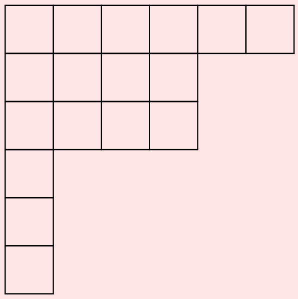
Contents

1 Day One

Definition 1.1. A ***partition*** $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a weakly decreasing sequence of positive integers.

1. The number $\ell = \ell(\lambda)$ is called the ***length*** of the partition.
2. The ***size*** of λ is $|\lambda| = \sum \lambda_i$. We also write $\lambda \vdash |\lambda|$.
3. The ***width*** of λ is λ_1.
4. The ***diagram*** (or Ferrers diagram, or Young diagram) of λ is given by putting boxes as parts. For example, 644111 has diagram

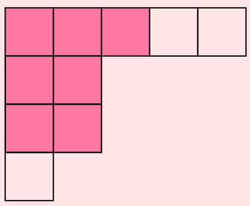


5. The ***conjugate*** of λ, denoted by λ', is the partition obtained by flip the diagram of λ. For example, conjugate of 644111 is 633311.
6. We also might write λ in following ways: for example, if $\lambda = 644111$ then

$$\lambda = 64^2 1^3 = (6, 4, 4, 1, 1, 1, 0, 0, \ldots) = 1^2 2^0 3^0 4^2 5^0 6^1 7^0 \ldots$$

7. We write $\mathcal{P}$ be the set of all partitions, $\mathcal{P}_{\ell,w} = \{\lambda \in \mathcal{P} : \ell(\lambda) \leq \ell, \lambda_1 \leq w\}$. For example, $\mathcal{P}_{\ell,\infty} = \{\lambda \in \mathcal{P} : \ell(\lambda) \leq \ell\}$ and $\mathcal{P}_{\infty,w} = \{\lambda \in \mathcal{P} : \lambda_1 \leq w\}$.

Example 1.2 (Containment Order/Young's Lattice)**.** We denote this order by $\subseteq$ and we say $\mu \subseteq \lambda$ iff $\mu_i \leq \lambda_i$ for all i. This is called containment order because $\mu \subseteq \lambda$ then the diagram of λ would contain μ. For example, if $\mu = 322, \lambda = 5221$ then we have



where the shaded area is μ.

Example 1.3 (Dominance Order)**.** We say $\mu \leq \lambda$ iff $\sum_{j=1}^{i} \mu_j \leq \sum_{j=1}^{i} \lambda_i$ for all i.

Example 1.4 (Lexicographic Order)**.** We say $\mu \leq_{lex} \lambda$ iff there exists $m \geq 1$ such that $\mu_i = \lambda_i$ for $i \leq m - 1$ and $\lambda_m > \mu_m$.

Example 1.5 (Refinement Order)**.** For $\lambda, \mu \vdash n$, we say $\mu \leq_{ref} \lambda$ iff there exists a surjective function $\phi : [\ell(\mu)] \to [\ell(\lambda)]$ such that $\lambda_i = \sum_{j \in \phi^{-1}(i)} \mu_j$.

To see this is not in fact complicated, say $\mu = 54421$ and $\lambda = 871$. Then we see we can use $4+4$ to make 8, $5+2$ to make 7 and 1 to 1, hence our function is simply $\phi(1) = \phi(4) = 2$, $\phi(2) = \phi(3) = 1$ and $\phi(5) = 3$.

Remark 1.6. We see

$$\mu \leq \lambda \Rightarrow \mu \leq_{lex} \lambda$$

$$\mu \leq_{ref} \lambda \Rightarrow \mu \leq \lambda$$

Remark 1.7. For the purpose of this course, when we talk about generating functions, we mean

$$\sum_{S \in \mathcal{S}} f(S)$$

where f is some function.

For example, the generating function for $\mathcal{P}_{\infty,w}$ with weight function $|\lambda|$ is given by

$$P_{\infty,w}(x) = \sum_{\lambda \in \mathcal{P}_{\infty,w}} x^{|\lambda|} = \prod_{j=1}^{w} \frac{1}{1-x^j}$$

Hence, the generating function for $\mathcal{P}_{\infty,\infty}$ is just (we are using the topology on formal power series)

$$P_{\infty,\infty}(x) = \lim_{w\to\infty} \prod_{j=1}^{w} \frac{1}{1-x^j} = \prod_{j=0}^{\infty} \frac{1}{1-q^j}$$

As for $\mathcal{P}_{\ell,\infty}$, we note there is a bijection (by conjugate) between $\mathcal{P}_{\ell,\infty}$ and $\mathcal{P}_{\infty,\ell}$ and hence it has the same generating function as $\mathcal{P}_{\infty,\ell}$.

Now, what is $\mathcal{P}_{\ell,w}$? We have to use q-analogous for this. We recall $n_q = 1 + q + q^2 + \ldots + q^{n-1}$ and $n!_q = n_q(n-1)_q \ldots (2)_q(1)_q$ and

$$\binom{n}{k}_q = \frac{n!_q}{(n-k)!_q k!_q}$$

Then we have the following theorem.

Definition 1.8. If $\mathbb{F}$ is a field, V a vector space over $\mathbb{F}$, then we define the ***Grassmannian***, $\mathrm{Grass}_{d,V}$, be the set of d-dimensinoal linear subspaces of V.

Theorem 1.9.

$$\sum_{\lambda \in \mathcal{P}_{\ell,w}} q^{|\lambda|} = \binom{\ell+w}{w}_q = \binom{\ell+w}{\ell}_q$$

Proof. The idea is that we are going to consider Grassmannian over finite field. In this case $\mathrm{Grass}(k, V)$ is a finite set. To show the equation, first observe both sides are rational functions of q. Thus it suffices to prove it for infinitely many values of q.

If q is a prime power, then there exists finite field $\mathbb{F}_q$ with q elements. We will count

$$\mathrm{Grass}(\ell, \mathbb{F}_q^{\ell+w})$$

in two ways.

Method 1: the first way to count, we look at basis of size ℓ in $\mathbb{F}_q^{\ell+w}$. We see every ℓ-dimensional subspace of $\mathbb{F}_q^{\ell+w}$ is the row space of a rank ℓ $\ell+w$ matrix. Each row must not in the span of previous rows, and hence there are

$$(q^{\ell+w}-1)(q^{\ell+w}-q)(q^{\ell+w}-q^2)\ldots(q^{\ell+w}-q^{\ell-1})$$

many such matrices. Two such matrices A, B have the same row space iff $A = KB$ for some (unique) invertible $\ell \times \ell$ matrix. There are $(q^\ell-1)(q^\ell-q)\ldots(q^\ell-q^{\ell-1})$ such matrices by what we just did (we are looking at basis of length ℓ of $\mathbb{F}_q^\ell$ here). Hence, by method 1, we get the answer is

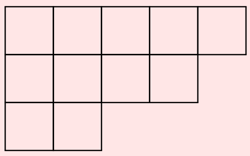
$$\frac{(q^{\ell+w}-1)(q^{\ell+w}-q)(q^{\ell+w}-q^2)\ldots(q^{\ell+w}-q^{\ell-1})}{(q^\ell-1)(q^\ell-q)\ldots(q^\ell-q^{\ell-1})} = \binom{\ell+w}{\ell}_q$$

This solves one side of our problem, now whats the second method?

Method 2: the second way to count, we note that every subspace is the row space of a unique matrix in reduced row echelon form (RREF). However, by definition of RREF we see they are in bijection with partitions in $\mathcal{P}_{\ell,w}$. Here is an example: for the matrix

$$\begin{bmatrix} 0 & 1 & * & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

we would get partition



Thus possible choices for pivot columns correspond to partitions of $\mathcal{P}_{\ell,w}$ and for each $\lambda \in \mathcal{P}_{\ell,w}$ the number of subspaces with pivot columns for λ is $q^{|\lambda|}$ Hence the number of subspaces is

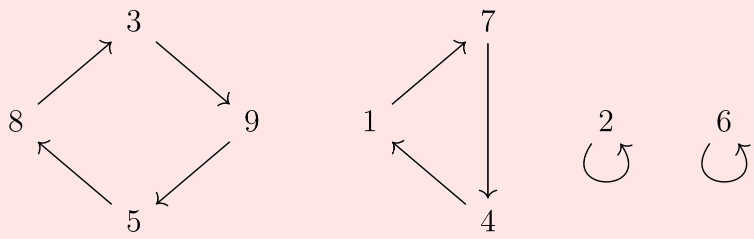
$$\sum_{\lambda \in \mathcal{P}_{\ell,w}} q^{|\lambda|}$$

♡

2 Day Two

We recall some notations for permutations.

1. We have one line notation, for example, [729186435] means we map 1 to 7 and so on.
2. We can use the cycle decompostion diagram, which looks like



   This correspond to 729186435 for [9].
3. We can use cycle notation, for example the above one would be (174)(2)(3958)(6).

Definition 2.1. The ***cycle type*** of a permutation is the partition $\lambda(\sigma) = (\lambda_1, ..., \lambda_\ell)$ such that λ_i is the length of the cycles. For example, (174)(3958)(2)(6) has cycle type 4311. We define $\mathcal{C}_\lambda = \{\sigma \in S_n : \lambda(\sigma) = \lambda\}$ as the collection of all permutations of cycle type λ.

Theorem 2.2. *$\mathcal{C}_\lambda$ are conjugacy classes for S_n, i.e. $\mathcal{C}_\lambda = \{\tau\sigma\tau^{-1} : \tau \in S_n\}$ for any $\sigma \in \mathcal{C}_\lambda$.*

Definition 2.3. If $\lambda = 1^{j_1}2^{j_2}3^{j_3}...$ is a partition, then we define the natural number

$$z_\lambda := (1^{j_1}2^{j_2}3^{j_3}....)(j_1! \cdot j_2! \cdot j_3! \cdot ...)$$

where this time the product means actual exponential map.

Proposition 2.4. *$\#\mathcal{C}_\lambda = \frac{n!}{z_\lambda}$. In particular, $z_\lambda = \#Z_{S_n}(\sigma)$ for any $\sigma \in S_n$, and Z_{S_n} means the centralizer.*

Also, recall we would define $\tau_i = (i, i+1)$ in cycle notation for $1 \le i \le n-1$, then those τ_i generates S_n. Those are called ***simple transpositions***, or ***coxeter generators***.

Theorem 2.5. *S_n is the group generated by $\tau_1, ..., \tau_{n-1}$ with relations:*

1. $\tau_i^2 = \mathrm{Id}$.
2. $\tau_i\tau_j = \tau_j\tau_i$ *with* $|i-j| \ge 2$.
3. $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ *with* $1 \le i \le n-2$.

Definition 2.6. For $\sigma \in S_n$, we say (i, j) is an ***inversion*** (or ***coxeter length***) of σ if $1 \le i < j \le n$ and $\sigma(j) < \sigma(i)$. We write $\mathrm{inv}(\sigma)$ as the number of inversions. We also note $\mathrm{inv}(\sigma)$ is the minimum k such that $\sigma = \tau_{i_1}...\tau_{i_k}$.

Definition 2.7. We also define $\operatorname{sgn}(\sigma)$ as $(-1)^{\operatorname{inv}(\sigma)}$ and we call σ is ***even/odd*** if $\operatorname{inv}(\sigma)$ is even/odd.

We note $\operatorname{inv}(\mathrm{Id}) = 0$ and $\operatorname{inv}([n, n-1, \dots, 1]) = \binom{n}{2}$ are the extreme bound on the inversion function.

Definition 2.8. For $\sigma \in S_n$, we define $M(\sigma)$ as the matrix such that

$$M(\sigma)_{ij} = \begin{cases} 1, & \text{if } \sigma(j) = i \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.9. Let G be a group, a ***G-action*** on set X is a group homomorphism $\rho_X : G \to S(X)$ where $S(X)$ is the permutation group on X. For $g \in G, x \in X$ we write $gx = \rho_X(g)(x)$.

We note this is just the old usual definition of group action, as one should verify. For S_n, we often get S_n-actions by taking X to be a set of combinatorial structures on $[n]$. For example, we can let $\mathcal{G}$ be the species of graphs, then we get a S_n-action on $\mathcal{G}_n = \mathcal{G}([n])$ by permute the labelled vertices.

Definition 2.10. Let G be a group and X a set with G-action, then the ***fixed point set*** of the action is $X^G = \{x \in X : \forall \sigma \in G, \sigma x = x\}$.

For graphs, we have $\mathcal{G}_n^{S_n} = \{K_n, K_n^c\}$ where K_n is the complete graph. This is not interesting, so we might consider sets with infinite many elements.

If X has additional structures, we can ask our group action respects that structure.

Definition 2.11. A G-action of X is (any adjective) if for all $g \in G$, $\rho_X(g)$ is (any adjective).

Example 2.12. If X is a topological space, then we say the G-action ρ_X is continuous if $\rho_X(g)$ is continuous for all $g \in G$. Similarly if X is a vector space, we say G-action is linear if $\rho_X(g)$ is linear for all $g \in G$. If X is a ring, we say G-action is ***by automorphisms*** if $\rho_X(g) : X \to X$ is a ring automorphism for all $g \in G$.

In general, we can get a linear action by consider its associated free vector space. Suppose X is a set, $\mathbb{F}$ a field, $\mathbb{F}X$ the free vector space on X. Given an G-action on X, then we get a linear G-action on $\mathbb{F}X$ by acts on the elements, i.e. $g(\sum a_i x_i) = \sum a_i(gx_i)$ for $a_i \in \mathbb{F}, x_i \in X$.

A couple of facts:

1. If X is a vector space with a linear G-action, then X^G is a subspace.
2. If X is a ring, and G acts by automorphisms on X, then X^G is a subring.

Definition 2.13. For commutative ring A, we get A-algebra $A[x_1, ..., x_n]$, the A-polynomials.

If $A = \mathbb{Q}$, then $\mathbb{Q}[x_1, ..., x_n]$ as $\mathbb{Q}$-vector space has basis $M^{(d)} = \{x^d : d \in \mathbb{Z}^n_{\geq 0}\}$ where for $x = (x_1, ..., x_n)$ and $d = (d_1, ..., d_n) \in \mathbb{Z}^n_{\geq 0}$ we use x^d to denote $x_1^{d_1} ... x_n^{d_n}$.

We see the symmetric group S_n acts on $M^{(d)}$ by permute the variables, e.g. $(1, 2)$ would map $x_1^2 x_2$ to $x_2^2 x_1$. In particular this means S_n acts on the polynomial ring.

Definition 2.14. We define $\Lambda(x_1, ..., x_d) = \Lambda^{(d)} := \mathbb{Q}[x_1, ..., x_d]^{S_d}$.

3 Day Three

Example 3.1. For $d = 3$, the following are symmetric polynomials:

1. 1
2. $x_1 + x_2 + x_3$
3. $x_1^n + x_2^n + x_3^n$
4. $x_1 x_2^2 + x_1^2 x_2 + x_1 x_3^2 + x_1^2 x_3 + x_2 x_3^2 + x_2^2 x_3$
5. $5x_1 x_2 + 5x_1 x_3 + 5x_2 x_3 + 21x_1^2 x_2^2 x_3^2$

Its not hard to see this algebra admits a grading by homogeneous polynomials, i.e. if we define $\Lambda_n^{(d)}$ as the subspace of homogeneous elements of degree n, then

$$\Lambda^{(d)} = \bigoplus_{n \geq 0} \Lambda_n^{(d)}$$

The first question we are going to ask is, what is the dimension of $\Lambda_n^{(d)}$? For that we look at the monomial basis.

Definition 3.2. If $\alpha \in \mathbb{N}^n = \mathbb{Z}^n_{\geq 0}$ with $\alpha = (\alpha_1, ..., \alpha_n)$, we define $\operatorname{sort}(\alpha)$ be the partition obtained by sorting non-zero α's in decreasing order. For example, $\operatorname{sort}(0, 1, 0, 0, 5, 0, 2, 1) = 5211$.

Definition 3.3. Let $\lambda \vdash n$ with $\ell(\lambda) \leq d$, we define

$$m_\lambda = m_\lambda(x_1, ..., x_d) = \sum_{\substack{\alpha \in \mathbb{N}^d \\ \operatorname{sort}(\alpha) = \lambda}} X^\alpha$$

where we use X^α to denote $x_1^{\alpha_1} ... x_d^{\alpha_d}$.

Proposition 3.4. *The set $\{m_\lambda : \lambda \vdash n, \ell(\lambda) \leq d\}$ is a basis for $\Lambda_n^{(d)}$.*

Proof. The set $M_n^{(d)} = \{X^\alpha : \alpha \in \mathbb{N}^d, \sum \alpha_i = n\}$ is a basis for $\mathbb{Q}[x_1, \ldots, x_d]_n$. Since every such monomial appears in at exactly one m_λ, the m_λ's are linearly independent.

To see they are spanning, consider map

$$P : \mathbb{Q}[x_1, \ldots, x_d]_n \to \Lambda^{(d)}$$
$$f \mapsto \frac{1}{d!} \sum_{\sigma \in S_d} \sigma f$$

One can show P is a projection onto $\Lambda^{(d)}$, i.e. $P(f) \in \Lambda^{(d)}$ for all f and $f \in \Lambda^{(d)}$ then $P(f) = f$.

In particular, we see $P(X^\alpha)$ is a non-zero multiply of some m_λ, with $\lambda = \mathrm{sort}(\alpha)$. The image of a basis under a projection always spans the image of the projection, hence m_λ spans $\Lambda^{(d)}$. ♡

Corollary 3.4.1. *$\{m_\lambda : \lambda \in \mathcal{P}_{d,\infty}\}$ is a basis for $\Lambda^{(d)}$.*

Proof. Because $\Lambda^{(d)} = \bigoplus_{n \geq 0} \Lambda_n^{(d)}$. ♡

Corollary 3.4.2. $\dim(\Lambda_n^{(d)}) = \#\{\lambda \in \mathcal{P} : \ell(\lambda) \leq d, \lambda \vdash n\}$.

Next, we want to study the structure coefficients of this (sub)algebra. In particular, to understand multiplication in any $\mathbb{Q}$-algebra, it suffices to understand product of basis elements.

In our case, we have

$$m_\mu m_\nu = \sum_\lambda c_{\mu\nu}^\lambda m_\lambda$$

where $c_{\mu\nu}^\lambda$ are the structure coefficients. If we know $c_{\mu\nu}^\lambda$, then we can compute $fg = (\sum_\mu f_\mu m_\mu)(\sum_\nu g_\nu m_\nu)$ as

$$fg = \sum_\mu \sum_\nu f_\mu g_\nu (m_\mu m_\nu) = \sum_\lambda \left(\sum_\mu \sum_\nu f_\mu g_\nu c_{\mu\nu}^\lambda \right) m_\lambda$$

The first thing we want to claim is that there is a universal multiplication formula that works for all d, i.e. $c_{\mu\nu}^\lambda$ does not depend on d. To make this sensible, we make the convention to define $m_\lambda = 0$ if $\ell(\lambda) > d$.

We have homomorphism $\phi^{(d)} : \Lambda^{(d+1)} \to \Lambda^{(d)}$ by $f(x_1, \ldots, x_{d+1}) = f(x_1, \ldots, x_d, 0)$. Moreover, this map takes $\phi(m_\lambda(x_1, \ldots, x_{d+1})) = m_\lambda(x_1, \ldots, x_d)$.

Example 3.5. Consider m_1^3. If $d = 3$, then

$$m_1^3 = (x_1 + x_2 + x_3)^3 = m_3 + 3m_{21} + 6m_{111}$$

as one can verify. On the other hand, if $d = 4$, we know $m_1^3 = am_3 + bm_{21} + cm_{111}$ where a, b, c are unknown, but then apply $\phi^{(3)}$ to both side we see we must have $a = 1, b = 3, c = 6$.

However, if $d = 2$, then

$$m_1^3 = (x_1 + x_2)^2 = x_1^3 + x_2^3 + 3(x_1^2 x_2 + x_1 x_2^2) = m_3 + 3m_{21}$$

This means if we don't have enough variable, then we are in bad situation. Hence we get the following definition.

Definition 3.6. Define Λ to be the $\mathbb{Q}$-algebra as:

1. $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$.
2. Λ_n has a basis $\{m_\lambda : \lambda \vdash n\}$.
3. The structure constant are given by universal multiplication formula.

In other word, $\Lambda = \varprojlim \Lambda^{(d)}$ in the category of $\mathbb{Q}$-algebra.

On the other hand, we think of Λ as the algebra of symmetric functions in infinitely many variables. For example,

$$m_{31} = x_1^3 x_2 + x_1 x_2^3 + x_1^3 x_3 + x_1 x_3^3 + x_2^3 x_3 + \ldots$$

and hence if we take multiplication, we are getting something like

$$\begin{aligned} m_1^3 &= (x_1 + x_2 + x_3 + x_4 + x_5 + \ldots)^3 \\ &= (x_1^3 + x_2^3 + x_3^3 + \ldots) + 3(x_1^2 x_2 + \ldots) + 6(x_1 x_2 x_3 + \ldots) \\ &= m_3 + 3m_{21} + 6m_{111} \end{aligned}$$

However, to work with infinitely many variables, weird things could happen, and we don't want that. Thus, we propose that, a formula is true in ∞ many variables iff it is true in $\Lambda^{(d)}$ for all $d \in \mathbb{N}$.

To this end, we have specialization maps from infinitely many variables to finitely many variables, which is given by $\phi^{(d)} : \Lambda \to \Lambda^{(d)}$ by map m_λ to $m_\lambda(x_1, \ldots, x_d)$. Hence, our aim will be to study Λ, as every claim in Λ will be true in $\Lambda^{(d)}$ by using specialization maps.

Next, we study more bases of Λ, and we will start with multiplicative bases:

1. Elementary symmetric functions. In Λ, we just have

$$e_n := m_{1^n} = \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1} x_{i_2} \ldots x_{i_n}$$

   For example, $e_1 = x_1 + x_2 + x_3 + \ldots$, $e_2 = x_1 x_2 + x_1 x_3 + x_2 x_3 + \ldots$.
2. Complete homogeneous symmetric functions. In Λ, we define

$$h_n = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \leq i_2 \leq \ldots \leq i_n} x_{i_1} x_{i_2} \ldots x_{i_n}$$

3. Power sum symmetric functions. In λ, we define

$$p_n := m_n = \sum_{i \geq 1} x_i^n$$

   For example, $p_1 = x_1 + x_2 + x_3 + \ldots$, $p_2 = x_1^2 + x_2^2 + x_3^2 + \ldots$.

The point of those symmetric functions is that, we don't study them, but products of them (hence the name multiplicative).

Definition 3.7. For any partition $\lambda = (\lambda_1, ..., \lambda_d)$ with d parts, we define the the corresponding multiplicative bases as:

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} ... e_{\lambda_d}$$

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} ... p_{\lambda_d}$$

$$h_\lambda = h_{\lambda_1} ... h_{\lambda_d}$$

Remark 3.8. It is reasonable to define $e_0 = h_0 = 1$, but for p, we cannot. In a vague sense, e_λ and h_λ don't "care" about number of partitions, but p_λ does and p_0 is undefined and should not be defined.

Theorem 3.9. *$\{p_\lambda : \lambda \vdash n\}$ is a $\mathbb{Q}$-basis for Λ_n.*

Proof. We have

$$p_\lambda = (x_1^{\lambda_1} + x_2^{\lambda_1} + ...)(x_1^{\lambda_2} + x_2^{\lambda_2} + ...)(x_1^{\lambda_3} + x_2^{\lambda_3} + ...)...$$

Then to expand it out, we pick one element from each product and sum together. For example, if we pick $x_2^{\lambda_1}$ in the first one, $x_3^{\lambda_2}$ in second, $x_2^{\lambda_3}$ in third, and so on, then we get $x_2^{\lambda_1+\lambda_3+...}$ times $x_3^{\lambda_2+...}$ and so on. After some observation, we see the only m_μ going to appear in the expansion is μ such that $\mu \geq_{ref} \lambda$. Thus we see

$$p_\lambda = M_{\lambda\lambda} m_\lambda + \sum_{\mu >_{ref} \lambda} M_{\lambda\mu} m_\mu$$

where $M_{\lambda\lambda}, M_{\lambda\mu}$ are coefficients, where we know for certain $M_{\lambda\lambda} > 0$, i.e. the change of basis matrix relating p_λ and m_λ has non-zero determinant, which shows p_λ is a basis. ♡

Corollary 3.9.1. *$\{p_\lambda : \lambda \in \mathcal{P}\}$ is a basis for Λ.*

Corollary 3.9.2. *$\Lambda = \mathbb{Q}[p_1, p_2, p_3, ...]$, i.e. $p_1, p_2, ...$ are algebraically independent.*

Remark 3.10. In three multiplicative bases, this is the only one that's just $\mathbb{Q}$-basis, the other e_λ, h_λ are both $\mathbb{Z}$-bases. Viz, we can't write every symmetric functions as integer combinations of p_λ.

Theorem 3.11. *$\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.*

Proof. This time the "triangular system" is $e_{\lambda'} = 1m_\lambda + \sum_{\mu<\lambda} m_\mu$ where recall λ' means conjugate of λ. Thus e_λ indeed form a basis for Λ_n. ♡

Corollary 3.11.1. *$\{e_\lambda : \lambda \in \mathcal{P}\}$ is a basis for Λ.*

Corollary 3.11.2. $\Lambda = \mathbb{Q}[e_1, e_2, e_3, \dots]$

4 Four

Today we look at h_λ. First we compute some examples:

$$h_{11} = 2m_{11} + 1m_2$$

$$h_2 = 1m_{11} + 1m_2$$

This is not triangular in any sense as the matrix is

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Next, we see

$$h_{111} = 6m_{111} + 3m_{21} + 1m_3$$
$$h_{21} = 3m_{111} + 2m_{21} + 1m_3$$
$$h_3 = 1m_{111} + 1m_{21} + 1m_3$$

which gives matrix

$$\begin{bmatrix} 6 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Note those both matrices have determinant equal 1, they are symmetric, and are positive definite.

Before we prove all of those, we need to show h_λ form a basis.

We note elements of Λ are $\sum_\lambda f_\lambda m_\lambda$, which are finite linear combinations of basis elements. But what if we want infinite combinations? Well, we can look at formal power series, i.e. we can consider $f(t) = \sum_{i \geq 0} f_i t^i \in \Lambda[[t]]$ where $f_i \in \Lambda_i$ for all i. This is just the completion of Λ (in the commutative algebra sense), and is denoted by $\hat{\Lambda}$.

In particular, we can look at the two generating functions

$$H(t) = \sum_{n \geq 0} h_n t^n$$

$$E(t) = \sum_{n \geq 0} e_n t^n$$

In fact, we can compute them directly as follows

$$H(t) = \prod_{i \geq 1} \frac{1}{1 - x_i t}$$

$$E(t) = \prod_{i\geq 1}(1 + x_i t)$$

To prove the first one, we note

$$\prod_{i\geq 1} \frac{1}{1 - x_i t} = (1 + x_1 t + x_1^2 t^2 + \ldots)(1 + x_2 t + x_2^2 t^2 + \ldots)(1 + x_3 t + x_3^2 t^2 + \ldots)\ldots$$

but if we pick each term from each product, we see we can get every possible monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} t^{\sum \alpha_i}$, but this is just what $H(t)$ is, hence the proof follows. A similar argument shows the formula for $E(t)$.

