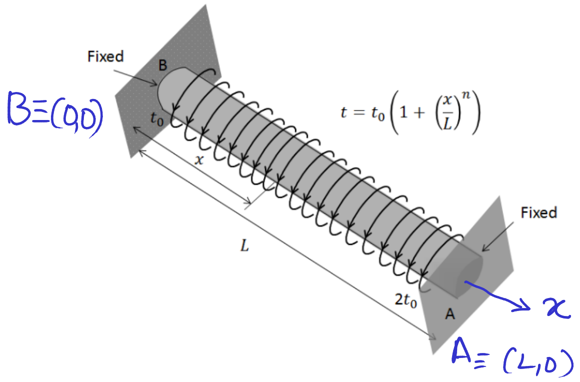
Name: Gurram Sai Sri Ram Hruthik
Roll. No: 20ME10045



1. The shaft shown in the figure is subjected to a distributed torque, $t = t_0\left(1 + \left(\frac{x}{L}\right)^n\right)$, where $n \geq 2$ is a positive integer. The two ends of the shaft are fixed to walls. Determine the reactions at the ends A and B. **[10 marks]**




Solution :

GIVEN THAT: A distributed torque is acted upon the shaft, and is given by, $t = t_0\left(1+\left(\frac{x}{L}\right)^n\right)$

POINT TO OBSERVE: The shaft is being fixed at both the ends to walls and hence, $\phi_{clockwise} = \phi_{anti\text{-}clockwise}$ — (1)

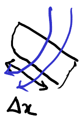
and $\phi_{clockwise} + \phi_{anticlockwise} = 0$ — (2)

PRE-REQUISITES THAT WE HAVE TO SOLVE THIS PROBLEM: If we have a shaft of length 'L' and is acted upon by a torque T and shear Modulus 'G' and the angle of twist '$\phi$'. The torque 'T' can be found in terms of all other variables as:

$$\boxed{T = \frac{GJ\phi}{L}} \quad \text{— (3)}$$

PROCEEDING FROM BASICS:

In the basic case, we don't have a distributed torque on the body.
But here we have a disributed torque which is varying with length.




If we consider a small portion of length $\Delta x$, we have a small angle of twist $\Delta\phi$ and the Torque is

$$\boxed{T = \frac{GJ\Delta\phi}{\Delta x}}$$

In the limit $\Delta x \to 0$, $\boxed{T = GJ\frac{d\phi}{dx}}$ — (4)

But this is not sufficient enough to solve the problem as the torque is distributed over the entire shaft AND ALSO it is varying with length. It's variation with length (Torque per unit length is given as 't')

$\therefore$ By differentiating again, we have

$$\frac{dT}{dx} = t = GJ\frac{d^2\phi}{dx^2}$$

$$\therefore \boxed{t = GJ\frac{d^2\phi}{dx^2}} \quad \text{— (5)}$$

So, now we can jump to solve the problem mathematically, as the problem is completely understood and sufficient pre-requisites are present.

Given, $t = t_0\left(1+\left(\frac{x}{L}\right)^n\right)$

From (5), for convinience, we represent '$\phi$' as function of $x \rightarrow \phi(x)$

$$GJ\frac{d^2\phi(x)}{dx^2} = t_0\left(1+\left(\frac{x}{L}\right)^n\right)$$

First integration;

$$GJ\frac{d\phi(x)}{dx} = \int t_0\left(1+\left(\frac{x}{L}\right)^n\right)dx$$

$$\boxed{GJ\frac{d\phi(x)}{dx} = t_0 x + \frac{t_0 x^{n+1}}{(n+1)L^n} + C_1 = T(x)} \quad [\text{from } (4)] \longrightarrow (6)$$

Second integration,

$$\boxed{GJ\phi(x) = \frac{t_0 x^2}{2} + \frac{t_0 x^{n+2}}{(n+1)(n+2)L^n} + C_1 x + C_2} \longrightarrow (7)$$

From (1) and (2) we can conclude that

$\phi_A$ and $\phi_B$ ie; $\phi(0)$ and $\phi(L)$ both are zeroes

($\because$ attached to fixed walls)

Upon substituting $x=0$ in (6)

$$0 = 0 + 0 + 0 + C_2$$

$$\Rightarrow \boxed{C_2 = 0}$$

Upon substituting $x=L$ in (7)

$$0 = \frac{t_0 L^2}{2} + \frac{t_0 L^{n+2}}{(n+1)(n+2)L^n} + C_1 L + 0$$

$$\boxed{C_1 = \frac{-t_0 L}{2} - \frac{t_0 L^{n+1}}{(n+2)(n+1)L^n}} \longrightarrow (8)$$

