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Section:2

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Partial Differential equations ASSIGNMENT-1

1. **[6+3+3=12M]**

a) For each of the following PDE, write down the order and degree [$6 \times 1 = 6$M]

(i) $p + q = z + xyz$, (ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\frac{\partial z}{\partial x}$, (iii) $r + (1+p)^2 = 1$,

(iv) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2}$, (v) $y(p^2 + q^2) = qz$, (vi) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$

b) Form a PDE by eliminating arbitrary constants a and b from $z = axe^y + \frac{1}{2}a^2e^{2y} + b$.

c) Form a PDE by eliminating arbitrary functions f and g from $z = yf(x) + xg(y)$.

Solution:-

a) Order of PDE is order of highest order derivative

Degree of PDE is exponent of highest order derivative

(i) $p+q = z+xyz \equiv \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xyz\right)$

$\boxed{\text{Order} = 1, \ \text{Degree} = 1}$

(ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\frac{\partial z}{\partial x}$

$\boxed{\text{Order} = 3, \ \text{Degree} = 1}$

(iii) $r + (1+p)^2 = 1$

$\equiv \frac{\partial^2 z}{\partial x^2} + \left(1 + \frac{\partial z}{\partial x}\right)^2 = 1$

$\boxed{\text{Order} = 2, \ \text{Degree} = 1}$

(iv) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2}$

$\left(\frac{\partial^2 z}{\partial x^2}\right)^2 = \left(1 + \frac{\partial z}{\partial y}\right)$

$\boxed{\text{Order} = 2, \ \text{Degree} = 2}$

(v) $y(p^2+q^2) = qz$

$y\left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right) = qz$

$\boxed{\text{Order} = 1, \ \text{Degree} = 2}$

(vi) $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=xyz$

$\boxed{\text{Order}=1,\ \text{Degree}=1}$

b) Given equation $z=axe^{y}+\frac{1}{2}a^2e^{2y}+b$

Partial differentiating wrt x;

$$p\equiv\frac{\partial z}{\partial x}=ae^{y} \quad \text{—— (1)}$$

Partial differentiating wrt y;

$$q\equiv\frac{\partial z}{\partial y}=axe^{y}+\frac{1}{2}a^2(2)e^{2y}$$

$$q=axe^{y}+a^2e^{2y} \quad \text{—— (2)}$$

From (1) $a=p/e^{y}$

Substitute 'a' in equation (2)

$$q=\frac{p}{e^{y}}\cdot x\cdot e^{y}+\frac{p^2}{e^{2y}}\cdot e^{2y}$$

$$\Rightarrow \boxed{\begin{aligned} &q=px+p^2\\ &(\text{or})\quad p^2+px-q=0\\ &(\text{or})\quad \left(\frac{\partial z}{\partial x}\right)^2+x\left(\frac{\partial z}{\partial x}\right)-\left(\frac{\partial z}{\partial y}\right)=0\end{aligned}}$$

—— PDE after eliminating arbitrary constants

c) $z=yf(x)+xg(y)$

Partial diff. wrt $x \Rightarrow p\equiv\frac{\partial z}{\partial x}=yf'(x)+g(y)$ —— (1)

$$\Rightarrow \quad f'(x)=\frac{1}{y}(p-g(y))$$

Partial diff. wrt $y \Rightarrow q\equiv\frac{\partial z}{\partial y}=f(x)+xg'(y)$ —— (2)

$$\Rightarrow \quad g'(y)=\frac{1}{x}(q-f(x))$$

Partial differentiate (1) wrt $y \Rightarrow \frac{\partial^2 z}{\partial x\partial y}=f'(x)+g'(y)$ —— (3)

$$s=f'(x)+g'(y)$$

Partial differentiate ② wrt $x \Rightarrow \frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$.

Substitute ① and ② in ③ $\Rightarrow \frac{\partial^2 z}{\partial x \partial y} \equiv s = \frac{1}{y}(p - g(y)) + \frac{1}{x}(q - f(x))$

$$s = \frac{1}{y}(p - g(y)) + \frac{1}{x}(q - f(x))$$

$$= \left(\frac{px - xg(y) + qy - yf(x)}{xy}\right)$$

$$s = \frac{1}{xy}(px + qy - z)$$

$$xys = px + qy - z$$

$$\boxed{\begin{aligned} &xys - px - qy + z = 0 \\ &xy\frac{\partial^2 z}{\partial x \partial y} - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} + z = 0 \end{aligned}}$$

PDE after eliminating arbitrary functions

2. **[12+6=18M]**

a) Solve the following PDEs by Lagrange method: [6 × 2 = 12M]

i. $(x + 2z)p + (4zx - y)q = 2x^2 + y$

ii. $\frac{y^2 z}{x}p + xzq = y^2, x \neq 0$

b) Find the integral surface of $x^2p + y^2q + z^2 = 0$ which passes through the curve $xy = x + y, z = 1$.

