Projective Transformation

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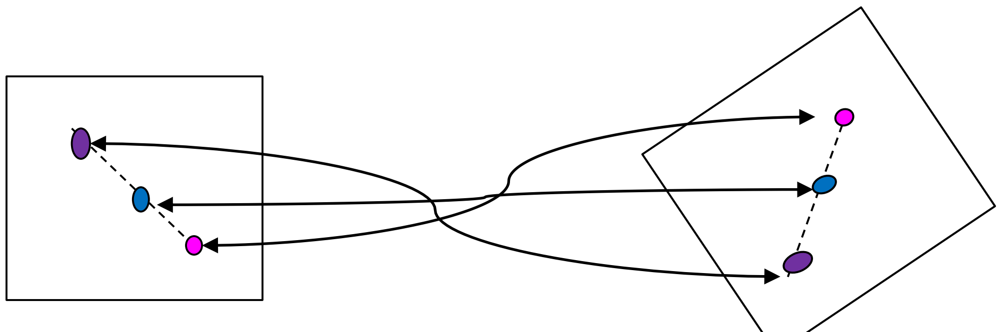
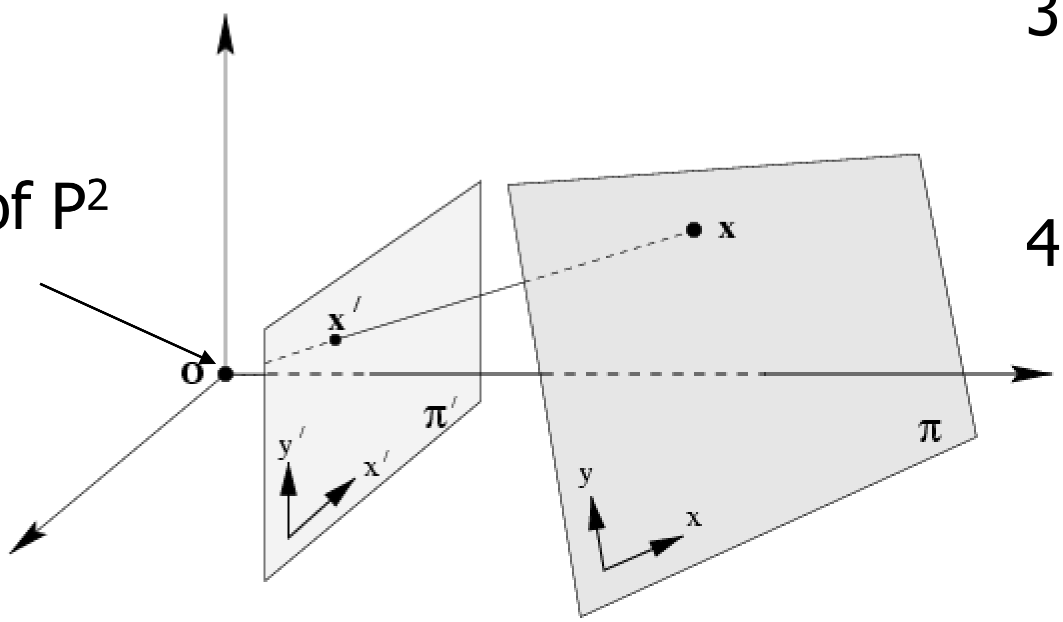
Projective transformation

- h: $P^2 \rightarrow P^2$
- Invertible
- Collinearity of every three points to be preserved, i.e. three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.



An example: change of coordinate convention

Origin of P^2 fixed.

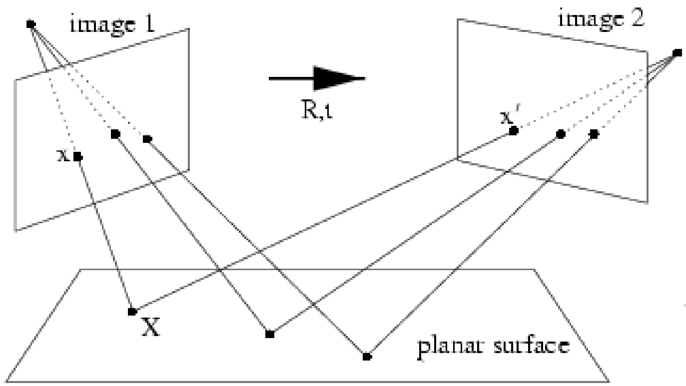


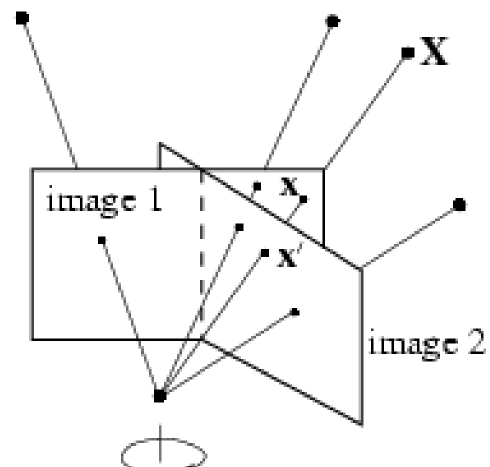
1. Rotation of axes.
2. Change of scale.
3. Translation of origin in planar coordinate system.
4. Use of Affine coordinate system in plane.

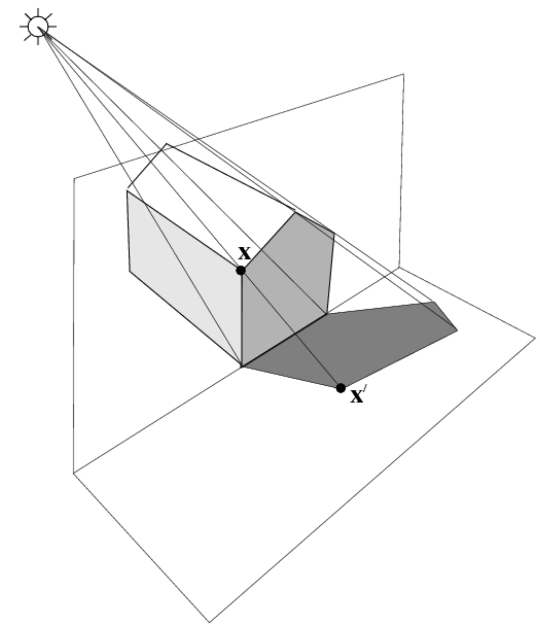
From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

More examples

Shift of origin of P^2







From Hartley and Zisserman, “Multiple view geometry in computer vision”, Cambridge Univ. Press (2000)

Form of h

- Only one form possible.
- It is linear and invertible.

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

8DOF

$$\mathbf{X}' = \mathbf{H}\mathbf{X} \equiv k\mathbf{H}\mathbf{X}$$

Also called homography and **H** is the homography matrix.

Hx preserves collinearity.

- Let $\boldsymbol{l}$ be a line in P^2.
- A point $\mathbf{x}$ on $\boldsymbol{l}$ satisfies

$$\boldsymbol{l}^T\mathbf{x}=\mathbf{0}$$

$$\rightarrow \boldsymbol{l}^T\mathbf{H}^{-1}\mathbf{H}\mathbf{x}=\mathbf{0}$$

$$\rightarrow (\mathbf{H}^{-T}\boldsymbol{l})^T\mathbf{H}\mathbf{X}=\mathbf{0}$$

- $\mathbf{H}^{-T}\boldsymbol{l}$ is the transformed line of $\boldsymbol{l}$.