Using (8), solve for $T(0)$ in (7)

$$T(0) = T_B = 0 + 0 + C_1 = \frac{-t_0 L}{2} - \frac{t_0 L^{n+1}}{(n+2)(n+1)L^n}$$

$$= -\frac{t_0 L}{2} - \frac{t_0 L}{(n+2)(n+1)}$$

$$= -t_0 L\left(\frac{1}{2} + \frac{1}{(n+1)(n+2)}\right)$$

$$\boxed{T_B = -\frac{t_0 L}{2}\left(\frac{(n+1)(n+2)+2}{(n+1)(n+2)}\right)}$$

*** Here, negative sign indicates the reverse sense of torque and nothing else.
Here in the question the sense is ANTI-CLOCKWISE (which is often taken as positive)

*THE OBTAINED ANSWER IS IN CLOCKWISE direction.

Finding $T_A$ from (6) and (8)

$$T_A = T(L) = t_0 L + \frac{t_0 L^{n+1}}{(n+1)L^n} + C_1$$

$$= t_0 L + \frac{t_0 L}{(n+1)} + \left(-\frac{t_0 L}{2} - \frac{t_0 L^{n+1}}{(n+1)(n+2)L^n}\right)$$

$$= \frac{t_0 L}{2} + \frac{t_0 L}{(n+1)} - \frac{t_0 L}{(n+1)(n+2)}$$

$$= \frac{t_0 L}{2} + \frac{t_0 L}{(n+1)}\left[1 - \frac{1}{n+2}\right]$$

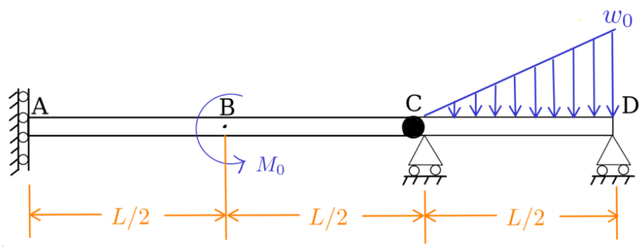
$$= \frac{t_0 L}{2} + \frac{t_0 L}{n+1}\left[\frac{n+1}{n+2}\right]$$

$$= \frac{t_0 L}{2} + \frac{t_0 L}{n+2} = t_0 L\left(\frac{n+2+2}{2(n+2)}\right)$$

$$\boxed{T_A = t_0 L\left(\frac{n+4}{2(n+2)}\right)}$$

$$\boxed{\text{Finally, } T_A = \frac{t_0 L(n+4)}{2(n+2)} \text{ and } T_B = \frac{t_0 L}{2}\left(\frac{(n+1)(n+2)+2}{(n+1)(n+2)}\right) \text{ magnitude wise}}$$

2. The compound beam made of the beams AC and CD are simply supported at C and D. The support at A allows free sliding along the vertical guide. Beams AC and CD are joined by a hinge just to the left of the simple support at C. Draw the shear force and bending moment diagrams of the entire compound beam. Take $w_0 = \frac{P}{L}$ and $M_0 = PL$. In your diagrams, clearly label all critical values of the shear force and bending moment and also the locations where they are zero. **[8 marks]**



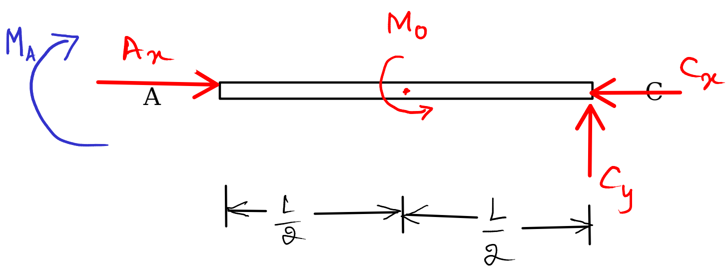

Solution : The very first thing is that the system is in equilibrium and hence the forces and moments must be balanced.

