EE445 DIGITAL SIGNAL PROCESSING

SOLUTIONS

1 (a) $H(z) = \dfrac{0.3 - 0.7z^{-1} + 0.5z^{-2}}{1 + 0.3z^{-1} - 0.4z^{-2}}$

Difference equation:

$y(n) = 0.3x(n) - 0.7x(n-1) + 0.5x(n-2) - 0.3y(n-1) + 0.4y(n-2)$ 1

Input signal : $x(n) =$ $u(n)\left[-0.8 + 0.5^n\right]$

First five samples

$= \{0.2, -0.3, -0.55, -0.675, -0.7375\}$ 1½

Tabular format to show calculations

Output $y(n) = \{0.06, -0.248, 0.2434, -0.1397\}$ 3½

(b) $H(\theta) = H(z)\big|_{z=e^{j\theta}}$

$H(z) = 0.4 + 0.6z^{-1} + 0.4z^{-2}$ 1

$H(\theta) = 0.4 + 0.6e^{-j\theta} + 0.4e^{-j2\theta}$

$= e^{-j\theta}\left[0.4e^{j\theta} + 1 + 0.4e^{-j\theta}\right]$

$= e^{-j\theta}\left[1 + (0.4)2\cos\theta\right]$

$= e^{-j\theta}\left[1 + 0.8\cos\theta\right]$ 2

$|H(\theta)| = 1 + 0.8\cos\theta$

$\angle H(\theta) = -\theta$ Note: linear phase response ½

$\frac{F_s}{4} \Rightarrow \theta = \frac{\pi}{2}$

$|H(\theta)|_{\theta=\frac{\pi}{2}} = 1 + 0.8\cos\left(\frac{\pi}{2}\right) = 1$ ½

$\angle H(\theta)\big|_{\theta=\frac{\pi}{2}} = -\frac{\pi}{2}$ ½

Group delay = 1 sample ½

(c) Time-domain convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- show calculation of y(n)

$$y(n) = \{3, 10, 4, 14, 5, -3\}$$ 3

Check length: 3+4−1 = 6 samples ½

- details for y(2) 1

- coeffs not symmetric (equal to samples of h(n)) ⇒ non-linear phase ½

(d) Aliasing

Nyquist rate = 2 × 450 Hz = 900 Hz 1

Actual rate = 800 Hz

$\frac{f_s}{2}$ = 400 Hz

Components in sampled signal: 150, 300, 350 1

aliased component 1

In radians: 0.7854, 2.3562, 2.7489 1

$\frac{2\pi f}{800}$

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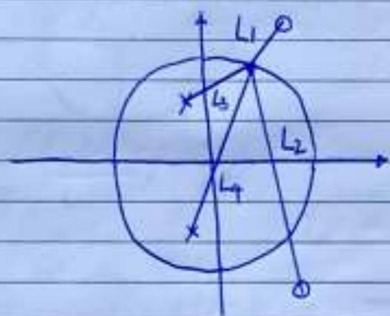
2. (a) Convolution property of z-transform 1

- z-transforms of $h(n)$ and $x(n)$ 1
- z-transform of output 3
- inverse z-transform

$$y(n) = \{3, 4, -9, 4, 10, -5, -5, 6\}$$

starting @ $n = 1$ 1

(b)



$f_x = 100\,Hz$ 1

$\theta_x = 2\pi \frac{100}{500} = 0.4\pi$ 1

$\cos(0.4\pi) + j\sin(0.4\pi)$
$= 0.309 + j0.951$ on the unit circle 1

$$|H(\theta_x)| = \frac{L_1 L_2}{L_3 L_4}$$

$L_1 = 0.7365$
$L_2 = 2.4997$
$L_3 = 0.6183$
$L_4 = 1.6324$ 1

$$|H(\theta_x)| = 1.8242$$

Minimum phase: reflect zero in the unit circle 1

$z_1 = 0.8 + j1.5 = 1.7e^{j1.0808}$

$r_2 = \frac{1}{r_1} = 0.5882$ 1

New zeros @ $0.5882\,e^{\pm j1.0808}$

$= 0.2768 \pm j0.519$ 1

(c) Resonator design

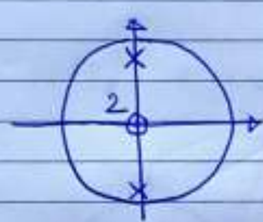
$$\theta_0 = 2\pi \frac{100}{400} = \frac{\pi}{2}$$

