Projective Geometry

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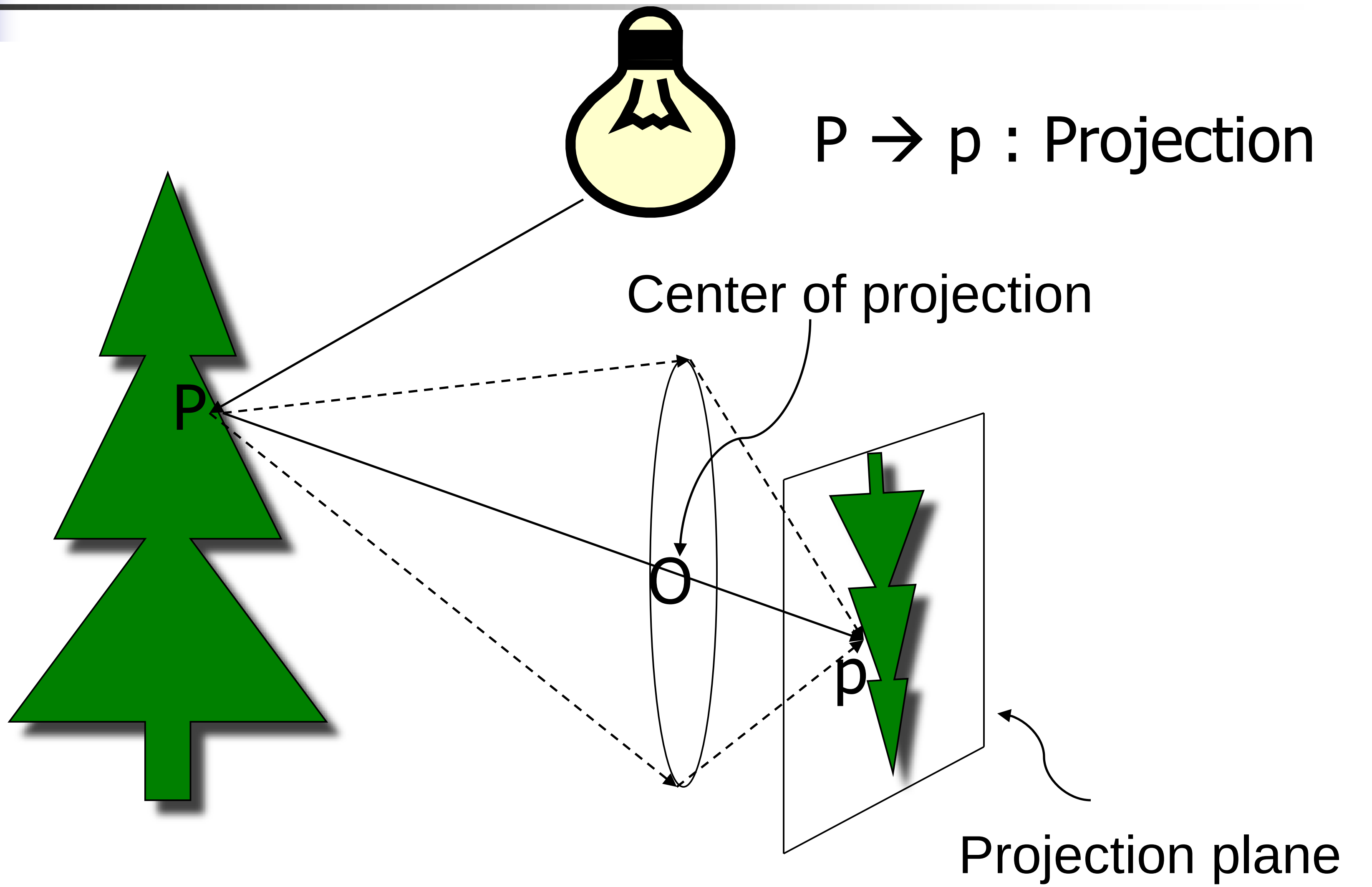
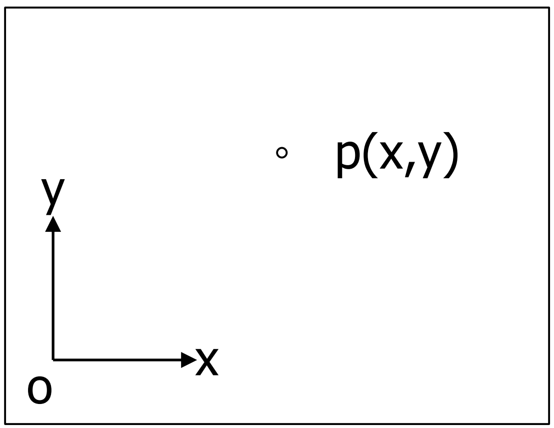
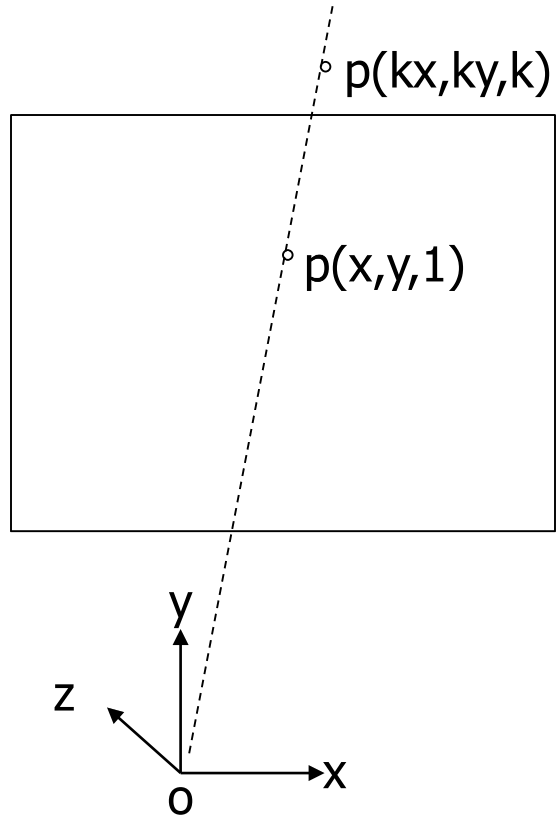
Image formation in optical camera



