SUMMARY OF THE SURVEY TALK AT 2ND 3J SEMINAR: TOPICS ON HYPERPLANE ARRANGEMENTS

YE LIU

A *hyperplane arrangement* $\mathcal{A}$ over a field K is simply a finite collection of (affine) hyperplanes in K^{ell}. The *intersection poset* $\mathcal{L}(\mathcal{A})$ is the set of nonempty intersections of subarrangements of $\mathcal{A}$, partially ordered by reverse inclusion. One theme of the study of hyperplane arrangements is to explore what can be said about $\mathcal{A}$ by looking at only $\mathcal{L}(\mathcal{A})$. For example, if $\mathcal{A}$ is an arrangement in a complex vector space, then the cohomology ring of the complement $\mathcal{M}(\mathcal{A})$ to the union of hyperplanes in $\mathcal{A}$ is completely determined by $\mathcal{L}(\mathcal{A})$, known as the *Orlik-Solomon algebra*. On the other hand, the fundamental group of $\mathcal{M}(\mathcal{A})$ is not determined solely by $\mathcal{L}(\mathcal{A})$, known from *Rybnikovs counterexample*. Ye Liu's survey addresses the following long-standing open problems.

Question 1. If $\mathcal{A}$ is an arrangement in a real vector space V, then $\mathcal{A}$ yields a *stratification* of V. The strata are also called *faces* of $\mathcal{A}$. Among the faces, the ones of codimension 0 are called *chambers*. The *face post* $\mathcal{F}(\mathcal{A})$ is the set of all faces partially ordered by reverse inclusion (of closures). By M. Salvettis work, $\mathcal{F}(\mathcal{A})$ determines the homotopy type of $\mathcal{M}(\mathcal{A}_{\mathbb{C}})$, where $\mathcal{A}_{\mathbb{C}}$ is the complexification of $\mathcal{A}$. So for a real hyperplane arrangement $\mathcal{A}$, $\mathcal{F}(\mathcal{A})$ contains more information than $\mathcal{L}(\mathcal{A})$. Rybnikovs counterexample is a pair of complex hyperplane arrangements which are not the complexification of any real arrangements. Thus the following question is still open. Can one find a pair of real arrangements, with isomorphic intersection posets, but non-isomorphic face posets?

Question 2. For a central complex arrangement $\mathcal{A}$, we fix for each hyperplane H in $\mathcal{A}$ a linear form whose kernel is H and let Q be the product of these forms. Then Q defines the *Milnor fibration* $\mathcal{M}(\mathcal{A}) \to \mathbb{C}^*$. The *Milnor fiber* $F = Q^{-1}(1)$ is a rather less known space compared to $\mathcal{M}(\mathcal{A})$. What topological information of F is determined by $\mathcal{L}(\mathcal{A})$? This question is widely open in dimension greater than 2. Even for the first Betti number of F, it is unknown if it is determined by $\mathcal{L}(\mathcal{A})$.

Question 3. Using *Tits representation*, every *Coxeter group* is faithfully realized as a reflection group acting on a real vector space. The collection of reflection hyperplanes forms the reflection arrangement, which can be infinite. Take a chamber as the fundamental chamber and collect the Coxeter group action orbit of the closure of the fundamental chamber. The result is a convex cone in the original real vector space. Consider the complexification and remove the reflection hyperplanes, we obtain an interesting space M. The celebrated $K(\pi, 1)$ *conjecture*, due to Arnold, Pham and Thom, asserts that M is an *Eilenberg-MacLane space*, for every Coxeter group. Among others, Deligne proved this conjecture for finite Coxeter groups. Charney-Davis proved for FC type, and 2-dimensional cases. Recently, Paolini-Salvetti proved for affine Coxeter groups. The space M admits a free and properly discontinuous action of the Coxeter group W, and the orbit space has

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fundamental group exactly the *Artin group* associated to W, due to Van der Lek. Proving the $K(\pi, 1)$ conjecture is then of great importance in the study of Artin groups.

Reference:

1. P. Orlik, H. Terao, Arrangements of hyperplanes.

2. A. Dimca, Hyperplane arrangements, an introduction.

3. L. Paris, $K(\pi, 1)$ conjecture for Artin groups.