Harder to show that **H** is the only form of homography.

Implications

- If there is a homography, there exists a unique **H**, which is a 3x3 invertible matrix.
- Functional form known, so easier to estimate.
- **H** and k**H** are equivalent, where k is a scalar constant.
- Number of unknowns in **H** = 8.

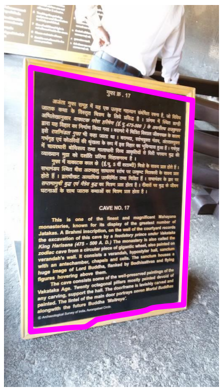
Estimation of **H**

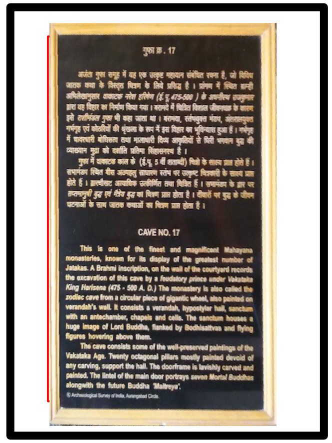
- Given point correspondences ($\mathbf{x_i}$, $\mathbf{x_i'}$) estimate **H** such that $\mathbf{x_i'}=\mathbf{H}\mathbf{x_i}$.
- There are 8 unknowns.
- $\mathbf{x'}=\mathbf{H}\mathbf{x}$ → Two independent equations.

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

- Minimum 4 point correspondences needed.

Removing projective distortion





1. Select four points in a plane with known coordinates.
2. Form equations.

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

(linear in h_{ij})

$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$

$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

3. Setting h_{33} at *1* solve them.

Remark: no calibration at all necessary.
Does not work if $h_{33}=0$ in **H**.

Form equations

$$x'(h_{31}x + h_{32}y + h_{33}) = h_{11}x + h_{12}y + h_{13}$$

$$y'(h_{31}x + h_{32}y + h_{33}) = h_{21}x + h_{22}y + h_{23}$$

(51,791) → (1, 900)
(63, 143) → (1,1)
(444 ,211) → (501,1)
(426, 719) → (501,900)

$-51\ h_{11}\ -791\ h_{12} - h_{13} + 51h_{31} \qquad + 791h_{32} \qquad = -1$

$-51\ h_{21}\ -791\ h_{22} - h_{23} + 45900\ h_{31} + 711900\ h_{32} = -900$

$-63\ h_{11}\ -143\ h_{12} - h_{13} + 63h_{31} \qquad + 43\ h_{32} \qquad = -1$

$-63\ h_{21}\ -143\ h_{22} - h_{23} + 63\ h_{31} \qquad +143\ h_{32} \qquad = -1$

$-444h_{11}\ -211h_{12} - h_{13} + 222444\ h_{31} + 105711h_{32} = -501$

$-444h_{21}\ -211\ h_{22} - h_{23} - 444\ h_{31} \qquad + 211\ h_{32} \qquad = -1$

$-426h_{11} - 719\ h_{12} - h_{13} + 213426\ h_{31} + 360219\ h_{32} = -501$

$-426h_{21}\ -719\ h_{22} - h_{23} + 383400\ h_{31} + 647100\ h_{32} = -900$

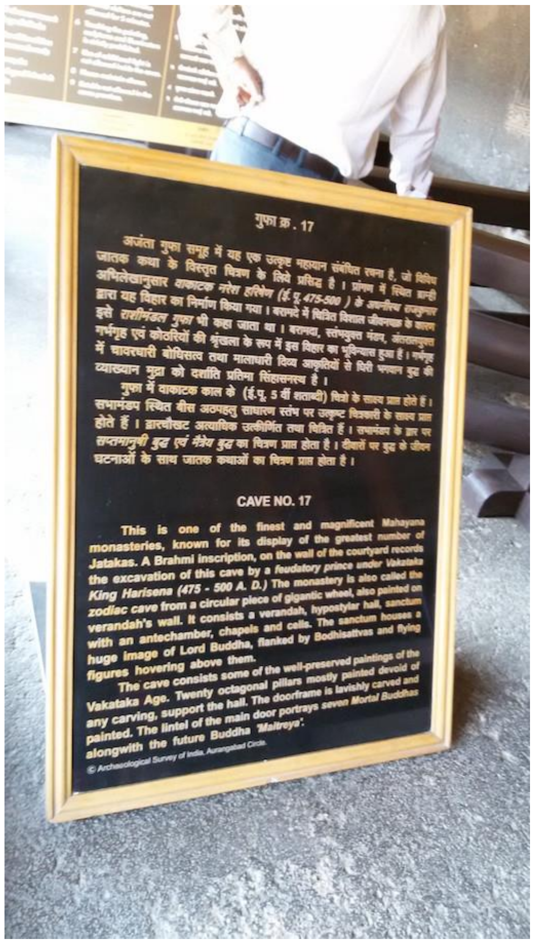
In matrix form

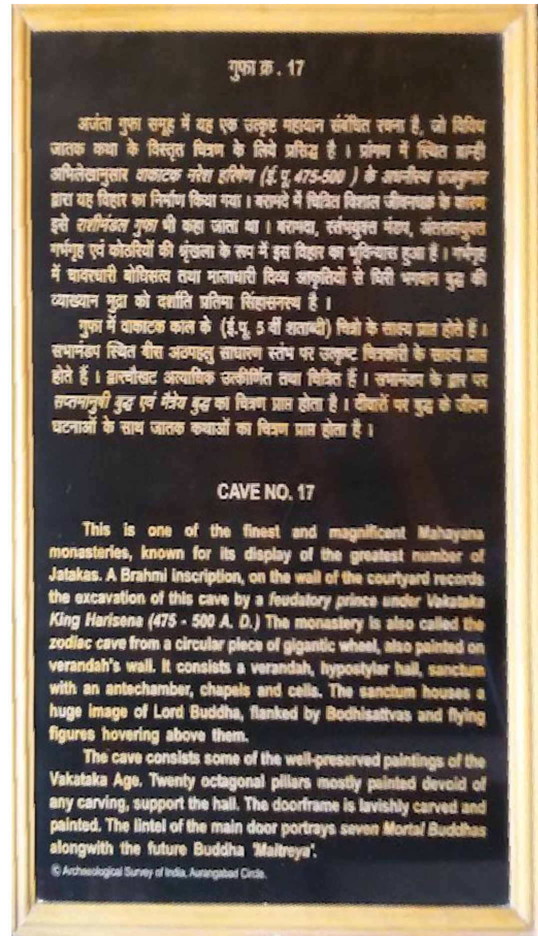
$$\begin{bmatrix} -51 & -791 & -1 & 0 & 0 & 0 & 51 & 791 \\ 0 & 0 & 0 & -51 & -791 & 1 & 45900 & 711900 \\ -63 & -143 & -1 & 0 & 0 & 0 & 63 & 143 \\ 0 & 0 & 0 & -63 & -143 & -1 & 63 & 143 \\ -444 & -211 & -1 & 0 & 0 & 0 & 222444 & 105711 \\ 0 & 0 & 0 & -444 & -211 & -1 & 444 & 211 \\ -426 & -719 & -1 & 0 & 0 & 0 & 213426 & 360219 \\ 0 & 0 & 0 & -426 & -719 & -1 & 383400 & 647100 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} -1 \\ -900 \\ -1 \\ -1 \\ -501 \\ -1 \\ -501 \\ -900 \end{bmatrix}$$

