Waves and Properties

Single Harmonic Oscillator -> Coupled Oscillators
Infinite number of Oscillators -> Continuum Limit -> Waves
Fourier decomposition of Waves -> Properties of Waves

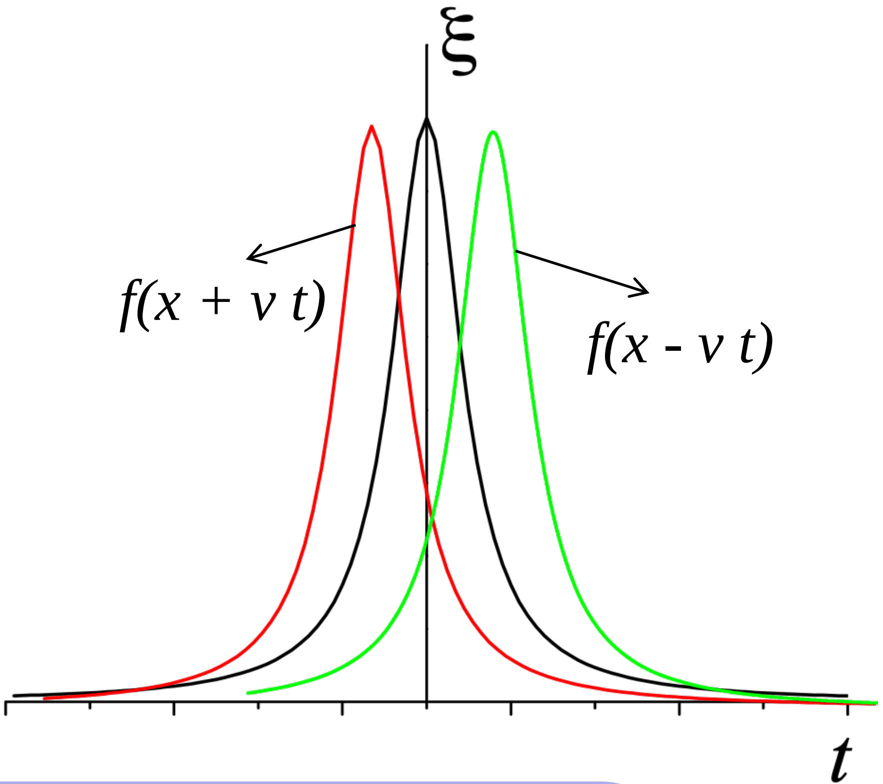
Wave Equation and Solution, Dispersion, Group velocity and Phase velocity, Pulse propagation

Wave equation:

$$\frac{\partial^2 \xi(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi(x,t)}{\partial t^2}$$

General solution:

$$\xi = f_1(ct - x) + f_2(ct + x)$$



Different forms of solutions :

$$\xi = a\cos\frac{2\pi}{\lambda}(ct - x) \qquad \xi = a\cos 2\pi\left(\nu t - \frac{x}{\lambda}\right)$$

$$\xi = a\cos\omega\left(t - \frac{x}{c}\right) \qquad \xi = a\cos(\omega t - kx)$$

$$\xi = ae^{i(\omega t - kx)}$$

Sinusoidal plane wave

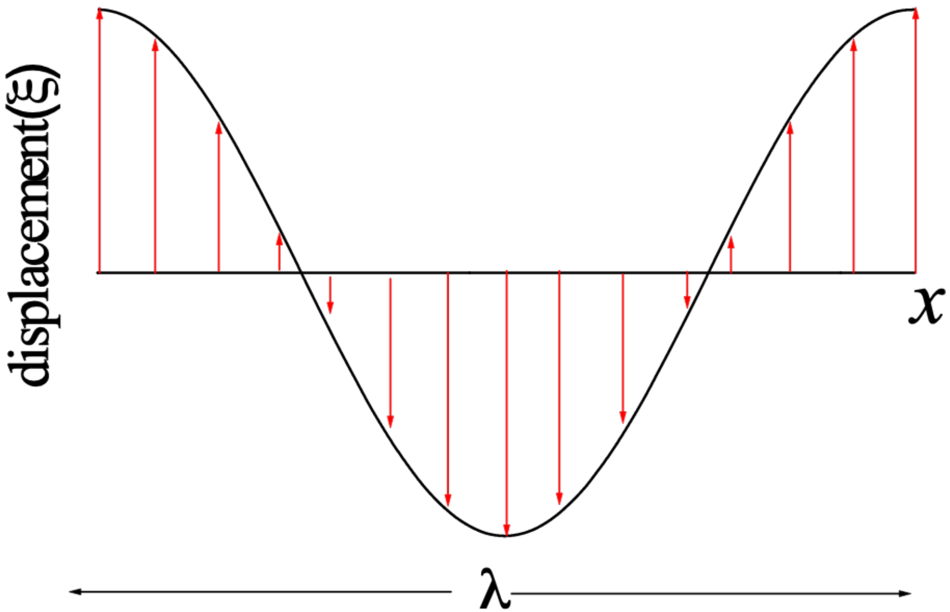
$$\xi = a\cos\frac{2\pi}{\lambda}(ct - x)$$

$$\frac{\lambda}{c} = \frac{1}{\nu} = \tau$$

τ: period of oscillation

$$\frac{2\pi c}{\lambda} = \omega = 2\pi\nu$$

$c = \nu\lambda$ $\quad c$: wave velocity



for $\quad x = n\lambda \quad$ pattern repeats

λ : Wavelength

$k = \dfrac{2\pi}{\lambda}$:wavenumber

Dispersion Relation

a link between spatial and temporal oscillations

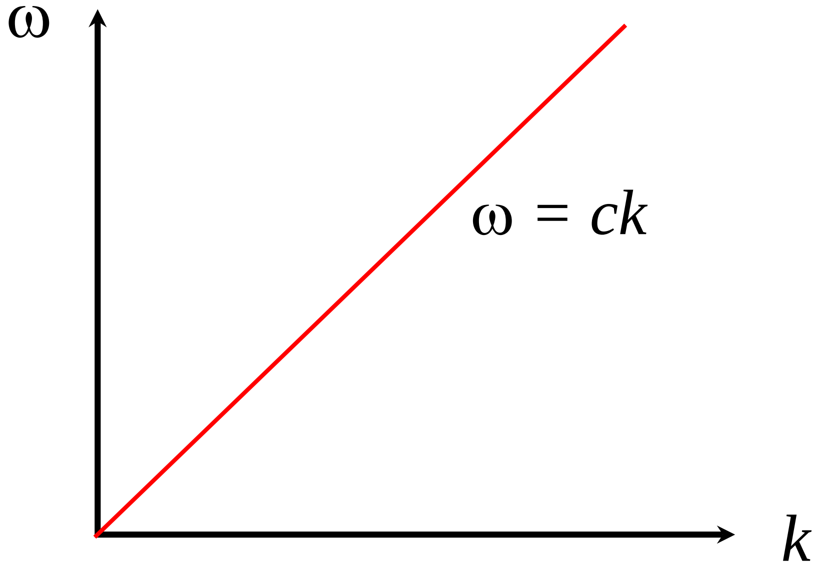
For the wave equation $\Rightarrow$ $$\boxed{\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}}$$

Plane wave solution $\Rightarrow$ $$\boxed{\xi = f\left[\frac{2\pi}{\lambda}(ct - x)\right]}$$

Oscillation frequency $\Rightarrow$ $$\boxed{\omega = \frac{2\pi c}{\lambda} = ck}$$

For monochromatic wave in a non-dispersive medium

$$\omega = ck$$

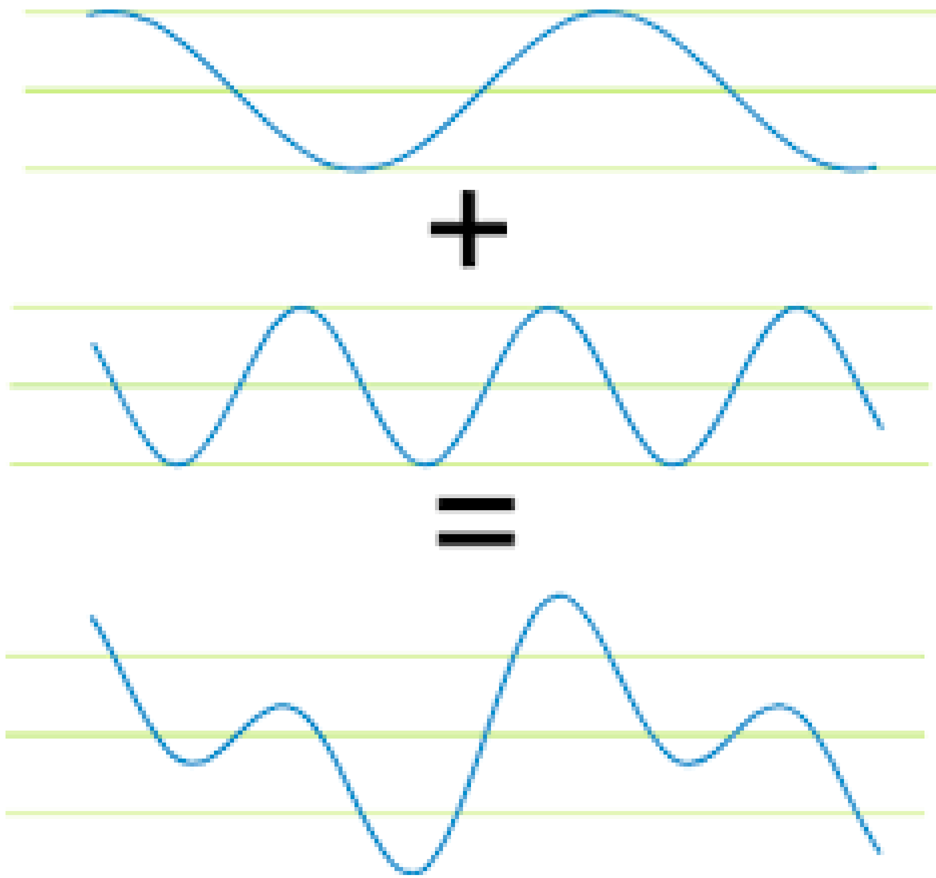


slope (c) : phase velocity of the wave

Fourier Decomposition of Waves

Fourier Series

Fourier Decomposition



Fourier Series:

$$0 < t < T$$

$$f(t) = A_0 + A_1 \cos(\omega t) + A_2 \cos(2\omega t) + A_3 \cos(3\omega t) + ... + B_1 \sin(\omega t) + B_2 \sin(2\omega t) + B_3 \sin(3\omega t) + ...$$

$$f(t) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\omega t) + B_n \sin(n\omega t)]$$

$$A_n = \frac{2}{T}\int_0^T f(t) \cos(n\,\omega t) dt$$

$$A_0 = \frac{1}{T}\int_0^T f(t)\, dt$$

$$B_n = \frac{2}{T}\int_0^T f(t) \sin(n\,\omega t) dt$$

Orthogonality conditions

If n and m are different, then

$$\cos n \qquad \sin m \qquad = 0$$

$$\cos n \qquad \cos m \qquad = 0$$

$$\sin n \qquad \sin m \qquad = 0$$

$$
\begin{aligned}
A_0 \;&+\; \frac{A_1}{2}[e^{i\omega t} + e^{-i\omega t}] + \frac{A_2}{2}[e^{i2\omega t} + e^{-i2\omega t}] \\
&+\; \frac{A_3}{2}[e^{i3\omega t} + e^{-i3\omega t}] \qquad \cdots\cdots \\
&+\; \frac{B_1}{2i}[e^{i\omega t} - e^{-i\omega t}] + \frac{B_2}{2i}[e^{i2\omega t} - e^{-i2\omega t}] \\
&+\; \frac{B_3}{2i}[e^{i3\omega t} - e^{-i3\omega t}] \qquad \cdots\cdots \\
&=\; \sum_{n=0}^{+\infty}[C_n e^{in\omega t} + C_n^* e^{-in\omega t}]
\end{aligned}
$$

$$
C_n = (A_n - iB_n)/2
$$

Dirichlet Conditions

» A periodic signal ***x*(*t*)**, has a Fourier series if it satisfies the following conditions:

1. ***x*(*t*)** is absolutely integrable over any period, namely

$$\int_{a}^{a+T} |\, x(t) \,|\, dt < \infty,$$

2. ***x*(*t*)** has only a finite number of maxima and minima over any period

3. ***x*(*t*)** has only a finite number of discontinuities over any period

Fourier Transform

- We have seen that periodic signals can be represented with the Fourier series

- Can aperiodic signals be analyzed in terms of frequency components?

- Yes, and the Fourier transform provides the tool for this analysis

- The major difference w.r.t. the line spectra of periodic signals is that the spectra of aperiodic signals are defined for all real values of the frequency variable ω not just for a discrete set of values

Properties of the Fourier Transform

- *Linearity:* $x(t) \leftrightarrow X(\omega) \qquad y(t) \leftrightarrow Y(\omega)$

$$\alpha x(t) + \beta y(t) \leftrightarrow \alpha X(\omega) + \beta Y(\omega)$$

- *Left or Right Shift in Time:* $x(t - t_0) \leftrightarrow X(\omega) e^{-j\omega t_0}$

- *Time Scaling:* $x(at) \leftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$

Properties of the Fourier Transform

» *Time Reversal:* $x(-t) \leftrightarrow X(-\omega)$

» *Multiplication by a Power of t:*

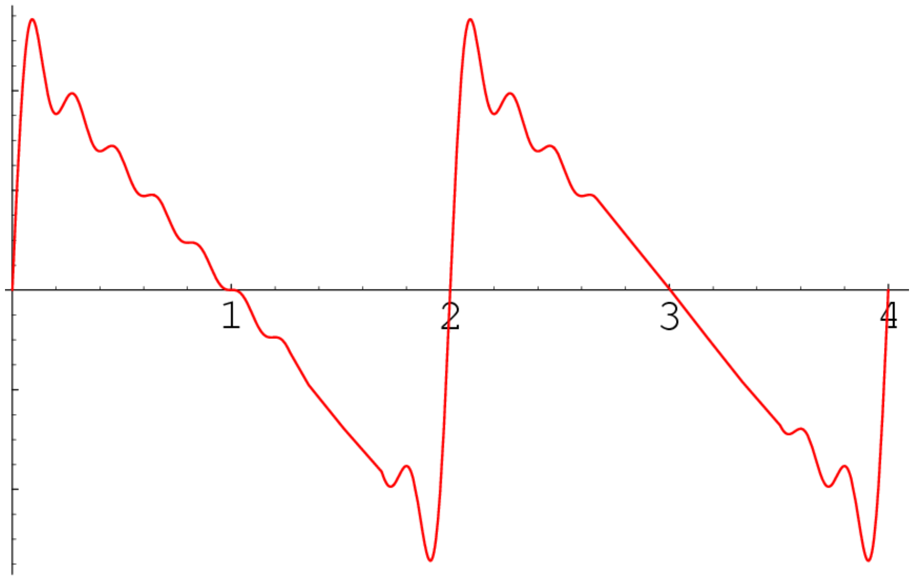
$$t^n x(t) \leftrightarrow (j)^n \frac{d^n}{d\omega^n} X(\omega)$$

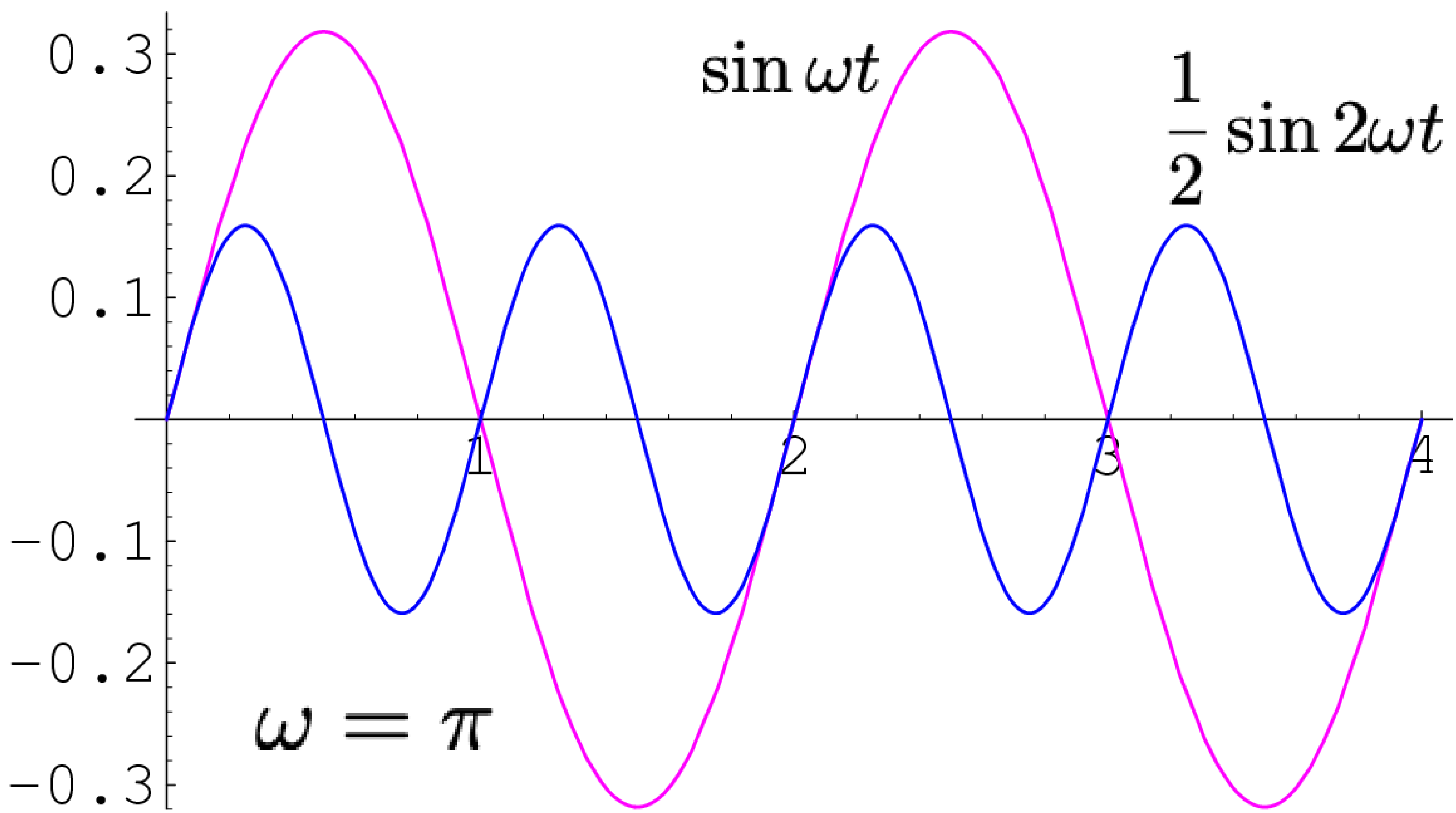
» *Multiplication by a Complex Exponential:*

$$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

Examples

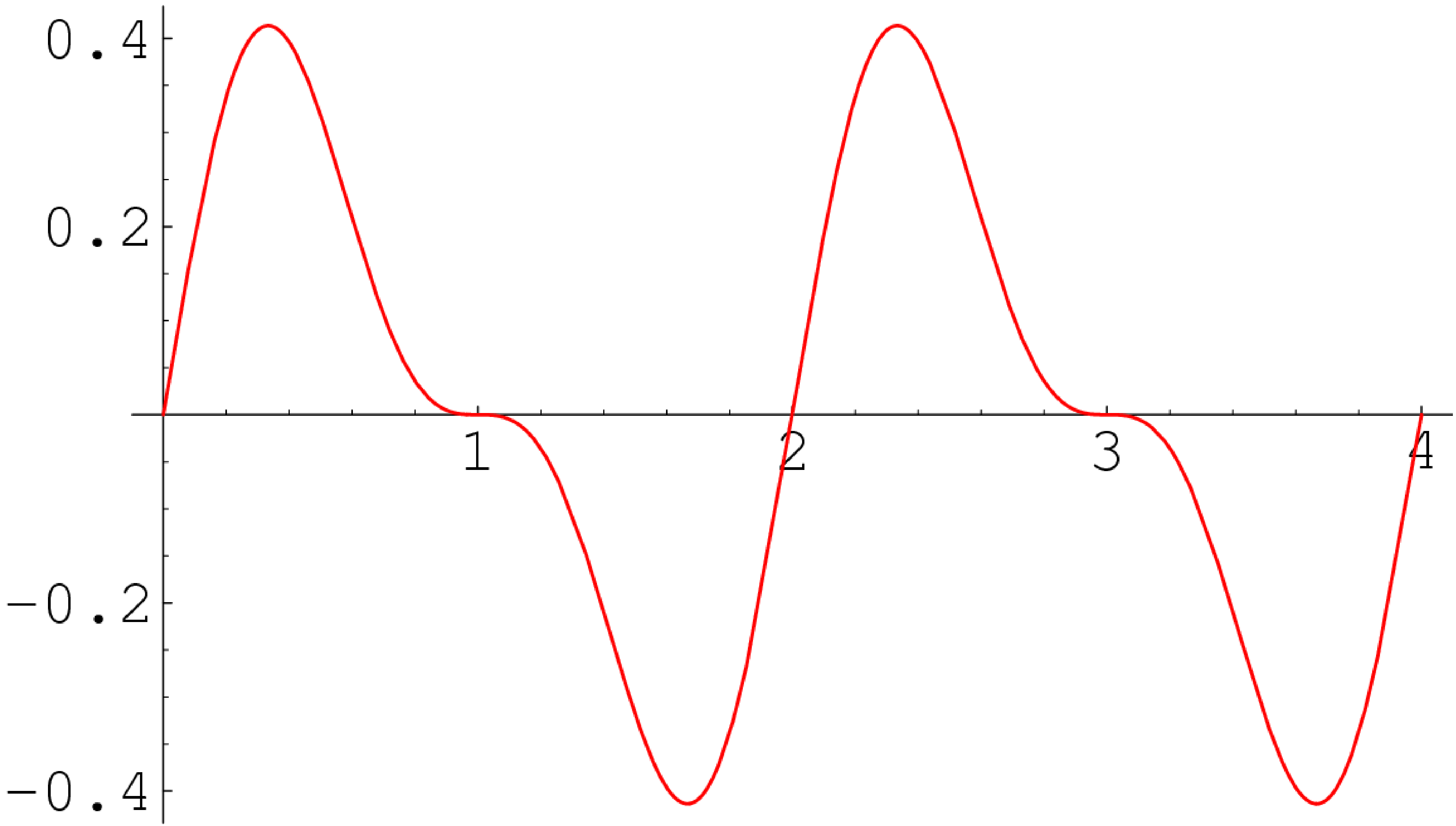
Fourier Series: Shaw-tooth Wave



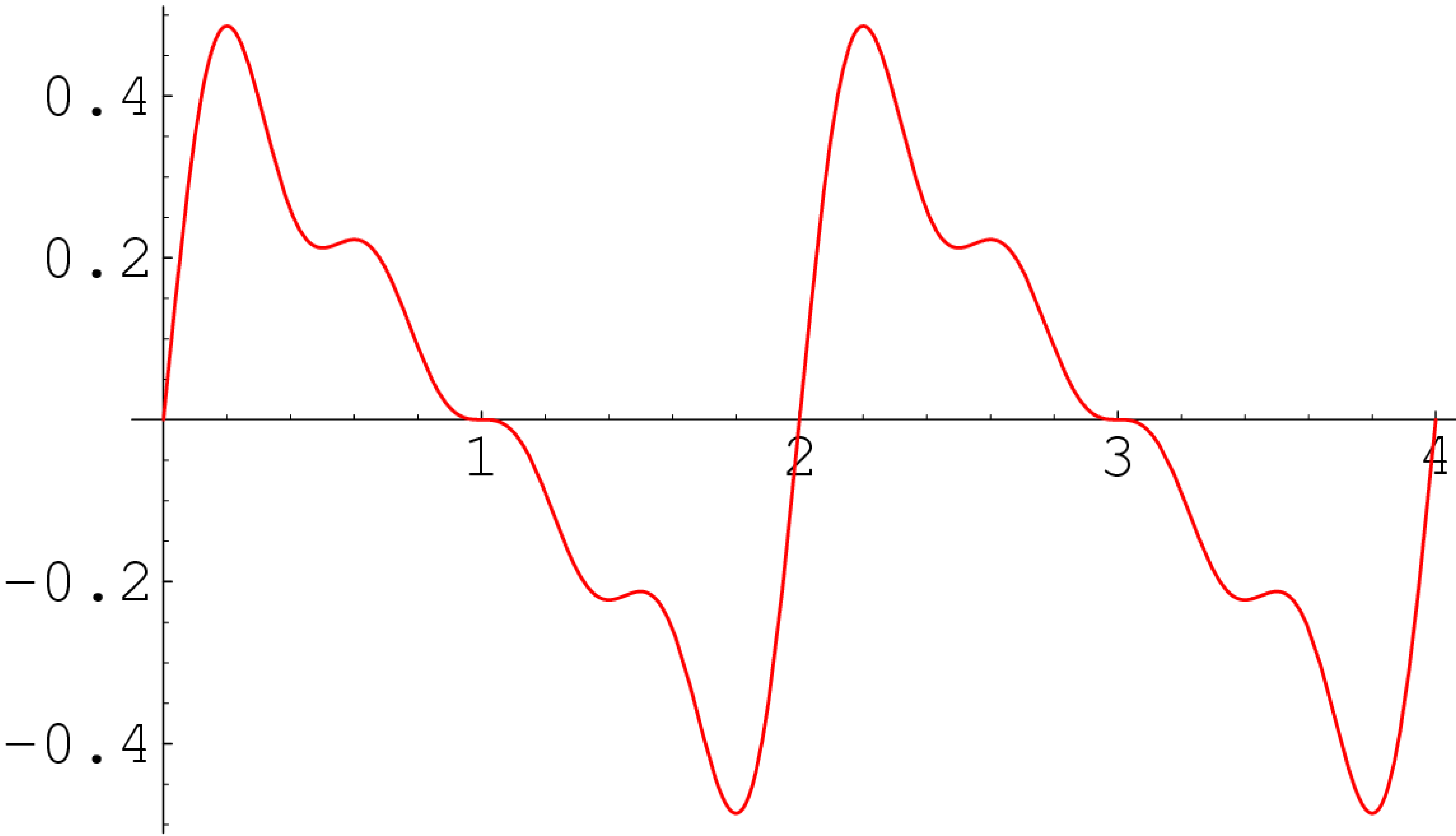


0.3
0.2
0.1
-0.1
-0.2
-0.3
1
2
3
4
$\sin \omega t$
$\frac{1}{2} \sin 2\omega t$
$\omega = \pi$

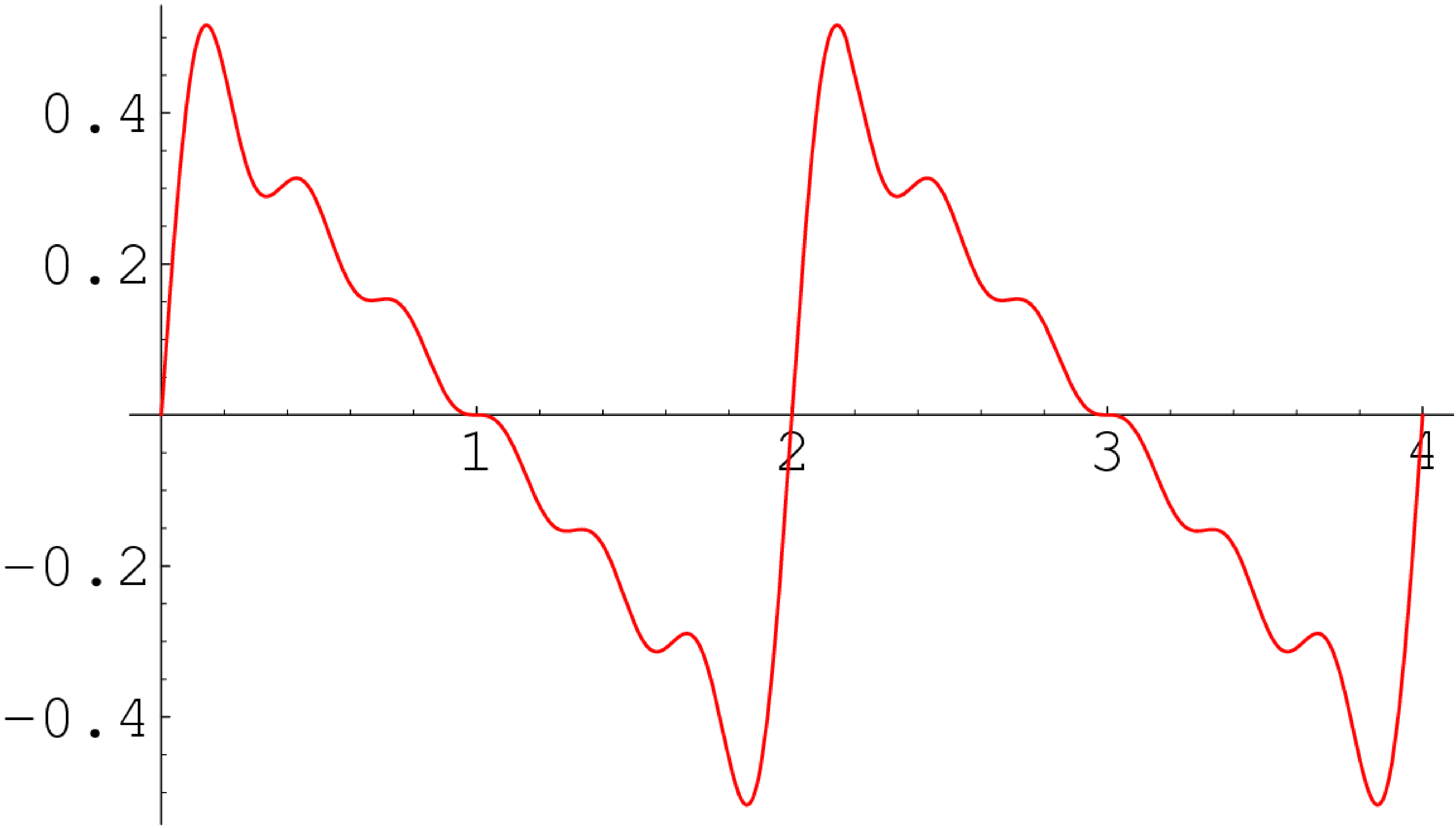
$$\sin \omega t + \frac{1}{2} \sin 2\omega t$$



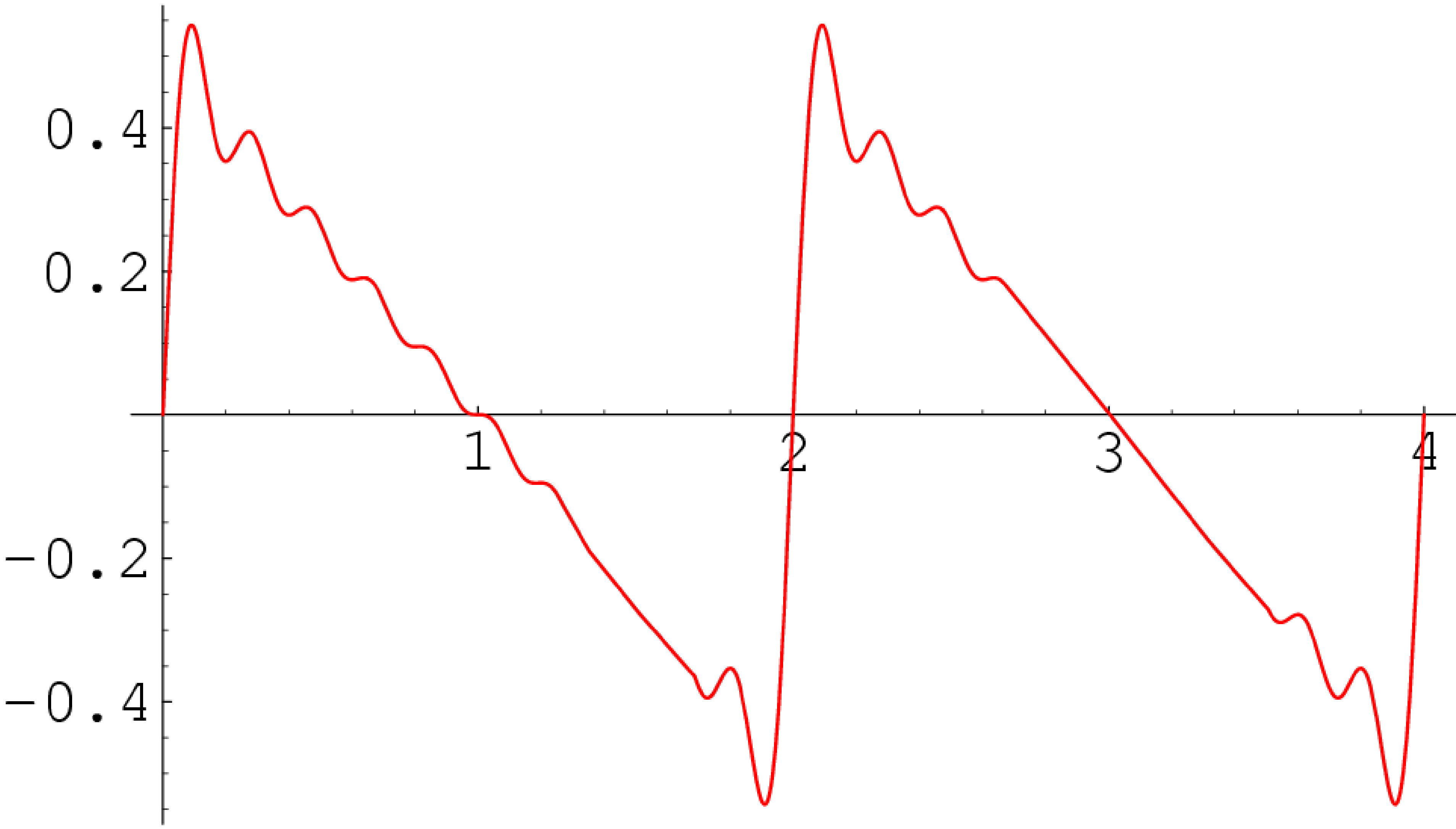
$$\sin \omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{4}\sin 4\omega t$$



