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EE445 – Digital Signal Processing

Digital Filter Structure

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Introduction

- Continue study of digital filters
- Effect of pole-zero placement on system frequency response
- Various “architectures” for implementing digital filters
- Cascade and Parallel systems
- “Special” digital filters – resonators and oscillators

Effect of Pole-Zero Placement on Frequency Response

- Suppose we have a transfer function:

$$H(z) = \frac{\sum_{k=0}^{K} a_k z^{-k}}{1 + \sum_{n=1}^{N} b_n z^{-n}} = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \ldots + a_K z^{-K}}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \ldots + b_N z^{-N}}$$

- Determine poles and zeros (factorise) to get:

$$H(z) = z^{N-K} \frac{a_0 (z - z_1)(z - z_2)(z - z_3)\ldots(z - z_K)}{(z - p_1)(z - p_2)(z - p_3)\ldots(z - p_N)}$$

- Frequency response of a system may be obtained by making the substitution $z = e^{j\theta}$:

Effect of Pole-Zero Placement on Frequency Response

$$H(\theta) = e^{j(N-K)\theta}\frac{a_0(e^{j\theta}-z_1)(e^{j\theta}-z_2).....(e^{j\theta}-z_K)}{(e^{j\theta}-p_1)(e^{j\theta}-p_2).....(e^{j\theta}-p_N)}$$

- Calculate the magnitude of $H(\theta)$

$$|H(\theta)| = \left|e^{j(N-K)\theta}\frac{a_0(e^{j\theta}-z_1)(e^{j\theta}-z_2).....(e^{j\theta}-z_K)}{(e^{j\theta}-p_1)(e^{j\theta}-p_2).....(e^{j\theta}-p_K)}\right|$$

- $e^{j\theta}$ represents a point on the unit circle, at angle θ with respect to the positive real axis

$$|H(\theta)| = a_0\left|e^{j(N-K)\theta}\right|\frac{\left|e^{j\theta}-z_1\right|\left|e^{j\theta}-z_2\right|.....\left|e^{j\theta}-z_K\right|}{\left|e^{j\theta}-p_1\right|\left|e^{j\theta}-p_2\right|.....\left|e^{j\theta}-p_N\right|}$$

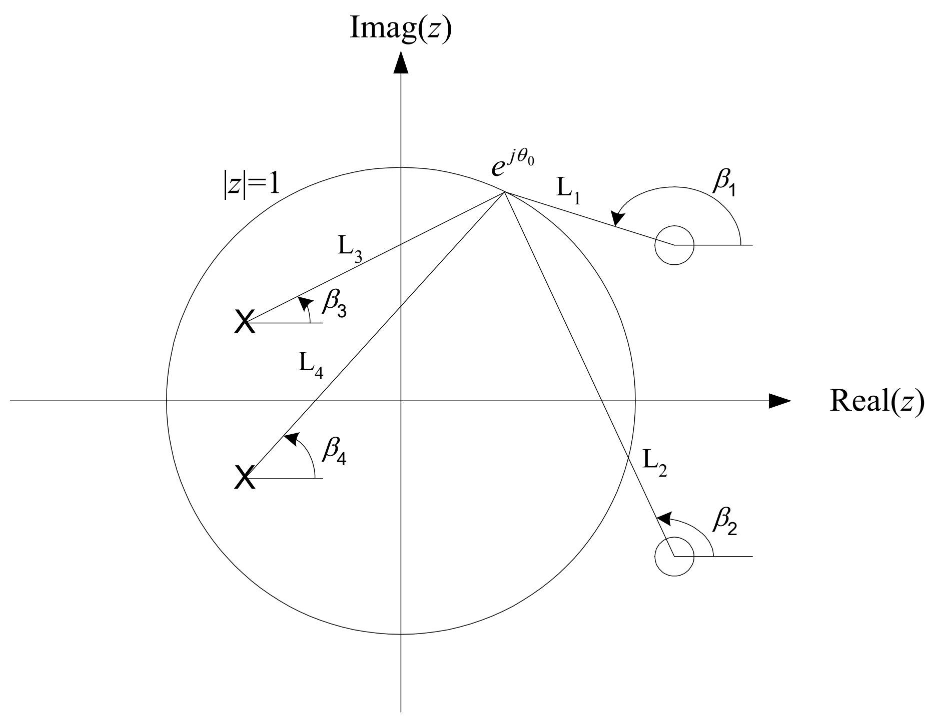
$$|H(\theta)| = a_0\frac{\prod_{i=1}^{K}\left|e^{j\theta}-z_i\right|}{\prod_{k=1}^{N}\left|e^{j\theta}-p_k\right|}$$

Effect of Pole-Zero Placement on Frequency Response

- z_i and p_k represent the system zeros and poles
- Therefore, $(e^{j\theta} - p_k)$ and $(e^{j\theta} - z_i)$ represent vectors in the z-plane, drawn from $e^{j\theta}$ to the corresponding poles and zeros
- The terms $\left|e^{j\theta} - p_k\right|$ and $\left|e^{j\theta} - z_i\right|$ represent the magnitude of these vectors
- Hence, the magnitude response of the filter at a given frequency θ is equal to a_0 times the product of the distances from each zero to the point $e^{j\theta}$ divided by the product of the distance from each pole to the point $e^{j\theta}$.

Effect of Pole-Zero Placement on Frequency Response

- Similarly for the phase angle: the phase response of the filter at a given frequency θ is equal to the sum of the angles of the vectors from the zeros to the point $e^{j\theta}$ minus the sum of the angles of the vectors from the poles to the point $e^{j\theta}$.
- Analogous to the interpretation of frequency response for continuous-time systems



Effect of Pole-Zero Placement on Frequency Response

- In the diagram above, the magnitude response at the arbitrarily chosen frequency θ_0 is given by:

$$|H(\theta_0)| = \frac{L_1 L_2}{L_3 L_4}$$

- The phase response is given by:

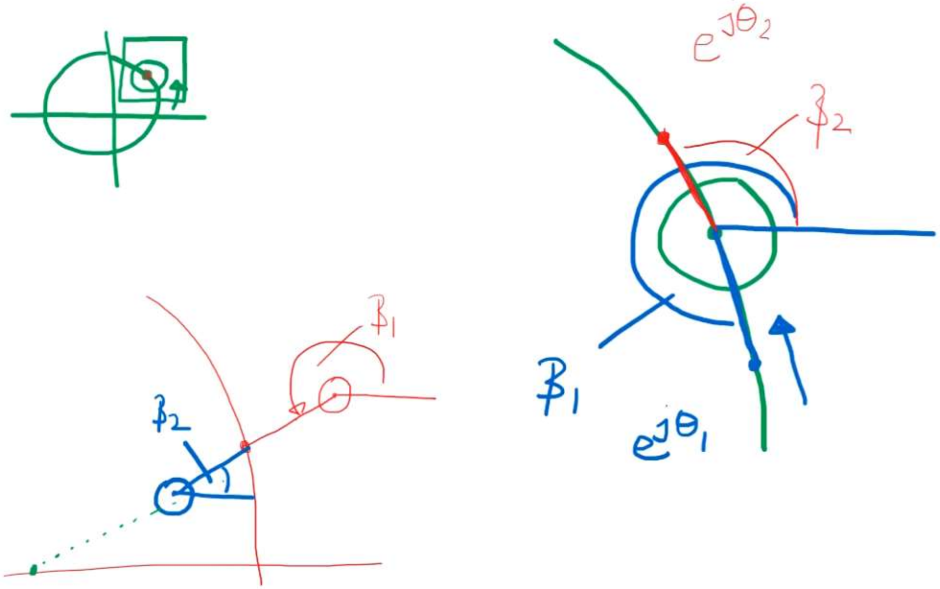
$$\phi(\theta_0) = \beta_1 + \beta_2 - \beta_3 - \beta_4$$

- Note the effect that poles and zeros have on the frequency response, particularly around frequencies that correspond to the poles and zeros
- These frequencies are often referred to as the "natural" frequencies of the system.

Effect of Pole-Zero Placement on Frequency Response

- The effect of a pole or zero on the frequency response increases the closer it is to the unit circle.
- In the extreme case of a zero actually on the unit circle, the magnitude response at the zero frequency is equal to zero, while the phase response jumps by π radians.
- Alternatively, if a pole is located on the unit circle, the magnitude response equals infinity, and again, the phase response jumps by π radians.
- If all of the zeros of a system are inside the unit circle, the system is said to be minimum phase, because zeros inside the unit circle contribute less to the phase response than zeros outside the unit circle.

