Notes on percolation theory

1 Lattice percolation problems

As we already know, lattice percolation problems were first discussed by S. Broadbent and J. Hammersley[1], and the discussion was motivated by a practical application (construction of effective gas masks for mine workers). Since then percolation problems were widely studied in physical and in mathematical literature, and can be considered as understood quite in detail [2]. The two standard problems of percolation theory correspond to Bernoulli bond and site percolation.

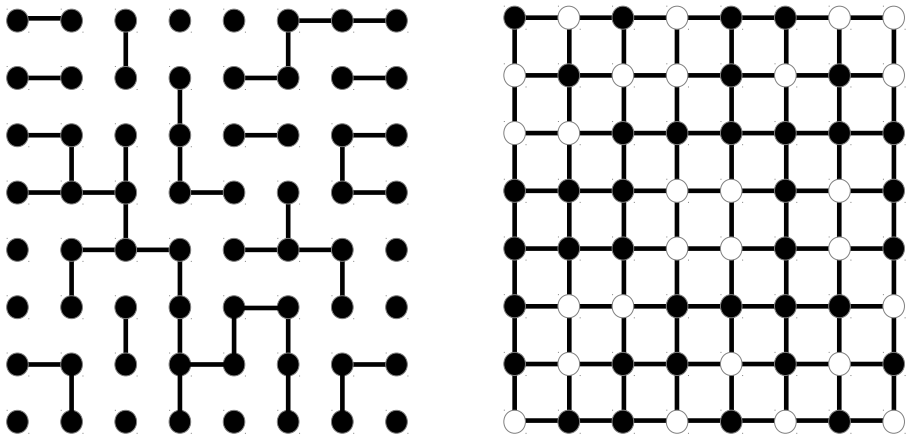

Figure 1: Two typical percolation problems: the bond problem (left) and the site problem (right). In the bond problem sites are always present, and a bond can be either present (or intact) with probability p, or absent (or broken) with probability $1-p$. In the site problem the bonds are always there, but the sites, representing the contacts between the bonds, may be intact with probability p or absent (blocked) with probability $1-p$.

The name Bernoulli stands here for the situations where bond or sites

[1]Percolation problems were first formulated in the work S.R. Broadbent and J.M. Hammersley, Proc. Cambridge Phil. Soc. **53** 629 (1957).

[2]A standard physical account on percolation theory is given in a book A. Aharony and D. Stauffer, *Introduction to Percolation Theory.*, Taylor and Francis, 1991, 1994

are present or absent independently on each other with a given probability p, which uniquely defines the probability space for the realizations of the percolation systems. The percolation transition corresponds to appearence of the *infinite cluster*[3] of intact connected sites, which contains ways connecting two opposite sides of a thermodynamically large system (say, in a form of a square or of a cube). These ways are the ones through which a fluid or electric current can flow (percolate) through the system. For $p = 0$ such a cluster (and such a way) is absent, and the system does not percolate, and for $p = 1$ all sites belong to the infinite cluster, and the system percolates. The percolation transition corresponds to some critical concentration $p_c \leq 1$ of intact sites (bonds).

First, it is not hard to show that the density of the infinite cluster is a non-decaying function of p. Therefore, if the cluster exists at some concentration p, it cannot disappear at higher concentrations. The uniqueness of the transition follows from the Kolmogorov's zero-one law for the probabilities of tail events. A tail event is the event which is probabilistically independent on each finite subset of random variables, in our case on switching on or off of the finite number of bonds or sites (this again can be proved). The probability that under given conditions an infinite cluster does exist is therefore 0 or 1. The statement is that under given conditions such a cluster either exists or not, almost surely, but doesn't tell us, under what conditions. Since for $p = 0$ the system does not percolate (percolation cluster is absent) and at $p = 1$ it does, there must be a transition somewhere [4].

1.1 Percolation concentrations

Now let us believe that the critical concentration p_c does exist and is uniquely defined (in thermodynamical limit). For $p < p_c$ the system is not conducting (no percolation over intact bonds or sites); for $p > p_c$ the system is conducting (it percolates). The critical concentrations for some common lattices are given in the Table 1.1. For these simple lattices only four results (denoted by asterisks) are exact, all other follow from numerical simulations. In the next paragraph we shortly discuss the three results for the two-dimensional bond problem, as following by duality arguments [5].

[3] A cluster is simply a set of connected sites

[4] For true mathematical proofs look e.g. in G. Grimmett, *Percolation*, Springer, 1999.

[5] The exact result for a site problem on a triangular lattice follows by so-called matching lattices argument, closely connected to duality.