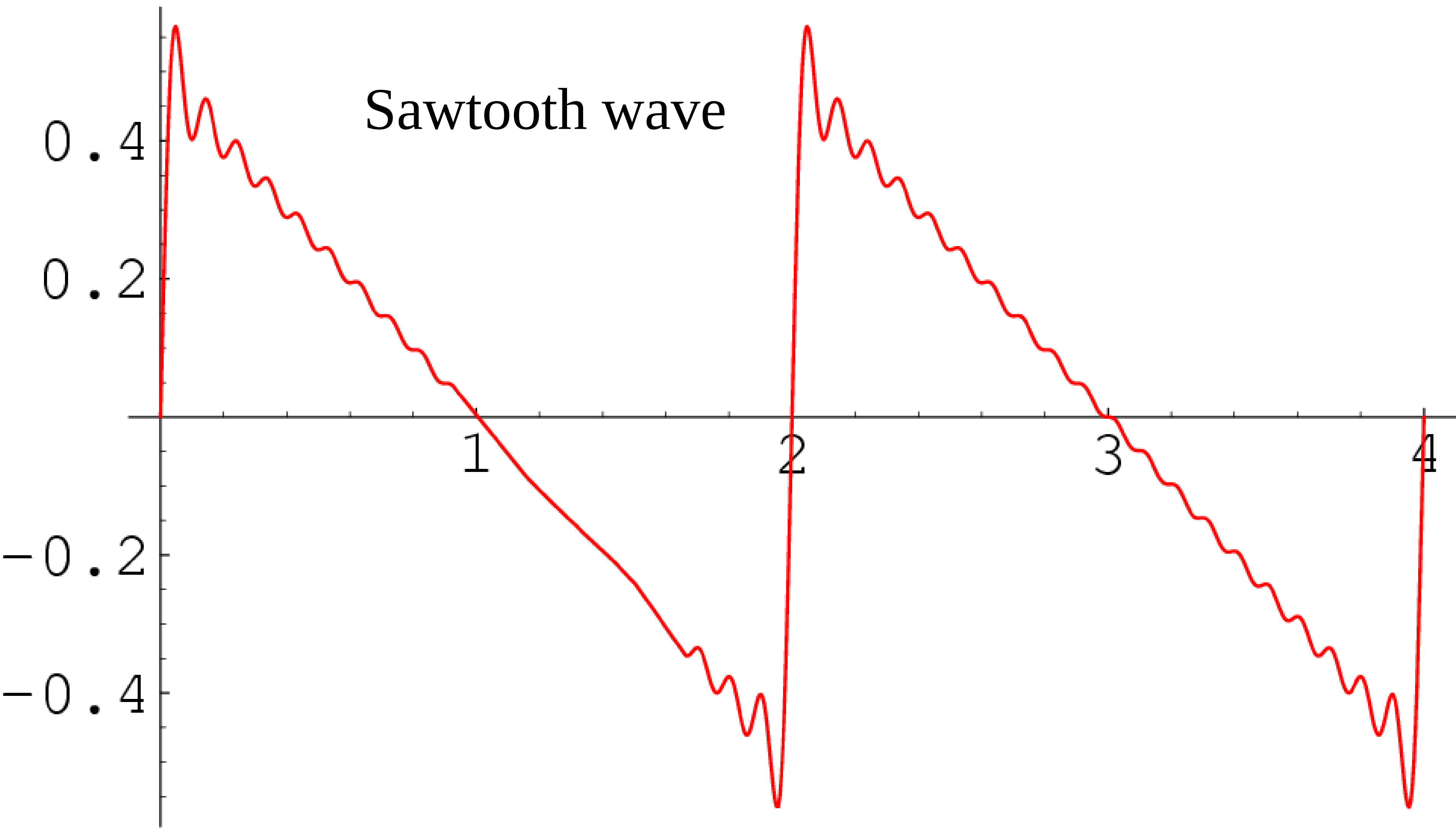
6 terms of the series



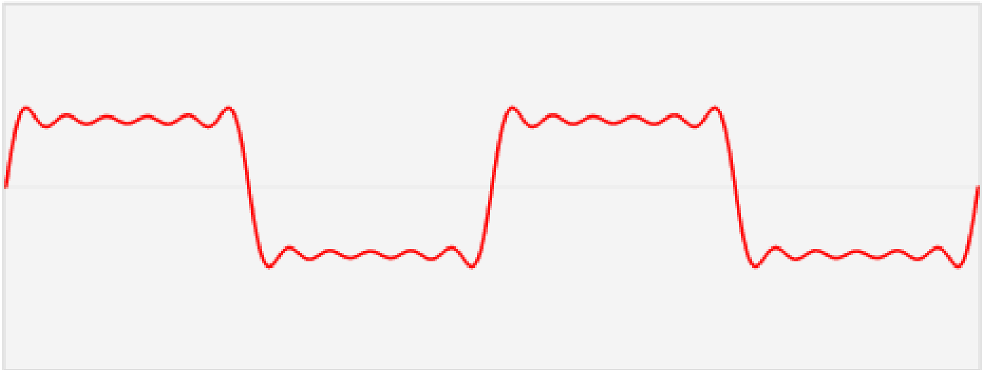
10 terms of the series

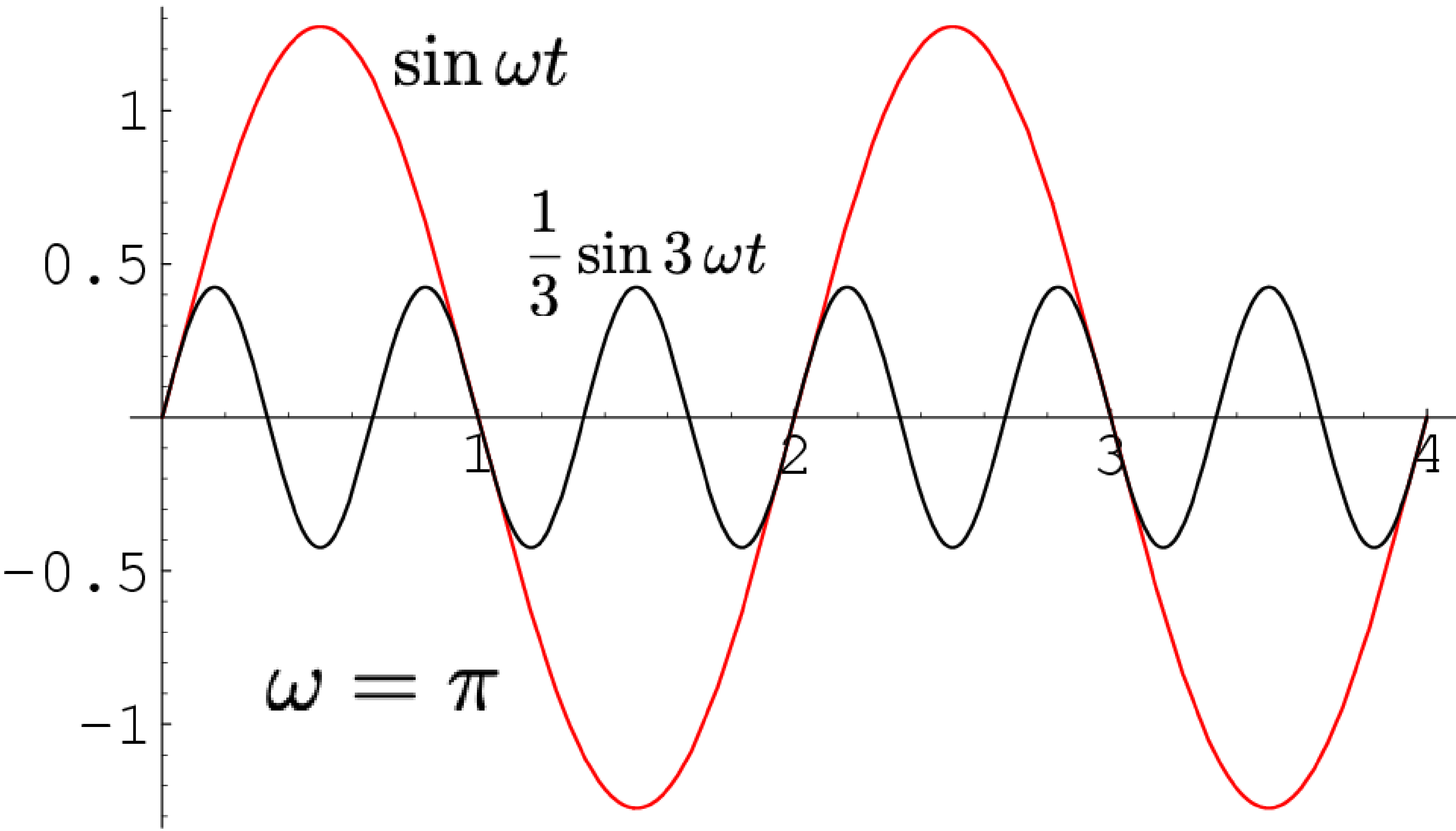


20 terms of the series



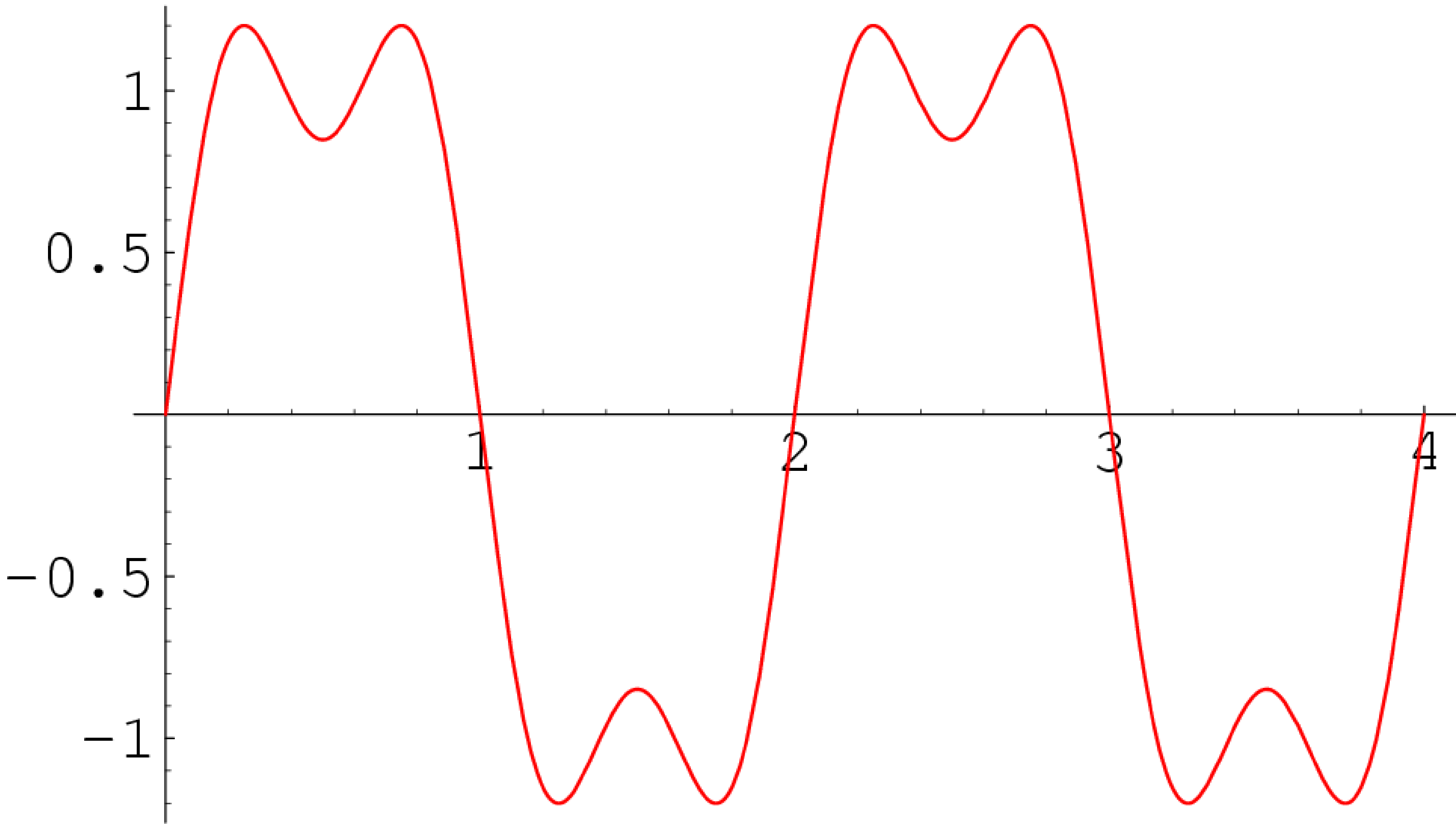
Fourier transform: Square Wave



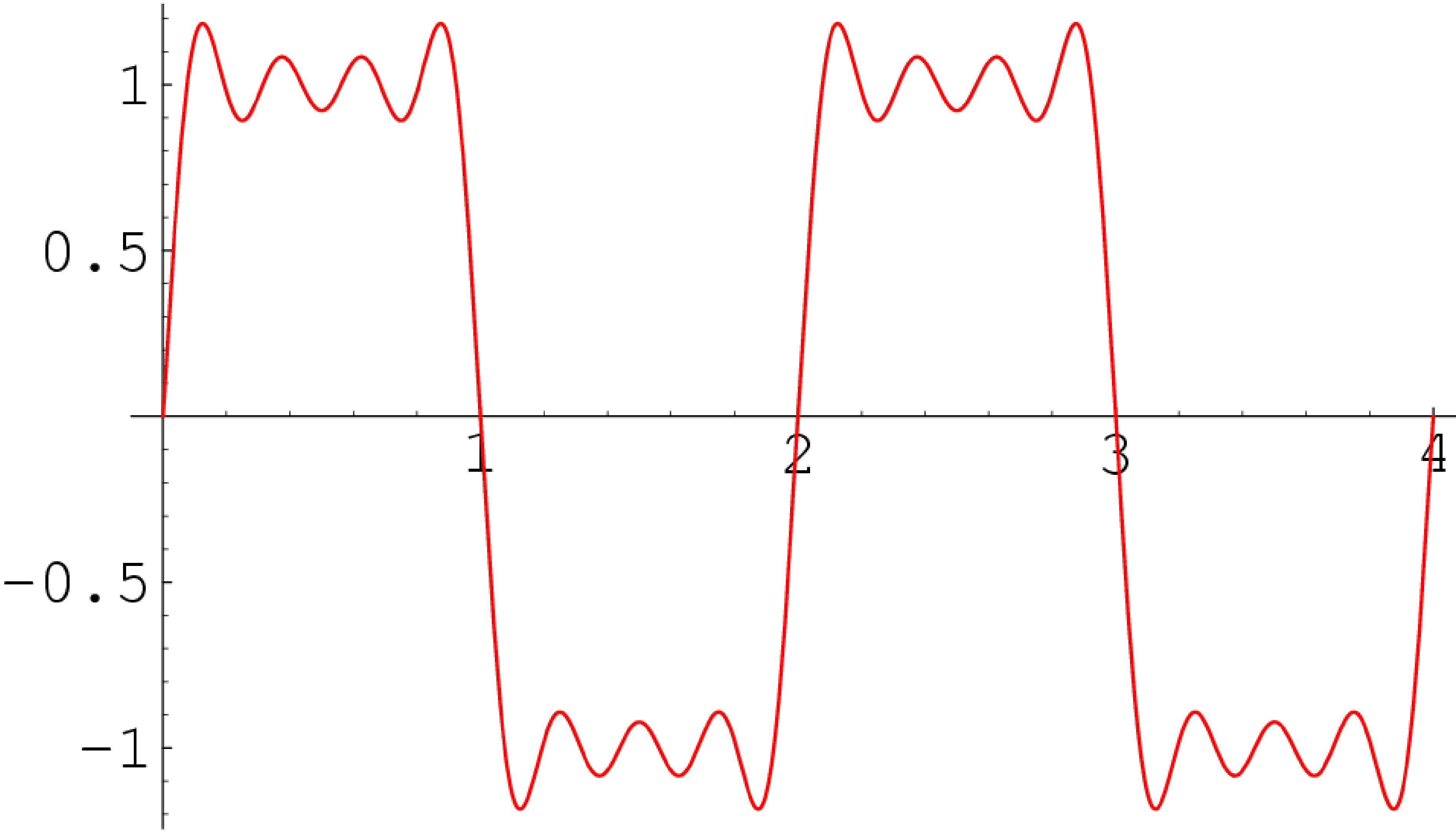


$\sin \omega t$
$\frac{1}{3} \sin 3\omega t$
$\omega = \pi$
1
0.5
-0.5
-1
1
2
3
4

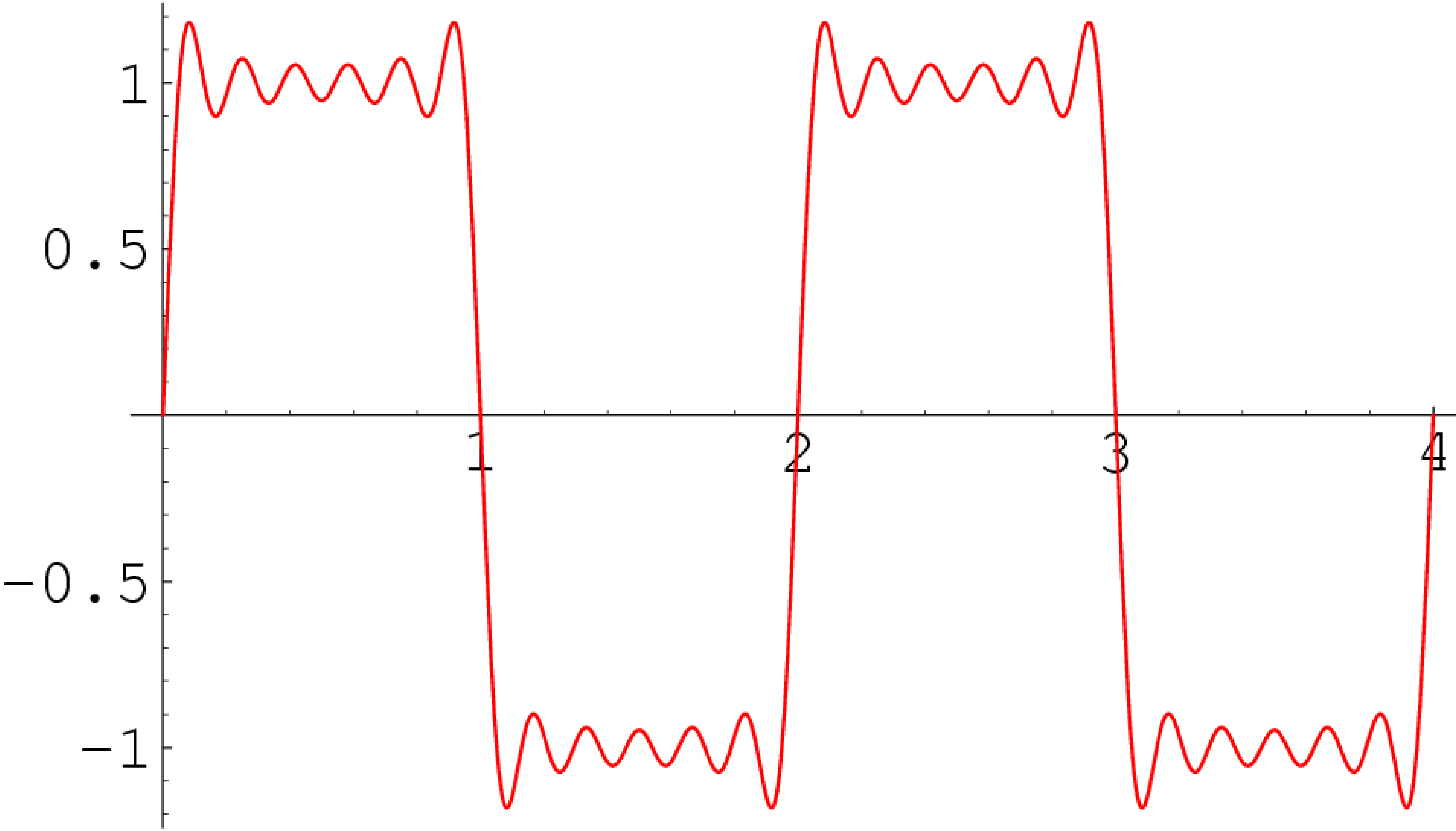
$$\sin \omega t + \frac{1}{3} \sin 3\,\omega t$$



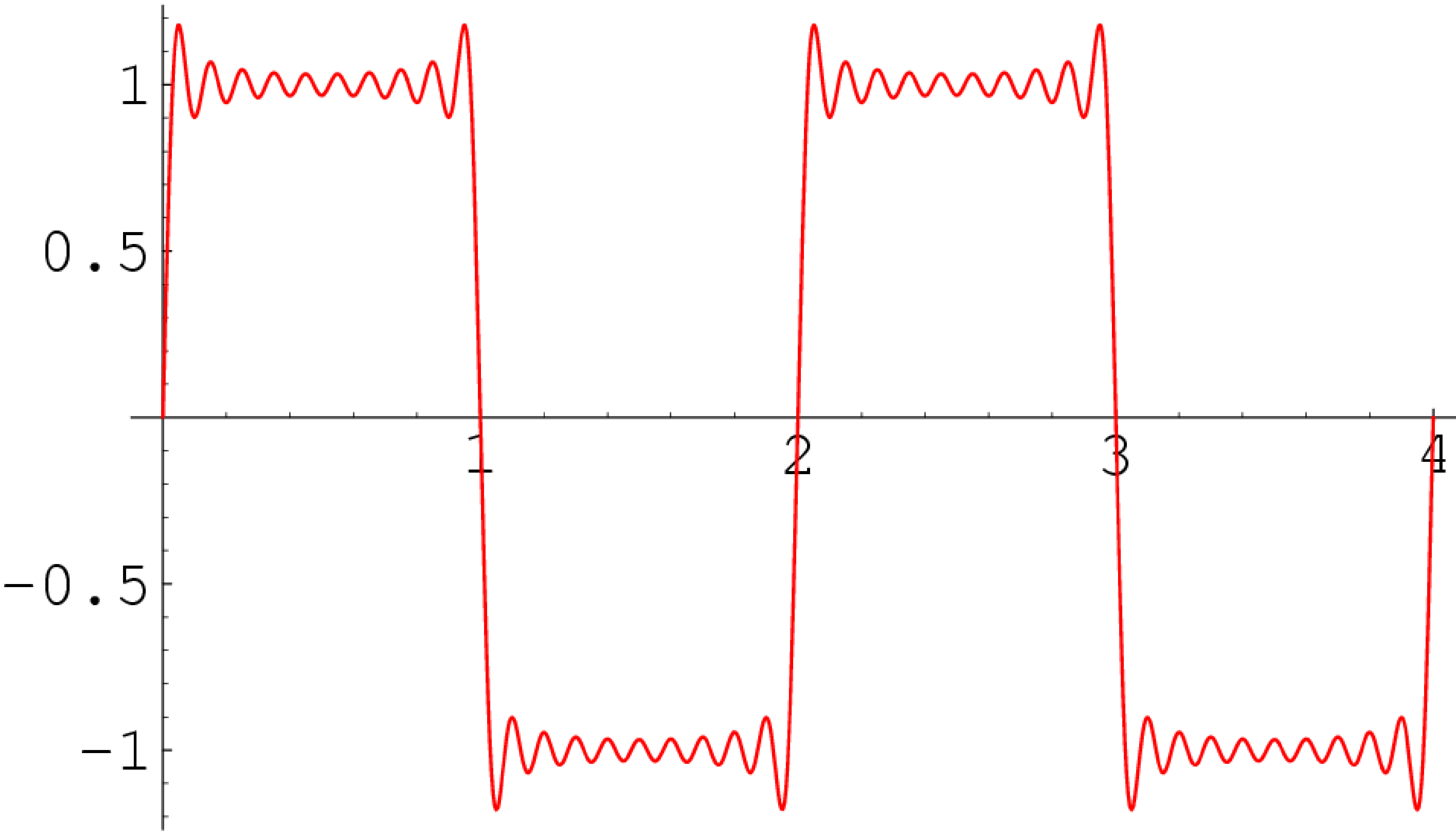
$$\sin \omega t+\frac{1}{3}\sin 3\,\omega t+\frac{1}{5}\sin 5\,\omega t+\frac{1}{7}\sin 7\,\omega t$$



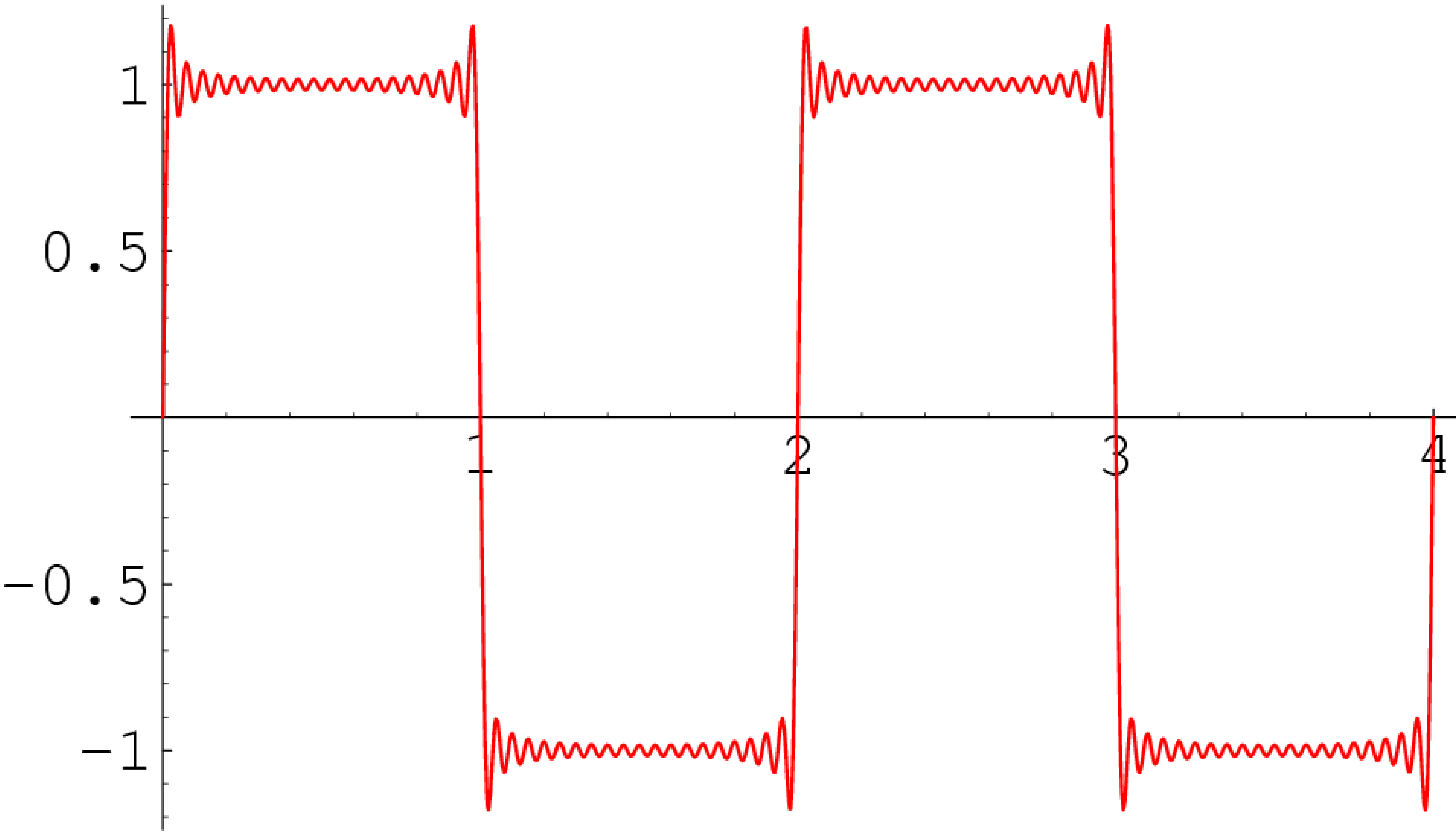
6 terms of the series



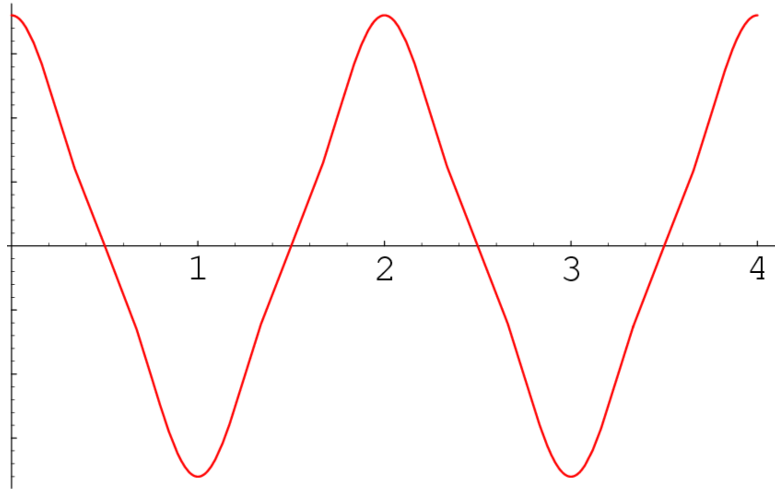
10 terms of the series



20 terms of the series

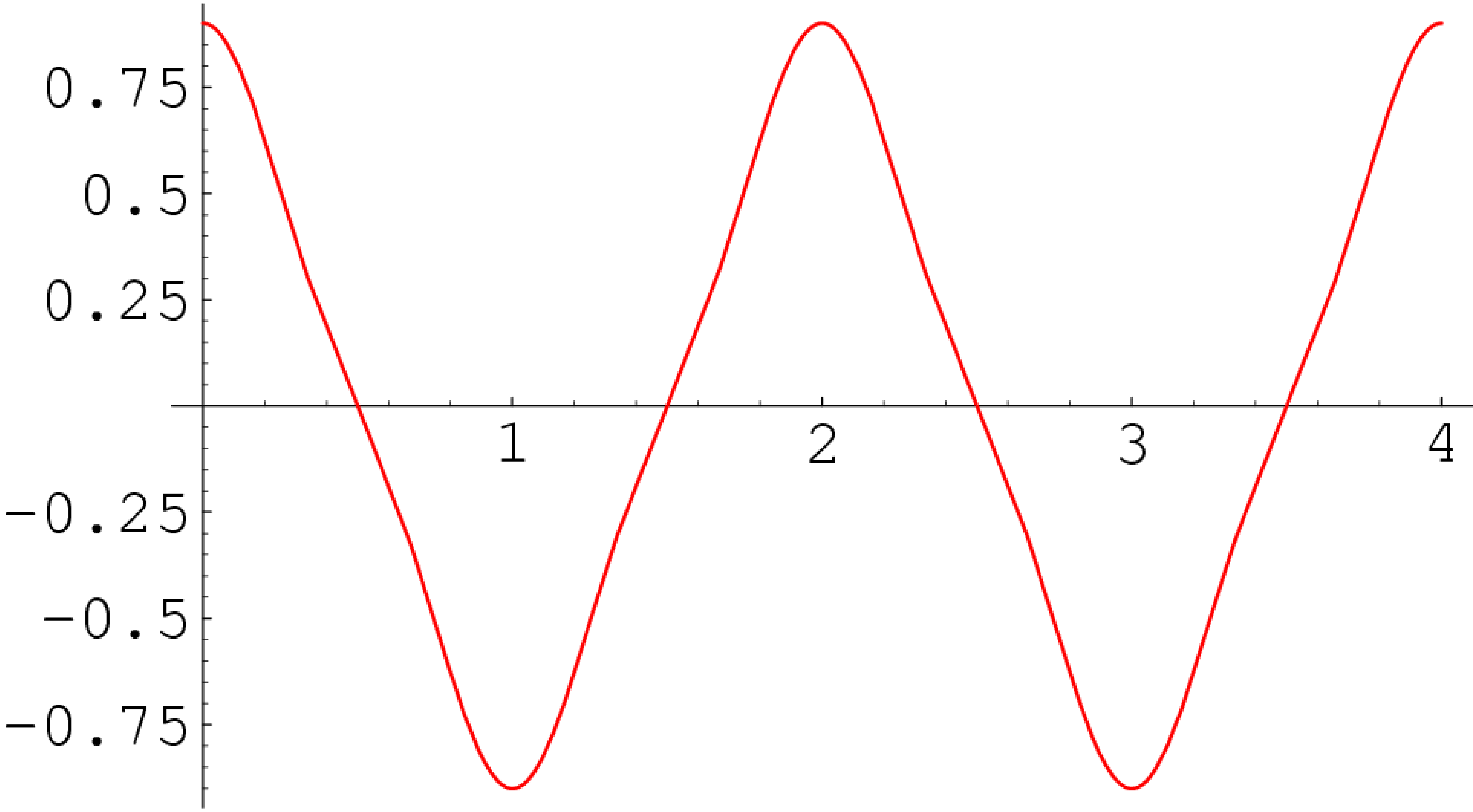


Fourier transform: Triangular Wave

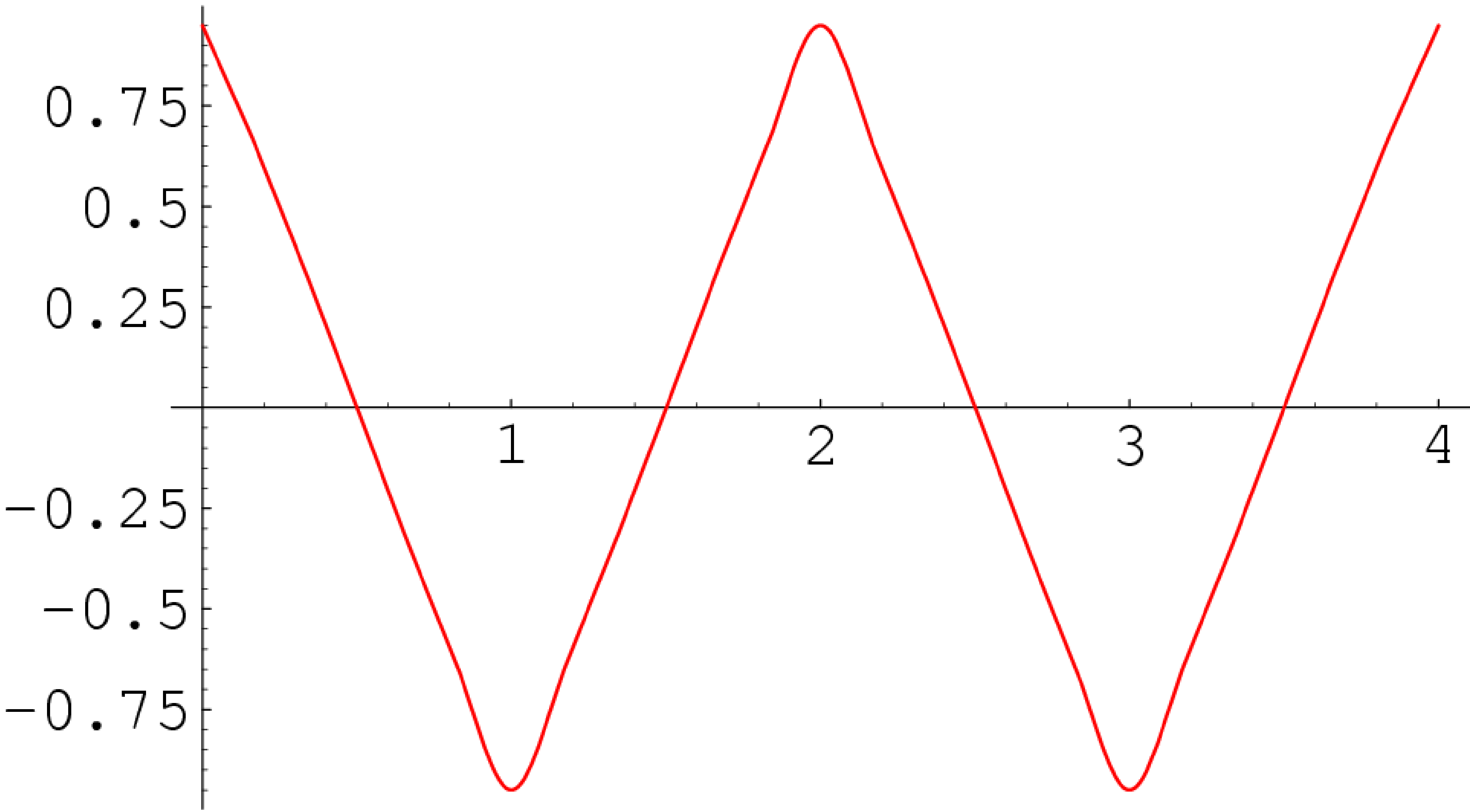


Triangular wave

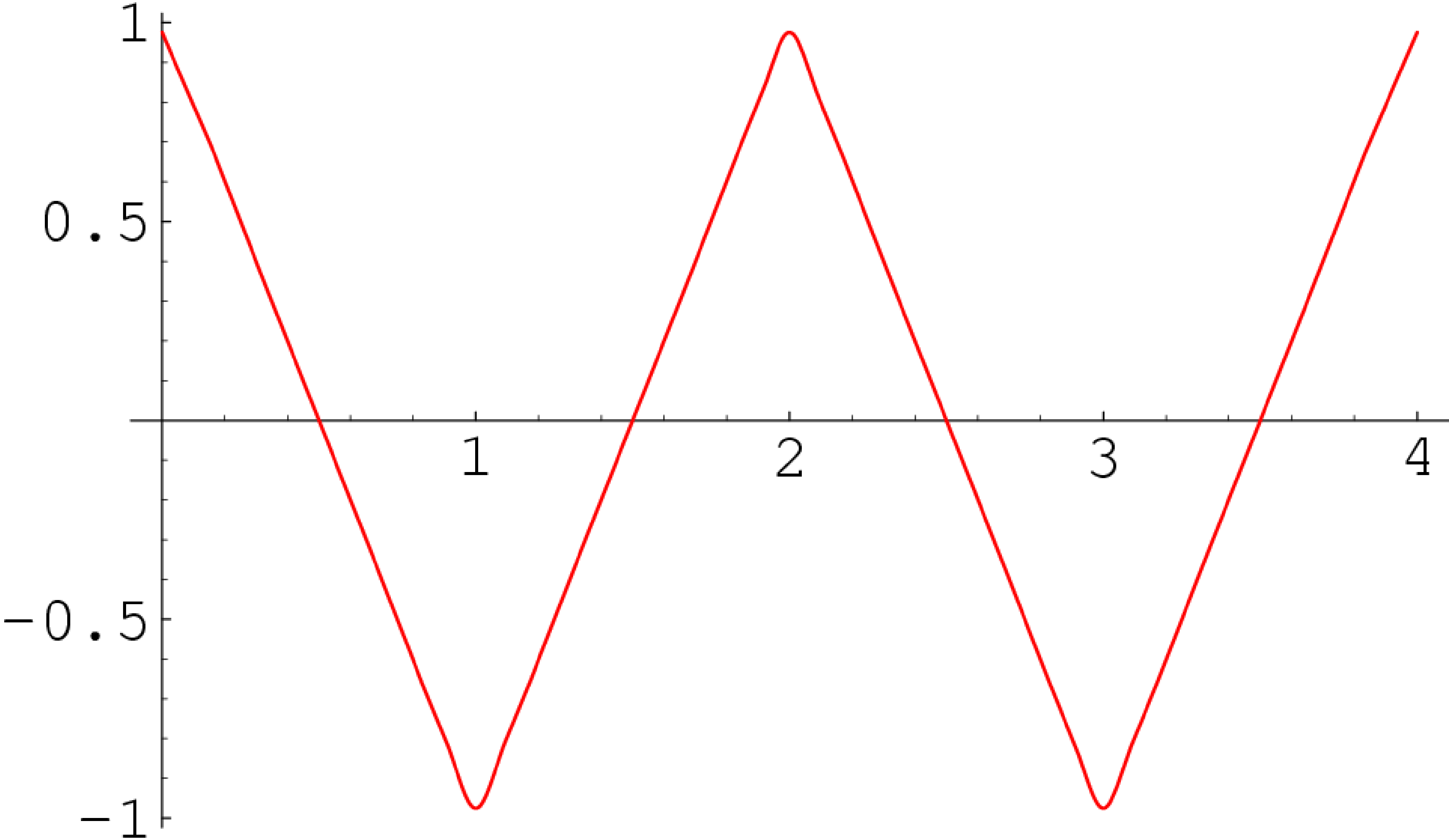
$$\cos \omega t + \frac{1}{9} \cos 3\omega t$$



$$\cos\omega t + \frac{1}{9}\cos 3\,\omega t + \frac{1}{25}\cos 5\,\omega t + \frac{1}{49}\cos 7\,\omega t$$