Last time, we showed $\Lambda = \mathbb{Q}[e_1, e_2, e_3, \ldots]$ and hence we can define a $\mathbb{Q}$-algebra homomorphism $\omega : \Lambda \to \Lambda$ by sending e_i to h_i, i.e. if $f = \sum_\lambda f_\lambda e_\lambda$ then $\omega(f) = \sum_\lambda f_\lambda h_\lambda$. This is called the ***fundamental involution***.

Theorem 4.1. *The ω is indeed an involution.*

Proof. Observe $E(t)H(-t) = 1$ by the product definition of $E(t)$ and $H(t)$. Now apply ω to the above equation by treating t as constant, then we get

$$\omega(E(t))\omega(H(-t)) = 1$$

However, we see $\omega(E(t)) = H(t)$ by definition and so we see

$$H(t) \sum_{n\geq 0} (-1)^n \omega(h_n) t^n = 1$$

Next we consider the homomorphism $t \mapsto -t$ applied to above equation, we get

$$H(-t) \sum_{n\geq 0} \omega(h_n) t^n = 1 \Rightarrow \sum_{n\geq 0} \omega(h_n) t^n = \frac{1}{H(-t)}$$

However, we know $\frac{1}{H(-t)} = E(t)$ and so

$$\sum_{n\geq 0} \omega(h_n) t^n = \sum_{n\geq 0} e_n t^n$$

which shows $\omega(h_n) = e_n$, which concludes ω is indeed involution. ♡

Corollary 4.1.1. *$\{h_\lambda\}$ is a basis for Λ.*

Proof. It is the image of a basis of Λ under an involution. ♡

Corollary 4.1.2. *$\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.*

Corollary 4.1.3. $\Lambda = \mathbb{Q}[h_1, h_2, \ldots]$

Next, we consider the change of basis formulas between the three bases.

Theorem 4.2.

1. $$\sum_{n\geq 1}\frac{p_n}{n}t^n = \log(\sum_{k\geq 0} h_k t^k) = \log(H(t))$$

2. $$\sum_{n\geq 1}(-1)^{n-1}\frac{p_n}{n}t^n = \log(\sum_{k\geq 0} e_k t^k) = \log(E(t))$$

3. $$h_n = \sum_{\lambda\vdash n}\frac{p_\lambda}{z_\lambda}$$

4. $$e_n = \sum_{\lambda\vdash n}(-1)^{|\lambda|-\ell(\lambda)}\frac{p_\lambda}{z_\lambda}$$

5. $$h_n = \begin{vmatrix} e_1 & e_2 & \dots & e_n \\ 1 & e_1 & \dots & e_{n-1} \\ 0 & 1 & \dots & e_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_1 \end{vmatrix}$$

6. $$e_n = \begin{vmatrix} h_1 & h_2 & \dots & h_n \\ 1 & h_1 & \dots & h_{n-1} \\ 0 & 1 & \dots & h_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & h_1 \end{vmatrix}$$

7. $$h_n = \frac{1}{n!}\begin{vmatrix} p_1 & p_2 & \dots & p_{n-1} & p_n \\ -1 & p_1 & \dots & p_{n-2} & p_{n-1} \\ 0 & -2 & \dots & p_{n-3} & p_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -n+1 & p_1 \end{vmatrix}$$

8. $$e_n = \frac{1}{n!}\begin{vmatrix} p_1 & -p_2 & \dots & (-1)^{n-2}p_{n-1} & (-1)^{n-1}p_n \\ -1 & p_1 & \dots & (-1)^{n-3}p_{n-2} & (-1)^{n-2}p_{n-1} \\ 0 & -2 & \dots & (-1)^{n-4}p_{n-3} & (-1)^{n-3}p_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -n+1 & p_1 \end{vmatrix}$$

Proof. We are not going to prove everything.

(1): We see

$$\begin{aligned}\log(\sum_{k\geq 0} h_k t^k) &= \log H(t)\\ &= \log \prod_{i\geq 1}(1-x_i t)^{-1}\\ &= \sum_{i\geq 1}\log(1-x_i t)^{-1}\\ &= \sum_{i\geq 1}\sum_{n\geq 1}\frac{x_i^n t^n}{n}\\ &= \sum_{n\geq 1}\sum_{i\geq 1}\frac{p_n}{n}t^n\end{aligned}$$

(2): It is similar but we are going to get some signs showing up.

(3), (4): Exercises.

(5): We use induction on n. Assume $n \geq 2$ and the result holds for values smaller than n. Then we see $H(t)E(-t) = 1$. Now expand this and compare the coefficient of t^n on both side and we get

$$h_n - e_1 h_{n-1} + e_2 h_{n-2} + \ldots + (-1)^n e_n = 0$$

and hence

$$h_n = e_1 h_{n-1} - e_2 h_{n-2} + \ldots + (-1)^{n-1} e_n$$

Now use inductive hypothesis, we see

$$h_n = e_1 \begin{vmatrix} e_1 & \ldots & e_{n-1} \\ 1 & \ldots & e_{n-2} \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e_1 \end{vmatrix} - e_2 \begin{vmatrix} e_1 & \ldots & e_{n-2} \\ \vdots & \ddots & \vdots \\ 0 & \ldots & e_1 \end{vmatrix} + \ldots + e_n$$

However, this is just "basically[1]" the expansion of the matrix in question along the first row.

(7): We showed $\log H(t) = \sum_{n\geq 1}\frac{p_n}{n}t^n$. Now take $\frac{d}{dt}$ on both side and we get

$$\frac{H'(t)}{H(t)} = \sum_{n\geq 1} p_n t^{n-1}$$

Thus $H'(t) = H(t)\sum_{n\geq 1} p_n t^{n-1}$ and now use induction similar to proof of (5). ♡

Corollary 4.2.1. *$\omega(p_n) = (-1)^{n-1}p_n$, i.e. p_n are eigenvectors of ω. More generally, $\omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)}p_\lambda$.*

[1] It is not just the expansion, as you expand along the row would not change the size of the matrix, but after some simplification it is what we are getting

Proof. Well, we see

$$\begin{aligned}\sum_{n\geq 1}\frac{\omega(p_n)}{n}t^n &= \omega\left(\sum_{n\geq 1}\frac{p_n}{n}t^n\right)\\ &= \omega(\log(H(t)))\\ &= \log(E(t))\\ &= \sum_{n\geq 1}(-1)^{n-1}\frac{p_n}{n}t^n\end{aligned}$$

and the result follows. ♡

Next, we want to construct an inner product on Λ built out of m's and h's.

Define a bilinear form $\langle\cdot,\cdot\rangle : \Lambda\times\Lambda\to\mathbb{Q}$ with the property that

$$\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu} := \begin{cases}1, & \lambda=\mu\\ 0, & \text{otherwise}\end{cases}$$

Next is some recall on tensor product of vector spaces.

Example 4.3.

1. If $V=\mathbb{F}^n, W=\mathbb{F}^m$ then $V\otimes W=\mathrm{Mat}_{m\times n}(\mathbb{F})$ with $(v,w)\mapsto vw^T := v\otimes w$.
2. If $V=M_{k\times\ell}(\mathbb{F}), W=\mathrm{Mat}_{m\times n}(\mathbb{F})$ then $V\otimes W=\mathrm{Mat}_{km\times ln}(\mathbb{F})$ where the product $(A,B)\mapsto A\otimes B$ is the Kronecker product.
3. If $V=\mathbb{F}[x], W=\mathbb{F}[y]$ then $V\otimes W=\mathbb{F}[x,y]$ with $(f(x),g(y))\mapsto f(x)g(y)$. Note this product is not commutative.
4. If $V=\Lambda^{(d_1)}, W=\Lambda^{(d_2)}$ then $V\otimes W=\mathbb{Q}[x_1,\ldots,x_{d_1},y_1,\ldots,y_{d_2}]^{S_{d_1}\times S_{d_2}}$ together with $(f,g)\mapsto f(x)g(y)$ is the bi-symmetric polynomials, i.e. they are symmetric in x variables and symmetric in y variables, separately.

5 Five

Theorem 5.1. *Let V be a finite dimensional vector space over $\mathbb{F}$. Let $[\cdot,\cdot] : V\times V\to\mathbb{F}$ be a non-degenerate bilinear form. Then there exists $K\in V\otimes V$ such that, for any two bases $(v_i)_{i\in S},(\overline{v}_i)_{i\in S}$ of V, $[v_i,\overline{v_j}]=\delta_{ij}$ if and only if $\sum_{i\in S}v_i\otimes\overline{v}_i=K$.*

Proof. Let $(u_i)_{i\in S}$ be any basis for V. By non-degeneracy, we can find a dual basis $(\overline{u_i})_{i\in S}$ such that $[u_i,\overline{u}_j]=\delta_{ij}$. Then define $K=\sum_{i\in S}u_i\otimes\overline{u}_i$.

Let $(v_i),(\overline{v}_i)$ be any two bases. Then we can find two change of basis matrices $Q,\overline{Q}$ such that $v_i=\sum_{k\in S}u_kQ_{ki}$ and $\overline{v_i}=\sum_{l\in S}\overline{u_l}\overline{Q}_{lj}$. Thus

$$[v_i,\overline{v_j}] = [\sum_{k\in S}u_kQ_{ki},\sum_{l\in S}\overline{u_l}\overline{Q}_{lj}] = \sum_{k\in S}Q_{ki}\overline{Q_{kj}} = (Q^T\overline{Q})_{ij}$$

ANd we have

$$\begin{aligned}\sum_{i\in S} v_i \otimes \overline{v_i} &= \sum_{i\in S}(\sum_{k\in S} u_k Q_{ki}) \otimes (\sum_{l\in S} \overline{u}_l \overline{Q}_{li}) \\ &= \sum_{i\in S}\sum_{k\in S}\sum_{l\in S} : Q_{ki}\overline{Q}_{li}(u_k \otimes \overline{u_l}) \\ &= \sum_{k\in S}\sum_{l\in S} (Q\overline{Q}^T)_{kl} u_k \otimes \overline{u_l}\end{aligned}$$

However, we see $[v_i, \overline{v_j}] = \delta_{ij}$ iff $Q^T\overline{Q} = \mathrm{Id}$ by the above computation. Similarly, we see $\sum_{i\in S} v_i \otimes \overline{v_i} = K = \sum_{i\in S} u_i \otimes \overline{u_i}$ iff $(Q\overline{Q}^T) = \mathrm{Id}$.

Thus, we see the two claims are equivlent as desired. ♡

Now back to symmetric functions, we have the following theorem.

Theorem 5.2. *If $(u_\lambda)_{\lambda\in\mathcal{P}}$ and $\{v_\lambda\}_{\lambda\in\mathcal{P}}$ be two bases consisting of homogeneous elements for Λ. Then the following are equivlent:*

1. *$\langle u_\lambda, v_\mu\rangle = \delta_{\lambda\mu}$ for all $\lambda, \mu \in \mathcal{P}$ where $\langle\cdot,\cdot\rangle$ is defined last section.*
2. *$\sum_\lambda u_\lambda(x)v_\lambda(y) = \prod_{i\geq 1}\prod_{j\geq 1}\frac{1}{1-x_iy_j}$.*

Proof. By previous theorem, we know there exists some $K(x,y)$ associated to $\langle\cdot,\cdot\rangle$ such that $[u_i, v_j] = \delta_{ij}$ iff $K(x,y) = \sum_\lambda u_\lambda(x)v_\lambda(y)$. We note we are applying the theorem to each Λ_n.

It suffices to compute $K(x,y)$. By definition of $\langle\cdot,\cdot\rangle$, we see $\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu}$. Thus we see we must have

$$K(x,y) = \sum_\lambda h_\lambda(x)m_\lambda(y) = \prod_{j\geq 1}\left(\sum_{k_j\geq 0} h_{k_j}(x)y_j^{k_j}\right) = \prod_{j\geq 1}(H(y_j)) = \prod_{j\geq 1}\prod_{i\geq 1}\frac{1}{1-x_iy_j}$$

♡

Corollary 5.2.1. *$\{p_\lambda\}_{\lambda\in\mathcal{P}}$ is an orthogonal basis with $\langle p_\lambda, p_\lambda\rangle = z_\lambda$. In particular this means p_λ and p_μ/z_μ is a dual basis.*

Proof. It suffices to show $\sum_\lambda p_\lambda(x) \cdot \frac{p_\lambda(y)}{z_\lambda} = \prod_{i\geq 1}\prod_{j\geq 1}\frac{1}{1-x_iy_j}$.

$$\begin{aligned}
\prod_{i,j\geq 1}\frac{1}{1-x_iy_j} &= \exp\sum_{i,j\geq 1}\log((1-x_iy_j)^{-1}) \\
&= \exp\sum_{i,j\geq 1}\sum_{k\geq 1}\frac{x_i^k y_j^k}{k} \\
&= \exp\left(\sum_{k\geq 1}\frac{p_k(x)p_k(y)}{k}\right) \\
&= \prod_{k\geq 1}\exp(\frac{p_k(x)p_k(y)}{k}) \\
&= \prod_{k\geq 1}\left(\sum_{i_k\geq 0}\frac{p_k(x)^{i_k}p_k(y)^{i_k}}{k^{i_k}(i_k)!}\right) \\
&= \sum_{i_1,i_2,i_3,\ldots\geq 0}\prod_{k\geq 1}\frac{p_k(x)^{i_k}p_k(y)^{i_k}}{k^{i_k}(i_k)!} \\
&= \sum_\lambda \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda}
\end{aligned}$$

♡

Corollary 5.2.2. *$\langle\cdot,\cdot\rangle$ is symmetric and positive definite. Moreover, ω is an isometry, i.e. $\langle\omega(f),\omega(g)\rangle = \langle f,g\rangle$ for all $f,g\in\Lambda$.*

Proof. Since (p_λ) is an orthogonal basis, then $\langle\cdot,\cdot\rangle$ must be symmetric. Moreover, since the orthogonal basis satisfies $\langle p_\lambda,p_\lambda\rangle > 0$, the inner product must be positive definite. Finally, p_λ are eigenvectors of ω with eigenvalues ± 1, hence ω must be isometry as desired as we see

$$\langle\omega(p_\lambda),\omega(p_\mu)\rangle = \langle(-1)^{|\lambda|-\ell(\lambda)}p_\lambda,(-1)^{|\mu|-\ell(\mu)}p_\mu\rangle = \begin{cases}0, & \lambda\neq\mu\\ \langle p_\lambda,p_\lambda\rangle, & \lambda=\mu\end{cases}$$

♡

Definition 5.3. The inner product we have defined, i.e. $\langle h_\lambda,m_\mu\rangle = \delta_{\lambda,\mu}$, is called the ***Hall inner product***.

Definition 5.4. The product $\prod_{i,j\geq 1}\frac{1}{1-x_iy_j}$ is called the ***Cauchy kernel***.

For any $f\in\Lambda$, we have the multiplication operation given by $g\mapsto fg$, which is linear. Now since we have an inner product, we can talk about the adjoint of f.

Definition 5.5. The adjoint of the multiplication by f map is denoted by $f^\perp$, and called the ***skewing operator***, i.e. $\langle fg,h\rangle = \langle g,f^\perp h\rangle$ for all $g,h\in\Lambda$.

Proposition 5.6. *Let $f_1, f_2 \in \Lambda$ and $c \in \mathbb{Q}$.*

1. $(f_1 + cf_2)^\perp = f_1^\perp + cf_2^\perp$.
2. $(f_1 f_2)^\perp = f_1^\perp \circ f_2^\perp = f_2^\perp \circ f_1^\perp$.
3. $$p_k^\perp\left(\frac{p_\lambda}{z_\lambda}\right) = \begin{cases} \frac{p_{\lambda^-}}{z_{\lambda^-}}, & \text{if } \lambda^- \text{ is } \lambda \text{ removing a part of size } k \\ 0, & \text{if } \lambda \text{ has no part of size } k \end{cases}$$
4. *Thinking of $\Lambda = \mathbb{Q}[p_1, p_2, \dots]$ then $p_k^\perp = k\frac{\partial}{\partial p_k}$ and so $f^\perp \in \mathbb{Q}[\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_3}, \dots]$. The point is, skewing operators play the role of derivatives in Λ in the p_k basis.*

We note the last point is not true in $\{e\}$ or $\{h\}$ bases. The reason is, if we look at the partial derivative of h, then we see

$$\frac{\partial}{\partial h_k} = \sum_{n \geq 1} \frac{\partial h_k}{\partial p_n} \frac{\partial}{\partial p_n}$$

where we have infinitely many of $\frac{\partial h_k}{\partial p_n}$ are non-zero, hence it does not even make sense to talk about $\frac{\partial}{\partial h_k}$.

6 Six

Last time we see we have

$$\begin{tikzcd} p_\lambda \arrow[r, leftrightarrow, "\langle \cdot, \cdot \rangle"'] \arrow[loop below, "\omega(\pm)"] & p_\lambda / z_\lambda \arrow[loop below, "\omega(\pm)"] \end{tikzcd} \qquad \begin{tikzcd} m_\lambda \arrow[r, leftrightarrow, "\langle \cdot, \cdot \rangle"'] \arrow[d, leftrightarrow, "\omega"'] & h_\lambda \arrow[d, leftrightarrow, "\omega"'] \\ \omega(m_\lambda) \arrow[r, leftrightarrow, "\langle \cdot, \cdot \rangle"'] & e_\lambda \end{tikzcd}$$

where the $\omega(m_\lambda)$ are called forgotten symmetric functions.

Next, we want to talk about some analogy between $\mathbb{Q}[x]$ and Λ.

	$\mathbb{Q}[x]$	Λ
Basis	$\{x^n\}$	$\{p_\lambda\}$
Basis	$\{x^n/n!\}$	$\{p_\lambda/z_\lambda\}$
Basis	Special families	Other bases
Derivative	$\frac{d^n}{dx^n}$	Skewing operation
Inner product	$[x^n]f(x)$	$\langle f, \cdot \rangle$
Completion	$\mathbb{Q}[[x]]$	$\hat{\Lambda}$
Involution	????	ω
Morphisms	Compostion	Plethysm

The last one is the thing we are going to talk about today. First, we introduce some homomorphism from Λ:

1. Evaluation maps: if R is $\mathbb{Q}$-algebra, $r_1, r_2, ..., r_d \in R$, then we define $\Lambda \to R$ by $f \mapsto f(r_1, r_2, ..., r_d)$.
2. Exponential Specialization: $\mathrm{ex} : \Lambda \to \mathbb{Q}[t]$ is given by $\mathrm{ex}(p_n) = \begin{cases} t, & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases}$

   where we consider $\Lambda = \mathbb{Q}[p_1, p_2, ...]$.

Proposition 6.1. *If* $f \in \Lambda$, *then* $\mathrm{ex}(f) = \langle f, \exp(tp_1) \rangle$.

Proof. It suffices to check this on basis $\{p_\lambda\}$. Take $f = p_\lambda$, then

$$\mathrm{ex}(p_\lambda) = \begin{cases} t^n & \text{if } \lambda = 1^n \\ 0, & \text{otherwise} \end{cases}$$

On the other hand, we see

$$\begin{aligned} \langle p_\lambda, \exp(tp_1) \rangle &= \sum_{n \ge 0} \langle p_\lambda, \frac{p_1^n}{n!} \rangle t^n \\ &= \mathrm{ex}(p_\lambda) \end{aligned}$$

where the last line we used the fact p_λ and p_μ / z_μ is a dual basis. ♡

Next we talk about plethysm. To start with, we note $\mathbb{R}$ is a ring, and functions $\mathbb{R} \to \mathbb{R}$ is a ring under pointwise multiplication, which is denoted by $\mathrm{End}(\mathbb{R})$. Clearly this extends to any ring R, and in particular we can replace $\mathbb{R}$ with $\mathrm{End}(\mathbb{R})$, and repeat...

We are not going to do this. We would take $R = \mathbb{Q}[u_1, ..., u_m]$, the polynomial ring in m variables, $R = \mathbb{Q}[[u_1, ..., u_m]]$, $R = \mathbb{Q}(u_1, ..., u_m)$, $R = \mathrm{Frac}(\mathbb{Q}[u_1, ..., u_m])$, and the list goes on.

Then, we are going to consider the ring $\mathrm{End}(R)$ with the pointwise multiplication and addition. We are going to call elements in $\mathrm{End}(R)$ to be ***transformations***.

Example 6.2. Some examples of transformations:

1. $\frac{d}{du_i} : \mathbb{Q}[u_1, ..., u_m] \to \mathbb{Q}[u_1, ..., u_m]$.
2. $\mathrm{sq} : \mathbb{Q}[u_1, ..., u_m] \to \mathbb{Q}[u_1, ..., u_m]$ given by $\mathrm{sq}(f) = f^2$.

We note that, multiplication in $\mathrm{End}(R)$ is not compostion. In particular,

$\mathrm{sq} \cdot \frac{d}{du_i}$ is not the second derivative. Indeed, it is the map

$$F \mapsto \left(\frac{d}{du_i} F\right)^2$$

Definition 6.3. We define ***plethysm***

$$\mathrm{pl} : \Lambda \to \mathrm{End}(R)$$

as the homomorphism (on the basis $\{p_n\}$)

$$\mathrm{pl}(p_n) = (F(u_1, \dots, u_m) \mapsto F(u_1^n, u_2^n, \dots, u_m^n))$$

Definition 6.4. We define ***plethystic evaluation*** as follows. If $F \in R$, $g \in \Lambda$, then we define

$$g[F] = \mathrm{pl}(g)(F)$$

and we say this is g ***plethystically evaluated at*** F.

Proposition 6.5. *For each $F \in R$, the map $\Lambda \to R$ given by $g \mapsto g[F]$ is a ring homomorphism.*

Remark 6.6. We note, on the other hand, if we fix g, then $g[\cdot] : R \to R$ is not a homomorphism in general.

Example 6.7. Let $R = \mathbb{Q}[[u]]$, let us try to compute $h_3[\frac{1}{1-u}]$.

Step 1: Compute h_3 in p_λ, which is

$$h_3 = \frac{1}{6} p_{1^3} + \frac{1}{2} p_{21} + \frac{1}{3} p_3$$

Step 2: plethystic evaluation $p_n[\frac{1}{1-u}]$. We see

$$p_1[\frac{1}{1-u}] = \frac{1}{1-u}$$

$$p_2[\frac{1}{1-u}] = \frac{1}{1-u^2}$$

$$p_3[\frac{1}{1-u}] = \frac{1}{1-u^3}$$

Step 3: Replace each p_i from step 1 with $p_i[\frac{1}{1-u}]$ from step 2. Thus we have

$$p_{1^3}[\frac{1}{1-u}] = \left(\frac{1}{1-u}\right)^3$$

$$p_{21}[\frac{1}{1-u}] = \frac{1}{1-u} \cdot \frac{1}{1-u^2}$$

$$p_3[\frac{1}{1-u}] = \frac{1}{1-u^3}$$

Hence together we get

$$h_3[\frac{1}{1-u}] = \frac{1}{6}\left(\frac{1}{1-u}\right)^3 + \frac{1}{2}\left(\frac{1}{1-u}\cdot\frac{1}{1-u^2}\right) + \frac{1}{3}\left(\frac{1}{1-u^3}\right)$$

Let's note that, if we take $p_n[u_1 + u_2 + ... + u_m]$, then we just get $u_1^n + ... + u_m^n$, which is exactly $p_n(u_1, ..., u_m)$ in the ordinary sense of evaluation. In particular, since $\Lambda \to R$ by $g \mapsto g[u_1 + ... + u_m]$ is a homomorphism, it follows that, for any symmetric function g we have $g[u_1 + ... + u_m] = g(u_1, ..., u_m)$.

Proposition 6.8. *If $F = \sum_{n\geq 1} M_i$ is an (infinite) sum of monomials. Then for all $g \in \Lambda$, we have*

$$g[F] = g(M_1, M_2, M_3, ...)$$

Proof. It suffices to show this for $g = p_n$, but that is trivial. ♡

We note in the above proposition, monomial means U^α for $\alpha \in \mathbb{Z}^m$ for some m and $U = (u_1, ..., u_m)$. Thus, $u_1^2 u_2^{-3}$ is a monomial but $2u_1^2 u_2$ is not monomial.

Example 6.9. Compute $e_3[u^2+2u^3]$. By the proposition, $u^2+2u^3 = u^2+u^3+u^3$, and so

$$e_3[u^2 + 2u^3] = e_3[u^2 + u^3 + u^3] = e_3(u^2, u^3, u^3) = u^8$$

where we used $e_3(x_1, x_2, x_3)$. We note if we just evaluate at $(u^2, 2u^3)$, then we get $e_3(u^2, 2u^3) = 0$, which is far away from u^8.

We remark that plethysm does not play nice with homomorphisms $R \to R$. If $g \in \Lambda$ and $\phi : R \to R$ is a homomorphism, then we do not expect $\mathrm{pl}(g) \circ \phi = \phi \circ \mathrm{pl}(g)$. This seems to be obvious, but it can get tricky. For example, if $\phi : R \to R$ is $F(u_1, ..., u_m) \mapsto F(r_1, ..., r_m)$ with $r_i \in R$, then this is a homomorphism, and we should not expect this to commute with the plethysm. In particular, we should expect $g[F](r_1, ..., r_m) \neq g[F(r_1, ..., r_m)]$. However, we have exceptions.

Lemma 6.10. *If $g \in \Lambda$, and $M_1, ..., M_m \in R$ are monomials (or zero). Then*

$$g[F](M_1, ..., M_m) = g[F(M_1, ..., M_m)]$$

Lemma 6.11. *If $g \in \Lambda$, $R = \mathbb{Q}[u_1, ..., u_m]$. Then $g[\circ]$ is algebraic, i.e. the coefficients of $g[F]$ are polynomials in the coefficients of F.*

Theorem 6.12. *If $f, g \in \Lambda$, then there exists unique symmetric function $f \circ g \in \Lambda$, such that $\mathrm{pl}(f) \circ \mathrm{pl}(g) = \mathrm{pl}(f \circ g)$ in $\mathbb{Q}[u_1, ..., u_m]$ for all m.*

Proof. There is a unique $h \in \Lambda$ such that $h(u_1, \ldots, u_m) = f[g(u_1, \ldots, u_m)]$. We need to show this is the correct thing. We must show that $f[g[F]] = h[F]$ for all $F \in \mathbb{Q}[u_1, \ldots, u_m]$.

Step 1: If $F = M_1 + M_2 + \ldots + M_r$ is a sum of monomials, then from the above equation about $h = f[g]$ we see $f[g[F]] = f[g(M_1, \ldots, M_r)] = h(M_1, \ldots, M_r) = h[F]$.

Step 2: Otherwise, $\mathrm{pl}(f) \circ \mathrm{pl}(g)$ and $\mathrm{pl}(h)$ are algebraic. Thus $[u^\alpha] f[g[F]] = [u^\alpha] h[F]$ is a polynomial identity that we need to prove. However, Step 1 proved that the above identity holds when the coefficients of F are non-negative integers. But since this polynomial identity holds in infinite many places, this must hold in general, and we are done. ♡

7 2021-09-30

Proposition 7.1. *Suppose $k, l \in \mathbb{Z}_+$.*

1. $p_k \circ p_l = p_{kl}$.
2. $p_k \circ g = g \circ p_k$.
3. $p_1 \circ f = f \circ p_1 = f$.
4. $p_k[\cdot]$ *is a homomorphism (not true for all $f \in \Lambda$).*
5. $\circ$ *is left-homomorphic, i.e. we have $(f \pm g) \circ h = f \circ h \pm g \circ h$, $(fg) \circ h = (f \circ h)(g \circ h)$, $c \circ h = c$ if $c \in \mathbb{Q}$, and $\alpha(f) \circ h = \alpha(f \circ h)$ if α is a polynomial in $\mathbb{Q}[t]$. We remark $\circ$ is not right-homomorphic.*
6. $f \circ (g \circ h) = (f \circ g) \circ h$ *and* $(f \circ g)(u_1, \ldots, u_m) = f[g(u_1, \ldots, u_m)]$.