$$r \approx 1 - \frac{\Delta f}{f_s}\pi = 0.8429$$ 1

$$b_1 = -2r\cos\theta_0 = 0 \qquad \text{DC gain} = \frac{1}{1+b_1+b_2}$$

$$b_2 = r^2 = 0.7105 \qquad = 0.5846$$ 1

$$H(z) = \frac{1.7105}{1 + 0.7105z^{-2}}$$



$$y(n) = 1.7105\,x(n) - 0.7105\,y(n-2)$$ 1

1

1

We want the gain of the resonator @ 100Hz

$$\theta = \frac{\pi}{2}$$

$$H(\theta) = \frac{1.7105}{1+0.7105e^{-j2\theta}}$$ 1

$$H(\theta)\Big|_{\theta=\frac{\pi}{2}} = \frac{1.7105}{1+0.7105(-1)} = \frac{1.7105}{0.2895}$$

$$= 5.9085$$ 1

if $|x(n)| = 1.6$, then $|y(n)| = 1.6(5.9085)$

$$= 9.4536$$ 1

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3. (a) Notch filter design

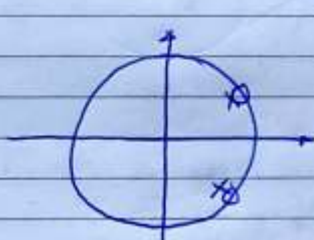
$$\theta_0 = 2\pi \frac{50}{500} = 0.2\pi$$ 1

Poles: $r \approx 1 - \frac{\Delta f}{f_s}\pi = 1 - \frac{25}{500}\pi = 0.8429$

$$b_1 = -2r\cos\theta_0 = -1.3638$$
$$b_2 = 0.7105$$ 1

Zeros: $a_1 = -2\cos\theta_0 = -1.618$
$$a_2 = 1$$

$$H(z) = \frac{1 - 1.618z^{-1} + z^{-2}}{1 - 1.3638z^{-1} + 0.7105z^{-2}}$$ 1



$$y(n) = x(n) - 1.618x(n-1) + x(n-2)$$ 1
$$+1.3638y(n-1) - 0.7105y(n-2)$$ 1

(b) $T_{win} = 30$ msec
$\Rightarrow N_{WIN} = (30 \text{ msec})(20 \text{ kHz})$
$= 600$ samples 1

$$\Delta f = \frac{f_s}{N} \leq 10 \text{ Hz} \Rightarrow N \geq \frac{f_s}{10} = 2000$$

Choose $N = 2048$
$\Rightarrow$ require zero-padding of 1448 samples

MPYs/frame $2N\log_2 N = 45056$ 1

10 seconds of signal $\Rightarrow$ (20 kHz)10 = 200,000 samples

w/out overlapping $\Rightarrow$ 333.33 frames 1

50% overlap $\Rightarrow$ 666.67 frames
Round to 667

$\Rightarrow$ total MPYs = 30,052,352 1

Hamming window:

Width of main lobe $= \frac{8\pi}{N}$ 1

$N = 2048 \Rightarrow$ width $= 0.0123$ rads 2

(c) FIR filter

linear phase $\Rightarrow$ 128 unique coeffs

Each output sample requires 128 MPY
256 ADD 1

10 seconds requires: 10(48kHz)128
= 61.44 million MPYs
122.88 " ADDs 1

Fast convolution: computation/frame

N = (10ms)(48kHz)		= 480. zero pad to <u>512</u>		2
Window	N	512		
FFT	$2N\log_2 N$	9216		
$H(\theta) \times X(\theta)$		1024	conjugate symmetry	1
IFFT		<u>9216</u>		
		19,968		1

No. of frames = 2,000
$\Rightarrow$ total MPY $\doteq$ 39.936 million MPY 1

Saving $= 1 - \frac{39.936}{61.44} = 0.35$ ie 35% 1

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4. (a) $H(s) = \frac{3}{(s+5)(s+4)} = \frac{A}{s+5} + \frac{B}{s+4}$

