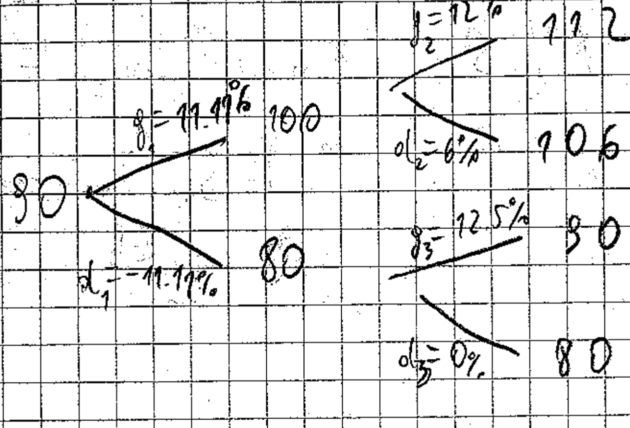
ZADACI 08 - FER

① $A(0)=100$, $A(1)=110$, $A(2)=121 \Rightarrow r=10\%$



a) Arbitraža nije moguća jer je u svakom čvoru $d < r < g$ $\left(d_i < r < g_i,\ i=1,2,3 \right)$

b) U slučaju plaćanja provizije investitor je u još gorem položaju, bilo da kupuje ili prodaje dionicu pa niti u ovom slučaju nema mogućnost arbitraže.

(2) $E[X_i] = \mu, \quad \operatorname{Var} X = \sigma^2, \quad X_1, X_2, \ldots,$ nejd.

$$S_n = \sum_{i=1}^{n} X_i, \quad n \geq 1$$

a)

$$E[S_{n+1} \mid \mathcal{F}_n] = E[S_n + X_{n+1} \mid \mathcal{F}_n] =$$

$$= E(S_n \mid \mathcal{F}_n) + E[X_{n+1} \mid \mathcal{F}_n] = S_n \quad \mathcal{F}_n\text{-měř}, \; X_{n+1} \text{ nez s } \mathcal{F}_n$$

$$= S_n + E(X_{n+1}) = S_n + \overset{=0}{\mu} = S_n$$

b)

$$M_n = S_n - n\mu, \quad n \geq 1$$

$$E[M_{n+1} \mid \mathcal{F}_n] = E[S_{n+1} - (n+1)\mu \mid \mathcal{F}_n] =$$

$$= E[S_n + X_{n+1} - (n+1)\mu \mid \mathcal{F}_n] =$$

$$= E[S_n \mid \mathcal{F}_n] + E[X_{n+1} \mid \mathcal{F}_n] - (n+1)\mu =$$

$$= S_n + E[X_{n+1}] - (n+1)\mu =$$

$$= S_n + \mu - (n+1)\mu = S_n - n \cdot \mu = M_n$$

c)

$$V_n = (S_n - n\mu)^2 - n\sigma^2, \quad n \geq 1$$

$$E[V_{n+1} \mid \mathcal{F}_n] = E[(S_{n+1} - (n+1)\mu)^2 - (n+1)\sigma^2 \mid \mathcal{F}_n]$$

$$= E[((S_n - n\mu) + (X_{n+1} - \mu))^2 - (n+1)\sigma^2 \mid \mathcal{F}_n] =$$

$$= E[(S_n - n\mu)^2 + 2(S_n - n\mu)(X_{n+1} - \mu) + (X_{n+1} - \mu)^2 - (n+1)\sigma^2 \mid \ldots] =$$

$$= E[\overbrace{(S_n - n\mu)^2}^{\mathcal{F}_n\text{-měř}} \mid \mathcal{F}_n] + 2 \cdot E[\overbrace{(S_n - n\mu)}^{\mathcal{F}_n\text{-měř}}(X_{n+1} - \mu) \mid \mathcal{F}_n] +$$

$$+ E[\underbrace{(X_{n+1} - \mu)^2}_{\text{nez s } \mathcal{F}_n} \mid \mathcal{F}_n] - E[\underbrace{(n+1)\sigma^2}_{\text{konst, ne sl. vel.}} \mid \mathcal{F}_n] =$$

$$= (S_n - n\mu)^2 + 2\cdot(S_n - n\mu)\cdot E[X_{n+1}-\mu \mid \mathcal{F}_n] +$$
$$+ E[(X_{n+1}-\mu)^2] - (n+1)\sigma^2$$

$$= (S_n - n\mu)^2 + 2\cdot(S_n - n\mu)\cdot \underbrace{E[X_{n+1}-\mu]}_{=E[X_{n+1}]-\mu=\mu-\mu=0} +$$
$$+ \sigma^2 - (n+1)\sigma^2 =$$

$$= (S_n - n\mu)^2 - n\sigma^2 = V_n$$

d) $X_i \sim B(p) \Rightarrow X_i \sim \begin{pmatrix} 1 & 0 \\ p & 1-p \end{pmatrix}$