Example 7.2. We have

$$\begin{aligned}
p_{32} \circ (p_{32} + 2p_5) &= p_3 p_2 \circ (p_{32} + 2p_5) \\
&= (p_3 \circ (p_{32} + 2p_5)) \cdot (p_2 \circ (p_{32} + 2p_5)) \\
&= ((p_{32} + 2p_5) \circ p_3) \cdot ((p_{32} + 2p_5) \circ p_2) \\
&= ((p_3 \circ p_3) \cdot (p_2 \circ p_3) + 2p_5 \circ p_3) \cdot ((p_3 \circ p_2)(p_2 \circ p_2) + 2p_5 \circ p_2) \\
&= (p_9 p_6 + 2p_{15})(p_6 p_4 + 2p_{10})
\end{aligned}$$

Proposition 7.3. *We have:*

1. $\mathrm{ex}(f \circ g) = \mathrm{ex}(f) \circ \mathrm{ex}(g)$ *where the latter is polynomial compostion.*
2. *If $f \in \Lambda_n$, then $f \circ (-p_1) = (-1)^n \omega(f)$.*

Let $\mathfrak{X}$ be a finite set with a S_n-action, then for $x \in \mathfrak{X}$, we use $O(x) = \mathrm{Orb}(x)$, $\mathrm{Stab}(x)$ to denote its orbit and stablizer. Next, we use $\mathfrak{X}^\sigma$ to denote the fixer, i.e. $\mathfrak{X}^\sigma = \{x \in \mathfrak{X} : \sigma x = x\}$. We use $\tilde{\mathfrak{X}}$ to denote the set of orbits of $\mathfrak{X}$.

Lemma 7.4.

1. $|\mathrm{Orb}(x)| \cdot |\mathrm{Stab}(x)| = |S_n|$
2. $|\tilde{\mathfrak{X}}| = \frac{1}{|S_n|} \sum_{\sigma \in S_n} |\mathfrak{X}^\sigma|$

Proof. We only prove (2).

$$\begin{aligned}
\frac{1}{n!} \sum_{\sigma \in S_n} |\mathfrak{X}^\sigma| &= \frac{1}{n!} \sum_{\substack{(\sigma,x)\in S_n \times \mathfrak{X} \\ \sigma x = x}} 1 \\
&= \frac{1}{n!} \sum_{x \in \mathfrak{X}} |\mathrm{Stab}(x)| \\
&= \frac{1}{n!} \sum_{x \in \mathfrak{X}} \frac{n!}{|\mathrm{Orb}(x)|} \\
&= \sum_{x \in \mathfrak{X}} \frac{1}{|\mathrm{Orb}(x)|} = |\tilde{\mathfrak{X}}|
\end{aligned}$$

♡

Next, we can put more information into this formula.

Definition 7.5. We define

$$Z_{\mathfrak{X}} := \frac{1}{n!} \sum_{\sigma \in S_n} |\mathfrak{X}^\sigma| \cdot p_{\lambda(\sigma)}$$

where $\lambda(\sigma)$ is the cycle type of σ.

Note:

1. $|\tilde{\mathfrak{X}}| = Z_{\mathfrak{X}}[1] = Z_{\mathfrak{X}}(1)$
2. $|\mathfrak{X}| = |\mathfrak{X}^{\mathrm{Id}}| = \langle Z_{\mathfrak{X}}, p_{1^n} \rangle$ and in particular this means $\mathrm{ex}(Z_{\mathfrak{X}}) = |\mathfrak{X}| \cdot \frac{t^n}{n!}$.

Recall a species $\mathcal{A}$ is a functor from the category $\mathscr{B}$ to $\mathscr{B}$, where $\mathscr{B}$ is the category of finite sets, i.e. objects are finite sets, and morphisms are bijections between finite sets.

Example 7.6. Examples of species includes:

1. Graphs, trees, connected graphs, etc.
2. We have the species of permutations $\mathcal{S}$, which is given by $X \mapsto S_X$, the set of permutations of X. The morphisms $f : X \to Y$ maps to $f_*(\sigma) := f \circ \sigma \circ f^{-1}$ where $\sigma \in S_X$.
3. Species of linear order: $\mathcal{L}(X) = \{(x_1, ..., x_n) : \{x_1, ..., x_n\} = X, |X| = n\}$ and $\mathcal{L}(f)(x_1, ..., x_n) = (f(x_1), ..., f(x_n)) \in \mathcal{L}(Y)$.
4. Species of sets $\mathcal{E}$: We may $\mathcal{E}(X) = \{X\}$ and $\mathcal{E}(f)(X) = Y$.
5. The zero species: $X \mapsto \varnothing$ and f sends to the unique map from empty set

to empty set.

We get an action of S_n on the set $\mathcal{A}_{[n]}$ as follows. Every $\sigma \in S_n$ is a bijection from $[n]$ to $[n]$, and hence we get $\mathcal{A}(\sigma) = \sigma_* : \mathcal{A}_{[n]} \to \mathcal{A}_{[n]}$, which is an element of $S_{\mathcal{A}_{[n]}}$, and that's exactly a action on $\mathcal{A}_{[n]}$.

Example 7.7.

1. If the species is $\mathcal{S}$, then we get $\mathcal{S}_n \to S_n$ and $\sigma \cdot \pi : X_n \times \mathcal{S}_{[n]}$ is given by $\sigma\pi\sigma^{-1}$.
2. If the species is $\mathcal{L}$, then S_n should act on $\mathcal{L}_{[n]}$ by left multiplication, where $\mathcal{L}_{[n]}$ elements are considered as one-line notation of permutations.
3. $\mathcal{E}$ correspond to the trivial action of S_n.

We can also think species as give labels to unlabelled objects. In this case, we use $\tilde{\mathcal{A}}$ to denote the unlabelled structures, which is just the isomorphism types, which is just S_n-orbits. We note that, in this case, we need to know how to interpret the diagram.

Definition 7.8. The cycle index function of a species $\mathcal{A}$ is

$$Z_{\mathcal{A}} := \sum_{n \geq 0} Z_{\mathcal{A}_{[n]}} \in \hat{\Lambda}$$

8 2021-10-05

We start with some operations on species.

Definition 8.1. The ***filtering***. Let $\mathcal{A}$ be a species, $\mathcal{A}_n$ be the species defined as $(\mathfrak{A}_n)_X = \mathfrak{A}_X$ if $|X| = n$ and $\varnothing$ otherwise.

Definition 8.2. The ***species product***. We denote this by $\mathcal{A} * \mathcal{B}$. Let $\mathcal{A}, \mathcal{B}$ be species, we define $(\mathcal{A} * \mathcal{B})_X = \biguplus_{W \subseteq X} \mathcal{A}_W \times \mathcal{B}_{X \setminus W}$ where we use $\biguplus$ is disjoint union.

In terms of unlabelled structures, we have $\widetilde{\mathcal{A} * \mathcal{B}} = \tilde{\mathcal{A}} * \tilde{\mathcal{B}}$.

Example 8.3. For example, consider $\mathcal{E} * \mathcal{T}$, the product of sets and trees. Then the unlabelled structure for $\tilde{\mathcal{E}}$ contains all the complete graphs, and $\tilde{\mathcal{T}}$ contains all trees. Thus, to get labelled version, we pick one complete grah, one tree, then label them using X.

Theorem 8.4.

$$Z_{\mathcal{A} \times \mathcal{B}} = Z_{\mathcal{A}} Z_{\mathcal{B}}$$

Example 8.5. If $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition. We can define a new species $\mathcal{H}^\lambda = \mathcal{E}_{\lambda_1} * \ldots * \mathcal{E}_{\lambda_l}$. To compute the cycle index function of $\mathcal{H}^\lambda$, we start with $\mathcal{E}_n$. We see

$$Z_{\mathcal{E}_n} = \frac{1}{n!} \sum_{\sigma \in S_n} 1 \cdot p_{\lambda(\sigma)} = \frac{1}{n!} \sum_{\lambda \vdash n} |\mathcal{C}_\lambda| p_\lambda = h_n$$

Thus we conclude

$$Z_{\mathcal{H}^\lambda} = h_\lambda$$

Definition 8.6. The ***composition***. Let $\mathcal{A}, \mathcal{B}$ be two species, we denote this by $\mathcal{A} \circ \mathcal{B}$. Assume $\mathcal{B}_\varnothing = \varnothing$. Then $(\mathcal{A} \circ \mathcal{B})_X = \dot{\bigcup}_{P \vdash X} \mathcal{A}_P \times \mathcal{B}_{P_1} \times \ldots \times \mathcal{B}_{P_{\ell(P)}}$ where $P \vdash X$ is a set partition of X where the product is unordered.

Example 8.7. Consider $\mathcal{E} \circ \mathcal{G}^c$ where $\mathcal{G}^c$ is the species of connected graphs. Then the compostion of unlabelled structure is just replace each node of $\mathcal{E}$ by an unlabelled structures of $\mathcal{G}^c$. In particular, this gives $\mathcal{E} \circ \mathcal{G}^c = \mathcal{G}$.

Theorem 8.8.

$$Z_{\mathcal{A} \circ \mathcal{B}} = Z_{\mathcal{A}} \circ Z_{\mathcal{B}}$$

where the left is species compostion and the right is plethystic compostion.

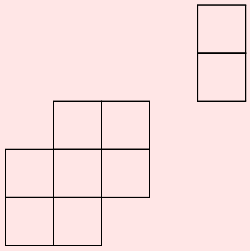
Now a sidenote about enumerations. Say $\mathcal{A}$ is a species, then typical type of questions asked are:

1. What is $|\mathcal{A}_{[n]}|$? To answer this, we define EGF by $A(t) = \sum_{n \geq 0} |\mathcal{A}_{[n]}| t^n / n!$. In particular, $A(t) = \mathrm{ex}(Z_{\mathfrak{A}})$. For example, if $\mathcal{G}$ is graph, then $G(t) = \sum_{n \geq 0} 2^{\binom{n}{2}} t^n / n!$. If $\mathcal{E}$ is the set, then $E(t) = \exp(t)$. In particular, then we get $E(G^c(t)) = G(t)$ and so $G^c(T) = \log(G(t))$.
2. What is $|\tilde{\mathcal{A}}_{[n]}|$? Again, we define the type generating function $\tilde{A}(t) = \sum_{n \geq 0} |\tilde{\mathcal{A}}_{[n]}| t^n$. Again, the answer is encoded in cycle index function as well, i.e. $\tilde{A}(t) = Z_{\mathfrak{A}}[t]$. Viz, the type generating function is the cycle index function plethystically evaluated at t. In particular, we get a theorem $Z_{\mathcal{A} \circ \mathcal{B}}[t] = Z_{\mathcal{A}}[Z_{\mathcal{B}}[t]] = Z_{\mathcal{A}}[\tilde{B}(t)]$. This is actually the "Polya enumeration theorem"

Now we are done with species and we jump to Schur functions.

Definition 8.9. Let $\mu \subseteq \lambda$ be two partitions where μ contained in λ. Then we write λ / μ to denote the pair and call this a ***skew partition***. The diagram of λ / μ is the difference of the two diagrams.

Example 8.10. Consider 55332/441, then the diagram should be



Definition 8.11. A ***semistandard Young tableau*** (SSYT) of ***shape*** λ/μ is a filling of boxes of a (skew) partition λ/μ such that the numbers weakly increasing from left to right, and strictly increasing top to bottom (in English convention). We use $\mathrm{SSYT}(\lambda/\mu)$ to denote the set of SSYT of shape λ/μ.

Definition 8.12. We say T is ***straight*** if it is of shape $\lambda/\varnothing$, and ***skew*** if it is of the shape λ/μ with $\mu \neq \varnothing$.

Example 8.13. An example of SSYT of shape 88442/21 is

$$\begin{array}{cccccccc} & & 2 & 2 & 2 & 2 & 2 & 4 \\ & 1 & 3 & 3 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 6 & & & & \\ 6 & 6 & 6 & 7 & & & & \\ 7 & 8 & & & & & & \end{array}$$

Definition 8.14. For a SSYT T, we define the ***content*** $c(T) = (c_1(T), c_2(T), c_3(T), \ldots)$ where $c_i(T)$ is the number of i in T.

Example 8.15. For example, if we have

$$\begin{array}{cccccccc} & & 2 & 2 & 2 & 2 & 2 & 4 \\ & 1 & 3 & 3 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 6 & & & & \\ 6 & 6 & 6 & 7 & & & & \\ 7 & 8 & & & & & & \end{array}$$

then $c(T) = (1, 6, 4, 3, 1, 5, 2, 1, 0, \ldots)$

Definition 8.16. We say SSYT $T \in \mathrm{SSYT}(\lambda/\mu)$ is ***standard*** if the content is equal $c(T) = (1, 1, 1, \ldots, 1, 0, 0, \ldots)$ where we have $|\lambda/\mu|$ many 1. We denote $\mathrm{SYT}(\lambda/\mu)$ to denote the set of standard Young tableaux. We use $f^{\lambda/\mu} := |\mathrm{SYT}(\lambda/\mu)|$.

Definition 8.17. The ***Schur function*** s_λ as

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{c(T)}$$

where $x^{c(T)} = x_1^{c_1(T)} x_2^{c_2(T)} \dots$.

Note it is clear we can also define

$$s_{\lambda/\mu} = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} x^{c(T)}$$

In fact, the Schur functions are special case of $s_{\lambda/\mu}$.

Example 8.18. Well,

$$s_1 = m_1 = e_1 = h_1 = p_1$$

More generally, we note

$$s_n = h_n$$

And of course,

$$s_{1^n} = e_n$$

Next, consider $s_{21}(x_1, x_2, x_3)$. We can work out the total possibilities are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Hence

$$s_{21}(x_1, x_2, x_3) = m_{21} + 2m_{111}$$

Proposition 8.19. *$s_{\lambda/\mu} \in \Lambda_n$ if $|\lambda/\mu| = n$.*

Proof. It should be clear $s_{\lambda/\mu}$ is homogeneous of degree n. We just need to show it is symmetric. It suffices to show $\tau_i \cdot s_{\lambda/\mu} = s_{\lambda/\mu}$ where $\tau_i = (i, i+1)$.

We will give a bijection between SSYT with content $(c_1, \dots, c_i, c_{i+1}, \dots)$ and SSYT with content $(c_1, \dots, c_{i+1}, c_i, \dots)$. This is easy, we just swap, at each row, the i and $i+1$'s that are free, and we are done. Here, free for i means there is no $i+1$ above or below, and free for $i+1$ means there is no i above or below. The proof follows. ♡

The above map is called the Bender Knuth involution. We note this does not induce a S_n-action on the set of SSYT as we have more than one way to write permutation as product of τ_i, and they lead to different output.

9 2021-10-07

Last time we defined the Schur functions and proved they are indeed symmetric. This time we want to show s_λ is a basis. In particular, we note if λ, μ are both partitions, then

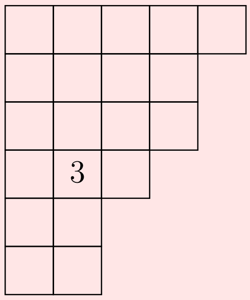
$$\begin{aligned}\#\{T \in \mathrm{SSYT}(\lambda) : c(T) = \mu\} &= [x_1^{\mu_1} x_2^{\mu_2} \dots] s_\lambda(x_1, x_2, \dots) \\ &= \text{coefficients of } m_\mu \text{ in } s_\lambda \\ &= \langle s_\lambda, h_\mu \rangle\end{aligned}$$

Definition 9.1. For partitions λ, μ, we define the Kostka numbers as

$$K_{\lambda,\mu} = \langle s_\lambda, h_\mu \rangle$$

Thus, if $\lambda \vdash n$, then $s_\lambda = \sum_{\mu \vdash n} K_{\lambda\mu} m_\mu$. Before we prove they form a basis, we look at an example.

Example 9.2. Consider the following filling



and we observe the above is impossible, as the columns need to be strictly increasing.

More generally, all the i's must be in the first i rows. If T is a SSYT of shape $\lambda \vdash n$, and content $\mu \vdash n$, then all entries $\leq i$ must be in the first i rows. In other word, we have

$$\mu_1 + \mu_2 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i$$

In other word, we get

$$s_\lambda = \sum_{\mu \leq \lambda} K_{\lambda\mu} m_\mu$$

but this is exactly the triangular relation we wanted, and clearly $K_{\lambda\lambda} = 1$, and we are done proving s_λ gives a basis.

Theorem 9.3. *$\{s_\lambda : \lambda \vdash n\}$ is a basis for Λ_n and hence $\{s_\lambda\}$ is a basis for Λ.*

Next, we note

$$K_{\lambda,1^n} = f^\lambda = \#\mathrm{SYT}(\lambda)$$

Also, observe we have

$$\begin{aligned}\mathrm{ex}(s_\lambda) &= \sum_{\mu \vdash n} K_{\lambda\mu}\,\mathrm{ex}(m_\mu) \\ &= K_{\lambda,1^n}\frac{t^n}{n!} = \frac{f^\lambda}{|\lambda|!}t^n\end{aligned}$$

This will actually lead us to a formula of f^λ.

Next, we are going to study things we can do about SSYT. The first topic is going to be row insertion algorithm.

Example 9.4 (Row Insertion Algorithm).

```
Input: T a SSYT(λ) and a ∈ ℤ₊
Output: T with a new entry with content a added
i=1
f=a
Repeat:
    if f > a for all entries in row i:
        add f to the end of row i
        Break
    else:
        g=the left most entry in row i that is > f.
        swap g and f
        i=i+1
```

Proposition 9.5. *$T \leftarrow a$ is a SSYT of shape λ^+ where λ^+ greater than λ in containment order and there is nothing in between them. We use $\lambda \Subset \lambda^+$ to denote this relation.*

Proposition 9.6. *If $U \in \mathrm{SSYT}(\lambda^+)$ and $\lambda \Subset \lambda^+$ is given, then there exists a unique $T \in \mathrm{SSYT}(\lambda)$ and $a \in \mathbb{Z}_+$ such that $U = T \leftarrow a$.*

Corollary 9.6.1. $s_\lambda s_1 = \sum_{\lambda^+ \Supset \lambda} s_{\lambda^+}$

Proposition 9.7. *Suppose $T \in \mathrm{SSYT}(\lambda), T \leftarrow a \in \mathrm{SSYT}(\lambda^+)$ and $T \leftarrow a \leftarrow b \in \mathrm{SSYT}(\lambda^{++})$.*

1. *If $a \leq b$ then the unique box of λ^{++}/λ^+ is weakly above and strictly right of the unique box of λ^+/λ.*
2. *If $a < b$ then the unique box of λ^{++}/λ^+ is strictly below and weakly left of the unique box of λ^+/λ.*

We will use this row insertion to define a monoid (this is a group without inverse) structure on the set of all SSYT of straight shape.

Definition 9.8. If T is a SSYT, we define the ***row word*** of T as the word obtained by reading entries of T from left to right along the row, starting at the bottom and moving up. This is denoted by $w(T)$.

Definition 9.9. If T, U are SSYT of straight shape, we define $T * U$ as follows. Let $w(U) = w_1 ... w_k$ and we let $T * W = T \leftarrow w_1 \leftarrow w_2 \leftarrow ... =: T \leftarrow w(U)$.

It is not obvious that the multiplication is associative.

Theorem 9.10. *The $*$ is associative.*

Next we are going to give a sketch of proof.

If $u = u_1 ... u_m \in \mathbb{Z}_+^*$ is a word, then we will write $T \leftarrow u$ to mean insert u_i one at a time. Then one can check if $T \in \mathrm{SSYT}(\lambda)$ then $\epsilon \leftarrow w(T) = T$ where ϵ is the empty tableau.

Definition 9.11. If $u, v \in \mathbb{Z}_+^*$, we say u and v are Knuth-equivalent (write $u \sim v$) if $\epsilon \leftarrow u = \epsilon \leftarrow v$.

Hence the Knuth equivalence classes are in bijection with SSYT of straight shape. The bijection is given by insertion and taking row words of T.

In particular, we note if $u = u_1 ... abc ... u_n$ and $u' = u_1 ... a'b'c' ... u_n$ where two words differ in at most 3 consecutive positions, and $abc \sim a'b'c'$ then we say u and u' are related by an elementary Knuth transformation.

Theorem 9.12. *$u \sim v$ iff there exists a sequence of elementary Knuth transformations taking u to v.*

Proof. First we need to determine when $abc \sim a'b'c'$. Once we know this, we need to show u, u' are related by a elementary Knuth transformation, then $u \sim u'$. Next, we deinfe an explicit sequence of elementary Knuth transformation relating $w(T \leftarrow a)$ to $w(T)a$. Now just use induction to show $u \sim w(\epsilon \leftarrow u)$. ♡

Corollary 9.12.1. *If $u \sim u'$ and $v \sim v'$ then $uv \sim u'v'$.*

Theorem 9.13. *$*$ is associative: the corollary above says the map $(\mathbb{Z}_+^*, \cdot) \to (\mathbb{Z}_+^* / \sim, \cdot) = (\mathrm{SSYT}, *)$ is a homomorphism.*

Definition 9.14. $(\mathrm{SSYT}, *)$ is called the Plactic monoid.

10 2021-10-19

Last time we learned row insertion $T \leftarrow a$, and it has some nice properties:

1. if $a \le b$ then shape "grows to right" in $T \leftarrow a \leftarrow b$.
2. if $a > b$ then shape "grows down" in $T \leftarrow a \leftarrow b$.

The main topic of today will be Robinson-Schensted-Knuth correspondence.

The main goal of this correspondence is to prove the following theorem.

Theorem 10.1. *The Schur functions are an orthonormal basis for Λ.*

Proof. We want to show $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. Hence, we need to show

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \prod_{i,j\ge 1} \frac{1}{1 - x_i y_j}$$

Well, in assignment, we proved

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \prod_{i,j\ge 1} \frac{1}{1 - x_i y_j} = \sum_{n\ge 0} \sum_{(a,b)\in\mathcal{B}_n} x_{a_1}\dots x_{a_n} y_{b_1}\dots y_{b_n}$$

where $\mathcal{B}_n$ is the set of bi-words.

On the other hand, we see

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \sum_\lambda \sum_{(P,Q)\in \mathrm{SSYT}(\lambda)^2} x^{c(P)} y^{c(Q)}$$

Thus we need to find a weight preserving bijection $\dot\bigcup_{\lambda\vdash n} \mathrm{SSYT}(\lambda)^2$ and $\mathcal{B}_n$.

This bijection is exactly called Robinson-Schensted-Knuth correspondence. ♡

We will describe RSK as an algorithm.

```
Input: A biword (a,b)
Output: A pair (P,Q) ∈ SSYT(λ)^2 for some λ

Start with P_0 = Q_0 = ϵ the empty tableau.
for 1 ≤ k ≤ n:
    P_k = P_{k-1} ← a_k
Define Q_k to be the tableau obtained by starting with Q_{k-1} and
    appending b_k, so it has the same shape as P_k.

Return P = P_n, Q = Q_n
```

Example 10.2. Take biwords $a = 121132, b = 113334$.

Then $P_0 = \epsilon = Q_0$. Next,

$$P_1 = \begin{ytableau} 1 \end{ytableau} \quad Q_1 = \begin{ytableau} 1 \end{ytableau}$$

$$P_2 = P_1 \leftarrow 2 = \begin{ytableau} 1 & 2 \end{ytableau} \quad Q_2 = \begin{ytableau} 1 & 1 \end{ytableau}$$

$$P_3 = \begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \quad Q_3 = \begin{ytableau} 1 & 1 \\ 3 \end{ytableau}$$

$$P_4 = \begin{ytableau} 1 & 1 & 1 \\ 2 \end{ytableau} \quad Q_4 = \begin{ytableau} 1 & 1 & 3 \\ 3 \end{ytableau}$$

$$P_5 = \begin{ytableau} 1 & 1 & 1 & 3 \\ 2 \end{ytableau} \quad Q_5 = \begin{ytableau} 1 & 1 & 3 & 3 \\ 3 \end{ytableau}$$

$$P_6 = \begin{ytableau} 1 & 1 & 1 & 2 \\ 2 & 3 \end{ytableau} \quad Q_6 = \begin{ytableau} 1 & 1 & 3 & 3 \\ 3 & 4 \end{ytableau}$$

where we take $P = P_6$ and $Q = Q_6$.

It is clear that:

1. P is SSYT.
2. P has entries $a_1, a_2, \ldots, a_n$ by design.
3. Q has the same shape as P.
4. Q has entries $b_1, \ldots, b_n$.

The things that are not clear:

1. Q is a SSYT.
2. This is a bijection.

To see why Q is SSYT, we see by definition $b_1 \leq b_2 \leq \ldots \leq b_n$. It is clear that the rows and columns are weakly increasing. We just need to prove the columns are strictly increasing.

If we have a tie $b_i = b_{i-1}$ in b, then we know $a_{i-1} \leq a_i$, then $P_i = P_{i-2} \leftarrow p_{i-1} \leftarrow p_i$ and so the new boxes added move to the right strictly. Thus the b_i and b_{i-1} would appear on the same row and hence we cannot have the same number in the same column. This proves Q is SSYT.

To see why it is a bijection, we just need to show the algorithm is reversible.

Given $(P, Q) = (P_n, Q_n)$. Then by the above observation, we look for the largest entry in Q_n that's also the rightmost, which is b_n. This tells us how to get Q_{n-1}. But then we also know P_{n-1} because we know the shape of P_{n-1} from Q_{n-1}. The proof follows (this shows one-side inverse, and the other direction also holds).

This RSK correspondence gives the bijection required to prove orthonormality of $\{s_\lambda\}$.

Proposition 10.3.

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$$

Proof. We saw

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

However, we see

$$\begin{aligned} h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda &\Leftrightarrow \langle h_\mu, s_\lambda\rangle = K_{\lambda\mu} \\ &\Leftrightarrow \langle s_\lambda, h_\mu\rangle = K_{\lambda\mu} \\ &\Leftrightarrow s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu \end{aligned}$$

♡

One thing we note is that we can also prove the above proposition using the Pieri rule.

Write $u \xrightarrow{k} \lambda$ to mean λ/μ is a horizontal strip with $|\lambda/\mu| = k$, i.e. no two boxes in same column. Then the Pieri rule says

$$s_\lambda \cdot h_k = \sum_{\substack{\lambda \\ \mu \xrightarrow{k} \lambda}} s_\lambda$$

Using this, we see $h_\mu = (((1 \cdot h_{\mu_1}) \cdot h_{\mu_2}) \cdot \ldots \cdot h_{\mu_m})$, where we can inductively compute h_μ in terms of s_λ. If you think about this, we are exactly construct SSYT with prescribed shape and content: at each h_{μ_i} multiplicatoin, we are adding one horizontal strip filled with i, and when we take all possible ways, we just get $K_{\lambda\mu}$ possibilities.

