EE445 – Digital Signal Processing

Frequency-Domain Analysis of Discrete-Time Signals and Systems

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Introduction

- Section 1 covered time-domain analysis
- Section 2 covered transform domain analysis
- Here, we cover frequency-domain analysis
- Same concepts as for continuous-time signals and systems …
- … but one important addition – signals are sampled

Fourier Transform of Sampled Signals

- Fourier Transform of a signal $x(t)$:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

- Inverse Fourier Transform is defined by:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

- Also have Fourier Series for periodic signals – but we will use the more "general" transform

Fourier Transform of Sampled Signals

- Fourier Transform for discrete-time signals is:

$$X(\theta) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\theta}$$

- Inverse Fourier Transform is

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)e^{jn\theta}\,d\theta$$

- For causal signals, *n* in the summation for the Fourier Transform starts at 0
- Notation:

$$X(\theta) = \Im[x(n)],\ x(n) = \Im^{-1}[X(\theta)]$$

Fourier Transform of Sampled Signals

- It can be shown that $X(\theta)$ is periodic, with period 2π (critical point!).
- Start with a continuous-time signal $x(t)$ (band limited). Sample at a rate that is at least twice the bandwidth of $x(t)$
- Represent the sampled signal in terms of the Fourier Transform of the continuous-time signal as follows:

$$x(n) = x(t)\Big|_{t=nT} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega\Bigg|_{t=nT} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega nT}\,d\omega$$

Fourier Transform of Sampled Signals

- If we "divide" the frequency axis into segments of length 2π, we can rewrite $x(n)$ as:

$$x(n) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{k2\pi-\pi}^{k2\pi+\pi} X(\omega) e^{j\omega nT} \frac{d\omega T}{T}$$

- If we now change the argument, we obtain:

$$x(n) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{1}{T} \int_{-\pi}^{\pi} X(\omega + k\frac{2\pi}{T}) e^{j(\omega + k\frac{2\pi}{T})nT} d\omega T$$

- The function $e^{j\omega nT}$ is periodic, with period 2π, so we can write:

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T}) \right] e^{j\omega nT} d\omega T$$

Fourier Transform of Sampled Signals

- Finally, we note that $\theta = \omega T$, and therefore

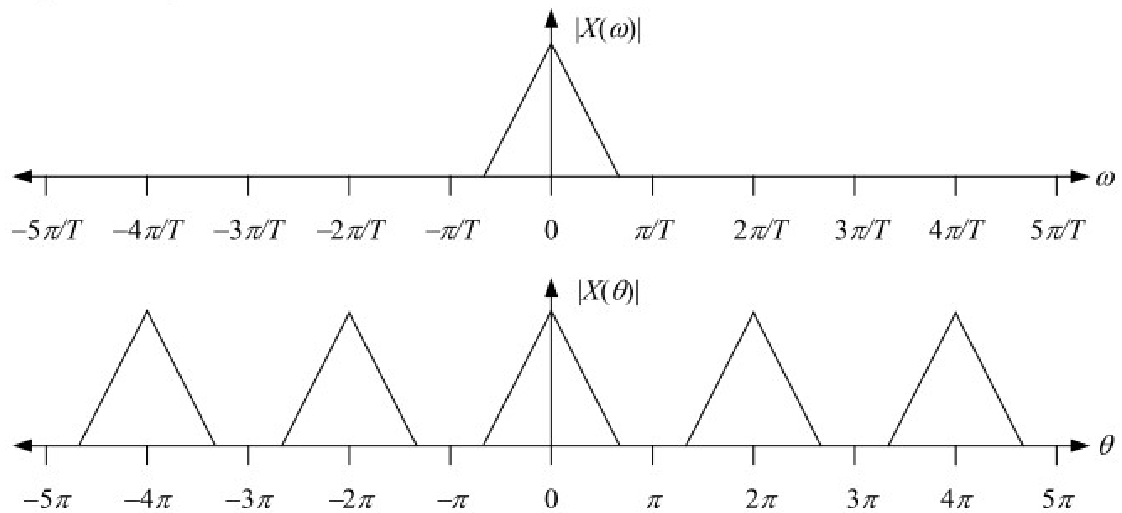
$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T})\right]e^{jn\theta}\,d\theta$$

- Compare this with the earlier expression for the Inverse Fourier Transform of a sampled signal – the expression inside the square brackets is equal to $X(\theta)$, i.e.

