $\lambda \cdot x = \lambda x$. Then $[\mathbb{A}^1/\mathbb{G}_m]$ has 2 points. It has a $\mathbb{G}_m$-stablizer 0 and a trivial stablizer η. In particular, $0 \in \overline{\eta}$. We note $\mathfrak{X}$ has no coarse space as if $\pi : \mathfrak{X} \to X$ is a map into algebraic space, then $\pi(\eta)$ and $\pi(0)$ are both k-points, i.e. they have the same residue field. But then we also have $\pi(0) \in \overline{\pi(\eta)}$, it just doesn't happen.

21 Good Moduli Space

Recall last time we defined coarse spaces, where we say $\pi : \mathfrak{X} \to X$ with X algebraic space is coarse space if π is universal for maps to algebraic spaces and for all $k = \overline{k}$ we have a bijection $\pi : \mathfrak{X}(k)/\mathrm{iso} \to X(k)$.

Theorem 21.1 (Keel-Mori Theorem)**.** *Let S be locally Noetherian, and suppose $\mathfrak{X}/S$ have finite diagonal (in char 0, this is basically equivalent to $\mathfrak{X}$ is DM). Then $\mathfrak{X}$ has a corase space $\pi : \mathfrak{X} \to X$. Furthermore,*

1. *If $\mathfrak{X}$ is of finite type, then X is of finite type.*
2. *π is proper and $\pi_* \mathscr{O}_{\mathfrak{X}} = \mathscr{O}_X$.*
3. *π commutes with flat base change, i.e. if we have flat map $X' \to X$ and*

diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "\pi"] \\ X' \arrow[r, "flat"] & X \end{tikzcd}$$

then $\pi' : \mathfrak{X}' \to X'$ *is coarse space.*

Remark 21.2. Even if S is not locally Noetherian, all but (1) still holds.

Remark 21.3. We know what this theorem says, but how do we think about the statement $\pi_* \mathscr{O}_{\mathfrak{X}} = \mathscr{O}_X$?

Well, it means functions on $\mathfrak{X}$ are functions on X, e.g. if

$$\pi : \mathfrak{X} := [\operatorname{Spec} A/G] \to X = \operatorname{Spec} A^G$$

then $\Gamma(\mathscr{O}_{\mathfrak{X}}) = (\Gamma \mathscr{O}_A)^G = A^G = \Gamma(\mathscr{O}_X)$.

We note KM (Keel-Mori) theorem only handles DM stacks in char 0. What about Artin stacks?

Typically, Artin stacks don't have corase space, e.g. $[\mathbb{A}^1/\mathbb{G}_m]$. Heuristically, this is because Artin stacks are rarely separated (and if we have corase space, it implies its proper and hence separated).

Jarod Alper in 2009 or 2010 introduced a notion of corase space for Artin stacks: they are called good moduli spaces (gms). We will talk about the idea.

Definition 21.4. We say a map $f : \mathfrak{X} \to \mathfrak{Y}$ is ***cohomologically affine*** if $f_* : (\mathrm{Qcoh})(\mathfrak{X}) \to (\mathrm{Qcoh})(\mathfrak{Y})$ is exact.

Theorem 21.5 (Serre). *If $f : \mathfrak{X} \to \mathfrak{Y}$ is a represetable map, then its cohomologically affine if and only if it is affine.*

Definition 21.6. We say $f : \mathfrak{X} \to \mathfrak{Y}$ is ***Stein*** if $f_* \mathscr{O}_{\mathfrak{X}} = \mathscr{O}_{\mathfrak{Y}}$.

Example 21.7.

1. KM theorem says if $\mathfrak{X}$ has finite diagonal, then coarse map is Stein.
2. If X is normal (e.g. smooth) scheme and $U \subseteq X$ open with $\operatorname{codim}(X \backslash U) \geq 2$ then the inclusion map $i : U \to X$ is Stein.
3. If $f : X \to Y$ is proper with connected fibers, e.g. any projective variety $X \to \operatorname{Spec} k$, then f is Stein.