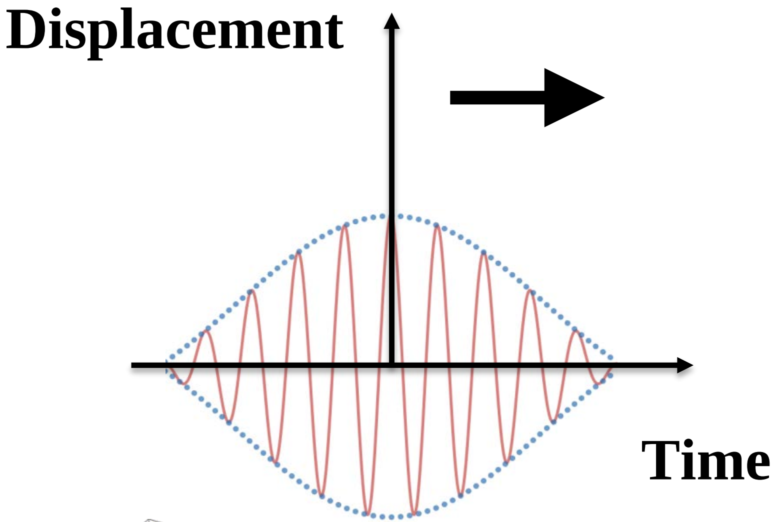


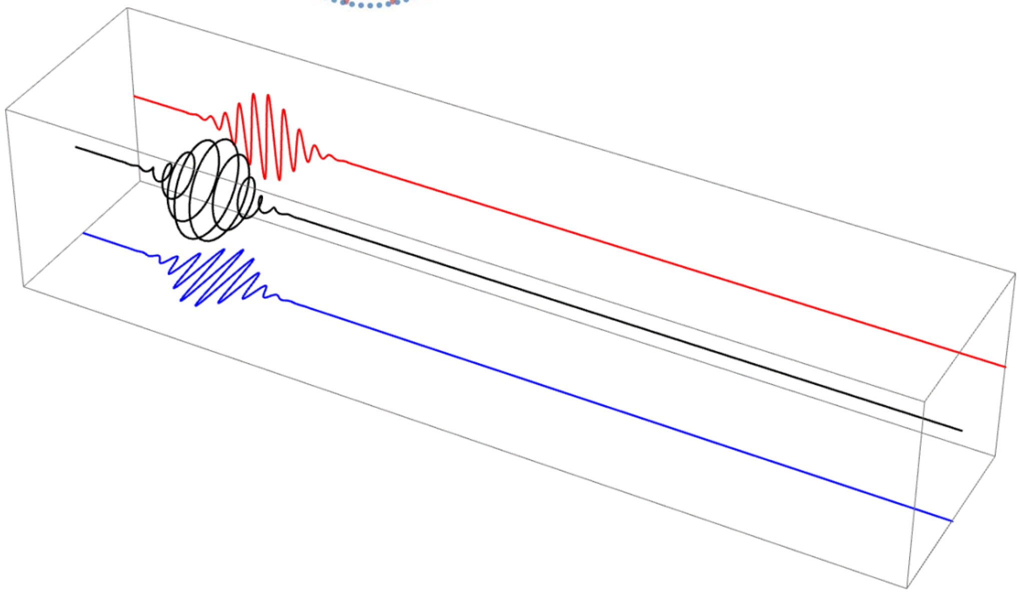
8 terms of the series



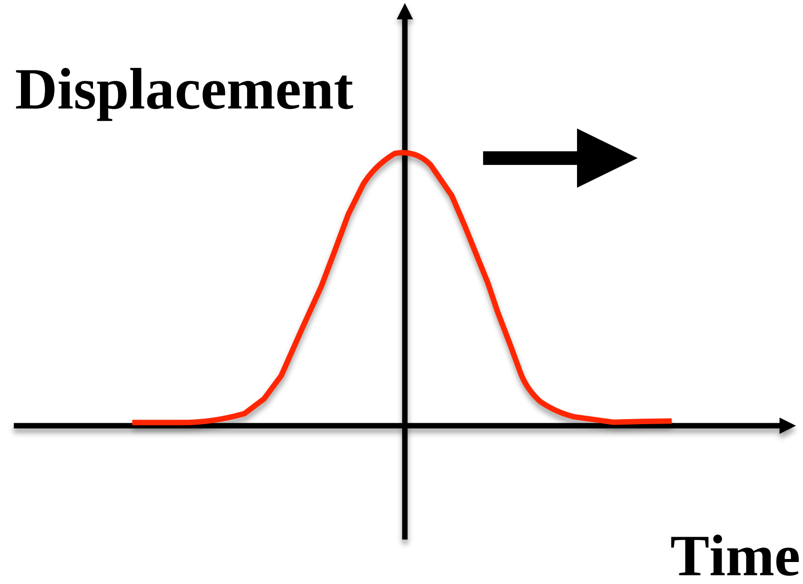
Wave packets and pulses

Wave packet

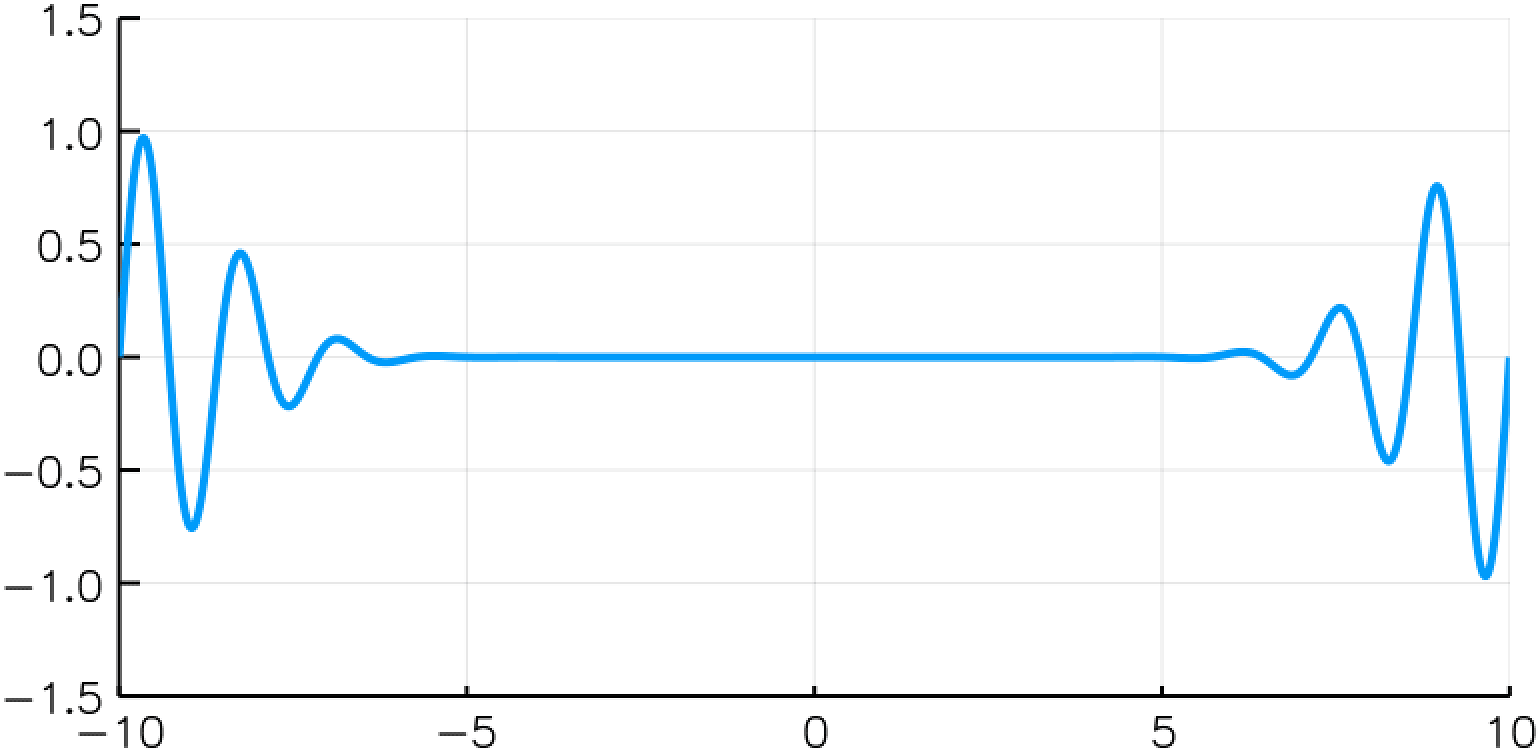




Pulses



Wave packet propagation



Let us synthesize a wave train by superposing a number of sinusoidal oscillations spanning a continuum window of frequencies

$$y(t) = \int \cos(\omega\, t)\, dA(\omega)$$

Spectral density : $\rho(\omega) = \dfrac{dA(\omega)}{d\omega}$

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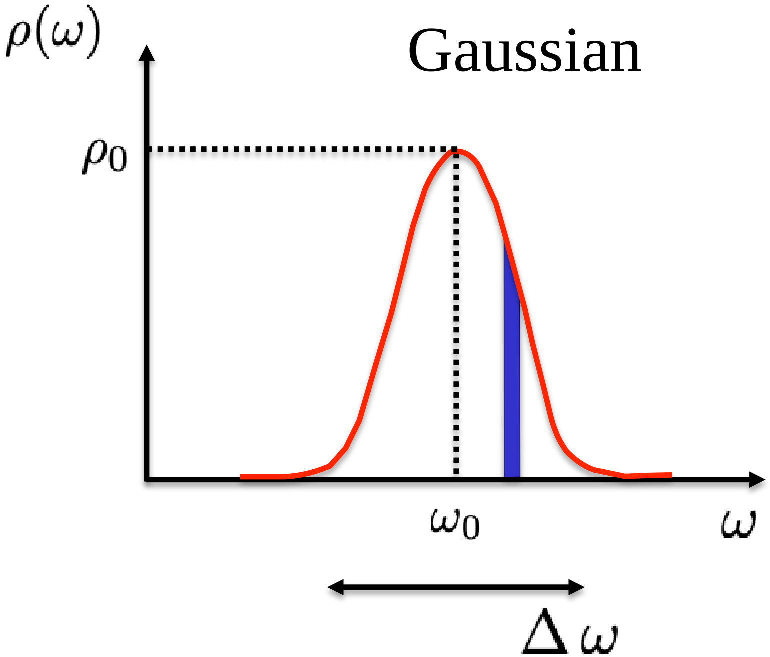
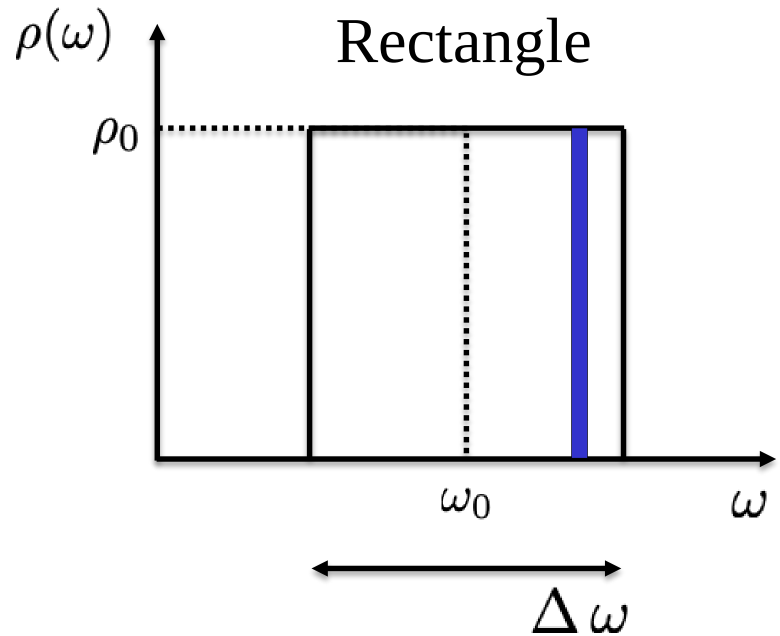


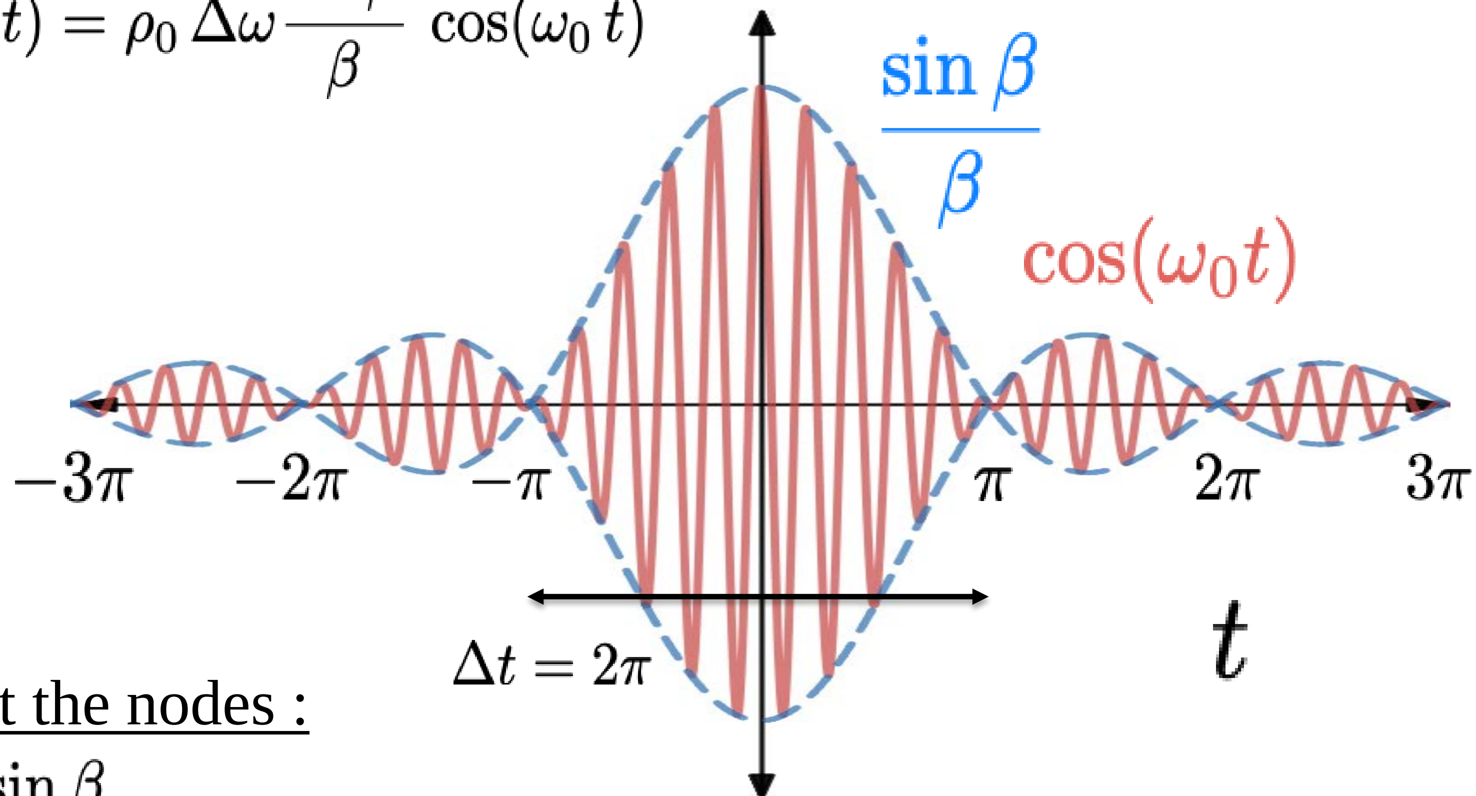


$$\beta = \frac{\Delta\omega\, t}{2}$$

$$y(t) = \int_{\omega_0 - \frac{\Delta\omega}{2}}^{\omega_0 + \frac{\Delta\omega}{2}} \rho_0\, \cos(\omega\, t)\, d\omega$$

$$= \rho_0\, \Delta\omega \frac{\sin\beta}{\beta}\, \cos(\omega_0\, t)$$

$$y(t) = \rho_0 \, \Delta\omega \frac{\sin\beta}{\beta} \, \cos(\omega_0 \, t)$$



<u>At the nodes :</u>

$$\frac{\sin\beta}{\beta} = 0$$

$$\sin\beta = 0$$

$$\beta = n\,\pi$$

$$\boxed{\beta = \frac{\Delta\omega\, t}{2}}$$

<u>Uncertainty product :</u>

$$\Delta\omega \cdot \Delta t = \text{const.} \approx 4\pi$$

Coherence Time and Coherence Length

For non-dispersive waves $\omega = c\,k$

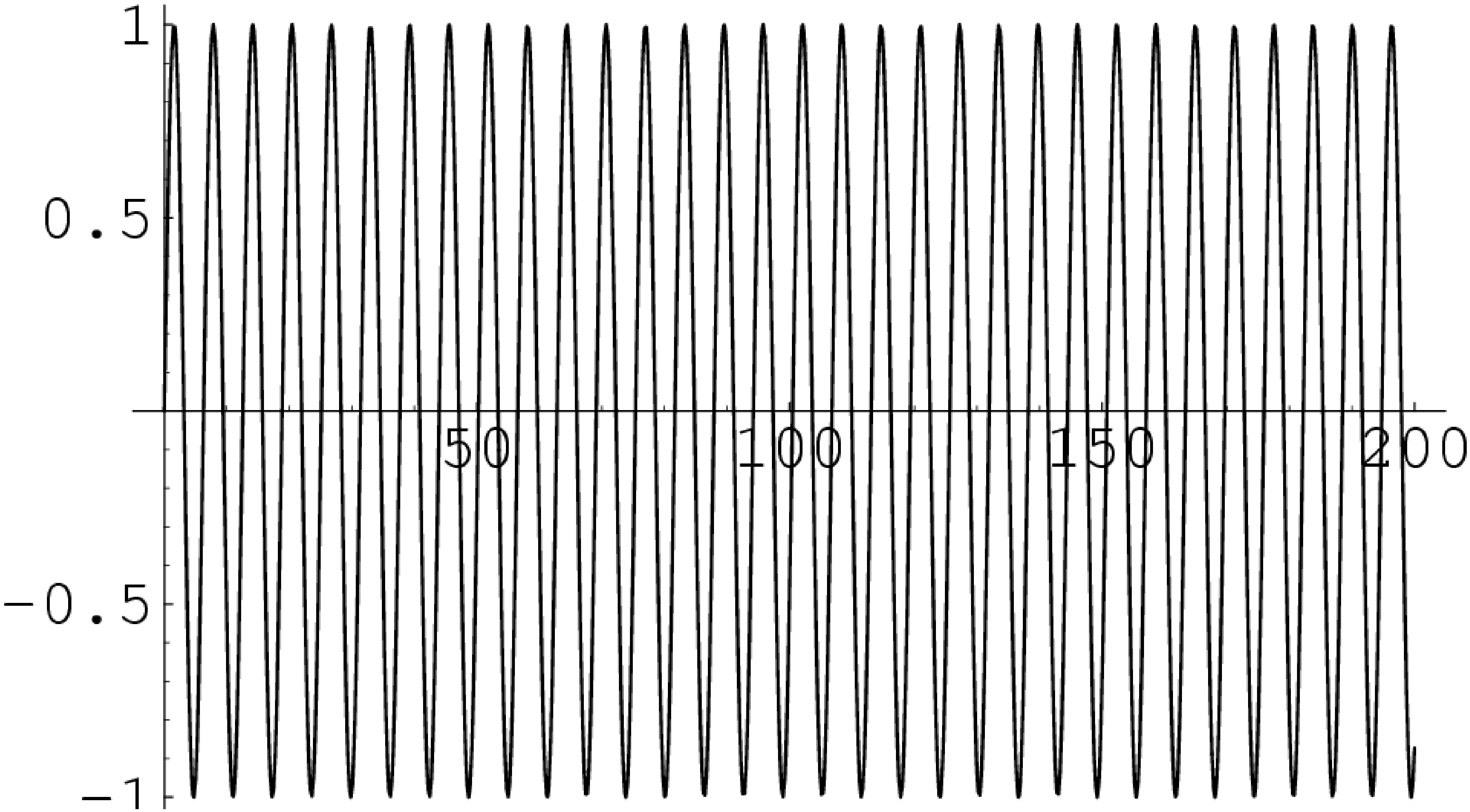
Signal emitted over a time interval Δt

Travels a distance $\Delta x = c\Delta t$

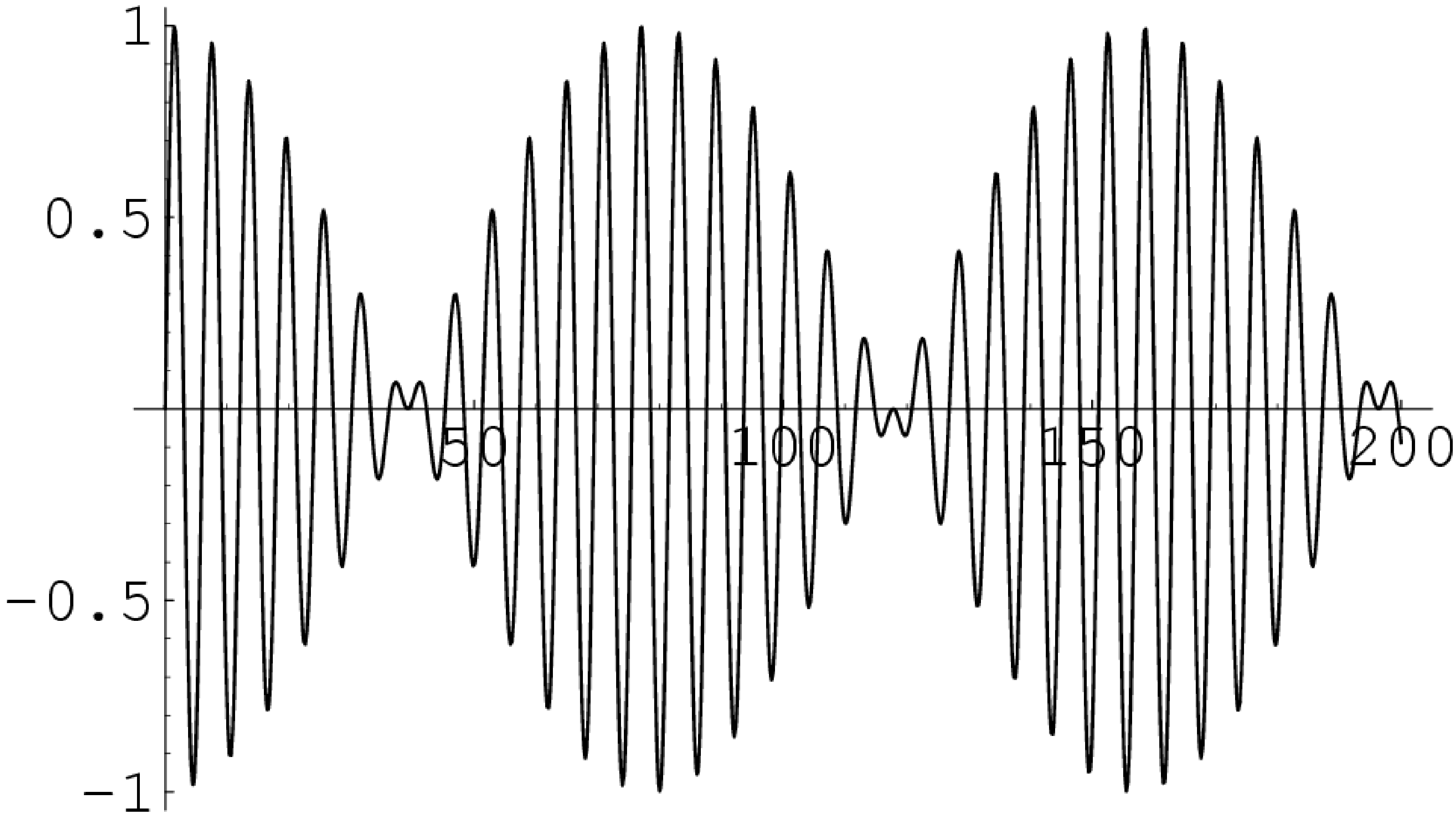
<u>Uncertainty product :</u>

$$\Delta\omega \cdot \Delta t = c\Delta k \cdot \Delta t = \boxed{\Delta k \cdot \Delta x \approx \pi}$$

superposition of waves having same amplitudes but different frequencies y(t) = Sin t   0 < t < 200