Effect of Pole-Zero Placement on Frequency Response



$\beta_1 \approx 280^\circ$

$\beta_2 \approx 100^\circ$

$\beta_1 - \beta_2 \approx 180^\circ$

Effect of Pole-Zero Placement on Frequency Response

- By "reflecting" the zeros of a filter in the unit circle, we can change the phase characteristics of the system, while leaving the magnitude response unchanged.
  Reflection means that a zero at $z = re^{j\theta_0}$ becomes zero at $z = \left(\frac{1}{r}\right) e^{j\theta_0}$

- Exercise 4.1
  - Plot the magnitude and phase responses, and hence find the group delay, of the filer with zeros and poles at the following locations:
    - $z_1 = 1.2 + j0.7, z_2 = 1.2 - j0.7$
    - $p_1 = 0.7 + j0.4, p_2 = 0.7 - j0.4$
  - Do the same calculations for the case where the zeros are reflected in the unit circle.
  - Note: scale to get a DC gain = 1.0 (i.e. such that the frequency dependent behaviour is unaffect)

Example – how to calculate DC gain of a system (ex. 4.1)

- Suppose the transfer function of the system is given by

$$H(z) = \frac{3 + 2z^{-1}}{1 + 0.6z^{-1}}$$

Substitute $e^{j\theta}$,

$$H(z) = \frac{3 + 2e^{-j\theta}}{1 + 0.6e^{-j\theta}}$$

Set $\theta = 0$

$$H(\theta)\Big|_{\theta=0} = \frac{3 + 2}{1 + 0.6} = 3.125$$

Effect of Pole-Zero Placement on Frequency Response – all-pass filter and linear phase filter

- An all-pass filter is a filter for which $|H(\theta)| = 1$, for all values of θ.
- This is achieved by ensuring that all poles have a corresponding zero, located at the reflection of the pole in the unit circle.
- Note, however, that the phase response of such a filter may be such that significant distortion can still occur
- A linear phase filter is one that contains only zeros, where the zeros occur in pairs that are reflected in the unit circle (there may also be zeros on the unit circle itself).
- An all-zero system is an FIR filter.

Effect of Pole-Zero Placement on Frequency Response

- Exercise 4.2
  - For the filter in Exercise 4.1, choose zeros such that the filter becomes an all-pass filter, and calculate and plot the frequency response
- Exercise 4.3
  - For the system in Exercise 4.1, replace the poles with zeros such that the resulting filter has linear phase; calculate and plot the frequency response.

Non-Recursive Filter Structures

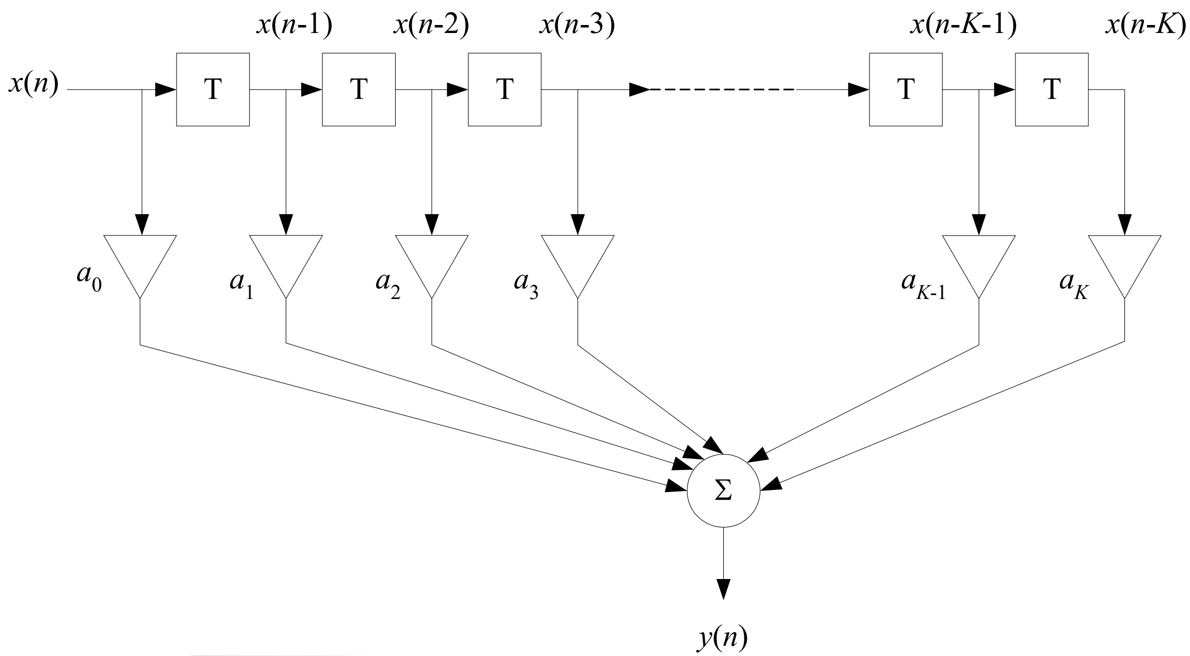
- The transfer function of a system may be expressed as:

$$H(z) = \frac{\sum_{k=0}^{K} a_k z^{-k}}{1 + \sum_{n=1}^{N} b_n z^{-n}} = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \ldots + a_K z^{-K}}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \ldots + b_N z^{-N}}$$

- However, there are many ways to implement....

Non-Recursive Filter Structures

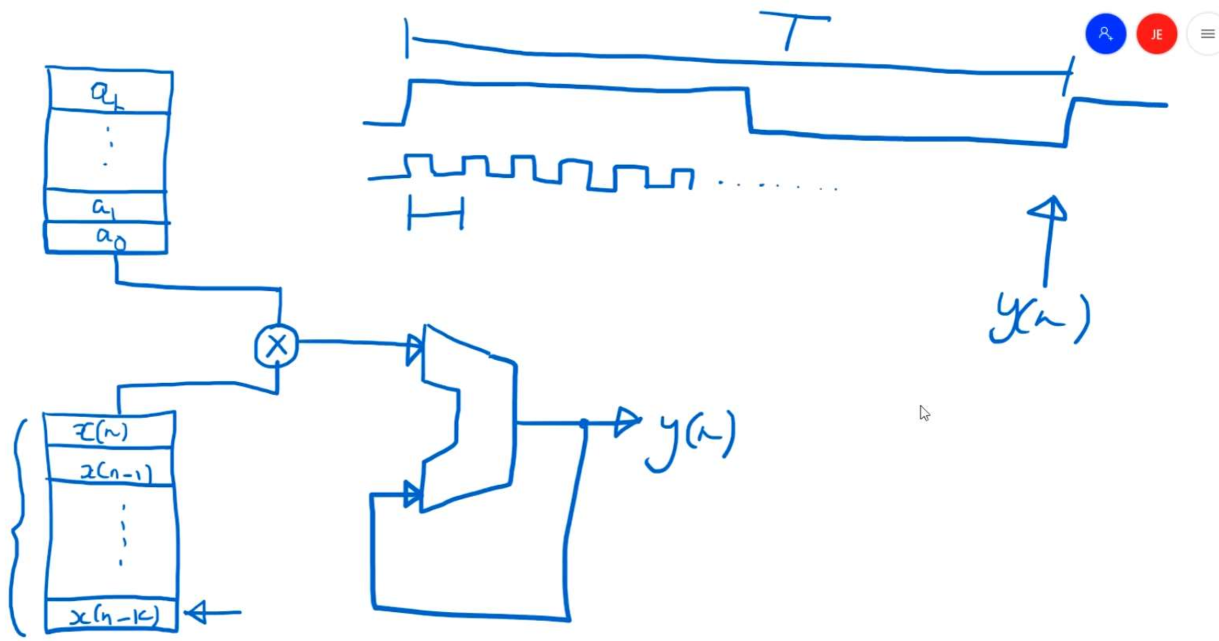
- FIR filter (N=0), use so-called “transversal” filter:



Non-Recursive Filter Structures

- This architecture is also referred to as a “tapped delay line”.
- Each of the delay blocks in the transversal filter could be implemented by means of a multi-bit shift register, or perhaps a memory location in an embedded RAM
- The gain blocks would be implemented by a multiplier (perhaps a single multiplier shared among all of the required coefficients).

