→ A general PDE − $f(x, y, t, \ldots \theta, \theta_x, \theta_y \ldots \theta_{zt}, \ldots) = 0$

→ Notation : $p = z_x, q = z_y$

→ $F(u,v) = 0$, $\underbrace{u = u(x,y,z),\ v = v(x,y,z)}_{\text{known fns}}$, F - arbitrary

$\Downarrow$

$$\frac{\partial F}{\partial u}(u_x + pu_z) + \frac{\partial F}{\partial v}(v_x + pu_z) = 0$$

$$\frac{\partial F}{\partial u}(u_y + qu_z) + \frac{\partial F}{\partial v}(v_y + pv_z) = 0$$

$$p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(x,z)} = \frac{\partial(u,v)}{\partial(x,y)}$$

$$\Rightarrow \boxed{\frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - v_x u_y} \leftarrow \text{Jacobian}$$

→ If $u = u(x,y)$, $v = v(x,y)$ but $u = H(v)$ then $\frac{\partial(u,v)}{\partial(x,y)} = 0$ *.

Euler's Equation for a homogeneous function :- $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

$$\boxed{x f_x + y f_y = nf} \ *$$

Classification of first order PDEs :-

1) Linear : $P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y)$

Eg: $yp - xq = xyz + x$

2) Semilinear: $P(x,y)p + Q(x,y)q = R(x,y,z)$

Eg: $e^x p - yxq = xz^2$

3) Quasi-linear : $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$

Eg: $(x^2 + z^2)p - xyq = z^3 x + y^2$

4) Non-linear: $f(x,y,z,p,q) = 0$ which doesn't come under any of above 3 types

Classification of Integrals :-

First order PDE − $f(x,y,z,p,q) = 0$

1) Complete Integral :- A two-parameter family of solns $z = F(x,y,a,b)$ — ① is 'complete integral' if in the region considered, the rank of matrix $M = \begin{pmatrix} F_a & F_{xa} & F_{ya} \\ F_b & F_{xb} & F_{yb} \end{pmatrix}$ is '2'

2) General Integral :- In ① $\Rightarrow b = \phi(a) \Rightarrow$ $z = F(x,y,a,\phi(a))$

$F(a) + F_b \phi'(a) = 0$; $a = a(x,y)$

$\Rightarrow z = F(x,y, a(x,y), \phi(a(x,y)))$

Lemma 1: Let $z = F(x,y,a)$ be a one-parameter family of solutions of (1) then the envelope of this family, if it exist is also a solution of (1)

3) Singular integral :- In addition to general integral, we sometimes obtain yet another soln by finding the envelope of two parameter family. This can be obtained by eliminating a & b from the eq^{n}s

$$z = F(x,y,a,b) \quad F_a = 0 \quad F_b = 0$$

and is called 'Singular Integral' of $z = f(x,y,z,a,b)$

Lemma 2: The singular integral is also a solution

Lemma 3: The singular integral of $f(x,y,z,p,q) = 0$ satisfies the following equations

$$\left.\begin{aligned} & f(x,y,z,p,q) = 0 \\ & f_p(x,y,z,p,q) = 0 \\ & f_q(x,y,z,p,q) = 0 \end{aligned}\right\}$$

Special Integral:- Which are not in any of the 3 above mentioned categories

Linear Equations of the first order :-

The GS of $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$ ——— (Lagrange's equation)

P, Q, R - continuously differentiable functions of x, y, z and not vanishing simultaneously is $F(u,v) = 0$

where F is an arbitrary differentiable function of u, v and

$$u(x,y,z) = c_1 \text{ and } v(x,y,z) = c_2$$

are two independent solutions of the system.

$$\boxed{\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}}$$

If the number of independent variables is greater than '2' :-

Theorem:- If $u_i(x_1, x_2, \ldots x_n, z) = c_i$, $(i = 1,2,3 \ldots n)$ are independent solutions of the equations

$$\frac{dx}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \cdots \frac{dx_n}{P_n} = \frac{dz}{R}$$

where $P_1, P_2, P_3 \cdots P_n, R$ are continuously differentiable functions of $x_1, x_2, x_3 \cdots x_n, z$, not simultaneously zero, then the relation is a general solution of the quasi-linear PDE

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \cdots + P_n \frac{\partial z}{\partial x_n} = R$$

Pfaffian Differential equations:-

A Pfaffian DE is of the form

$$F_1(x_1, \ldots x_n)\, dx_1 + F_2(x_1, \ldots x_n)\, dx_2 + \cdots + F_n(x_1, \ldots x_n)\, dx_n = 0$$

where F_i's are continuous functions.

→ A Pfaffian DE is said to be $\boxed{\text{exact}}$ if we find a continuously differentiable function $u(x_1, x_2 \ldots x_n)$ such that

$$du = F_1(x_1, x_2, x_3 \ldots x_n)\, dx_1 + F_2(x_1, x_2, x_3 \ldots x_n)\, dx_2 + \cdots + F_n(x_1, x_2, x_3 \ldots x_n)\, dx_n$$

→ A Pfaffian DE is said to be $\boxed{\text{integrable}}$ if there exist non-zero differentiable function $\mu(x_1, x_2, x_3 \ldots x_n)$ such that Pfaffian differential form $\mu\left(F_1(x_1, \ldots x_n)\, dx_1 + \cdots + F_n(x_1, x_2 \ldots x_n)\, dx_n\right)$ is $\boxed{\text{Exact}}$

The function $\mu(x_1, x_2 \ldots x_n)$ is called the integrating factor and $u(x_1, x_2, x_3 \ldots x_n) = c$ is called integral of corresponding Pfaffian DE.