Real Space and Projective Space (2D)

$p \in R^2$





$p \in P^2$

p(x,y) ⇔ p(kx,ky,k)

Homogeneous Coordinate system

Homogeneous Representation

A point in R^2: $\vec{x} \equiv \begin{bmatrix} x \\ y \end{bmatrix}$ $\Leftrightarrow$ A point in P^2: $\vec{X} \equiv \begin{bmatrix} kx \\ ky \\ k \end{bmatrix}$

$$P^2 = R^3 - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Singular point in the projective space.

Homogeneous Representation

In R^2: ? $\Leftrightarrow$ In P^2: $\vec{X} \equiv \begin{bmatrix} 25 \\ 30 \\ 5 \end{bmatrix}$

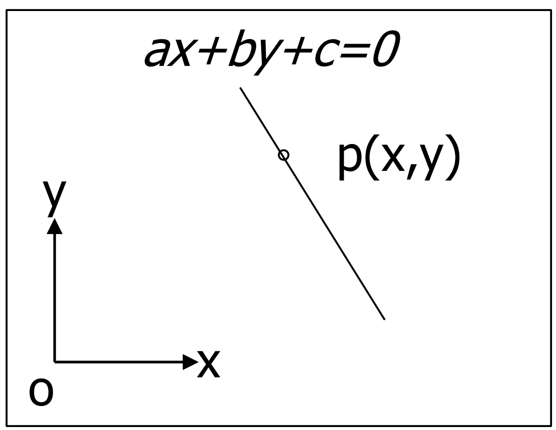
The point in R^2: $\vec{x} \equiv \begin{bmatrix} \frac{25}{5} \\ \frac{30}{5} \end{bmatrix} \equiv \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

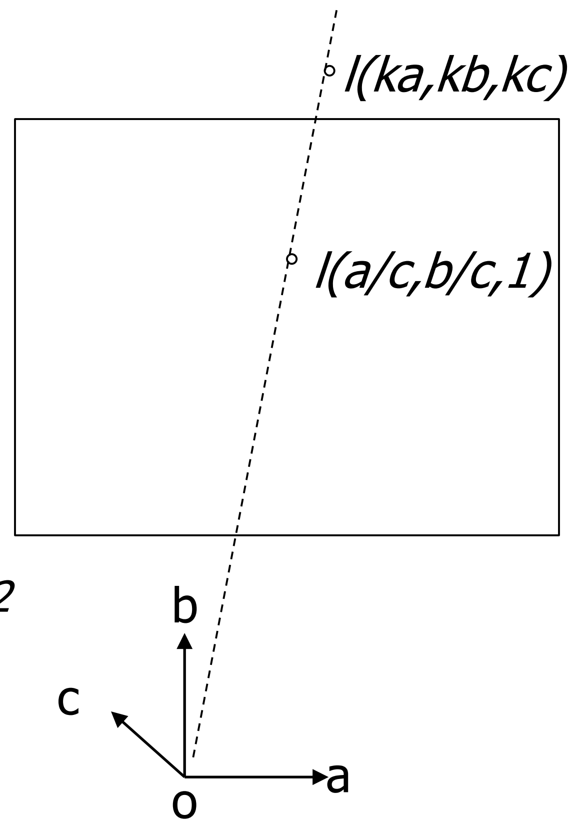
In P^2: $\vec{X} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\Leftrightarrow$ In R^2: ?

Does not belong to P^2.