Tapped delay line – hardware implementation



Recursive Filter Structures

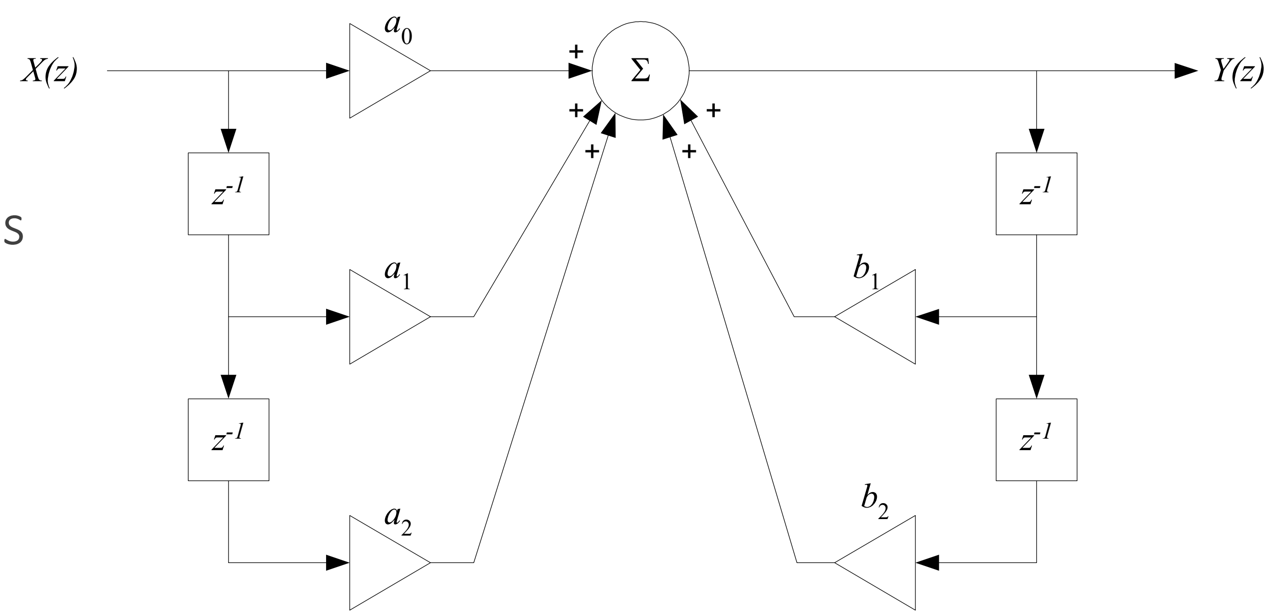
- A number of trade-offs exist …
- Various structures differ in terms of their "processing" requirements (number of memory elements required, speed of multipliers etc.),
- Sensitivity to quantisation of coefficients and data
- A general IIR filter transfer function may be written as shown opposite:
- In one case, zeros are implemented first, followed by the poles
- In the other case, the reverse is true.

$$H(z) = \frac{\sum_{k=0}^{K} a_k z^{-k}}{1 + \sum_{n=1}^{N} b_n z^{-n}} = \underbrace{\sum_{k=0}^{K} a_k z^{-k}}_{\text{zeros}} \; \underbrace{\frac{1}{1 + \sum_{n=1}^{N} b_n z^{-n}}}_{\text{poles}}$$

$$= \underbrace{\frac{1}{1 + \sum_{n=1}^{N} b_n z^{-n}}}_{\text{poles}} \; \underbrace{\sum_{k=0}^{K} a_k z^{-k}}_{\text{zeros}}$$

Direct Form I

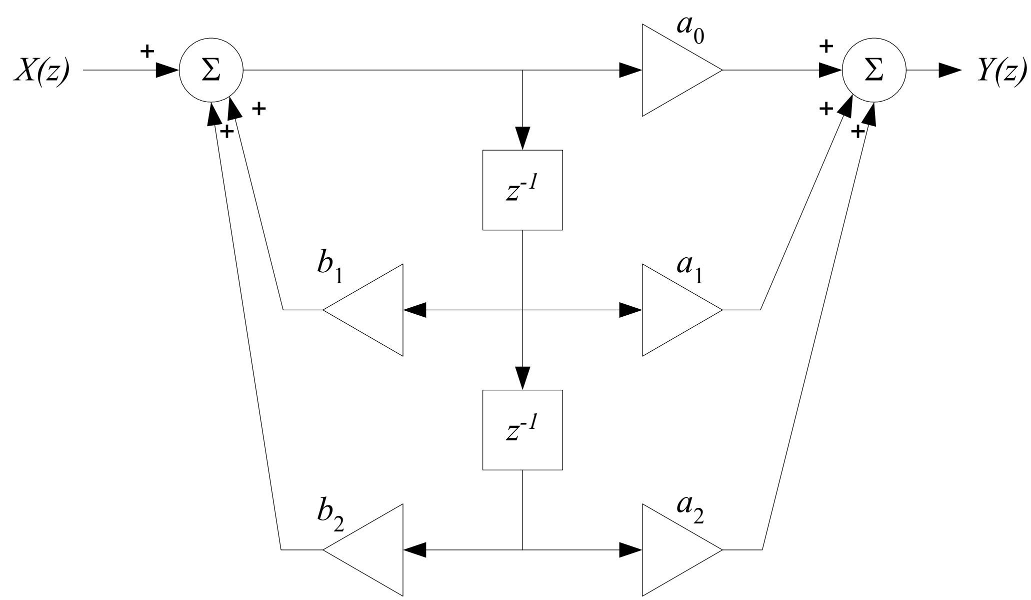
- Zeros first
- Four memory locations required
- Single “accumulator”



Direct Form 1 implementation of a 2nd-Order IIR filter

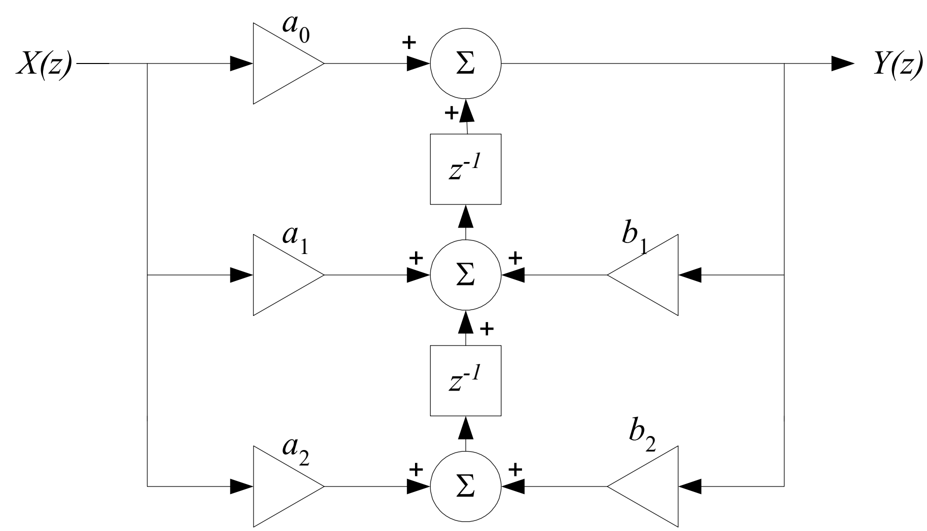
Direct Form II

- Only two memory elements are required, but two “accumulators” are needed.
- Orders of the numerator and denominator are the same => canonic network.



Direct Form II IIR filter

Transpose Form



Transpose form

Implementing High Order IIR filters

- Digital filters may be much more complex than second-order …
- Generally decomposed into a number of second-order and first-order stages that are connected either in cascade or in parallel.
- “Generic” first-order and second-order sections:

$$H(z) = \frac{c_0 + c_1 z^1}{1 + d_1 z^{-1}}$$

$$H(Z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

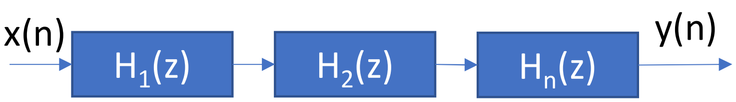
Note: (sometimes, the signs of the denominator coefficients are negative)

Implementing High Order IIR filters

- Filters are generally easier to analyse and design when they are represented as second-order or first-order sections – it's easier to see how these simpler sections influence system characteristics like impulse response and frequency response.
- From an implementation point of view, a DSP designer may have already designed a software routine or VHDL block to implement a first- or second-order filter => can be re-used.
- When implemented using limited numbers of bits for coefficients and data, high-order filters (especially with feedback) are usually much more sensitive to quantisation effects when implemented with a single section, than when implemented with multiple simpler sections.
- Individual second-order sections may be implemented using one of the structures discussed in previously