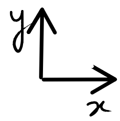
Pre- requisites and START: $\Sigma \vec{F} = 0$ and $\Sigma \vec{M} = 0$ for each beam AC and CD

Drawing Free Body Diagram for AC:

A is supported by a roller support and hence only the horizontal force exists

AC and CD are connected by a hinge at C and hence only Horizontal and vertical forces exist but no moment




Moment $\rightarrow$ (+)

$$\Sigma \vec{F_x} = 0 \Rightarrow A_x - C_x = 0 \Rightarrow A_x = C_x \longrightarrow (1)$$

$$\Sigma \vec{F_y} = 0 \Rightarrow C_y = 0 \longrightarrow (2)$$

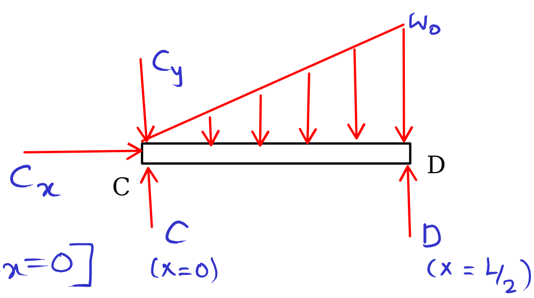
$\Sigma \vec{M_z} = 0 \Rightarrow$ **The very important point is due to the normal forces acting on the end A by the vertical slot, a moment is created,the phenomenon named 'SHEAR RELEASE'

This moment balances the $M_0$

$$\therefore \quad M_0 - M_A = 0 \Rightarrow M_A = M_0 \longrightarrow (3)$$

Drawing Free Body Diagram for CD

Considering x=0 at C and x=L/2 at D




$$\Sigma \vec{F_x} = 0 \Rightarrow C_x = 0 \longrightarrow (4) \quad [\because A_x = 0]$$

$$\Sigma \vec{F_y} = 0 \Rightarrow C + D - C_y - \int_0^{L/2} W(x)\, dx = 0 \longrightarrow (5)$$

From question, $W(x) = \dfrac{W_0 x}{(L/2)}$

We also know that $W(x) = \dfrac{dF(x)}{dx} = \dfrac{2W_0 x}{L}$

$$F(x) = \int_0^x W(x)\,dx$$

$$= \int_0^x \frac{2W_0 x}{L}\,dx$$

$$F(x) = \frac{2W_0}{L}\frac{x^2}{2}$$

$$\Rightarrow \boxed{F(x) = \frac{W_0 x^2}{L}} \longrightarrow 6$$

Using 5 and 6 we can conclude that,

$$C + D - C_y - \frac{W_0 L^2}{4L} = 0$$

$$\Rightarrow C + D - 0 - \frac{W_0 L}{4} = 0 \qquad [\because C_y = 0 \text{ from } 2]$$

$$\Rightarrow \boxed{C + D = \frac{W_0 L}{4}} \longrightarrow 7$$

$\Sigma \vec{M} = 0$ about an axis passing through 'C' (i.e, y-axis) (+↺)

$$-\frac{DL}{2} + \int_0^{L/2} \frac{2W_0 x}{L}.x\,dx = 0 \qquad W(x).x.dx$$

$$\Rightarrow -\frac{DL}{2} + \int_0^{L/2} \frac{2W_0 x^2}{L}\,dx = 0$$

$$\Rightarrow -\frac{DL}{2} + \left(\frac{2W_0 x^3}{3L}\right)_0^{L/2} = 0 \qquad \left[\therefore \text{Torque function } T(x) = \frac{2W_0 x^3}{3L}\right] \longrightarrow 10$$

$$\Rightarrow -\frac{DL}{2} + \frac{2W_0}{3L} \times \frac{L^3}{8} = 0$$

$$\Rightarrow -12DL + 2W_0 L^2 = 0$$

$$\Rightarrow \boxed{D = \frac{W_0 L}{6}} \longrightarrow 8$$

From 7 and 8, we can conclude that

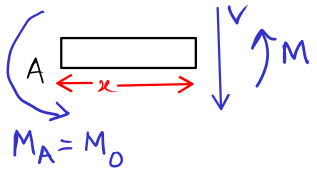
$$C = \frac{W_0 L}{4} - \frac{W_0 L}{6}$$

$$\boxed{C = \frac{W_0 L}{12}} \longrightarrow 9$$

Now we are good to go and solve for Shear force diagram and Bending moment diagrams