Homogeneous representation of a line in a plane

$p \in R^2$





$l \in P^2$

A point in P^2

A line in P^2

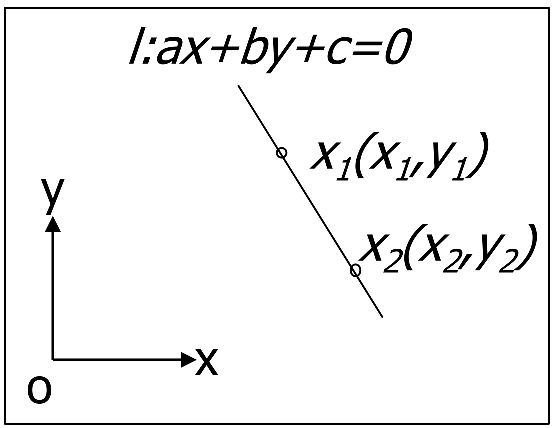
$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

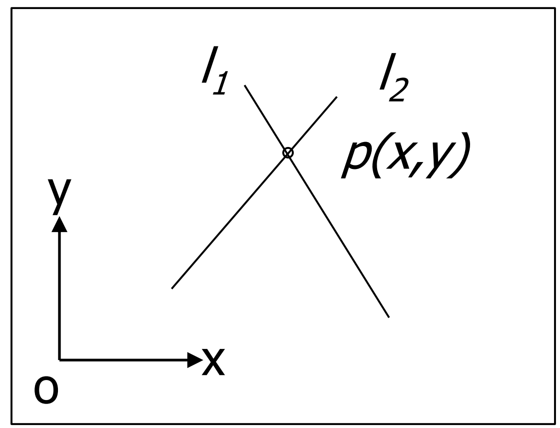
Point containment in P^2 $\Rightarrow$ $\vec{X}^T . \vec{l} = 0 \Leftrightarrow \vec{l}^T . \vec{X} = 0$

Points and lines in P^2

$$X_1 \times X_2 = \begin{vmatrix} i & j & k \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$$

$x_1 \in R^2$





$$\vec{l} = \vec{X}_1 \times \vec{X}_2$$

$$\vec{P} = \vec{l}_1 \times \vec{l}_2$$

Exactly one line through two points.

Exactly one point at intersection of two lines.

Examples

1. Compute the line passing through (3,5) and (5,0) in a plane.

$$\vec{l}=\begin{bmatrix}3\\5\\1\end{bmatrix}\times\begin{bmatrix}5\\0\\1\end{bmatrix}=\begin{bmatrix}5\\2\\-25\end{bmatrix}$$

2. Compute the point of intersection of the lines: 5x-2y+4=0 and 6x-7y-3=0.

$$\vec{P}=\begin{bmatrix}5\\-2\\4\end{bmatrix}\mathrm{x}\begin{bmatrix}6\\-7\\-3\end{bmatrix}=\begin{bmatrix}34\\39\\-23\end{bmatrix} \Rightarrow \left(-\frac{34}{23},-\frac{39}{23}\right)$$

Duality

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

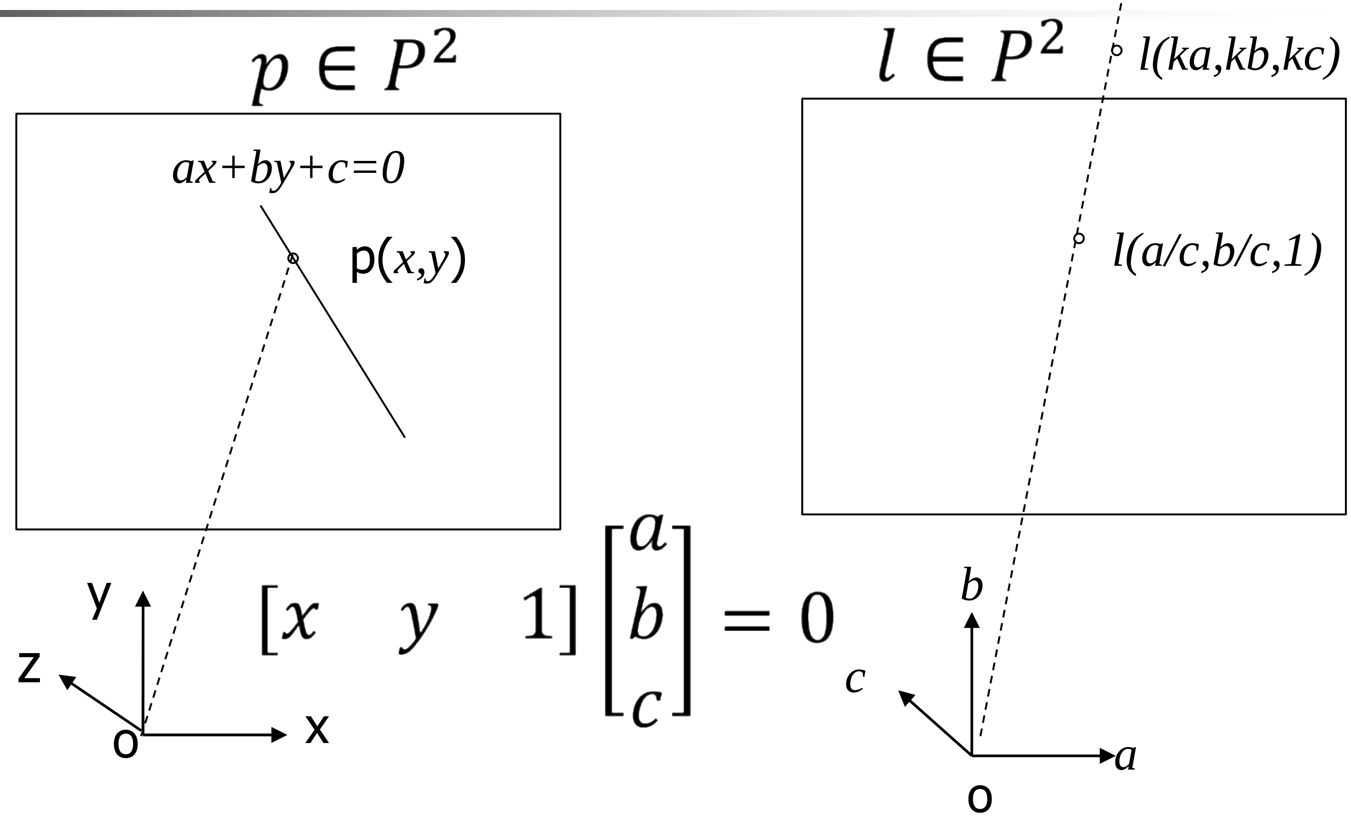
$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \longleftrightarrow \mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