Table 1.1: Some percolation concentrations

	Lattice	Z	p_c^{site}	p_c^{bond}
Two-dimensional	Honeycomb	3	0.6962	0.65271...*
	Quadratic	4	0.59275	0.5*
	Triangular	6	0.5*	0.34729...*
Three-dimensional	Diamond	4	0.43	0.388
	SC	6	0.3116	0.2488
	BCC	8	0.246	0.1803
	FCC	12	0.198	0.119
Hyper-cubic	$d = 4$	8	0.197	0.1601
	$d = 5$	10	0.141	0.1182
	$d = 6$	12	0.107	0.0942
	$d = 7$	14	0.089	0.0787

From the table we see that for all lattices with well-defined number of neighbors Z one has $p_c Z > 1$ (this inequality indeed can be proved). The following *approximate* relation holds for bond percolation in d dimensions:

$$p_c Z \approx \frac{d}{d-1}$$

(a Scher-Zallen "invariant"). An analogous approximate relation for site percolation involves the total volume of touching spheres drawn around the sites. At percolation concentration the portion of the total volume inside the spheres drawn around the intact sites is approximately the same for all simple d-dimensional lattices (but depends on d).

1.1.1 The results for the bond model on square, triangular, and honeycomb lattices

In the lattices shown in Fig.1 the dual lattices can be defined by connecting the centers of the neighboring cells. In the figure, the bonds of the dual lattice cut those of the initial lattice in their middle. If the original lattice percolates, there is a way in this lattice, say, from top to bottom, and no way over the dual lattice from left to right might be present. Therefore, both lattices cannot percolate simultaneously. Therefore $p_c + q_c \geq 1$. Here p_c is

the percolation concentration on the initial lattice and q_c is the percolation concentration on the dual one. This inequality is however a strict equality:

$$p_c + q_c = 1, \tag{1}$$

absence of percolation via initial lattice *implies* percolation in the dual one and vice versa [6].

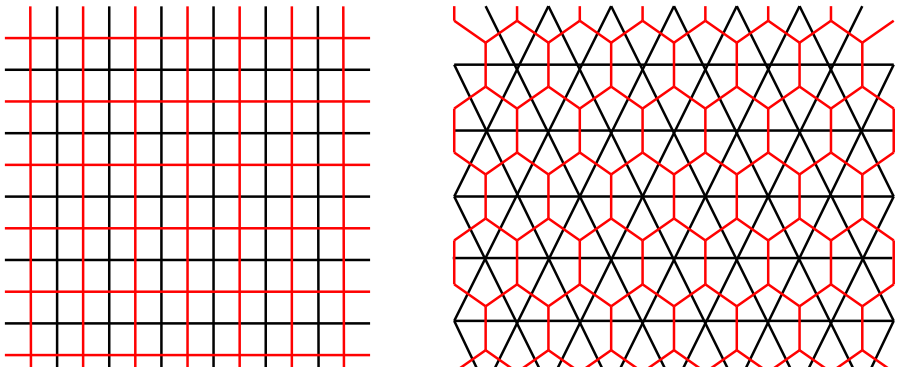

Figure 2: Simple planar lattices and their dual ones.

The square lattice is self-dual: $p_c = q_c$ and thus, according to Eq.(1) $p_c = 1/2$.

The triangular and the honeycomb lattices are dual to each other. Therefore $p_c^t + p_c^h = 1$.

For calculating the conductivity (or simply stating the presence or absence of the percolation) in the triangular lattice this can be considered as an arrangement of three-pole circuits (triangles) connected with each other at their poles (note that in this representation all triangles have the same orientation, say, with the vertex up).

Let us chose a vertex of a honeycomb lattice, and consider its three neighbors as poles of a three-pole circuit, now represented as a star. Performing

[6] Although this sounds trivial, and was announced and used by M. F. Sykes and J. W. Essam in J. Math. Phys. **5**, 1117 (1964), mathematicians needed almost 20 years to give a full proof, see H. Kesten, Comm. Math. Phys. 74, 41-59 (1982)!

the triangle-star-transformation, we can represent the stars as triangles, and return to the triangular lattice (shifted or mirrored with respect to the initial one).

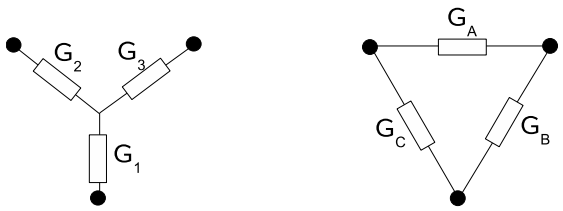

Figure 3: In electrotechnics each passive three-terminal device can be equivalently presented as a star-like or as a triangle-like circuit.