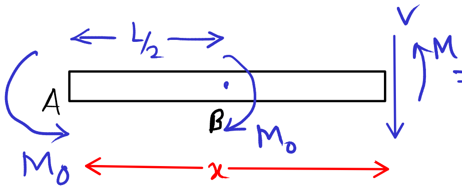
Shear Force diagram and Bending Moment Diagrams:

$0 < x < \frac{L}{2}$ :



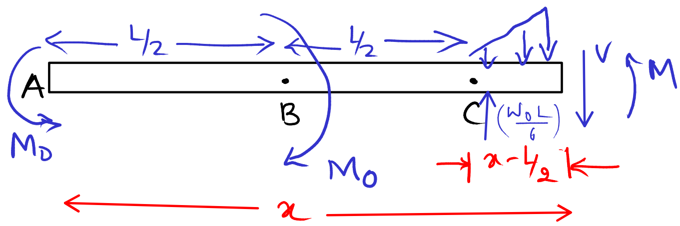

$$\Rightarrow \boxed{V = 0,\ M = M_0}$$

$\frac{L}{2} < x < L$ :




$$\Rightarrow \boxed{V = 0,\ M = 0}$$

$\frac{3L}{2} < x < L$ :




$$V = -F(x-L) + \frac{W_0 L}{12}$$

[get F(x) from 6]

$$V = -\frac{W_0 (x-L)^2}{L} + \frac{W_0 L}{12}$$

$$V = -\frac{W_0}{L}\left(x^2 + L^2 - 2xL - L^2/12\right)$$

$$\boxed{V = -\frac{W_0}{L}\left(x^2 - 2xL + \frac{11L^2}{12}\right)}$$

$$M = T(x-L) - V\frac{L}{2}$$

[get T(x) from 10]

$$M = \frac{2W_0(x-L)^3}{3L} - \frac{W_0(x-L)^3}{L} + \frac{W_0 L(x-L)}{12}$$

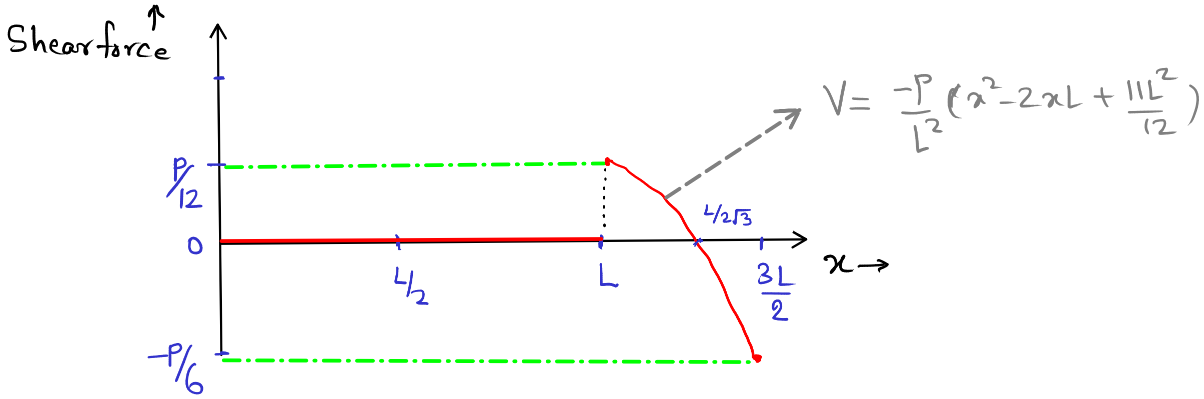
$$M = -\frac{1}{3}\frac{W_0(x-L)^3}{L} + \frac{W_0 L}{12}(x-L)$$

Given that $W_0 = P/L$ and $M_0 = PL$

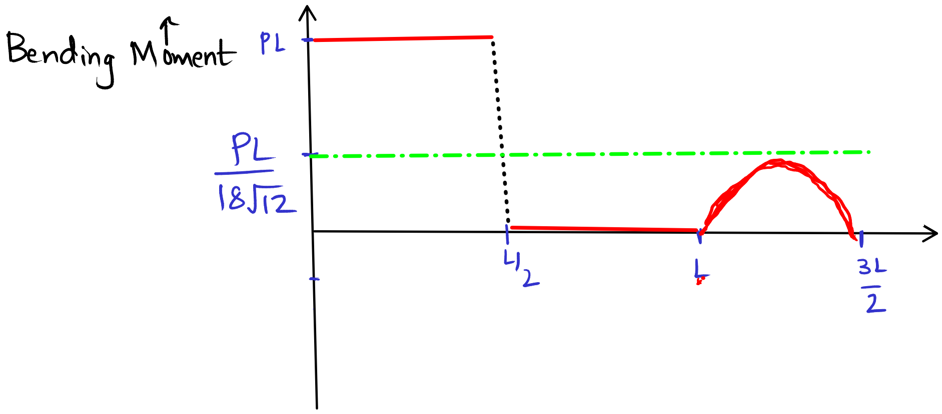
$$\therefore V = -\frac{P}{L^2}\left(x^2 - 2xL + \frac{11L^2}{12}\right)$$