Remark 21.8. If $f : X \to Y$ with X, Y schemes, and f is affine and Stein, then f is an isomorphism. More generally, this is true if $f : \mathfrak{X} \to \mathfrak{Y}$ is represetable.

Definition 21.9. Let $\pi : \mathfrak{X} \to X$ with X algebraic space. Then we say π is ***good moduli space*** if π is cohomologically affine and Stein.

Example 21.10. Let G be an affine algebraic group over k. Then a coarse space map $\pi : BG \to \operatorname{Spec} k$ is Stein. Hence it is gms iff it is cohomologically affine. In particular, note $\pi_* : (\mathrm{Qcoh})(BG) \to (\mathrm{Qcoh})(\operatorname{Spec} k)$ correspond to a map from k-vector spaces with G-action to k-vector spaces, i.e. π_* is given by $G \circlearrowleft V \mapsto V^G$. By definition, π is cohomologically affine iff taking G-invariants is exact.

These are called ***linearly reductive groups***.

Example 21.11. Let G be finite group, then G is linearly reductive iff $\operatorname{Char} k \nmid |G|$. Recall when we prove every representation of G can be decompose into irreducible, we did something like $0 \to V' \to V \to V'' \to 0$ and we need to find a G-equivariant splitting. To do this, you need to divide by $|G|$ (hence we want $\operatorname{Char} k \nmid |G|$).

Example 21.12. The following groups are all linearly reductive: SL_n, GL_n, Sp_n, SO_n, tori. A non-example would be $\mathbb{G}_a = \mathbb{A}^1$ considered as a group under addition. To see this, consider $\mathbb{G}_a \to \mathrm{GL}_2$ via the map

$$x \mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

this correspond to a sequence $0 \to k \to k^{\oplus 2} \to k \to 0$ but the map is not diagonalizable and hence it does not split, i.e. $\mathbb{G}_a$ is not linearly reductive.

Example 21.13. Let G be linearly reductive group scheme and let G acts on $\operatorname{Spec} A$. Then $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is gms map. For example, $[\mathbb{A}^1/\mathbb{G}_m] \to \operatorname{Spec} k[x]^{\mathbb{G}_m} = \operatorname{Spec} k$ is a gms. Thus, we see the notion of gms is sort of containing the notion of corase space, but that's not always the case.

We also note, this example shows that if we have gms $\pi : \mathfrak{X} \to X$, the points of $\mathfrak{X}(k)$ up to isomorphism are not in bijection with $X(k)$.

Example 21.14. Consider the corase map $B\mathbb{G}_a \to \operatorname{Spec} k$. However, this is not gms.

Theorem 21.15 (Alper)**.**

1. *gms are universal maps to algebraic spaces if the base is locally Noetherian.*
2. *gms commutes with arbitrary base change.*
3. *let $\pi : \mathfrak{X} \to X$ be gms, then for all $x : \operatorname{Spec} k \to X$ with $k = \overline{k}$, there exists unique closed point in $\mathfrak{X}$ above x.*

There is now a version of KM theorem for gms, i.e. for Artin stacks. This is a result by Alper, Halpern-Leistner, Heinloth in 2019.

Remark 21.16. If $\mathfrak{X}$ has finite diagonal and $\mathfrak{X}$ is DM, then the corase space $\pi : \mathfrak{X} \to X$ is a good moduli space if and only if all stablizer groups are prime to characteristic.

Theorem 21.17. *Let S be locally Noetherian, $\mathfrak{X}/S$ be DM with finite diagonal. Let $\pi : \mathfrak{X} \to X$ be coarse space. Let $\overline{x} : \operatorname{Spec} k \to X$ with $k = \overline{k}$. Since π is coarse, there exists a unique lift $\tilde{x} \in \mathfrak{X}(k)$. Let $G_{\tilde{x}}$ be the automorphism group, which is finite because $\Delta_{\mathfrak{X}/S}$ is finite.*

Then, there exists etale neighbourhood $U \xrightarrow{et} X$, such that the pullback is given by

$$\begin{tikzcd} {[V/G_{\tilde{x}}]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ U \arrow[r, "et"] & X \end{tikzcd}$$

with $V \to U$ finite and $G_{\tilde{x}} \circlearrowright V$.