Duality principle:

To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem.

Points and lines in a plane

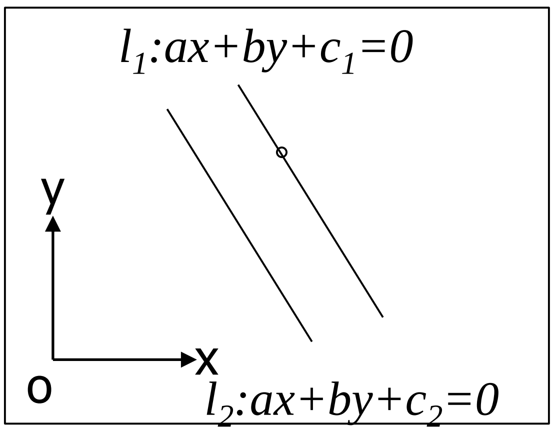


$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

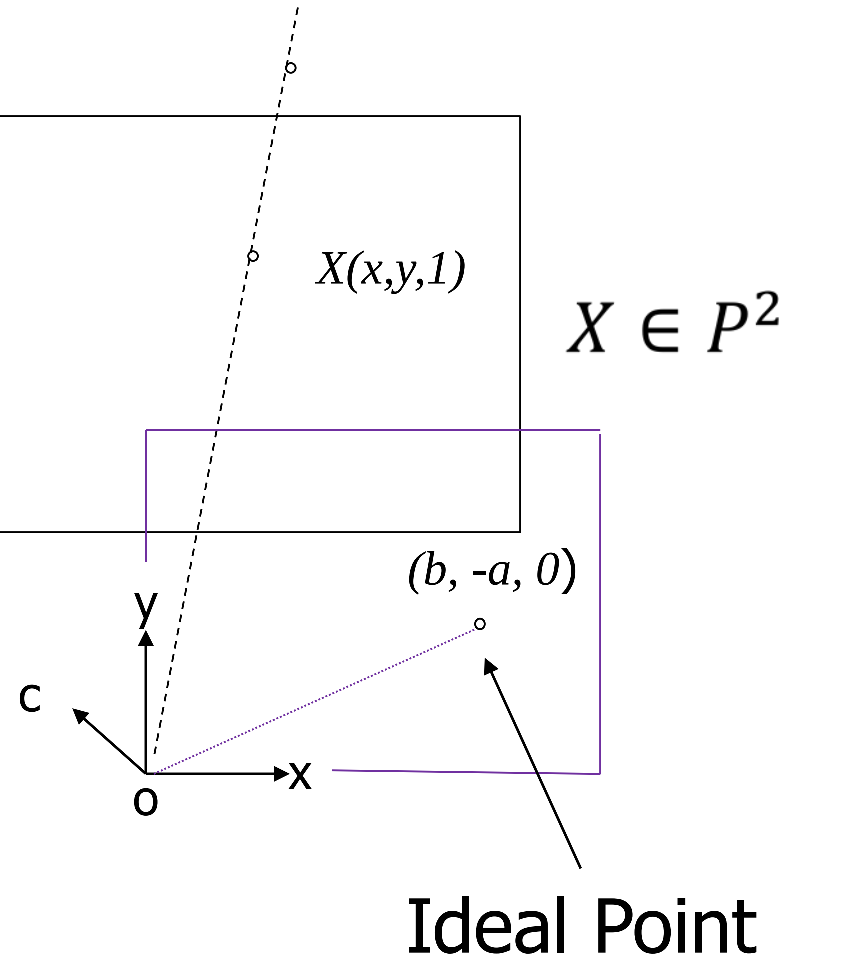
$$\mathrm{x}^{\mathsf{T}}\mathrm{l} = 0 \quad \longleftrightarrow \quad \mathrm{l}^{\mathsf{T}}\mathrm{x} = 0$$

$$\mathrm{x} = \mathrm{l} \times \mathrm{l}' \quad \longleftrightarrow \quad \mathrm{l} = \mathrm{x} \times \mathrm{x}'$$

Intersection of parallel lines



$$\vec{l}_1 \times \vec{l}_2 = (c_2 - c_1)\begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$