More generally,

$$s_\mu \cdot h_\nu = (((s_\mu \cdot h_{\nu_1}) \cdot h_{\nu_2}) \ldots h_{\nu_m}) = \sum_{\lambda \geq \mu} K_{(\lambda/\mu),\nu} s_\lambda$$

where $K_{(\lambda/\mu),\nu}$ is the number of SSYT with shape λ/μ and content ν.

This is just the iterated Pieri rule, and it actually tells us

$$\langle s_\mu \cdot h_\nu, s_\lambda\rangle = K_{(\lambda/\mu),\nu}$$

for all ν, μ, λ. This is the same as

$$\langle h_\nu, s_\mu^\perp(s_\lambda)\rangle = K_{(\lambda/\mu),\nu}$$

for all ν. But we also have $K_{(\lambda/\mu),\nu}$ is the coefficient in expanding $s_{\lambda/\mu}$ in terms of m_ν, i.e.

$$s_{\lambda/\mu} = \sum_\nu K_{\lambda/\mu,\nu} m_\nu$$

But we can take inner product of the above and conclude

$$\langle h_\nu, s_{\lambda/\mu}\rangle = K_{\lambda/\mu,\nu}$$

Hence, we conclude

$$\langle h_\nu, s_\mu^\perp(s_\lambda)\rangle = \langle h_\nu, s_{\lambda/\mu}\rangle$$

In other word, the skew operator $s_\mu^\perp$ acts on s_λ is just $s_{\lambda/\mu}$, i.e. the skew operator is literally skewing things.

Theorem 10.4.

$$s_\mu^\perp(s_\lambda) = s_{\lambda/\mu}$$

Proposition 10.5 (Properties of RSK)**.**

1. *For each $\sigma \in S_n$, identify σ with biword*

$$((\sigma(1), ..., \sigma(n)), (1, 2, ..., n)) \in \mathcal{B}_n$$

Then RSK restricting to S_n gives a bijection between permutation $\sigma \in S_n$ and $(P_\sigma, Q_\sigma) \in \bigcup_{\lambda\vdash n} \mathrm{SYT}(\lambda)^2$.
2. *In particular this means $n! = \sum_{\lambda\vdash n}(f^\lambda)^2$.*
3. *If (a, b) correspond to (P, Q), then $\mathrm{sort}(b, a) = (Q, P)$ where $\mathrm{sort}(b, a)$ is the biword obtained by sorting (b, a). In particular, if σ correspond to (P, Q) then σ^{-1} correspond to (Q, P).*
4. *Define $\tilde{P}_\sigma$ to be the SYT obtained by reflect along diagonal. Then $\tilde{P}_\sigma = P_{\sigma w_0}$ and $\tilde{Q}_\sigma = Q_{w_0\sigma}$ where we recall $w_0 = n(n-1)...21$ is the longest element of S_n.*
5. *There is an involution called "evacuation" on $\mathrm{SYT}(\lambda)$ such that evac such that $P_{w_0\sigma w_0} = evac(P_\sigma)$ and $Q_{w_0\sigma w_0} = evac(Q_\sigma)$. We might talk about this next time.*
6. *(Greene's theorem): If $\lambda = \mathrm{sh}(P)$. Then:*
   (a) *λ_1 is the length of the longest weakly increasing subsequence in $a_1a_2...a_n$.*
   (b) *$\lambda_1+\lambda_2$ is the combined length of the longest pair of disjoint increasing subsequence and so on.*
   (c) *For strictly decreasing subsequences, use conjugate.*

It is clear from above that there are more symmetry than we expected from the above properties. This is called the Viennot's shadow diagram interpretations of the RS correspondence for $\sigma \in S_n$.

11 2021-10-26

We are going to talk about some applications.

The first one is MacMahen's theorem, which is about enumeration of plane partitions.

Definition 11.1. A ***plane partition*** is a tableau in which the entries weakly decrease to the right and down.

Example 11.2. For example, we have

5	5	5	3	1	1
4	4	3	1		
4	3	3	1		
2	2				
2	1				

Definition 11.3. If π is a plane partition, then we define $|\pi|$ equal the sum of entries.

The goal would be to compute the generating function $\sum_\pi q^{|\pi|}$, i.e. we want to find the number of plane partitions of size n for all n.

Here is an example of what we are going to do. Take the running example

5	5	5	3	1	1
4	4	3	1		
4	3	3	1		
2	2				
2	1				

Then we get a bunch of partitions by choosing the diagonal and go from right. For example, by look at the diagram, we get $\alpha_0 = 543, \alpha_1 = 531, \alpha_2 = 51, \alpha_3 = 3, \alpha_4 = 1, \alpha_5 = 1, \alpha_6 = \epsilon$. Then, using α_0 we get a Young diagram

Next, we put i in each box of α_{i-1}/α_i. For example, we should put

			1	
	1	1		

for $i = 1$, and

$$\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline & 2 & 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-4} 2 & 1 & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

for $i = 2$. In the end we get

$$P_\pi := \begin{array}{|c|c|c|c|c|}\hline 6 & 4 & 4 & 3 & 3 \\ \hline 3 & 2 & 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-4} 2 & 1 & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

This is a reverse SSYT with each entries filled with i a horizontal strip, i.e. α_{i-1}/α_i is a horizontal strip.

Next, we do the same for the lower half of the diagonal. This gives $\beta_0 = 543, \beta_1 = 43, \beta_2 = 42, \beta_3 = 21, \beta_4 = 2$. Next, we put i in β_{i-1}/β_i. We denote this by Q_π. For the running example, we get

$$Q_\pi = \begin{array}{|c|c|c|c|c|}\hline 5 & 5 & 3 & 3 & 1 \\ \hline 4 & 3 & 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-4} 1 & 1 & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

Then, one can show

$$|\pi| = |P_\pi| + |Q_\pi| - |\lambda_\pi|$$

where $\lambda_\pi = \alpha_0 = \beta_0$.

This is a bijection. Thus we get

$$\sum_\pi q^{|\pi|} = \sum_\lambda \sum_{(P,Q) \in Rev\,\mathrm{SSYT}(\lambda)^2} q^{|P|+|Q|-|\lambda|}$$

But there is a bijection between the reversed SSYT to SSYT. Thus

$$\sum_\pi q^{|\pi|} = \sum_\lambda \sum_{(P,Q) \in \mathrm{SSYT}(\lambda)^2} q^{|P|+|Q|-|\lambda|}$$

We know

$$\sum_\lambda \sum_{(P,Q) \in \mathrm{SSYT}(\lambda)^2} x^{c(P)} y^{c(Q)} = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(x) s_\lambda(y)$$

Next, evaluate at $x_i = q^i$ and $y_i = q^{i-1}$ would exactly give us $x^{c(P)} y^{c(Q)} = q^{|P|+|Q|-|\lambda|}$.

Hence, we conclude

$$\sum_\pi q^{|\pi|} = \prod_{i,j \geq 1} \frac{1}{1 - q^{i+j-1}} = \prod_{n \geq 0} \frac{1}{(1 - q^n)^n}$$

Theorem 11.4 (MacMahen's Theorem). *The number of plane partitions of size k is given by*

$$[q^k] \prod_{n \geq 1} \frac{1}{(1-q^n)^n}$$

The next topic is classical definition of Schur functions.

To start out, we talk about Cauchy-Binet formula. This states that, if we have two $k \times n$ matrices A, B, and we want to compute

$$\det(AB^T)$$

This can be done using the dot product of $k \times k$ minors of A and $k \times k$ minors of B.

We will use partitions to index minors. Given partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we assign it to $\{\lambda_1 + k, \lambda_2 + k - 1, \ldots, \lambda_k + 1\}$, which is a set of k positive integers.

Thus A_λ is the k minor of A with

$$\{\lambda_1 + k, \lambda_2 + k - 1, \ldots, \lambda_k + 1\}$$

columns.

Now consider the $k \times \infty$ matrix

$$H = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & \ldots \\ 0 & h_0 & h_1 & h_2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \ldots \end{bmatrix}$$

where we end at h_0 at column k, and we get

$$H_\lambda = \begin{vmatrix} h_{\lambda_1+k-1} & h_{\lambda_2+k-2} & \ldots & h_{\lambda_k} \\ h_{\lambda_1+k-2} & h_{\lambda_2+k-3} & \ldots & h_{\lambda_k-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_1} & h_{\lambda_2-1} & \ldots & h_{\lambda_k-k+1} \end{vmatrix}$$

This is exactly the JT determinant rotated 90 degree counterclock-wise and hence it only differ by a sign $\pm$.

Next, consider the $k \times \infty$ matrix

$$A = \begin{bmatrix} 1 & y_1 & y_1^2 & y_1^3 & \ldots \\ 1 & y_2 & y_2^2 & y_2^3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & y_k & y_k^2 & y_k^3 & \ldots \end{bmatrix}$$

Then, we see we get

$$HA^T = \begin{bmatrix} H(y_1) & H(y_2) & H(y_3) & \ldots & H(y_k) \\ y_1 H(y_1) & y_2 H(y_2) & y_3 H(y_3) & \ldots & y_k H(y_k) \\ y_1^2 H(y_1) & y_2^2 H(y_2) & y_3^2 H(y_3) & \ldots & y_k^2 H(y_k) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{k-1} H(y_1) & y_2^{k-1} H(y_2) & y_3^{k-1} H(y_3) & \ldots & y_k^{k-1} H(y_k) \end{bmatrix}$$

When compute the determinant, we see this is given by

$$\det(HA^T) = \prod_{i=1}^{k} H(y_i) \cdot \pm \prod_{i<j} (y_i - y_j)$$

as we can pull out the $H(y_i)$ in each column. But, we have

$$\begin{aligned}\det(HA^T) &= \prod_{i=1}^{k} H(y_i) \cdot \left(\pm \prod_{i<j} (y_i - y_j) \right) \\ &= \prod_{i\geq 1} \prod_{j=1}^{k} \frac{1}{1 - x_i y_j} \cdot \left(\pm \prod_{i<j} (y_i - y_j) \right)\end{aligned}$$

By Cauchy-Binet, we have

$$\det(HA^T) = \sum_{\lambda} |H_\lambda| \cdot |A_\lambda| = \sum_{\lambda} \pm s_\lambda |A_\lambda|$$

Now divide both side by the determinant of Vandermonde matrix, we end up with

$$\prod_{i\geq 1} \prod_{j=1}^{k} \frac{1}{1 - x_i y_j} = \sum_{\lambda} (\pm) s_\lambda(x) \frac{|A_\lambda|}{\pm \prod_{i<j} (y_i - y_j)}$$

But we also have

$$\prod_{i\geq 1} \prod_{j=1}^{k} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x) s_\lambda(y_1, \ldots, y_k)$$

Hence, we conclude

$$s_\lambda(y_1, \ldots, y_k) = \frac{|A_\lambda|}{\prod_{i<j(y_i - y_j)}}$$

Example 11.5. Let $\lambda = 52$. Then, we have

$$s_\lambda(y_1, y_2, y_3) = \frac{\begin{vmatrix} 1 & y_1^3 & y_1^7 \\ 1 & y_2^3 & y_2^7 \\ 1 & y_3^3 & y_3^7 \end{vmatrix}}{-(y_1 - y_2)(y_1 - y_3)(y_2 - y_3)}$$

The final topic is going to be a positive combinatoric rule for Schur functions.

First, we are going to talk about ***Jeu de Taquin***.

Given $T \in \mathrm{SSYT}(\lambda/\mu)$ and a corner box of μ (marked box). We can perform a Jeu de Taquin slide, which means marked box switches with other box to its right or box below, whichever "works". This means it exists and preserve weakly increasing rows and strictly increasing columns. We end if we cannot move anymore.

Example 11.6. Consider

		1	1	1	2	2
	$\bullet$	2	2	3	3	
2	2	3	4	4		
3	4	4	5	5		
5	5	6				

where $\bullet$ means the marked box. When we do jdt slide, we cannot move to the right because when we swap $\bullet$ with 2 we get a 2 above a 2, which is bad. At the end, we get

		1	1	1	2	2
	2	2	2	3	3	
2	3	4	4	4		
3	4	5	5	$\bullet$		
5	5	6				

We note this is totally reversible.

Starting with an "outer corner" of λ, we can perform a reverse jdt slide which inverts the jdt slide.

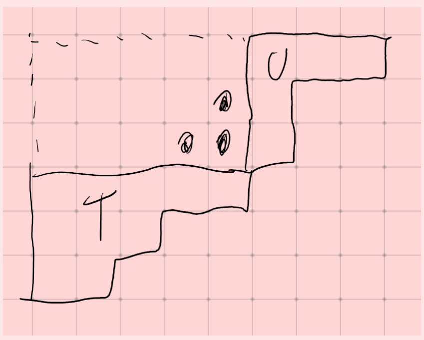
In the above, we call the starting corner to be α, the outer corner α'. Then, we will use $(T, \alpha) \to slide(T, \alpha) = (T', \alpha')$ to denote the jdt slide.

Theorem 11.7. *If $(T', \alpha') = slide(T, \alpha)$, then $w(T) \sim w(T')$ where the $\sim$ denotes Knuth equivalence.*

Proof. There exists a sequence of elementary Knuth transformations. The point is, each time we move one step of the box, we get one elementary Knuth transformation. ♡

Corollary 11.7.1. *$T * U$ can be computed by "rectification", i.e. we put T and U together, and we keep use jdt slide to remove inner boxes, and at the end we must get $T * U$.*

Here is an example of rectification:



We stack U on top of T, then those $\bullet$ are empty boxes, where we can use jdt moves to pull out to the outer area, and in the end we get a non-skewed tableaux.

Example 11.8. Here is an actual example with numbers in it. Consider

$$\begin{array}{ccc} & & 2 \\ \bullet & 1 & 5 \\ 3 & 4 & \end{array}$$

Then we can use jdt move and get

$$\begin{array}{ccc} & \bullet & 2 \\ 1 & 4 & 5 \\ 3 & & \end{array}$$

Now use jdt again, we get

$$\begin{array}{ccc} \bullet & 2 & 5 \\ 1 & 4 & \\ 3 & & \end{array}$$

One last time, we end up with

$$\begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \end{array}$$

On the other hand, we can start with

$$\begin{array}{ccc} & & 2 \\ & 1 & 5 \\ 3 & 4 & \end{array}$$

but we start with a different marked box. In particular, we can start with

$$\begin{array}{ccc} & \bullet & 2 \\ & 1 & 5 \\ 3 & 4 & \end{array}$$

and end up with

$$\begin{array}{ccc} & 1 & 2 \\ \bullet & 4 & 5 \\ 3 & & \end{array} \to \begin{array}{ccc} \bullet & 1 & 2 \\ 3 & 4 & 5 \end{array} \to \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 4 & \end{array}$$

We see even during the process we have different tableaux, at the end we get the same thing. The non-trivial theorem is, this process is well-defined, i.e. no matter which empty box we picked up at the start, we get the same output.

The consequence of the fact rectification is well-defined is that, we can use this to give an alternate definition of Knuth equivalence. To be concrete, given two words u, v, we can form a tableau with $u_1, \ldots, u_n$ in anti-diagonal, then it correspond to some unique tableau after rectification. We say $u \sim v$ iff they have the same rectification.

12 2021-10-28

Last time, we defined Jeu de Taquin. This changes the shape in smallest possible way, and it does not change the content.

Today we are going to talk about ***crystal structure***, which does not change the shape, but change the content in smallest possible way. Specifically, we change an i to an $i+1$ or an $i+1$ to an i.

We note the crystal structure is actually first introduced in the context of representations of quantum groups.

Now comes the definition.

Let $\mathbb{Z}_+^*$ be set of words in positive integers. We define (partially defined) operators

$$E_i : \mathbb{Z}_+^* \to \mathbb{Z}_+^* \cup \{\varnothing\}$$

where $E_i(w) = \varnothing$ means this is undefined. Similarly we will define

$$F_i : \mathbb{Z}_+^* \to \mathbb{Z}_+^* \cup \{\varnothing\}$$

and

$$R_i : \mathbb{Z}_+^* \to \mathbb{Z}_+^*$$

The E_i are called the ***raising operators/crystal operators***, F_i are called ***lowering operators***, and R_i are called ***reflection operators***.

Definition 12.1. Let w be a word, then we hide every letter that is not $\{i, i+1\}$. Then, if we see $i+1$ followed by an i, we hide both of those. We repeat this until there is no $i+1$ followed by i, i.e. we are left with a sequence of i's followed by $(i+1)$'s. Say we are left with a many i's and b many $i+1$, i.e. $iiii\ldots(i+1)(i+1)\ldots$.

Then, we define E_i to changes this sequence to $a+1$ many i and $b-1$ many $i+1$, where if $b=0$ then E_i is undefined.

We define F_i to change this sequence to $a-1$ many i and $b+1$ many $i+1$, where $a=0$ then R_i is undefined.

We define R_i to change this sequence of b many i and a many $i+1$.

Then finally we unhide everything as the actual output of the operator.

Definition 12.2. We say u is ***dual Knuth equivlent*** to v, denoted by $u \sim^* v$, if there exists a sequence of crystal operators taking u to v.

Let us now consider an example.

Example 12.3. Let $w = 1223431232423$. We compute $F_2(w)$. First, hide everything that's not 2 or 3, i.e. we get

$$*223 * 3 * 2 * 3 * 2 * 23$$

Next, we hide all 3 followed 2 and repeat. This gives

$$*223 * (3 * 2)(32) * 23$$

where the things in bracket get hidden, and get

$$*223 * * * * * * * 23$$

now we look again to hide any 3 then 2, and we end up with

$$*22 * * * * * * * * * 3$$

Then, apply $F_2(w)$, we change one 2 into a 3, and we get 233 as result, hence the finally output is given by unhide everything, i.e.

$$F_2(w) = 12\textcolor{red}{3}3431232423$$

where the red 3 is the only changed thing.

We can also define E_i, F_i, R_i for a SSYT T by perform the operation on the reading word of T in place.

Example 12.4. Suppose

$$T = \begin{array}{cccccccc} & & & & \boxed{1} & \boxed{1} & \boxed{2} & \boxed{3} \\ & & \boxed{2} & \boxed{2} & \boxed{2} & \boxed{4} & & \\ \boxed{1} & \boxed{1} & \boxed{3} & \boxed{3} & & & & \\ \boxed{2} & \boxed{4} & & & & & & \end{array}$$

and perform $F_2(T)$.

This time, we first hide all the things that are not $2, 3$. We get

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\cline{5-8}
\multicolumn{4}{c|}{} & \star & \star & 2 & 3 \\
\cline{3-8}
\multicolumn{2}{c|}{} & 2 & 2 & 2 & \star & \multicolumn{2}{c}{} \\
\cline{1-6}
\star & \star & 3 & 3 & \multicolumn{4}{c}{} \\
\cline{1-4}
2 & \star & \multicolumn{6}{c}{} \\
\cline{1-2}
\end{array}$$

Then cross out any 3 followed by a 2, we get

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\cline{5-8}
\multicolumn{4}{c|}{} & \star & \star & 2 & 3 \\
\cline{3-8}
\multicolumn{2}{c|}{} & \star & \star & 2 & \star & \multicolumn{2}{c}{} \\
\cline{1-6}
\star & \star & \star & \star & \multicolumn{4}{c}{} \\
\cline{1-4}
2 & \star & \multicolumn{6}{c}{} \\
\cline{1-2}
\end{array}$$

Then we change the top 2 to 3 as the output, i.e. we get

$$F_2(T) = \begin{array}{|c|c|c|c|c|c|c|c|}
\cline{5-8}
\multicolumn{4}{c|}{} & 1 & 1 & 3 & 3 \\
\cline{3-8}
\multicolumn{2}{c|}{} & 2 & 2 & 2 & 4 & \multicolumn{2}{c}{} \\
\cline{1-6}
1 & 1 & 3 & 3 & \multicolumn{4}{c}{} \\
\cline{1-4}
2 & 4 & \multicolumn{6}{c}{} \\
\cline{1-2}
\end{array}$$

What we notice is that, whenever we have $i+1$ directly below i, both will get hidden. Thus, apply E_i, F_i to T does not ruin semistandardness.

Proposition 12.5. *If $T \in \mathrm{SSYT}(\lambda/\mu)$ then $E_i(T), F_i(T) \in \mathrm{SSYT}(\lambda/\mu)$ if this is defined. In particular, since $R_i(T)$ can be obtained using a sequence of E_i and F_i, we get $R_i(T) \in \mathrm{SSYT}(\lambda/\mu)$.*

Note that in w and $E_i(w)/F_i(w)$, we have exactly the same subsequence get hidden.

We also note R_i gives a bijection between $\mathrm{SSYT}(\lambda/\mu)$ of content $(..., u_i, u_{i+1}, ...)$ and content $(..., u_{i+1}, u_i, ...)$, just like the Bender Knuth involution. But this is different from the Bender Knuth. In particular, this R_i gives an S_n-action on the set of $\mathrm{SSYT}(\lambda/\mu)$, i.e. R_i satisfy the braid relations (while the Bender Knuth does not satisfy), i.e. $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$. This is going to be in the assignment...

Next, we discuss the relation between Jeu de Taquin.

Theorem 12.6. *Crystal operators commutes with Jeu de Taquin slides, i.e. start with T, we have $C_i(S_{k,j}(T)) = S_{k,j}(C_i(T))$ where $S_{k,j}$ is jdt slide at position i, j and $C_k \in \{E_i, F_i, R_i\}$.*

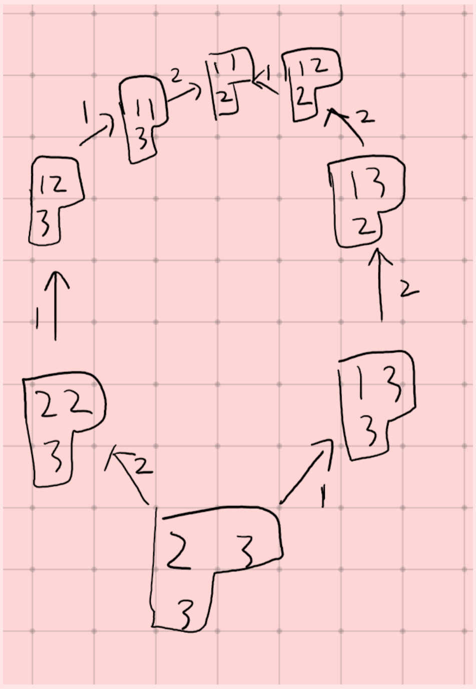
Proof. We do $C_i = E_i$. Look at slide through T vs $E_i(T)$. The slide movement is exactly the same until the marked box from jdt hits $i/i+1$. Next, if slide nowhere near entry that changes, rest of the slide is same too. Final case, if marked box does go near entry that changes, it is a simple case analysis. ♡

Definition 12.7. The ***crystal*** $\Gamma(\lambda/\mu)$ is the directed graph with coloured edges $1, 2, 3, \ldots$, where:

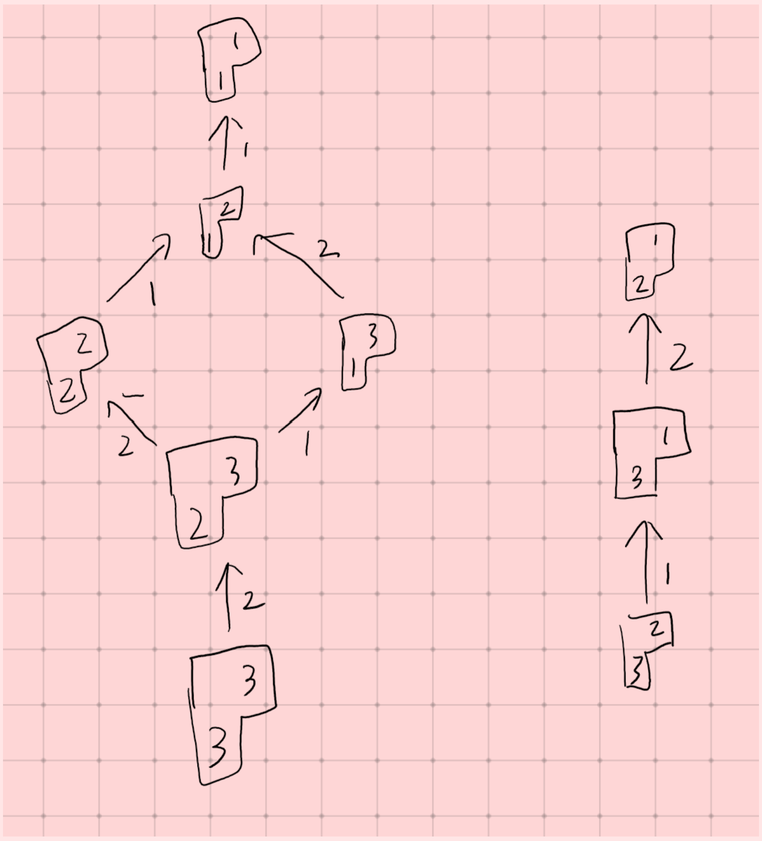
1. $V(\Gamma(\lambda/\mu)) = \mathrm{SSYT}(\lambda/\mu)$
2. We get directed edge coloured by i, from T to T', if $E_i(T) = T'$.

We note the (weakly) components of $\Gamma(\lambda/\mu)$ are the dual Knuth equivalence classes restricted to $\mathrm{SSYT}(\lambda/\mu)$.

Example 12.8. We draw an example of a finite crystal (note crystal is infintie graph and we cannot draw such thing). In particular, we will consider $\mathrm{SSYT}(21)_{\leq 3}$ with entries ≤ 3 and edges coloured $1, 2$.



Next, we draw $\Gamma(21/1)_{\leq 3}$. This is given by



Definition 12.9. A ***Littlewood-Richardson tableau*** of shape λ/μ is a tableau $T \in \mathrm{SSYT}(\lambda/\mu)$ such that $E_i(T) = \varnothing$ for all i. Equivalently, these are sinks in $\Gamma(\lambda/\mu)$.

Definition 12.10. Let $LR(\lambda/\mu, \nu)$ denote the set of all Littlewood-Richardson tableaux of shape λ/μ and content ν.

Theorem 12.11 (Littlewood-Richardson,V1)**.**

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

where $c^\lambda_{\mu\nu} = \#LR(\lambda/\mu; \nu)$

13 2021-11-02

Last time we talked about crystal operations commute with jdt slides (without proof). Thus, let's start with an example.

Example 13.1. Consider

$$\begin{ytableau} \star & 1 & 2 & 3 & 3 \\ 2 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{ytableau}$$

Then using E_2 we get

$$\begin{ytableau} \star & 1 & 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{ytableau}$$

Next, let's use jdt slide to the above two diagrams with the $\star$-box, we get

$$\begin{ytableau} \star & 1 & 2 & 3 & 3 \\ 2 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{ytableau} \mapsto \begin{ytableau} 1 & 2 & 2 & 3 & 3 & 3 \\ 2 & 3 & 3 & 4 \\ 4 & 4 & 4 & \star \end{ytableau}$$

$$\begin{ytableau} \star & 1 & 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{ytableau} \mapsto \begin{ytableau} 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 3 & 3 & 4 \\ 4 & 4 & 4 & \star \end{ytableau}$$

Finally, apply E_2 to

$$\begin{ytableau} 1 & 2 & 2 & 3 & 3 & 3 \\ 2 & 3 & 3 & 4 \\ 4 & 4 & 4 & \star \end{ytableau}$$

we get

$$\begin{ytableau} 1 & 2 & 2 & 2 & 3 & 3 \\ 2 & 3 & 3 & 4 \\ 4 & 4 & 4 & \star \end{ytableau}$$

as well.