Proof. We give a sketch proof.

It is enough to prove this where U is the "etale stalk of X". The etale stalk is $\operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,x}$ where sh stands for strict Henselization (it is sort of like a completion/extension that makes Hensel's lemma holds).

Then, there exists etale $V \twoheadrightarrow \mathfrak{X}$ such that the sequence $V \to \mathfrak{X} \to X$ is quasi-finite in neighbourhood of x. Now take pullback, we get

$$\begin{tikzcd} V_{\overline{x}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ \mathfrak{X}_{\overline{x}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}} \arrow[r] & X \end{tikzcd}$$

Now we import a black box statement without proof: $V_{\overline{x}} \to \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}}$ is quasi-finite, so $V_{\overline{x}} = W_1 \coprod W_2$ where $W_1 \to \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}}$ is finite and $W_2 \to \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}}$ misses $\overline{x}$.

Let V' be one connected component of W_1. Then we have $V' \subseteq V_{\overline{x}} \twoheadrightarrow \mathfrak{X}_{\overline{x}}$ where we note $V' \twoheadrightarrow \mathfrak{X}_{\overline{x}}$ is etale cover because it hits closed point. Thus we get the following sequence

$$\begin{tikzcd} \mathfrak{Z}' = V' \times_{\mathfrak{X}_{\overline{x}}} V' \arrow[r, shift left, "et"] \arrow[r, shift right, "et"'] & V' \arrow[r, "et"'] \arrow[rr, bend left, "finite"] & \mathfrak{X}_{\overline{x}} \arrow[r] & \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\overline{x}} \end{tikzcd}$$

and it becomes

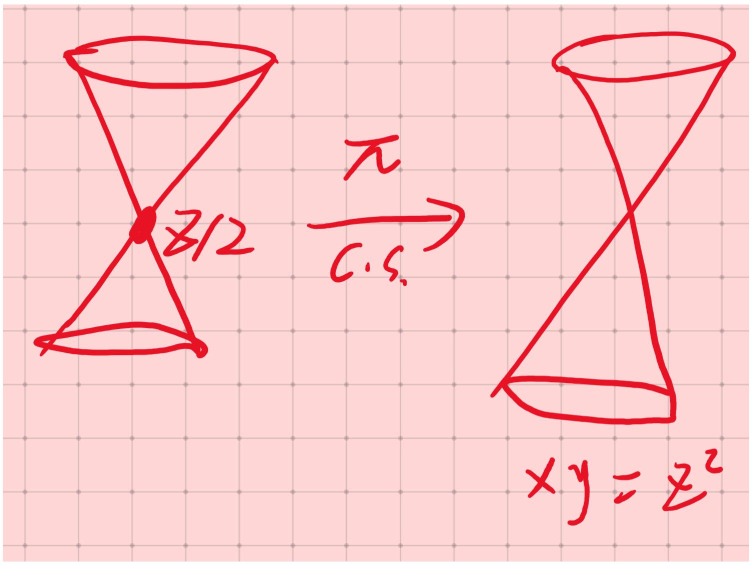
$$\begin{tikzcd} \mathfrak{Z}' = V' \times_{\mathfrak{X}_{\bar{x}}} V' \arrow[r, shift left, "et"] \arrow[r, shift right, "et"'] & V' \arrow[r, "et"'] \arrow[rr, bend left, "finite"] & \mathfrak{X}_{\bar{x}} \arrow[r] & \operatorname{Spec} \mathscr{O}^{\mathrm{sh}}_{X,\bar{x}} \\ G_{\tilde{x}} \times V'_k \arrow[u] \arrow[r, shift left] \arrow[r, shift right] \arrow[ur, phantom, "\square"] & V'_k \arrow[r] & \mathfrak{X}_k \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \operatorname{Spec} k \arrow[u] \end{tikzcd}$$

By deformation theory, there exists a deformation of $G_{\tilde{x}} \times V'_k \to V'_k$ over V', therefore, $\mathfrak{Z}' \cong V' \times G_{\tilde{x}} \to V'$ (this is called invariance of the etale site).