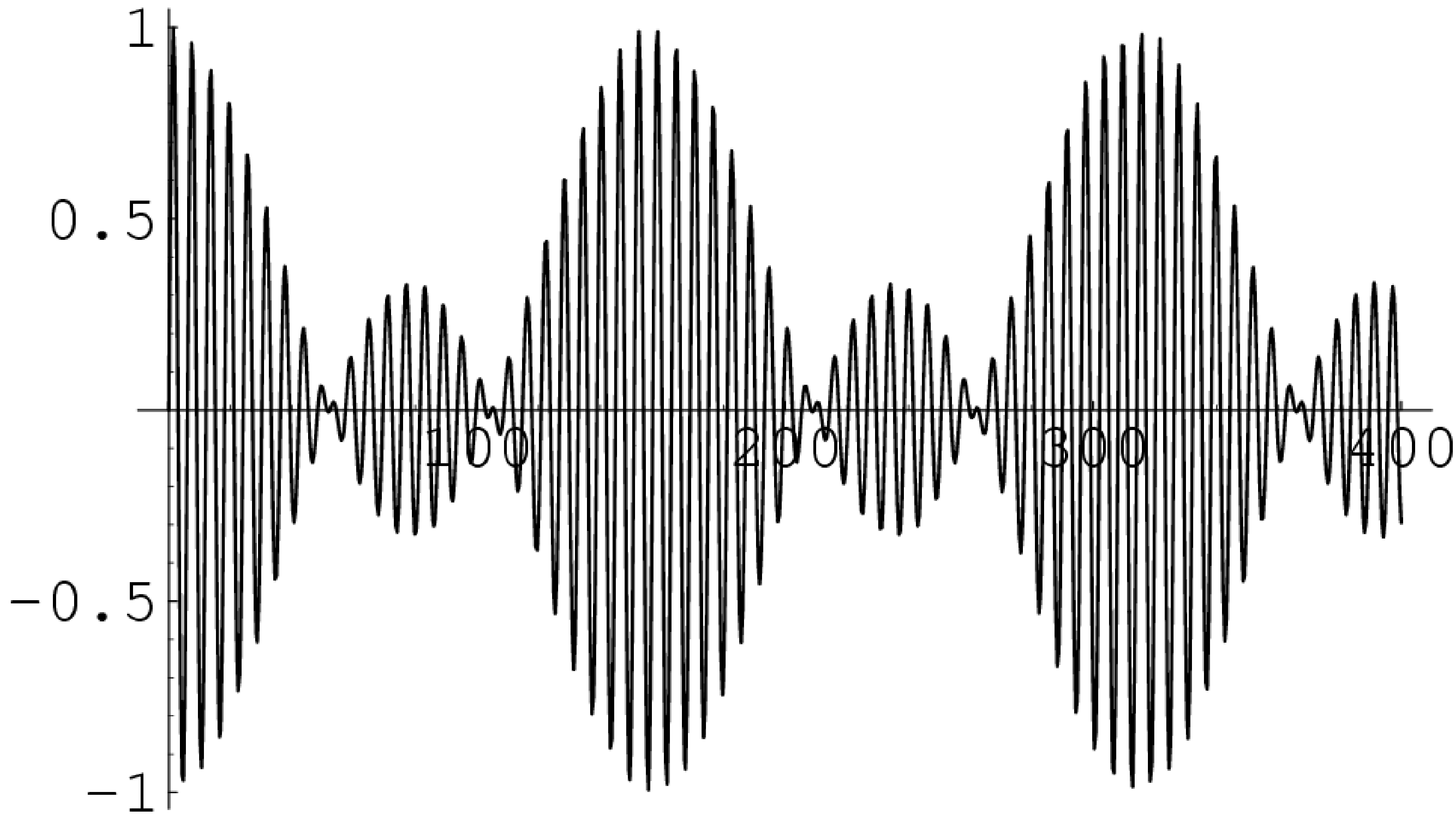


y(t) = [Sin t + Sin (1.08 t)]/2   0 < t < 200

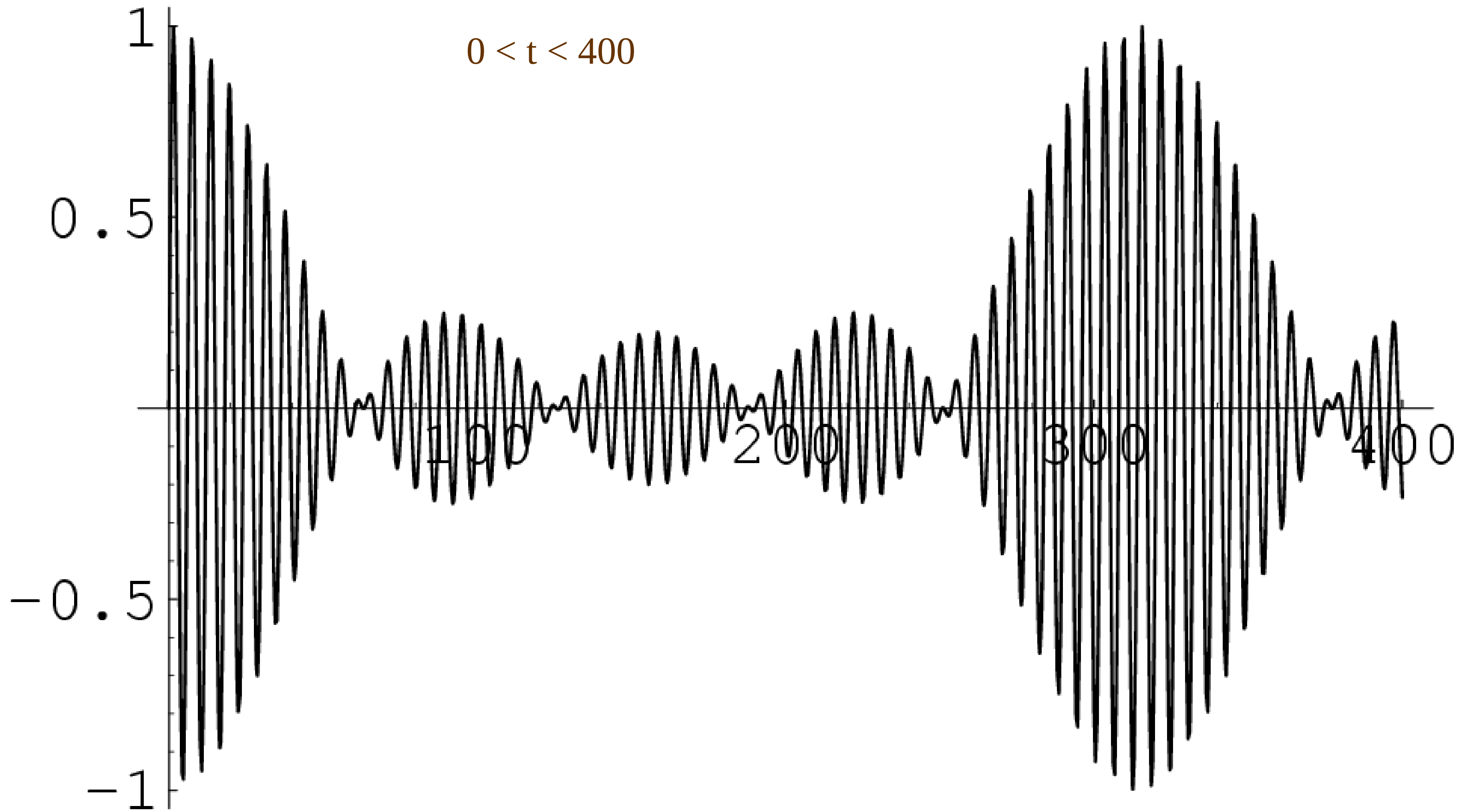


y(t) = [Sin t + Sin(1.04 t) + Sin (1.08 t)]/3

0 < t < 400

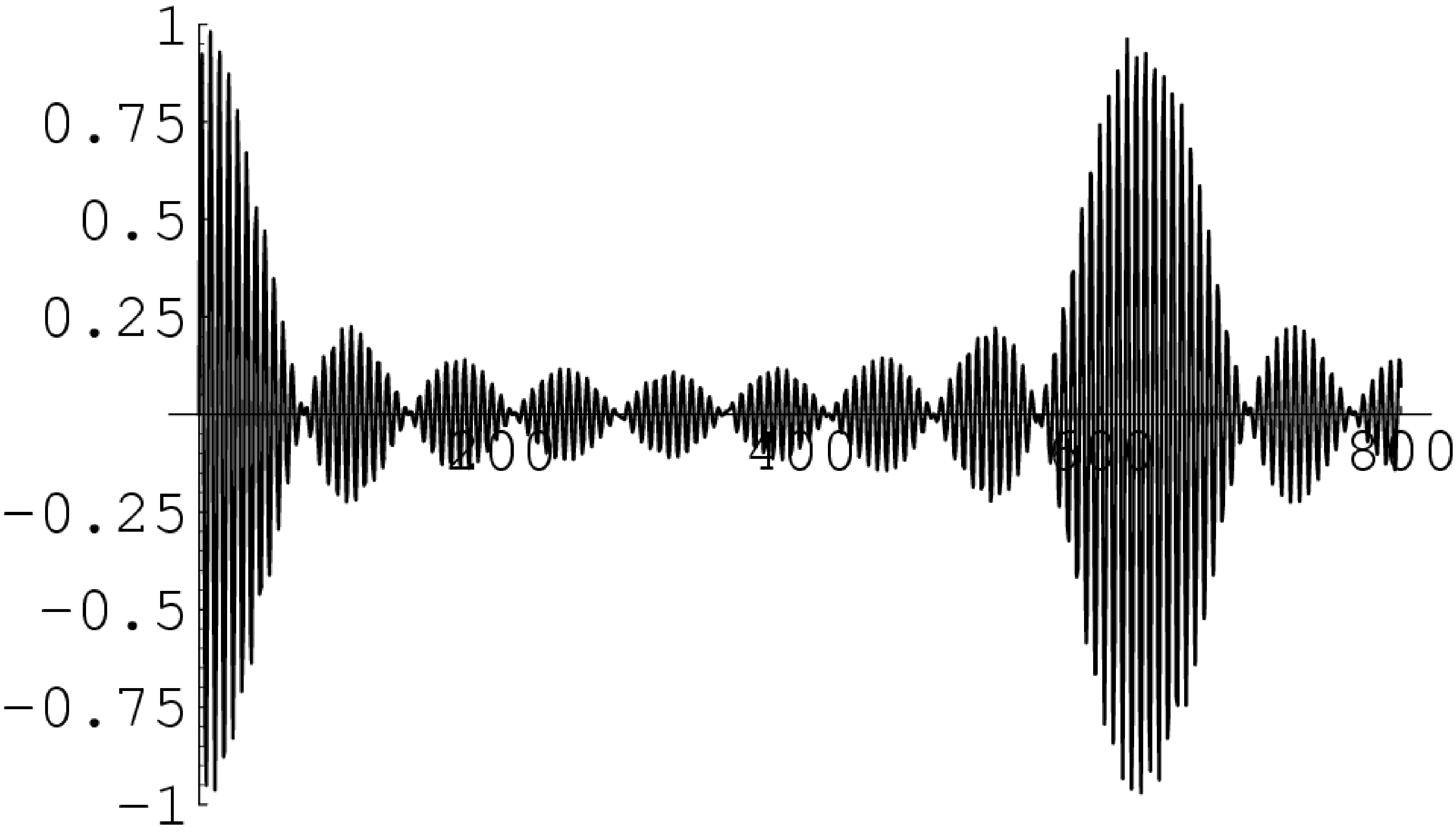


$$y(t) = [\mathrm{Sin}\ t + \mathrm{Sin}(1.02\ t) + \mathrm{Sin}\ (1.04\ t) + \mathrm{Sin}(1.06\ t) + \mathrm{Sin}\ (1.08\ t)]/5$$



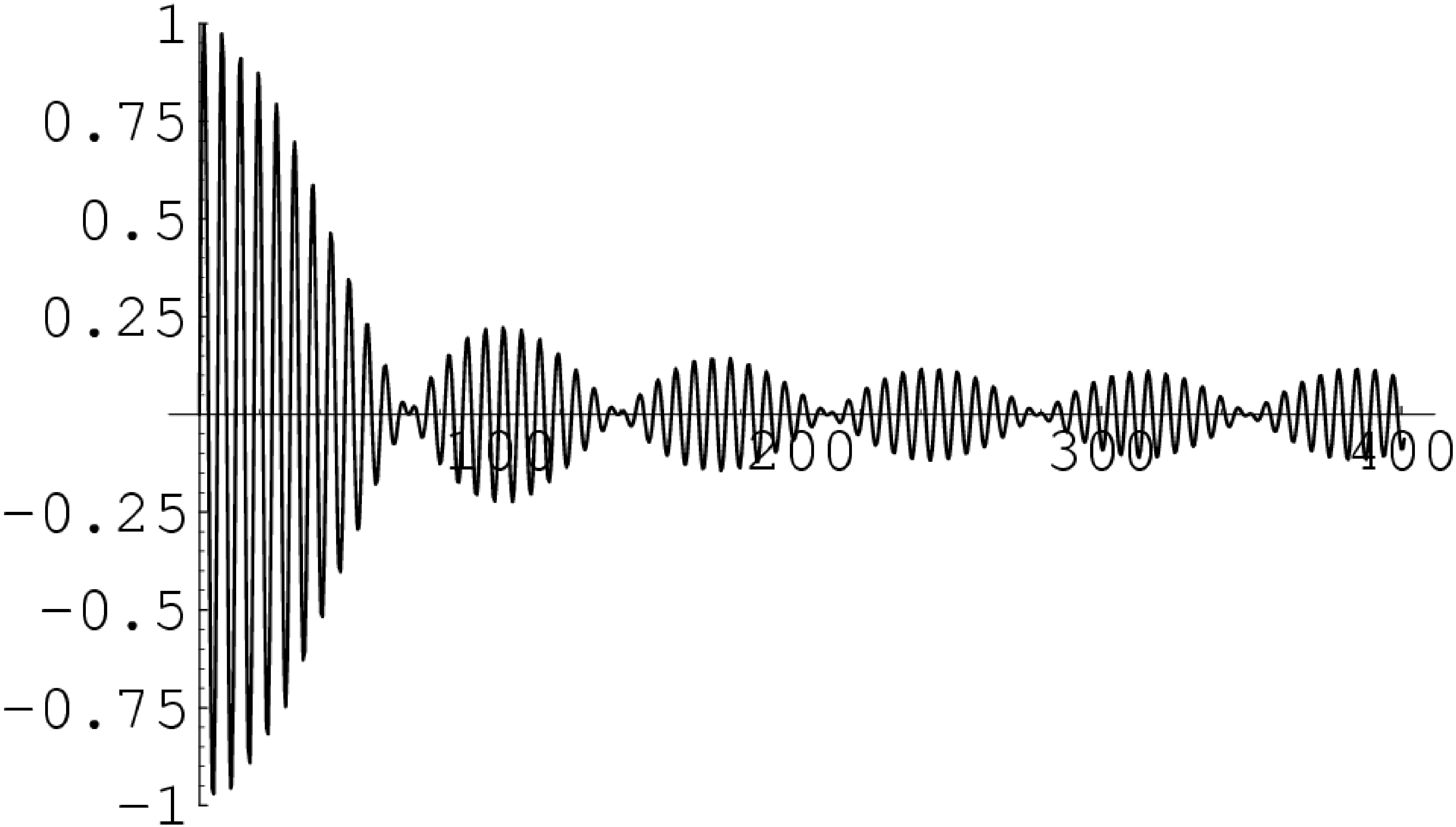
$$y(t) = [\sin t + \sin(1.01\,t) + \sin(1.02\,t) + \sin(1.03\,t) + \sin(1.04\,t) + \sin(1.05\,t) + \sin(1.06\,t) + \sin(1.07\,t) + \sin(1.08\,t)]/9$$

$$0 < t < 800$$

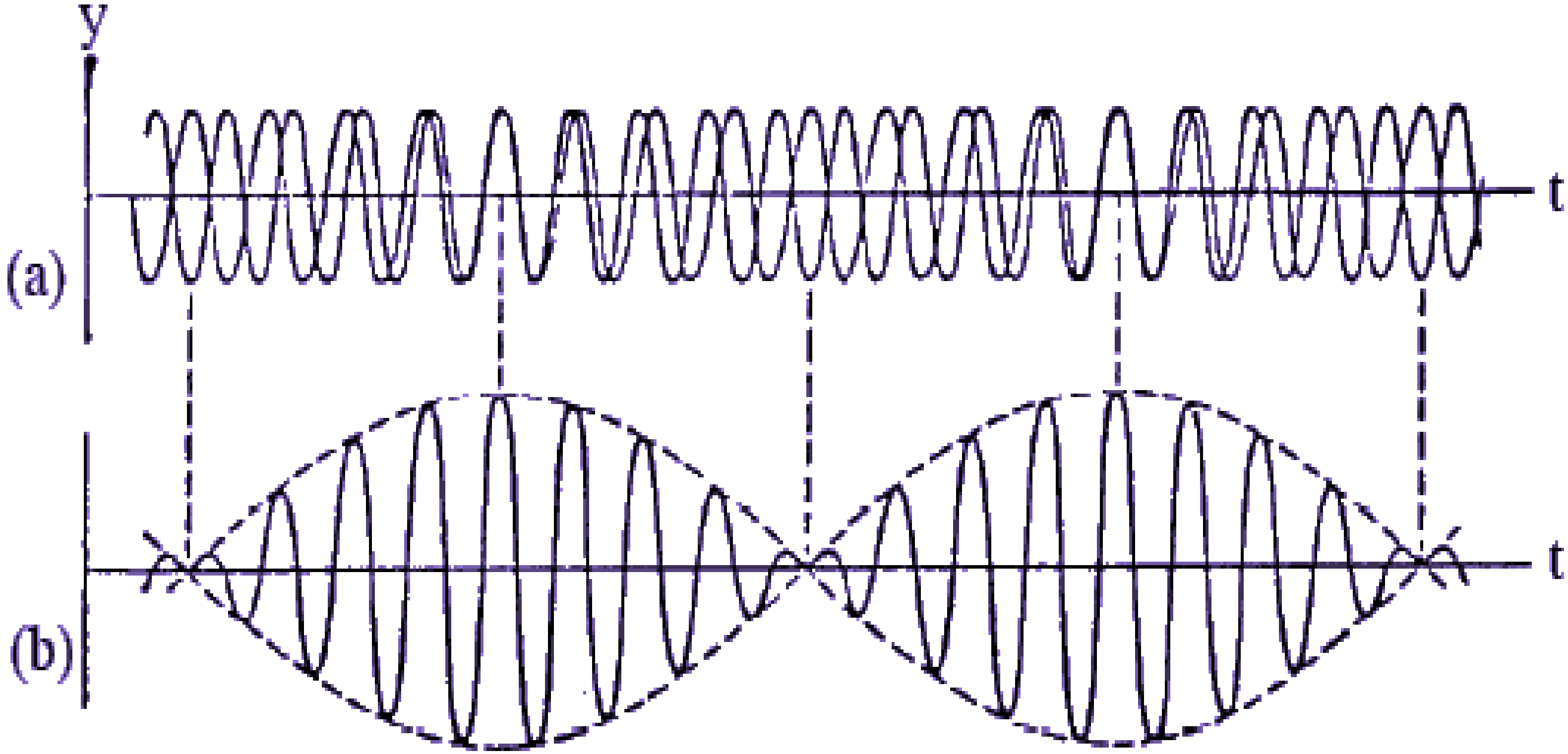


$$y(t) = [\mathrm{Sin}\, t + \mathrm{Sin}(1.01\, t) + \mathrm{Sin}\,(1.02\, t) + \mathrm{Sin}(1.03\, t) + \mathrm{Sin}\,(1.04\, t) + \mathrm{Sin}\,(1.05\, t) + \mathrm{Sin}\,(1.06\, t) + \mathrm{Sin}\,(1.07\, t) + \mathrm{Sin}\,(1.08\, t)]/9$$

$$0 < t < 400$$



Why does it happen?



Waves

Phase velocity

Group velocity

Sinusoidal waves

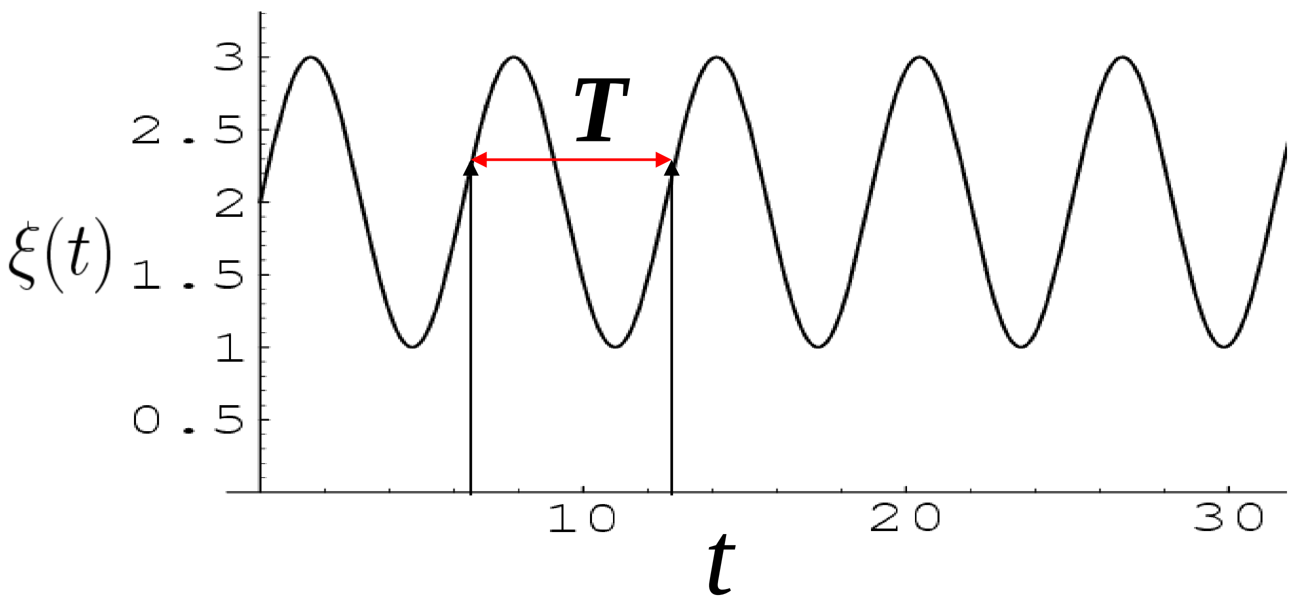
They are Progressive Waves

This is standing wave

$$\psi = -2i \exp\left(i\omega t\right) \sin kx$$

$$\omega T = 2\pi, \qquad t = T$$

$$\xi(t) = \xi(t+T)$$

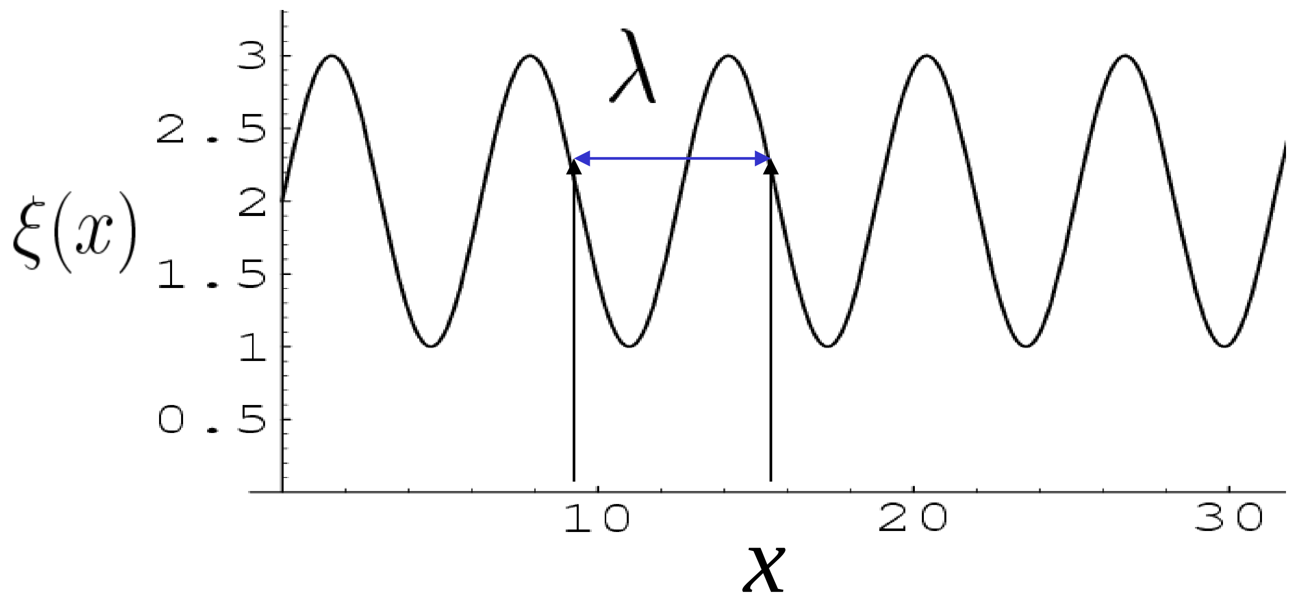


At, $t = t_0$

$$\xi(x) = D\exp(ikx), \qquad D = A\exp(-i\omega t_0)$$

$$k\lambda = 2\pi, \qquad x = \lambda$$

$$\xi(x) = \xi(x+\lambda)$$



$$\phi(x,t) = kx - \omega t$$

$$\phi = 0, \qquad x = 0,\ t = 0$$

New position of $\phi = 0,$ at Δt

Phase velocity = the speed with which the constant phase moves

$$\omega = \left| \frac{\partial \phi(x,t)}{\partial t} \right|$$

$$k = \left| \frac{\partial \phi(x,t)}{\partial x} \right|$$

Group velocity

$$\psi(x,t) = A\cos(k_1\, x - \omega_1\, t)$$

$$+ A\cos(k_2\, x - \omega_2\, t)$$

$$\psi(x,t) = 2A\cos(k\,x - \omega\,t)\cdot\cos(\Delta k\,x - \Delta\omega\,t)$$

Phase velocity $$v_p = \frac{\omega}{k}$$

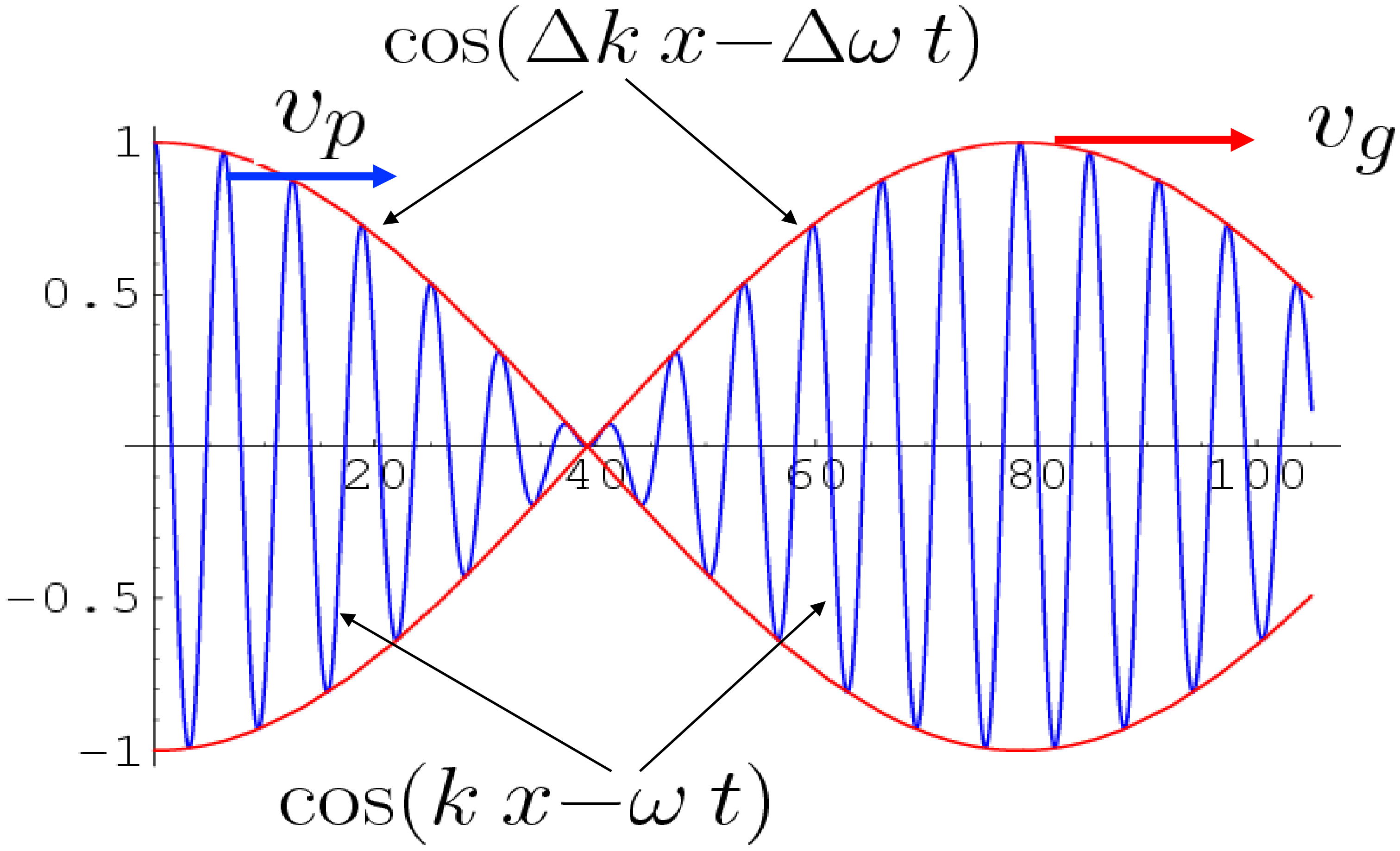
Group velocity $$v_g = \frac{\Delta\omega}{\Delta k} \longrightarrow \frac{\partial\omega}{\partial k}$$

$$\Delta k \rightarrow 0$$

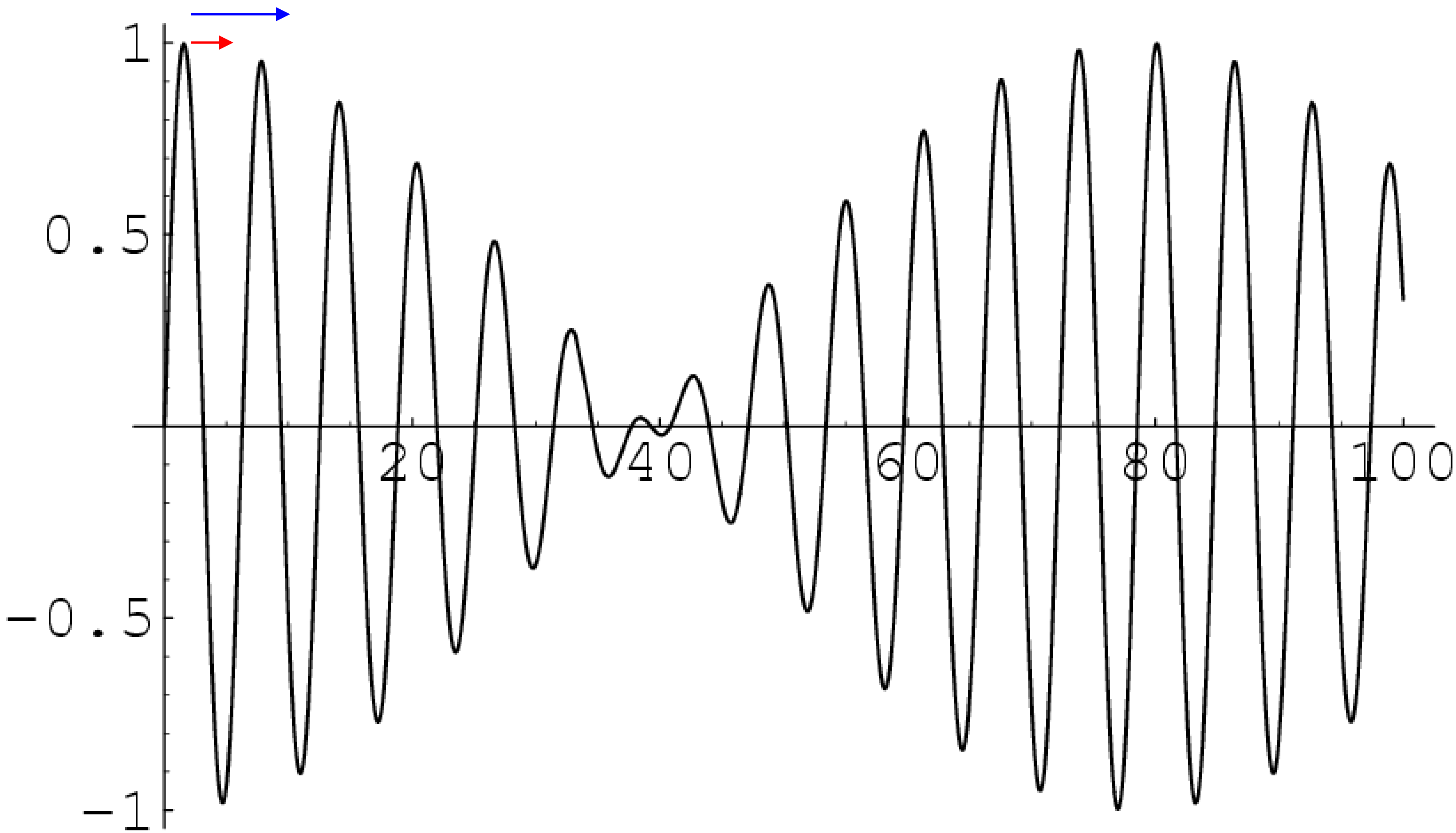
Wave packet consist of individual waves whose amplitude is modulated by an envelop

Speed of envelop=Group velocity (**v_g**)

Speed of wavelets=Phase velocity (**v_p**)