Since the absence or presence of the percolation depends only on the probability with which each site (say the lower site of the triangle or of the star) is connected with none, one or both other sites, let us express these probabilities (ϕ_0, ϕ_1 and ϕ_2, respectively) via the probability p that a side of a triangle is intact, or probability q that the ray of a star is intact. These probabilities are given in the next table:

ϕ_i	Δ	Y
ϕ_0	$(1-p)^2$	$(1-q)+q(1-q)^2$
ϕ_1	$2p(1-p)^2$	$2q^2(1-q)$
ϕ_2	$p^3+3p^2(1-p)$	q^3

Since at a critical concentration $q = p_c^h$ for the honeycomb lattice, its triangular representation also has to be at a critical concentration p_c^t, which, according to Eq.(1) is equal to $p_c^t = 1-q$, we get from the expression for ϕ_0 with $q = 1-p$ the equation for the critical concentration

$$(1-p)^2 = p + (1-p)p^2$$

i.e.

$$1 - 3p + p^3 = 0.$$

The equations for ϕ_1 and ϕ_2 lead to the same equation for the critical concentration, so that all three are fulfilled simultaneously.

The cubic equation can be explicitly solved using the Cardano formula leading to three real roots, from which only one lies in (0,1). This one can be expressed as

$$p_c^t = 2 \sin \frac{\pi}{18},$$

which was one of the results of Sykes and Essam's work cited above (to see this express p^3 through the trigonometrical function of a triple angle). The critical concentration on the honeycomb lattice is therefore

$$p_c^h = 1 - 2 \sin \frac{\pi}{18}.$$

1.2 Scaling at percolation threshold

Below percolation threshold only finite clusters of sites connected by intact bonds (in bond percolation) or only finite clusters of intact sites (in site percolation) exist. Above percolation concentration an infinite cluster of corresponding sites appears; there exists a way through the whole system which runs over this infinite cluster. Percolation implies the existence of such an infinite path. If the percolation cluster exists (on "normal" lattices in d dimensions), it is (almost surely) unique[7].

Looking at the percolation system at $p > p_c$, we see that it consists of the infinite cluster (a large connected part of the lattice taking the finite portion of all sites) with holes, in which the smaller clusters sit. A portion of the lattice sites belonging to the infinite cluster is called its density P_∞. The density of an infinite cluster is the probability that a site picked up at random (is intact and) belongs to this cluster. In what follows we discuss the bond problem (the only difference to the site problem is that the words "is intact and ..." are omitted).

Below percolation threshold the infinite cluster doesn't exist, and $P_\infty = 0$. Above the threshold but close to it the density of the infinite cluster behaves as a power law

$$P_\infty \propto (p - p_c)^\beta$$

where the values of critical exponent β are known (at least numerically) in all relevant dimensions, see Table 1.2.

[7]This was clear to physicists since late seventies, but a mathematical proof was given considerably later: M. Aizenman, H. Kesten and C.M. Newman, Comm. Math. Phys. **111**, 505 (1987); R.M. Burton and M.S. Keane, Comm. Math. Phys. **121**, 501 (1989)

Below the percolation threshold, only finite clusters exist. Their characteristic size (i.e. the mean gyration radius) defines the *correlation length* ξ in the system. Close to the threshold the typical size of the finite cluster grows (at the threshold the largest ones get connected to form the infinite one) and ξ diverges. Above the threshold ξ is again considered as a mean gyration radius of finite clusters; the infinite cluster is disregarded. In this case the finite clusters sit in "holes" of the infinite one. Since, when approaching the critical point from above, the density of the percolation cluster decays, which means that the holes in it get larger, ξ diverges also on the other side of the transition. On both sides of it the divergence is given by a power law

$$\xi \propto |p - p_c|^{-\nu}$$

with a critical exponent ν.

Let $n_s(p)$ be the probability distribution of the sizes of finite clusters. Then the probability that a site picked up at random belongs to a cluster of s sites is $p_s = sn_s$. Below the percolation threshold a site definitely belongs to one of the finite clusters and therefore $\sum_s sn_s = 1$. Above the threshold $\sum_s sn_s = 1 - P_\infty$. For site percolation the results have to be normalized by p, the probability that the corresponding site is intact.

The mean cluster size is defined as a mean size of a cluster which is picked by choosing one of its sites at random, and is therefore the *second* moment of n_s:

$$S(p) = \sum_{s=1}^{\infty} s^2 n_s(p).$$

This mean size (like the mean gyration radius) diverges on both sides of the transition and behaves as

$$S \propto |p - p_c|^{-\gamma}.$$

The density of the infinite cluster, the correlation length and the mean cluster size in percolation theory are analogues of the order parameter, correlation length and susceptibility in the theory of magnetic phase transitions (and the concentration p plays the role of temperature T)[8].