$$Z_n = \left(\frac{1-p}{p}\right)^{2S_n - n}, \quad n \geq 0$$

$$E[Z_{n+1} \mid \mathcal{F}_n] = E\left[\left(\frac{1-p}{p}\right)^{2S_{n+1}-n-1} \Big| \mathcal{F}_n\right] =$$
$$= E\left[\left(\frac{1-p}{p}\right)^{2\cdot S_n + 2X_{n+1} - n - 1} \Big| \mathcal{F}_n\right] =$$
$$= E\left[\underbrace{\left(\frac{1-p}{p}\right)^{2\cdot S_n - n - 1}}_{\mathcal{F}_n\text{-izmj.}} \cdot \left(\frac{1-p}{p}\right)^{2X_{n+1}} \Big| \mathcal{F}_n\right] =$$
$$= \left(\frac{1-p}{p}\right)^{2S_n - n - 1} \cdot E\left[\underbrace{\left(\frac{1-p}{p}\right)^{2X_{n+1}}}_{\text{nez. o. } \mathcal{F}_n} \Big| \mathcal{F}_n\right] =$$
$$= \left(\frac{1-p}{p}\right)^{2S_n - n - 1} \cdot E\left[\left(\frac{1-p}{p}\right)^{2X_{n+1}}\right] =$$
$$= \left(\frac{1-p}{p}\right)^{2S_n - n - 1} \cdot \left[\left(\frac{1-p}{p}\right)^{2\cdot 1} p + \left(\frac{1-p}{p}\right)^{2\cdot 0}(1-p)\right] =$$
$$= \left(\frac{1-p}{p}\right)^{2S_n - n - 1} \cdot \left[\frac{(1-p)^2}{p} + 1-p\right] = \left(\frac{1-p}{p}\right)^{2S_n - n - \cancel{1}} \cdot \cancel{\frac{1-p}{p}} = Z_n$$

e) $$K_n = \frac{e^{yS_n}}{\varphi(y)^n}, \quad n \geq 1$$

$$\Rightarrow K_n = \frac{e^{yS_n}}{\left(E\left[e^{yX_1}\right]\right)^n}$$

$$E\left[K_{n+1} \mid \mathcal{F}_n\right] = E\left[\frac{e^{yS_{n+1}}}{E\left[e^{yX_1}\right]^{n+1}} \mid \mathcal{F}_n\right] =$$

$$= \frac{1}{E\left[e^{yX_1}\right]^{n+1}} \cdot E\left[\underbrace{e^{yS_n}}_{\mathcal{F}_n\text{-izmj.}} \cdot \underbrace{e^{yX_{n+1}}}_{\text{nez. od } \mathcal{F}_n} \mid \mathcal{F}_n\right] =$$

$$= \frac{1}{E\left[e^{yX_1}\right]^{n+1}} \cdot e^{yS_n} \cdot E\left[e^{yX_{n+1}} \mid \mathcal{F}_n\right] =$$

$$= \frac{e^{yS_n}}{E\left[e^{yX_1}\right]^{\cancel{n+1}}} \cdot \cancel{\underbrace{E\left[e^{yX_{n+1}}\right]}_{= E\left[e^{yX_1}\right]}} = \frac{e^{yS_n}}{E\left[e^{yX_1}\right]^n} = K_n$$

Q.E.D.

3. $$X_n = E\left[X \mid \mathcal{F}_n\right], \quad \mathcal{F}_n \subseteq \mathcal{F}_{n+1}$$

$$E\left[X_{n+1} \mid \mathcal{F}_n\right] = E\left[E\left[X \mid \mathcal{F}_{n+1}\right] \mid \mathcal{F}_n\right] =$$

$$= E\left[X \mid \mathcal{F}_n\right] = X_n$$

Q.E.D.

(4.) a) isplata u n-tom bacanju je $Y_i = W_i \cdot X_i$,

$$X_i \sim \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \ (W_i)_i \text{ predvidiva}$$

b) $S_n = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} W_i X_i$

c) $E[S_{n+1} | \mathcal{F}_n] = E[S_n + W_{n+1} \cdot X_{n+1} | \mathcal{F}_n] =$

$$= E[\underbrace{S_n}_{\mathcal{F}_n\text{-izmj.}} | \mathcal{F}_n] + E[\underbrace{W_{n+1}}_{\mathcal{F}_n\text{-izmj. jer je } (W_i)_i \text{ predvidiva}} \cdot X_{n+1} | \mathcal{F}_n]$$

$$= S_n + W_{n+1} \cdot E[\underbrace{X_{n+1}}_{\text{nez. od } \mathcal{F}_n} | \mathcal{F}_n] =$$

$$= S_n + W_{n+1} \cdot \underbrace{E[X_{n+1}]}_{=\frac{1}{2}-\frac{1}{2}=0} = S_n + W_{n+1} \cdot 0 =$$

$$= S_n$$

Q.E.D

(5.) $(S^D_n)_n$ je martingal ako vrijedi

$$E[S^D_{n+1} | \mathcal{F}_n] = S^D_n$$

$$E[S^D_{n+1} | \mathcal{F}_n] = E\left[\frac{S_{n+1}}{(1+r)^{n+1}} \middle| \mathcal{F}_n\right] = \frac{1}{(1+r)^{n+1}} \cdot E[S_{n+1} | \mathcal{F}_n]$$

$$= \frac{1}{(1+r)^{n+1}} \cdot \left(S_n(1+g) \cdot q_g + S_n(1+d) \cdot q_d\right) = \frac{S_n}{(1+r)^n} \Longleftrightarrow$$

$$\Longleftrightarrow (1+g) \cdot q_g + (1+d) \cdot q_d = 1+r$$

$$\Longleftrightarrow (1+g) q_g + (1+d) \cdot (1-q_g) = 1+r \Longleftrightarrow \boxed{q_g = \frac{r-d}{g-d}}$$

$$\Rightarrow \alpha_d = 1 - \alpha_g = 1 - \frac{r-d}{g-d} = \boxed{\frac{g-r}{g-d}}$$

← vjer. pada cijene tj. recimo je, vjer. manjeg rasta.

Kako ne postoji mogućnost arbitraže, iz Prop 1. slijedi $d < r < g$

$$\Rightarrow \alpha_g = \frac{r-d}{g-d} \in \langle 0, 1 \rangle,$$

pa se zaista radi o vjerojatnosti.

Q.E.D.