Last time we also observed crystals sometimes have more than one components. Also, we defined T is Littlewood-Richardson tableau to be $E_i(T) = \varnothing$ for all i. This is equivalent to $w(T) = w_1 w_2 ... w_n$ holds the reverse ballot condition, which means, for all k, the number of 1's in $w_k w_{k+1} ... w_n \geq$ number of 2's in $w_k w_{k+1} ... w_n \geq$ number of 3's in $w_k w_{k+1} ... w_n \geq$ and so on.

The goal today is to prove Littlewood-Richardson.

Lemma 13.2. *Let C be a component of $\Gamma(\lambda/\mu)$. Let α be a corner of μ. Then $T \mapsto slide(T, \alpha)$ defines an isomorphism between C and C', where C' is a component of $\Gamma(\lambda^-/\mu^-)$ for some $\lambda^- \Subset \lambda$ and $\mu^- \Subset \mu$.*

We note this isomorphism preserves content. In particular, by repeating this, we see C is isomorphic to $\Gamma(\mu)$ for some component of $\Gamma(\mu)$ for some partition μ.

Lemma 13.3. *$\Gamma(\nu)$ has a unique LR tableau. It is the unique SSYT of shape ν and content ν.*

This implies $\Gamma(\nu)$ is connected (if we have two components, we would have two LR tableaux, one for each component). It also implies $\Gamma(\nu)$ has no non-trivial automorphism. Now we consider the proof.

Proof. Suppose $T \in \mathrm{SSYT}(\nu)$ is LR. Consider rightmost entry in 1st row of T. By ballot condition it must be 1. So the whole first row must contain 1. Now go to the rightmost entry in the 2nd row. It must be a 2 (its either 1 or 2 but it cannot be 1 as we have a 1 above it), and so the whole second row must be 2. The proof follows by continue to do this for all rows. ♡

Use Lemma 13.2 and Lemma 13.3, we get the following theorem.

Theorem 13.4. *If C is a component of $\Gamma(\lambda/\mu)$, then:*

1. *There exists a partition ν_C such that C is isomorphic to $\Gamma(\nu_C)$.*
2. *The isomorphism is unique and preserves content.*
3. *C contains a unique LR tableau.*
4. *The isomorphism can be computed by performing rectification, in any order.*
5. *ν_C is the content of the unique LR tableau in C.*

Corollary 13.4.1. *Rectification is well-defined.*

Theorem 13.5 (Littlewood-Richardson for Skew Schur)**.**

$$s_{\lambda/\mu} = \sum_{\nu} (\#LR(\lambda/\mu, \nu)) s_\nu$$

Proof. By definition,

$$s_{\lambda/\mu} = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} x^{c(T)}$$

but then we can also sum this over all components of the crystal graph. Thus we get

$$s_{\lambda/\mu} = \sum_{C} \sum_{T \in V(C)} x^{c(T)}$$

where the first sum sum over all components C of $\Gamma(\lambda/\mu)$. Thus we see

$$s_{\lambda/\mu} = \sum_{C} \sum_{T \in \mathrm{SSYT}(\nu_C)} x^{c(T)} = \sum_{C} s_{\nu_C}$$

But then the # of times s_ν appear is the # of times ν_C equal the content of unique LR tableau in C which equal $\#LR(\lambda/\mu, \nu)$. ♡

Theorem 13.6 (Littlewood-Richardson for Product, V1)**.**

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

where $c^\lambda_{\mu\nu}$ is $\#LR(\lambda/\mu,\nu)$, which is also equal $\#LR(\lambda/\nu,\mu)$.

Proof. By the above theorem, we see

$$\begin{aligned}\langle s_{\lambda/\mu}, s_\nu\rangle &= \#LR(\lambda/\mu,\nu)\\ &= \langle s_\nu^\perp(s_\lambda), s_\nu\rangle\\ &= \langle s_\lambda, s_\mu s_\nu\rangle\end{aligned}$$

but s_λ form orthonormal basis and so $\langle s_\lambda, s_\mu s_\nu\rangle = c^\lambda_{\nu\mu}$ by definition. The proof follows. ♡

Theorem 13.7 (Littlewood-Richardson for Product, V2)**.** *For partitions μ,ν, define $\mu * \nu = (\nu_1+\mu_1, \nu_2+\mu_1, ..., \mu_1, \mu_2)/(\mu_1)^{\ell(\nu)}$, which is just we stack ν on top of μ just as in rectification, note $s_{\mu*\nu} = s_\mu s_\nu$. Then*

$$c^\lambda_{\mu\nu} = \#LR(\mu*\nu,\lambda)$$

where the $c^\lambda_{\mu\nu}$ are structure coefficients for s_λ.

Proof. We see $\#LR(\mu*\nu,\lambda) = \langle s_{\mu*\nu, s_\lambda}\rangle = \langle s_\mu s_\nu, s_\lambda\rangle$. ♡

To finish off, we give some other criterion for checking if a tableau is LR.

We have two ways to check at the moment:

1. $E_i(T) = \varnothing$ for all i.
2. reverse ballot condition for $w(T)$.
3. $(\epsilon \leftarrow w(T))$ is of shape ν and content ν for some ν.
4. $T \in LR(\lambda/\mu,\nu)$ iff there exists $T \in \mathrm{SSYT}(\lambda/\mu)$ and there exists $T^* \in \mathrm{SSYT}(\nu)$ such that the # of i's in row j of T is equal the number of j's in row i of T^*.

Example 13.8. Consider

$$T = \begin{array}{|c|c|c|}\hline \cdot & 1 & 1\\ \hline 1 & 2 & \\ \hline 3 & & \\ \hline\end{array}$$

Then T^* would be

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2\\ \hline 2 & & \\ \hline 3 & & \\ \hline\end{array}$$

This shows T is LR. On the other hand, if we let

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

then

$$T^* = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

T^* is not SSYT, hence T is not LR.

The next goal is to study the representation theory of S_n. This means we are going to cover the basic representation theory...

Definition 13.9. A ***representation*** of finite group G is a (finite-dimensional) vector space V over $\mathbb{C}$ with a linear G-action.

The above definition is not the most general one, but it is enough for the purpose of this course.

We note this is the same as a group homomorphism $\rho_V : G \to \mathrm{GL}(V)$.

Definition 13.10. Let ρ_V be a representation of G. A subspace $W \subseteq V$ is called a ***subrepresentation*** if W is $\rho_v(g)$-invariant for all $g \in G$. This is equivalent to for all $w \in W$ and $g \in G$, $gw \in W$.

Example 13.11.

1. 0 is a subrepresentation of any representation V.
2. V is a subrepresentation of any representation V.
3. $V^G = \{v \in V : gv = v \forall g \in G\}$ is a subrepresentation of V.

Definition 13.12. V is an ***irreducible representation*** if V has exactly 2 distinct subrepresentations 0 and V.

Example 13.13. The following are representations of S_n:

1. Permutation representation of S_n. Take $V = \mathbb{C}^n$ with linear action given by $\sigma v = M_\sigma v$ where M_σ is the permutation matrix correspond to σ (under basis $e_1, e_2, \dots, e_n$). This is the regular representation of S_n and in particular this is not irreducible. For example, $T = \mathrm{Span}(e_1 + e_2 + \dots + e_n)$ is a non-trivial subrepresentation. On the other hand, $T^\perp$, the orthogonal complement of T, is also a non-trivial subrepresentation.
2. Take $V = \mathrm{Mat}_{n\times n}(\mathbb{C})$ with linear action $\sigma A = M_\sigma A M_\sigma^{-1}$. This is again not irreducible. For example, the following are not irreducible:
   (a) diagonal matrices.
   (b) trace zero matrices

(c) symmetric matrices
(d) skew symmetric matrices

14 2021-11-04

Last time, we had some examples of representations, include the permutation representation of S_n and the conjugation representation. Next, we give more examples.

Example 14.1.

1. The sign representation: in this case $\mathbb{A} = \mathbb{C}^1$, and $\rho_{\mathbb{A}}(\sigma) = \operatorname{sgn}(\sigma)$, i.e. $\sigma \cdot x = (\operatorname{sgn}\sigma)x$. This is an irreducible representation.
2. Given any finite set $\mathfrak{X} = \{x_1, ..., x_n\}$ with an S_n-action, we get $\mathbb{C}\mathfrak{X}$ is the free vector space on $\mathfrak{X}$. This has a linear action $\rho_{\mathbb{C}\mathfrak{X}} : \mathbb{C}\mathfrak{X} \to \mathbb{C}\mathfrak{X}$ given by $\sigma(a_1 x_1 + ... a_n x_n) = a_1\sigma(x_1) + ... + a_n\sigma(x_n)$. This is a representation of S_n.

In particular, we see the action on $\mathbb{C}\mathfrak{X}$ in terms of matrices is just the permutation matrix of $\rho_{\mathfrak{X}}(\sigma) \in S_{\mathfrak{X}}$. In particular, if $x \in \mathfrak{X}, \sigma \in S_n$, then $x \in \mathfrak{X}^\sigma$ iff diagonal entry correspond to x in this matrix is equal 1.

In particular, this implies $\operatorname{Tr}(\rho_{\mathfrak{X}}(\sigma))$ is equal $|\mathfrak{X}^\sigma|$. We note $(\mathbb{C}\mathfrak{X})^{S_n}$ has a basis correspond to S_n orbits, i.e. if the set of orbits is given by $\tilde{\mathfrak{X}} = \{O_1, ..., O_k\}$, then consider $y_i = \sum_{y \in O_i} y$. Then $y_1, ..., y_k$ is a basis for $(\mathbb{C}\mathfrak{X})^{S_n}$. Hence, $\dim((\mathbb{C}\mathfrak{X})^{S_n}) = |\tilde{\mathfrak{X}}|$.

Since with each S_n action on fintie set gives a representation of S_n, we see all species give a representation of S_n as well. For example, take $\mathcal{S}$ be the species of permutations, then $\mathcal{S}_{[n]}$ is the set S_n with S_n-action by conjugation. Thus we get $\mathbb{C}\mathcal{S}_{[n]}$, which gives a representation of S_n. In particular $\dim((\mathbb{C}\mathcal{S}_n)^{S_n})$ is equal the number of conjugacy classes of S_n, which is equal the number of partitions of n.

Next, recall $\mathcal{H}^\lambda$, which is the species given by $\mathcal{H}^\lambda = \mathcal{E}_{\lambda_1} * ... * \mathcal{E}_{\lambda_{\ell(\lambda)}}$. We will view the elements of $\mathcal{H}_n^\lambda$ as tabloids. For example, if $\lambda = 4421$, then a tabloid is given by

4	5	8	9
1	3	6	10
2	7		
1	1		

where we forget the order in each row, i.e. we can freely swap the elements in the same row and remain the same tabloid. In this case, when we apply some permutation, we just swap the elements.

Definition 14.2. A ***tabloid*** is a filling of λ with $1, ..., n$, each used once, and the order within rows is irrelevant.

Some special cases of this includes:

1. $\lambda = (n)$, which gives the trivial representation.
2. $\lambda = 1^n$, which gives the left regular representation.
3. $\lambda = (n-1, 1)$, which is isomorphic to the permutation representation.

Let G be a finite group, let V be a representation of G.

Theorem 14.3. *Let V be a representation of G, then $V = V_1 \oplus V_2 \oplus \ldots \oplus V_r$ where V_i are irreducible subrepresentations.*

We note this is not true for infintie groups, and the irreducible components are not unique. To see it is false for infinite groups, consider $(\mathbb{Z}, +)$ with $V = \mathbb{C}^2$ given by $\rho_V(n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. V has three subrepresentations, namely 0, V and $\begin{bmatrix} * \\ 0 \end{bmatrix}$.

Lemma 14.4. *There exists a G-invariant Hermitian inner product $H_V(\cdot, \cdot)$ on V, i.e. $H_V(gv, gw) = H_V(v, w)$.*

Proof. Take any Hermitian inner product $\langle \cdot, \cdot \rangle$ on V, then define

$$H_V(v, w) = \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle$$

♡

Corollary 14.4.1. *$\rho_v(g)$ is unitary with respect to $H_V(\cdot, \cdot)$.*

Lemma 14.5 (Maschke's). *If $W \subseteq V$ is a subrepresentation, then so is $W^\perp = \{v \in V : \forall w \in W, H_V(v, w) = 0\}$.*

Proof. If $v \in W^\perp$, $g \in G$, then $H_v(gv, w) = H_v(v, g^{-1}w) = 0$ as $g^{-1}w \in W$ since W is G-invariant. Since this holds for all $w \in W$ we conclude $gv \in W^\perp$ as desired. ♡

Theorem 14.6. *Let V be a representation of G, then $V = V_1 \oplus V_2 \oplus \ldots \oplus V_r$ where V_i are irreducible subrepresentations.*

Proof. If V is irreducible we are done. If not, we can find a non-trivial subrepresentation V_1, then $V_1^\perp$ is also a non-trivial subrepresentation, with both smaller dimensions. Now use induction on the dimension we are done. ♡

Theorem 14.7. *The operator $\rho_V : V \to V$ given by $\rho_V(v) = \frac{1}{|G|} \sum_{g \in G} gv$ is a projection from V onto V^G.*

Proof. Not hard. ♡

Just like in linear algebra, we have many constructions for representations.

Example 14.8. Let V, W be two G-representations.

1. Direct sum: $V \oplus W$ with the natural linear G-action given by $g(v \oplus w) = (gv) \oplus (gw)$.
2. Dual space: V^* is a representation with G-action given by $g \cdot f : V \to \mathbb{C}$ via $v \mapsto f(g^{-1}v)$.
3. Tensor product: $V \otimes W$ with G-action given by $g(v \otimes w) = (gv) \otimes (gw)$ and extend by linearity.
4. Hom space: $\operatorname{Hom}(V, W) = W \otimes V^*$ with G-action given by $g\phi : V \to W$ via $v \mapsto g\phi(g^{-1}v)$. We note $(\operatorname{Hom}(V, W))^G$ are linear maps ϕ such that $g\phi = \phi$ for all g. In particular, we see

$$\begin{aligned} \phi(v) = g\phi(g^{-1}v) \quad \forall g \in G &\Leftrightarrow g^{-1}\phi(v) = \phi(g^{-1}v) \quad \forall g \in G \\ &\Leftrightarrow g\phi(v) = \phi(gv) \quad \forall g \in G \end{aligned}$$

Definition 14.9. Let V, W be G-representations. We call linear map $\phi : V \to W$ to be ***G-equivariant*** if for all g we have $g\phi(v) = \phi(gv)$. We denote the set of G-equivariant maps to be $\operatorname{Hom}_G(V, W)$.

Definition 14.10. A ***G-isomorphism*** is a G-equivariant homomorphism $\phi : V \to W$ which is invertible.

Theorem 14.11 (Schur's Lemma)**.** *Let V, W be irreducible representations.*

1. *If V, W are isomorphic, then* $\dim(\operatorname{Hom}_G(V, W)) = 1$.
2. *If V, W are not isomorphic, then* $\dim(\operatorname{Hom}_G(V, W)) = 0$.

Proof. Take $\phi \in \operatorname{Hom}_G(V, W)$. Then $\operatorname{Ker}\phi$ and $\operatorname{Im}\phi$ are subrepresentations of V and W, respectively. Since V, W are both irreducible, thus we either have $\operatorname{Ker}\phi = 0$ or $\operatorname{Ker}\phi = V$, and $\operatorname{Im}\phi = W$ or $\operatorname{Im}\phi = 0$. By rank-nullity theorem, we have either $\operatorname{Ker}\phi = 0$ and $\operatorname{Im}\phi = W$, or $\operatorname{Ker}\phi = V$ and $\operatorname{Im}\phi = 0$. The first case says ϕ is isomorphism, the second case says ϕ is the zero map. This proves 2.

To see (1), suppose $\psi : V \to W$ is G-equivariant isomorphism. Consider $\psi^{-1} \circ \phi : V \to V$. Any eigenspace of this map is a subrepresentation of V, but since V is irreducible, we only have one eigenspace, which is all of V. Hence $\psi^{-1} \circ \phi = \alpha I$ for some $\alpha \in \mathbb{C}\backslash 0$. Hence $\psi \in \operatorname{Span}(\phi)$, i.e. $\dim(\operatorname{Hom}_G(V, W)) = 1$ as desired. ♡

15 2021-11-09

We start with a list of properties of trace:

1. $\operatorname{Tr}(A) = \operatorname{Tr}(B^{-1}AB)$
2. $\operatorname{Tr}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \operatorname{Tr}(A) + \operatorname{Tr}(B)$

3. $\mathrm{Tr}(A^T) = \mathrm{Tr}(A)$
4. If A is unitary, then $\mathrm{Tr}(A^{-1}) = \overline{\mathrm{Tr}(A)}$
5. $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$
6. If $A^2 = A$ then $\mathrm{Tr}(A) = \mathrm{Rank}(A)$

Next, we recall last time we talked about Schur's lemma, which says, for irreducible representations, $\dim \mathrm{Hom}_G(V, W)$ is 1 or 0, depends on V and W are isomorphic or not.

If V, W are not irreducible, then we can sum V as irreducibles, and similarly for W. We can do this because $\mathrm{Hom}_G(\cdot,\cdot)$ is bilinear in the sense that

$$\mathrm{Hom}_G(V_1 \oplus V_2, W_1 \oplus W_2) = \bigoplus_{i,j} \mathrm{Hom}_G(V_i, W_j)$$

This clearly extend to any finite direct sums.

Why we care?

Consider the case V is irreducible and W is not irreducible. Then we know $W = \bigoplus_i W_i$ as a sum of irreducible (of course it is not unique). By what we did, we see $\dim \mathrm{Hom}_G(V, W) = \sum_i \dim \mathrm{Hom}_G(V, W_i) = k$ where k is the number of W_i isomorphic to V. Now note the left hand side does not depend on the particular decompostion, hence we see the right hand side does not depend on the particular decompostion either.

Thus, we conclude the irreducible decompostion of W is unique up to isomorphism. Hence, we proved the following.

Proposition 15.1. *Any two decompostion of W into irreducibles have same number of copies of each irreducible V, which is equal* $\dim \mathrm{Hom}_G(V, W)$.

Another consequence of the above is the following.

Proposition 15.2. *W is isomorphic to W' if and only if* $\dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}_G(V, W')$ *for all irreducible V.*

Proof. If $\phi : W \to W'$ is isomorphism then we see $\mathrm{Hom}_G(V, W) \to \mathrm{Hom}_G(V, W')$ given by $\psi \mapsto \phi \circ \psi$ is isomorphism. This proves $(\Rightarrow)$.

If $\dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}_G(V, W')$ for all irreducible V, we conclude they have the same number of copies of each irreducible, hence they are isomorphic. ♡

Proposition 15.3. *V is irreducible if and only if* $\dim \mathrm{Hom}_G(V, V) = 1$.

Proof. $(\Rightarrow)$: By Schur's lemma.

$(\Leftarrow)$: If $V = 0$ then $\dim \mathrm{Hom}_G(V, V) = 0$. If $V = V_1 \oplus V_2$ as sum of two irreducible then $\dim \mathrm{Hom}_G(V, V) = 1 + 1 = 2$. If V is more sum of irreducibles, the sum only gets bigger. ♡

Hence, $\dim \mathrm{Hom}_G(V,W)$ is a good quantity that we want to care about. There is only one problem left, as follows.

How do we compute this $\dim \mathrm{Hom}_G(V,W)$? This is not an easy task, e.g. if we want to look at rep of S_n, we are looking at a system of equations with $(n!)^2$ variables.

To that end, we use character theory.

So, what is character? Well, we have the analogy of subrepresentations and eigenvectors of matrices. To compute eigenvectors, it will be a lot easier if we know the eigenvalues. The characters are the analogy of eigenvalues to subrepresentations.

Definition 15.4. Let $F(G,\mathbb{C})$ be the vector space of functions $\alpha : G \to \mathbb{C}$. This is an inner product space with

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}$$

Definition 15.5. Let $\alpha \in F(G,\mathbb{C})$, we say α is ***class function*** if $\alpha(g) = \alpha(hgh^{-1})$ for all $g, h \in G$, i.e. it is constant on conjugacy classes. We use $CF(G,\mathbb{C})$ to denote the subspace of class functions.

For $G = S_n$, we have the conjugacy classes correspond to cycle types $\mathcal{C}_\mu$. Then, we see $\alpha \in CF(G,\mathbb{C})$ is the same as $\alpha(\mu) := \alpha(\sigma)$ where $\mu \vdash n$ are partitions and $\sigma \in \mathcal{C}_\mu$.

Definition 15.6. If (V,ρ) is G-representation, we define $\chi_V : G \to \mathbb{C}$ as

$$\chi_V(g) = \mathrm{Tr}(\rho_V(g))$$

We remark that even it seems like we have thrown away a lot of information, we still kept some essential information. For example, in assignment one, we had the following problem:

Proposition 15.7. *Let A be n by n $\mathbb{C}$-matrix, then*

$$\det(tI - A) = \sum_\lambda \frac{(-1)^{\ell(\lambda)} t^{n-|\lambda|}}{z_\lambda} \mathrm{Tr}(A^{\lambda_1})\,\mathrm{Tr}(A^{\lambda_2}) \ldots \mathrm{Tr}(A^{\lambda_{\ell(\lambda)}})$$

Hence, by knowing $\mathrm{Tr}(\rho_V(g))$ along, we also get the characterstic polynomials of $\rho_V(g)$.

Proposition 15.8 (Basic Properties of Characters)**.**

1. *If $V \cong_G W$ then $\chi_V = \chi_W$. The converse also holds but its harder to prove.*
2. *$\chi_V \in CF(G,\mathbb{C})$.*
3. *$\chi_V(\mathrm{Id}) = \dim(V)$.*

4. $\chi_{V\oplus W} = \chi_V + \chi_W$.
5. $\chi_{V^*}(g) = \overline{\chi_V(g)}$ for all $g \in G$.
6. $\chi_{V\otimes W} = \chi_V \cdot \chi_W$.
7. $\chi_{\mathrm{Hom}(V,W)} = \chi_W(g)\overline{\chi_V(g)}$ as we note $\mathrm{Hom}(V,W) \cong W \otimes V^*$.

Proof. All the proofs are basically translate the statement to linear algebra.

(1): Suppose $\phi : V \to W$ is isomorphism. Take basis (v_i) of V we get a basis $(\phi(v_i))$ of W. But then $[\rho_v(g)]_{(v_i)} = [\rho_w(g)]_{(\phi(v_i))}$ and hence take trace we are done.

(2):

$$\chi_V(g) = \mathrm{Tr}(\rho_V(g)) = \mathrm{Tr}(\rho_V(hgh^{-1})) = \mathrm{Tr}(\rho_V(h)\rho_V(g)\rho_V(h)^{-1}) = \mathrm{Tr}(\rho_V(g))$$

and we are done.

(3): $\chi_V(\mathrm{Id}) = \chi_V(\rho_G(\mathrm{Id})) = \mathrm{Rank}(\rho_G(\mathrm{Id})) = \dim(V)$

(4): Take matrix form, we get

$$\rho_{V\oplus W}(g) = \begin{bmatrix} \rho_V(g) & 0 \\ 0 & \rho_W(g) \end{bmatrix}$$

and take trace we are done.

(5): Take (v_i) as a basis of V, take (f_i) as the dual basis of V^*, i.e. $f_i(v_j) = \delta_{ij}$. Then we see

$$[\rho_{V^*}(g)]_{(f_i)} = ([\rho_V(g)]^{-1}_{(v_i)})^T$$

Thus the result follows as $\rho_{V^*}(g)$ is unitary.

(6): We note $\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g)$.

(7): We see $\mathrm{Hom}(V,W) \cong W \otimes V^*$, hence combine (5) and (6). ♡

Theorem 15.9.

$$\dim \mathrm{Hom}_G(V,W) = \langle \chi_W, \chi_V \rangle$$

Proof.

$$\begin{aligned}
\langle \chi_W, \chi_V \rangle &= \frac{1}{|G|}\sum_{g\in G} \chi_W(g)\overline{\chi_V(g)} \\
&= \frac{1}{|G|}\sum_{g\in G} \chi_{\mathrm{Hom}(V,W)}(g) \\
&= \frac{1}{|G|}\sum_{g\in G} \mathrm{Tr}(\rho_{\mathrm{Hom}(V,W)}(g)) \\
&= \mathrm{Tr}\left(\frac{1}{|G|}\sum_{g\in G} \rho_{\mathrm{Hom}(V,W)}(g)\right) \\
&= \mathrm{Tr}\left(P_{\mathrm{Hom}(V,W)}\right) \\
&= \mathrm{Rank}(P_{\mathrm{Hom}(V,W)}) \\
&= \mathrm{Rank}(\mathrm{Hom}(V,W)^G) = \dim \mathrm{Hom}_G(V,W)
\end{aligned}$$

♡

Proposition 15.10 (Advanced Properties of Characters)**.**

1. *If V, W are irreducible, then $\langle \chi_V, \chi_W \rangle = 1$ if $V \cong W$ and 0 otherwise.*
2. *V is irreducible iff $\langle \chi_V, \chi_V \rangle = 1$.*
3. *If $V_1, V_2, \ldots, V_k$ are distinct (non-isomorphic) irreducible representations, then $\chi_{V_1}, \ldots, \chi_{V_k}$ are orthonormal in $CF(G, \mathbb{C})$. In particular they are linearly independent.*
4. *$V \cong_G W$ iff $\chi_V = \chi_W$.*

Now we take $G = S_n$, since this is the group we care about. In this case, we see if α, β are class functions, then we see

$$\begin{aligned}\langle \alpha, \beta \rangle &= \frac{1}{n!} \sum_{\sigma \in S_n} \alpha(\sigma)\overline{\beta(\sigma)} \\ &= \frac{1}{n!} \sum_{\lambda \vdash n} |\mathcal{C}_\lambda| \alpha(\lambda)\overline{\beta(\lambda)} \\ &= \sum_{\lambda \vdash n} \frac{\alpha(\lambda)\overline{\beta(\lambda)}}{z_\lambda}\end{aligned}$$

This should look similar to power sum symmetric functions. Hence we will exploit that.

Thus, we will define the Frobenius character map as follows.

Definition 15.11. We define the ***Frobenius character map***

$$\mathrm{Frob} : CF(S_n, \mathbb{C}) \to \Lambda_n^{\mathbb{C}}$$

given by

$$\mathrm{Frob}(\alpha) = \sum_{\mu \vdash n} \alpha(\mu) \frac{p_\mu}{z_\mu}$$

A few things to note here.

First, we used $\Lambda_n^{\mathbb{C}}$ instead of the normal Λ_n with coefficients in $\mathbb{Q}$. However, it will not matter as we develop more theory.

Second, we have $\alpha(\mu) = \langle \mathrm{Frob}(\alpha), p_\mu \rangle$, i.e. we can recover class function α by know its $\mathrm{Frob}(\alpha)$.