Solution:

a) i) $(x+2z)p + (4zx - y)q = 2x^2 + y$

$$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$$

then the auxiliary equation $\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$

$$\frac{dx}{x+2z} = \frac{dy}{4zx-y} = \frac{dz}{2x^2+y}$$

then $\lambda = \frac{dx + dy + dz}{(x+2z) + (4zx-y) + (2x^2+y)} = \frac{\mu_1 dx + \mu_2 dy + \mu_3 dz}{\mu_1(x+2z) + \mu_2(4zx-y) + \mu_3(2x^2+y)}$

Upon observation, we get $\mu_1 = 2x, \mu_2 = -1, \mu_3 = -1$

$$\therefore \underbrace{2x\,dx - dy - dz = 0}_{\text{Pfaffian Differential equation}} \text{ — } ①$$

$P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0$ (General Pfaffian DE)

Integrating eqn 1 yields $\Rightarrow$

$$2x^2/2 - y - z \pm c_1 = 0$$

$$x^2 - y - z \pm C_1 = 0$$

$$\Rightarrow \boxed{C_1 = u = \pm(x^2 - y - z)}$$

From another observation, we get $\mu_1 = y$, $\mu_2 = x$, $\mu_3 = -2z$

$$\therefore\ y\,dx + x\,dy - 2z\,dz = 0 \quad \text{——} (2)$$

Integrating eqn 2 yields

$$\int d(xy) - 2z\,dz = 0$$

$$\Rightarrow \quad xy - z^2 \pm c_2 = 0$$

$$\Rightarrow \boxed{c_2 = v = \pm(xy - z^2)}$$

$\therefore$ The general integral is given by $F(u,v) = 0$

$$F\left(\pm(x^2-y-z),\ \pm(xy-z^2)\right) = 0$$

ii) $\dfrac{y^2 z}{x}\,p + xzq = y^2,\ x \neq 0$

$P(x,y,z)\,p + Q(x,y,z)\,q = y^2$

Auxiliary equation: $\dfrac{dx}{\left(\frac{y^2 z}{x}\right)} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$

then $\lambda = \dfrac{dx + dy + dz}{\frac{y^2 z}{x} + xz + y^2} = \dfrac{\mu_1 dx + \mu_2 dy + \mu_3 dz}{\mu_1 \frac{y^2 z}{x} + \mu_2 xz + \mu_3 y^2}$

(OR)

$$\frac{dx}{\left(\frac{y^2 \cancel{z}}{x}\right)} = \frac{dy}{x\cancel{z}} \Rightarrow x^2\,dx = y^2\,dy$$

Integrate both sides,

$$C_1 = \pm(x^3 - y^3) = u$$

$$\frac{dx}{\left(\frac{\cancel{y^2} z}{x}\right)} = \frac{dz}{\cancel{y^2}} \Rightarrow x\,dx = z\,dz$$

$$\Rightarrow C_2 = \pm(x^2 - z^2) = v$$

$\therefore$ The general integral is given by $F(u, v) = 0$

$$\Rightarrow \boxed{F\left(\pm(x^3 - y^3), \pm(x^2 - z^2)\right) = 0}$$

$\rightarrow$ We can also arrive at the same solution by generating two Pfaffian Differential equations.

From observation, we can get $\mu_1 = \frac{x}{z}$, $\mu_2 = 0$, $\mu_3 = -1$

$$\frac{x}{z}dx - dz = 0 \Rightarrow x\,dx = z\,dz$$

$$\Rightarrow \pm(x^2 - z^2) = C_1 \;(= v)$$

Another observation, $\mu_1 = \frac{x^2}{y^2}$, $\mu_2 = -1$, $\mu_3 = 0$

$$\frac{x^2}{y^2}dx + (-1)\,dy + 0\,dz = 0$$

$$\Rightarrow x^2\,dx = y^2\,dy$$

$$\Rightarrow C_2 = \pm(x^3 - y^3) = u$$

From this also, we get the same general integral.