Solve by matrix inversion. $h_{33}=1$

$$H = \begin{bmatrix} 0.9791 & 0.0181 & -63.3104 \\ -0.2303 & 1.2874 & -168.6295 \\ -0.0005 & -0.0001 & 1.0000 \end{bmatrix}$$

Apply homography





Direct Linear Transformation (DLT)

$$\mathbf{x}'_i = \left(x'_i, y'_i, w'_i\right)^\mathsf{T}$$

$$\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$$

$$\mathbf{H}\mathbf{x}_i = \begin{pmatrix} \mathbf{h}^{1\mathsf{T}}\mathbf{x}_i \\ \mathbf{h}^{2\mathsf{T}}\mathbf{x}_i \\ \mathbf{h}^{3\mathsf{T}}\mathbf{x}_i \end{pmatrix}$$

$$\mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = 0$$

$$H = \begin{bmatrix} h^{1T} \\ h^{2T} \\ h^{3T} \end{bmatrix}$$

$$\mathbf{x}'_i \times \mathbf{H}\mathbf{x}_i = \begin{pmatrix} y'_i\mathbf{h}^{3\mathsf{T}}\mathbf{x}_i - w'_i\mathbf{h}^{2\mathsf{T}}\mathbf{x}_i \\ w'_i\mathbf{h}^{1\mathsf{T}}\mathbf{x}_i - x'_i\mathbf{h}^{3\mathsf{T}}\mathbf{x}_i \\ x'_i\mathbf{h}^{2\mathsf{T}}\mathbf{x}_i - y'_i\mathbf{h}^{1\mathsf{T}}\mathbf{x}_i \end{pmatrix} = 0$$

Redundant: $x_i'(1)+y_i'(2)=(3)$

Direct Linear Transformation (DLT)

$$\begin{bmatrix} 0^{\mathsf{T}} & -w_i'\mathrm{x}_i^{\mathsf{T}} & y_i'\mathrm{x}_i^{\mathsf{T}} \\ w_i'\mathrm{x}_i^{\mathsf{T}} & 0^{\mathsf{T}} & -x_i'\mathrm{x}_i^{\mathsf{T}} \\ -y_i'\mathrm{x}_i^{\mathsf{T}} & x_i'\mathrm{x}_i^{\mathsf{T}} & 0^{\mathsf{T}} \end{bmatrix} \begin{pmatrix} \mathrm{h}^1 \\ \mathrm{h}^2 \\ \mathrm{h}^3 \end{pmatrix} = 0$$

$\mathrm{A}_i\mathrm{h} = 0$ where $A_i = \begin{bmatrix} 0^T & -w'_i x_i^T & y'_i x_i^T \\ w'_i x_i^T & 0^T & -x'_i x_i^T \end{bmatrix}$

Dimension of A_i : 2 x 9.

Direct Linear Transformation (DLT): Non-homogeneous Equations

- Solving for H by setting h_{33}=1.

$$h = \begin{bmatrix} \tilde{h} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & -x_i w'_i & -y_i w'_i & -w_i w'_i & x_i y'_i & y_i y'_i \\ x_i w'_i & y_i w'_i & w_i w'_i & 0 & 0 & 0 & -x_i x'_i & -y_i x'_i \end{bmatrix} \tilde{h} = \begin{bmatrix} -w_i y'_i \\ w_i x'_i \end{bmatrix}$$

$$\tilde{A}_i \tilde{h} = b_i$$

$$A\tilde{h} = b$$

$$Minimize \left\| A\tilde{h} - b \right\|$$

Solution: $\tilde{h} = (A^T A)^{-1} A^T b$

Dimension of A: 2nx8

Rank: 8

Dimension of h: 8x1

Dimension of b: 2nx1

Caution: If h_{33}=0, no multiplication scale exists, and no solution obtained.

Direct Linear Transformation (DLT): Homogeneous Equations

- Solving for **H**: $\mathrm{Ah} = 0$

$$\mathrm{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix}$$

Dimension of *A*: 2nx9

Rank: 8

Dimension of *h*: 9x1

Dimension of *Ah*: 2nx1

Minimize $||Ah||$ such that $\|\mathrm{h}\| = 1$

Solution: Unit eigen vector of smallest eigen value of A^TA.

Other error criteria

- Algebraic error: Error term in DLT.
- Geometric error: $\sum d_e^2\left(\mathbf{x}', \mathbf{H}\mathbf{x}\right)$ Euclidean distance
- Geometric error with reprojection:

$$\sum\left(d_e^2\left(\mathbf{x}', \mathbf{H}\mathbf{x}\right) + d_e^2\left(\mathbf{H}^{-1}\mathbf{x}', \mathbf{x}\right)\right)$$

- Use of nonlinear iterative optimization techniques such as Newton iteration, Levenberg-Marquardt (LM) method, etc.

Transformation invariance and normalization

- Problem: To estimate **H** given a set of ($\mathbf{x}_i$, $\mathbf{x}_i'$) .
- Consider, $\mathbf{y}_i=\mathbf{T}\mathbf{x}_i$ and $\mathbf{y}'_i=\mathbf{T}'\mathbf{x}_i'$ for known **T** and **T'**, which are invertible.
- Now estimate homography **G** from ($\mathbf{y}_i$, $\mathbf{y}'_i$).
- Can you estimate **H** from **G**?

$$
\begin{aligned}
\mathbf{x}' &= \mathbf{H}\mathbf{x} \\
\Rightarrow \mathbf{T}'^{-1}\mathbf{y}' &= \mathbf{H}\mathbf{T}^{-1}\mathbf{y} \\
\Rightarrow \mathbf{y}' &= \underbrace{\mathbf{T}'\mathbf{H}\mathbf{T}^{-1}}_{\mathbf{G}}\ \mathbf{y}
\end{aligned}
$$

Caution: For DLT it is not equivalent.
As the constraint ||g||=1 is not equivalent to ||h||=1.