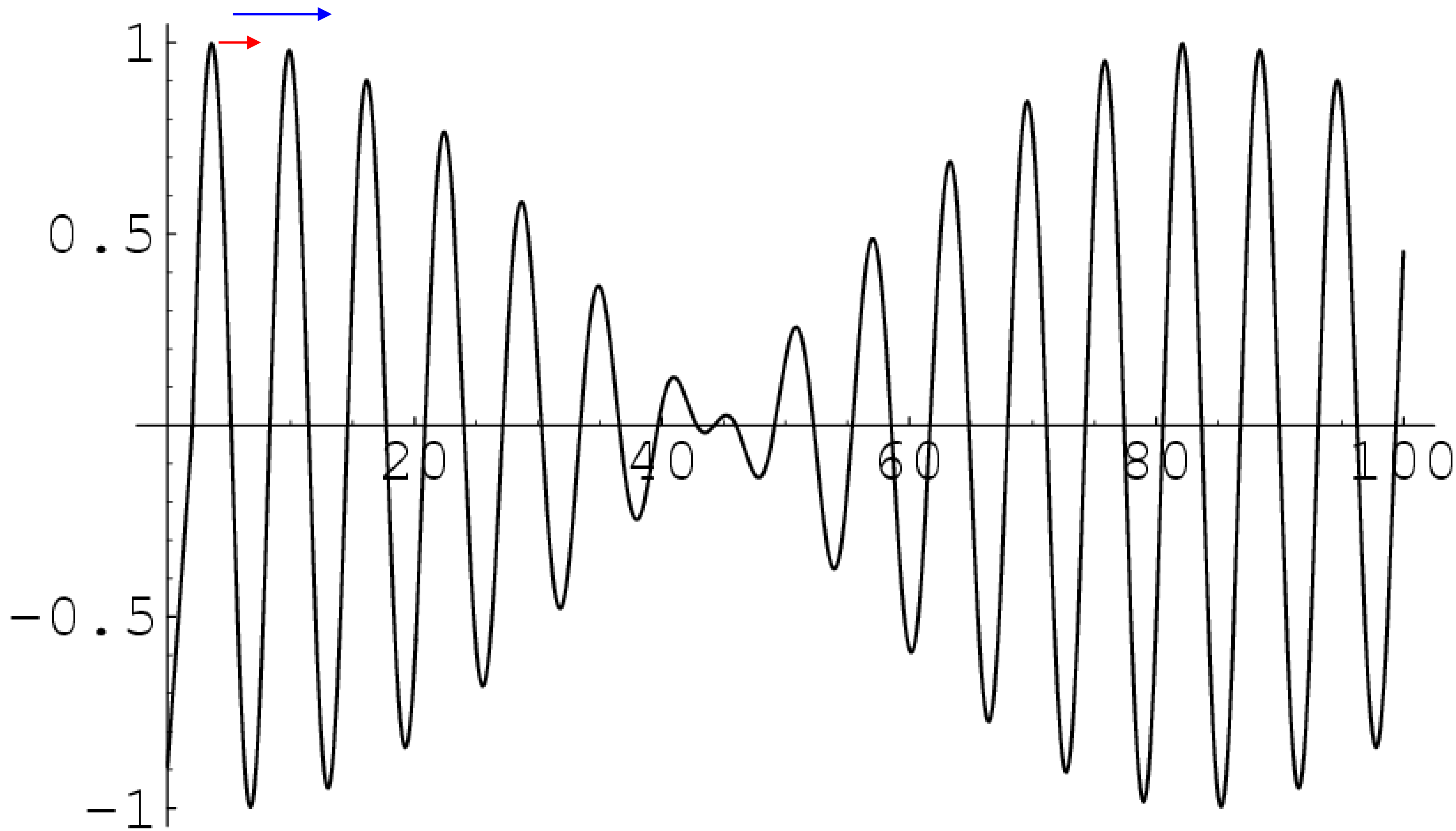
$\mathbf{V}_p < \mathbf{V}_g$



t = 0

$$\sin(1.00\ x - 2.0\ t)\cos(0.04\ x - 0.2\ t)$$

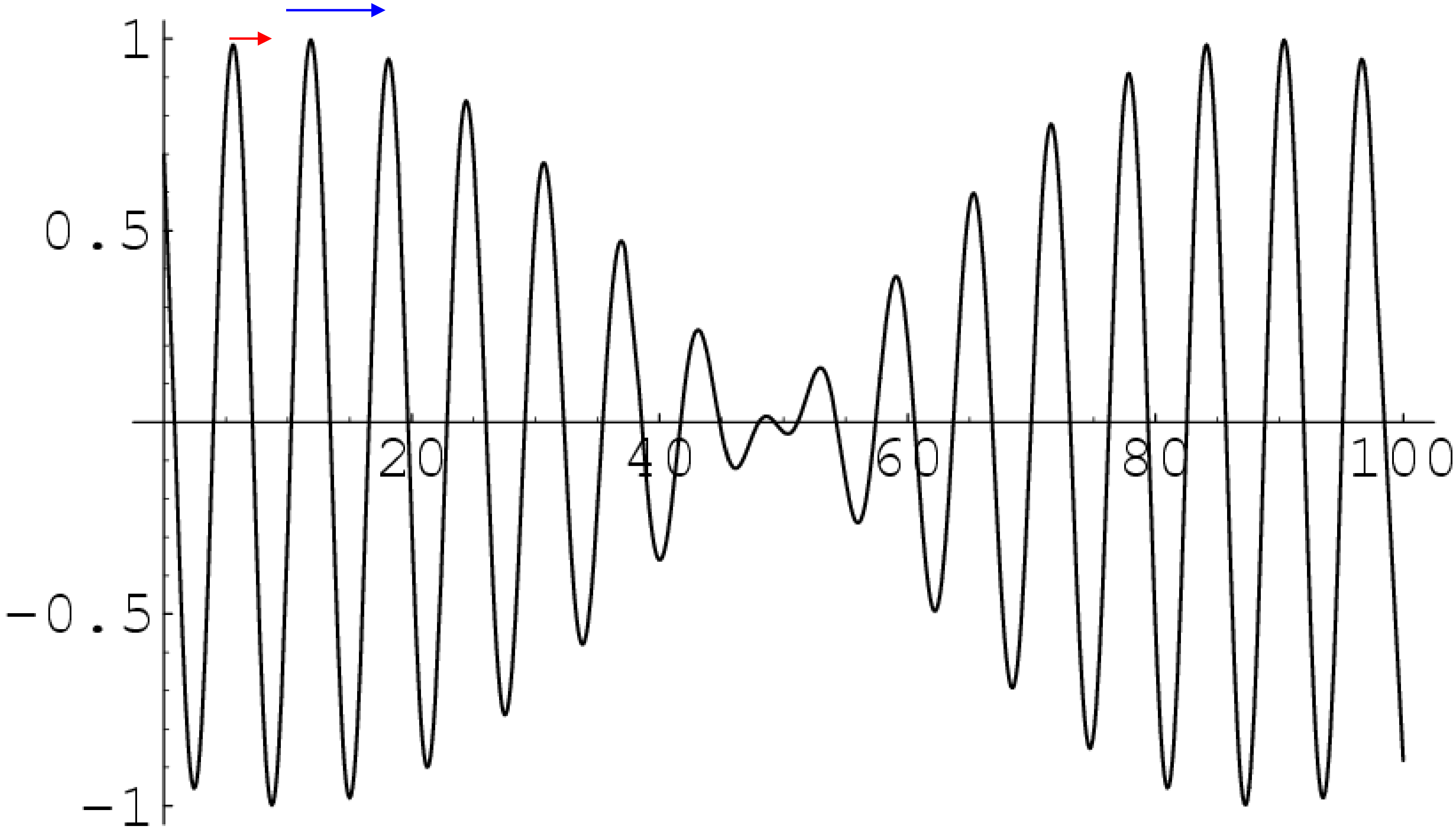
$\mathbf{V}_p < \mathbf{V}_g$



$t = 1$

$$\sin(1.00\,x - 2.0\,t)\cos(0.04\,x - 0.2\,t)$$

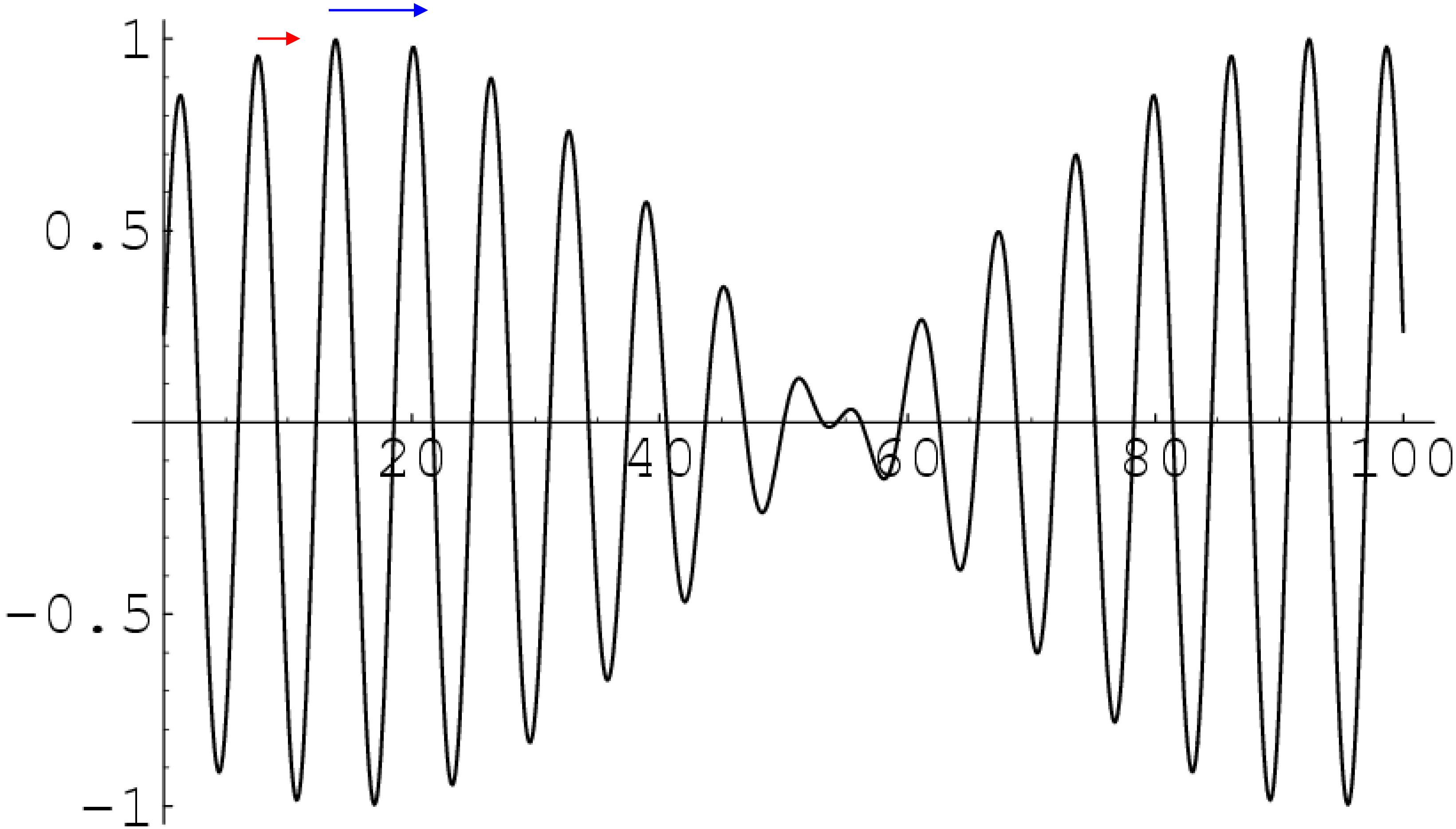
$V_p < V_g$



$t = 2$

$$\sin(1.00\ x - 2.0\ t)\cos(0.04\ x - 0.2\ t)$$

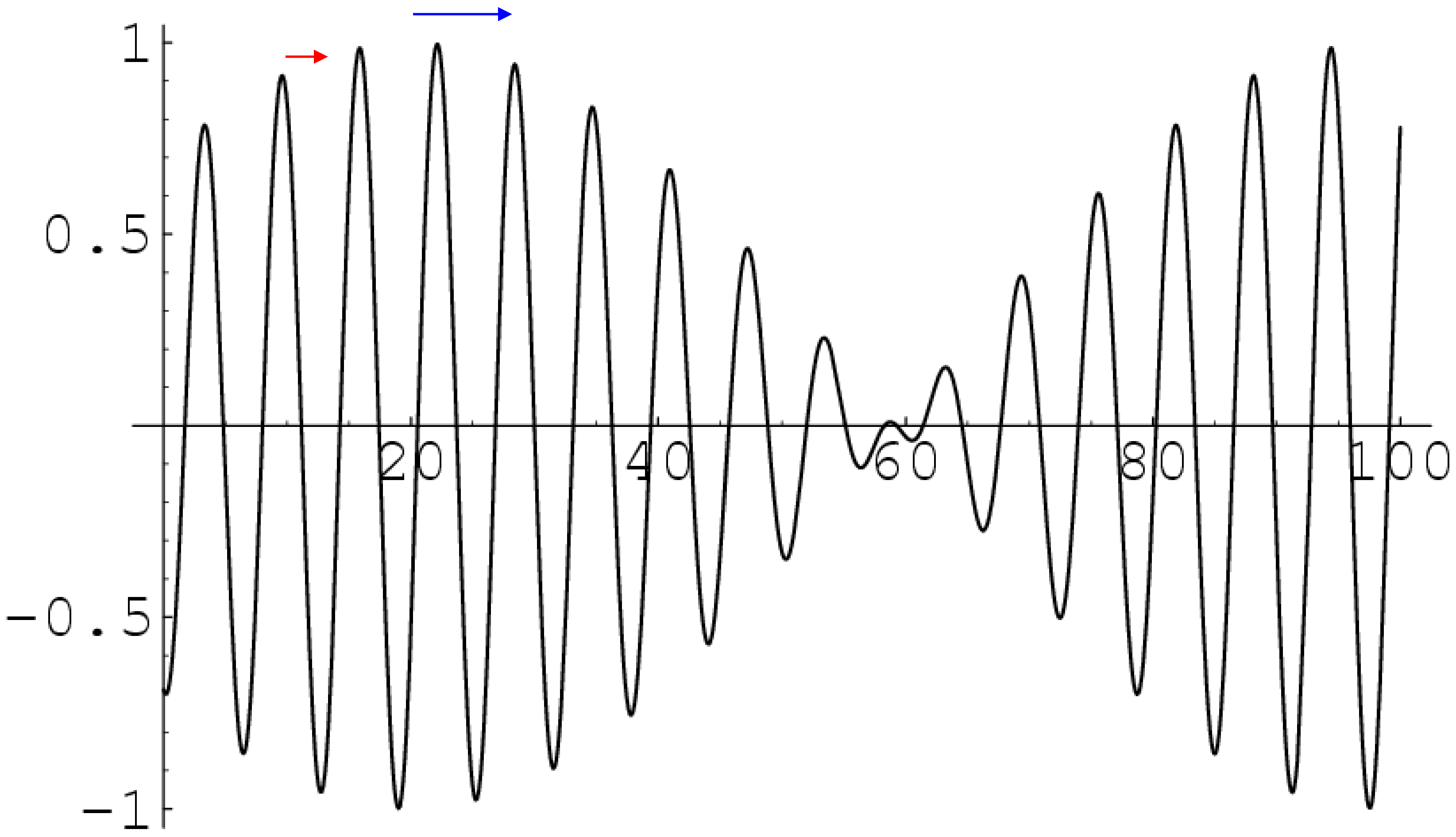
$V_p < V_g$



$t = 3$

$$\sin(1.00\ x - 2.0\ t)\cos(0.04\ x - 0.2\ t)$$

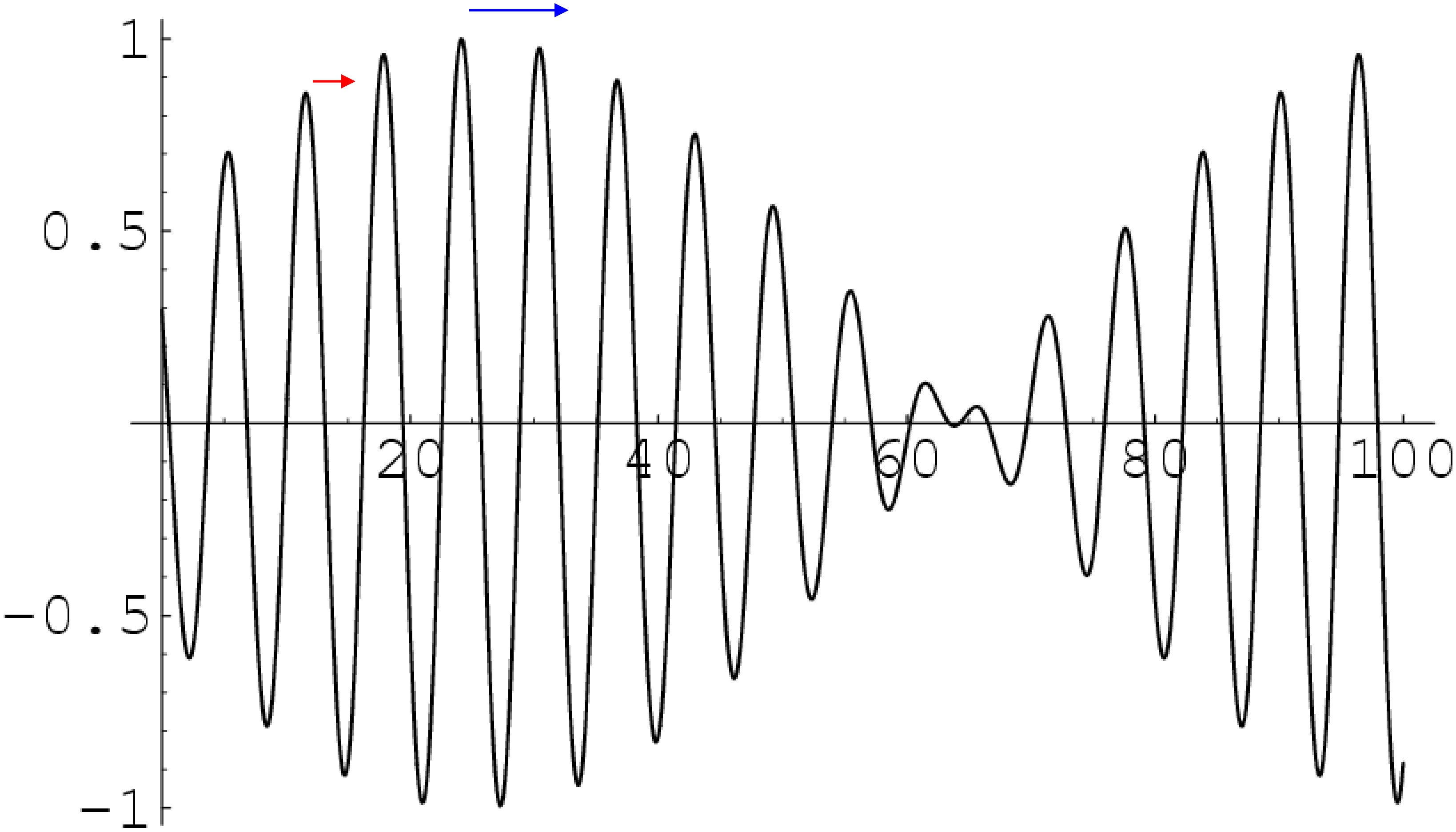
$\mathbf{V}_p < \mathbf{V}_g$



$t = 4$

$$\sin(1.00\,x - 2.0\,t)\cos(0.04\,x - 0.2\,t)$$

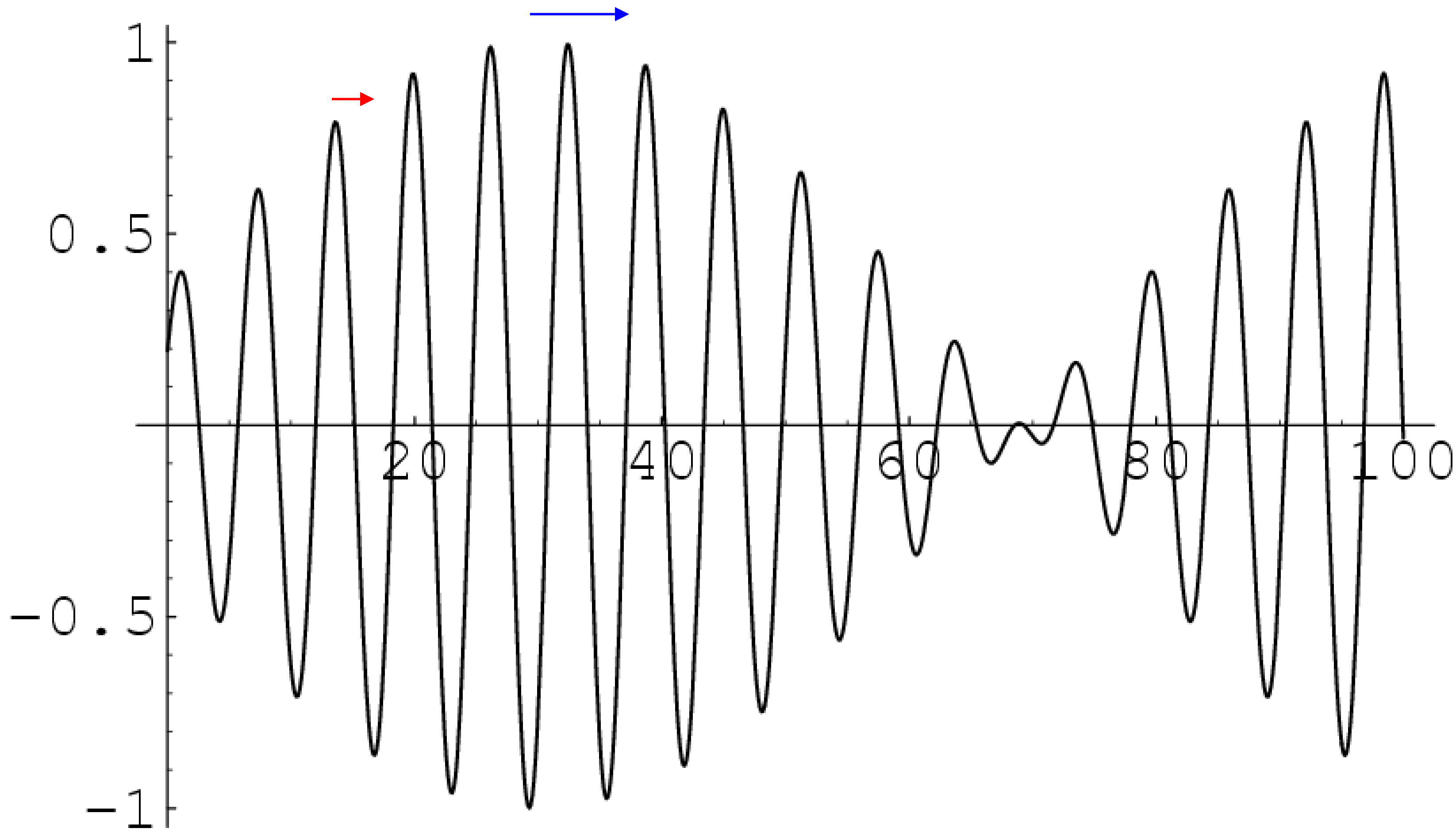
$V_p < V_g$



$t = 5$

$$\sin(1.00\,x - 2.0\,t)\cos(0.04\,x - 0.2\,t)$$

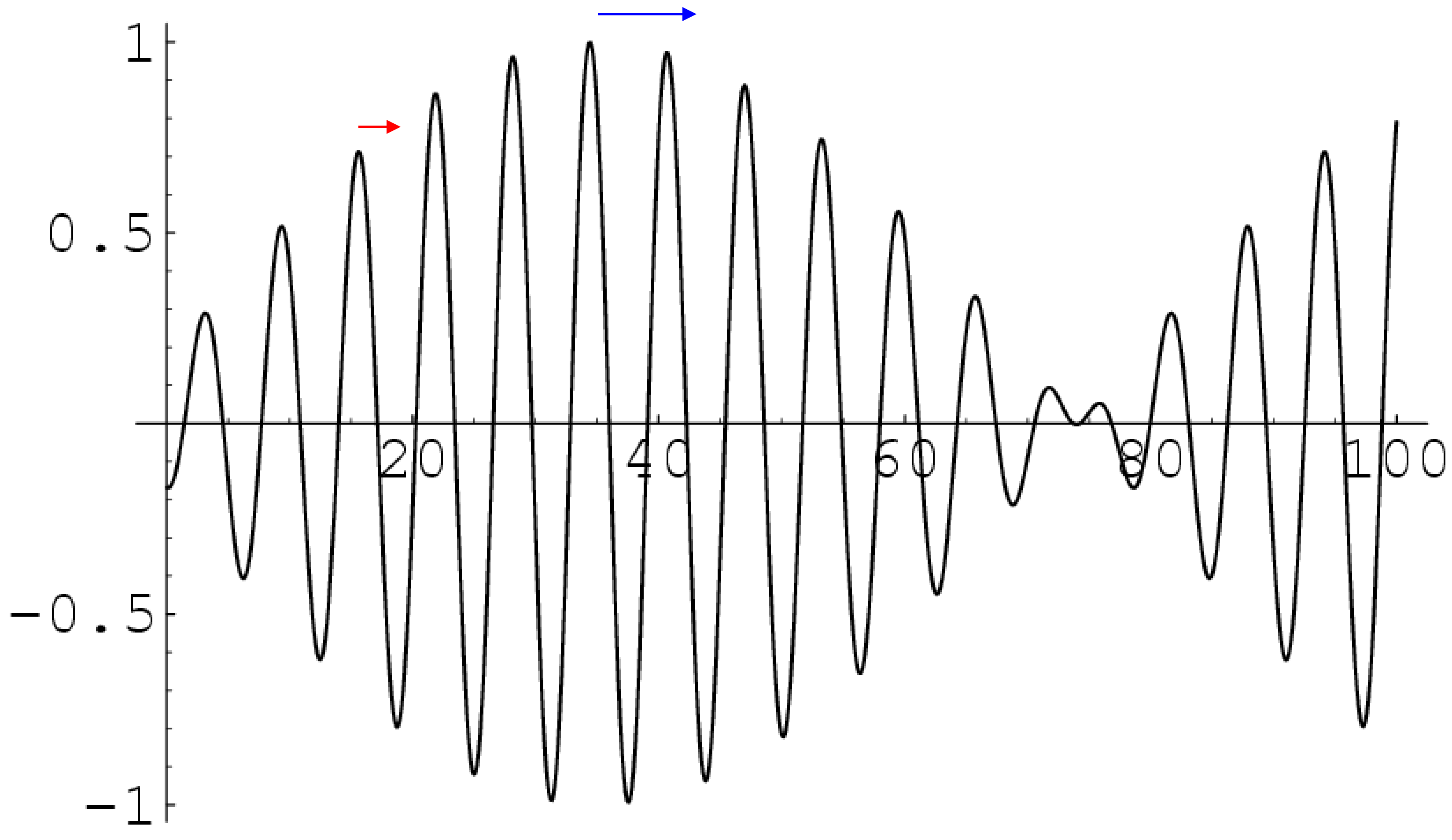
$\mathbf{V}_p < \mathbf{V}_g$



t = 6

$$\sin(1.00\,x - 2.0\,t)\cos(0.04\,x - 0.2\,t)$$

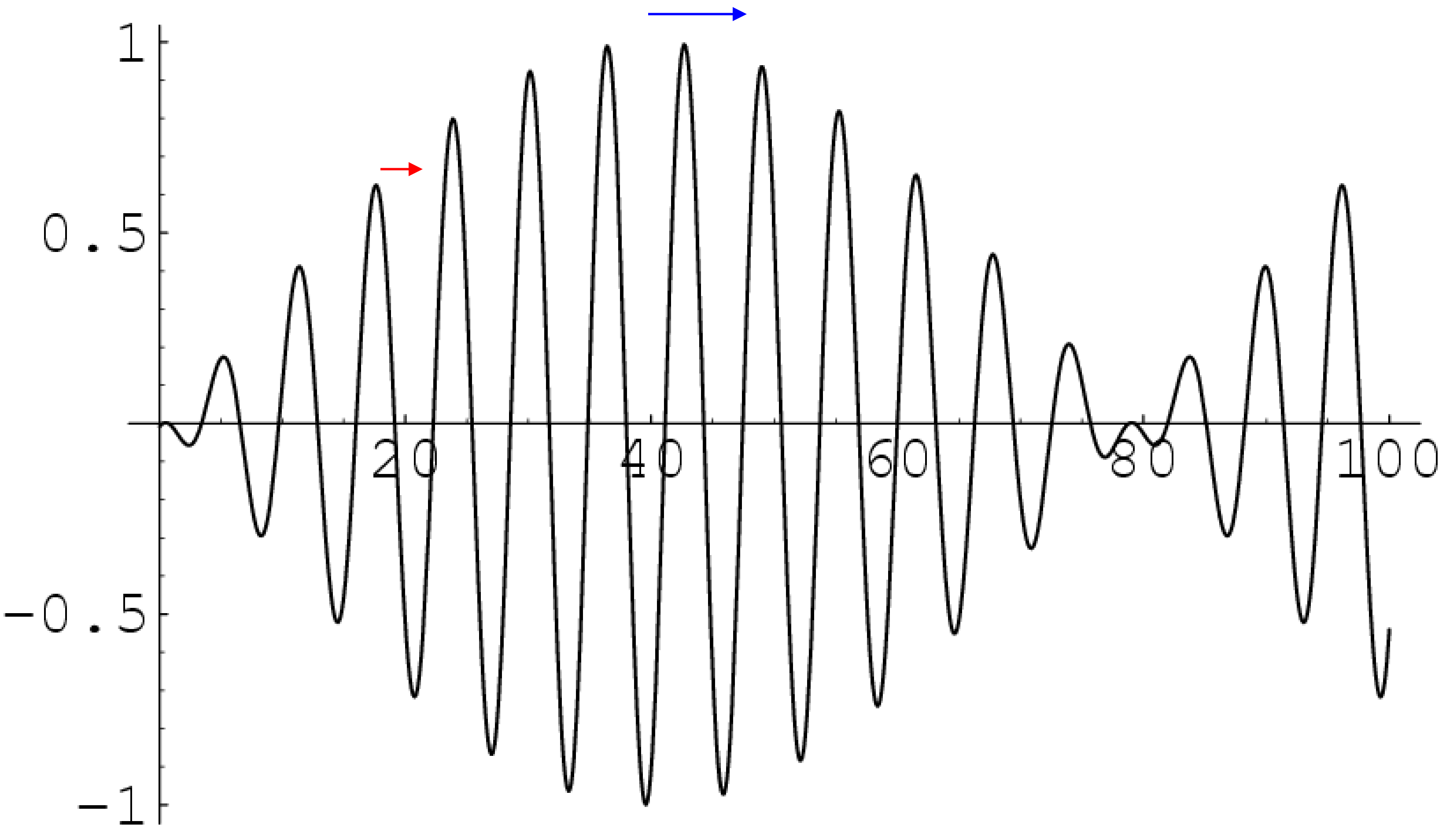
$\mathbf{V}_p < \mathbf{V}_g$



t = 7

$$\sin(1.00\,x - 2.0\,t)\cos(0.04\,x - 0.2\,t)$$

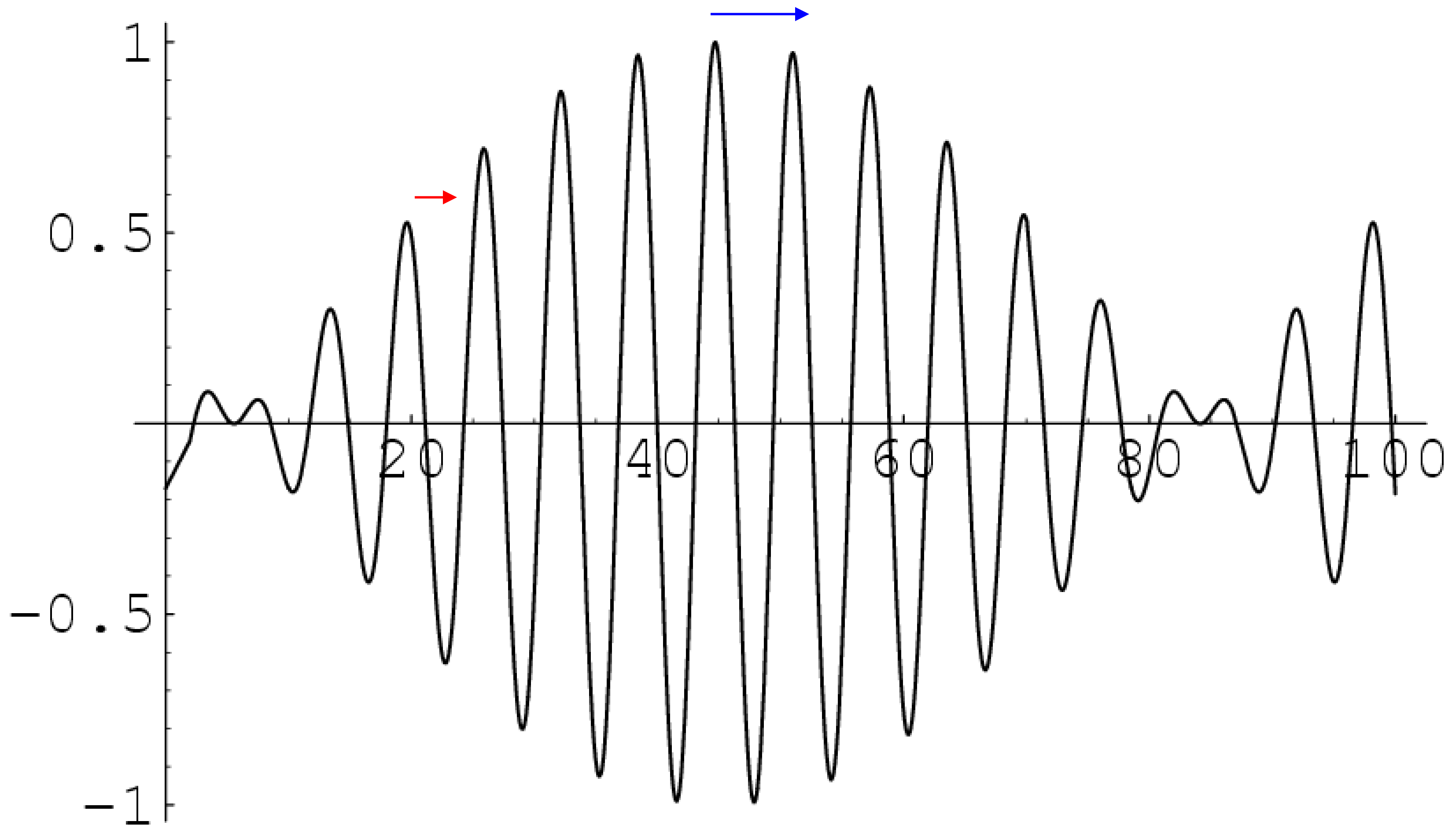
$\mathbf{V}_p < \mathbf{V}_g$



t = 8

$$\sin(1.00\ x - 2.0\ t)\cos(0.04\ x - 0.2\ t)$$