b) integral surface of $x^2p + y^2q + z^2 = 0$ which passes through the curve $xy = x + y$, $z = 1$

$$x^2p + y^2q = -z^2$$

The auxiliary equation: $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \frac{1}{x} = \frac{1}{y} \pm C_1$$

$$\Rightarrow C_1 = u = \pm\left(\frac{1}{x} - \frac{1}{y}\right) \quad \text{—— (1)}$$

$$\frac{dx}{x^2} = \frac{dz}{-z^2} \Rightarrow -\frac{1}{x} = \frac{1}{z} \pm c_2$$

$$\Rightarrow c_2 = v = \pm\left(\frac{1}{x}+\frac{1}{z}\right) \quad — (2)$$

Let us take both positive for a while :-

$$c_1 = u = \left(\frac{1}{x}-\frac{1}{y}\right) \quad \text{and} \quad c_2 = v = \left(\frac{1}{x}+\frac{1}{z}\right)$$

Now let us put $z=1$ in u, v

$$u = \frac{1}{x}-\frac{1}{y} \quad \text{and} \quad v = \frac{1}{x}+1$$

$$\Rightarrow \frac{1}{y} = \frac{1}{x}-u \qquad \Rightarrow \boxed{x = \frac{1}{v-1}} \quad — (3)$$

$$\Rightarrow \frac{1}{y} = v-1-u$$

$$\Rightarrow \boxed{y = \frac{1}{v-1-u}} \quad — (4)$$

Put (1) & (2) in $xy = x+y$

$$\Rightarrow \frac{1}{x}+\frac{1}{y} = 1$$

$$\Rightarrow (v-1)+(v-u-1) = 1$$

$$\Rightarrow 2v-u-2 = 1$$

$$\Rightarrow 2v-u = 3$$

From (1) & (2) substitute u, v values here

$$2\left[\left(\frac{1}{x}+\frac{1}{z}\right)\right] - \left(\frac{1}{x}-\frac{1}{y}\right) = 3$$

$$\Rightarrow \frac{2}{x}+\frac{2}{z}-\frac{1}{x}+\frac{1}{y} = 3$$

$$\Rightarrow \boxed{\frac{1}{x}+\frac{1}{y}+\frac{2}{z} = 3}$$ → Integral surface of given equation.

3. **[5+10=15M]**

Consider a family of PDE $(p^2z^a + q^2)y^\alpha = b + \beta q z^\gamma, \alpha, \beta, \gamma, a, b \in \mathbb{R}$. The complete integral of this PDE-family is denoted by $z = z(x, y; c, d)$, where c, d are arbitrary constants.

a) For $\alpha = \beta = a = 0, b = 25, \gamma \in \mathbb{R}$, given that $z(x, y; c, d) > 10, \forall x, y, c, d \in \mathbb{R}$, where c is the value of p at any point (x, y), and d is the value of the solution at $(x, y) = (0,0)$. Compute the value of $|z(2,1; 4,1)|$. [5M]

b) For $\alpha = \beta = \gamma = 1, a = b = 0$, given that $|z(x, y; c, d)| < 3, \forall x, y, c, d \in \mathbb{R}$, where c is the value of sum of squares of p and q at any point (x, y), and d^2 is the value of the square of the solution at $(x, y) = (0,0)$. Find the value of $|z(1,1; 2, -1)|^2$. [10M]

Solution:- $(p^2 z^a + q^2)y^\alpha = b + \beta q z^r \qquad \alpha, \beta, r, a, b \in \mathbb{R}$

Complete integral of this PDE-family is $z = z(x, y; c, d)$

a) $\alpha = \beta = a = 0$, $b = 25$, $r \in \mathbb{R}$ and given that $z(x, y; c, d) > 10$

$|z(2,1; 4,1)| = ?$

Put $\alpha = \beta = a = 0$, $b = 25$ in $(p^2 z^a + q^2)y^\alpha = b + \beta q z^r$

$$(p^2 + q^2) = 25$$

$$\therefore \quad f(x, y, z, p, q) = p^2 + q^2 - 25 = 0 \quad \text{—— (1)}$$

The auxiliary equation :-

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

$$\Rightarrow \frac{dx}{2p} = \frac{dy}{2q} = \frac{dz}{2p^2 + 2q^2} = \frac{dp}{-(0)} = \frac{dq}{-(0)}$$

$$\frac{dx}{2p} = \frac{dp}{0} \Rightarrow dp = 0 \Rightarrow p = c \quad \text{—— (2)}$$