$$M = \frac{P}{L}\left(-\frac{(x-L)^3}{3L} + \frac{L}{12}(x-L)\right)$$

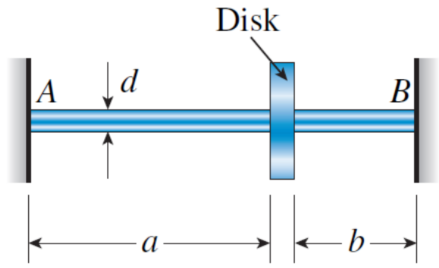
SHEAR FORCE DIAGRAM:




BENDING MOMENT DIAGRAM:



3. A solid circular shaft AB of diameter $d$ is fixed against rotation at both ends (see figure). A rigid circular disk is attached to the shaft at the location shown. What is the largest per missible angle of rotation $\phi_{\max}$ of the disk if the allowable shear stress in the shaft is $\tau_{\text{allow}}$? (Assume $a > b$.) **[8 marks]**



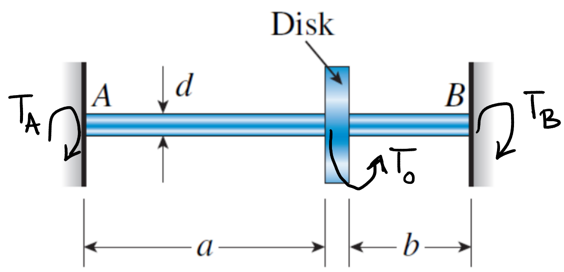

Solution:
Given that the shaft is fixed at both the ends of the walls and in between a disk is attached to the shaft
Also it is given that a>b

Pre-requistes: If a shaft is attached at both ends and a torque is applied in between,
angle of twist(right) = angle of twist(left)

$$\Rightarrow \phi_A = \phi_B$$

$$\frac{T_A a}{GJ} = \frac{T_B b}{GJ}$$

and also $T_A + T_B = T_o$




$$\frac{T_A}{T_B} = \frac{b}{a} \Rightarrow T_B = \frac{T_A a}{b}$$

$$\therefore T_A\left(1+\frac{a}{b}\right) = T$$

$$\Rightarrow T_A = \frac{T_o b}{a+b}$$

and $T_B = \dfrac{T_o a}{a+b}$

$\boxed{1A, 1B}$

This $T_B$ value is the maximum value of the torque.

Now, upon calculating the maximum shear stress which is at point B (since a>b),

$$\tau_{max} = \frac{T_B d/2}{J}$$

[Using the basic formula, $\tau = \frac{T\rho}{J}$]

$$\Rightarrow \tau_{max} = \frac{\frac{T_o a}{a+b} \cdot \frac{d}{2}}{J}$$

$$\tau_{max} = \frac{T_o a d}{2J(a+b)}$$

Since, shear stress is maximum (allowable), $T_o$ corresponding to this is maximum.

i.e., $$(T_o)_{max} = \frac{\tau_{max} \cdot 2J(a+b)}{ad} \quad (\tau_{max} \equiv \tau_{allow})$$

Here, again by using basic formula; $\phi = \frac{TL}{GJ}$

$$\phi_{max} = \frac{(T_B)(b)}{GJ}$$

$$= \frac{\frac{(T_o)_{max} \cdot a}{a+b} \cdot b}{GJ} \quad [\text{From } 1B]$$

$$\phi_{max} = \frac{(T_o)_{max} \cdot ab}{GJ(a+b)}$$

$$= \frac{\frac{\tau_{allow} \cdot 2\cancel{J}\cancel{(a+b)}}{\cancel{a}d} \cdot \cancel{a}b}{G\cancel{J}\cancel{(a+b)}}$$

$$\boxed{\phi_{max} = \frac{2b\,\tau_{allow}}{Gd}}$$