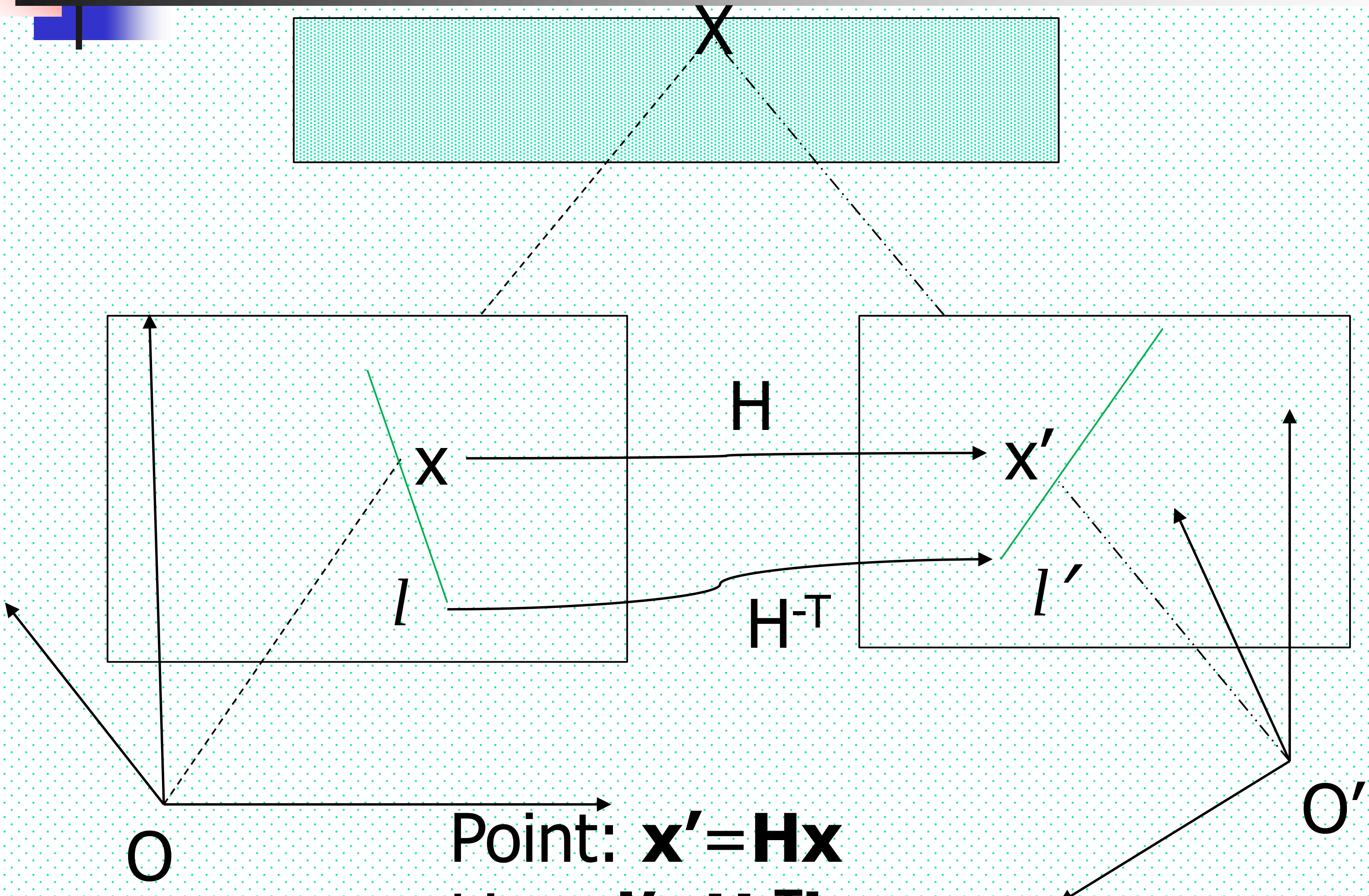
Robust computation through Normalization of data

- Transform the point set so that its center becomes origin (in the plane) and avg. distance from it is $\sqrt{2}$.

$$x_i^{(n)} = \frac{x_i - \bar{x}}{\sigma_x} \qquad y_i^{(n)} = \frac{y_i - \bar{y}}{\sigma_y}$$

- Apply DLT on transformed point.
- Recover homography from the homography of transformed point sets.

Projective transformation induced by a plane

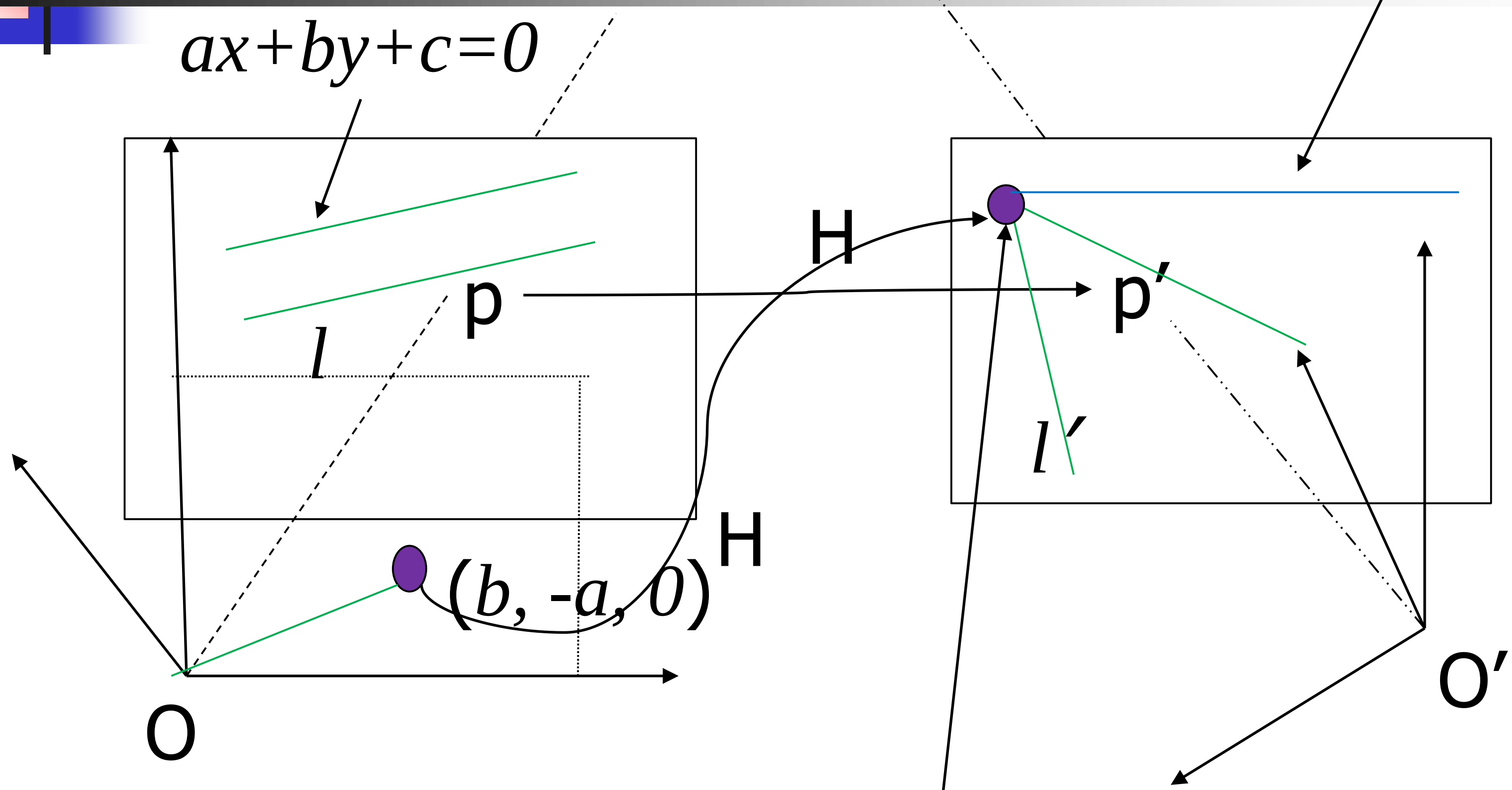


Point: $\mathbf{x'}=\mathbf{Hx}$

Line: $\mathbf{l'}=\mathbf{H^{-T}l}$

Vanishing line=$H^{-T}\mathbf{l}_{\alpha} = H^{-T}(0, 0,1)^{T}$

Vanishing point and line



Vanishing point: $\mathbf{v}_{\mathbf{l}}=H\,(b, -a, 0)^{T}$

Point and line transformation

- Point transformation:
  - $\mathbf{x}'=\mathbf{H}\mathbf{x}$
- Line transformation:
  - $\mathbf{l}'=\mathbf{H}^{-T}\mathbf{l}$
- Vanishing point for lines parallel to $\mathbf{l}=(a,b,c)^T$:
  - $\mathbf{v}_\mathbf{l}=\mathbf{H}\ (b, -a, 0)^T$
- Vanishing line:
  - $\mathbf{l}_\mathbf{H}=\mathbf{H}^{-T}\mathbf{l}_\alpha$
    $=\mathbf{H}^{-T}\ (0, 0,1)^T$