And from (1) we can write $c^2 + q^2 = 25$

$$q = \pm\sqrt{25 - c^2} \quad \text{—— (3)}$$

And since z is a function of x & y we can write

$$dz = \left(\frac{\partial z}{\partial x}\right)dx + \left(\frac{\partial z}{\partial y}\right)dy$$

$$\Rightarrow dz = p\,dx + q\,dy$$

From 2 & 3, this equation becomes

$$dz = c\,dx + \sqrt{25-c^2}\,dy$$

Upon integrating this eq^n,

$$z = cx + \sqrt{25-c^2}\,y + b$$

When $(x,y) = (0,0)$, $z = d$ (from question)

and $z = b$ at $(0,0)$

$$\therefore\ b = d$$

$$\therefore\ z = cx + \sqrt{25-c^2}\,y + d$$

And also given that $p = \frac{\partial z}{\partial x} = c$

$$\frac{\partial z}{\partial x} = c$$

$$\therefore\ |z(2,1;4,1)| = |\ 4(2) + \sqrt{25-4^2}\,(1) + 1|$$
$$= |8 + 3 + 1|$$
$$= |12|$$

$$\therefore\ \boxed{|z(2,1;4,1)| = 12}$$ $\rightarrow$ This has also satisfied $z(x,y;c,d) > 10$

Let us also check $q = -\sqrt{25-c^2}$

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = c\,dx - \sqrt{25-c^2}\,dy$$

$$\Rightarrow z = cx - \sqrt{25-c^2}\,y + b$$

and when $(x,y) = (0,0)$, $z = d$

$$\Rightarrow b = d$$

$$\therefore\ z = cx - \sqrt{25-c^2}\,y + d \qquad \& \quad \frac{\partial z}{\partial x} = p = c$$

$$\therefore\ |z(2,1;4,1)| = |\ 4(2) - \sqrt{25-4^2}\,(1) + 1|$$
$$= |8 - 3 + 1| = 6$$ (which is less than 10)

$\therefore$ This value is not valid

$$\therefore \quad \boxed{|z(2,1;4,1)| = 12}$$

b) For $\alpha=\beta=\gamma=1$, $a=b=0$ & $|z(x,y;c,d)| < 3 \ \forall x,y,c,d \in \mathbb{R}$

$c = p^2+q^2$ at any point (x,y)

when $(x,y) = (0,0)$, value of square of soln $= d^2$

Then $|z(1,1;2,-1)|^2 = ?$

$$(p^2+q^2)y = qz$$

$$f(x,y,z,p,q) = (p^2+q^2)y - qz = 0 \quad \text{--- } 1$$

$\therefore$ the auxiliary equation:

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-(f_x+pf_z)} = \frac{dq}{-(f_y+qf_z)}$$

$$\Rightarrow \frac{dx}{2py} = \frac{dy}{2qy} = \frac{dz}{2p^2y+q[2qy-z]} = \frac{dp}{-(p(-q))} = \frac{dq}{-((p^2+q^2)+q(-q))}$$

$$\Rightarrow \frac{dx}{2py} = \frac{dy}{2qy} = \frac{dz}{2y(p^2+q^2)-qz} = \frac{dp}{pq} = \frac{dq}{-p^2}$$

$$\frac{dp}{pq} = \frac{dq}{-p^2} \Rightarrow \frac{dp}{q} = \frac{dq}{-p}$$

$$\Rightarrow -pdp = qdq$$

$$\Rightarrow pdp+qdq = 0$$

$$\Rightarrow p^2+q^2 = c \quad \text{--- } 2$$

$$\Rightarrow p = \pm\sqrt{c-q^2}$$

Put 2 in 1 $\Rightarrow$

$$cy - qz = 0$$

$$\Rightarrow \quad \boxed{q = \frac{cy}{z}}$$

and $p = \pm\sqrt{c - \frac{c^2y^2}{z^2}}$

$$\boxed{p = \pm \frac{1}{z}\sqrt{cz^2 - c^2y^2}}$$

And also $\quad dz = p\,dx + q\,dy$

$$dz = \frac{1}{z}\sqrt{cz^2 - c^2y^2}\,dx + \frac{cy}{z}\,dy$$

$$\Rightarrow \quad z\,dz = \sqrt{cz^2 - c^2y^2}\,dx + cy\,dy$$

$$\Rightarrow \quad \frac{z\,dz - cy\,dy}{\sqrt{cz^2 - c^2y^2}} = dx$$

$$\Rightarrow \quad \frac{1}{2\sqrt{c}}\,\frac{d(z^2 - cy^2)}{\sqrt{z^2 - cy^2}} = dx$$

