LAAIML (Test - 1)

Q.1) Consider standard unit vectors e_1, e_2, e_3 in $\mathbb{R}^3$. What is the trace of matrix of linear transformation which acts on e_1, e_2, e_3 as follows:

$$e_1 \mapsto \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; \quad e_2 \mapsto \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}; \quad e_3 \mapsto \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Answer: The matrix of transformation $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$

$$\text{trau}(A) = 1 + 1 + 0 = 2$$

Q.2) Let $a = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct rotators Q_1 and Q_2 such that $Q_2^T Q_1^T a = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$.

Compute $Q = Q_2^T Q_1^T$.

Further, compute L_{21} and L_{31}, lower triangular matrices such that

$$L_{31} L_{21} a = \begin{bmatrix} * \\ 0 \\ 0 \end{bmatrix}$$

Compute $L = L_{31} L_{21}$.

(Clearly show all the steps and final answers).

Q.2) Ans:

$$Q_1^T \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{5} \\ 0 \\ 2 \end{bmatrix}$$

$$Q_1^T = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \cos\theta_1 = \frac{1}{\sqrt{5}}, \quad \sin\theta_1 = \frac{2}{\sqrt{5}}$$

$$Q_2^T \begin{bmatrix} \sqrt{5} \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad Q_2^T = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & & \cos\theta_2 \end{bmatrix}$$

$$\cos\theta_2 = \frac{\sqrt{5}}{3}; \quad \sin\theta_2 = \frac{2}{3}$$

$$Q = Q_2^T Q_1^T = \begin{bmatrix} \sqrt{5}/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ -2/3 & 0 & \sqrt{5}/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -2/3\sqrt{5} & -4/3\sqrt{5} & \sqrt{5}/3 \end{bmatrix}$$

$$L_{21}\, a = \begin{bmatrix} * \\ 0 \\ 2 \end{bmatrix} \qquad L_{21} = I - l_k e_k^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_{21}\, a = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Now $L_{31}(L_{21}\, a) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$L_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow L_{32} L_{21}\, a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$L = L_{32} L_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Q.3) Let $A \in \mathbb{R}^{3\times 2}$ be a matrix with full column rank. Let $\eta \in \mathbb{R}^{3\times 1}$ be a nonzero vector such that $\eta^T A = 0$. Then prove that for any $b \in \mathbb{R}^{3\times 1}$ such that $\eta^T b = 0$, the system of equations $Ax = b$ always has a unique solution.

Note that $\eta^T A = 0 \Rightarrow \eta$ is orthogonal to both the columns of A.

$\Rightarrow \eta$ is orthogonal to the column span of A.

Since columns of A are linearly independent, they span a 2-dimensional subspace in $\mathbb{R}^3$ (Geometrically a plane)

For a given vector $b \in \mathbb{R}^3$, if $\eta^T b = 0$

$\Rightarrow b$ is orthogonal to η

$\Rightarrow b \in \text{colspace}(A) \Rightarrow Ax = b$ solvable.

Q.4) Let $\mathcal{L} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \ x_1 + x_2 + x_3 = 0 \right\} \subseteq \mathbb{R}^3$.

i) Prove that $\mathcal{L}$ is a subspace of $\mathbb{R}^3$.

ii) Construct a reflector Q which reflects every vector of $\mathbb{R}^3$ through $\mathcal{L}$.

i) $\mathcal{L} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1 + x_2 + x_3 = 0 \right\}$

for any two vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathcal{L}$ and

$\alpha, \beta \in \mathbb{R}$,

we want to show $\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathcal{L}$

Note: $\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathcal{L}$ as $\alpha(x_1 + x_2 + x_3) = 0$

& $\beta \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathcal{L}$ as $\beta(y_1 + y_2 + y_3) = 0$

$\Rightarrow \alpha(x_1 + x_2 + x_3) + \beta(y_1 + y_2 + y_3) = 0$

$\Rightarrow \alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathcal{L}$

iii) The reflector $Q = I - 2uu^T$

where $u \in \mathcal{L}^2$ and $\|u\|_2 = 1$

$$u = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$Q = I - 2uu^T = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix}$$

Q.5) For $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$, check whether the following transformations are linear ?? Justify your answer in each case.

(i)
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} x_1 \\ \frac{x_1+x_2}{2} \\ x_2 \\ \frac{x_2+x_3}{2} \\ \vdots \\ \frac{x_{n-1}+x_n}{2} \\ x_n \end{bmatrix}$$

(ii)
$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \sum_{i=1}^{n} |x_i|$$

(iii)
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} \max\{x_1, 0\} \\ \max\{x_2, 0\} \\ \vdots \\ \max\{x_n, 0\} \end{bmatrix}$$

i) $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \overset{T}{\longmapsto} \begin{bmatrix} x_1 \\ \frac{x_1+x_2}{2} \\ x_2 \\ \vdots \\ \frac{x_{n-1}+x_n}{2} \\ x_n \end{bmatrix}$ T is linear.

$$T\left(\alpha \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) = T\begin{pmatrix} \alpha x_1 + y_1 \\ \vdots \\ \alpha x_n + y_n \end{pmatrix} = \begin{bmatrix} \alpha x_1 + y_1 \\ \frac{\alpha x_1 + y_1 + \alpha x_2 + y_2}{2} \\ \alpha x_2 + y_2 \\ \vdots \\ \frac{\alpha x_{n-1} + y_{n-1} + \alpha x_n + y_n}{2} \end{bmatrix}$$

$$= \alpha \begin{bmatrix} x_1 \\ \frac{x_1+x_2}{2} \\ x_2 \\ \vdots \\ \frac{x_{n-1}+x_n}{2} \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \frac{y_1+y_2}{2} \\ y_2 \\ \vdots \\ \frac{y_{n-1}+y_n}{2} \\ y_n \end{bmatrix} = \alpha T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + T\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

ii) $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \xmapsto{T} \sum_{i=1}^{n} |x_i|$. **NOT** linear

$$T\left(\alpha \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \sum_{i=1}^{n} |\alpha x_i| = |\alpha| \sum_{i=1}^{n} |x_i|$$

$$= |\alpha| \, T\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\neq \alpha \, T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

iii) $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \xmapsto{T} \begin{bmatrix} \max\{x_1, 0\} \\ \vdots \\ \max\{x_n, 0\} \end{bmatrix}$ NOT linear

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) \neq T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + T\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Ex: $T(e_1 - e_1) = T(0) = 0$

$$T(e_1) + T(-e_1) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = e_1$$

$$T(e_1 - e_1) \neq T(e_1) + T(-e_1)$$