However, note we have $G_{\tilde{x}} \times V' \rightrightarrows V'$ where one arrow is the projection, and the second arrow is σ. What is σ?

The groupoid structure on $\mathfrak{X}_{\bar{x}}$ turns σ into a group action map, i.e. $\mathfrak{X}_{\bar{x}} = [V'/G_{\tilde{x}}]$ ♡

Example 21.18. Recall we had example $[\mathbb{A}^2/\mathbb{Z}/2]$ which looks like a cone



We know π is proper, coarse space map. If we consider the pullback

$$\begin{tikzcd} \, \arrow[r, "\subseteq"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} = \text{smooth} \arrow[d] \\ X^{sm} \arrow[r] & X \end{tikzcd}$$

So π is birational, proper map from a smooth stack. So π is a "stacky resolution".

Now, let's ask what can we say about X by looking at $\mathfrak{X}$. Hodge theory says that over $\mathbb{C}$, we have decompostion aobut $H^*(Y)$ with Y projective over $\mathbb{C}$, i.e. $H^n(Y)$ breaks up into finer invariants isomorphic to $\bigoplus_{p+q=n} H^q(\Omega^p_{Y/\mathbb{C}})$. This is called Hodge decomposition.

Mirror symmetry is about: we start with Calabi-Yau Y, and we can show there exists a mirror Y^* where the $\dim H^q(\Omega^p_{Y^*})$ are $\dim H^q(\Omega^p_Y)$ up to a "flip".

Now, in our example, we have a variety X with $\mathfrak{X}$ lying above. It turns out, we can develop a version of Hodge theory of $\mathfrak{X}$ and it gives the Hodge theory of X (this is the work of Steenbrink, where he called those things V-manifolds).

Question: is there a smooth DM stack $\mathfrak{X} \xrightarrow[c.s.]{\pi} \operatorname{Spec} k[x,y,z]/(xy - z^2)$ in char 2?

The answer is no. However, there exists a smooth Artin stack $\mathfrak{X}$ with finite Δ with X as coarse space.

It is given by $[\mathbb{A}^2/\mu_2]$ where μ_2 is 2nd roots of unity, where in Char(2) it is different from $\mathbb{Z}/2$, i.e. $\#\{\mu_2(\mathbb{F}_2[x]/x^2)\} \neq 2$. In particular, this is a singular group scheme and it is linearly reductive.

22 Canonical And Root Stacks

In this section we consider two constructions, called canonical stacks and root stacks.

If G is a finite group with $\operatorname{Char} k \nmid |G|$ and if $G \circlearrowright \mathbb{A}^n_k$ is a linear representation, then there is a characterization of when $\mathbb{A}^n_k/G$ is smooth.

Example 22.1. Let $\mathbb{Z}/2$ acts on $\mathbb{A}^2_k$ via $(x,y) \mapsto (x,-y)$. Then

$$\mathbb{A}^2_k/(\mathbb{Z}/2) = \operatorname{Spec} k[x,y]^{\mathbb{Z}/2} = \operatorname{Spec} k[x,y^2] \cong \mathbb{A}^2_k$$

This is smooth.

On the other hand, let $\mathbb{Z}/2$ acts on $\mathbb{A}^2_k$ via $(x,y) \mapsto (-x,-y)$. Then we get

$$\mathbb{A}^2_k/(\mathbb{Z}/2) = \operatorname{Spec} k[x^2,xy,y^2] = \operatorname{Spec} k[a,b,c]/(ac - b^2)$$

This is singular.

Let's compare this two examples. In example 1, the fixed locus of the action is $y = 0$, a hyperplane. In example 2, the fixed locus is $x = y = 0$.

Definition 22.2. We say $1 \neq g \in G$ is a ***pseudo-reflection*** if its fixed locus is a hyperplane, i.e. if g has all but one eigenvalue 1.