Theorem:- There always exist an integration factor for a Pfaffian DE in two variables.

Lemma:- Let $u(x,y)$, $v(x,y)$ be two functions of x & y such that $\frac{\partial v}{\partial y} \neq 0$, if further $\frac{\partial(u,v)}{\partial(x,y)} = 0$, then there exist a relation $F(u,v) = 0$ b/w u & v not involving x & y explicitly

Compatible systems for first order PDEs:-

The eq^{n}s $f(x,y,z,p,q) = 0$ ①, $g(x,y,z,p,q) = 0$ ② are compatible on a domain D if

(i) $J = \frac{\partial(f,g)}{\partial(p,q)} \neq 0$ on D ③ (ii) $p = \phi(x,y,z)$, $q = \psi(x,y,z)$

obtained by solving ① & ②, render $dz = \phi(x,y,z)\,dx + \psi(x,y,z)\,dy$, ④ integrable.

Theorem:- A necessary and sufficient condition for integrability of ④ is $[f,g] = \dfrac{\partial(f,g)}{\partial(x,p)} + p\dfrac{\partial(f,p)}{\partial(z,p)} + \dfrac{\partial(f,g)}{\partial(y,q)} + q\dfrac{\partial(f,g)}{\partial(z,q)} = 0$

Charpit's Method:- Complete integral of $f(x,y,z,p,q) = 0$

A family of PDEs $g(x,y,z,p,q) = 0$

Since $f=0$, $g=0$ are compatible we have $[f,g]=0$ i.e,

$$f_p\frac{\partial g}{\partial x} + f_q\frac{\partial g}{\partial y} + (pf_p + qf_q)\frac{\partial g}{\partial z} - (f_x + pf_z)\frac{\partial g}{\partial p} - (f_y + qf_z)\frac{\partial g}{\partial q} = 0$$

This is a quasi-linear first order PDE for g with x,y,z,p,q as independent variables.

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{f_x + pf_z} = \frac{-dq}{f_y + qf_z}$$

$P = \phi(x,y,z,a)$, $Q = \psi(x,y,z,a)$ then $dz = \phi\,dx + \psi\,dy$

$\downarrow$ is of the form

$F(x,y,z,a,b) = 0$

Some Standard types

Type 1:- $f(p,q)=0$, the auxiliary equations are $\dfrac{dx}{f_p} = \dfrac{dy}{f_q} = \dfrac{dz}{pf_p+qf_q} = \dfrac{dp}{0} = \dfrac{dq}{0}$

Upon solving this we either get $p=0$ (or $q=0$)

Then we solve for $f(a,q)=0$ [or $f(p,a)=0$ for $q=Q(a)$ (or $p=P(a)$)]

then $dz = a\,dx + Q(a)\,dy$

$\Rightarrow z = ax + Q(a)y + b$

[or $dz = P(a)\,dx + a\,dy \Rightarrow z = P(a)x + ay + b$]

Type-2: $f(z,p,q)=0$

AE are:- $\dfrac{dx}{f_p} = \dfrac{dy}{f_q} = \dfrac{dz}{pf_p+qf_q} = \dfrac{dp}{-pf_z} = \dfrac{dq}{-qf_z}$

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow p = aq$$

$\therefore f(z, az, q) = 0$ or $q = Q(a,z)$ and also $p = aQ(a,z)$

$$\Rightarrow dz = p\,dx + q\,dy = Q(a,z)\,(a\,dx + dy)$$

$$\Rightarrow \int \frac{dz}{Q(a,z)} = ax + y + b$$

Type 3: $g(x,p) = h(y,q) \longrightarrow$ separable form

AE $$\frac{dx}{g_p} = \frac{dy}{-h_q} = \frac{dz}{pg_p - qh_q} = \frac{dp}{-g_x} = \frac{dq}{h_y}$$

$g_x\,dx + g_p\,dp = 0 \Longrightarrow g(x,p) = a$ & hence $h(y,q) = a$

$\therefore p = G(a,x)$, $q = H(a,y)$

Then $dz = p\,dx + q\,dy \Rightarrow z = \int G(a,x)\,dx + \int H(a,y)\,dy + b$

Type-4: Clairaut Form

$$z = px + qy + g(p,q)$$

Here, a complete integral is given by $z = ax + by + g(a,b)$ for it is a solution and the matrix $\begin{pmatrix} x + g_a & 1 & 0 \\ y + g_b & 0 & 1 \end{pmatrix}$

Compatibility of two given first order PDEs:-

$$f(x,y,z,p,q) = 0 \quad \text{and} \quad g(x,y,z,p,q) = 0$$

$$\boxed{(f_x g_p - f_p g_x) + (f_y g_q - f_q g_y) + p(f_z g_p - f_p g_z) + q(f_z g_q - f_q g_z) = 0}$$