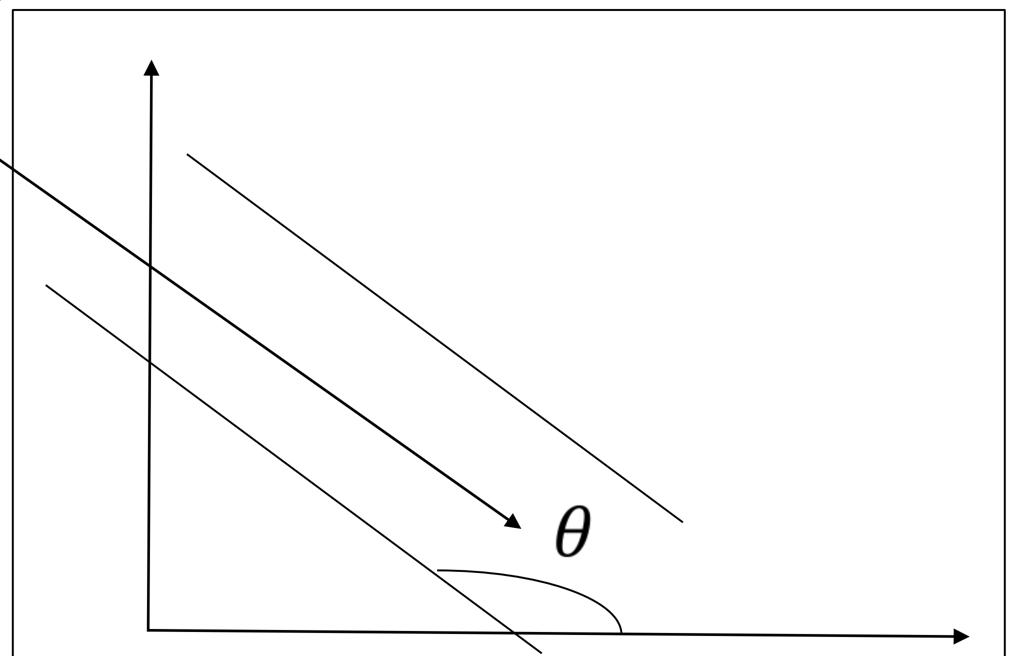
Meaning of an ideal point

$$ax + by + c = 0 \quad \Rightarrow \quad y = -\frac{a}{b}x + \frac{c}{b} = \tan(\theta)\, x + c'$$

Intersection point

$$\downarrow$$

$(b, -a, 0)$



A direction !

Ideal points

Ideal points: Points on the X-Y plane or principal plane parallel to projection plane.

For canonical coordinate system, they are of the form:

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

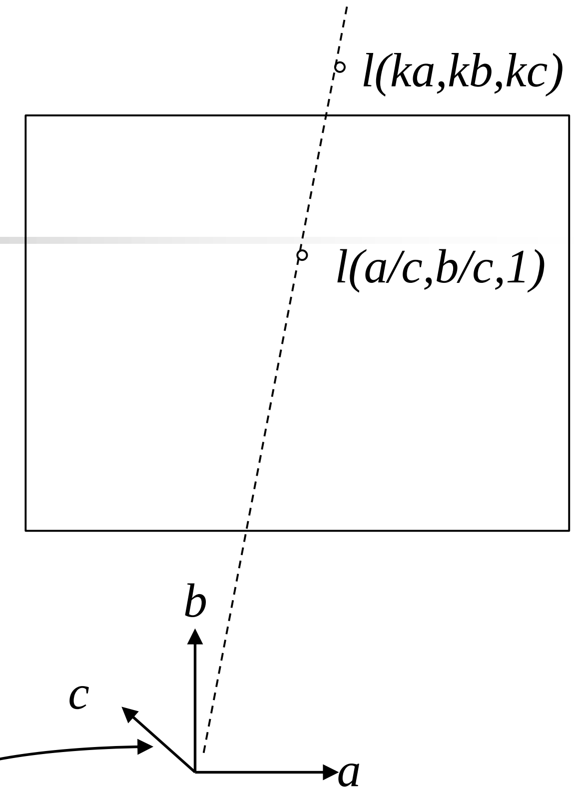
An ideal point denotes a direction toward infinity!

Line at infinity

$$[x \quad y \quad 0]\begin{bmatrix}0\\0\\1\end{bmatrix} = 0$$

Any ideal point

$$\begin{bmatrix}0\\0\\1\end{bmatrix}$$



Line at infinity (l_∞): Line containing every ideal point. In canonical system, it is $\begin{bmatrix}0\\0\\1\end{bmatrix}$.