Theorem 15.12. *Let $\mathfrak{X}$ be a finite set with an S_n-action with $\mathbb{C}\mathfrak{X}$ the S_n-representation. Then*

$$\mathrm{Frob}\, \chi_{\mathbb{C}\mathfrak{X}} = Z_{\mathfrak{X}}$$

Proof. Recall $\mathrm{Tr}\, \rho_{\mathbb{C}\mathfrak{X}}(\sigma) = |\mathfrak{X}^\sigma|$. But we also have $\mathrm{Tr}\, \rho_{\mathbb{C}\mathfrak{X}}(\sigma) = \chi_{\mathbb{C}\mathfrak{X}}(\sigma)$. Hence we see

$$\mathrm{Frob}(\chi_{\mathbb{C}\mathfrak{X}}) = \sum_{\mu \vdash n} \chi_{\mathbb{C}\mathfrak{X}}(\sigma) \frac{p_\mu}{z_\mu} = \sum_{\mu \vdash n} |\mathfrak{X}^\mu| \frac{p_\mu}{z_\mu} = Z_{\mathfrak{X}}$$

where $|\mathfrak{X}^\mu| = |\mathfrak{X}^\sigma|$ for any $\sigma \in \mathcal{C}_\mu$. ♡

16 2021-11-11

Last time we saw Frobenius character map, which is problematic because it maps from $CF(S_n, \mathbb{C})$ to $\Lambda_n^{\mathbb{C}}$, where we have the $\mathbb{C}$ stick on top of Λ_n. We want to get rid of this but that would mean we get have to get rid of $\mathbb{C}$ in $CF(S_n, \mathbb{C})$, but this makes character theory useless as we don't have eigenvalues anymore.

Definition 16.1. If V is an S_n representation, $\operatorname{Frob}\chi_V$ is called the ***Frobenius characterstic*** of V.

Last time, we saw if $\mathfrak{X}$ is a finite set with S_n-action, then $\operatorname{Frob}\chi_{\mathbb{C}\mathfrak{X}} = Z_{\mathfrak{X}}$.

Example 16.2. For tabloid representation $\mathcal{H}^\lambda$, we have

$$\operatorname{Frob}\chi_{\mathcal{H}^\lambda} = Z_{\mathcal{H}^\lambda} = h_\lambda$$

A special case of this is $\operatorname{Frob}\chi_{\mathcal{H}^{(n)}} = h_n$.

We note we may use $\mathcal{H}^\lambda$ and H^λ interchangeably (for normally I use $\mathcal{H}$ to mean a species, but for representations we use normal H, hence hence they seems to be mixed up at this point).

Theorem 16.3.

1. *There is one S_n-irreducible representation for every $\lambda \vdash n$, and no two are isomorphic.*
2. *Every S_n-irreducible representation is isomorphic to a subrepresentation of H^λ for some λ.*
3. *For every S_n-representation V, $\chi_V \in CF(S_n, \mathbb{Q})$, i.e. $\operatorname{Frob}\chi_V \in \Lambda_n$ with rational coefficients.*

Proof. We know there exists a decompostion $H^\lambda = W_1 \oplus \ldots \oplus W_l$ into irreducible representations. Thus $h_\lambda = \chi_{H^\lambda} = \chi_{W_1} + \ldots + \chi_{W_l}$. If $V_1, \ldots, V_k$ are the distinct irreducible representations of S_n, then h_λ is non-negative integer linear combinations of $\operatorname{Frob}\chi_{V_i}$.

But $\{h_\lambda\}$ form a basis for $\Lambda_n^{\mathbb{C}}$ and hence $\operatorname{Frob}\chi_{V_1}, \ldots, \operatorname{Frob}\chi_{V_k}$ span $\Lambda_n^{\mathbb{C}}$ as well. We also know they are linearly independent as χ_{V_i} are linearly independent. Thus $\{\operatorname{Frob}\chi_{V_i}\}$ is a basis and so the numbers match up, i.e. we have the $\#\{\lambda \vdash n\}$ many V_i. This proves 1.

To see (2), we note h_λ is a basis of $\Lambda_n^{\mathbb{C}}$. Hence if V_i is not a subrepresentation of H^λ for all $\lambda \vdash n$, then $\dim \operatorname{Hom}_{S_n}(V_i, H_\lambda) = 0 = \langle \chi_{V_i}, \chi_{H_\lambda} \rangle$ for all $\lambda \vdash n$. Hence we see $\chi_{H^\lambda} \in (\chi_{V_i})^\perp$, and hence χ_{H^λ} does not span $CF(S_n, \mathbb{C})$ and so h_λ does not span Λ_n, a contradiction.

For (3), we just saw that the change of basis matrix expresses h_λ in terms of $\operatorname{Frob}\chi_{V_i}$ has $\mathbb{Z}$-coefficients, hence the inverse change of basis matrix must have rational coefficients. Thus $\operatorname{Frob}\chi_{V_i}$ is $\mathbb{Q}$-linear combinations of h_λ's, and hence in Λ_n as desired. ♡

By this theorem, we see we don't really need to worry about $\mathbb{C}$ coefficients, i.e. they are automatically rational.

Theorem 16.4. *The* $\operatorname{Frob} : CF(S_n, \mathbb{Q}) \to \Lambda_n$ *is an isometric isomorphism, i.e.* $\langle \chi_W, \chi_V \rangle = \langle \operatorname{Frob} \chi_W, \operatorname{Frob} \chi_V \rangle$.

Proof. Check the definitions. Indeed, if α, β are class functions, then

$$\langle \alpha, \beta \rangle = \sum_{\mu \vdash n} \alpha(\mu) \frac{\beta(\mu)}{z_\mu}$$

In particular, we see

$$\begin{aligned} \langle \operatorname{Frob} \chi_W, \operatorname{Frob} \chi_V \rangle &= \langle \sum \alpha(\mu) \frac{p_\mu}{z_\mu}, \sum_\mu \beta(\mu) \frac{p_\mu}{z_\mu} \rangle \\ &= \sum_\mu \frac{\alpha(\mu)\beta(\mu)}{z_\mu} \end{aligned}$$

This proves Frob is an isometry, and since $\dim \Lambda_n = \dim CF(S_n, \mathbb{Q})$ (as $CF(S_n, \mathbb{Q})$ is equal the number of conjugacy classes and hence the number of partitions of n), we conclude Frob is an isomorphism. ♡

We want to create the irreducible representations of S_n, but before we do this, we need to introduce group algebra.

We know $\mathbb{C}[S_n]$ is the non-commutative algebra over $\mathbb{C}$. As vector space, it is $\mathbb{C}S_n$, the free vector space of S_n over $\mathbb{C}$. The multiplication is the unique multiplication thats bilinear and consistent with group multiplication of S_n, i.e. on basis we have $e_\alpha \cdot e_\beta = e_{\alpha\beta}$ where we use e_α, e_β as the elements of $\mathbb{C}S_n$. On arbitrary elements, we have

$$\left(\sum_{\sigma \in S_n} a_\sigma \sigma \right) \left(\sum_{\pi \in S_n} b_\pi \pi \right) = \sum_{\sigma \in S_n} \sum_{\pi \in S_n} a_\sigma b_\pi (\sigma\pi)$$

If V is an S_n-representation, extend $\rho_V : S_n \to \mathrm{GL}(V)$ to an algebra homomorphism $\rho_V : \mathbb{C}[S_n] \to \operatorname{End}(V)$ where $\operatorname{End}(V)$ is all linear maps from V to V. This can be done by setting $\rho_V(\sum_\sigma a_\sigma \sigma) = \sum_{\sigma \in S_n} a_\sigma \rho_V(\sigma)$.

In this way, V becomes a representation of $\mathbb{C}[S_n]$, which are exactly $\mathbb{C}[S_n]$-modules.

We will continue to write $av \in V$ to mean $\rho_V(a)(v)$, where $a \in \mathbb{C}[S_n]$.

Now its time to study irreducible representations of S_n, the Specht representations/modules.

Definition 16.5. A ***permutation tableau*** of shape λ is a tableau of shape λ with content $(1, 1, \ldots, 1, 0, 0, \ldots)$ where we have $|\lambda|$ many 1's, and we have no restriction on the rows and columns. The set of all permutation tableaux of shape λ is denoted by $\mathrm{PT}(\lambda)$.

Example 16.6. If $\lambda = 432$ then one permutation tableau would be

$$\begin{array}{|c|c|c|c|}\hline 3 & 7 & 2 & 6 \\ \hline 8 & 1 & 5 \\ \cline{1-3} 9 & 4 \\ \cline{1-2}\end{array}$$

Definition 16.7. To each $T \in \mathrm{PT}(\lambda)$, we define the ***Young subgroups*** as follows:

1. The ***row subgroup***:

$$A(T) = \{\sigma \in S_n : \forall i, \sigma(i) \text{ in the same row as } i\}$$

2. The ***column subgroup***:

$$B(T) = \{\sigma \in S_n : \forall i, \sigma(i) \text{ in the same column as } i\}$$

Definition 16.8. To each $T \in \mathrm{PT}(\lambda)$, we define the ***Young symmetrizers*** as follows

$$a_T := \sum_{\sigma \in A(T)} \sigma$$

$$b_T := \sum_{\sigma \in B(T)} \mathrm{sgn}(\sigma)\sigma$$

Definition 16.9. We define a ***tabloid*** $\{T\}$ as the tableau T but with each row are now considered as sets.

Definition 16.10. For $T \in \mathrm{PT}(\lambda)$, we define

$$v_T := b_T\{T\} \in H^\lambda$$

We note if $\sigma \in S_n$, $T \in \mathrm{PT}(\lambda)$, then $\sigma v_T = v_{\sigma T}$. Hence

$$\{v_T : T \in \mathrm{PT}(\lambda)\}$$

is an orbit of S_n.

Definition 16.11. We define the ***Specht representation***

$$S^\lambda := \mathrm{Span}\{v_T : T \in \mathrm{PT}(\lambda)\} \subseteq H^\lambda$$

Theorem 16.12. *S^λ is irreducible.*

Lemma 16.13. *Let $T \in \mathrm{PT}(\lambda)$, $U \in \mathrm{PT}(\mu)$, where $\mu, \lambda \vdash n$. If there exists $i \neq j$ in the same row of U and the same column of T, then $b_T\{U\} = 0$.*

Proof. We can write

$$b_T = \left(\sum_{\substack{\sigma \in B(T) \\ \mathrm{sgn}(\sigma)=1}} \sigma \right) (\mathrm{Id} - (i,j))$$

but $(i,j)\{U\} = \{U\}$, thus we see

$$b_T\{U\} = \left(\sum_{\substack{\sigma \in B(T) \\ \mathrm{sgn}(\sigma)=1}} \sigma \right) (\mathrm{Id} - (i,j))\,\{U\} = 0$$

as $(\mathrm{Id} - (i,j))\{U\} = 0$. ♡

Lemma 16.14. *Let $T \in \mathrm{PT}(\lambda), U \in \mathrm{PT}(\mu)$, with $\mu, \lambda \vdash n$.*

1. *If $\mu \not\trianglelefteq \lambda$ in dominance order. Then there exists $i \neq j$ in same row of U and same column of T.*
2. *If $\mu = \lambda$ then either there exists i, j in same row of U and same column of T, or there exists $\alpha \in A(U)$ and $\beta \in B(T)$ such that $\beta T = \alpha U$.*

Proof. Exercise. ♡

Lemma 16.15. *If $T \in \mathrm{PT}(\lambda)$, then for $W \in H^\lambda$, $b_T W$ is a multiple of v_T and $b_T v_T \neq 0$, i.e. b_T acting on H^λ is a rank one map onto $\mathrm{Span}\{v_T\}$.*

17 2021-11-16

From last time, we have $T \in \mathrm{PT}(\lambda)$, $U \in \mathrm{PT}(\mu)$, with $\lambda, \mu \vdash n$. Then, it might happen that there exists $i \neq j$ in same column of T and same row of U. We call this the star condition.

If this star condition holds, then $b_T\{U\} = 0$.

Next, we showed if $\mu \not\trianglelefteq \lambda$ then star condition holds, and if $\mu = \lambda$, either star condition holds or we can find $\alpha \in A(U), \beta \in B(T)$ and $\alpha U = \beta T$.

The last lemma we stated last time was, for all $W \in H^\lambda$, then $b_T W \in \mathrm{Span}(v_T)$, moreover, $b_T V_T \neq 0$.

More precisely, we have the following.

Lemma 17.1. *If $T \in \mathrm{PT}(\lambda)$, then for $W \in H^\lambda$, $b_T W$ is a multiple of v_T and $b_T v_T \neq 0$, i.e. b_T acting on H^λ is a rank one map onto* $\mathrm{Span}\{v_T\}$.

Proof. It suffices to prove the first claim for $W = \{U\}$, $U \in \mathrm{PT}(\lambda)$. If star condition holds, then by Lemma 16.13, we see $b_T\{W\} = 0$ and hence we are done. Otherwise, by Lemma 16.14 there exists $\alpha \in A(U)$ and $\beta \in B(T)$ such that $\alpha U = \beta T$. But then $\{U\} = \{\alpha U\} = \{\beta T\} = \beta\{T\}$. Thus we see

$$b_T\{U\} = b_T\beta\{T\} = \mathrm{sgn}(\beta) b_T\{T\}$$

Hence we see $b_T\{U\} = \mathrm{sgn}(\beta)b_T\{T\} = \mathrm{sgn}(\beta)v_T \in \mathrm{Span}(v_T)$.

Finally,

$$\begin{aligned}
b_T v_T &= \left(\sum_{\sigma\in B(T)} \mathrm{sgn}(\sigma)\sigma\right) \\
&= \left(\sum_{\sigma\in B(T)} \mathrm{sgn}(\sigma)\sigma\right)\left(\sum_{\pi\in B(T)} \mathrm{sgn}(\pi)\pi\{T\}\right) \\
&= \sum_{(\sigma,\pi)\in B(T)^2} \mathrm{sgn}(\sigma\pi)\sigma\pi\{T\} \\
&= \sum_{\sigma\in B(T)}\sum_{\tau\in B(T)} \mathrm{sgn}(\tau)\tau\{T\} \\
&= \sum_{\sigma\in B(T)} v_T \\
&= |B(T)|v_T \neq 0
\end{aligned}$$

as desired. ♡

So, in summary, we see all the three lemmas tells us how does b_T acts on H^μ

1. If $\mu \leq \lambda$ then b_T acts as 0.
2. If $\mu = \lambda$ then $\frac{b_T}{|B(T)|}$ is a rank 1 projection onto $\mathrm{Span}(v_T)$.
3. If $\mu < \lambda$ then $\frac{b_T}{|B(T)|}$ is a projection, but we don't know its rank.

Theorem 17.2. *S^λ is irreducible.*

Proof. Suppose to the contrary that $S^\lambda = W \oplus W'$, with $W, W' \neq \{0\}$ are subrepresentations. Pick any $T \in \mathrm{PT}(\lambda)$. Then $v_T = W + W'$ where $W \in W$ and $W' \in W'$. Then, apply b_T to v_T, we see $0 \neq b_T v_T = b_T W + b_T W'$. But $b_T W$ and $b_T W'$ are both multiple of v_T. WLOG we may assume $b_T W = c v_T$ with $c \neq 0$. But S^λ is the smallest subrepresentation of H^λ that contains v_T, hence $W = S^\lambda$ and so $W' = \{0\}$, a contradiction. ♡

Theorem 17.3. *If $\mu \ntrianglelefteq \lambda$, then* $\mathrm{Hom}_{S_n}(S^\lambda, H^\mu) = 0$.

Proof. Suppose to the contrary, we can find $\phi \in \operatorname{Hom}_{S_n}(S^\lambda, H^\mu)$ with $\phi \neq 0$. Then $\phi(v_T) \neq 0$ for some $T \in \mathrm{PT}(\lambda)$. Since ϕ is S_n-equivariant, so

$$b_T \phi(v_T) = \phi(b_T v_T) \neq 0$$

by Lemma 17.1 but we also have $b_T \phi(v_T) = 0$ by Lemma 16.13. ♡

Now, we showed S^λ are irreducible, but what is even the dimension of this? What about its character? Well, we do Gram-Schmidt[1].

Theorem 17.4 (Gram-Schmidt)**.** *Let V be a finite dimensional inner product space, with ordered basis (v_i). Then, there exist a unique basis $(w_1, \ldots, w_n)$ with the following properties:*

1. $\langle w_i, w_j \rangle = \delta_{ij}$.
2. $\langle v_i, w_j \rangle = 0$ *if* $i < j$.
3. $\langle v_i, w_i \rangle > 0$.

Next, we set up the following notations:

$$\rho^\lambda := \rho_{S^\lambda} : \mathbb{C}[S_n] \to \operatorname{End}(S_\lambda)$$

$$\chi^\lambda := \chi_{S^\lambda}$$

Theorem 17.5.

$$\operatorname{Frob} \chi^\lambda = s_\lambda$$

In other word, the Frobenius character map sends Specht representations to Schur functions.

Proof. Take $\{h_\lambda : \lambda \vdash n\}$ and order it in lexicographically decreasing order. This is an ordered basis and we can apply Gram-Schmidt to it.

Claim 1: $\{s_\lambda : \lambda \vdash n\}$ is the basis you get from apply Gram-Schmidt to $\{h_\lambda : \lambda \vdash n\}$. To prove this, we only need to check the three conditions above.

First, we see

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$$

before. Second, suppose $\mu >_{lex} \lambda$, then we see $\langle h_\mu, s_\lambda \rangle = K_{\lambda\mu} = 0$ by its combinatorial interpretation, since you cannot fill any SSYT of shape λ with content μ if $\mu >_{lex} \lambda$ (too many contents to fill). Third, we see $\langle h_\lambda, s_\lambda \rangle = 1 > 0$.

This proves the claim 1.

[1]Two nice properties of Gram-Schmidt: first, it is a continuous algorithm in the sense that if you have a family of continuously varying family of bases, then Gram-Schmidt returns a continuous family. Second, it has the "grabage in grabage out" property. This means, if you feed it with grabage input, you are guaranteed to get grabage output. This can be used to detect if your basis is orthogonal or not, i.e. if they are not, then apply Gram-Schmidt would lead to divide by 0 or some non-sense at some point.

Claim 2: $\operatorname{Frob}\chi^\lambda$ is the Gram-Schmidt basis obtained from apply GS to $\{h_\lambda : \lambda \vdash n\}$.

We check the three conditions. First, we see

$$\langle \operatorname{Frob}\chi^\lambda, \operatorname{Frob}\chi^\mu \rangle = \dim \operatorname{Hom}_{S_n}(S^\lambda, S^\mu) = \delta_{\lambda\mu}$$

by Schur's lemma. Next, we see

$$\langle h_\mu, \operatorname{Frob}\chi^\lambda \rangle = \dim \operatorname{Hom}_{S_n}(S^\lambda, H^\mu) = 0 \quad \text{if } \mu \ntrianglelefteq \lambda$$

In particular this implies

$$\langle h_\mu, \operatorname{Frob}\chi^\lambda \rangle = 0$$

if $\mu >_{lex} \lambda$. Finally, we see

$$\langle h_\lambda, \operatorname{Frob}\chi^\lambda \rangle = \dim \operatorname{Hom}_{S_n}(S^\lambda, H^\lambda) \geq 1$$

since S^λ is a subrepresentation of H^λ.

Now by uniqueness of Gram-Schmidt basis, the claim of the theorem holds. ♡

Corollary 17.5.1.

$$\chi^\lambda(\mu) = \langle s_\lambda, p_\mu \rangle$$

Corollary 17.5.2.

$$\dim S^\lambda = f^\lambda = \#\operatorname{SYT}(\lambda)$$

Proof.

$$\dim S^\lambda = \langle s_\lambda, p_{1^n} \rangle = \langle s_\lambda, h_{1^n} \rangle = f^\lambda$$

♡

Theorem 17.6. *$\{v_T : T \in \operatorname{SYT}(\lambda)\}$ is a basis of S^λ.*

Proof. We know the dimension matches. Thus we just need to either show span or linearly independent. We will show linearly independent. The point is, $v_T = \{T\} +$ other terms, and we want to argue $\{T\}$ is the "leading term" and the other terms are of lower order.

To do this, we put a lexicographical order on tabloids: from tabloid U, we get a permutation tableau by putting each row of U in increasing order, then we look at the row word $w(U)$. Thus, a lexicographical order on tabloids is given by compare the row word of U.

With this lexicographical order, we conclude linearly independent as desired. ♡

The next theorem is about the $\mathbb{C}[S_n]$-case of Artin-Wedderburn. To that end, we know $\rho^\lambda : \mathbb{C}[S_n] \to \mathrm{End}(S^\lambda)$, and hene we can define $\rho : \mathbb{C}[S_n] \to \bigoplus_{\lambda \vdash n} \mathrm{End}(S^\lambda)$.

Consider an example. We know S_3 has three irreducible representations, say S^3, S^{21} and S^{111}, where they have dimensions $1, 2, 1$, respectively. Then, by the Artin-Wedderburn, we would know

$$\mathbb{C}[S_3] \cong \mathrm{Mat}_{1\times 1}(\mathbb{C}) \oplus \mathrm{Mat}_{2\times 2}(\mathbb{C}) \oplus \mathrm{Mat}_{1\times 1}(\mathbb{C})$$

The following is the precise statement about $\mathbb{C}[S_n]$, and we remark this actually holds for general finite groups G.

Theorem 17.7 (Wedderburn Cecompostion)**.** *Define $\rho : \mathbb{C}[S_n] \to \bigoplus_{\lambda \vdash n} \mathrm{End}(S^\lambda)$ to be the induced map from $\rho^\lambda : \mathbb{C}[S_n] \to \mathrm{End}(S^\lambda)$. Then ρ is an $\mathbb{C}$-algebra isomorphism.*

Proof. First, we show this is injective. Suppose $a \in \mathbb{C}[S_n]$ with $a \in \operatorname{Ker} \rho$. Then $\rho^\lambda(a) = 0$ for all λ and we know every representation is a direct sum of irreducible representations, hence direct sum of S^λ's. Thus $\rho_V(a) = 0$ for every S_n-representation V. Now take $V = \mathbb{C}[S_n]$ to be the regular representation (with action being left-multiplication), we conclude $a = 0$ as desired, since this means $\rho_V(a) = 0$ and so $\rho_V(a)b = 0$ for all $b \in V$ and so $ab = 0$ for all $b \in V$ but take $b = 1 \cdot \mathrm{Id}$ we get $a = 0$. This shows $\operatorname{Ker} \rho = \{0\}$.

Next, we show surjective. To show this, we note $\dim(\bigoplus_{\lambda \vdash n} \mathrm{End}(S^\lambda)) = n! = \dim \mathbb{C}[S_n]$. Indeed, we see

$$\dim(\bigoplus_{\lambda \vdash n} \mathrm{End}(S^\lambda)) = \sum_{\lambda \vdash n} \dim(\mathrm{End}(S^\lambda)) = \sum_{\lambda \vdash n} (f^\lambda)^2 = n!$$

where the last equality is by Robinson-Schensted. Since the dimension matches, and it is injective, we conclude it is surjective as well. ♡

The next topic is going to be Frobenius reciprocity for S_n, i.e. the restriction to subgroups of S_n.

Suppose $G \subseteq S_n$ is a subgroup of S_n, then if V is S_n-representation, then we get $\rho_V : S_n \to \mathrm{GL}(V)$. But then V is also a G-representation by consider $\rho_V|_G : G \to \mathrm{GL}(V)$. In particular, we will write $V|_G$ to mean the representation of G obtained by restriction of the representation $\rho : S \to \mathrm{GL}(V)$.

What can we say about those? Well, we can compute their characters. Indeed,

$$\chi_{V|_G}(g) = \chi_V(g)$$

where the latter is happening in S_n.

To put more details to it, if we want to compute $\chi_{V|_G}$, for $g \in G$, we can embed $g \in S_n$. THus we can determine its cycle type in S_n, say μ. Then we can compute

$$\langle \mathrm{Frob}\, \chi_V, p_\mu \rangle = \chi_V|_G(g) = \chi_V(g)$$

The first example we want to consider is $G = S_{n-1} \leq S_n$. We see we can view S_{n-1} as $\{\sigma \in S_n : \sigma(n) = n\} \leq S_n$.

Thus, let's ask, fix $\lambda \vdash n$, and consider $V = S^\lambda$. How does $V|_{S_{n-1}}$ decompose into irreducible representations of S_{n-1}?

To do this, we want to compute the character. Suppose $\sigma \in S_{n-1} \subseteq S_n$, then σ has cycle type $\mu 1$ (this means add a part of size 1 to partition μ) in S_n, if σ has cycle type μ in S_{n-1}, if you think about it.

Thus we see

$$\begin{aligned}
\chi_{S^\lambda|_{S_{n-1}}}(\sigma) &= \langle \operatorname{Frob} \chi^\lambda, P_{\mu 1} \rangle \\
&= \langle s_\lambda, p_\mu p_1 \rangle \\
&= \langle s_\lambda, p_\mu s_1 \rangle \\
&= \langle s_{\lambda/1}, p_\mu \rangle
\end{aligned}$$

Thus, we conclude

$$\operatorname{Frob} \chi_{S^\lambda|_{S_{n-1}}} = s_{\lambda/1} = \sum_{\mu \Subset \lambda} s_\mu$$

This gives proof to the following theorem.

Theorem 17.8 (Branching Formula)**.**

$$S^\lambda|_{S_{n-1}} \cong_{S_{n-1}} \bigoplus_{\mu \Subset \lambda} S^\mu$$

18 2021-11-18

Last time, we ended at the branching formula. One application of this formula is to obtain a better basis of S^λ. The meaning of better here is, we end up with a orthonormal basis (as v_T is not orthonormal). To do this, we break S^λ into smaller pieces, where in those smaller spaces we use induction to find the basis, and put them back together, we obtain a orthonormal basis of S^λ. The basis obtained this way is the Young's orthonormal basis.

Next, we consider the subgroup $A(T)$. Take $T \in \mathrm{PT}(\lambda)$, $\lambda \vdash n$, recall we have $G = A(T) \subseteq S_n$, which is the row subgroup. In this case, we can ask, for S_n-rep V, what is $\dim(V^G)$? The answer is

$$\dim(V^G) = \langle \operatorname{Frob} \chi_V, h_\lambda \rangle$$

This is cool because it tells us the coefficient of Frobenius character in monomial basis, which is exactly $\dim(V^G)$.

