Indian Institute of Technology Kharagpur
Department of Mathematics
MA11003 - Advanced Calculus
Problem Sheet - 1
Autumn 2020

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1. Verify which functions satisfy the conditions of the Rolle's theorem and if satisfies find c which satisfy the conclusion of the Rolle's theorem:
(a) $f(x) = x^2 + \cos(x)$ on $[\frac{-\pi}{4}, \frac{\pi}{4}]$ (b)$f(x) = 1 - |x-1|$ on $[0, 2]$
(c) $f(x) = \sin(\frac{1}{x})$ on $[\frac{1}{3\pi}, \frac{1}{2\pi}]$ (d)$f(x) = 1 - (x-1)^{\frac{2}{3}}$ on $[0, 2]$.

2. Calculate $\xi \in (a, b)$ in cauchy MVT for each of the following pairs:
(a) $f(x) = \sin x,\ g(x) = \cos x$ on $[\frac{\pi}{4}, \frac{3\pi}{4}]$,
(b) $f(x) = (1 + x)^{\frac{3}{2}}$, $g(x) = \sqrt{1+x}$ on $[0, \frac{1}{2}]$.

3. Prove that between any two real roots of the equation $e^x \sin x + 1 = 0$ there is atleast one real root of the equation $\tan x + 1 = 0$.

4. Show that the formula in the Lagrange MVT can be written as follows:
$$\frac{f(x+h) - f(x)}{h} = f'(x + \theta h)$$
where $0 < \theta < 1$. Determine θ as a function of x and h when
(a)$f(x) = x^2$ (b)$f(x) = e^x$ (c)$f(x) = \log x$, $x > 0$.
Keep $x \neq 0$ fixed, and find $\lim\limits_{h \to 0} \theta$ in each case.

5. Let f be a function having a finite derivative f' in the half-open interval $0 < x \leq 1$ such that $|f'(x)| < 1$. Define $a_n = f(\frac{1}{n})$ for $n = 1, 2, 3, ...$Show that $\lim\limits_{n\to\infty} a_n$ exists.

6. Assume f has a finite derivative in (a, b) and is continuous on $[a, b]$ with $f(a) = f(b) = 0$. Prove that for every real λ there is some c in (a, b) such that $f'(c) = \lambda f(c)$.

7. Answer the followings:

   (a) Suppose, $f(x)$ is continuous on $[-7,0]$ and differentiable in $(-7,0)$ such that $f(-7) = -3$ and $|f'(x)| \leq 2$. Then, what is largest possible value of $f(0)$.
   (b) Use Lagrange MVT to estimate $\sqrt[3]{28}$.
   (c) If $f''(x) \geq 0$ on $[a,b]$ prove that $f\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{2}\left[f(x_1)+f(x_2)\right]$ for any two points x_1 and x_2 in $[a,b]$.

8. If f has a finite third derivative f''' in $[a,b]$ and if $f(a) = f(b) = f'(a) = f'(b) = 0$. Prove that $f'''(c) = 0$ for some c in (a,b).

9. Prove that

   (a) $\frac{2x}{\pi} < \sin x < x$ for $0 < x < \frac{\pi}{2}$.
   (b) $na^{n-1}(b-a) < b^n - a^n < nb^{n-1}(b-a)$ where $0 < a < b$ and $n > 1$.
   (c) $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$.

10. Use Rolle's theorem to prove the following:

    (a) Let $f : [1,3] \to \mathbb{R}$ be a continuous function such that $\int_1^2 f(x)dx = 2$ and $\int_1^3 f(x)dx = 3$. Then show that there exist some $c \in (2,3)$ such that
    $$cf(c) = \int_1^c f(x)dx.$$
    (b) Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and $f''(x)$ exists for all $x \in (a,b)$. Let $a < c < b$, then there exists a point ξ in (a,b) such that
    $$f(c) = \frac{b-c}{b-a}f(a) + \frac{c-a}{b-a}f(b) + \frac{1}{2}(c-a)(c-b)f''(\xi).$$

11. If $c_0 + \frac{c_1}{2} + \frac{c_2}{3} + ... + \frac{c_n}{n+1} = 0$ where $c_0, c_1, ..., c_n$ are real. Show that the equation $c_0 + c_1x + ... + c_nx^n = 0$ has atleast one real root between 0 and 1.

12. Answer the followings:

    (a) Assume f is continuous on $[a,b]$ and has a finite second derivative f'' in the open interval (a,b). Assume that the line segment joining the points $A = (a, f(a))$ and $B = (b, f(b))$ intersects the graph of f in a third point P different from A and B. Prove that $f''(c) = 0$ for some c in (a,b).
    (b) If f is differentiable on $[0,1]$ show by Cauchy's MVT that the equation $f(1) - f(0) = \frac{f'(x)}{2x}$ has atleast one solution in $(0,1)$.
    (c) Let, f be continuous on $[a,b]$ and differentiable on $[a,b]$. If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1 and c_2 in (a,b) such that $f'(c_1) + f'(c_2) = 2$.

13. Answer the followings:

(a) If $f(x)$ and $\phi(x)$ are continuous on $[a, b]$ and differentiable on (a, b), then show that

$$\begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix} = (b-a)\begin{vmatrix} f(b) & f'(c) \\ \phi(b) & \phi'(c) \end{vmatrix}, a < c < b.$$

(b) Let f be continuous on $[a, b]$ and differentiable on(a, b). Using Cauchy's MVT show that if $a \geq 0$ then there exist $x_1, x_2, x_3 \in (a, b)$ such that

$$f'(x_1) = (b+a)\frac{f'(x_2)}{2x_2} = (b^2 + ba + a^2)\frac{f'(x_3)}{3x_3^2}.$$

14. Use CMVT to prove the followings:

(a) Show that $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$.

(b) Let f be continuous on $[a, b]$, $a > 0$ and differentiable on (a, b). Prove that there exist $c \in (a, b)$ such that $\frac{b^2 f(a) - a^2 f(b)}{b^2 - a^2} = \frac{1}{2}[2cf(c) - c^2 f'(c)]$.

(c) Show that $\frac{2\ln x}{2\arcsin x - \pi} < \frac{\sqrt{1-x^2}}{x}$.

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