Examples

- Consider the following homography H between two projective spaces.

$$H=\begin{bmatrix}3 & 4 & -6\\ 1 & 3 & -8\\ 0 & 5 & 1\end{bmatrix}$$

Compute the transformation of the line formed by two points (2,4,2) and (6,9,3) in P^2.

$$H=\begin{bmatrix}3 & 4 & -6\\ 1 & 3 & -8\\ 0 & 5 & 1\end{bmatrix}$$

Method-I

- Compute transformed points of (2,4,2) and (6,9,3).
- Take their cross product to compute the transformed line.

$$\begin{bmatrix}2\\4\\2\end{bmatrix}\equiv\begin{bmatrix}1\\2\\1\end{bmatrix} \qquad \begin{bmatrix}6\\9\\3\end{bmatrix}\equiv\begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$H.\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}5\\-1\\11\end{bmatrix} \qquad H.\begin{bmatrix}2\\3\\1\end{bmatrix}=\begin{bmatrix}12\\3\\16\end{bmatrix}$$

$$H = \begin{bmatrix} 3 & 4 & -6 \\ 1 & 3 & -8 \\ 0 & 5 & 1 \end{bmatrix}$$

Method-I

- Take their cross product to compute the transformed line.

$$H.\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 11 \end{bmatrix} \qquad H.\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ -1 \\ 11 \end{bmatrix} \times \begin{bmatrix} 12 \\ 3 \\ 16 \end{bmatrix} = \begin{bmatrix} -49 \\ 52 \\ 27 \end{bmatrix}$$

Method-II

Compute the line and transform it.

The line between the points l:
(1,2,1) x (2,3,1)→ (-1,1-1)

Transformed line: $l' = H^{-T} l$

$$H^{-1} = \frac{1}{95}\begin{bmatrix} 43 & -34 & -14 \\ -1 & 3 & 18 \\ 5 & -15 & 5 \end{bmatrix} \qquad l' = \frac{1}{95}\begin{bmatrix} -49 \\ 52 \\ 27 \end{bmatrix}$$

$$H=\begin{bmatrix}3 & 4 & -6\\ 1 & 3 & -8\\ 0 & 5 & 1\end{bmatrix}$$

Example: Vanishing line

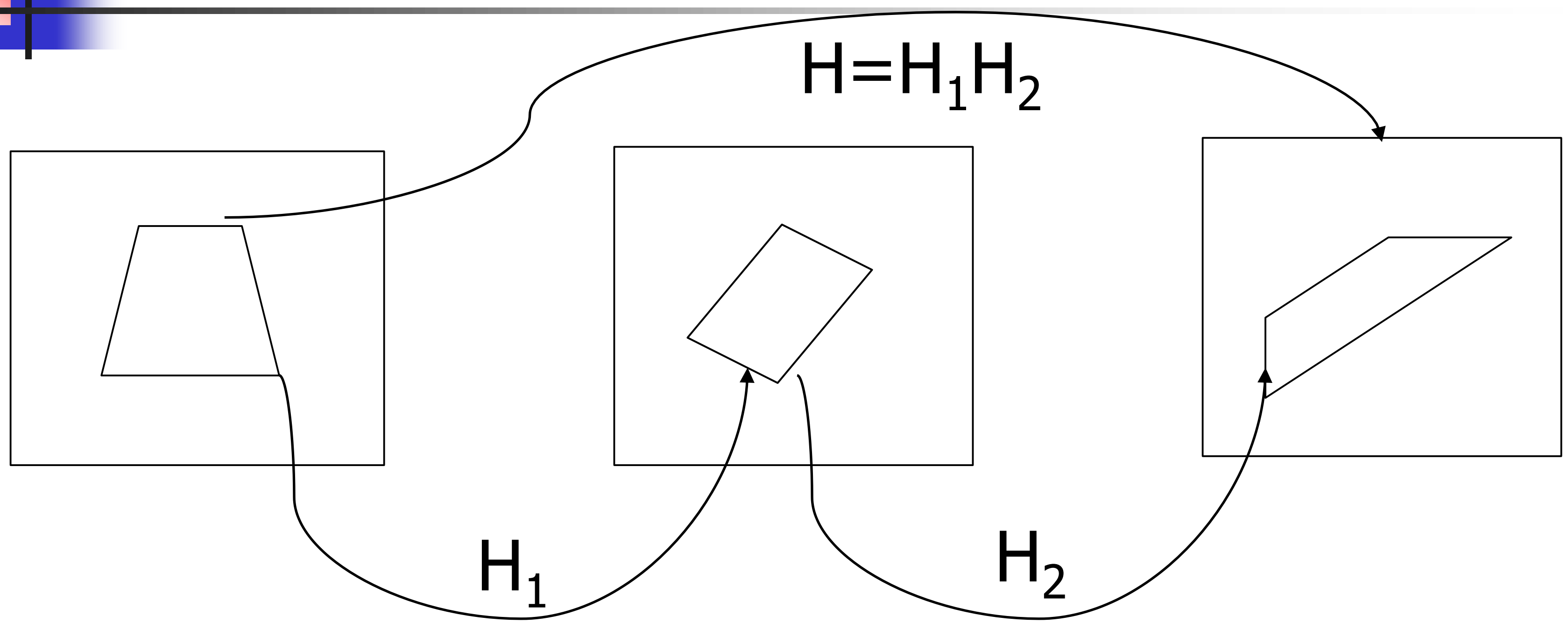
- Compute the vanishing line in the transformed space.

Transformed line: $l'_v = H^{-T}\, l_{\alpha}$

$$H^{-T}l_{\propto} = \frac{1}{95}\begin{bmatrix}43 & -1 & 5\\ -34 & 3 & -15\\ -14 & 18 & 5\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