Next, we consider the subgroup $G = S_k \times S_l$ with $k + l = n$. This is a subgroup of S_n. The irreducible representations for $S_k \times S_l$ would be $S^\mu \otimes S^\nu$ where $\mu \vdash k$ and

$\nu \vdash l$. Thus, the question is, given S^λ with $\lambda \vdash n$, what is

$$\dim \operatorname{Hom}_G(S^\mu \otimes S^\nu, S^\lambda|_G)$$

The answer is $c^\lambda_{\mu\nu}$, i.e. the Littlewood-Richardson numbers. The sketch of proof is given as follows. We can define Frobenius characterstic for products of symmetric groups. For $S_k \times S_l$, the Frobenius characterstic $\operatorname{Frob}\chi_V$ will be in $\Lambda_k \otimes \Lambda_l$ (symmetric functions with two set of variables). In this case, we would get

$$\operatorname{Frob}\chi_{S^\mu \otimes S^\nu} = s_\mu(x) s_\nu(y)$$

On the other hand, if we want to compute $\chi_{S^\lambda|_G}$, we get

$$\chi_{S^\lambda|_G}(\mu, \nu) = \langle s_\lambda, p_\mu p_\nu \rangle$$

Thus

$$\operatorname{Frob}\chi_{S^\lambda|_G} = \sum_{\mu,\nu} \langle s_\lambda, p_\mu p_\nu \rangle \frac{p_\mu(x)}{z_\mu} \frac{p_\nu(y)}{z_\nu}$$

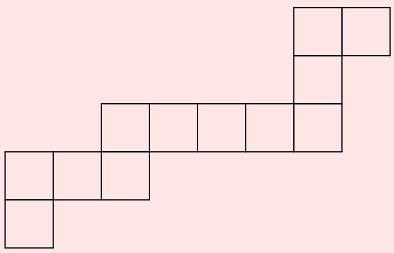
But, by magic of linear algebra, if we see a pair of basis and dual basis, we can change this to any other pair of basis and dual basis, thus we will change $p_\nu, \frac{p_\nu}{z_\nu}$ to s_ν, s_ν and so on. Hence we see

$$\operatorname{Frob}\chi_{S^\lambda|_G} = \sum_{\mu,\nu} \langle s_\lambda, s_\mu s_\nu \rangle s_\mu(x) s_\nu(y)$$

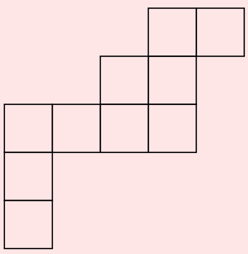
and this concludes the proof.

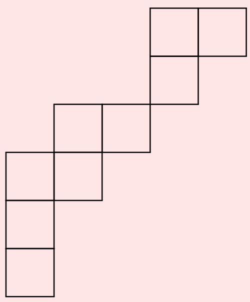
This concludes our study about subgroups of S_n, for now, and we go back to the topic of study the quantity $\chi^\lambda(\mu) = \langle s_\lambda, p_\mu \rangle$. One way to do this is called the Murnaghan-Nakayama rule.

We will look at partitions of shape similar to the following:



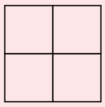
The partitions of the following are now allowed:





To say this formally, we look at skew shapes λ/μ in which:

1. No box shape



2. It is connected

In the above non-examples, the left one contains a box shape, and the right one is not connected. They have many names:

1. ***rim-hooks***
2. ***connected ribbons***
3. ***skew hooks***

We write $\mu \xrightarrow[rim]{k} \lambda$ to denote λ/μ is a rim-hook of size k. In this case, we use $\operatorname{sgn}(\lambda/\mu)$ to denote the number $(-1)^{(\#\text{ rows}(\lambda/\mu))-1}$, where $\operatorname{row}(\lambda/\mu)$ denote the number of non-empty rows in λ/μ.

Theorem 18.1 (Murnaghan-Nakayama Rule)**.**

$$s_\mu p_k = \sum_{\mu \xrightarrow[rim]{k} \lambda} \operatorname{sgn}(\lambda/\mu) s_\lambda$$

Proof. We see

$$\langle s_\lambda, s_\mu p_k \rangle = \begin{cases} \operatorname{sgn}(\lambda/\mu), & \text{if } \mu \xrightarrow[rim]{k} \lambda \\ 0, & \text{otherwise} \end{cases}$$

But we also have

$$\langle s_\lambda, s_\mu p_k \rangle = \langle s_\mu^\perp(s_\lambda), p_k \rangle = \langle s_{\lambda/\mu}, p_k \rangle = \langle s_{\lambda/\mu}, m_k \rangle$$

Thus it suffices to prove the equation

$$\langle s_{\lambda/\mu}, m_k \rangle = \begin{cases} \operatorname{sgn}(\lambda/\mu), & \text{if } \mu \xrightarrow[rim]{k} \lambda \\ 0, & \text{otherwise} \end{cases}$$

To do this, we use the Jacobi Trudi formula.

We may assume $\lambda_1 > \mu_1$ and $\lambda_\ell > \mu_\ell$, where ℓ is the length of both partitions. Also, we note if λ/μ is not connected, then $s_{\lambda/\mu} = s_{\lambda'/\mu'} \cdot s_{\lambda''/\mu''}$, where both $s_{\lambda'/\mu'}$ and $s_{\lambda''/\mu''}$ are linear combinations of h's, and when we multiply the two together, all the h's in the product would have at least to parts, and so we cannot pick up any single h_k. Thus in this case $\langle s_{\lambda/\mu}, m_k \rangle = 0$, and so we can assume λ/μ is connected.

In this case, we use Jacobi-Trudi and we see

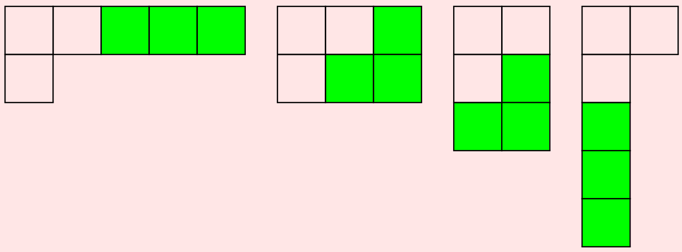
$$s_{\lambda/\mu} = \det\left(h_{\lambda_i - i - \mu_j + j}\right) = \det \begin{bmatrix} * & & & & & \\ * & * & & & & \\ & * & * & & & \\ & & * & * & & \\ & & & \ddots & \ddots & \\ & & & & * & * \end{bmatrix}$$

where the $*$ are all non-zero entries, plus the upper parts above the diagonal are also non-zero. Thus, there is only one way we can get h_k in such expansion, which is

$$\det\begin{bmatrix} * & & & & & h_k \\ 1 & * & & & & \\ & 1 & * & & & \\ & & 1 & * & & \\ & & & \ddots & \ddots & \\ & & & & 1 & * \end{bmatrix}$$

This immediately concludes λ/μ is rim-hook (of size k). But then we are done because now we know what the sign we are picking up, which is exactly the sign $\operatorname{sgn}(\lambda/\mu)$. ♡

Example 18.2. Take s_{21} and p_3, then we have four partitions, namely



where the green part is the rim-hook we are adding. This gives

$$s_{21}p_3 = s_{2,1,1,1,1} - s_{2,2,2} - s_{3,3} + s_{5,1}$$

Definition 18.3. A ***Murnaghan-Nakayama tableau*** of shape λ/μ is a filling of λ/μ in which entries equal i form a rim-hook and rows/columns weakly increasing. We use $MN(\lambda/\mu,\nu)$ to denote the set of all Murnaghan-Nakayama tableaux of shape λ/μ and content ν.

Theorem 18.4.

$$\langle s_{\lambda/\mu}, p_\nu\rangle = \sum_{T\in MN(\lambda/\mu,\nu)} \operatorname{sgn}(T)$$

where $\operatorname{sgn}(T)$ *is the product of signs of all the rim-hooks in* T.

Example 18.5. Consider the example

$$\langle s_{321}, p_{33}\rangle$$

We have two MN tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 & 2 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

They both have negative sign, and hence the answer should be -2.

Example 18.6. This time consider

$$\langle s_{321}, p_{321}\rangle$$

The MN tableaux are

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&3\\\cline{1-2} 2\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 1&3\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 1&2\\\cline{1-2} 1\\\cline{1-1}\end{array}$$

The signs are $+,-,+,-$ and hence the total sum is 0.

This is an unfortunate way to get 0. But, if we think about how we get the proof, we note it does not matter what order we are multiplying with. Thus, we must have

$$\langle s_{321}, p_{321}\rangle = \langle s_{321}, p_3p_1p_2\rangle$$

But, there are clearly no MN tableaux with content 312, thus we are done immediately. Thus, in practice, we can write p_λ in different order will often simplify the computation.

The next topic in this representation theory part of the course will be center of the group algebra. Clearly $Z(\mathbb{C}[S_n]) := \{a \in \mathbb{C}[S_n] : ab = ba \forall b \in \mathbb{C}[S_n]\}$. To check a commute with everything, it suffices to check $b = \sigma$ on the basis of $\mathbb{C}[S_n]$, and since $\sigma \in S_n$, we see $ab = ba$ for all $b \in \mathbb{C}[S_n]$ iff $\sigma a \sigma^{-1} = a$ for all $\sigma \in S_n$.

Well, this is just the fixed points of conjugation representation $\mathbb{C}\mathcal{S}_n$, where $\mathcal{S}_n$ is the species of permutations.

From this perspective, we get a natural basis

$$C_\mu = \sum_{\sigma \in \mathcal{C}_\mu} \sigma$$

But we also have isomorphism

$$\rho : \mathbb{C}[S_n] \to \bigoplus_{\lambda \vdash n} \mathrm{End}(S^\lambda)$$

where we know there is only one kind of element that commute with all matrices, namely $c\,\mathrm{Id}$ with $c \in \mathbb{C}$, i.e. we know $Z(\mathrm{End}(S^\lambda)) = \mathrm{Span}\{\mathrm{Id}_{S^\lambda}\}$. This gives another basis of $Z(\mathbb{C}[S_n])$, namely

$$F_\theta \in \bigoplus_{\lambda \vdash n} \mathrm{End}(S^\lambda), \quad \theta \vdash n$$

where F_θ acts as identity on S^λ if $\lambda = \theta$, and acts as 0 if $\lambda \neq \theta$. In other word, for $v \in S^\lambda$, we have

$$F_\theta v = \begin{cases} v & \text{if } \theta = \lambda \\ 0 & \text{otherwise} \end{cases}$$

A few facts about those two.

$$F_\lambda F_\theta = \begin{cases} 0 & \text{if } \lambda \neq \theta \\ F_\theta & \text{if } \lambda = \theta \end{cases}$$

19 2021-11-23

Today we finish the representation part of the course. Last time we talked about two bases of $Z(\mathbb{C}[S_n])$. The reason we want to consider both is that, F_θ is good to do algebra, while C_μ is good to do combinatorics. Thus, we need change of basis formulae.

Theorem 19.1 (Change of Basis).

1.
$$C_\mu = \sum_{\lambda \vdash n} \frac{n!}{z_\mu f^\lambda} \chi^\lambda(\mu) F_\lambda$$

2.
$$F_\lambda = \sum_{\mu \vdash n} \frac{f^\lambda}{n!} \chi^\lambda(\mu) C_\mu$$

Proof. Consider the action of C_μ on S^λ. Since $C_\mu \in Z(\mathbb{C}[S_n])$, it acts as a multiple of the identity I_{S^λ}. To see which multiple we are get, we compute its trace $\operatorname{Tr} \rho^\lambda(C_\mu)$. We see

$$\begin{aligned} \operatorname{Tr} \rho^\lambda(C_\mu) &= \operatorname{Tr}\left(\sum_{\sigma \in \mathcal{C}_\mu} \rho^\lambda(\sigma) \right) \\ &= \sum_{\sigma \in \mathcal{C}_\mu} \operatorname{Tr} \rho^\lambda(\sigma) \\ &= \sum_{\sigma \in \mathcal{C}_\mu} \chi^\lambda(\sigma) \\ &= |\mathcal{C}_\mu| \chi^\lambda(\mu) \\ &= \frac{n!}{z_\mu} \chi^\lambda(\mu) \end{aligned}$$

Therefore, $\rho^\lambda(C_\mu) = \frac{1}{f^\lambda}\left(\frac{n!}{z_\mu}\chi^\lambda(\mu)\right)$ where f^λ is the dimension of S^λ. Thus the proof follows for (1).

For (2), we just need to check this is the inverse of (1). Thus we need to show

$$\sum_{\lambda \vdash n} \left(\frac{n!}{z_\mu f^\lambda} \chi^\lambda(\mu) \right) \cdot \left(\frac{f^\lambda}{n!} \chi^\lambda(\nu) \right) = \delta_{\mu\nu}$$

Thus, we see we get

$$\begin{aligned}\sum_{\lambda\vdash n}\left(\frac{n!}{z_\mu f^\lambda}\chi^\lambda(\mu)\right)\cdot\left(\frac{f^\lambda}{n!}\chi^\lambda(\nu)\right) &= \sum_{\lambda\vdash n}\frac{\chi^\lambda(\mu)\chi^\lambda(\nu)}{z_\mu}\\ &= \sum_{\lambda\vdash n}\frac{\langle s_\lambda,p_\mu\rangle\langle s_\lambda,p_\nu\rangle}{z_\mu}\\ &= \frac{1}{z_\mu}\langle p_\mu,p_\nu\rangle\\ &= \begin{cases}1, & \text{if } \mu=\nu\\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

This concludes the proof of (2). ♡

Now we consider applications.

Suppose we want to count permutations with a given product. Let $\mu^1,\mu^2,\ldots,\mu^k\vdash n$ be k partitions, with $\gamma\in S_n$. We want to ask, how many k-tuples $(\sigma_1,\ldots,\sigma_k)$ with $\sigma_i\in\mathcal{C}_{\mu^i}$ such that $\sigma_1\sigma_2\ldots\sigma_k=\gamma$.

The generating function answer is $C_{\mu^1}C_{\mu^2}\ldots C_{\mu^k}$. This is because C_{μ^i} is the sum of all permutations with cycle type μ^i, and hence the product

$$C_{\mu^1}C_{\mu^2}\ldots C_{\mu^k}=\sum_{(\sigma_1,\ldots,\sigma_k)\in\mathcal{C}_{\mu^1}\times\ldots\times\mathcal{C}_{\mu^k}}\sigma_1\ldots\sigma_k$$

gives the number of $\sigma_1\ldots\sigma_k=\gamma$ if we count the coefficient of γ appear in $C_{\mu^1}\ldots C_{\mu^k}$. But this is in the center of the element, hence the number of γ is the same as the number of $C_{\lambda(\gamma)}$ appearing in the product, where $\lambda(\gamma)$ is the cycle type of γ. In other word, the actual answer is

$$[C_{\lambda(\gamma)}]C_{\mu^1}\ldots C_{\mu^k}$$

We focus on the case where we consider triple of permutations with product equal the identity.

Consider the species $\mathcal{A}$ of triples of permutations with product equal the identity. Thus, $\mathcal{A}_{[n]}=\{(\pi,\sigma,\tau)\in S_n^3:\pi\circ\sigma\circ\tau=\mathrm{Id}\}$. Then, consider the generating function

$$A(t;x,y,z):=\sum_{n\geq 0}\left(\sum_{(\pi,\sigma,\tau)\in\mathcal{A}_{[n]}}p_{\lambda(\pi)}(x)p_{\lambda(\tau)}(y)p_{\lambda(\sigma)}(z)\right)\frac{t^n}{n!}$$

Hence, coefficient of $p_\lambda(x)p_\mu(y)p_\nu(z)\frac{t^n}{n!}$ in $A(t;x,y,z)$ is the number of triples $(\pi,\sigma,\tau)\in\mathcal{C}_\lambda\times\mathcal{C}_\mu\times\mathcal{C}_\nu$ such that $\pi\sigma\tau=\mathrm{Id}$.

Thus, suppose $a_{\lambda\mu\nu}$ is the above number, then we can rewrite the above as

$$A(t;x,y,z)=\sum_{n\geq 0}\sum_{\lambda,\mu,\nu\vdash n}a_{\lambda\mu\nu}p_\lambda(x)p_\mu(y)p_\nu(z)\frac{t^n}{n!}$$

However, we sort of know what $a_{\lambda\mu\nu}$ is. It is given by

$$a_{\lambda\mu\nu} = [C_{1^n}]C_\lambda C_\mu C_\nu$$

In particular, with a long (but not so sophisticated) computation, we can conclude the following:

Theorem 19.2.

$$A(t; x, y, z) = \sum_{n\geq 0}\sum_{\theta\vdash n}\left(\frac{f^\theta}{n!}\right)^{-1} s_\theta(x)s_\theta(y)s_\theta(z)t^n$$

This formula generalizes to n tuples, where we add more s_θ terms, and change $(\frac{f^\theta}{n!})^{-1}$ to $(\frac{f^\theta}{n!})^{-n+2}$.

We can also consider $\mathcal{A}^c$, the species of triple of permutations (π, σ, τ) with $\pi\circ\sigma\circ\tau = \mathrm{Id}$, and the graph defined by these permutations is connected (the c denotes connected). In other word, we want (π, σ, τ) to be act transitively on the set $[n]$. Then, by general species non-sense, we get

$$\mathcal{A} \equiv \mathcal{E}\circ\mathcal{A}^c$$

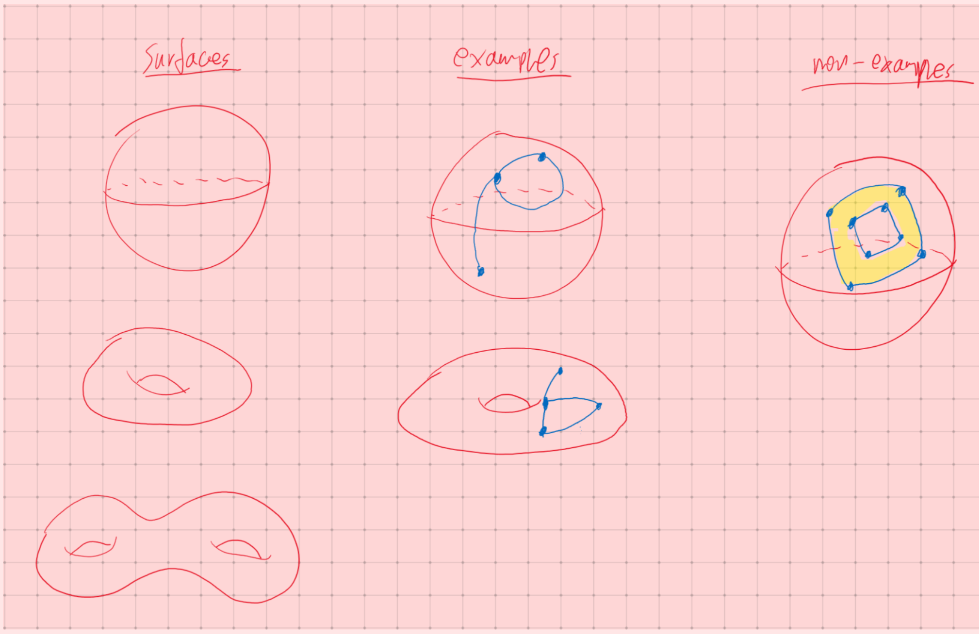
and hence the generating functions for $\mathcal{A}^c$ is

$$A^c(t; x, y, z) = \log\left(\sum_{n\geq 0}\sum_{\theta\vdash n}\left(\frac{f^\theta}{n!}\right)^{-1} s_\theta(x)s_\theta(y)s_\theta(z)t^n\right)$$

Let's now consider an actual enumeration application, i.e. the enumeration of maps.

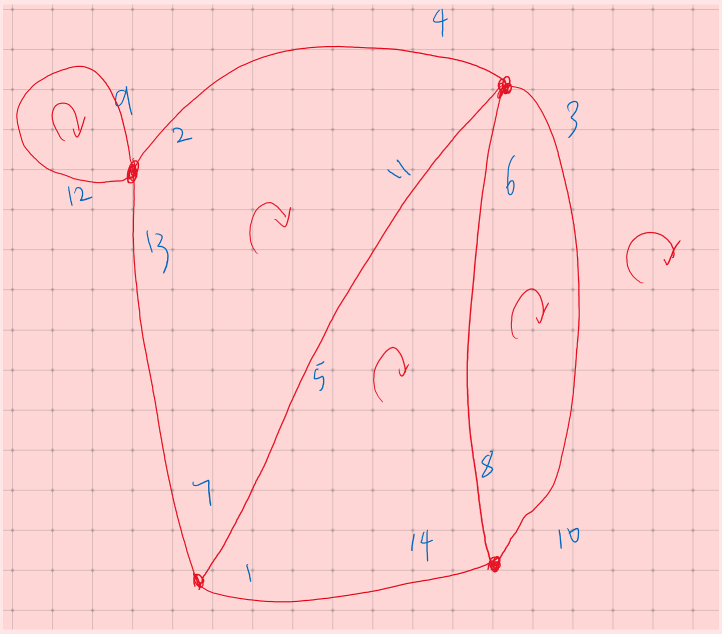
Example 19.3. We define a map to be a graph G embedded on an closed connected oriented surface, such that the interior of each face homeomorphic to an open disk in some $\mathbb{R}^n$.

The following are some surfaces, and examples of maps and non-maps:



The non-example is because the face highlighted with yellow is not homeomorphic to a open disk.

Next, we consider half-edge labelled maps:



For each of those half-edge labelled maps, we can encode this by 3 permutations:

1. The vertex permutation, which is read vertex clockwise at each vertex. In

the above example, we have

$$\tau^{-1} = (9,12,13,2)(1,5,7)(14,10,8)(3,4,11,6)$$

2. The edge permutation (clockwise). In the above example, we have

$$\sigma = (1,14)(2,4)(3,10)(5,11)(6,8)(7,13)(9,12)$$

3. The face permutation (clockwise). In the above example, we have

$$\pi = (7,2,11)(5,6,14)(3,8)(12)(1,10,4,9,13)$$

We note, those three permutations must satisfy the condition

$$\pi\sigma\tau = \mathrm{Id}$$

In particular, the cycle types of these permutations:

1. τ has cycle type the degree of vertices
2. σ has cycle type equal $2^{\#\text{ edges}}$
3. τ has cycle type degree of faces.

Finally, (π,σ,τ) must be connected.

Thus, the number of half-edge labelled maps with:

1. vertex degree $\lambda_1,\ldots,\lambda_p$
2. q edges
3. faces degree $\nu_1,\ldots,\nu_s$

is exactly equal to

$$[p_\lambda(x)p_{2^q}(y)p_\nu(z)\frac{t^{2q}}{(2q)!}]A^c(t;x,y,z)$$

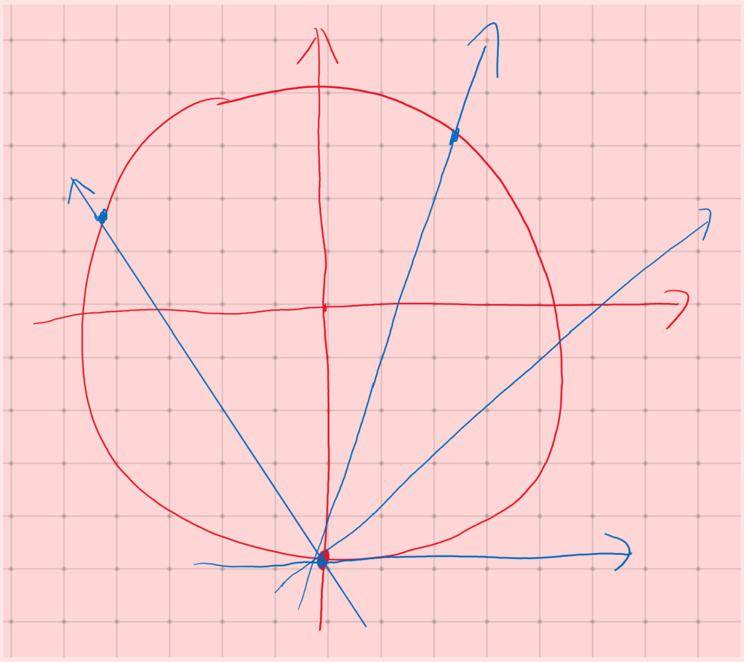
20 2021-11-25

The next few lectures will be about Schubert calculus. First, of course, we recall what $\mathrm{Grass}(k,V)$ is. This is given by

$$\mathrm{Grass}(k,V) := \{k\text{-dimensional subspace of } V\}$$

This set actually has nice geometry on it, and so we want to think of this as some space.

To get a grasp on the concept, let's consider $\mathbb{RP}^1 := \mathrm{Grass}(1,\mathbb{R}^2)$, the 1-dimensional projective space. We can think of this as a circle as follows:



Each blue line above correspond to a 1-dimensional subspace of $\mathbb{R}^2$. This is obtained by define a base point, in this case $(0,-1)$, and then each point on the circle correspond to a 1-dimensional space given by connect the base point with the point on the circle.

To get more symbolic, we can write $[x:y]$ to mean the span of (x,y) in $\mathbb{R}^2$. Clearly we have $[\alpha x:\alpha y]=[x:y]$ for $\alpha\neq 0$, and most lines can be described by a single real number, i.e. the slope $[1:m]$. The only exception is the line parallel to the y-axis, and that parallel line is $[0:1]$. On the other hand, we can also do the other way around, i.e. almost every point in $\mathrm{Grass}(1,\mathbb{R}^2)$ can be represented by $[\ell:1]$ for some $\ell\in\mathbb{R}$, except the line parallel to x-axis.

In other word, we see when we identify using the $[1:m]$, we get a copy of $\mathbb{R}$ plus a point at infinity, as well as the $[\ell:1]$ identification, which is another copy of $\mathbb{R}$ plus a point at infinity.

In other word, $\mathrm{Grass}(1,\mathbb{R}^2)$ can be thought as two copy of $\mathbb{R}$ glue together on the set $\mathbb{R}\backslash\{0\}$ via the map $t\mapsto\frac{1}{t}$.

Next, let's go up one dimension, say $\mathbb{RP}^2=\mathrm{Grass}(1,\mathbb{R}^3)$. We can mimic what we did above, and observe all elements can be represented by $[x:y:z]$ with $(x,y,z)\neq 0$. In this case $[x,y,z]:=\mathrm{Span}\{(x,y,z)\}$ and clearly $\alpha[x,y,z]=[x,y,z]$ if $\alpha\neq 0$. In this case, we get three affine patches

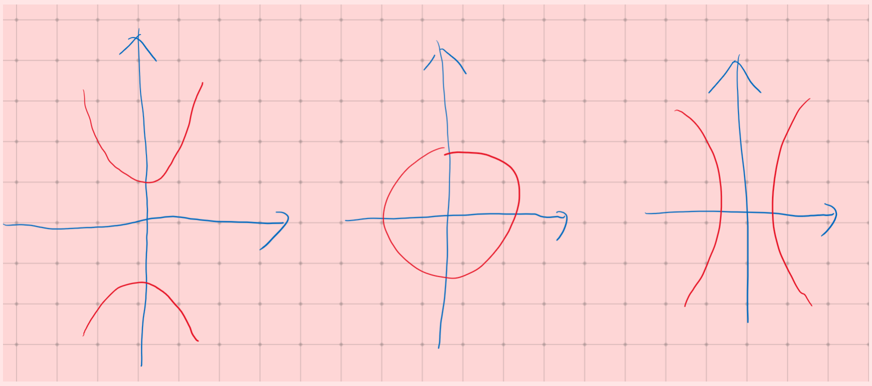
$$U_1=\{[x:y:1]\}\quad U_2=\{[x:1:z]\}\quad U_3=\{[1:y:z]\}$$

that covers $\mathbb{RP}^2$.