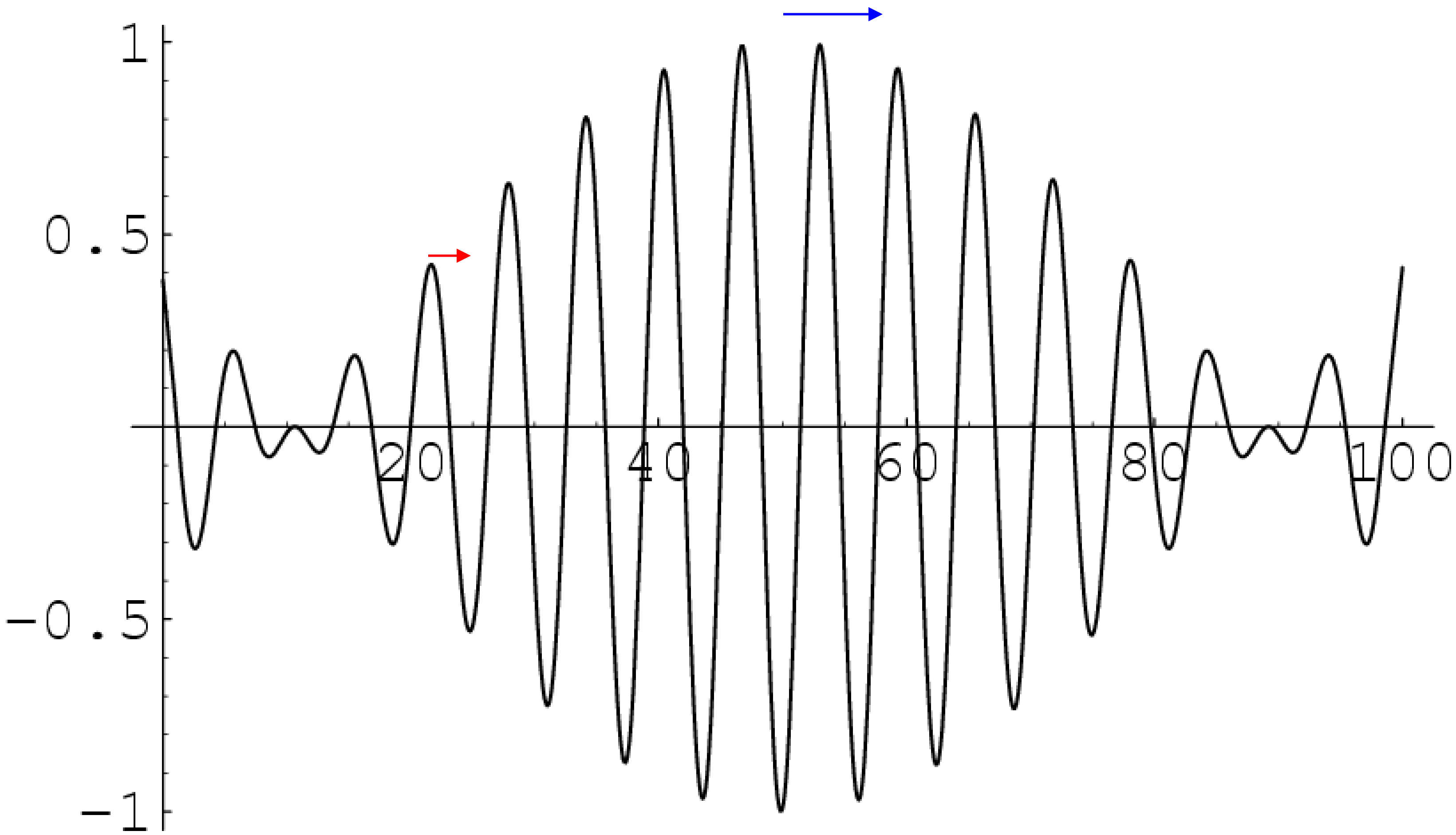
$V_p < V_g$



t = 9

$$\sin(1.00\ x - 2.0\ t)\cos(0.04\ x - 0.2\ t)$$

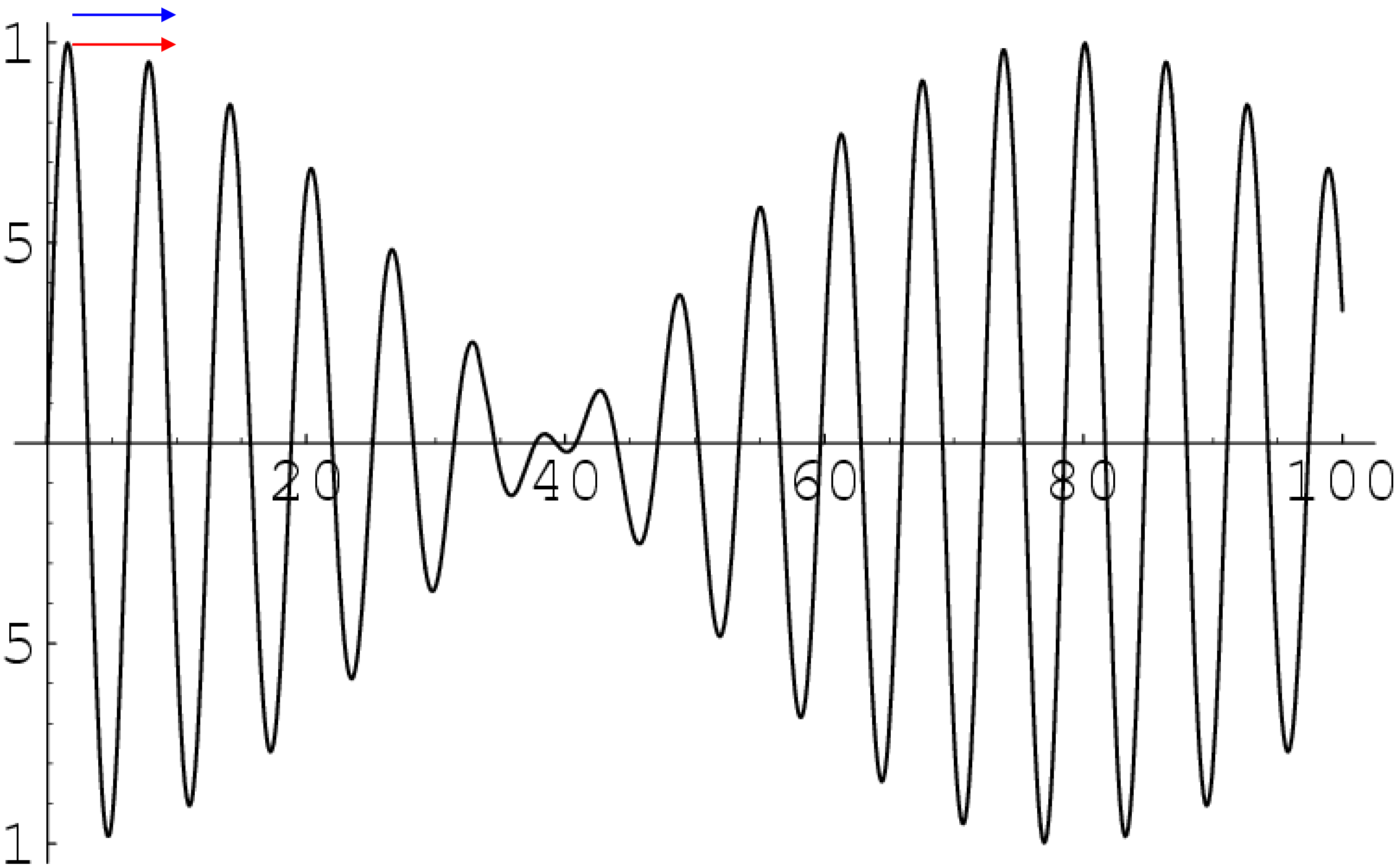
$\mathbf{V}_p < \mathbf{V}_g$



t = 10

$$\sin(1.00\ x - 2.0\ t)\cos(0.04\ x - 0.2\ t)$$

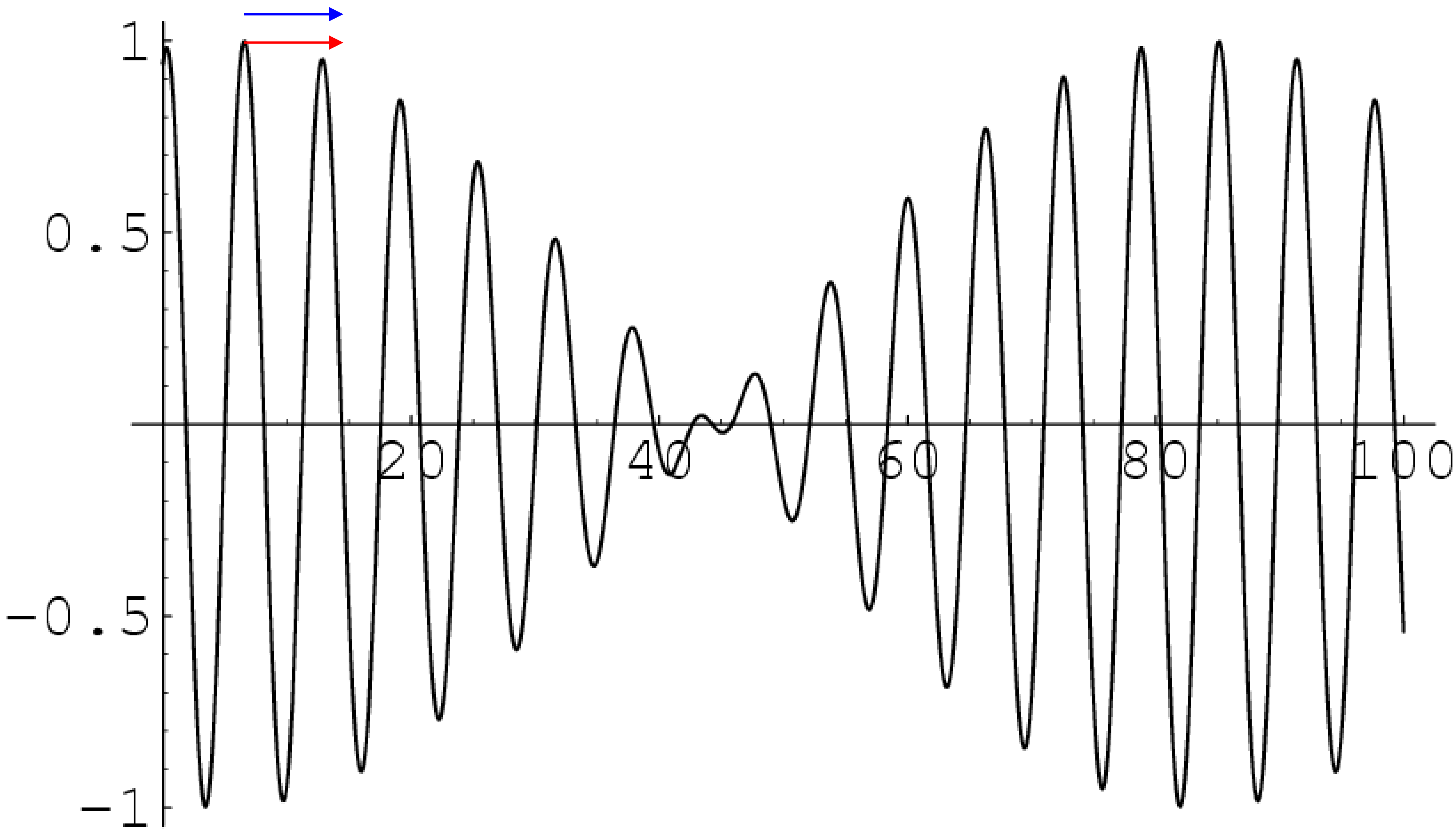
$\mathbf{V}_p = \mathbf{V}_g$



t = 0

$$\sin(1.00\,x - 5.0\,t)\cos(0.04\,x - 0.2\,t)$$

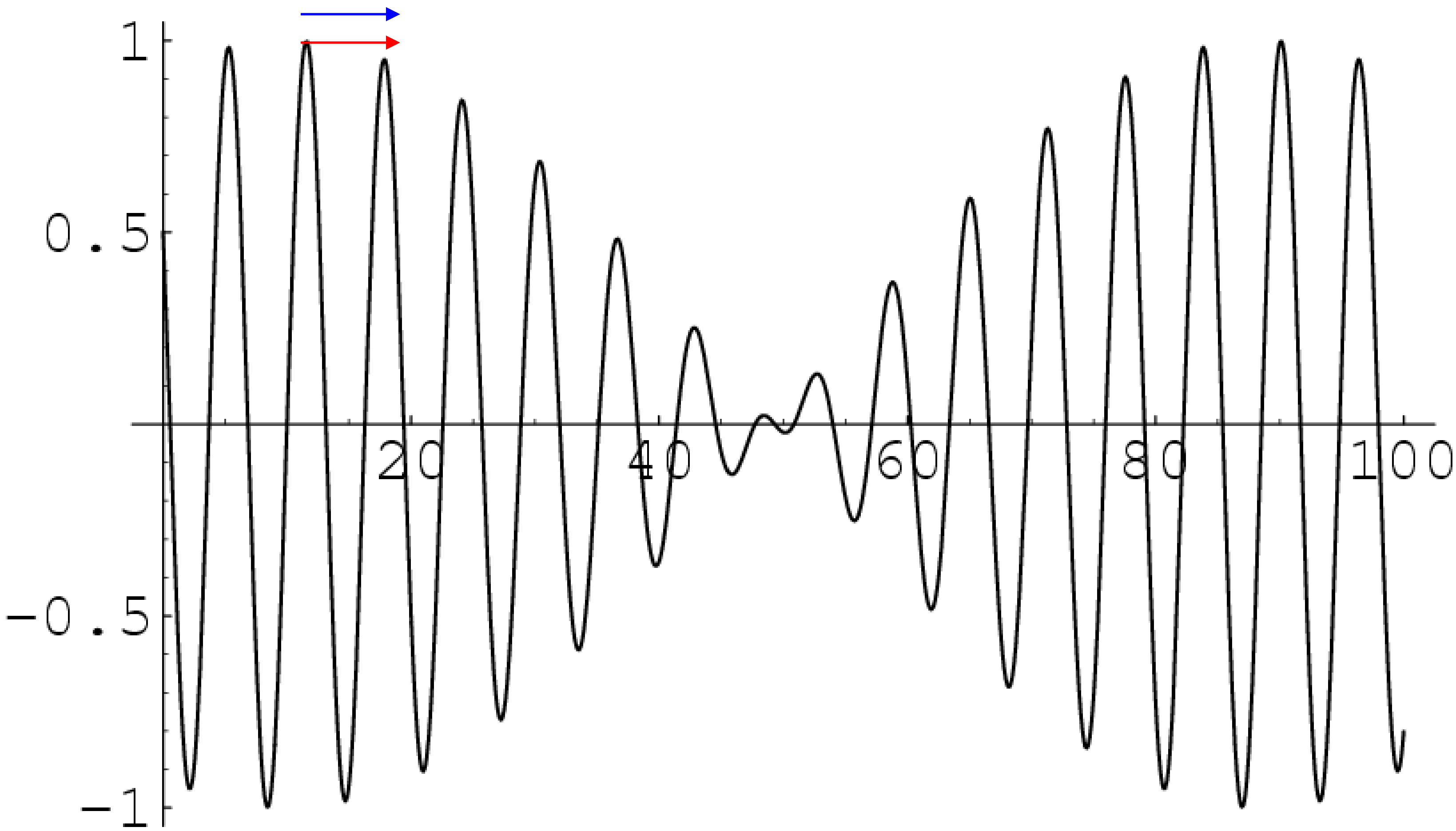
Vp = **V**g



t = 1

$$\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$$

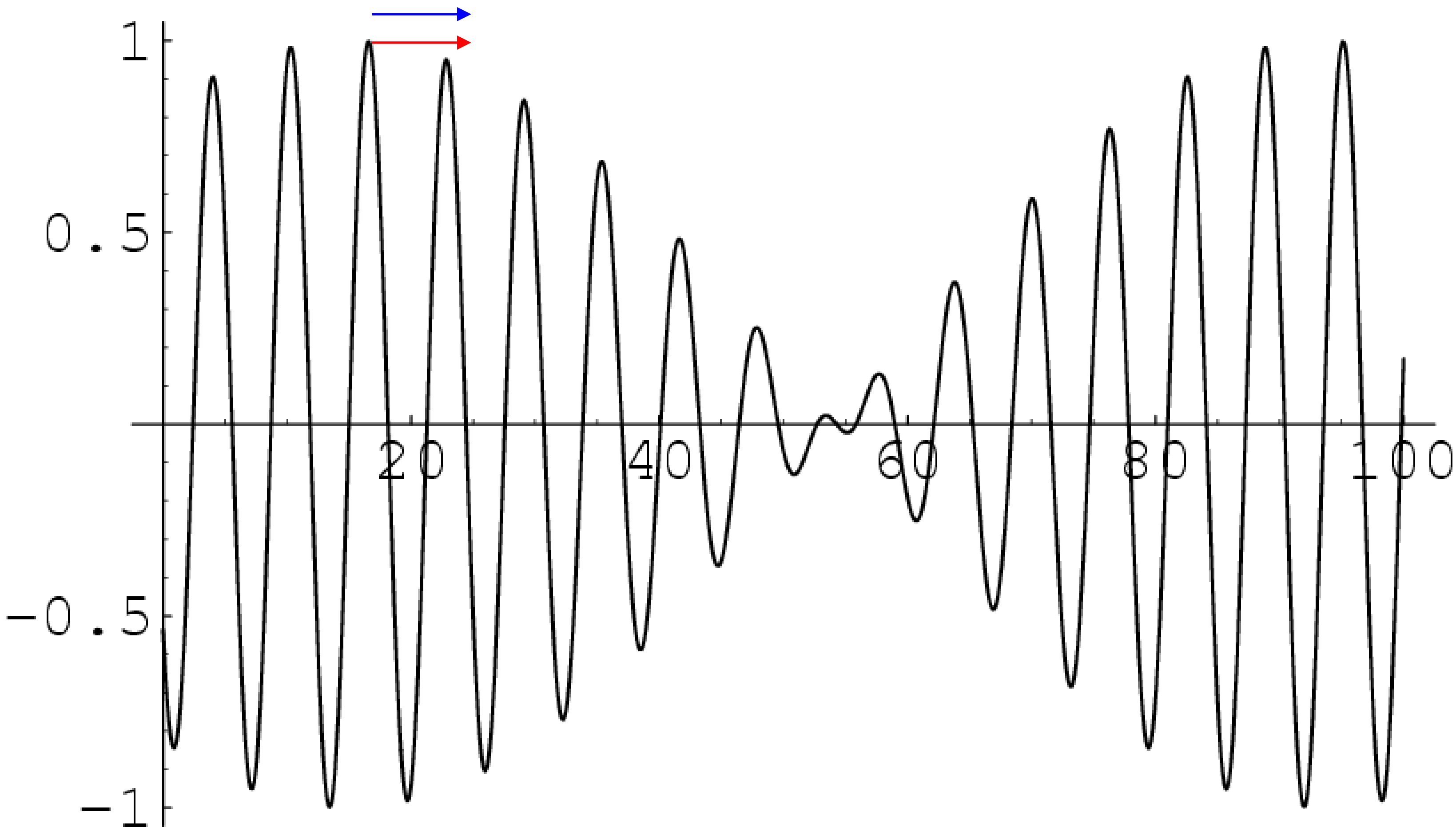
$\mathbf{V}_p = \mathbf{V}_g$



t = 2

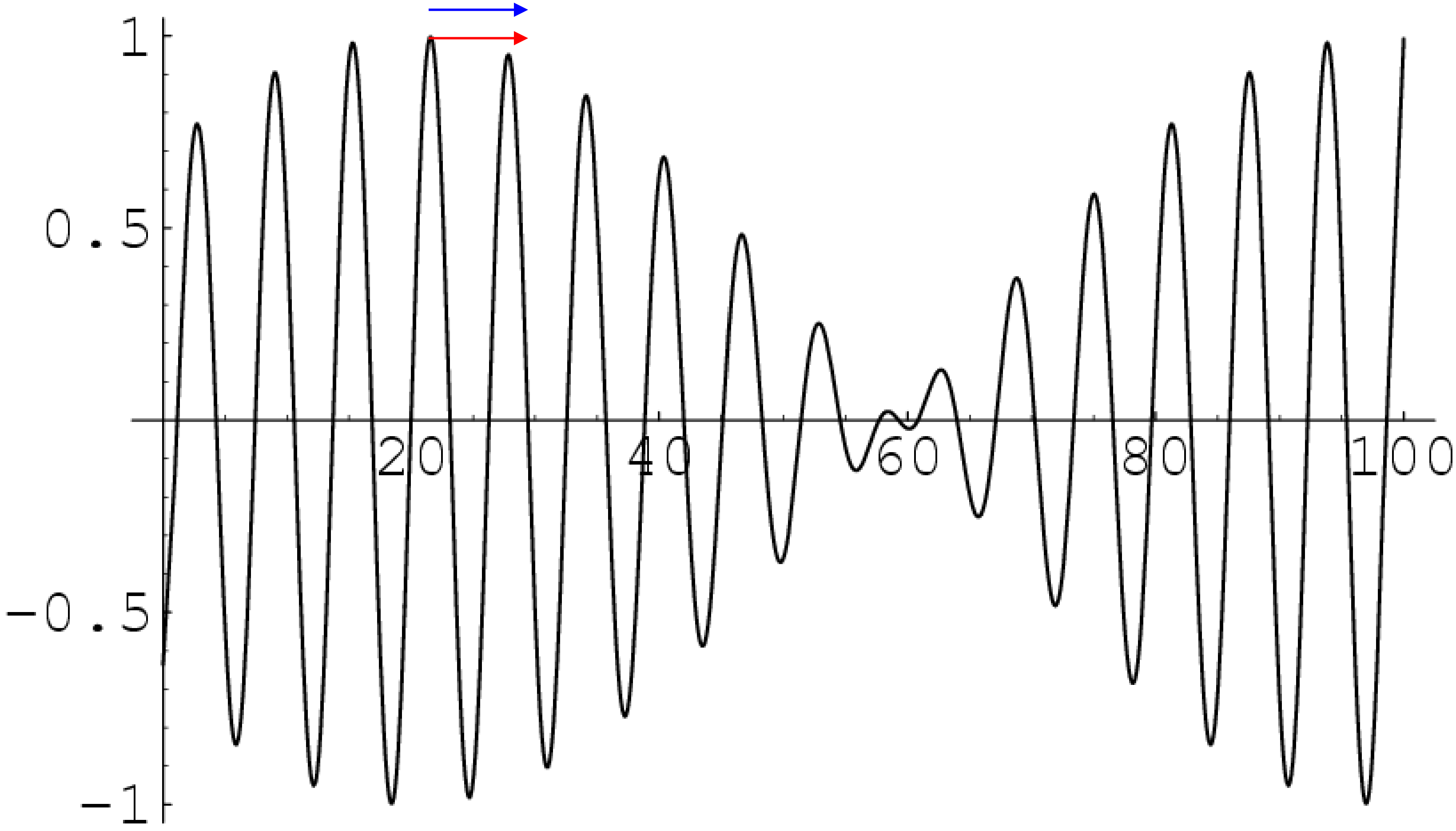
$$\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$$

$V_p = V_g$



$t = 3$ $\quad$ $\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$

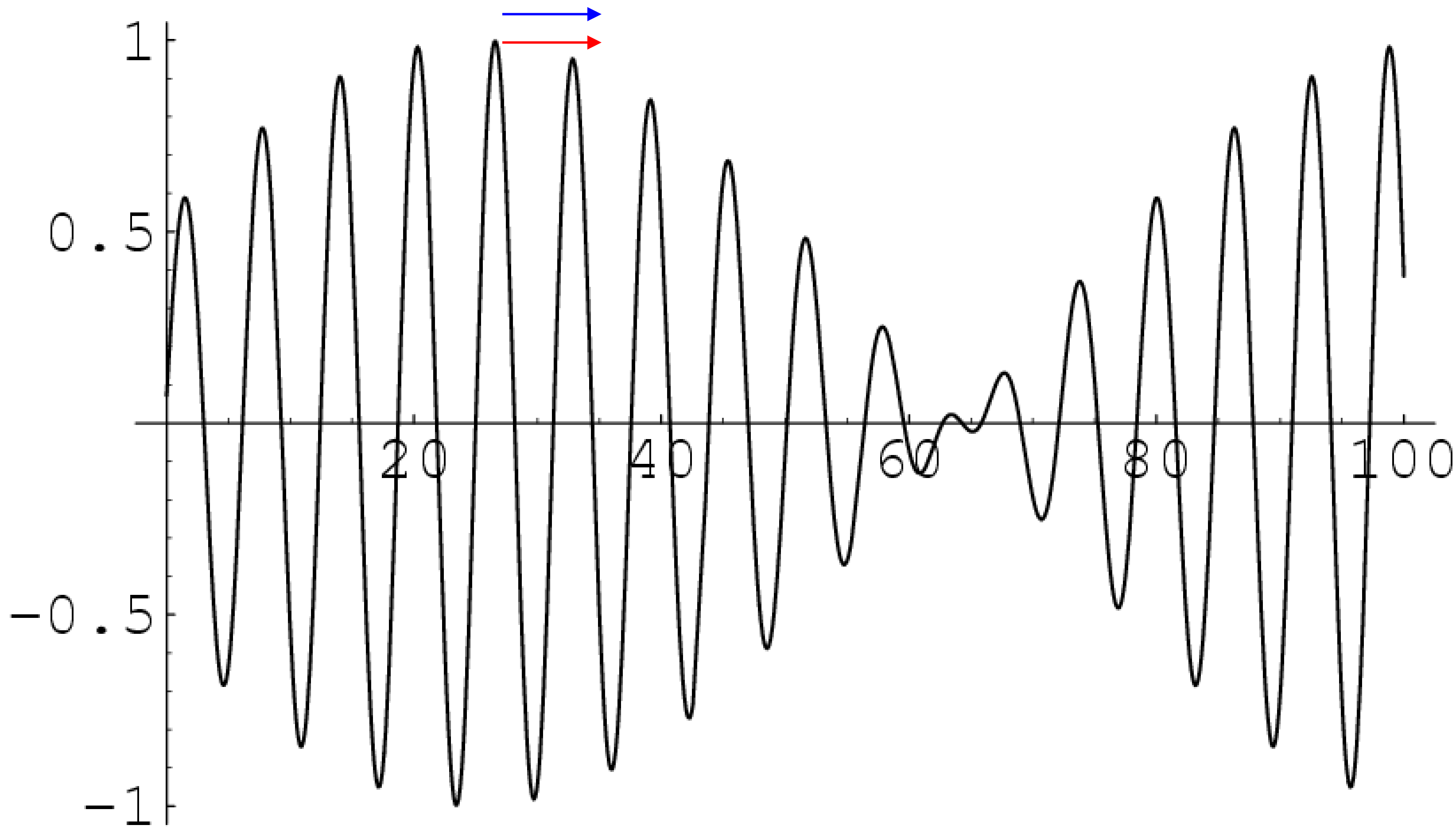
$\mathbf{V}_p = \mathbf{V}_g$



t = 4

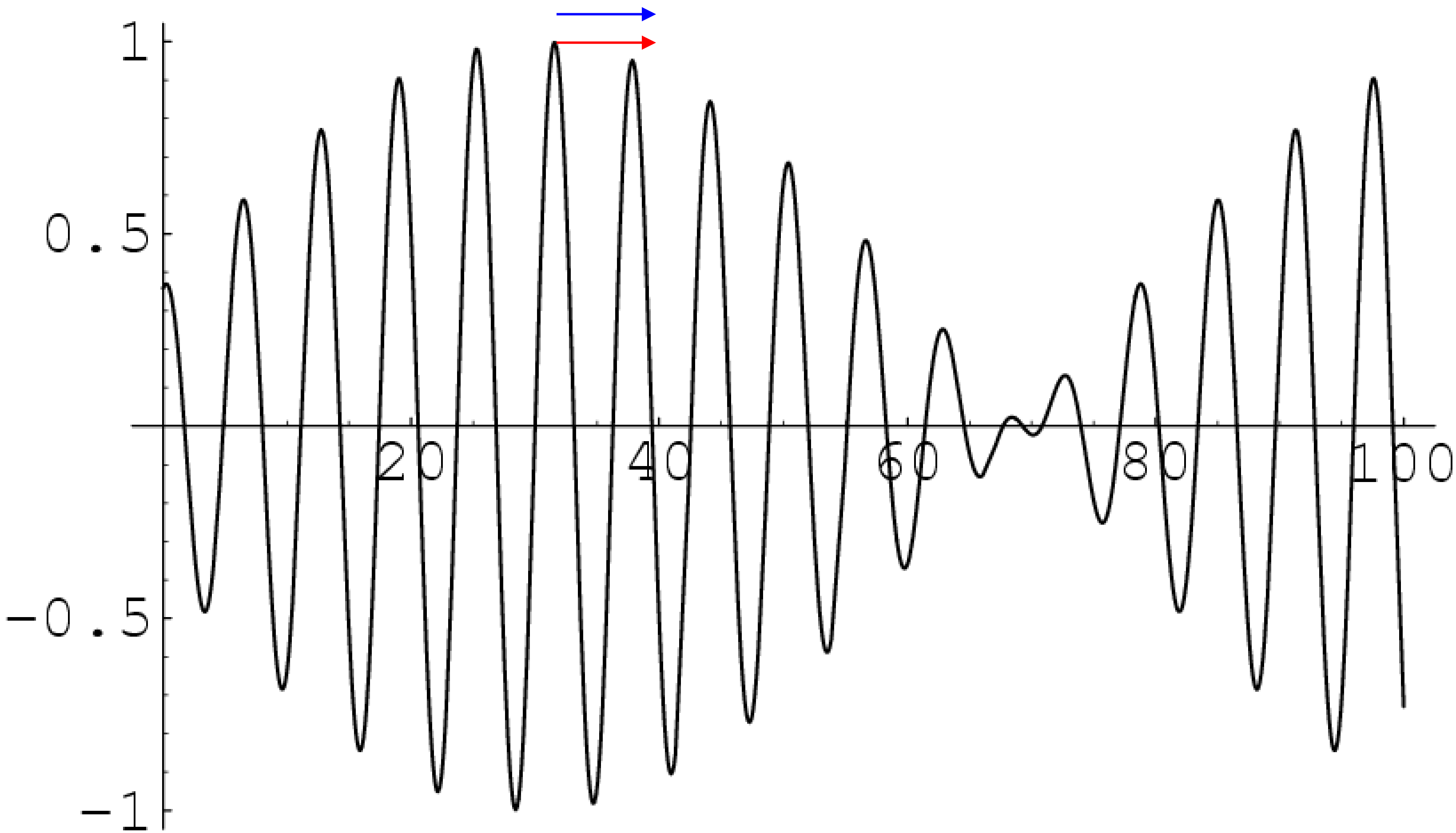
$$\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$$

$\mathbf{V}_p = \mathbf{V}_g$



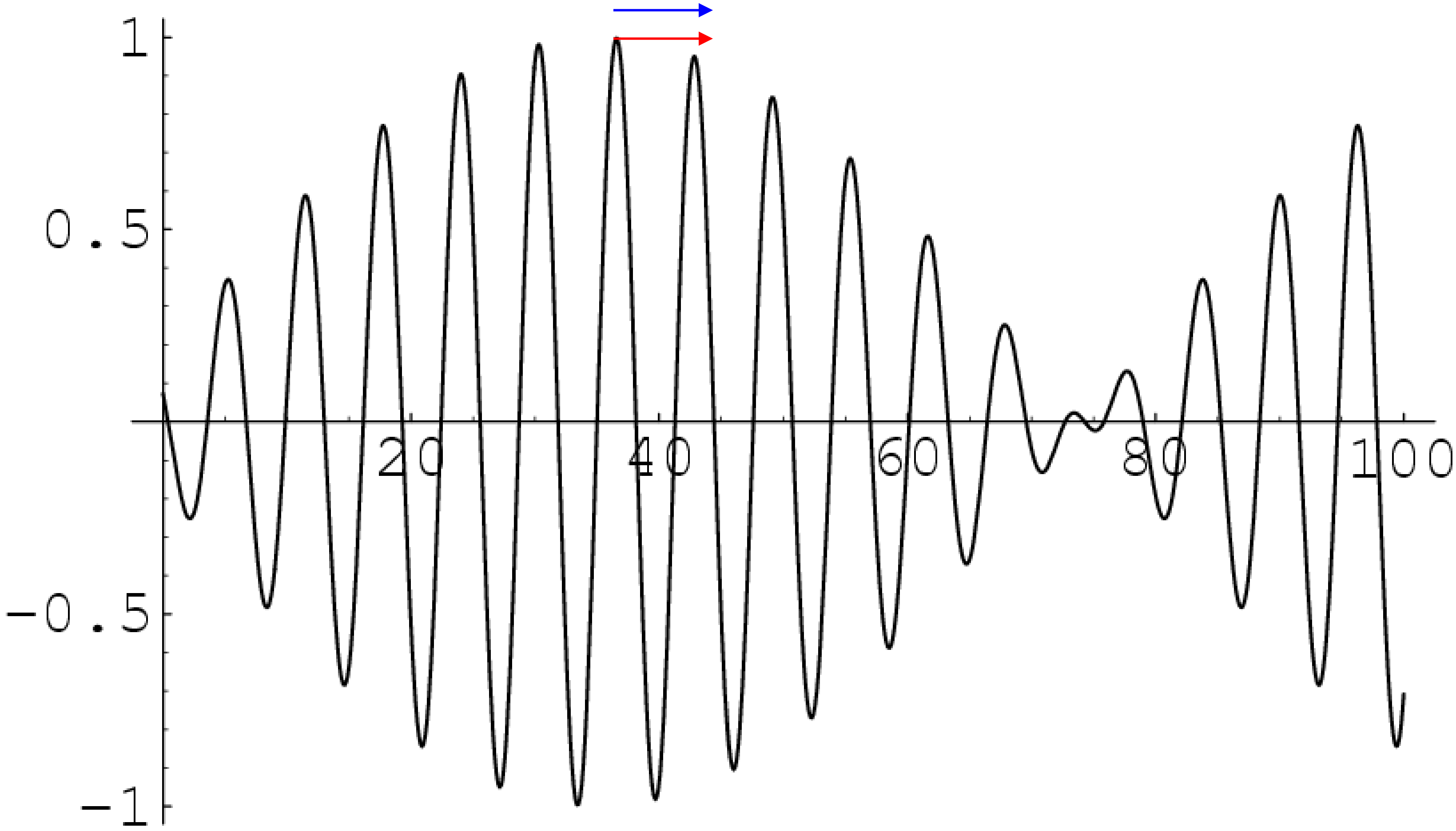
t = 5 $\quad \sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$