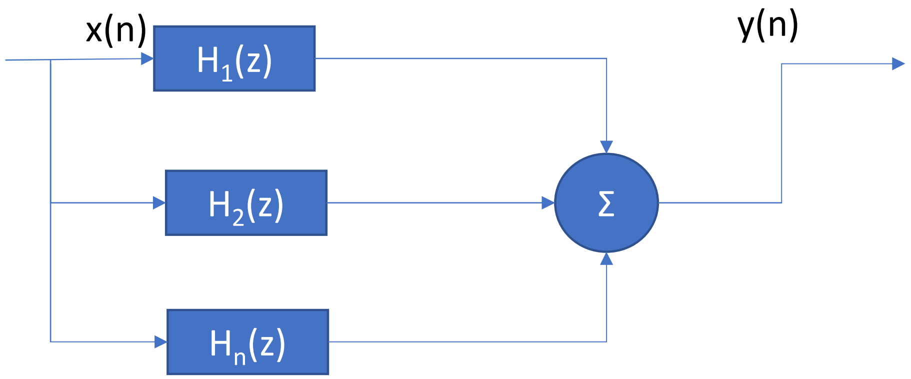
Implementing High Order IIR filtre

Cascade structure



$$H(z) = \prod_{i=1}^{L} H_i(z) = H_1(z)H_2(z) \ldots H_l(z)$$

Parallel structure



$$H(z) = Gz^{-k} + C + \sum_{i=1}^{L} H_i(z) = Gz^{-k} + C + H_1(z) + H_2(z) + H_3(z) + \cdots + H_L(z)$$

Note: each sub-block H_n is implemented using Direct form I or Direct form II

Cascade Structure

$$H(z) = \prod_{i=1}^{L} H_i(z) = H_1(z)H_2(z)\ldots H_l(z)$$

- $H_i(z)$ is either a second-order or first-order section.
- Overall order of the filter can be easily ascertained from the number and type of the individual sections
- Determination of the cascade form of a filter requires factorisation of the numerator and denominator of the "original" high-order transfer function into second-order and first-order sections.

Example 4.1

$$\mathrm{H}(z) = \frac{23 + 40z^{-1} + 36z^{-2} + 19z^{-3}}{10 + 9z^{-1} + 8z^{-2} + 3z^{-3}}$$

Factorize to obtain

$$H(z) = \frac{(1 + z^{-1})(23 + 17z^{-1} + 19z^{-2})}{(2 + z^{-1})(5 + 2z^{-1} + 3z^{-2})}$$

$$H(z) = \frac{(1 + z^{-1})}{(2 + z^{-1})} \frac{(23 + 17z^{-1} + 19z^{-2})}{(5 + 2z^{-1} + 3z^{-2})}$$

$$H(z) = \frac{(0.5 + 0.5z^{-1})}{(1 + 0.5z^{-1})} \frac{(4.6 + 3.4z^{-1} + 3.8z^{-2})}{(1 + 0.4z^{-1} + 0.6z^{-2})}$$

Hence, $H(z) = H_1(z)H_2(z), H_1\,(z) = \frac{(0.5+0.5z^{-1})}{(1+0.5z^{-1})}, H_2(z) = \frac{(4.6+3.4z^{-1}+3.8z^{-2})}{(1+0.4z^{-1}+0.6z^{-2})}$

Parallel Structure

$$H(z) = Gz^{-k} + C + \sum_{i=1}^{L} H_i(z) = Gz^{-k} + C + H_1(z) + H_2(z) + H_3(z) + \cdots + H_L(z)$$

- (this is a more generals case of the decomposition we studied in Section 2, when obtaining inverse z-Transforms)
- Gz^{-k} will exist only if the order of the numerator is greater than the order of the denominator.
- In this case, the numerator should be divided by the denominator to obtain Gz^{-k}.
- The remainder will be such that the order of the numerator will be less than or equal to the order of the denominator, so it can be expanded into parallel form using the method of partial fractions.
- The C term will be zero if the order of the numerator is less than the order of the denominator.

Parallel Structure

- Example 4.1 again:

$$\mathrm{H}(z) = \frac{23 + 40z^{-1} + 36z^{-2} + 19z^{-3}}{10 + 9z^{-1} + 8z^{-2} + 3z^{-3}}$$

- The orders of the numerator and denominator are equal, so we know that the transfer function will be of the form:

$$H(z) = C_0 + H_1(z) + H_2(z)$$

- $H_1(z)$ is first-order and $H_2(z)$ is second-order.

- The C_0 term is obtained from the ratio of the two terms containing the order of both the numerator and denominator polynomials (i.e. 3):

$$C_0 = \frac{19}{3}$$

- Factorize the denominator (already done):

$$H(z) = \frac{23 + 40z^{-1} + 36z^{-2} + 19z^{-3}}{10 + 9z^{-1} + 8z^{-2} + 3z^{-3}}$$

$$= \frac{23 + 40z^{-1} + 36z^{-2} + 19z^{-3}}{(2 + z^{-1})(5 + 2z^{-1} + 3z^{-2})}$$

Parallel structure

- Hence we can write $H(z)$ as

$$H(z) = \frac{19}{3} + H_1(z) + H_2(z) = \frac{19}{3} + \frac{A}{2+z^{-1}} + \frac{B+Cz^{-1}}{5+2z^{-1}+3z^{-2}}$$

$$= \frac{19(2+z^{-1})(5+2z^{-1}+3z^{-2}) + A(3)(5+2z^{-1}+3z^{-2}) + (B+Cz^{-1})(3)(2+z^{-1})}{3(2+z^{-1})(5+2z^{-1}+3z^{-2})}$$

$$H(z) = \frac{(190+15A+6B) + z^{-1}(171+6A+6C+3B) + z^{-2}(152+9A+3C) + 57z^{-3}}{3(2+z^{-1})(5+2z^{-1}+3z^{-2})}$$

- Comparing this with the original transfer function, we obtain:

$$H(z) = \frac{23+40z^{-1}+36z^{-2}+19z^{-3}}{(2+z^{-1})(5+2z^{-1}+3z^{-2})} = \frac{69+120z^{-1}+108z^{-2}+57z^{-3}}{3(2+z^{-1})(5+2z^{-1}+3z^{-2})}$$

Parallel structure

- This gives us three simultaneous equations:

$$190 + 15A + 6B = 69$$
$$171 + 6A + 3B + 6C = 120$$
$$152 + 9A + 3C = 108$$

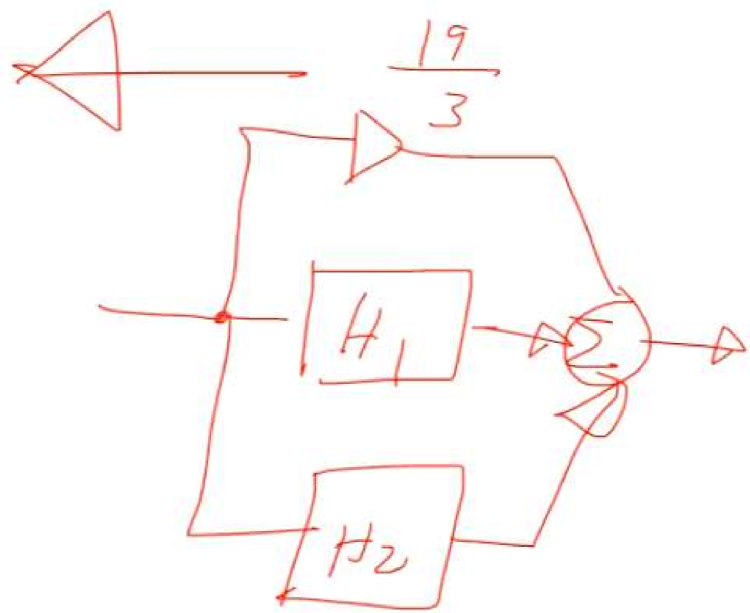
- The solution of these equation yields:

A = -5, B = -23/3 and C = 1/3

- Thus, the transfer function can be written in parallel form as

$$\begin{aligned} H(z) &= \frac{19}{3} + H_1(z) + H_2(z) \\ &= \frac{19}{3} - \frac{5}{2 + z^{-1}} - \frac{23 - z^{-1}}{3(5 + 2z^{-1} + 3z^{-2})} \\ &= \frac{19}{3} - \frac{2.5}{1 + 0.5z^{-1}} - \frac{\frac{23}{15} - \frac{1}{15}z^{-1}}{1 + 0.4z^{-1} + 0.6z^{-2}} \end{aligned}$$

Hardware implementation



Parallel Structure

- This can be quite tedious to do on paper…
- However, each method essentially involves manipulation of polynomials (factorisation etc.), so any computer-based tools that assist in this procedure can be used.
- Matlab function “roots” can be used to obtain the roots of a polynomial, which can then be used to obtain the cascade form (complex conjugate pairs of roots should be “combined” to give second-order sections with real coefficients).
- The function “residuez” may be used to obtain a partial fraction expansion of a transfer function (again, complex conjugate terms may be combined to give second-order terms with real coefficients).