[8]The analogous behavior of the correlation length is more or less easy to grasp. The order parameter in the magnetic system, the total magnetization, is essentially proportional to the fraction of spins belonging to an infinite cluster of spins of the same orientation. The percolation analogue of susceptibility gets clear after one understands what may play a role of the external field. The role of the external field is that that it provides the

Table 8.2: Critical exponents in percolation and in Ising model

Crit. exponent	$d=1$		$d=2$		$d=3$		$d>d_c$	
	Perc.	Magn.	Perc.	Magn.	Perc.	Magn.	Perc.	Magn.
β	0	0	5/36	1/8	0.41	0.32	1	1/2
ν	1	1	4/3	1	0.88	0.63	1/2	1/2
γ	1	1	43/18	7/4	1.80	1.24	1	1

The values of the corresponding indices differ however from the ones in Ising model, see table below. Note that definition of critical exponents for usual Ising model in 1d poses difficulties since $T_c = 0$. Moreover, the critical dimension for percolation is $d_c = 6$ while for the Ising magnet it is $d_c = 4$.

Let us recuperate what we have learned about percolation clusters (for simplicity we adopt the language of the site model here). Below the percolation concentration there are only finite clusters of intact sites, and their size distribution $n_s(p)$ (the probability to find a cluster of s sites among *all clusters*) can be obtained at least numerically. Above the critical concentration an infinite cluster appears. The density of this cluster P_∞ (i.e. the probability to find a site belonging to it among all sites) scales as $P_\infty \propto (p - p_c)^\beta$ with $0 < \beta \leq 1$. The typical size (gyration radius) of a finite cluster behaves as $\xi \propto |p - p_c|^{-\nu}$ on both sides of the transition. This ξ defines the scale above which the whole system can be considered as homogeneous, i.e. above p_c it also gives the size of the largest holes in the percolation cluster.

Above percolation concentration, the infinite cluster is characterized by finite density, and the number of sites ("mass") of the part of the cluster inside the ball of radius $L \gg \xi$ grows, for $L \gg \xi$ with this radius as

$$N \propto P_\infty L^d,$$

and its density $\rho(L) \simeq P_\infty$ is constant.

preferential orientation to any spin, creating some total magnetization. Now imagine that there is a very special site *outside* of the lattice, a "ghost", which with a very small probability can be connected to any of the lattice sites, by this creating an infinite cluster. This probability will be taken $q = 1 - \exp(-h)$ i.e. $\approx h$ if h is small. The parameter h serves as the percolation analogue of the external field in magnetic problems, and allows for defining the exponents γ and δ. The analogue of the free energy is the combination $F(p,h) = \sum_s n_s \exp(-sh)$. The value of $\tau = p - p_c$ then indeed plays a role of a dimensionless temperature, and allows for determining the exponent α of the "specific heat".

At length scales smaller than ξ the scale dependent density of the percolation cluster $\rho(L)$ measured as the number of sites N within a domain of radius L divided by L^d explicitly depends on L. It was (numerically) found that this dependence corresponds to a power law

$$N \propto L^{d_f},$$

which allows us to define the fractal (mass) dimension of the infinite cluster d_f which is smaller than d. This d_f is connected with critical indices β and ν via a simple scaling argument: The density of the cluster grows at $L < \xi$ as

$$\rho(L) \propto N/L^d = L^{d_f - d}.$$

At $L > \xi$ it stagnates and is equal to $P_\infty(p)$. Exactly a ξ we thus have $\xi^{d_f-d} \simeq P_\infty$. Expressing both ξ and P_∞ as functions of $p - p_c$ we get $(p - p_c)^{-\nu(d_f-d)} = (p - p_c)^\beta$ so that

$$d_f = d - \frac{\beta}{\nu}.$$

We thus see that just at percolation threshold the infinite cluster is fractal (has a dimension smaller than the one of the embedding space), i.e. is an extremely inhomogeneous structure characterized by holes on all scales. Such structures are quite peculiar and worth discussing.

The dependence of the mass on the size of the system is characteristic of the spatial dimension, which can be obtained, e.g., in an experiment by measuring N (or "mass" M) at different sizes L. Thus, for dense objects in d dimensions increasing L by a factor of 2 increases M by a factor 2^d (think about a segment, a square, or a cube), and

$$d = \frac{\ln[M(L_1)/M(L_2)]}{\ln(L_1/L_2)}.$$

We can argue that, instead of cutting parts of different sizes from the infinite system, we can add parts together and define the dimension as an exponent describing the growth of the mass of a finite system put together from smaller but similar parts. For example, a square can be put together from four similar squares of half the size, and therefore its dimension is $d = \ln 4/\ln 2 = 2$, while the dimension of a cube, which can be put together from eight cubes of half the size again, is $d = \ln 8/\ln 2 = 3$. The dimension defined through the mass

of the object (or through the number of similar parts that it is built of) is called the mass dimension.