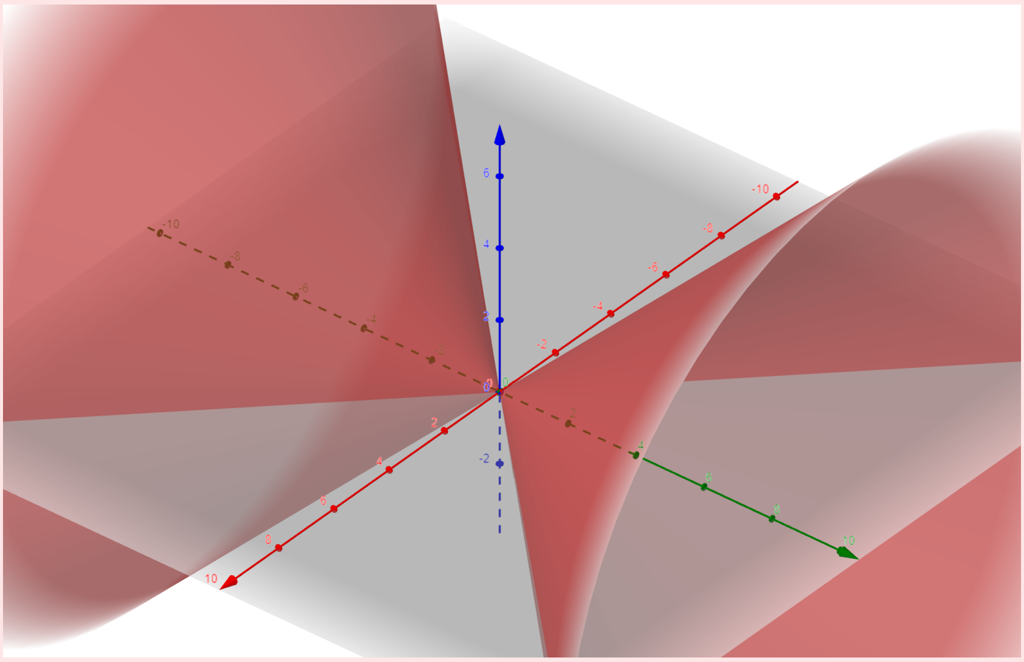
Example 20.1. Consider

$$\{[x:y:z]\in\mathbb{RP}^2 : x^2-y^2+z^2=0\}$$

If we intersect this with U_1, then the above set becomes $\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 + 1 = 0\}$. Similarly if we intersect the above set to U_2, we get $\{(x,z) \in \mathbb{R}^2 : x^2 - 1 + z^2 = 0\}$ and if intersect U_3 we get $\{(y,z) \in \mathbb{R}^2 : 1 - y^2 + z^2 = 0\}$. The following are U_1, U_2, U_3, respectively:

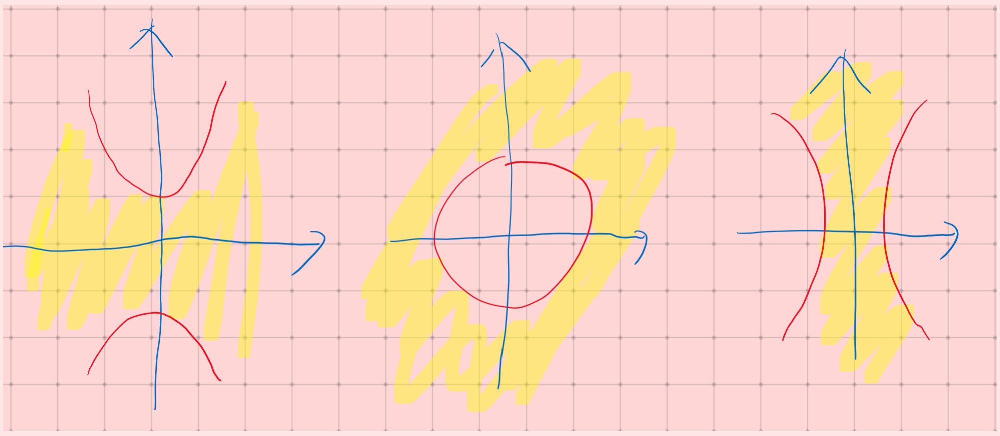


What happens is that in $\mathbb{R}^3$, we have $x^2 - y^2 + z^2 = 0$, which looks like



We note this is not the same as the set $\{[x : y : z] \in \mathbb{RP}^2 : x^2 - y^2 + z^2 = 0\}$. Then, the intersection with U_1, U_2, U_3 are snapshots of the above cone at the place $z = 1$, $y = 1$ and $x = 1$, respectively.

One fun fact is that this set $\{[x : y : z] \in \mathbb{RP}^2 : x^2 - y^2 + z^2 = 0\}$ is not orientable. This is because the following region is a Mobius strip:



Hence, this set contains a Mobius strip and so it is not orientable.

Next, we are going to focus on $\text{Grass}(k, \mathbb{C}^n)$. We have few ways to think about this:

1. The literal sense: each point is a subspace of $\mathbb{C}^n$ of dimension k.
2. Row spaces of matrices: each point is the row space of some rank k matrix A of size $k \times n$, mod out the relation $A \sim B$ iff $\exists K \in \text{GL}_k(\mathbb{C})$ such that $KA = B$. Viz, $\text{Grass}(k, \mathbb{C}^n) = \text{Mat}_{k\times n}^{\text{Rank}(k)}(\mathbb{C})/\text{GL}_k(\mathbb{C})$.
3. Affine patches: recall if $M \in \text{Mat}_{k\times n}(\mathbb{C})$ and $\lambda \subseteq (n-k)^k$, then we define M_λ as the $k \times k$ submatrix with columns $\lambda_k + 1, \lambda_{k-1} + 2, \ldots, \lambda_1 + k$. Next, we define
$$U_\lambda = \{M \in \text{Mat}_{k\times n}(\mathbb{C}) : M_\lambda = \text{Id}_k\}$$
For example, if $\lambda = 0$, then λ correspond to $1, 2, 3, \ldots, k$ and
$$U_\lambda = \begin{bmatrix} \text{Id}_k & * \end{bmatrix}$$
where $*$ means any matrix (of right size) is allowed. We note if $A, B \in U_\lambda$ with $A \neq B$, then the row space of A is not equal the row space of B. Then $\text{Grass}(k, \mathbb{C})$ is going to be unions of U_λ with $\lambda \subseteq (n-k)^k$.
4. $\text{Grass}(k, \mathbb{C}^n)$ is a homogeneous space. This means $\text{GL}_n(\mathbb{C})$ acts on $\text{Grass}(k, \mathbb{C}^n)$ by $g{\cdot}V = gV = \{gv : v \in V\}$ where $g \in \text{GL}_n(\mathbb{C})$ and $V \in \text{Grass}(k, \mathbb{C}^n)$. In terms of row spaces, if V is row space of A, then $gV = \text{Row}(Ag^T)$. This action is transitive, i.e. if $V, W \in \text{Grass}(k, \mathbb{C}^n)$ then we can find $g \in \text{GL}_n(\mathbb{C})$ such that $gV = W$. The consequence of this is that we can think $\text{Grass}(k, \mathbb{C}^n) = \text{GL}_n(\mathbb{C})/\text{Stab}(V)$ for any fixed $V \in \text{Grass}(k, \mathbb{C}^n)$.
5. We identify V with the orthogonal projection matrix $P \in \text{Mat}_{n\times n}(\mathbb{C})$ such that $P = P^2$ and $\text{Im}(P) = V$. In other word, $\text{Grass}(k, \mathbb{C}^n) = \{P \in \text{Mat}_{n\times n}(\mathbb{C}) : P = P^2, \text{Rank}(P) = k\}$.
6. Plucker coordinates: the row space of A correspond to sequence of all $k \times k$ minors of A, which is called the Plucker coordinates. Viz, $V = \text{Row}(A)$ then we can identify V with its Plucker coordinates (this is Plucker embedding, i.e.we embed $\text{Grass}(k, \mathbb{C}^n)$ into some projective space).

We are going to focus on perspective (2) and (3).

Some consequences of this:

1. From affine patches description, we see $\dim_{\mathbb{R}} \mathrm{Grass}(k, \mathbb{C}^n) = 2k(n-k)$ and $\dim_{\mathbb{C}} \mathrm{Grass}(k, \mathbb{C}^n) = k(n-k)$.
2. From (5), we see $\mathrm{Grass}(k, \mathbb{C}^n)$ is compact. We note it is also oriented.
3. From the Plucker embedding, we see $\mathrm{Grass}(k, \mathbb{C}^n)$ is a projective variety.
4. From the above, it is also a smooth manifold.

Now we can start to define Schubert varieties.

For $V \in \mathrm{Grass}(k, \mathbb{C}^n)$, there is a unique $\subseteq$-minimal λ such that $V \in U_\lambda$. To find it, pick a matrix A so $\mathrm{Row}(A) = V$. Put A in RREF, and the pivot determines λ.

We say λ is the ***Schubert position*** of V, if it is the unique $\subseteq$-minimal one.

Definition 20.2. The ***Schubert cell***

$$X_\lambda^o \subseteq U_\lambda$$

is the set of all V with Schubert position λ.

Definition 20.3. The ***Schubert variety*** X_λ is the set of all V with Schubert position μ with $\mu \supseteq \lambda$

Proposition 20.4. *X_λ is the closure of X_λ^0 in* $\mathrm{Grass}(k, \mathbb{C}^n)$.

We note from the definition, it follows

$$X_\lambda = \bigcap_{\mu \not\supseteq \lambda} (\mathrm{Grass}(k, \mathbb{C}^n) \backslash U_\mu)$$

Before we end, we talk a little bit about translated Schubert variety.

If $g \in \mathrm{GL}_n(\mathbb{C})$, we define

$$gX_\lambda = \{gV : g \in \mathrm{GL}_n(\mathbb{C})\}$$

to be a ***translated Schubert variety***. If $\sigma \in S_n$, then we write σX_λ to mean $M_\sigma X_\lambda$. This is called a ***permuted Schubert variety***.

In particular, if $\sigma = w_0 \in S_n$, then $w_0 X_\lambda$ is called an ***opposite Schubert variety***.

21 2021-11-30

Last time we talked about X_λ, which is basically change the 1's in X_λ^o with arbitrary elements. For example, if $\lambda = 331$ then it correspond to matrices

$$\begin{bmatrix} 0 & 1 \mapsto * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 \mapsto * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \mapsto * & * \end{bmatrix}$$

with $1 \mapsto *$ means we changed from 1 to allow any entries. We know we should look at this because the 1's are at $2, 5, 6$th column, and we have $1 + 1 = 2$, $3 + 2 = 5$ and $3 + 3 = 6$.

Last time we also looked at the opposite Schubert variety, which is given by flip the shape of the above diagram. For example, take $\lambda = 331$, we get $w_0 X_\lambda$ are represented by

$$\begin{bmatrix} * & * & * & * & * & * & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Those diagrams above will be called $*$-patterns.

However, we note this $*$-pattern of $w_0 X_\lambda$ is not very illustrating, and it is often make more sense to work with $\lambda^\vee$. In this case, we get $\lambda^\vee = (n - k - \lambda_k, \dots, n - k - \lambda_1)$, which is just flip λ by rotate 180 degrees. In this case, we get

$$w_0 X_{\lambda^\vee} = \begin{bmatrix} * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{bmatrix}$$

By doing this, we can read off the partition easier.

This looks pretty weird, and that's fair. Let's consider the special Schubert varieties, which are a lot easier to define, and a lot easier to think.

Start with a linear subspace $W \subseteq \mathbb{C}^n$, define $\Sigma(W)$ as

$$\Sigma(W) = \{V \in \mathrm{Grass}(k, \mathbb{C}^n) : V \cap W \neq \{0\}\}$$

This is a pretty obvious thing to think about, and this is where the whole Schubert calculus started.

Proposition 21.1. *$\Sigma(W)$ is a translated Schubert variety. Specifically, let $d = (n - k + 1) - \dim W$, then there exists $g \in \mathrm{GL}_n(\mathbb{C})$ such that*

$$\Sigma(W) = \begin{cases} gX_d & \text{if } d \geq 0 \\ \mathrm{Grass}(k, \mathbb{C}^n) & \text{if } d < 0 \end{cases}$$

Proof. Case 1: Suppose $W = \mathrm{Span}\{e_{k+d}, e_{k+d+1}, \dots, e_n\}$ be the span of basis vectors for $\mathbb{C}^n$. Suppose $V \in X_d$, then we see

$$V \leftrightarrow \begin{bmatrix} * & * & * & \dots & \dots & * \\ 0 & * & * & \dots & \dots & * \\ 0 & 0 & * & \dots & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & * & \dots & * \\ 0 & \dots & \dots & \dots & * & \dots \end{bmatrix} =: M$$

where we keep going down the diagonal, until the last row where we make the $*$ start at $k+d$th column instead of follow the diagonal. Then we see the last row of M is a non-zero intersection of V and W. On the other hand, if the intersection is non-trivial, then we can run the argument in reverse.

Suppose $V \in \Sigma(W)$, then we can find $0 \neq v \in V \cap W$. Take M to be a matrix where $\mathrm{Row}(M) = V$ and last row is v. Then perform row reduction, we end up with the desired $*$-pattern.

Case 2: In general, suppose $W' \subseteq \mathbb{C}^n$ with $\dim W' = \dim W$. Then there exists $g \in \mathrm{GL}_n(\mathbb{C})$ such that $gW = W'$. Then we see $\Sigma(W') = g\Sigma(W) = gX_d$ and the proof follows. ♡

THis motivates our study of Schubert varieties, and the next topic is Schubert intersection. Those Schubert varieties are pretty special, and hence if V intersect with a Schubert variety, we say it satisfies certain Schubert conditions. We want to look for those V that satisfies more than one Schubert conditions, and this is the same as to look the intersections of Schubert varieties.

Definition 21.2. For $g_1, \ldots, g_s \in \mathrm{GL}_n(\mathbb{C})$, suppose $\lambda_1, \ldots, \lambda_s \subseteq (n-k)^k$. The intersection

$$g_1 X_{\lambda_1} \cap \ldots \cap g_s X_{\lambda_s}$$

is called a ***Schubert intersection***.

We note $\mathrm{codim}_{\mathbb{C}}(X_\lambda) = |\lambda|$ as it is the closure of a codimension $|\lambda|$ Schubert cells X_λ^o. We might "expect" the codimension of Schubert intersections is just the sum of codimensions. This is obviously false in general, i.e. if $g_1 = g_2 = \ldots = g_s$ and $\lambda_1 = \ldots = \lambda_s$, then the claim is false.

Thus, the next question we can ask is, does it happen at least sometimes? Well, it usually does.

Theorem 21.3 (Kleiman Theorem)**.**

1. *For $g_1, \ldots, g_s \in \mathrm{GL}_n(\mathbb{C})$ generic, the intersection $g_1 X_{\lambda_1} \cap \ldots \cap g_s X_{\lambda_s}$ is transverse (in particular it implies intersection has codimension equal sum of codimension).*
2. *Let $Y_{\lambda_i} \subseteq X_{\lambda_i}$ be a dense open subset. For $g_1, \ldots, g_s$ generic, then $g_1 Y_{\lambda_1} \cap \ldots \cap g_s Y_{\lambda_s}$ is a dense open subset of $g_1 X_{\lambda_1} \cap \ldots \cap g_s X_{\lambda_s}$.*

Now, this just says points in general position, hence what if we are unlucky and the $g_1, \ldots, g_s$ we wrote down is not general? In other word, how do we use this theorem?

Well, first, we note we can always assume $g_1 = \mathrm{Id}_n$ (by apply g_1^{-1} to the whole intersection).

We can also assume $g_s = w_0 = \begin{bmatrix} 0 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 0 \end{bmatrix}$. To prove this, since g_s is generic, we may assume $g_s^{-1}w_0$ has LU decompostion (the set of matrices with LU decompostion is open dense). Hence $g_s^{-1} = \ell_s u_s$ where ℓ_s is lower triangular and u_s is upper triangular. Now, $u_s = w_0 \ell_s' w_0$ where ℓ_s' is another lower triangular matrix. Thus we see $g_s^{-1} = \ell_s w_0 \ell_s'$. Now take

$$X_{\lambda_1} \cap \ldots \cap g_s X_{\lambda_s}$$

and translate by ℓ_s, we get

$$\ell_s' X_{\lambda_1} \cap \ldots \cap \ell_s' g_s X_{\lambda_s}$$

but $\ell_s' g_s = w_0 \ell_s^{-1}$ and hence we get

$$\ell_s' X_{\lambda_1} \cap \ldots \cap w_0 \ell_s^{-1} X_{\lambda_s}$$

Finally, if $\ell \in \mathrm{GL}_n(\mathbb{C})$ is lower triangular then $\ell X_\lambda = X_\lambda$, and hence we see $\ell_s^{-1} X_{\lambda_s} = X_{\lambda_s}$ and $\ell_s' X_{\lambda_1} = X_{\lambda_1}$ and so we are left with

$$X_{\lambda_1} \cap \ldots \cap w_0 S_{\lambda_s}$$

Often, one or more of $g_i X_{\lambda_i}$ will be a special Schubert variety. In this case, the assumption g_i is generic is the same as W_i is generic.

So, what's the point? Well, we will consider the cohomology ring of the Grassmannian.

Suppose we have a compact manifold X, then we consider its rational cohomology ring $H^*(X;\mathbb{Q})$. Note we don't really care which cohomology we are using, as most of the cohomology would agree. Then, the cycles of the cohomology would correspond to element in the cohomology ring.

Some examples of cycles are:

1. oriented closed submanifolds of X
2. algebraic subvarieties
3. single points
4. formal $\mathbb{Q}$-linear combinations of cycles

Some basic facts about $H^*(\mathrm{Grass}(k, \mathbb{C}^n);\mathbb{Q})$:

1. Schubert varieties are cycles
2. $\{[X_\lambda] : \lambda \subseteq (n-k)^k\}$ is a basis
3. If $g \in \mathrm{GL}_n(\mathbb{C})$, then $[gX_\lambda] = [X_\lambda]$
4. If the intersection is transverse and finite, then the number of points of intersection is the coefficient of $[X_{\lambda_s^\vee}]$ in the product $[X_{\lambda_1}] \cdot \ldots \cdot [X_{\lambda_{s-1}}]$.

22 2021-12-02

Last time we talked about Schubert intersection. We talk about the multiplication in cohomology ring is the same as Schubert intersections if the intersection is transverse.

Let's make this a little bit more precise: The coefficient of $[X_{\lambda_s^\vee}]$ in $[X_{\lambda_1}]...[X_{\lambda_{s-1}}]$, is equal

$$\begin{cases} \#\text{ of points of the intersection if it is finite} \\ 0, \text{ if the points of intersection is infinite} \end{cases}$$

We note this cohomology module is really a ring, because the intersection is really similar to multiplication. Indeed, $\bigcap$ is associative, and distributes over $\bigcup$. Hence, from a set theory perspective, $\bigcap$ behaves like multiplication and $\bigcup$ behaves like addition.

Of course, cohomology is finer than just a set, and hence we have more details to take care of.

Example 22.1. Consider $H^*(\text{Grass}(2, \mathbb{C}^4); \mathbb{Q})$. This has a basis

$$[X_0], [X_1], [X_2], [X_{1,1}], [X_{2,1}], [X_{2,2}]$$

i.e. it is a six dimensional vector space. Thus, let's try to find $[X_1]^2$. The idea is really simple, we just need to compute $g_1X_1 \cap g_2X_1 \cap g_3X_\lambda$, where λ run over all partitions inside $(2,2)$. Thus, let's make a table:

λ	Intersection to consider	number of points
(0)	$g_1X_1 \cap g_2X_1 \cap g_3X_0$	∞ (two-dimensional)
(1)	$g_1X_1 \cap g_2X_1 \cap g_3X_1$	∞ (one-dimensional)
(2)	$g_1X_1 \cap g_2X_1 \cap g_3X_2$	1
$(1,1)$	$g_1X_1 \cap g_2X_1 \cap g_3X_{11}$	1
$(2,1)$	$g_1X_1 \cap g_2X_1 \cap g_3X_{21}$	$\varnothing$
$(2,2)$	$g_1X_1 \cap g_2X_1 \cap g_3X_{22}$	$\varnothing$

At the end of the day, we actually want to compute $\lambda^\vee$, but since we already have a complete list of values, we just need to invert the list, and get

$\lambda^\vee$	λ	Intersection to consider	number of points
$(2,2)$	(0)	$g_1X_1 \cap g_2X_1 \cap g_3X_0$	∞ (two-dimensional)
$(2,1)$	(1)	$g_1X_1 \cap g_2X_1 \cap g_3X_1$	∞ (one-dimensional)
(2)	(2)	$g_1X_1 \cap g_2X_1 \cap g_3X_2$	1
$(1,1)$	$(1,1)$	$g_1X_1 \cap g_2X_1 \cap g_3X_{11}$	1
(1)	$(2,1)$	$g_1X_1 \cap g_2X_1 \cap g_3X_{21}$	$\varnothing$
(0)	$(2,2)$	$g_1X_1 \cap g_2X_1 \cap g_3X_{22}$	$\varnothing$

Hence the final answer is

$$[X_1]^2 = [X_2] + [X_{11}]$$

Let's consider the following thing: the $\mathbb{Q}$-linear map $\phi : \Lambda \to H^*(\text{Grass}(k, \mathbb{C}^n); \mathbb{Q})$ given by

$$\phi(s_\lambda) = \begin{cases} [X_\lambda], & \text{if } \lambda \subseteq (n-k)^k \\ 0, & \text{otherwise} \end{cases}$$

Theorem 22.2. *The map ϕ is $\mathbb{Q}$-algebra homomorphism.*

Before we prove this, we consider an implications: this tells us the following quantities are the same:

1. $|g_1 X_{\lambda_1} \cap \ldots \cap \lambda_s X_{\lambda_s}|$ if the intersection is fintie and transverse.
2. coefficient of $[X_{\lambda_s^\vee}]$ in $[X_{\lambda_1}]\ldots[X_{\lambda_{s-1}}]$ in $H^*(\mathrm{Grass}(k, \mathbb{C}^n); \mathbb{Q})$.
3. $\langle s_{\lambda_s^\vee}, s_{\lambda_1} \ldots s_{\lambda_{s-1}} \rangle$.
4. $\langle s_{(n-k)^k}, s_{\lambda_1} \ldots s_{\lambda_s} \rangle$.

At the end of the day, this tells us we can study the Schubert intersection via Schur functions.

Proof of Theorem 22.2. Now let's prove the theorem. In the assignments, we proved that, it suffices to show

$$\phi(s_\mu)\phi(s_d) = \sum_{\mu \xrightarrow{d} \lambda} \phi(s_\lambda)$$

This is the same as to prove

$$[X_\mu][X_d] = \sum_{\substack{\mu \xrightarrow{d} \lambda \\ \lambda \subseteq (n-k)^k}} [X_\mu]$$

This happens iff

$$|g_1 X_\mu \cap g_2 X_d \cap g_3 X_{\lambda^\vee}| = \begin{cases} 1, & \mu \xrightarrow{d} \lambda \\ 0 \text{ or } \infty, & \text{otherwise} \end{cases}$$

Finally, we can use some simplifications: $g_1 = \mathrm{Id}$ and $g_3 = w_0$. What about g_2? Well, we cannot do anything about g_2, but last time we learned Schubert variety with one part is just special Schubert variety. Thus, we just need to prove the result about $|X_\mu \cap \Sigma(W) \cap w_0 X_{\lambda^\vee}|$ where $\dim W = n - k + 1 - d$, where W is a generic subspace of $\mathbb{C}^n$.

We are going to do this computation in two steps, and we are going to develop some theorems. ♡

Theorem 22.3. *Intersection of $*$-patterns for X_μ and $w_0 X_{\lambda^\vee}$ gives a dense subset of $X_\mu \cap w_0 X_{\lambda^\vee}$.*

For example, consider $\mathrm{Grass}(4, \mathbb{C}^{10})$ with $\mu = 4110$, $\lambda = 6533$. Then the $*$-pattern for μ is given by

$$\begin{bmatrix} * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \end{bmatrix}$$

On the other hand, for λ we get the opposite $\ast$-pattern is

$$\begin{bmatrix} * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * \end{bmatrix}$$

Then the intersection gives

$$\begin{bmatrix} * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * \end{bmatrix}$$

and the variety defined by the above $\ast$-pattern is an open dense subset of $X_\mu \cap w_0 X_{\lambda^\vee}$.

Proof of Theorem 22.3. To prove this theorem, we first consider the special case $\mu \nsubseteq \lambda$. In this case we claim $X_\mu \cap w_0 X_{\lambda^\vee} = \varnothing$. Suppose $X_\mu \cap w_0 X_{\lambda^\vee} \neq \varnothing$ for a contradiction. THen by Kleiman's Theorem 21.3.(2), we see $X_\mu^o \cap w_0 X_{\lambda^\vee}^o \neq \varnothing$ as $X_\lambda^o \subseteq X_\lambda$ is a dense open subset.

Let $V \in X_\mu^o \cap w_0 X_{\lambda^\vee}^o$. Since $V \in X_\mu^o$, μ is the unique smallest partition such that $V \in U_\mu$. Since $V \in w_0 X_{\lambda^\vee}^o$, we see λ is the unique largest partition such that $V \in U_\lambda$. In particular, the largest partition must be bigger than the smallest one, i.e. $\mu \subseteq \lambda$, a contradiction. This proves the special case.

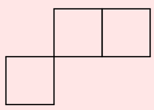
Next, we consider $\mu \subseteq \lambda$. In this case consider the set of all $V = \mathrm{Row}(A)$ where A has $\ast$-pattern for X_μ and A_λ (this is the minor of A defined by λ) has an LU decompostion.

Intersecting this set with $w_0 X_{\lambda^\vee}$, we get only points of $X_\mu \cap w_0 X_{\lambda^\vee}$ representable by intersections of $\ast$-patterns. This proves the general case, as the set of all A is a dense open subset of X_μ. ♡

Now to finish the proof of Theorem 22.2, we just use Theorem 22.3.

Proof of Theorem 22.2, Continued, By Example. Let $n = 7$, $k = 3$, and $\mu = 31$ and $\lambda = 331$. Then

$\lambda/\mu =$



Then, we see

$$X_\mu \cap w_0 X_{\lambda^\vee} \leftrightarrow \begin{bmatrix} * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 \end{bmatrix}$$

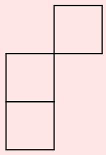
We want to intersect the above $\ast$-pattern with $\Sigma(W)$. Suppose V is inside the intersection, then we see $V \in \mathrm{Span}\{e_1, e_2, \ldots, e_6\}$. Since $\dim W$ is 2-dimensional, thus $W \cap \mathrm{Span}(e_1, ..., e_6)$ non-trivially. Take a non-zero vector $(w_1, w_2, ..., w_6, 0) \in W \cap \mathrm{Span}(e_1, ..., e_6)$, how do we get V? Well, V must contain the above vector too, and hence if we want $V \in \Sigma(W)$, then $(w_1, ..., w_6, 0) \in V$. This actually is the only non-zero vector, because the intersection of W with $\mathrm{Span}(e_1, ..., e_6)$ is 1-dimensional. Thus, there is only one possibility for V, which is

$$V = \mathrm{Row}\begin{bmatrix} w_1 & w_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & w_3 & w_4 & w_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w_6 & 0 \end{bmatrix}$$

This proof by example shows if we have a row strip then the intersection must be 1.

Next, suppose we don't have a row strip, say $\mu = 211$ and $\lambda = 331$. Then

$\lambda/\mu =$



Then

$$X_\mu \cap w_0 X_{\lambda^\vee} \leftrightarrow \begin{bmatrix} 0 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & 0 \end{bmatrix}$$

But this time we have $W \cap \mathrm{Span}\{e_2, ..., e_5\} = \{0\}$ as W is generic. By the same reasoning, we get $V \notin \Sigma(W)$ as desired. ♡