Exercise 4.4

- Use Matlab to obtain the cascade and parallel implementations of the third-order transfer function used in the above examples.

Resonators

- Looked at decomposing “big” filters into first- and second-order filters
- Second-order filter can be viewed as a basic building block in DSP systems … worth looking at in more detail.
- Complex conjugate pole pair causes resonance in the magnitude response => second-order system with such a pole pair often called a resonator.

Resonator Design

- Write the poles in polar form as:

$p_1 = re^{j\theta_0}$ and $p_2 = re^{-j\theta_0}$

- Resonators also have a double zero at the origin; however, these zeros have no effect on the magnitude response of the filter (equal distances from all points on the unit circle)
- Zeros at origin do affect phase response…

Resonator Design

- Transfer function can be written as:

$$
\begin{aligned}
H(z) &= \frac{z^2}{(z - re^{j\theta_0})(z - re^{-j\theta_0})} \\
&= \frac{z^2}{z^2 - r(e^{j\theta_0} + e^{-j\theta_0})z + r^2} \\
&= \frac{z^2}{z^2 - 2r\cos(\theta_0)z + r^2}
\end{aligned}
$$

- Divide numerator and denominator by the highest power of z

$$
\begin{aligned}
H(z) &= \frac{1}{1 - 2r\cos(\theta_0)z^{-1} + r^2 z^{-2}} \\
&= \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}}
\end{aligned}
$$

$$
\begin{aligned}
b_2 &= r^2 \\
b_1 &= -2r\cos(\theta_0) = -2\sqrt{b_2}\cos(\theta_0)
\end{aligned}
$$

Resonator design

- For stability, the pole radius r must be less than 1.
- The closer the poles are to the unit circle, the narrower and higher is the corresponding peak in the magnitude response
- Hence, r controls the “sharpness” of the peak.
- An empirical equation that is often used to choose a value for r is as follows:

$$r \approx 1 - \frac{\Delta f}{f_{samp}} \pi$$

- where Δf is the desired “bandwidth” of the resonance, and f_{samp} is the sampling frequency.
- The transfer function is often scaled by the term $1 + b_1 + b_2$, to ensure that the gain of the filter at DC (0 Hz) is equal to 1.
- Final transfer function

$$H(z) = \frac{1 + b_1 + b_2}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

- And the difference equation is

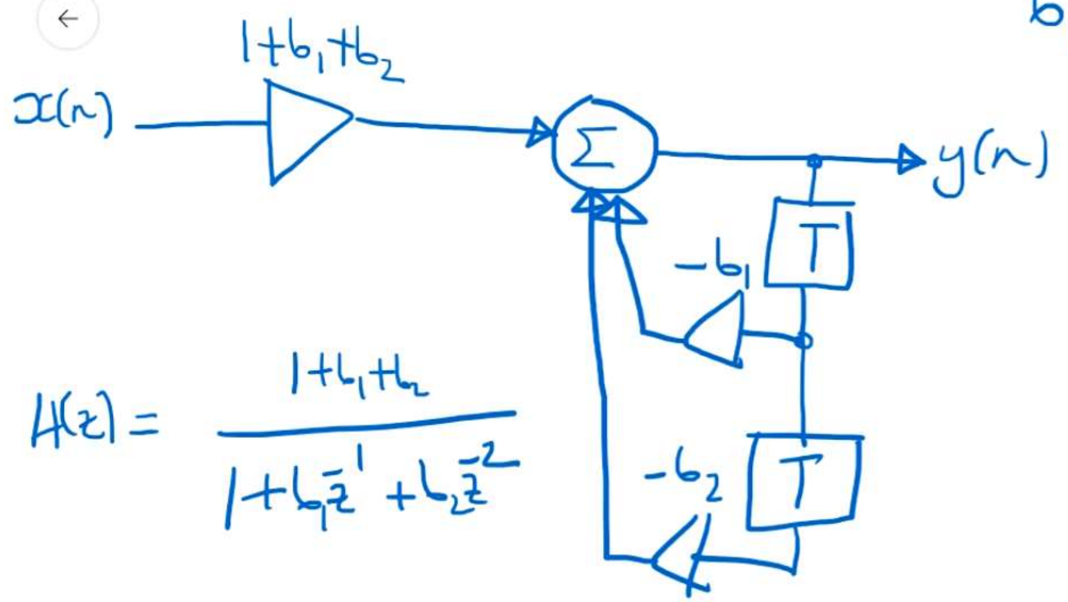
$$y(n) = (1 + b_1 + b_2)x(n) - b_1 y(n-1) - b_2 y(n-2)$$

Exercises

- Exercise 4.5
  - Draw the block diagram of a resonator
- Exercise 4.6
  - Show how the presence or absence of zeros at the origin of a resonator influences the frequency response.

Block diagram of Resonator

$$b_1 = -2r\cos\theta_0$$
$$b_2 = r^2$$



$$H(z) = \frac{1+b_1+b_2}{1+b_1 z^{-1}+b_2 z^{-2}}$$

Exercise 4.6 answer

$$H(z) = \frac{z^2}{z^2 + b_1 z + b_2} \leftarrow \text{double zero @ } z = 0$$

$$H(z) = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}} \quad \times z^{-2}$$

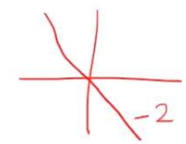
$$y(n) = x(n) - b_1 y(n-1) - b_2 y(n-2)$$

$$H(z) = \frac{1}{z^2 + b_1 z + b_2}$$

$$= \frac{z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

$$y(n) = \underbrace{x(n-2)}_{\text{delay by 2 samples}} - b_1 y(n-1) - b_2 y(n-2)$$

$$z^{-2} \qquad e^{-j2\theta}$$



Example 4.2

- Suppose we want to design a resonator with the following specification:
  - Centre (pole) frequency = 1200 Hz
  - Bandwidth = 75 Hz
  - Sampling rate = 9.6 kHz
  - DC gain = 1
- Calculate the pole frequency (in radians) and the pole radius

Example 4.2

$$\theta_0 = 2\pi \frac{f_0}{f_{samp}} = 2\pi \frac{1200}{9600} = \frac{\pi}{4}$$

$$r \approx 1 - \frac{\Delta f}{f_{samp}}\pi = 1 - \frac{75}{9600}\pi = 0.9754$$

- Then, the coefficients are calculated as follows:

$$b_2 = r^2 = 0.9515$$

$$b_1 = -2r\cos(\theta_0) = -2(0.9754)\cos\left(\frac{\pi}{4}\right) = -1.3794$$

$$1 + b_1 + b_2 = 1 - 1.3794 + 0.9515 = 0.5721$$

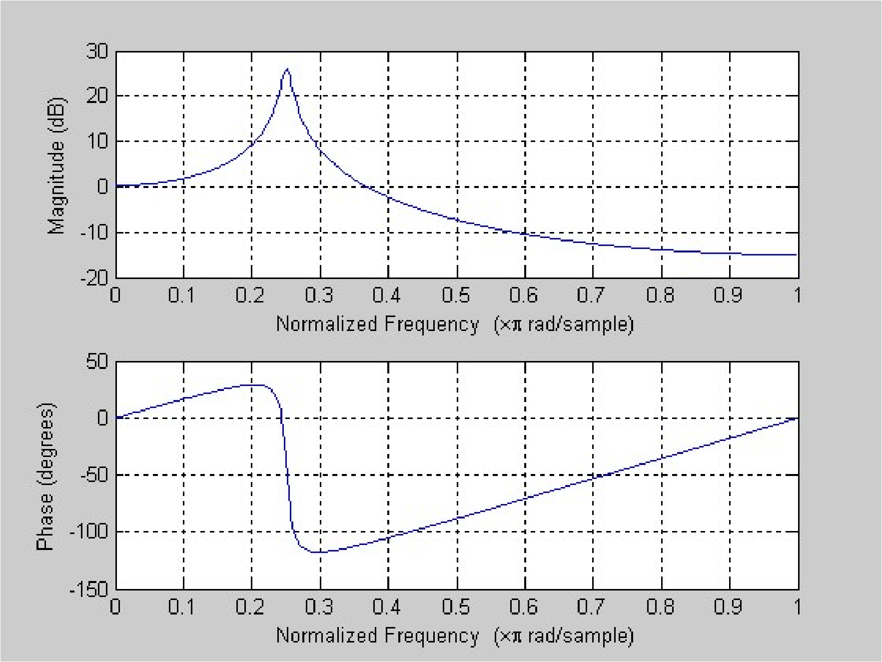
- Hence, the transfer function and difference equation can be written as:

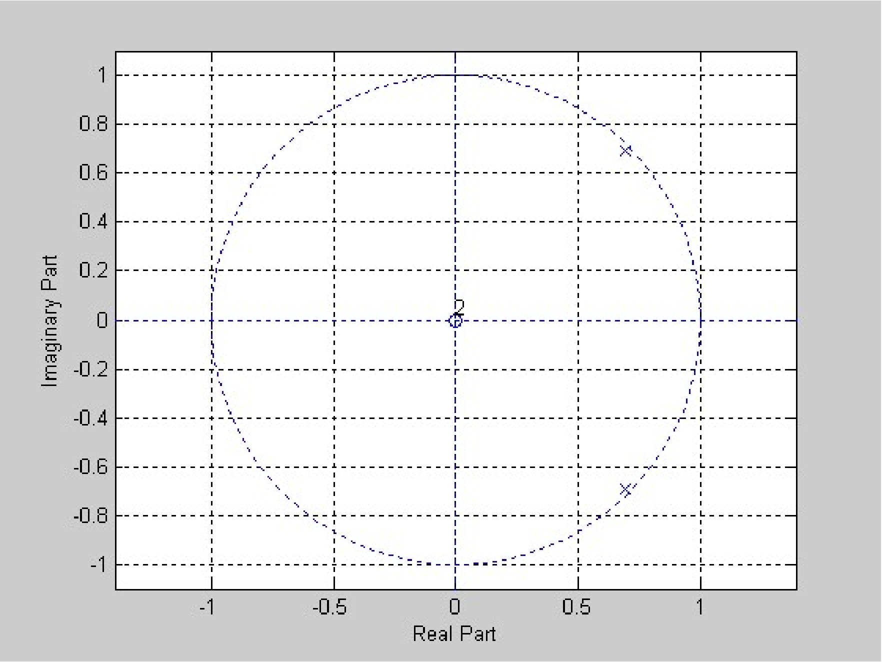
$$H(z) = \frac{0.5721}{1 - 1.3794z^{-1} + 0.9515z^{-2}}$$

Difference equation:

$$y(n) = 0.57210x(n) + 1.3794y(n-1) - 0.9515y(n-2)$$

Example 4.2 - frequency response and pole-zero map





Stability of second-order filter

- How do we determine the stability of a second-order filter from the coefficients b_1 and b_2?
- Poles must lie inside the unit circle, i.e. the pole radius must be less than 1.
- Coefficient b_2 must be less than 1, since $b_2 = r^2$. However, this strictly applies only in the case of complex conjugate poles.
- Different possibilities (real poles, complex poles).

Stability of second-order filter

- Given a transfer function

$$H(z) = \frac{1 + b_1 + b_2}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

- Determine the poles in terms of the coefficients by getting the roots of the denominator polynomial

$$z = \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2}$$

- The poles will form a complex conjugate pair if:

$$b_1^2 - 4b_2 < 0$$

- This is the situation we have examined so far with the “resonator”.
- However, we could also have a pair of real poles – in this case, however, it will not actually be a “resonator”.
- Same stability requirement holds – poles must lie between +1 and -1…

Stability of second-order filter

$$-1 < \frac{-b_1 \pm \sqrt{b_1^2 - 4b_2}}{2} < 1$$

$$\Rightarrow -2 + b_1 < \pm\sqrt{b_1^2 - 4b_2} < 2 + b_1$$

$$\Rightarrow -2 + b_1 < -\sqrt{b_1^2 - 4b_2} \text{ and } \sqrt{b_1^2 - 4b_2} < 2 + b_1$$

$$\Rightarrow (-2 + b_1)^2 > b_1^2 - 4b_2 \text{ and } b_1^2 - 4b_2 < (2 + b_1)^2$$

$$\therefore b_1 - b_2 - 1 < 0 \text{ and } b_1 + b_2 + 1 > 0$$

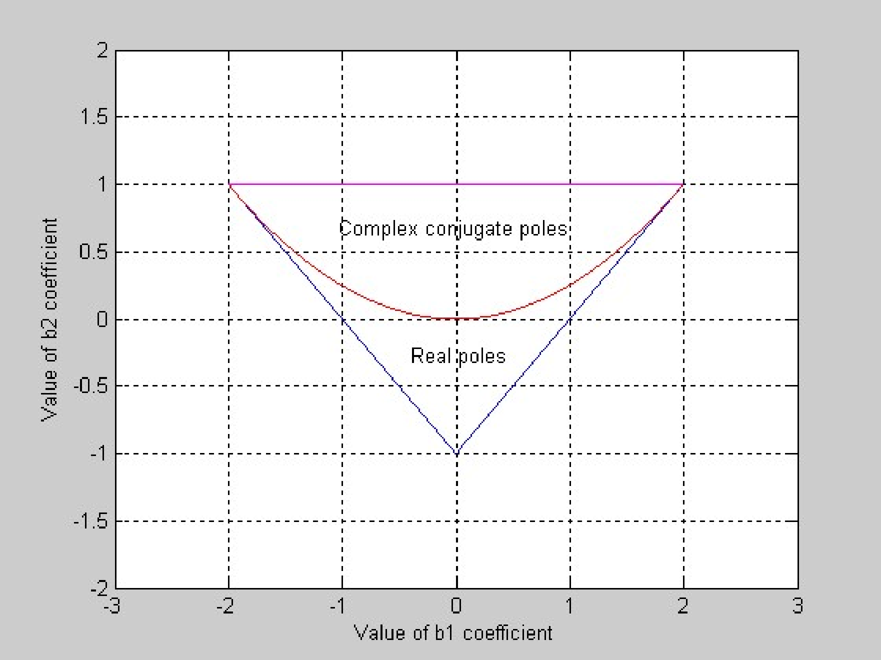
- These equations define two lines in the "coefficient plane" whose co-ordinates are b_1 and b_2:

$$b_2 = b_1 - 1 \qquad \text{and} \qquad b_2 = -b_1 - 1$$

- Region formed by intersection is called the "stability triangle"…

Stability of second-order filter

- A second-order filter is stable only if the point (b_1, b_2) lies inside this triangle.
- Points below the parabola correspond to real and distinct poles.
- Points on the parabola correspond to real and equal (double) poles
- Points above the parabola correspond to complex conjugate poles.



Simple Filter Design using Pole-Zero Placement

- Previous discussion basically involves determining poles locations explicitly – special case of more general “pole-zero placement” technique for filter design.
- For zeros, instead of a peak in the magnitude response, we get a “notch” in the magnitude response close to the zero frequency.
- A zero placed on the unit circle results in complete rejection of any signal components at the zero frequency
- The general procedure for calculating the numerator coefficients in this case is the same as for calculating denominator coefficients (but stability is not dependent on zero locations)

Example 4.3

- Using pole-zero placement, design a filter with the following specification:
  - Complete rejection of DC inputs
  - Narrow pass band at 200 Hz, with bandwidth of 15 Hz
  - Sampling rate = 500 Hz

- We calculate the pole terms as before:

$$\theta_0 = 2\pi \frac{f_0}{f_{samp}} = 2\pi \frac{200}{500} = 0.8\pi$$

$$r \approx 1 - \frac{\Delta f}{f_{samp}}\pi = 1 - \frac{15}{500}\pi = 0.9057$$

- The coefficients are calculated as follows:

$$b_2 = r^2 = 0.8204$$

$$b_1 = -2r\cos(\theta_c) = -2(0.9057)\cos(0.8\pi) = 1.4654$$

Example 4.3

- For the zero, the requirement for complete rejection of DC means we need a zero at $s = 1$. Therefore the numerator polynomial will be $(z - 1)$
- Overall transfer function of the filter:

$$H(z) = \frac{z-1}{z^2 + 1.4654z + 0.8204}$$

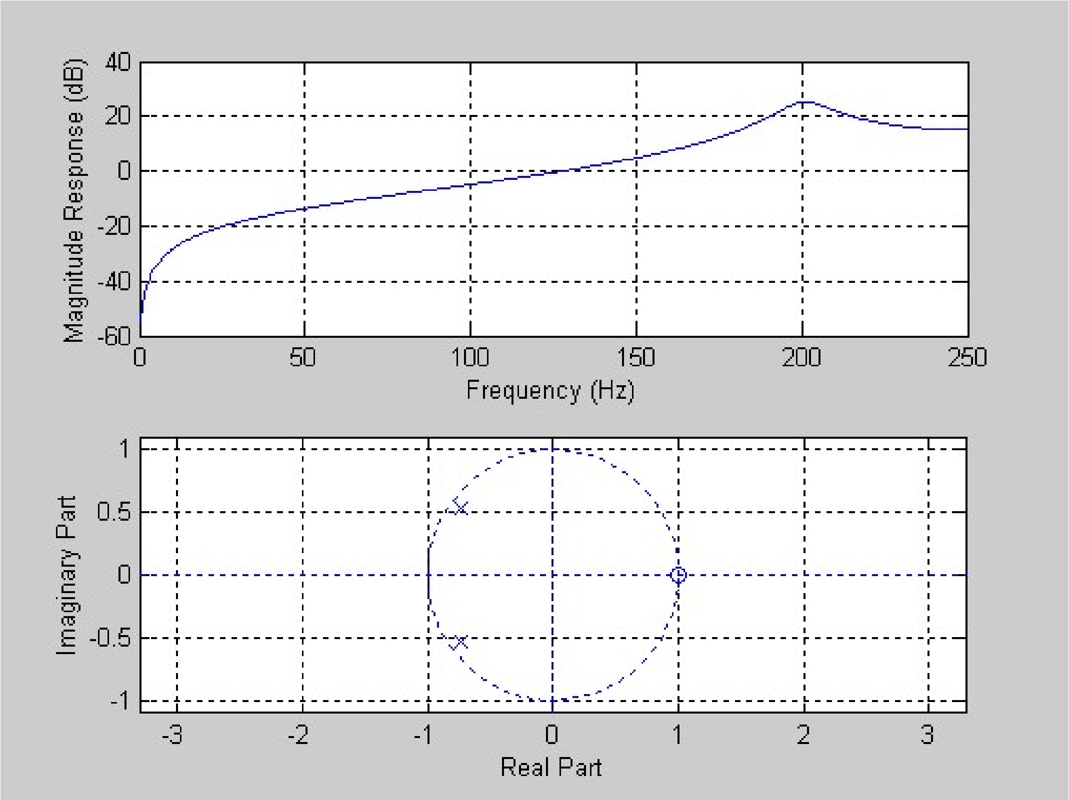
$$= \frac{z^{-1} - z^{-2}}{1 + 1.4654z^{-1} + 0.8204z^{-2}}$$

- The magnitude response and pole-zero map are plotted on the next slide

Example 4.3

Magnitude plot

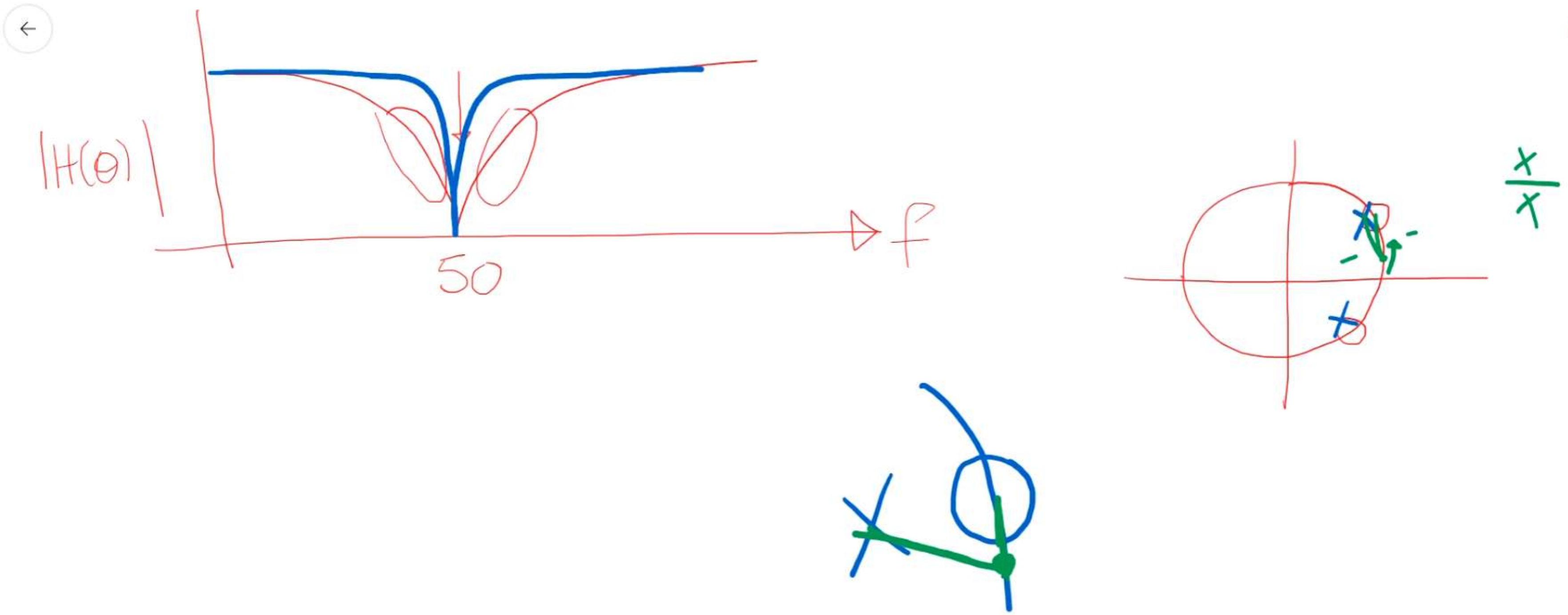
Pole-zero plot



Notch filter design considerations

- Some applications require complete rejection of a very narrow band of frequencies in the input signal, e.g. removing mains interference in biomedical instrumentation.
- We need to design a notch filter, with a zero at 50 Hz, to effect complete rejection of the unwanted interference.
- However, we want to avoid having the notch too “broad” (so “desired” signal will not be affected too much)
- To achieve this, we normally place a pair of complex conjugate poles at the same frequency as the zeros, and close to the unit circle.
- The effect of these poles is to “cancel out” the zeros at frequencies other than those close to the zero frequency.

Using poles and zeros to control the notch shape



Example 4.4

- A biomedical instrumentation application requires the removal of mains hum (50Hz) from a signal that is sampled at 300 Hz.
- Design a notch filter to achieve this. A notch of width 10 Hz should be sufficient.

- Calculate pole and zero frequencies, and pole radius (using same empirical equation as before):

$$\theta_0 = 2\pi \frac{f_0}{f_{samp}} = 2\pi \frac{50}{300} = 0.333\pi$$

$$r \approx 1 - \frac{\Delta f}{f_{samp}}\pi = 1 - \frac{10}{300}\pi = 0.8953$$

Example 4.4

- The denominator coefficients are:

$$b_2 = r^2 = 0.8015$$
$$b_1 = -2r\cos(\theta_c) = -2(0.8953)\cos\left(0.333\pi\right) = -0.8969$$

- While the numerator coefficients are

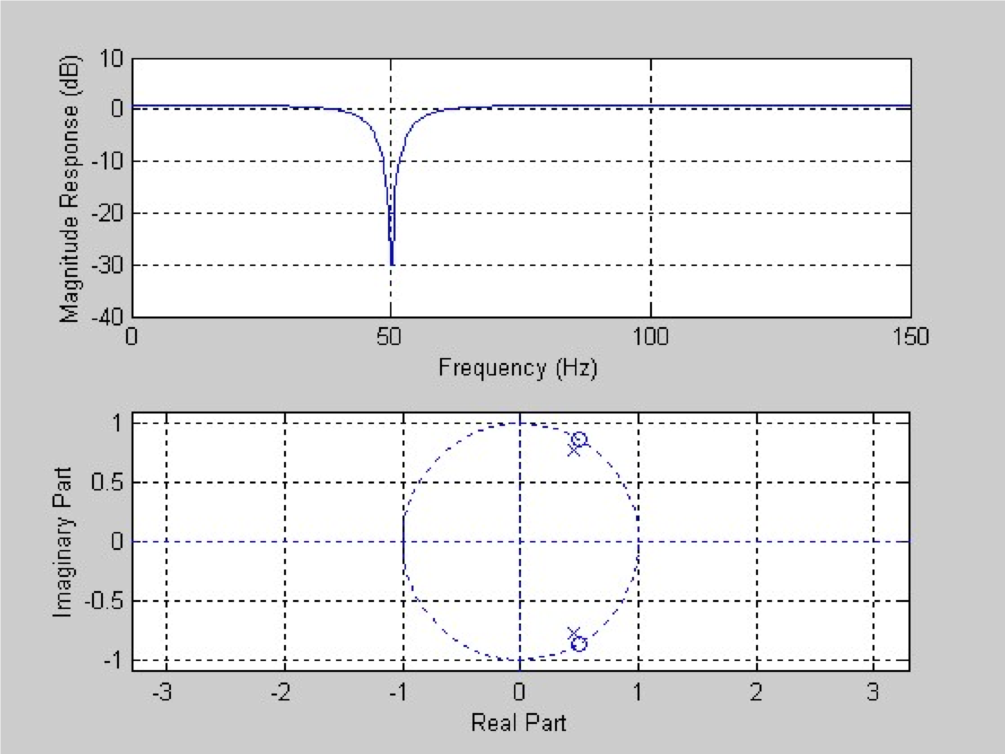
$$a_2 = r^2 = 1$$
$$a_1 = -2r\cos(\theta_c) = -2(1)\cos\left(0.333\pi\right) = -1.0018$$

- Hence, the transfer function is given by:

$$H(z) = \frac{1 - 1.0018z^{-1} + z^{-1}}{1 - 0.8969z^{-1} + 0.8015z^{-2}}$$

- The magnitude response and pole-zero map are plotted in the next slide...