A simple object in the figure below, built according to the evident rules of putting parts together, is called a *Sierpinski Gasket*. Here you should only look at the two leftmost panels, the two other ones will be discussed later.

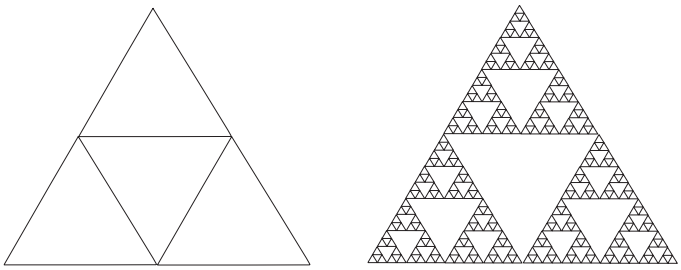

Figure 4: A Sierpinski gasket: (a) its generator (b) a structure after 4th iteration. The lattice is rescaled by the factor of 4 to fit into the picture.

A Sierpinski gasket is constructed from three similar objects of half the size and therefore has a dimension $d = \ln 3/\ln 2 \approx 1.585...$, which is not a whole number. The objects with non-integer (fractional) mass dimensions are called fractals. Mass dimension is one of the family of differently defined fractal dimensions, and is considered as the fractal dimension in what follows[9].

1.3 Percolation on a tree

To gain some intuition on what can happen close to the transition we first discuss the percolation on a tree (a model discussed by polymer scientists even

[9]Mathematicians introduce dimensions and related properties (capacities) by covering procedures, e.g., by counting how the number of boxes, or balls, etc., covering the system, will grow when the boxs or balls size tends to zero. There are mathematical reasons to do so (e.g., to let the dimension be invariant under smooth coordinate transformations), but the differences will not play any role in our discussion below. For the introduction to fractals see: B Mandelbrot, *The fractal geometry of nature*, Freeman, N.Y. 1977 (there are many editions, also later ones).

before the percolation theory was formulated). A tree is a topologically one-dimensional structure, and many problems of phase transitions (including percolation) on a tree are exactly solvable.

1.3.1 Percolation concentration

Percolation on a tree is the oldest percolation model and corresponds to a classical model of a polymer network due to Flory and Stockmayer (1943). Since trees are infinitely dimensional lattices (since $M \propto Z^L$ where L is the length of the path connecting the central site (root) with the periphery (leafs)) they do not grow infinite in any finite dimension (you know what happens with dendrimers!), this is not a very good model for percolation in a three-dimensional world. There are also some differences in the behavior of the percolation system on a tree and in lower dimensions. For example, the number of infinite clusters (if any) is infinite on a tree (it is enough to cut one bond to separate the system into two independent parts) and is one in the "normal" case.

However, the percolation model on a tree is an exactly solvable one. The fact that it is exactly solvable is connected with the absence of the loops which permits us, after defining some (arbitrary) intact site of the tree as its root, to give each bond a direction (from the root outwards) and to classify all sites according to their generation (distance from the root). If Z is the coordination number of the tree, the root possesses exactly Z outgoing bonds and each site other than the root has one incoming and $Z-1$ outgoing bonds.

Note also that for a tree the bond and the site problems are equivalent: the absence of the site is the same as the absence of the incoming bond (the last one on the way from the root to this site). Let P_∞ be the probability that the root belongs to an infinite cluster (i.e. that there is an infinite way starting at a root): due to the arbitrariness of our choice for the root, P_∞ is exactly the density of (all) infinite cluster(s). An infinite way starting at the root only exists if there is an infinite way from one of Z neighbors of the root. Denoting the probability that a given site other then the root possesses an infinite way starting on it by Q we get:

$$1 - P_\infty = (1 - pQ)^Z.$$

(the probability that there is no infinite way from the root is the probability that there is no infinite way starting with each of its outgoing bonds). The outgoing bond starts an infinite way if the site it is pointing at is intact, and

there is an infinite way starting from the site it is showing to. If $Q = 0$ then P_∞ vanishes as well, and if $Q > 0$ the value of P_∞ is nonzero and positive. Similar discussion also holds for any site other then the root, with the only difference that there are not Z but $Z - 1$ outgoing bonds.

$$1 - Q = (1 - pQ)^{Z-1}.$$

Expanding the binomial in the right hand side of this equation we get

$$Q = p(Z-1)Q - p^2\frac{(Z-1)(Z-2)}{2}Q^2 + ...$$

This equation always possesses a trivial solution $Q = 0$, but may also possess a nontrivial one. To look for this we note that Q is always non-negative and that close to the critical concentration Q is small (it is enough to keep the terms up to the second order, just as written above). We thus get

$$Q = 2\frac{p(Z-1)-1}{p^2(Z-1)(Z-2)},$$

and the non-negative solution only exists for $p(Z-1)-1 > 0$, i.e. formulation

$$p > p_c = \frac{1}{Z-1}.$$

Note that close to p_c the value of Q goes approximately as $Q \simeq const \cdot (p - p_c)$ and thus also

$$P_\infty \simeq const_1 \cdot (p - p_c)$$

(to see this just expand the binomial in the equation for P_∞). Thus, on a tree we have $\beta = 1$.