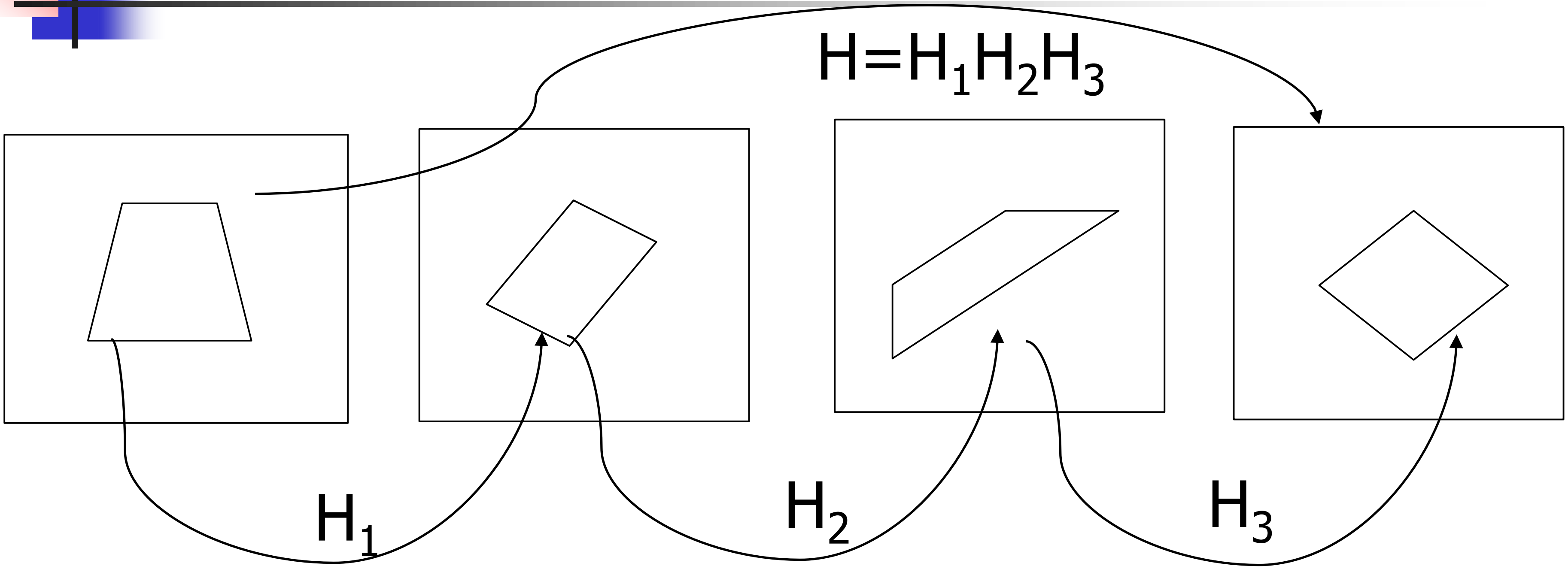
$$l'_v = \begin{bmatrix}5\\ -15\\ 5\end{bmatrix}$$

Projective linear group



A cascade of transformation can be replaced by a single transformation.

Different compositions



A series of transformation could be performed in different composition.

$$H=H_1(H_2H_3)=(H_1H_2)H_3$$

Subgroups and hierarchy

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Projective linear group

$\Uparrow$

Affine group (last row (0,0,1))

$\Uparrow$

Euclidean group (upper left 2x2 orthogonal)

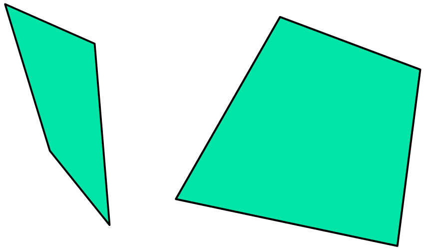
$\Uparrow$

Oriented Euclidean group (upper left 2x2 det 1)

From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

Projective Group

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



$$\mathrm{x}' = \mathbf{H}_P\, \mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \mathrm{x}$$

$$\mathrm{v} = (v_1, v_2)^\mathsf{T}$$

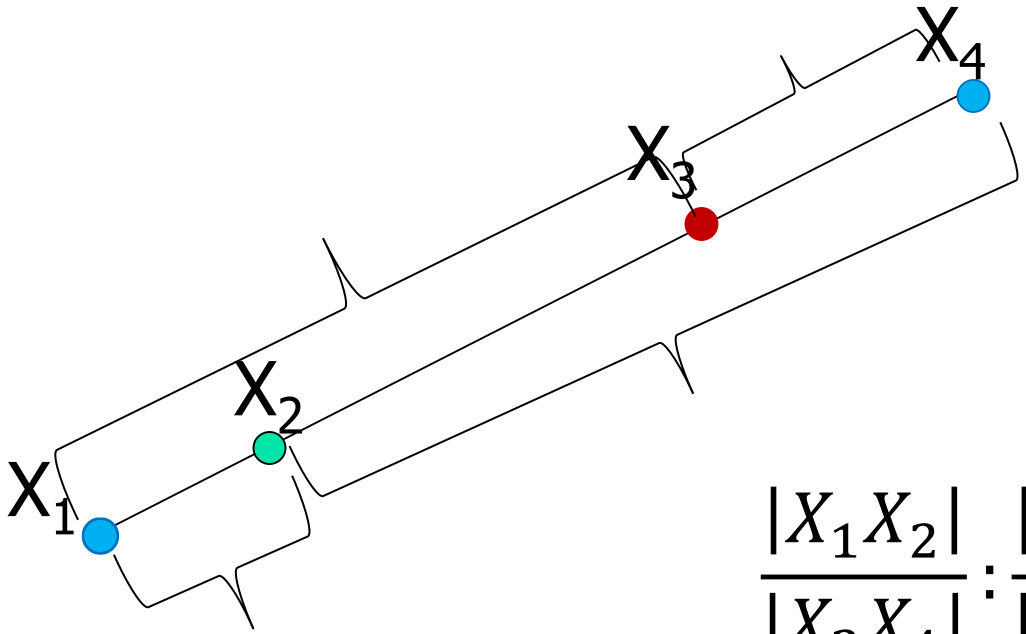
dof=8: 2 scale, 2 rotation, 2 translation, 2 line at infinity)

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite, allows to observe vanishing points, horizon.

Concurrency, collinearity, order of contacts, cross ratio (ratio of ratio).

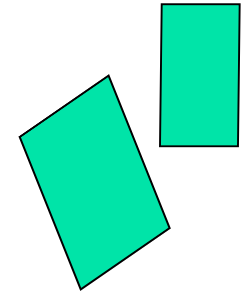
Cross Ratio



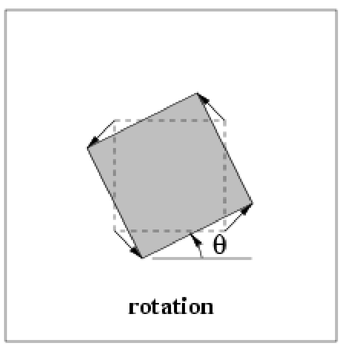
$$\frac{|X_1X_2|}{|X_2X_4|} : \frac{|X_1X_3|}{|X_3X_4|}$$

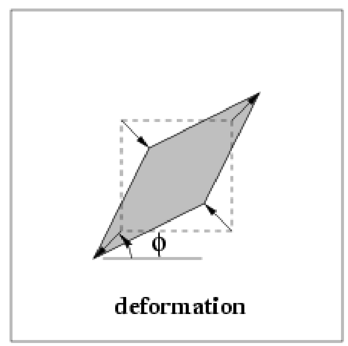
Affine group

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mathrm{x}' = \mathbf{H}_A\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x}$$





dof=6

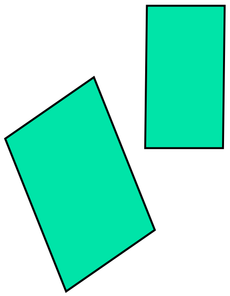
$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi)$$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

From Hartley and Zisserman, “Multiple view geometry in computer vision”, Cambridge Univ. Press (2000)

Affine group

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mathrm{x}' = \mathbf{H}_A\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix}\mathrm{x}$$

dof=6

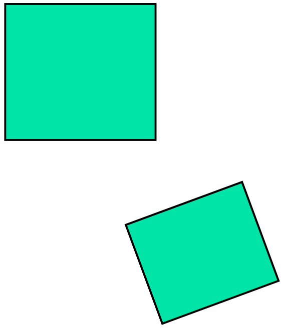
$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & v \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity, but points move along line.