Note reflection means all but one eigenvalue equal 1 and the last eigenvalue equal -1. Then pseudo-reflection relaxes the last condition, i.e. we allow the last eigenvalue to be some root of unity.

Theorem 22.3 (Chevalley–Shephard–Todd)**.** *Let G be a finite group, $G \circlearrowright \mathbb{A}^n_k$ with $\operatorname{Char} k \nmid |G|$. Then $\mathbb{A}^n_k/G \cong \mathbb{A}^n_k$ iff $\mathbb{A}^n_k/G$ is smooth iff the subgroup $H \subseteq G$ generated by pseudo-reflections is all of G.*

What this means is if $X = \mathbb{A}^n_k/G$ for some finite group G with $\operatorname{Char} k \nmid |G|$, then there exists a minimal way to write X in this form.

For example, $\mathbb{A}^2_k \cong \mathbb{A}^2_k/(\mathbb{Z}/2)$ as in the first example. But there was a more minimal way to write $\mathbb{A}^2_k$ as a quotient by a finite group, namely $\mathbb{A}^2_k \cong \mathbb{A}^2_k/$trivial group.

Definition 22.4. A variety X/k has ***quotient singularity*** if there exists a etale cover $\{X_i \xrightarrow{et} X\}$ such that $X_i = V_i/G_i$ with V_i smooth over k and G_i finite.

We proved in last class that if $\mathfrak{X}$ is smooth DM, then its coarse space X has quotient singularity since we showed

$$\begin{tikzcd}
{[V/G] \cong \mathfrak{X}'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "\pi"] \\
X' \arrow[r, two heads, "et"] & X
\end{tikzcd}$$

with V smooth and G finite.

Conversely, in Vistoli's thesis, he showed the following

Theorem 22.5 (Vistoli)**.** *For all X with quotient singularity prime to the characteristic, there exists a canonical smooth DM stack $\mathfrak{X}$ with coarse space X. Furthermore, $\mathfrak{X} \xrightarrow{\pi} X$ is an isomorphism over X^{sm} the smooth locus of X.*

Vistoli proves this by looking locally where $X_i = V_i/G_i$ and G has no pseudo-reflections. Then he lets $\mathfrak{X}_i = [V_i/G_i]$ and proves the $\mathfrak{X}_i$ glues.

This is called the ***canonical stack*** of X, we denote it by X^{can}.

Next, we consider the root stacks.

Given a Cartier divisor D on X, its cut out by $s = 0$ where $s \in \mathscr{L}$ is a line bundle. There is a stack, the ***nth root stack***, denoted by $\sqrt[n]{X/D} \xrightarrow{\pi} X$ which is an isomorphism over $X \setminus D$ and along D it adds $\mathbb{Z}/n$-stablizers.

It has a moduli interpretation, namely $\sqrt[n]{X/D}(T)$ is given by the collection of the following data: a map $T \xrightarrow{f} X$, a line bundle $\mathscr{M}$, a section $t \in \mathscr{M}$, and an isomorphism $z : \mathscr{M}^{\otimes n} \xrightarrow{\sim} f^*\mathscr{L}$ such that $z(t^{\otimes n}) = s$.

We see that root stacks are a way of putting stacky structure in codimension 1 although it only adds Abelian stablizers.

It turns out that every smooth DM stack with trivial generic stablizer can be built out of these two operations: canonical stacks and root stacks.

Theorem 22.6 (The Bottom-Up Theorem, Geraschenko & Satriano)**.** *If $\mathfrak{X}$ is a smooth separated DM stack with trivial generic stablizer and if X is coarse space of $\mathfrak{X}$. Then $\mathfrak{X}$ is a canonical stack over a root stack over the canonical stack of X.*

More specifically, if $\pi : \mathfrak{X} \to X$ has ramification divisor $D = D_1 \cup D_2 \ldots \cup D_n \subseteq X$ and π is ramified to order e_i along D_i, then $\mathfrak{R} = (\sqrt[e]{X^{\mathrm{can}}/D})^{\mathrm{can}}$ where $\sqrt[e]{X^{\mathrm{can}}/D}$ means take the e_ith root stack along D_i.