$A = (s+5)H(s)\big|_{s=-5} = \frac{3}{s+4}\Big|_{s=-5} = -3$

$B = (s+4)H(s)\big|_{s=-4} = \frac{3}{s+5}\Big|_{s=-4} = 3$

$\Rightarrow H(s) = \frac{-3}{s+5} + \frac{3}{s+4}$ 1

IIT $\frac{1}{s+a} \longrightarrow \frac{1}{1-e^{-aT}z^{-1}}$

$\Rightarrow H(z) = \frac{-3}{1-e^{-5T}z^{-1}} + \frac{3}{1-e^{-4T}z^{-1}} = \frac{-3(1-e^{-4T}z^{-1}) + 3(1-e^{-5T}z^{-1})}{1-e^{-4T}z^{-1}-e^{-5T}z^{-1}+e^{-9T}z^{-2}}$ 1

$= \frac{\cancel{-3}-3e^{-4T}z^{-1}\cancel{+3}-3e^{-5T}z^{-1}}{1-(e^{-4T}+e^{-5T})z^{-1}+e^{-9T}z^{-2}}$

$= \frac{-3(e^{-4T}+e^{-5T})z^{-1}}{1-(e^{-4T}+e^{-5T})z^{-1}+e^{-9T}z^{-2}}$ 1

$f_s = 10 \times \left(\frac{5}{2\pi}\right) = 7.958 \text{ Hz}$

$\Rightarrow T = 0.126 s$

$\Rightarrow H(z) = \frac{-3(1.1367)z^{-1}}{1-1.1367z^{-1}+0.3217z^{-2}} = \frac{-3.4101z^{-1}}{1-1.1367z^{-1}+0.3217z^{-2}}$ 1

Add zero @ $z=0$ to normalise orders of numerator and denominator

$H(z) = \frac{-3.4101}{1-1.1367z^{-1}+0.3217z^{-2}}$ 2

(b) $H(s) = \frac{\omega_c}{s+\omega_c}$

Target cutoff frequency:

$\theta_c = 2\pi \frac{3}{10} = 0.6\pi$ 1

Pre-warp: $\omega_c = \frac{2}{T}\tan\left(\frac{\theta_c}{2}\right)$

$= 2(10\text{kHz})\tan(0.3\pi)$

$= 27{,}527.6 \text{ rad/s}$ 1

Bilinear Transform:

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = \frac{w_c(1+z^{-1})}{(2f_s+w_c)+(w_c-2f_s)z^{-1}}$$

$$= \frac{27527.6\,z^{-1}}{47528+7527.6z^{-1}} \qquad 1$$

$$= \frac{0.5792z^{-1}}{1+0.1584z^{-1}}$$

Without pre-warping:

$$\theta_d = 2\tan^{-1}\left(\frac{w_cT}{2}\right)$$

$$= 2\tan^{-1}\left(\frac{18{,}849.6}{2(10kHz)}\right)$$

$$= 1.5115 \text{ rad}$$

$$= 2\pi\frac{f_d}{f_s}$$

$$\Rightarrow f_d = \frac{\theta_d f_s}{2\pi} = 2405.6 \text{ Hz} \qquad 1$$

Less than 3 kHz, as expected 1

(c) FIR filter design

Derive $h(n) = \begin{cases} \frac{1}{3} & n=0 \\ \frac{1}{3}\dfrac{\sin\left(\frac{n\pi}{3}\right)}{\frac{n\pi}{3}} & \end{cases}$ 3

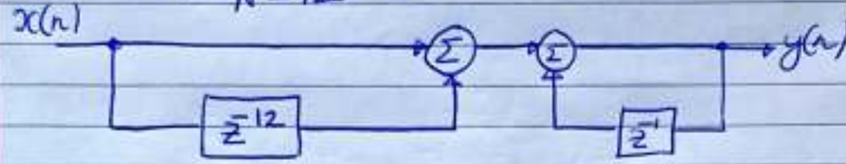
Group delay = 13 msec = 78 samples $= \frac{N-1}{2}$ 1

$$\Rightarrow N = 2(78)+1 = 157$$

$$\therefore h(n) = \frac{1}{3}\frac{\sin\left([n-78]\frac{\pi}{3}\right)}{[n-78]\frac{\pi}{3}} \quad n=0,1,\ldots 156 \qquad 1$$

(d) FIR filter implementation 1

$N = 12$



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zeroes @ $z = e^{j\frac{2\pi}{12}k}$ $k = 0, 1, \dots, 11$ 1

pole @ $z = 1$

Group delay = 5.5 samples 1

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