Parallelism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids). **The line at infinity $\mathbf{l}_\infty$.**

Similarity Group

$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mathbf{x}' = \mathbf{H}_S\,\mathbf{x} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}\mathbf{x}$$

$$\mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

dof=4 (1 scale, 1 rotation, 2 translation)

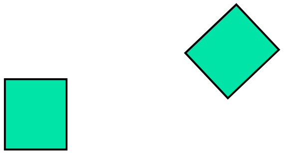
$$\mathbf{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathbf{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$\mathrm{I}' = \mathbf{H}_S\,\mathrm{I} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

Ratios of lengths, angles. **The circular points I,J.**

Isometry

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\varepsilon = \pm 1$$

Orientation preserving: $\varepsilon = 1$

Orientation reversing: $\varepsilon = -1$

dof=3 (1 rotation, 2 translation)

Invariants: length, angle, area

Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^\mathsf{T}$$

$\mathbf{K}$ Upper-triangular

$\det \mathbf{K} = 1 \quad v \neq 0$

Decomposition unique (if chosen s>0)

QR decomposition:
Any square matrix decomposed as a product of an orthogonal matrix (Q) and an upper triangular matrix (R)

Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

$$v \neq 0$$

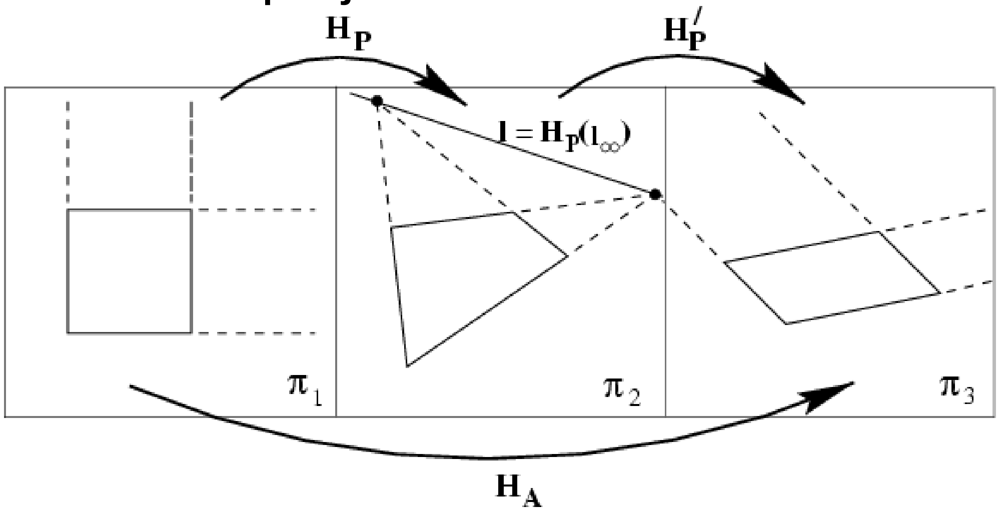
$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^\mathsf{T} \qquad \det \mathbf{K} = 1$$

Example:

$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Affine properties from images



$$\mathbf{l}_\infty = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^\mathsf{T}, l_3 \neq 0$$

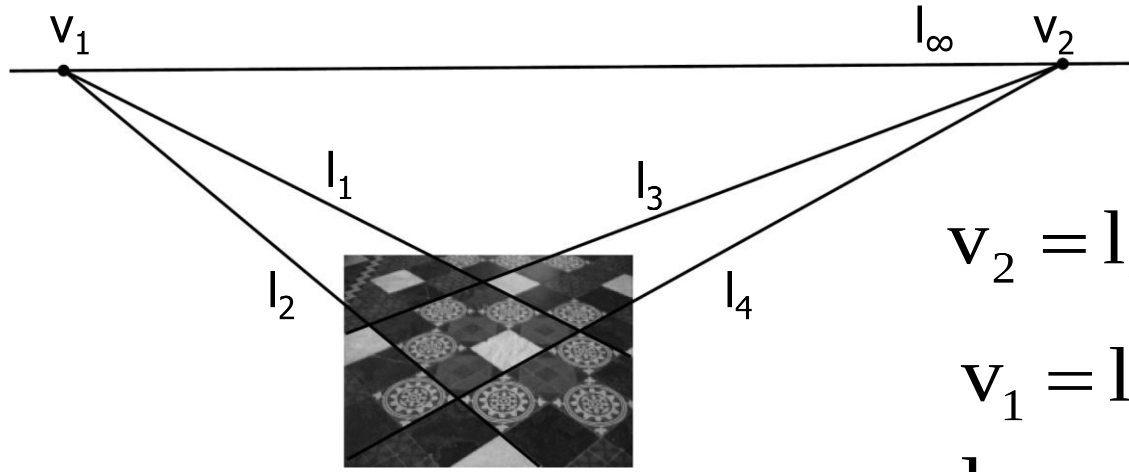
$$H'_p = H_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

For any affine H_A.

$$H'^{-T}_p \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

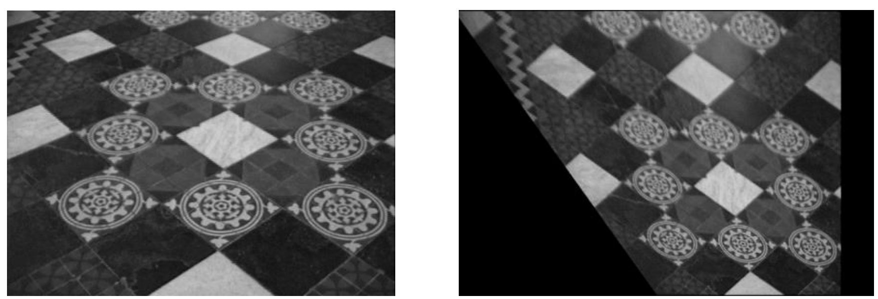
Affine rectification



$$\mathbf{v}_2 = \mathbf{l}_3 \times \mathbf{l}_4$$

$$\mathbf{v}_1 = \mathbf{l}_1 \times \mathbf{l}_2$$

$$\mathbf{l}_\infty = \mathbf{v}_1 \times \mathbf{v}_2$$



From Hartley and Zisserman, “Multiple view geometry in computer vision”, Cambridge Univ. Press (2000)

An example

(104,69,1) X (380, 71,1)

l_1=(-2, 276, -18836)

(122,226,1) x (366,228,1)

l_2=(-2, 244 , -54900)

(88,254,1) x (62,49,1)

l_3=(205,-26,-11436)

(390,250,1) x (406,53,1)

l_4=(197,16,-80830)

v_1=l_1 x l_2 =(-10556416, -72128, 64)

v_2=l_3 x l_4= (2284556,14317258,8402)

Vanishing Line: l_v= 1.0e+14 * (0,0.0009, -1.5097)

An example

Vanishing Line: l_v= 1.0e+14 * (0,0.0009, -1.5097)

Scaled l_v= (0, -0.0006,1)