1.3.2 The correlation length

Let us now concentrate on the correlation function $g(r)$, giving the mean number of sites at the distance r from the root which are connected with the root by a path over intact sites (or bonds). For the tree this can be easily calculated in the *chemical space* (where the distance r is measured as the number of steps (bonds) of the path connecting each two points; on a tree there is exactly one such path):

$$g(r) = Z(Z-1)^{r-1}p^r$$

where $Z(Z-1)^{r-1}$ is exactly the number of paths of length r. This gives us the possibility to calculate the correlation length in the chemical space

$$\tilde{\xi}^2(p) = \frac{\sum_{r=0}^{\infty} r^2 g(r)}{\sum_{r=0}^{\infty} g(r)}$$

(at least below p_c when both sums converge). The trick used for such a calculation is quite standard: we note that

$$\sum_{r=0}^{\infty} g(r) = \frac{Z}{Z-1} \sum_{r=0}^{\infty} [(Z-1)p]^r = \frac{Z}{Z-1} \frac{1}{1-(Z-1)p}$$

and denote $(Z-1)p = x$ so that

$$\sum_{r=0}^{\infty} g(r) = \frac{Z}{Z-1} \frac{1}{1-x}$$

We moreover note that

$$\sum_{r=0}^{\infty} r^2 g(r) = \frac{Z}{Z-1} \sum_{r=0}^{\infty} r^2 [(Z-1)p]^r$$

so that

$$\begin{aligned} \sum_{r=0}^{\infty} r^2 g(r) &= \sum_{r=0}^{\infty} r^2 x^r = \frac{d}{dx} x \frac{d}{dx} \frac{Z}{Z-1} \sum_{r=0}^{\infty} [x]^r \\ &= \frac{Z}{Z-1} \frac{d}{dx} x \frac{d}{dx} \frac{1}{1-x} = \frac{Z}{Z-1} \frac{1+x}{(1-x)^3}. \end{aligned}$$

We finally get

$$\begin{aligned} \tilde{\xi}^2(p) &= \frac{1+x}{(1-x)^2} = \frac{1+(Z-1)p}{[1-(Z-1)p]^2} \\ &= (Z-1)^{-1} \frac{(Z-1)^{-1}+p}{[(Z-1)^{-1}-p^2]} = p_c \frac{p_c+p}{(p_c-p)^2}. \end{aligned}$$

The same expression holds essentially also on the other side of transition (note that in this case the infinite cluster has to be disregarded). Thus, $\tilde{\xi} \propto |p-p_c|^{-1}$. In the (infinitely dimensional) Euclidean space in which the tree lives each such path corresponds to a trajectory of a random walk, and its "real" length ξ scales as a root of its chemical length and therefore $\xi \propto \tilde{\xi}^{1/2} \propto (p-p_c)^{1/2}$, so that $\nu = 1/2$.

1.4 Cluster size distribution

Let us consider a cluster with s sites. For this cluster its t *perimeters sites* (i.e. immediate neighbors of the cluster sites not belonging to the cluster) has to be blocked, otherwise the cluster would be larger. Therefore the probability to find such a cluster is

$$n_s = N_{s,t} p^s (1-p)^t,$$

where $N_{s,t}$ is the number of different geometric configurations of a cluster with s sites and t perimeter sites (such different configurations are called *lattice animals*). On a whatever lattice other than a tree calculation of t and $N_{s,t}$ is a hard job. On a tree t is a simple function of s:

$$t(s) = Z + (Z-2)(s-1).$$

To see this it is enough to note that $t(1) = Z$ and that

$$t(s+1) = t(s) - 1 + (Z-1) = t(s) + (Z-2)$$

(passing from a cluster with s sites to a one with $s+1$ sites we change one of the perimeter sites for an internal one and add $Z-1$ new perimeter sites to block its neighbors), and to use mathematical induction. Therefore $N_{s,t}$ is essentially a function of s only. This will be denoted by N_s in what follows.