$\mathbf{V}_p = \mathbf{V}_g$



t = 6 $\sin(1.00\,x - 5.0\,t)\cos(0.04\,x - 0.2\,t)$

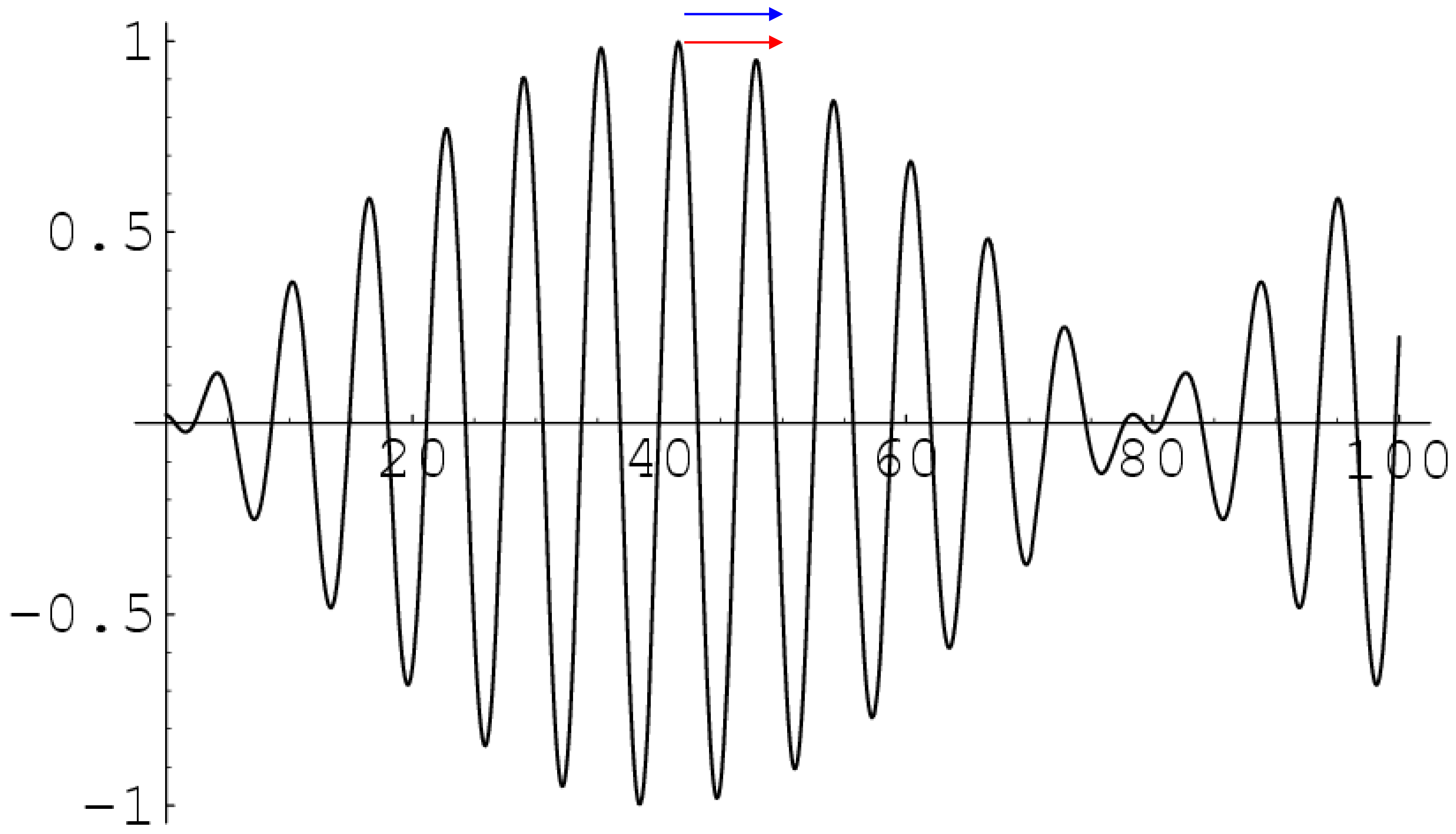
$\mathbf{V}_p = \mathbf{V}_g$



t = 7

$$\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$$

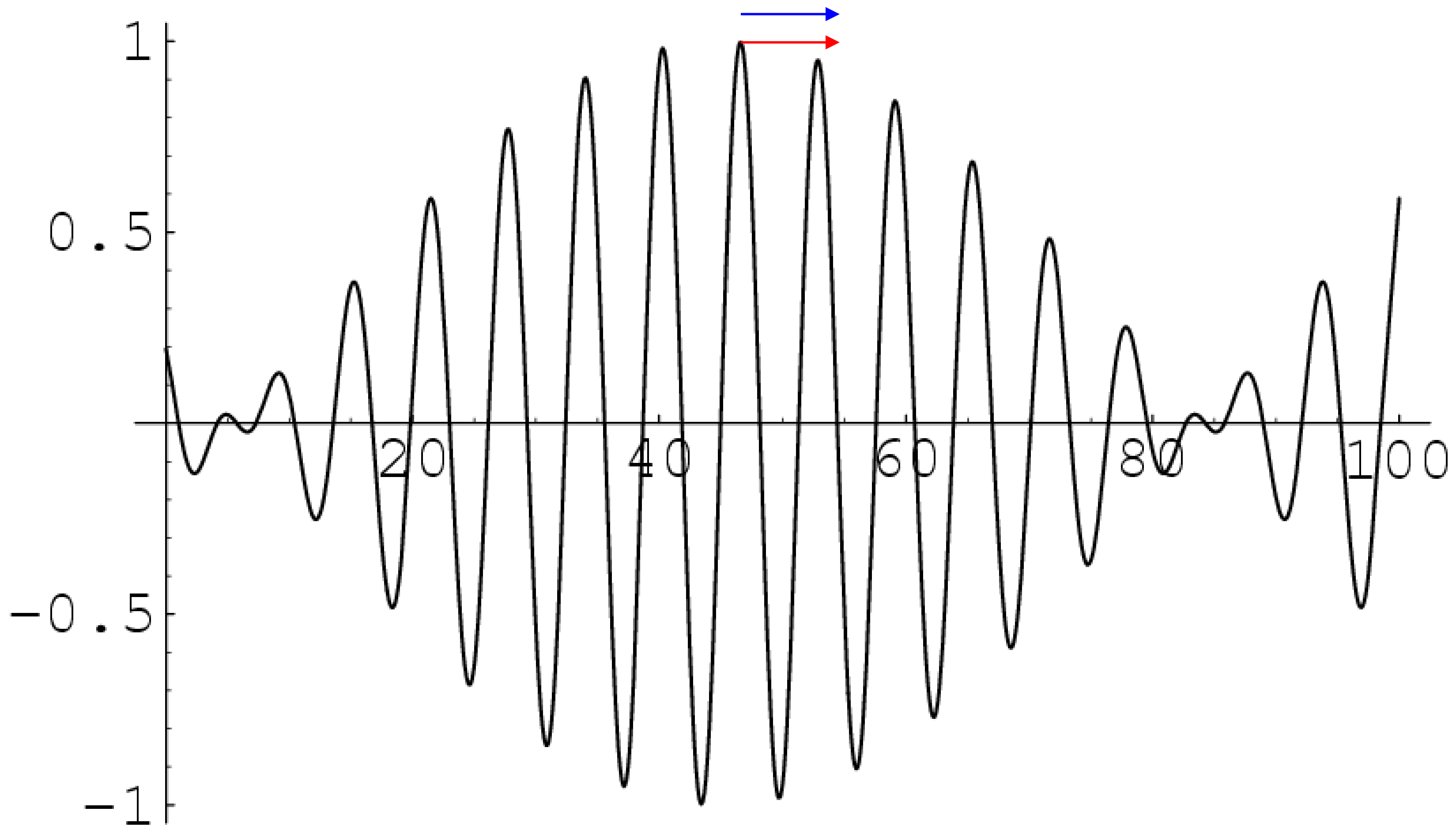
$\mathbf{V}_p = \mathbf{V}_g$



t = 8

$$\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$$

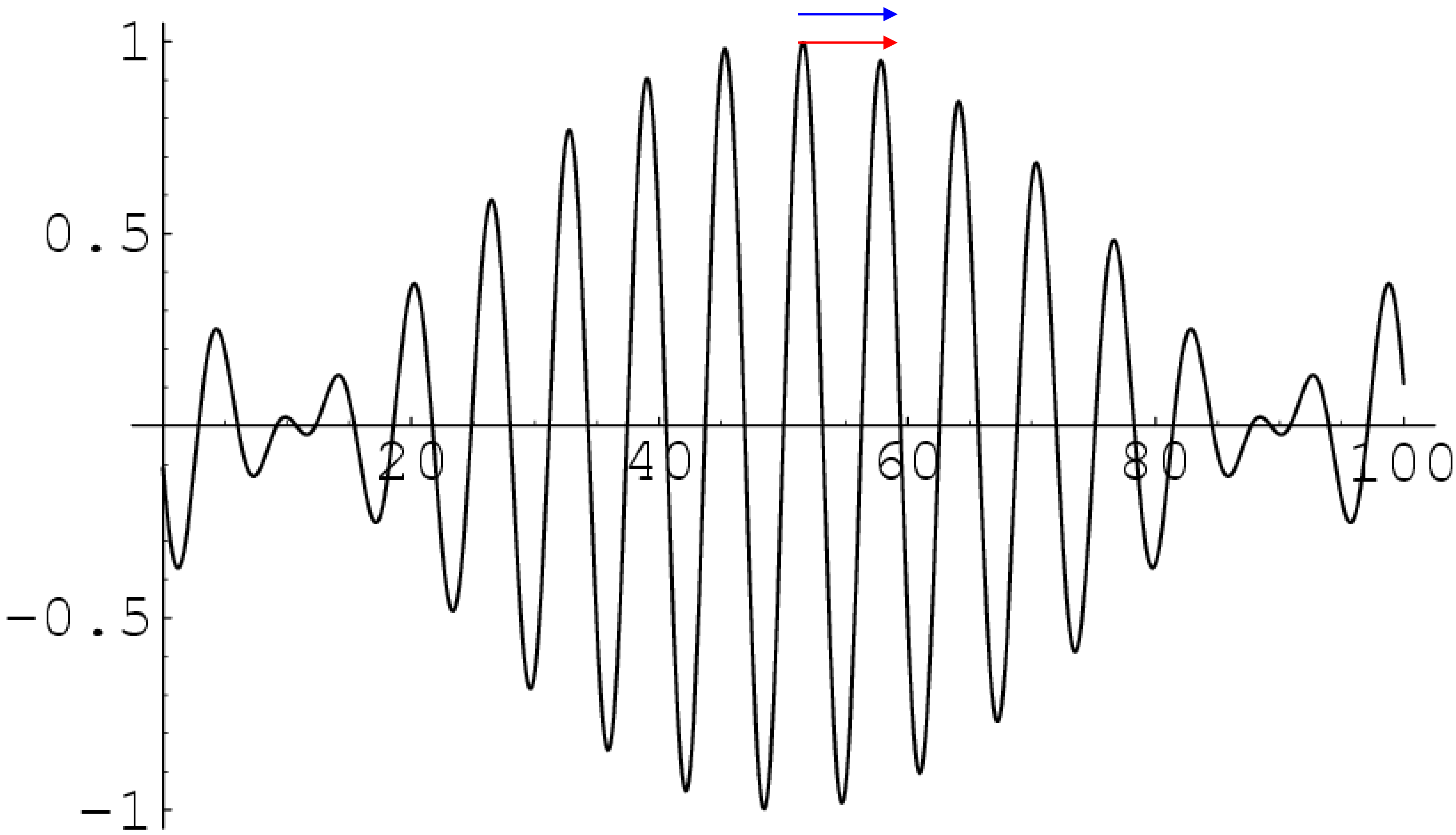
$\mathbf{V}_p = \mathbf{V}_g$



t = 9

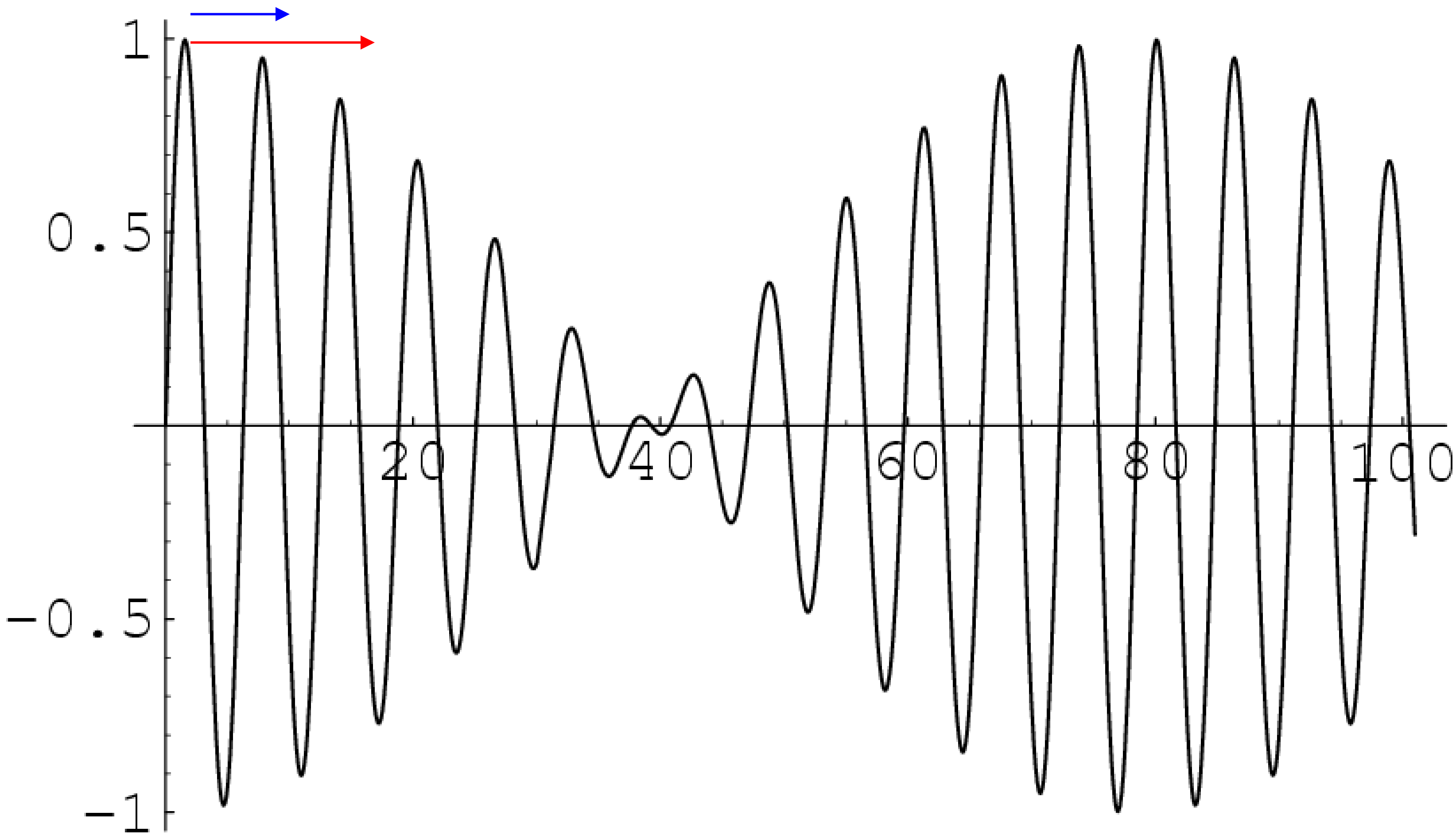
$$\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$$

$\mathbf{V}_p = \mathbf{V}_g$



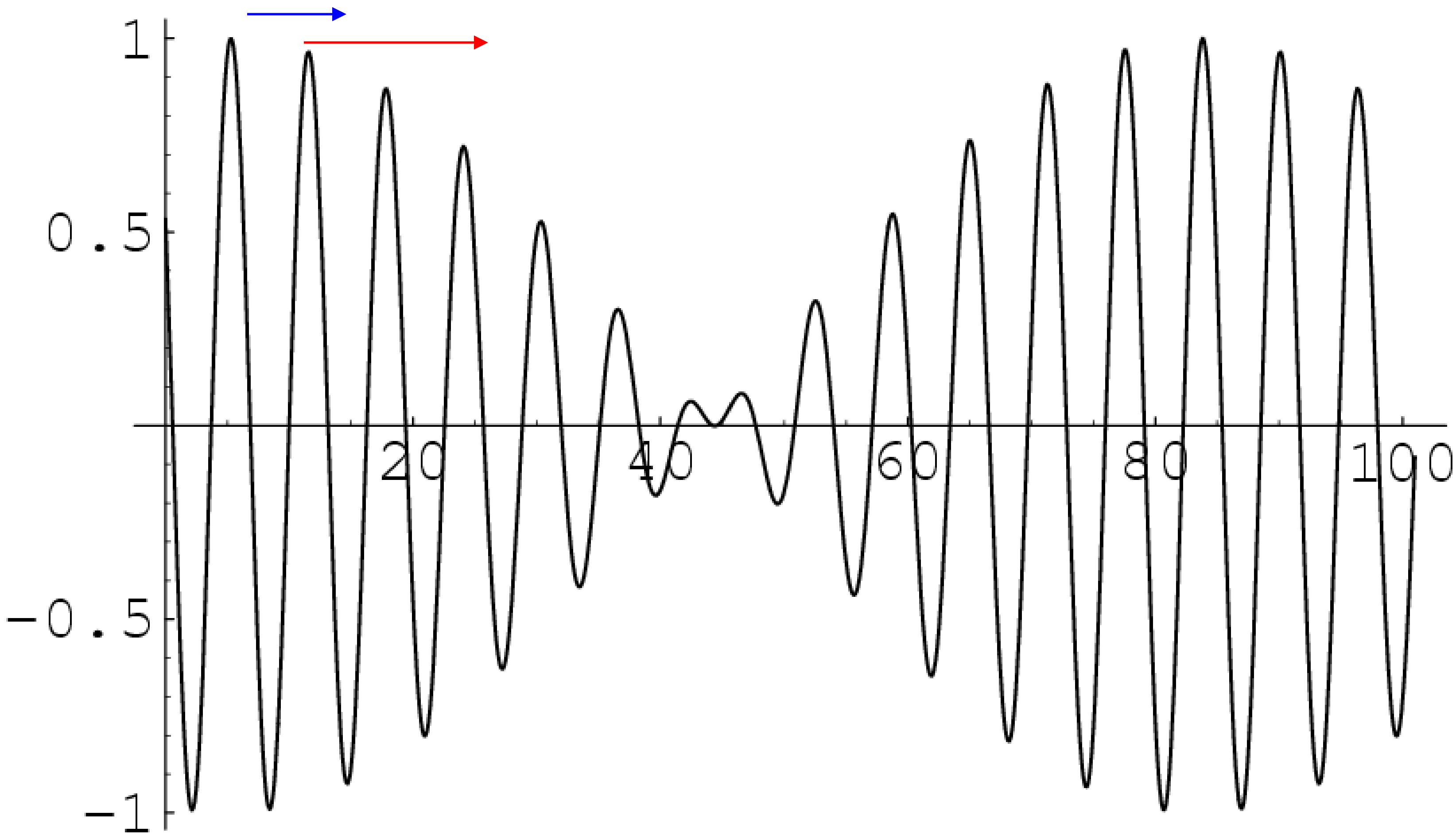
$t = 10$ $\sin(1.00\ x - 5.0\ t)\cos(0.04\ x - 0.2\ t)$

Vp > **V**g



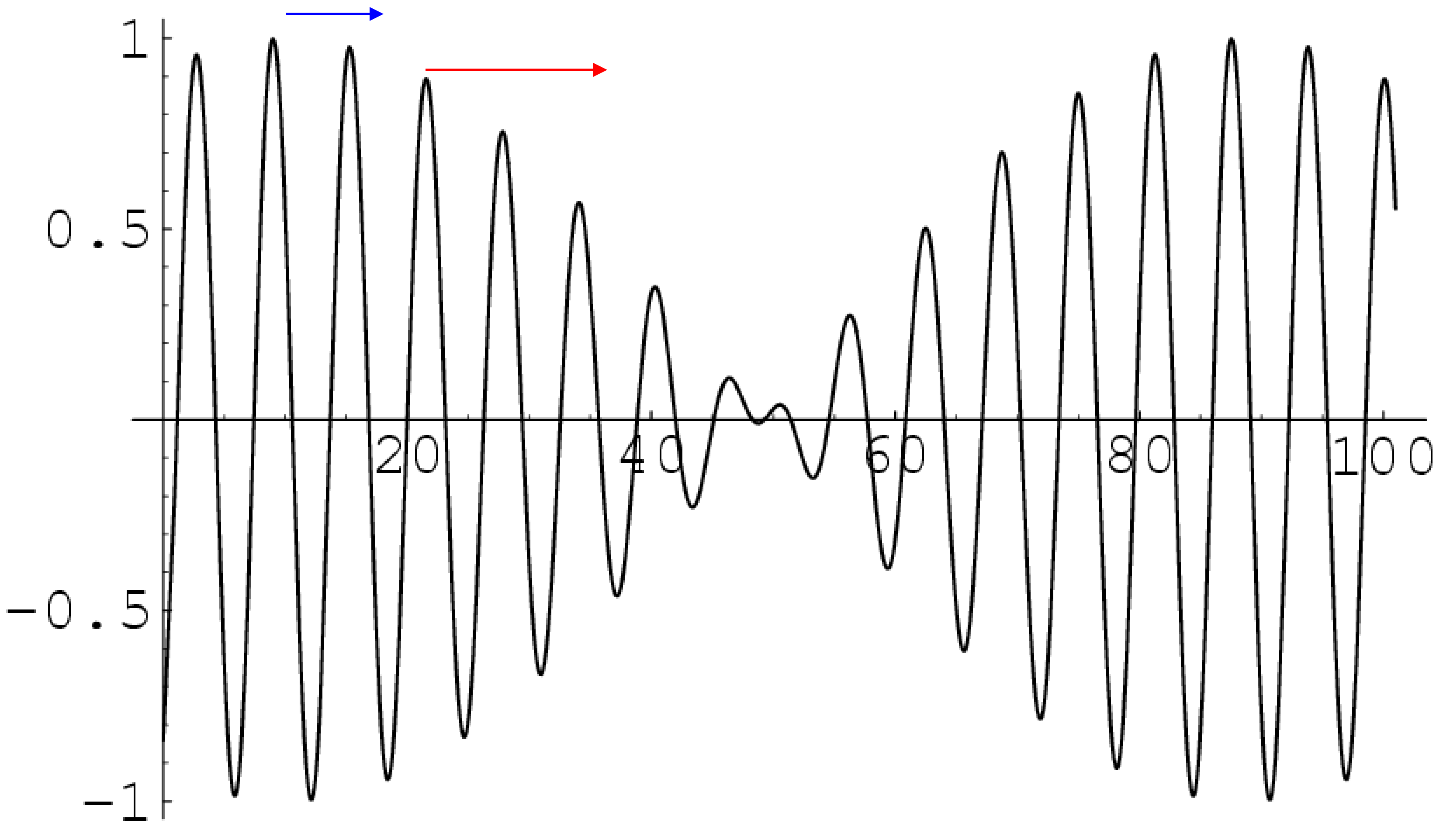
t = 0 $\sin(1.00\ x - 10.0\ t) \cos(0.04\ x - 0.2\ t)$

$\mathbf{V}_p > \mathbf{V}_g$



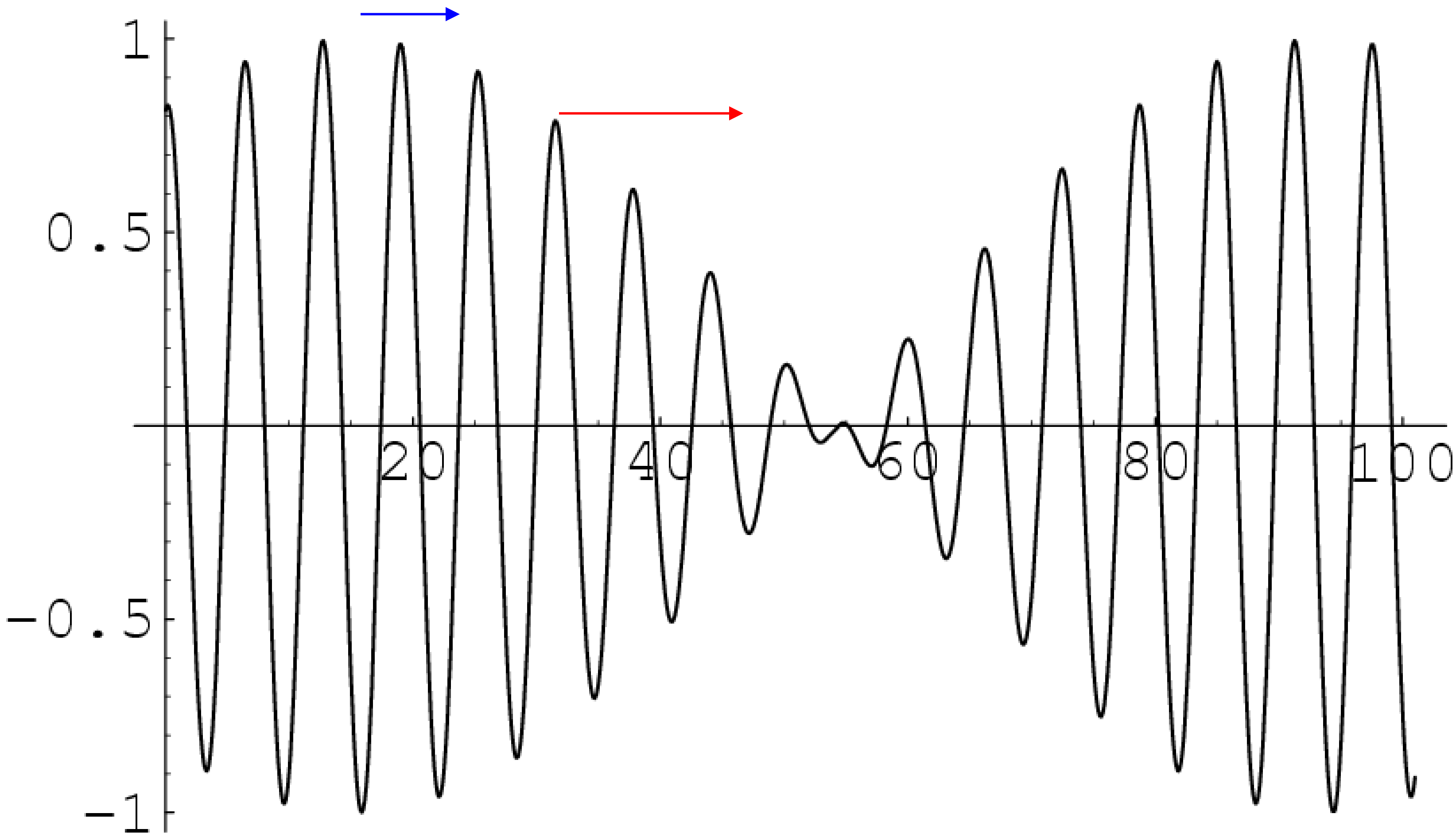
$t = 1$ $\sin(1.00\ x - 10.0\ t)\cos(0.04\ x - 0.2\ t)$

V$_p$ > **V**$_g$



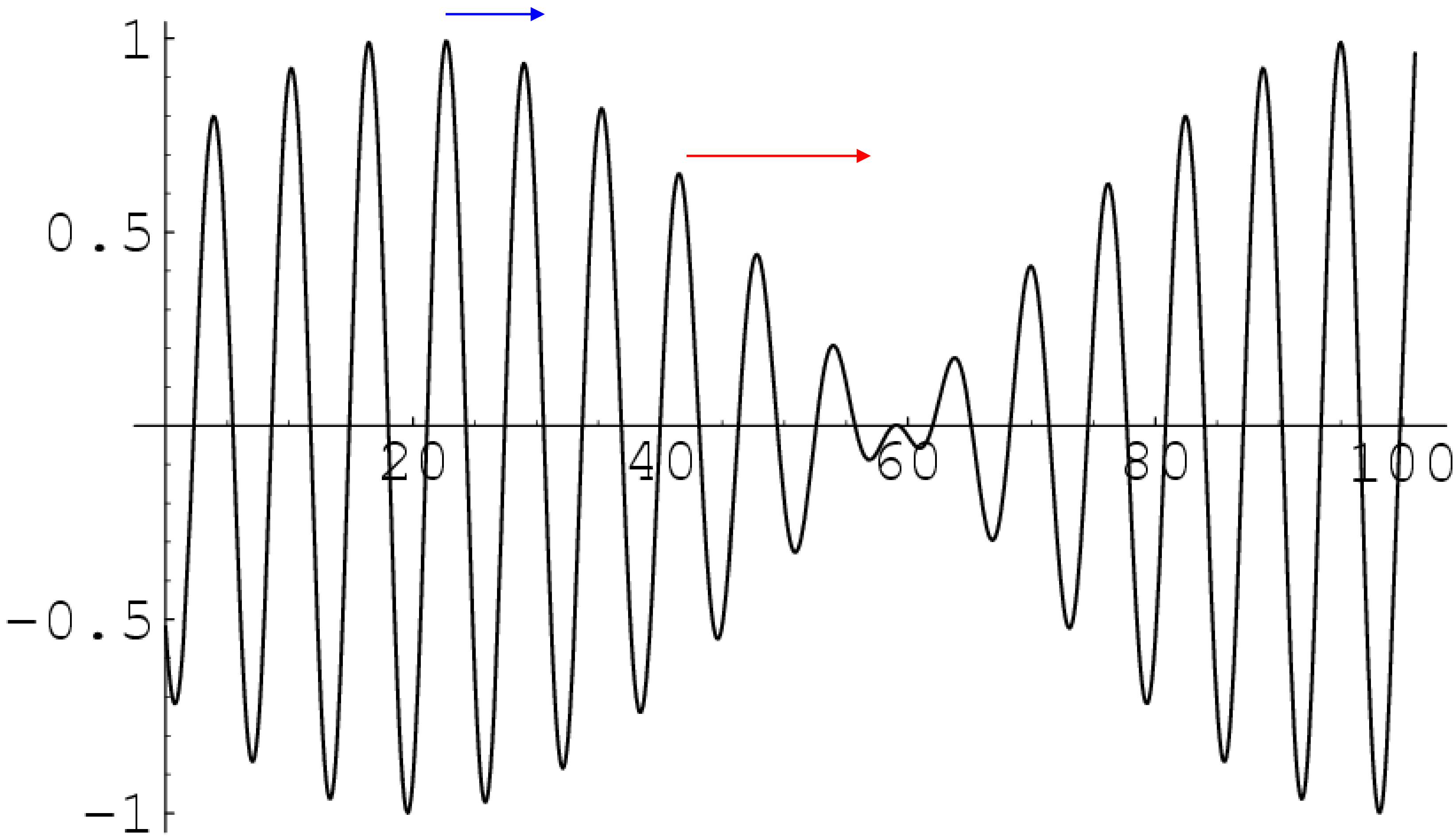
t = 2 $\sin(1.00\ x - 10.0\ t)\cos(0.04\ x - 0.2\ t)$

$\mathbf{V}_p > \mathbf{V}_g$



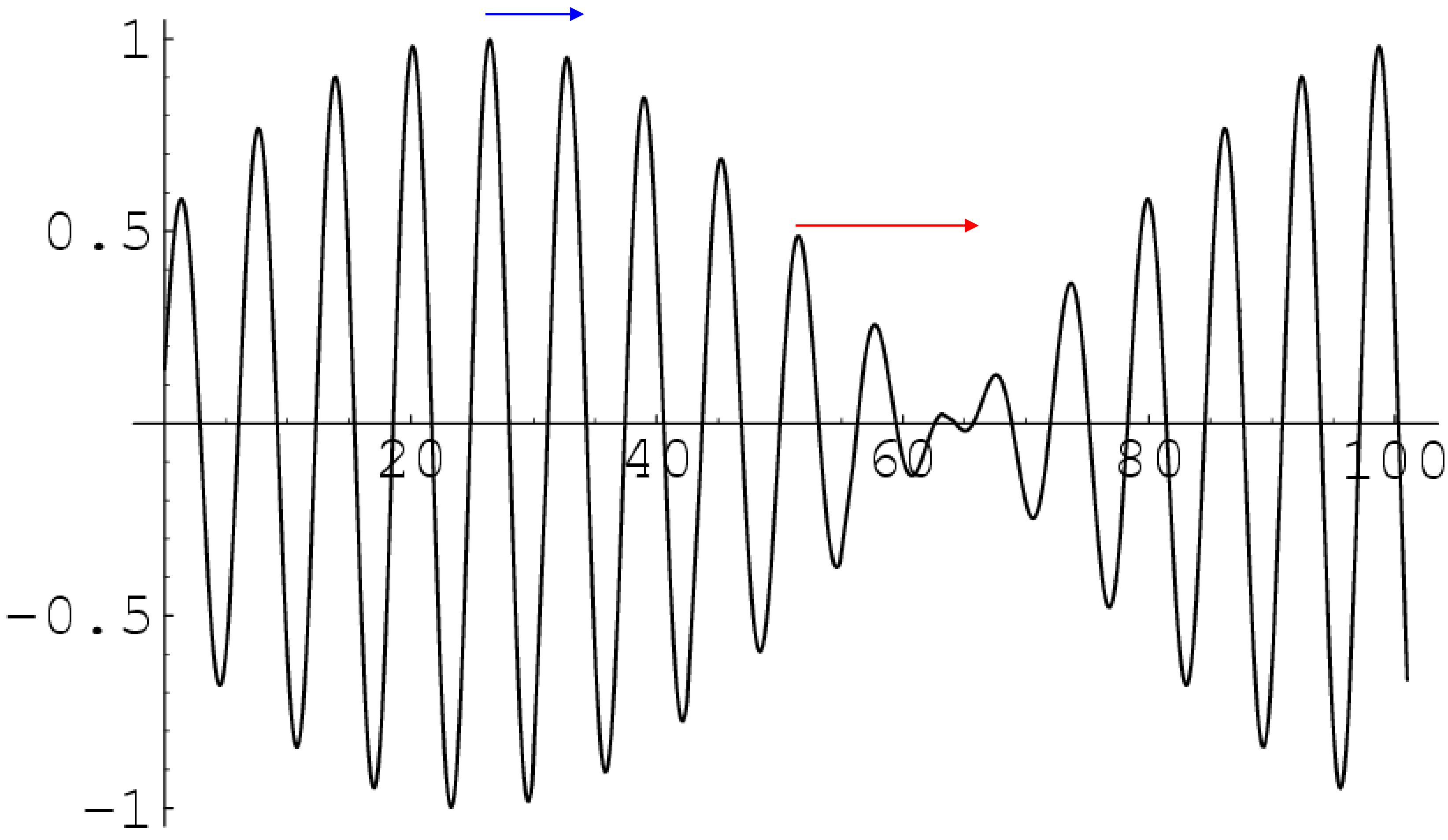
t = 3 $\sin(1.00\ x - 10.0\ t)\cos(0.04\ x - 0.2\ t)$

$\mathbf{V}_p > \mathbf{V}_g$



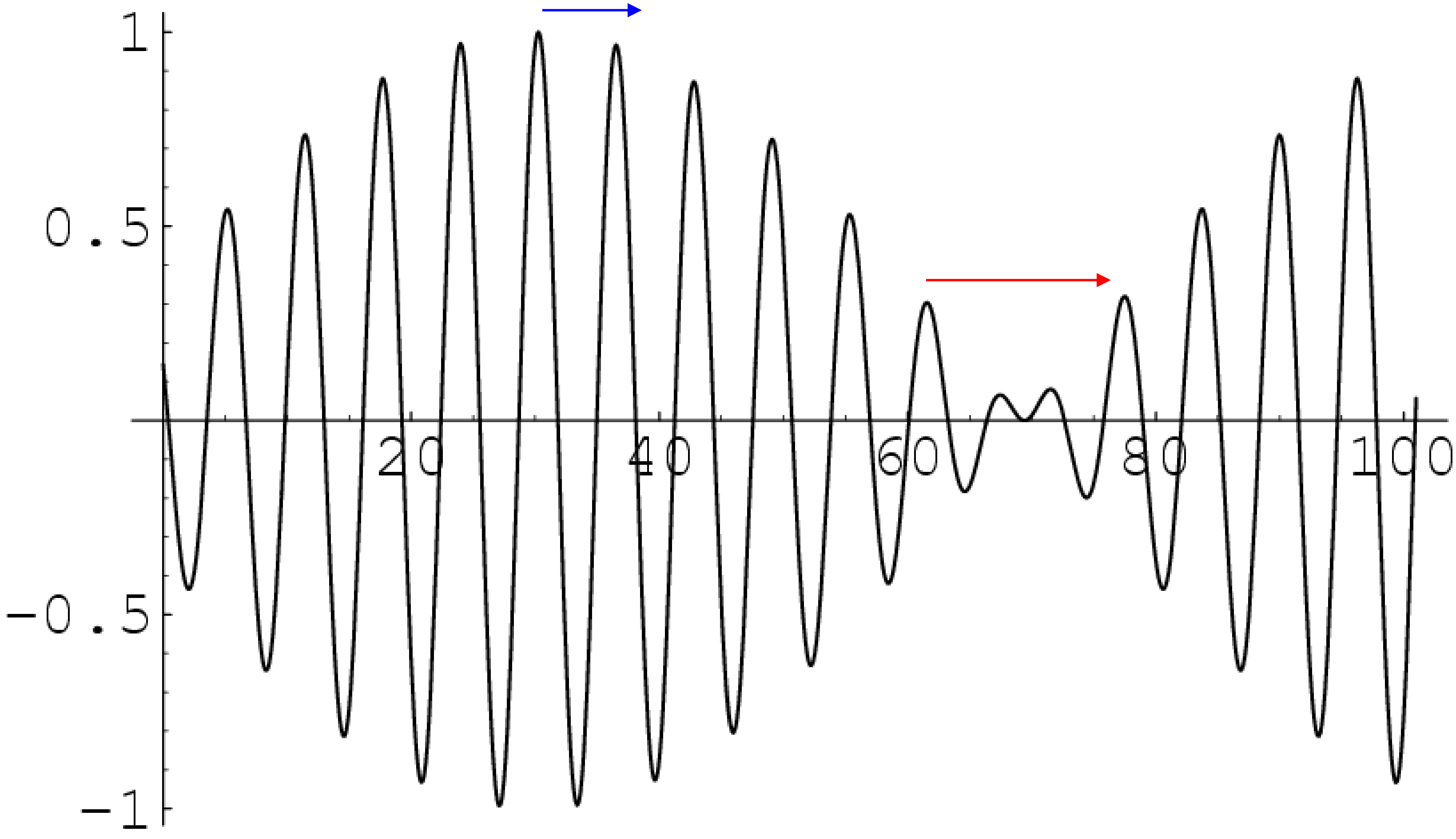
t = 4 $\sin(1.00\ x - 10.0\ t)\cos(0.04\ x - 0.2\ t)$

$V_p > V_g$



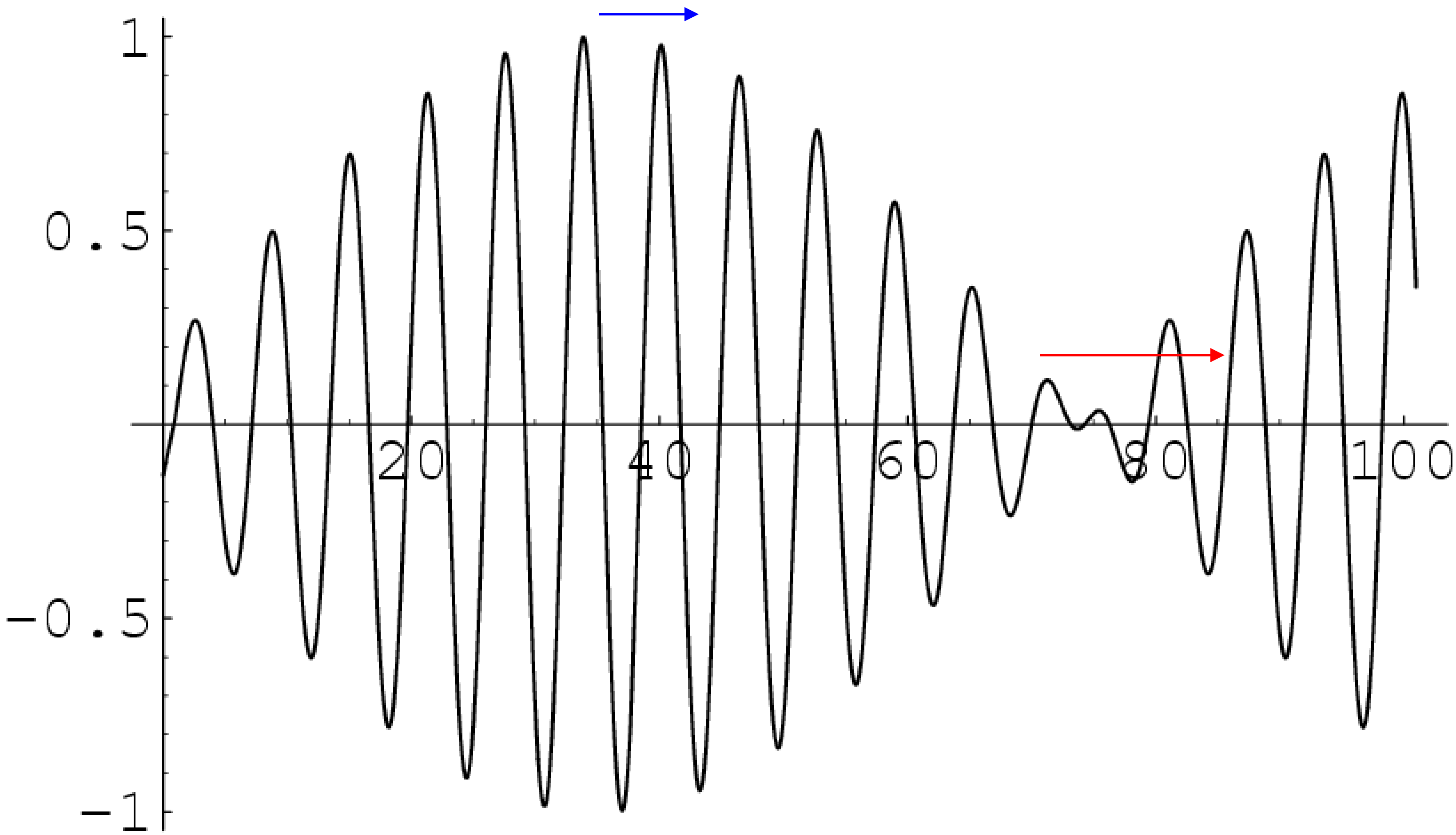
t = 5 $\sin(1.00\ x - 10.0\ t)\cos(0.04\ x - 0.2\ t)$