$$\Rightarrow \quad \frac{d(z^2 - cy^2)}{\sqrt{z^2 - cy^2}} = 2\sqrt{c}\,dx$$

$$\Rightarrow \quad 2\sqrt{z^2 - cy^2} = 2\sqrt{c}\,x + b$$

$z^2\Big|_{(x,y)=(0,0)} = d^2 \quad \Rightarrow \quad 2\sqrt{z^2_{(0,0)}} = b$

$$\Rightarrow \quad 4z^2_{(0,0)} = b^2$$

$$\Rightarrow \quad 4d^2 = b^2$$

$$\Rightarrow \quad b = \pm 2d$$

$$2\sqrt{z^2 - cy^2} = 2\sqrt{c}\,x + 2d$$

$$\Rightarrow \quad \sqrt{z^2 - cy^2} = \sqrt{c}\,x + d$$

$$z^2 - cy^2 = (\sqrt{c}\,x + d)^2$$

$$|z(1,1;2,-1)|^2 - 2(1)^2 = [\sqrt{2}(1) - 1]^2$$

$$z^2 - 2 = (\sqrt{2} - 1)^2$$

$$\Rightarrow \quad z^2 = 2 + 2 + 1 - 2\sqrt{2}$$

$$z^2 = 5 - 2\sqrt{2} \quad (< 3)$$

$$\therefore \quad \boxed{|z(1,1;2,-1)|^2 = 5 - 2\sqrt{2}}$$

4. **[10+5=15M]**

Consider the pair of PDEs $\alpha(px - qy) - \beta z = 0\ (\alpha\beta \neq 0)$, $x^2yp + xy^2q - xyz = 0$.

a) Show that two PDEs are compatible for all non-zero real values of α, β. [10M]

b) If $z(x, y; c)$ is the common solution where c is the value of the square of the solution at $(x, y) = (1,1)$, then for all real α, β, compute the value/s of $z(\alpha\beta, \alpha\beta; \frac{1}{\alpha^2\beta^2})$.

THE END

Solution:

a) $\alpha(px - qy) - \beta z = 0 \quad (\alpha\beta \neq 0)$

$x^2yp + xy^2q - xyz = 0$

$f(x, y, z, p, q) = 0 \longrightarrow \alpha(px - qy) - \beta z = 0$

$g(x, y, z, p, q) = 0 \longrightarrow x^2yp + xy^2q - xyz = 0$

Compatibility condition $\Rightarrow$

$$(f_x g_p - f_p g_x) + (f_y g_q - f_q g_y) + p(f_z g_p - f_p g_z) + q(f_z g_q - f_q g_z) = 0$$

$$(f_x g_p - f_p g_x) = (\alpha p)(x^2y) - (\alpha x)(2xyp + y^2q - yz)$$
$$= \alpha p x^2 y - 2\alpha x^2 yp - \alpha x y^2 q + \alpha xyz$$

$$(f_y g_q - f_q g_y) = (-\alpha q)(xy^2) - (-\alpha y)(x^2p + 2xyq - xz)$$
$$= -\alpha xy^2 q + \alpha x^2 yp + 2\alpha xy^2 q - \alpha xyz$$

$$p(f_z g_p - f_p g_z) = p[(-\beta)(x^2y) - (\alpha x)(-xy)]$$
$$= -p\beta x^2 y + p\alpha x^2 y$$

$$q(f_z g_q - f_q g_z) = q[(-\beta)(xy^2) - (-\alpha y)(-xy)]$$
$$= -q\beta xy^2 - q\alpha xy^2$$

$$\cancel{\alpha p x^2 y} - \cancel{2\alpha x^2 yp} - \cancel{\alpha xy^2 q} + \cancel{\alpha xyz}$$
$$- \cancel{\alpha xy^2 q} + \cancel{\alpha x^2 yp} + \cancel{2\alpha xy^2 q} - \cancel{\alpha xyz}$$
$$\underbrace{-p\beta x^2 y} + \underbrace{p\alpha x^2 y} \underbrace{- q\beta xy^2} - \underbrace{q\alpha xy^2}$$

$$= -p\beta x^2 y - q\beta xy^2 - q\alpha xy^2 + p\alpha x^2 y$$

$$= -\beta(x^2yp + xy^2q) + \alpha(px^2y - qxy^2)$$

$$= -\beta(xyz) + \alpha(xy)(py - qx)$$

$$= -\beta(xyz) + \alpha xy\left(\frac{\beta z}{\alpha}\right) \text{ ——— From } g(x,y,z,p,q)=0$$

$$= -\beta \cancel{xyz} + \beta \cancel{xyz}$$

$$\boxed{=0}$$

∴ The two given pair of PDEs are compatible for all non-zero α, β.