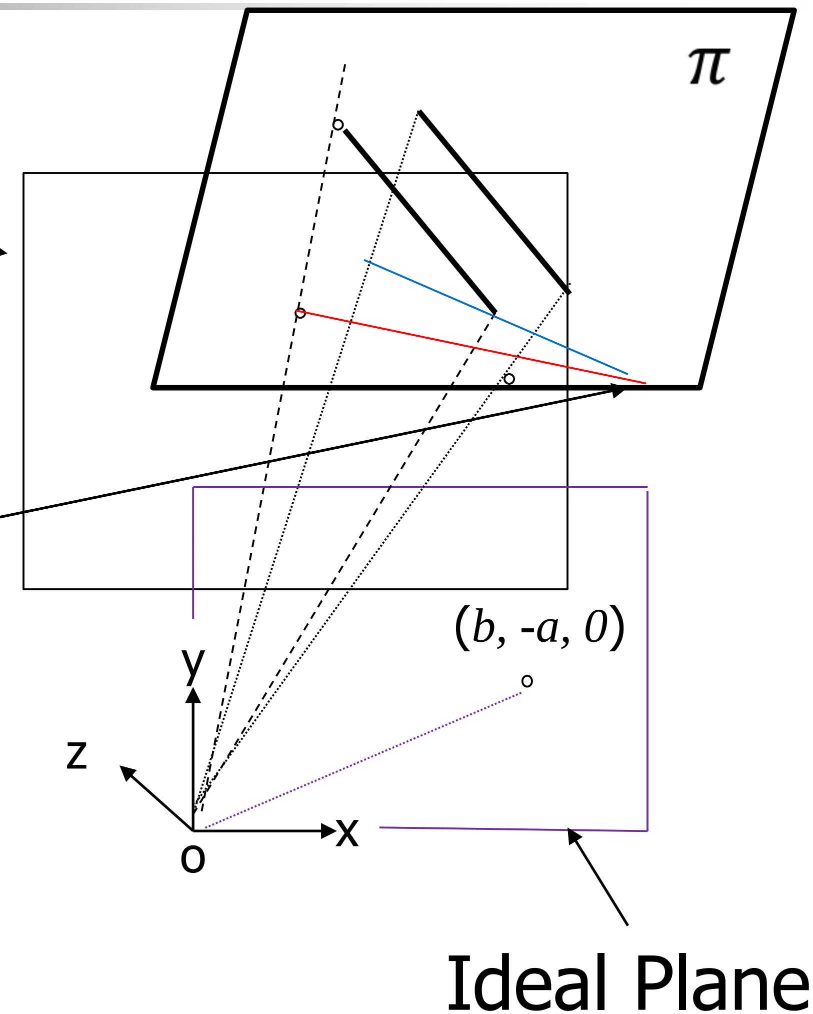
Projection of parallel lines from any arbitrary plane

Canonical projection plane (CPP)