Vp > **V**g



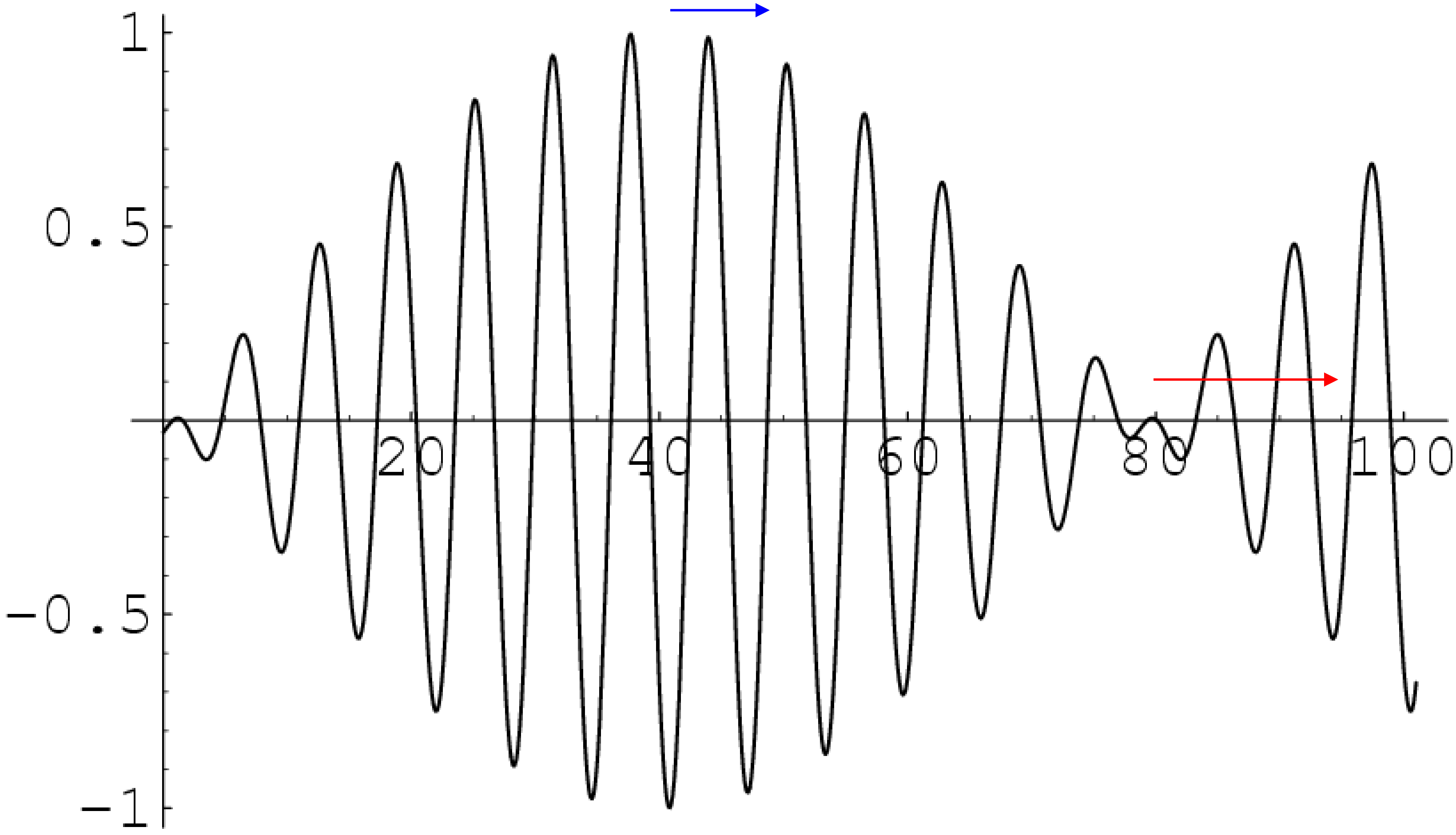
t = 6 $\sin(1.00\,x - 10.0\,t)\cos(0.04\,x - 0.2\,t)$

$\mathbf{V}_p > \mathbf{V}_g$



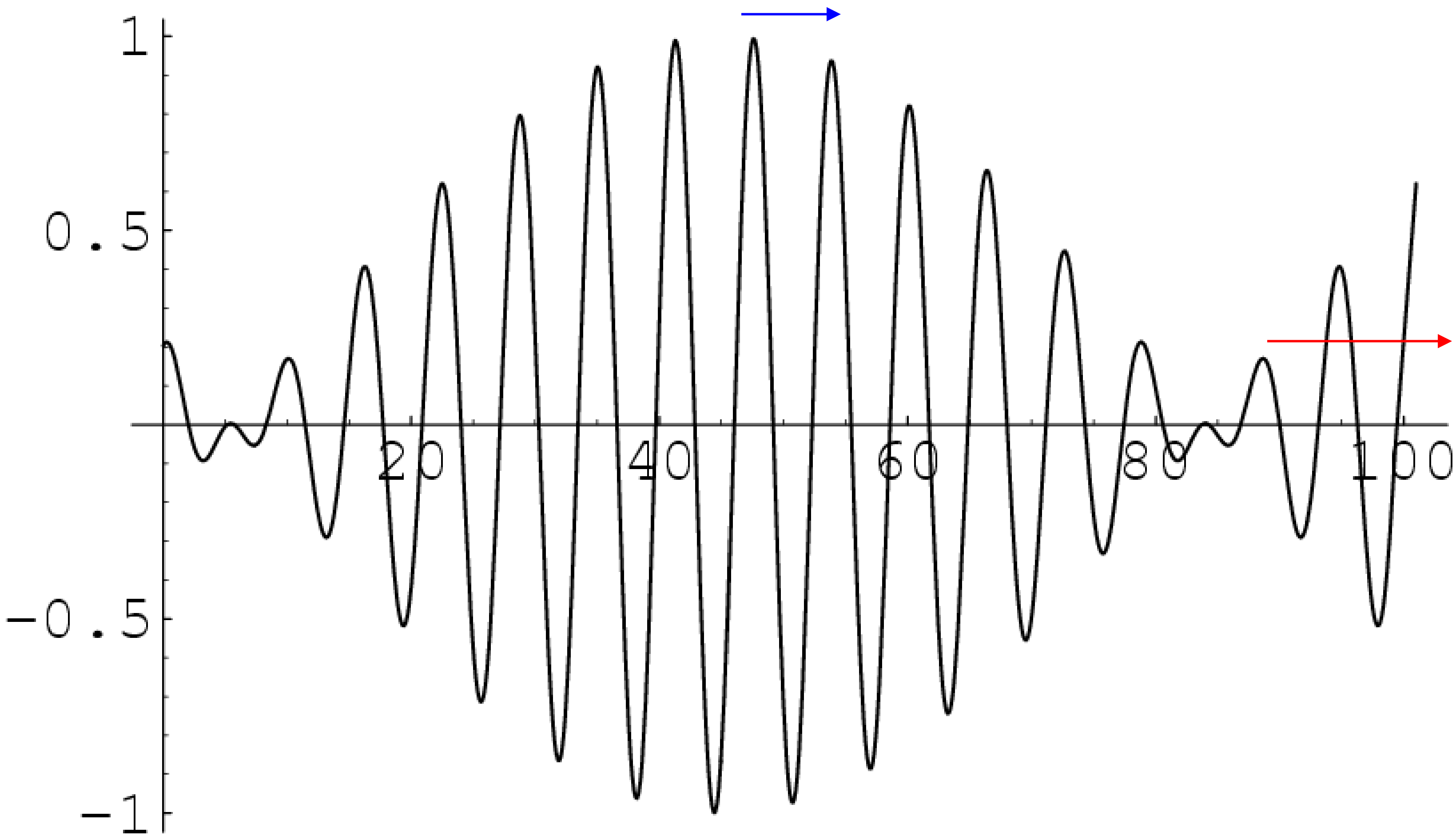
t = 7 $\quad \sin(1.00\,x - 10.0\,t)\cos(0.04\,x - 0.2\,t)$

$V_p > V_g$



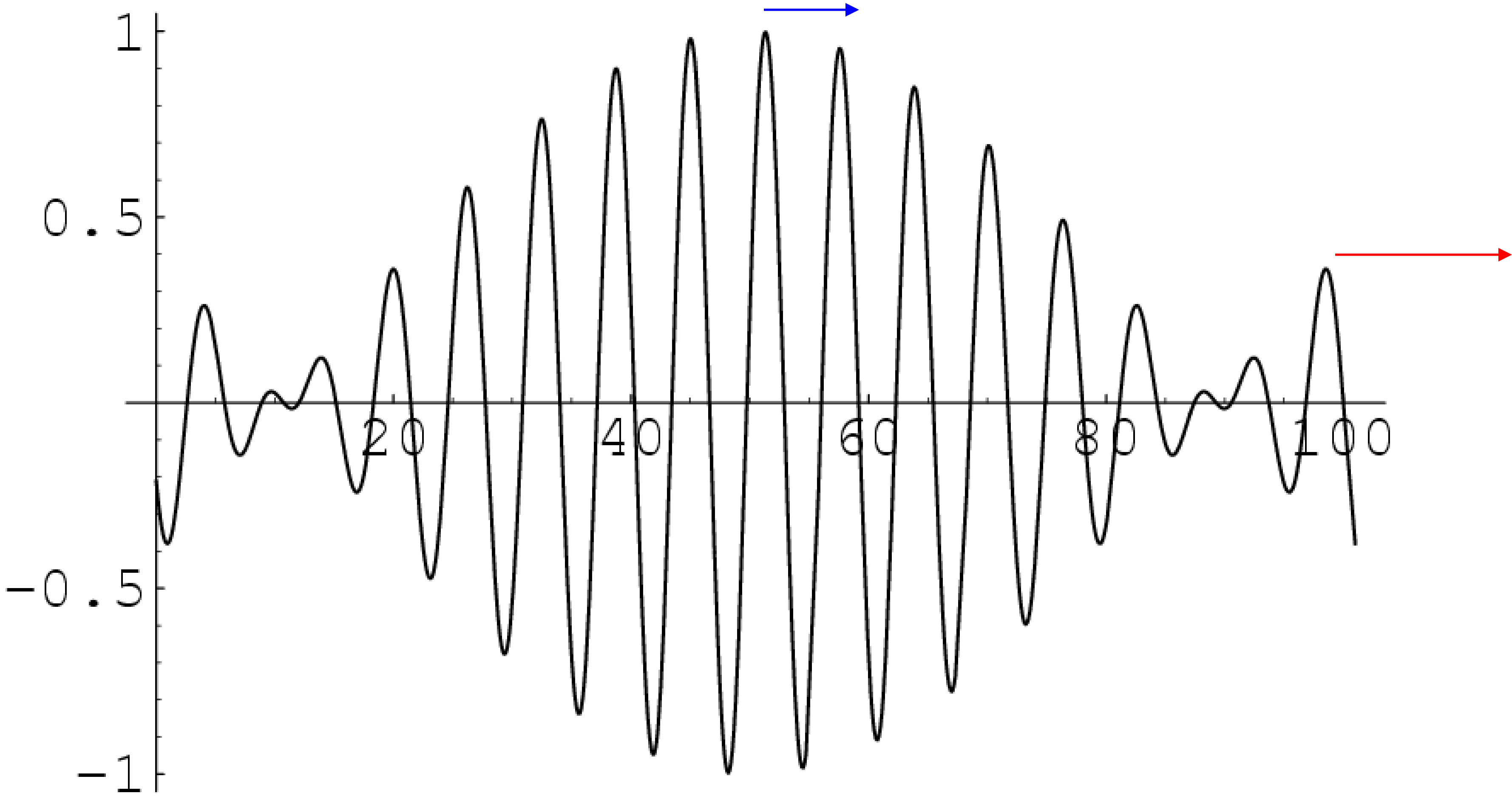
t = 8 $\sin(1.00\ x - 10.0\ t)\cos(0.04\ x - 0.2\ t)$

$\mathbf{V}_p > \mathbf{V}_g$

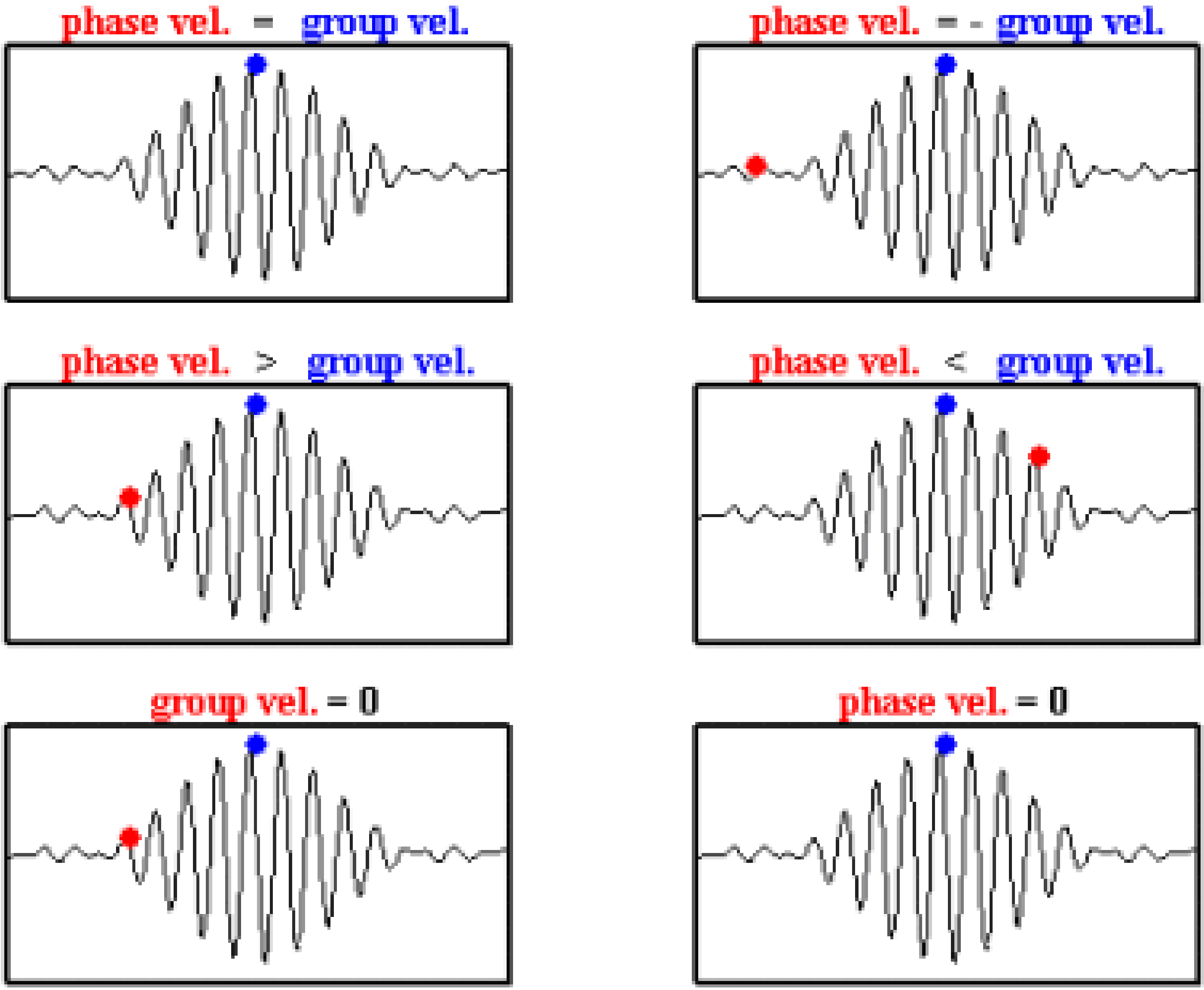


t = 9 $\sin(1.00\,x - 10.0\,t)\cos(0.04\,x - 0.2\,t)$

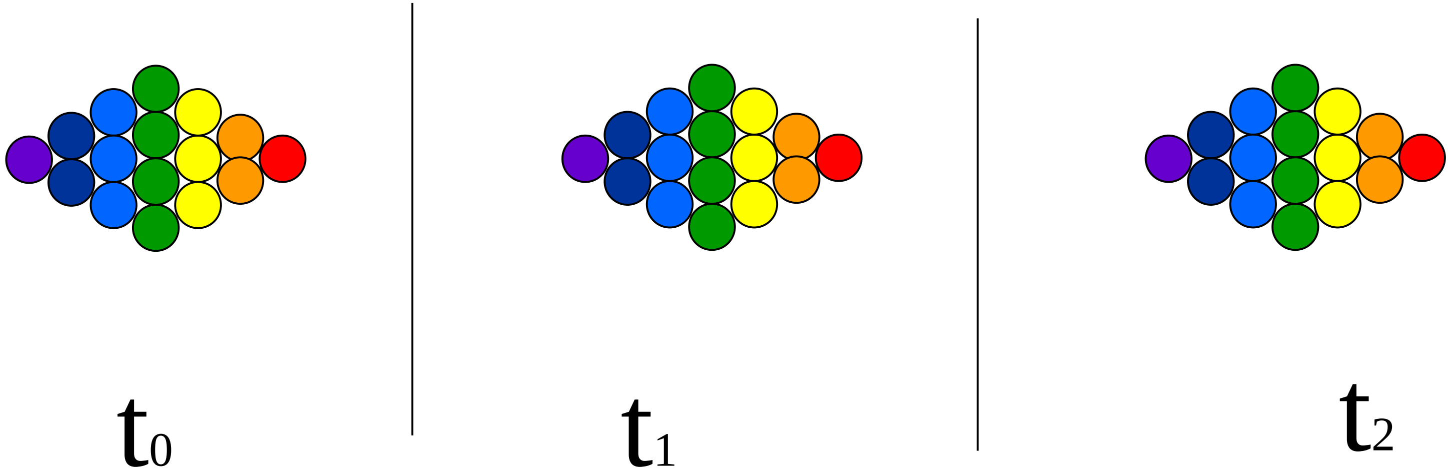
$V_p > V_g$



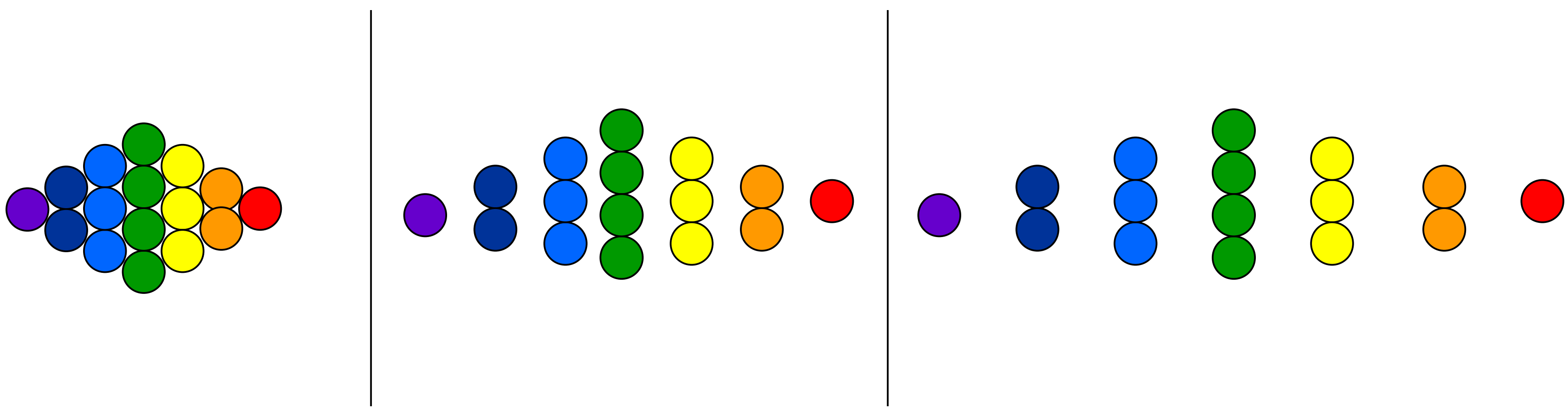
t = 10 $\sin(1.00\,x - 10.0\,t)\cos(0.04\,x - 0.2\,t)$

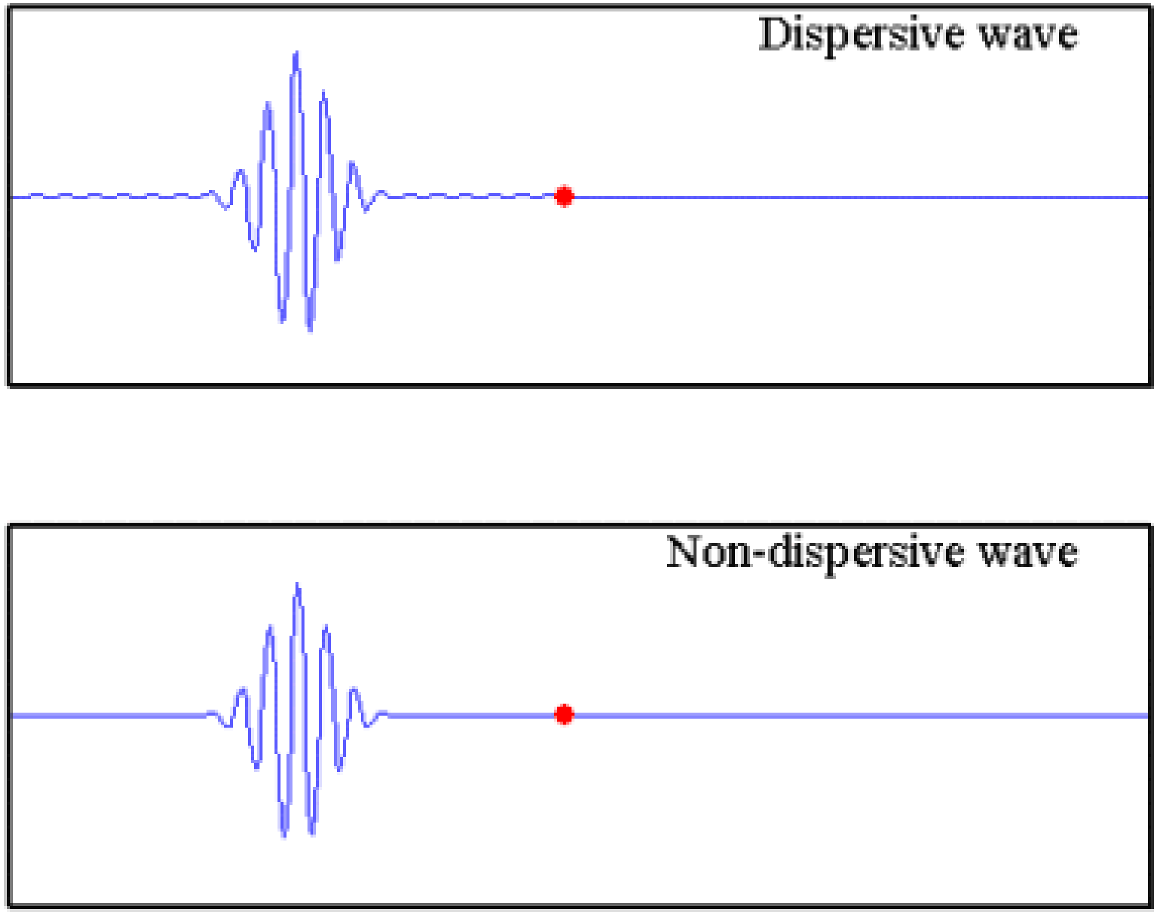

phase vel. = group vel.
phase vel. = - group vel.
phase vel. > group vel.
phase vel. < group vel.
group vel. = 0
phase vel. = 0

Nondispersive : All colours moving with same speed

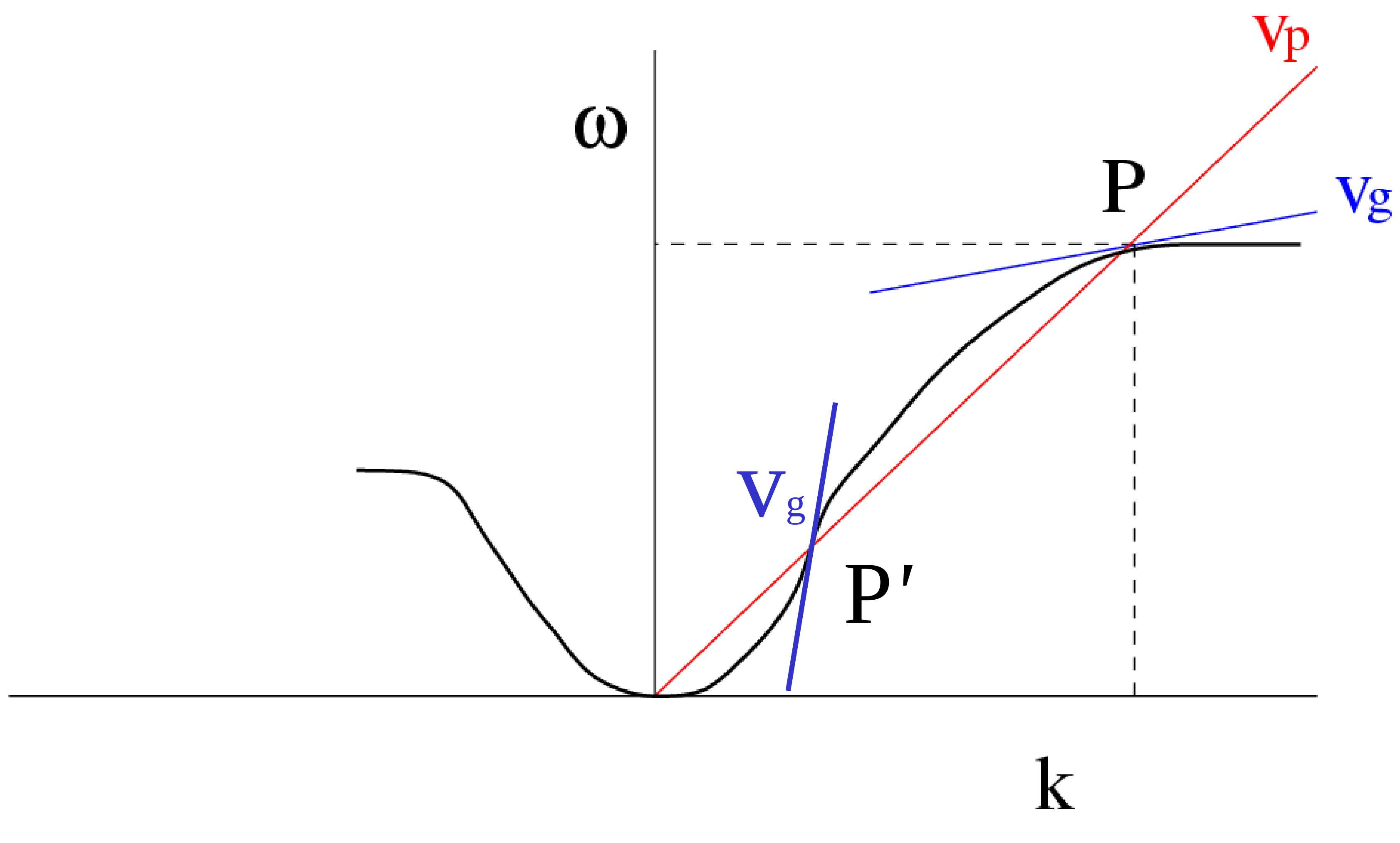


Dispersive : Red moving faster than the blue ones




Dispersive wave
Non-dispersive wave

Phase velocity and Group velocity



For nondispersive waves v_p = constant

Signal is propagated without distortion

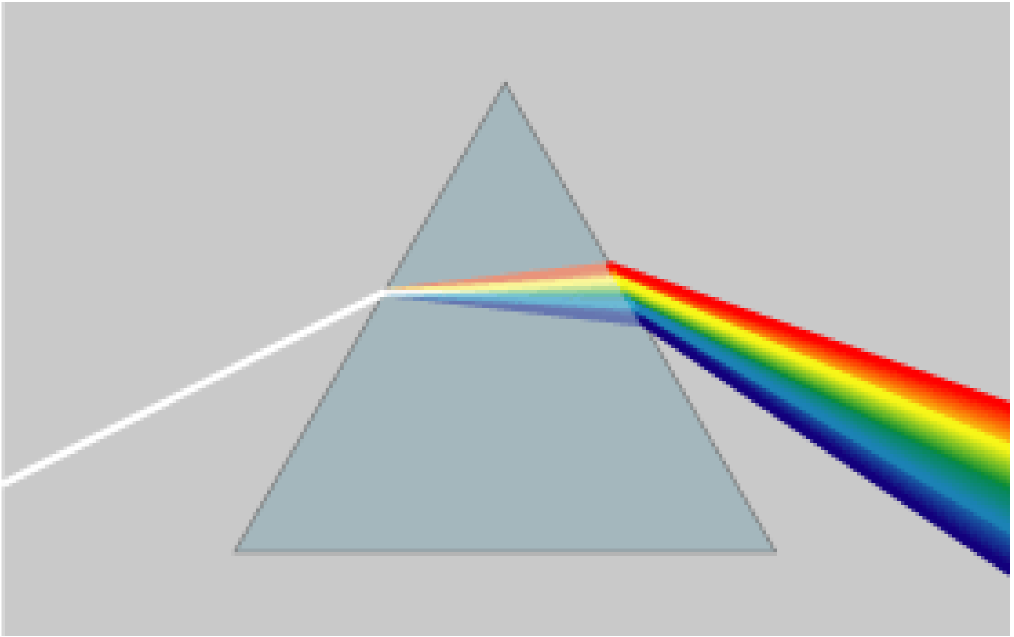
More generally v_p is a function of (or k)

$$\boxed{v_p = \frac{\omega}{k}}$$

Prism Experiment in your 1st year lab

$$\mu(\lambda) = \frac{\sin\left(\frac{A+\delta_m}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

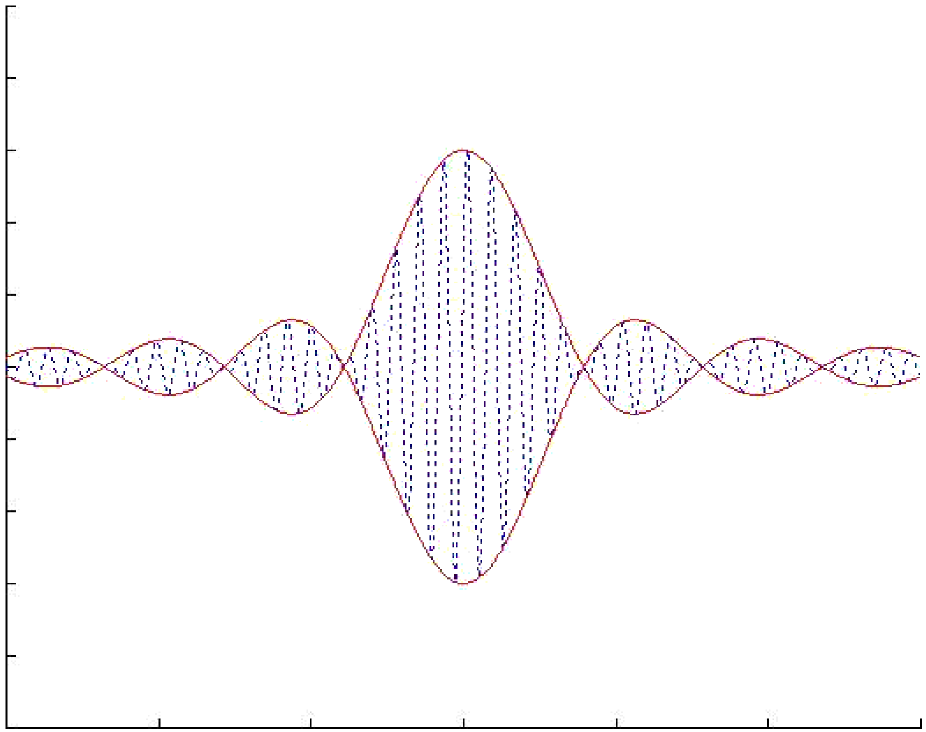
$$\omega = \frac{\mu_2 - \mu_1}{\mu - 1}$$

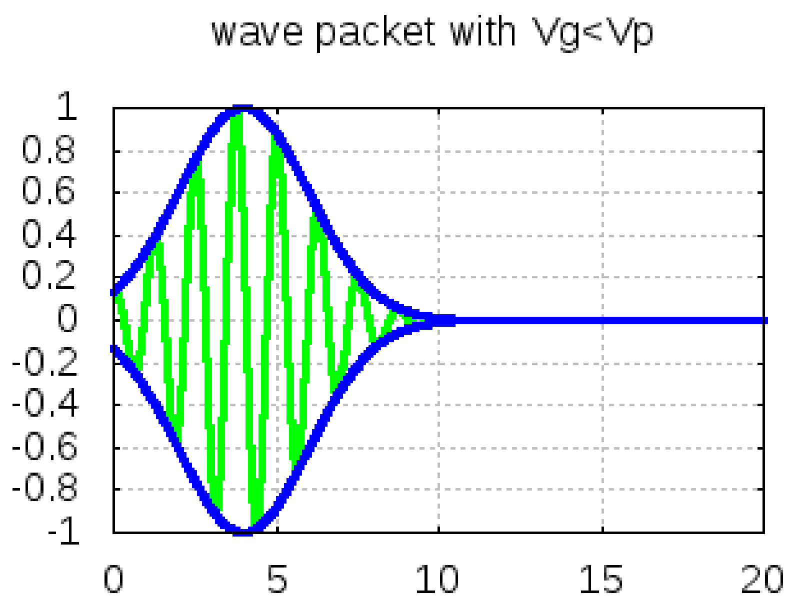


Reference

1. LECTURE NOTES FOR PHYSICS I
SASTRY AND SARASWAT

2. THE PHYSICS OF VIBRATIONS AND WAVES
AUTHOR: H.J. PAIN
IIT KGP Central Library
Class no. 530.124 PAI/P




wave packet with Vg<Vp
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20

Solution of 3D wave equation

In Cartesian coordinates

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Separation of variables

$$\psi(x, y, z, t) = X(x)Y(y)Z(z)T(t)$$

Substituting for ψ we obtain

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=\frac{1}{c^2}\left(\frac{1}{T}\frac{\partial^2 T}{\partial t^2}\right)$$

Variables are separated out

Each variable-term independent

And must be a constant

So we may write

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2; \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2;$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k_z^2; \quad \left(\frac{1}{T}\frac{\partial^2 T}{\partial t^2}\right) = -\omega^2$$

where we use

$$\omega^2 / c^2 = k_x^2 + k_y^2 + k_z^2 = k^2$$

Solutions are then

$$X(x) = e^{\pm ik_x x}; \quad Y(y) = e^{\pm ik_y y};$$

$$Z(z) = e^{\pm ik_z z}; \quad T(t) = e^{\pm i\omega t}$$

Total Solution is

$$\psi(x, y, z, t) = X(x)Y(y)Z(z)T(t)$$

$$= Ae^{i[\omega t \mp (k_x x + k_y y + k_z z)]}$$

$$= Ae^{i[\omega t \mp \vec{k}.\vec{r}]}$$ plane wave