Example 4.4



Magnitude Response (dB)
10
0
-10
-20
-30
-40
0
50
100
150
Frequency (Hz)
Imaginary Part
1
0.5
0
-0.5
-1
-3
-2
-1
0
1
2
3
Real Part

Exercise

- Exercise 4.7
  - Plot the frequency response and pole-zero map for the case where no poles are included in the transfer function.

Oscillators

- Oscillators have poles sitting on the unit circle => filter is marginally stable, and its output oscillates at a frequency corresponding to the pole frequency.
- One form of equation for the output signal of such an oscillator is:

$$p(n) = A\cos(n\theta_0)$$

- where A is the amplitude of the cosine output, and θ_0 is its (digital) frequency in radians (could use sine instead of cosine).

Oscillators

- The z-transform of $p(n)$ is:

$$P(z) = \frac{1-\cos(\theta_0)z^{-1}}{1-2\cos(\theta_0)z^{-1}+z^{-2}}$$

- If we view this as the oscillator "transfer function", and take the inverse z-Transform:

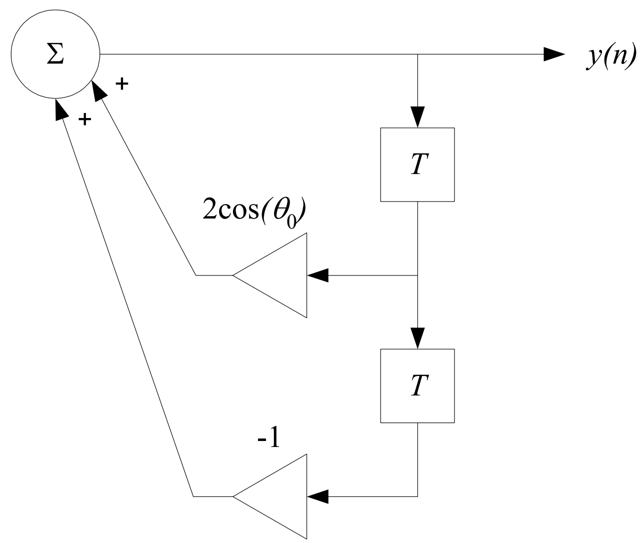
$$y(n) = [x(n)-\cos(\theta_0)x(n-1)] + 2\cos(\theta_0)y(n-1) - y(n-2)$$

- However, by definition, an oscillator will "oscillate" without any input signal, so we set the terms involving $x(n)$ to zero to obtain:

$$y(n) = 2\cos(\theta_0)y(n-1) - y(n-2)$$

- The block diagram is shown on the next slide...

Oscillators



- Initial conditions:

$$y(-1) = A\,cos(-\theta_0)$$
$$y(-2) = A\,cos(-2\theta_0)$$

- Note that y(-1) and y(-2) represent y(n-1) and y(n-2) for n = 0. However, we can readily choose another starting sample index if needed for the application.
- Also, note that the usual causality assumption for digital filters is not made in this case
- Initial conditions are critical (otherwise, how could oscillation be sustained?)

Oscillators

- Oscillator is a particular case of a resonator, with poles sitting on the unit circle (i.e. r = 1)

$$H(z) = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

- where $b_2 = 1$, and the coefficient b_1 is related to the oscillation frequency by:

$$b_1 = -2\cos(\theta_0)$$

- This is the same equation as for a resonator, of course, but with $r = 1$
- Again, note how the denominator coefficients change sign when going from the transfer function to the difference equation

Exercises

- Exercise 4.8
  - Calculate the coefficients of an oscillator whose frequency of oscillation is 100 Hz (the sampling frequency is 10 kHz). Write Matlab code to calculate the first 1000 samples of the oscillator output signal, and verify that the frequency of oscillation is as expected.
- Exercise 4.9
  - Design an oscillator (including initial conditions) to produce

$$p(n) = A\sin(n\theta_0)$$

Hint – start with the z-Transform of $\sin(\theta_0)$

Oscillators - applications

- An important application of oscillators is in digital modulators for communication systems.
- Modulation is the process by which an information signal is modulated or shifted to a higher frequency to facilitate transmission.
- Usually involves multiplication of the information signal by a sine or cosine carrier.

Oscillators - applications

- In many communication systems, both sine and cosine carriers are needed at the same time.
- A structure to achieve this in digital communications systems can be derived as follows.
- From basic trigonometry, we know that:

$$\cos(n+1)\theta = \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta)$$

If we let $y(n) = \cos(n\theta)$ and $x(n) = \sin(n\theta)$, we obtain

$$y(n+1) = \cos(\theta)y(n) - \sin(\theta)x(n)$$

Oscillators - applications

- Delaying both sides of this equation by one sample interval yields:

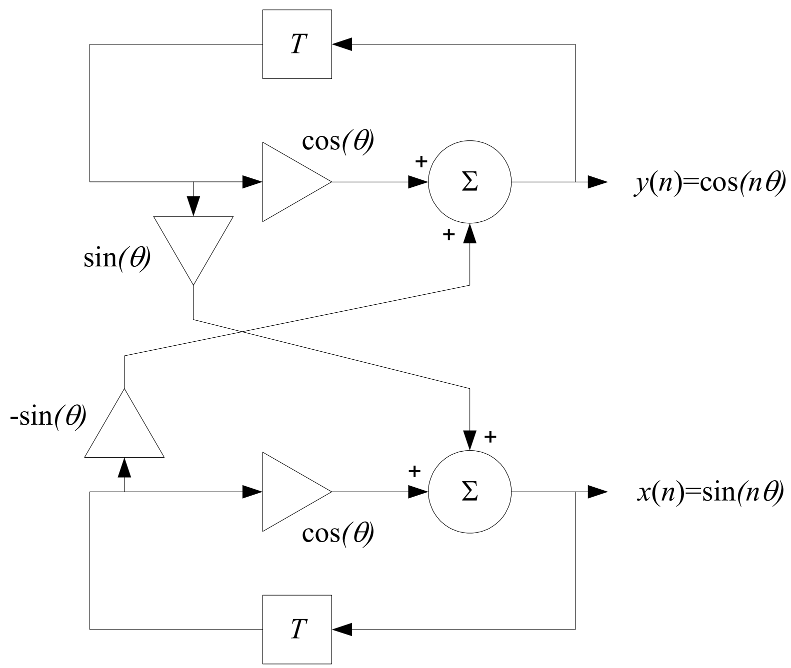
$$y(n) = \cos(\theta)y(n-1) - \sin(\theta)x(n-1)$$

- Similarly

$$\begin{aligned} \sin(n+1)\theta \quad &= \cos(n\theta)\sin(\theta) + \sin(n\theta)\cos(\theta) \\ \Rightarrow \quad x(n+1) \quad &= \sin(\theta)y(n) + \cos(\theta)x(n) \\ \Rightarrow \quad x(n) \quad &= \sin(\theta)y(n-1) + \cos(\theta)x(n-1) \end{aligned}$$

- • Combining these two equations results in the “coupled” oscillator structure shown on the next slide…

Oscillators - applications



Coupled oscillator to produce sine and cosine carriers.

Exercises

- Exercise 4.10
  - Determine suitable initial conditions for the coupled oscillator in Figure 4.12.
- Exercise 4.11
  - For an oscillator whose output is $p(n) = \sin(n\theta_0)$, where θ_0 is determined by the coefficient b_1, show that a small change in oscillation frequency Δf_0 is related to a change Δb_1 in the coefficient b_1 by the following expression:

$$\Delta f_0 = \frac{-f_{samp}\Delta b_1}{4\pi \cos\left(2\pi \dfrac{f_0}{f_{samp}}\right)}$$

where f_0 is the frequency of oscillation in Hz, and f_{samp} is the sampling frequency.