Vanishing Point

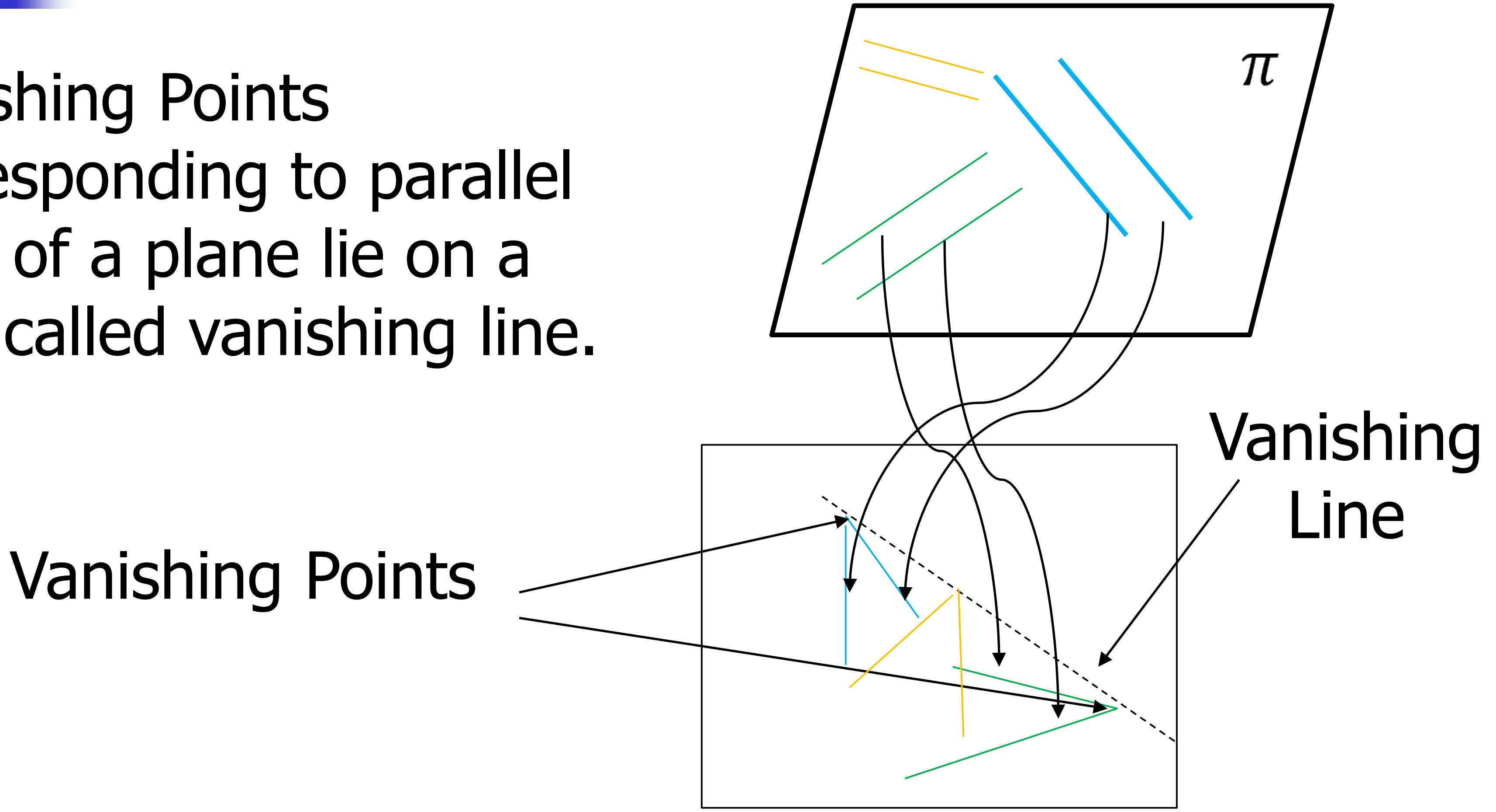
⇩

Point of intersection of parallel lines on π.

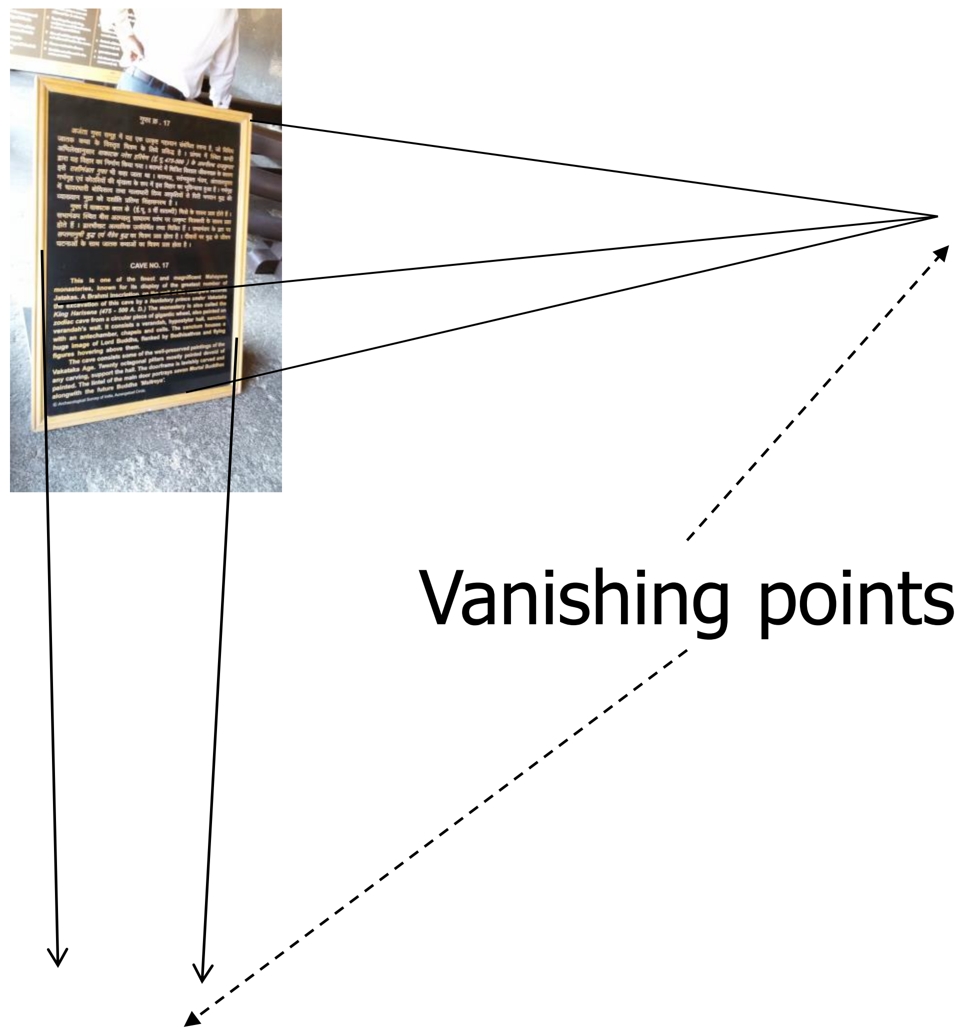


Vanishing points

Vanishing Points corresponding to parallel lines of a plane lie on a line, called vanishing line.



A real life example



A journey toward infinity ….

Conics in P^2

- Curves described by 2^{nd} degree equation in the plane.