Therefore

$$n_s = N_s p^s (1-p)^{Z+(Z-2)(s-1)} = N_s (1-p)^2 [p(1-p)^{Z-2}]^s.$$

Let us assume that exactly at p_c the value of n_s

$$n_s(p_c) = N_s (1-p_c)^2 [p_c(1-p_c)^{Z-2}]^s$$

is known (and follows for large s a power law $n_s(p_c) \propto s^{-\tau}$, which can be verified later). Then, at any other concentration than p_c we can write

$$n_s(p) = N_s (1-p)^2 [p(1-p)^{Z-2}]^s = n_s(p_c) \frac{(1-p)^2 [p(1-p)^{Z-2}]^s}{(1-p_c)^2 [p_c(1-p_c)^{Z-2}]^s}$$

i.e. we see that $n_s(p) = n_s(p_c) f_s(p)$ where the function

$$f_s(p) = \frac{(1-p)^2}{(1-p_c)^2} \left[\frac{p(1-p)^{Z-2}}{p_c(1-p_c)^{Z-2}} \right]^s$$

tends to unity for $r \to p_c$ for any s. Now we substitute $p_c = 1/(Z-1)$ take $p = p_c + \delta p = (Z-1)^{-1} + \delta p$, and expand the function in the square brackets in powers of δp. We get

$$f_s(p) \simeq 1 - \frac{1}{2}\frac{(Z-1)^3}{Z-2}(\delta p)^2 = 1 - \frac{1}{2p_c^2(1-p_c)}(p-p_c)^2.$$

(the first oder in δp vanishes). Therefore close to p_c and for s large

$$f_s(p) \simeq \left[1 - \frac{1}{2p_c^2(1-p_c)}(p-p_c)^2\right]^s \approx \exp\left[-\frac{(p-p_c)^2}{2p_c^2(1-p_c)}s\right],$$

a strongly decaying (exponential) function. This shows that the cluster size distribution at p follows the one at p_c for smaller clusters, but has a cutoff at the maximal cluster size $s_c \propto |p-p_c|^{-2}$.

Now we use the fact that the mean size of a finite cluster

$$S(p) \simeq \sum_{s=1}^{\infty} s \cdot s n_s(p),$$

approximate the sum by the integral and calculate it using our expression for $n_s(p) \propto s^{-\tau}\exp[-\text{const} \cdot (|p-p_c|^2 s)]$:

$$S(p) \simeq \int_1^{\infty} s^{2-\tau} e^{-\text{const}\cdot(|p-p_c|^2 s)} ds \approx \int_0^{\infty} s^{2-\tau} e^{-\text{const}\cdot(|p-p_c|^2 s)} ds.$$

We may evaluate the integral explicitly or simply note that the exponential introduces an effective upper cutoff (i.e. the effective upper bound of integration at $s \simeq |p-p_c|^{-2}$):

$$S(p) \simeq \int_0^{|p-p_c|^{-2}} s^{2-\tau} \propto |p-p_c|^{2(\tau-3)}.$$

On the other hand,

$$S(p) = 1 + \sum_{r=1}^{\infty} g(r) = p_c \frac{1+p}{p_c - p} \propto (p_c - p)^{-1}$$

(at least below the percolation concentration). Therefore $2(\tau-3) = 1$ i.e.

$$\tau = \frac{5}{2},$$

so that for larger clusters

$$n_s(p) \propto s^{-5/2} e^{-\mathrm{const}\cdot(|p-p_c|^2 s)}.$$

There is strong numerical evidence that also in general

$$n_s(p) = s^{-\tau} f_{\pm}\left(|p-p_c|^{1/\sigma} s\right) \tag{2}$$

where the values of critical exponents τ and σ may depend on the lattice (essentially only on its dimensionality), and the functions f may differ below and above transition. In our case

$$\sigma = \frac{1}{2}.$$

Now we concentrate on the site problem (as we already know, the problems slightly differ in normalization constants). The total balance of probabilities is given by

$$1 = (1-p) + P_\infty + p\sum_s s n_s(p)$$

(the first term in the r.h.s. is the probability that the site is broken, the second one is the probability that it is intact and belongs to an infinite cluster, and the third one corresponds to the probability that it is intact and belongs to a finite cluster. Therefore

$$P_\infty = p - p\sum_s s n_s(p),$$

so that

$$\frac{P_\infty}{p} = 1 - \sum_s s n_s(p). \tag{3}$$

(for the bond problem there will be no p in the denominator in the l.h.s.). Exactly at p_c the density P_∞ vanishes, and

$$0 = p_c - p_c\sum_s s n_s(p_c),$$

so that $1 = \sum_s s n_s(p_c)$. Changing unity for this expression in Eq.(3) we get

$$\begin{aligned} \frac{P_\infty}{p} &= \sum_s s[n_s(p_c) - n_s(p)] = \sum_s s n_s(p_c)[1 - f_s(p)] \\ &\simeq \int_0^\infty s s^{-\tau}(1 - e^{-\mathrm{const}\cdot(|p-p_c|^2 s)})ds. \end{aligned}$$