b) $z(x,y;c) \longrightarrow$ common soln

value of solution square $= c$ when $(x,y) = (1,1)$

$$z\left(\alpha\beta, \alpha\beta; \frac{1}{\alpha^2\beta^2}\right) = ? \quad \forall \alpha, \beta \in \mathbb{R}$$

$$f(x,y,z,p,q)=0 \rightarrow \alpha(px - qy) - \beta z = 0 \quad \text{——— (1)}$$

$$g(x,y,z,p,q)=0 \longrightarrow x^2y\,p + xy^2q - xyz = 0 \quad \text{——— (2)}$$

(1) $\times\ xy$ + (2) $\times\ \alpha$ and add

$$\alpha px^2y - \alpha qxy^2 - \beta xyz = 0$$

$$(+)\quad \alpha px^2y + \alpha qxy^2 - \alpha xyz = 0$$

$$2\alpha px^2y - (\alpha+\beta)xyz = 0$$

$$\Rightarrow p = \frac{(\alpha+\beta)\cancel{x}\cancel{y}z}{2\alpha x^2\cancel{y}}$$

$$\Rightarrow p = \frac{(\alpha+\beta)z}{2\alpha x} \quad \text{——— (3)}$$

and put 'p' in eqn (1)

$$\alpha\left[\frac{(\alpha+\beta)z}{2\alpha x} - qy\right] - \beta z = 0$$

$$\Rightarrow q = \frac{(\alpha-\beta)z}{2\alpha y} \quad \text{——— (4)}$$

And, $dz = p\,dx + q\,dy$

$$dz = \frac{(\alpha+\beta)z}{2\alpha x}\,dx + \frac{(\alpha-\beta)z}{2\alpha y}\,dy$$

$$\frac{\alpha+\beta}{2\alpha x}\,dx + \frac{(\alpha-\beta)\,dy}{2\alpha y} = \frac{dz}{z}$$

$$\frac{(\alpha+\beta)}{2\alpha}\ln x + \left(\frac{\alpha-\beta}{2\alpha}\right)\ln y = \ln z + C_2$$

$$(\alpha+\beta)\ln x + (\alpha-\beta)\ln y = 2\alpha\ln z + C_1$$

$$x^{\alpha+\beta}\cdot y^{\alpha-\beta} = C_1 z^{2\alpha}$$

$$\Rightarrow\quad z = \left[\frac{x^{\alpha+\beta}\,y^{\alpha-\beta}}{C_1}\right]^{1/2\alpha}$$

given that soln square $= c^2$ at $(x,y) = (1,1)$

$$\therefore\quad \left[\frac{1\cdot 1}{C_1}\right]^{1/\alpha} = c$$

$$\Rightarrow\quad (C_1)^{\frac{1}{\alpha}} = \frac{1}{c}$$

$$\Rightarrow\quad C_1 = \left(\frac{1}{c}\right)^{\alpha} \quad\text{—— } 5$$

$$z = \left(\frac{x^{\alpha+\beta}\,y^{\alpha-\beta}}{(1/c)^{\alpha}}\right)^{1/2\alpha}$$

$$z(x,y;c) = \left(c^{\alpha}\,x^{\alpha+\beta}\,y^{\alpha-\beta}\right)^{\frac{1}{2\alpha}}$$

$$z\left(\alpha\beta,\alpha\beta;\frac{1}{\alpha^2\beta^2}\right) = \left(\left(\frac{1}{\alpha^2\beta^2}\right)^{\alpha}(\alpha\beta)^{(\alpha+\beta)}(\alpha\beta)^{\alpha-\beta}\right)^{\frac{1}{2\alpha}}$$

$$= \left(\frac{1}{\alpha^2\beta^2}\right)^{\frac{1}{2}}\left(\alpha^{2\alpha}\beta^{2\alpha}\right)^{\frac{1}{2\alpha}}$$

$$= \frac{1}{\alpha\beta}\cdot\alpha\beta = 1 \quad\Rightarrow\quad \boxed{z\left(\alpha\beta,\alpha\beta;\frac{1}{\alpha^2\beta^2}\right) = 1}$$