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

- A point in homogeneous coordinate: (x_1, x_2, x_3)

$$\rightarrow (x_1/x_3, x_2/x_3)$$

$$a\left(\frac{x_1}{x_3}\right)^2 + \mathrm{b}\left(\frac{x_1}{x_3}\right)\left(\frac{x_2}{x_3}\right) + \mathrm{c}\left(\frac{x_2}{x_3}\right)^2 + \mathrm{d}\left(\frac{x_1}{x_3}\right) + \mathrm{e}\left(\frac{x_2}{x_3}\right) + \mathrm{f} = 0$$

$$\Longrightarrow ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

Conics in P^2

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$
$$\Longrightarrow X^TCX = 0$$

Where

$$C = \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix}$$

Conics identified by C with 5 d.o.f. (a:b:c:d:e:f)

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_i y_i + cy_i^2 + dx_i + ey_i + f = 0$$

$$\Rightarrow \quad \left(x_i^2, x_i y_i, y_i^2, x_i, y_i, f\right)\mathbf{c} = 0$$

$$\mathbf{c} = \left(a, b, c, d, e, f\right)^{\mathsf{T}}$$

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

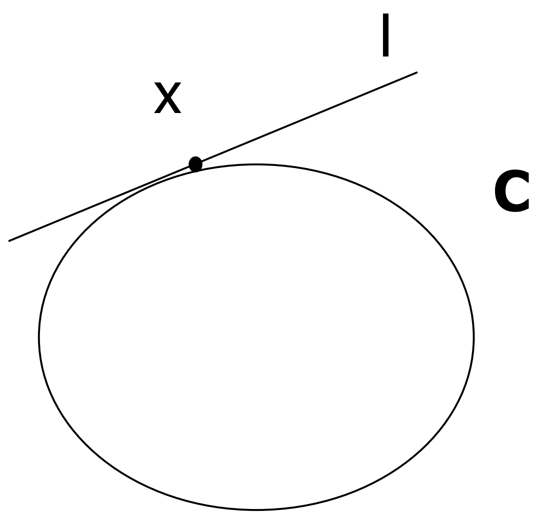
Stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

- Rank deficient **C**
  → degenerate conic
  two lines (of rank 2)
  a repeated line (of rank 1).

Tangent lines to conics

The line **l** tangent to **C** at point x on **C** is given by **l**=**C**x



From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

Dual conics

A line tangent to the conic **C** satisfies $\mathbf{l}^{\mathsf{T}}\mathbf{C}^{*}\mathbf{l} = 0$

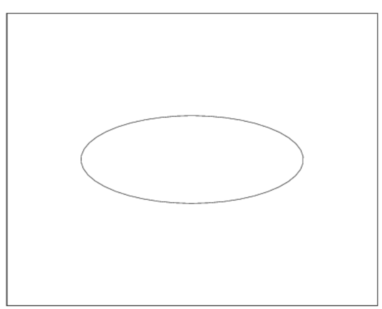
$$\mathbf{l} = \mathbf{C}\mathbf{x} \;\Rightarrow\; \mathbf{x} = \mathbf{C}^{-1}\mathbf{l}$$

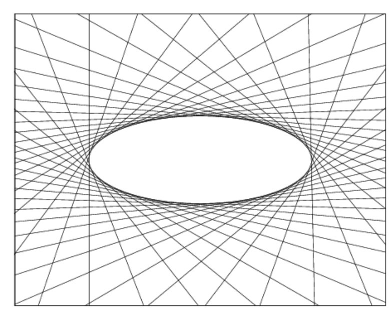
$$\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x} = \mathbf{0} \;\Rightarrow\; \left(\mathbf{C}^{-1}\mathbf{l}\right)^{\mathsf{T}}\mathbf{C}\left(\mathbf{C}^{-1}\mathbf{l}\right) = \mathbf{0}$$

$$\Rightarrow\; \mathbf{l}^{\mathsf{T}}\left(\mathbf{C}^{-1}\right)^{\mathsf{T}}\mathbf{C}\mathbf{C}^{-1}\mathbf{l} = \mathbf{0} \;\Rightarrow\; \mathbf{l}^{\mathsf{T}}\mathbf{C}^{-\mathsf{T}}\mathbf{l} = \mathbf{0}$$

As **C** is symmetric, $\mathbf{C}^{-1}$

Dual conics = line conics = conic envelopes





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Degenerate Conics

- Rank of C <3
- Rank 2 $\rightarrow$ Two lines / points
- Rank 1 $\rightarrow$ One repeated lines / points
- Degenerate point conic:

  $\mathbf{C}=\mathbf{l}.\mathbf{m}^{T}+\mathbf{m}.\mathbf{l}^{T}$ rank 2, if $\mathbf{l} <> \mathbf{m}$
- Degenerate dual line conic:

  $\mathbf{C}^{*}=\mathbf{x}.\mathbf{y}^{T}+\mathbf{y}.\mathbf{x}^{T}$ rank 2, if $\mathbf{x} <> \mathbf{y}$

- $\mathbf{x}^T\mathbf{l}=0$, and $\mathbf{l}^T\mathbf{x}=0$
- $\mathbf{x}=\mathbf{l} \times \mathbf{l}'$, and $\mathbf{l}=\mathbf{x} \times \mathbf{x}'$

Summary

- A point in a 2-D projective space represents a ray passing through origin of an implicit 3D space.
  - Requires additional dimension for representation.
    - Homogeneous Coordinate Representation
- Straight lines in R^2 are elements of a 2D projective space.
- Points and lines hold duality theorem.
- Conics are represented by a 3x3 symmetric matrix.
  - Every conic has a dual conic or line conic as an envelop of its tangents.

Thank you!