$Y=10^4/6$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -.0006 & 1 \end{bmatrix}$$

Conics in P^2

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

$$X^T C X = 0$$

$$C = \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix}$$

Conics identified by C
with 5 d.o.f. $(a:b:c:d:e:f)$

A line tangent to the conic C satisfies $\quad \mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0$

Dual conic

C^{-1}

Transformation of conics under homography **H**

- $\mathbf{X'}=\mathbf{HX}$
- $\mathbf{X}^{T}\mathbf{CX}=0$

  $\rightarrow (\mathbf{H}^{-1}\mathbf{X'})^{T}\mathbf{C}(\mathbf{H}^{-1}\mathbf{X'})=0$

  $\rightarrow \mathbf{X'}^{T}\,\mathbf{H}^{-T}\mathbf{CH}^{-1}\,\mathbf{X'}=0$

  $\rightarrow \mathbf{X'}^{T}\,\mathbf{C'}\,\mathbf{X'}=0$

A conic remains a conic under homography.

where transformed conics $\mathbf{C'}= \mathbf{H}^{-T}\mathbf{CH}^{-1}$

- $\mathbf{C'}^{*}=\mathbf{C'}^{-1}=(\mathbf{H}^{-T}\mathbf{CH}^{-1})^{-1}=\mathbf{HC}^{-1}\mathbf{H}^{T}$

The circular points

$$I = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad J = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad I' = \mathbf{H}_S I = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = I$$

The circular points I, J are fixed points under the projective transformation **H** iff **H** is a similarity. They are also on $\mathbf{l}_\alpha$.

Every circle intersects $\mathbf{l}_\alpha$ at I and J.

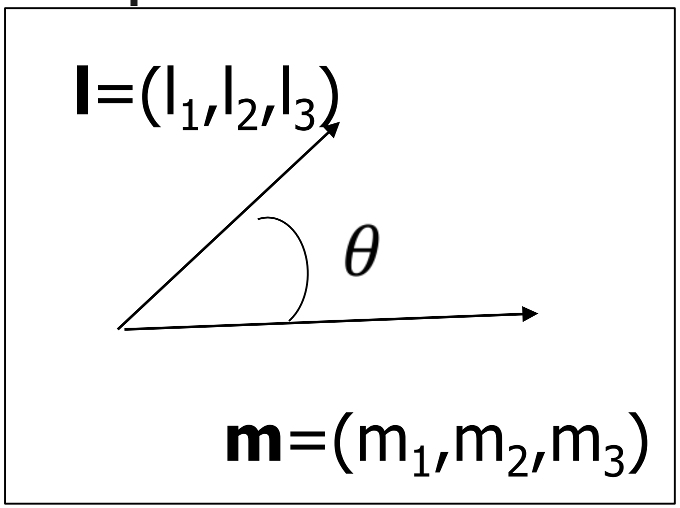
Circle: $x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$

Setting x_3=0, $x_1^2 + x_2^2$=0. (I and J satisfies it)

Conic dual to the circular points (C_α^*)

- $C_\alpha^* = I.J^T + J.I^T$ (line conic)
- $C_\alpha^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- As I and J are fixed under similarity C_α^* is also fixed, i.e. $C_\alpha^{*\prime} = H_s\, C_\alpha^* H_s^T = C_\alpha^*$
- C_α^* is fixed iff H is a similarity.
- D.o.f. of transformed C_α^* is 4 and det. =0.
- l_α is the NULL vector of C_α^*.

Measurement of angle under homography



$$cos(\theta) = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

Invariant under homography

$$\cos(\theta) = \frac{\boldsymbol{l}^T C_\infty^* \boldsymbol{m}}{\sqrt{(\boldsymbol{l}^T C_\infty^* \boldsymbol{l})(\boldsymbol{m}^T C_\infty^* \boldsymbol{m})}}$$

$C_\infty^{*\prime} = H C_\infty^* H^T$ and $\boldsymbol{l}' = H^{-T} \boldsymbol{l}$

$$\boldsymbol{l}'^T C_\infty^{*\prime} \boldsymbol{m}'$$
$$= \boldsymbol{l}^T H^{-1} H C_\infty^* H^T H^{-T} \boldsymbol{m}$$
$$= \boldsymbol{l}^T \boldsymbol{C}_\infty^* \boldsymbol{m}$$

Once $C_\alpha^{*\prime}$ is obtained Euclidean angle could be recovered.

If $\boldsymbol{l}$ and $\boldsymbol{m}$ orthogonal, $\boldsymbol{l}^T C_\alpha^{*\prime} \boldsymbol{m} = 0$.

Estimation of $C_\alpha^{*\prime}$

- Use the property of orthogonal lines.
  - $\mathbf{l}^T \mathbf{C}_\alpha^{*\prime} \mathbf{m} = 0$
- Minimum 5 such orthogonal pairs needed.
  - A typical equation

$$\left[l_1 m_1 \quad \frac{1}{2}(l_1 m_2 + l_2 m_1) \quad l_2 m_2 \quad \frac{1}{2}(l_1 m_3 + l_3 m_1) \quad \frac{1}{2}(l_2 m_3 + l_3 m_2) \quad l_3 m_3 \right] C = 0$$

  - Where C represented by $(a,b,c,d,e,f)^T$.
- Apply direct linear transform (LSE method) to solve a set of homogeneous equations to get C.
- Make it ($C_\alpha^{*\prime}$) a rank 2 matrix using SVD on C.

Recovery of metric properties

- Compute H from $C_\alpha^{*\prime}$ upto similarity.
  - Matrix decomposition method $C_\propto^{*\prime} = H C_\propto^{*} H^T$

$$C_\propto^{*\prime} = \underset{H}{U} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T$$

- Apply H^{-1} to the image.

Summary

- A projective transformation, invertible and preserving collinearity, is always in a linear form.
  - $\mathbf{x}'=\mathbf{H}\mathbf{x}$
- A computational problem: estimation of homography given a set of point correspondences.
  - Minimum 4 point correspondences needed.
  - Direct linear transformation (using LSE).
  - Use of linear transformation of points to make the computation robust.

Summary (contd.)

- Various transformation under homography
  - $\mathbf{x}'=\mathbf{H}\mathbf{x}$
  - $\mathbf{l}'=\mathbf{H}^{-T}\,\mathbf{l}$
  - $\mathbf{C}'= \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$
  - $\mathbf{C}'^{*}=\mathbf{C}'^{-1}=\mathbf{H}\mathbf{C}^{-1}\mathbf{H}^{T}$
- Projective linear group, its subgroup and hierarchy
  - Projective linear group (8 D.O.F)
  - Affine group (6 D.O.F)
  - Euclidean group (4 D.O.F.)
  - Oriented Euclidean group (3 D.O.F)

Summary (contd.)

- Conic dual to circular points (C_α^*)
  - Invariant under similarity transform.
  - l_α is the zero (NULL) vector.
  - Preserves cosine of angle of two lines under transformation

$$\cos(\theta) = \frac{\boldsymbol{l}^T C_\infty^* \boldsymbol{m}}{\sqrt{(\boldsymbol{l}^T C_\infty^* \boldsymbol{l})(\boldsymbol{m}^T C_\infty^* \boldsymbol{m})}}$$

- Use of homography
  - Affine rectification
  - Stratification (recovery of metric properties)

Thank you!