Introducing the new integration variable $z = |p - p_c|^2 s$ and assuming that the integral in

$$\frac{P_\infty}{p} \simeq |p - p_c|^{2(\tau-2)} \int_0^\infty s s^{-\tau} (1 - e^{-\text{const}\cdot(|p-p_c|^2 s)}) ds$$

converges, we compare now this expression with our previous one for $P_\infty \propto (p - p_c)^1$, to obtain the same value of τ as before: $2\tau - 4 = 1$, i.e. $\tau = 5/2$. In general for σ different from $1/2$ and for β different from one we will get

$$\beta = \frac{\tau - 2}{\sigma},$$

an expression of general validity. Haven learned about cluster properties in a tree and assuming that the overall behavior of the cluster size distribution is always given by Eq.(2) we can turn to systems other then the tree and obtain general relations between the critical exponents.

2 Scaling theory of percolation clusters

The expression for $n_s(p)$, Eq.(2) allows to obtain the moments of the cluster sizes:

$$\begin{aligned} M_k &= \sum_{s=1}^{\infty} s^k n_s(p) = \sum_{s=1}^{\infty} s^{k-\tau} f_\pm(|p - p_c|^{\frac{1}{\sigma}} s) \\ &\sim \sum_{s=1}^{|p-p_c|^{-\frac{1}{\sigma}}} s^{k-\tau} \simeq \int_1^{|p-p_c|^{-\frac{1}{\sigma}}} s^{k-\tau} ds \\ &\simeq s^{k+1-\tau} \Big|_1^{|p-p_c|^{-\frac{1}{\sigma}}} . \end{aligned}$$

Depending on k (which does not have to be a whole number) this expression can either diverge for $p \to p_c$ or converge to a constant. Thus, for $k > \tau - 1$

$$M_k \propto |p - p_c|^{-\frac{k+1-\tau}{\sigma}}$$

and for $k < \tau - 1$

$$M_k = \text{const} - A|p - p_c|^{-\frac{k+1-\tau}{\sigma}}$$

with A being some prefactor. The first moment

$$M_1 = \sum_{s=1}^{\infty} s n_s(p) = \text{const} - A|p - p_c|^{-\frac{2-\tau}{\sigma}}$$

Comparing this with the expression (3) from the previous section leading us to

$$\sum_s s n_s(p) = 1 - P_\infty$$

(for the bond problem; for the site problem the r.h.s. will be $1 - P_\infty/p$) gives us the value of the constant and the expression for β already discussed. The second moment M_2 giving

$$M_2 = \sum_{s=1}^{\infty} s^2 n_s(p) \propto |p - p_c|^{-\frac{3-\tau}{\sigma}}$$

defines the value of the critical exponent γ:

$$\gamma = \frac{3-\tau}{\sigma}.$$

Our next aim will be to connect the exponent ν of the correlation length with τ and σ. On one hand, ξ (at least below p_c) is defined as

$$\xi^2(p) = \frac{\sum_{r=0}^{\infty} r^2 g(r)}{\sum_{r=0}^{\infty} g(r)},$$

on the other hand, as the mean gyration radius of the finite cluster, it can be obtained via

$$\xi^2 \simeq \frac{\sum_{r=0}^{\infty} r^2(s) s^2 n(s)}{\sum_{r=0}^{\infty} s^2 n(s)}$$

where $r^2(s)$ is a mean squared distance between the two sites of a cluster of s sites, and $s^2 n(s)$ is proportional to the probability that the two sites belong to the same cluster. Assuming that large finite clusters have the same fractal geometry as the infinite one we can assume that the geometrical size of the cluster

$$r(s) \propto s^{1/d_f}$$

so that

$$\xi^2 = \frac{M_{2+2/d_f}}{M_2} \simeq |p - p_c|^{-\frac{2}{d_f \sigma}}$$

Therefore

$$\nu = \frac{2}{d_f \sigma} = \frac{2}{(d - \beta/\nu)\sigma}.$$

In the last equation the relation $d_f = d - \beta/\nu$ was used. Resolving this last equation as an equation for ν we get

$$\nu = \frac{\tau - 1}{d\sigma}.$$

Combining this relation with the ones for β and γ we get

$$d\nu = 2\beta + \gamma.$$

Introducing the "classical" values of critical exponents (the one for the tree) $\beta = 1$, $\gamma = 1$ and $\nu = 1/2$ we obtain that these ones correspond to $d = 6$ which is the upper critical dimension for percolation problems.