$$X(\theta) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T})$$

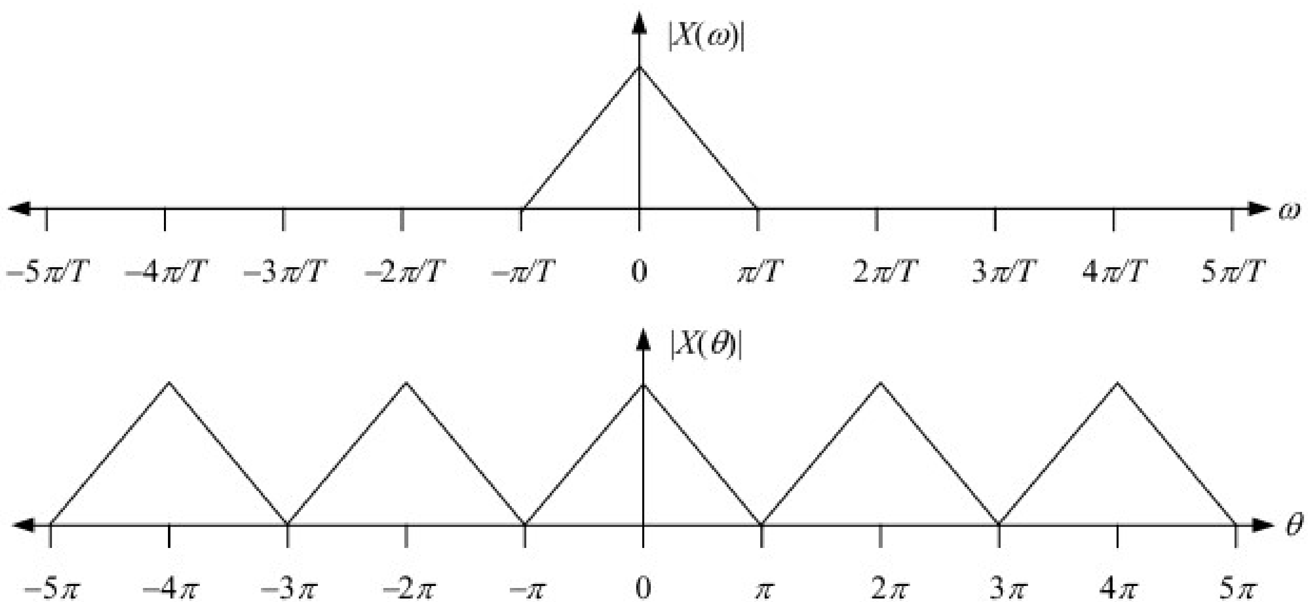
Fourier Transform of Sampled Signals

- In other words, the Fourier Transform of a discrete-time signal, obtained by sampling a continuous-time signal, is equal to an infinite set of "copies" of the original continuous-time spectrum, repeating at multiples of $2\pi/T$
- Spectrally, this looks like the figure opposite



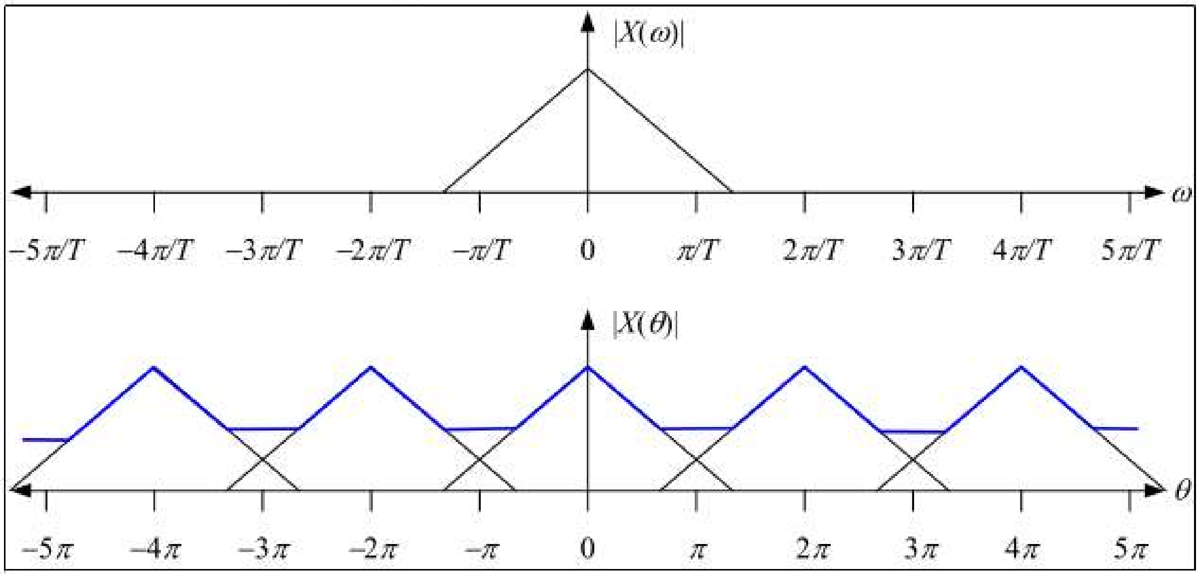
Fourier Transform of Sampled Signals

- If the sampling frequency is exactly equal to twice the bandwidth of $x(t)$:



Fourier Transform of Sampled Signals

- However, if the sampling frequency is less than twice the bandwidth of $x(t)$… problems!



Relationship between the z-Transform and Fourier Transform

- Recall

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- Since z is a complex variable, we can express it in polar form as $z = r^{ej\theta}$. Thus:

$$X(z)\Big|_{z=re^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-jn\theta}$$

Relationship between the z-Transform and Fourier Transform

- Suppose we let $r = 1$. Then:

$$X(z)\Big|_{z=e^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\theta} = X(\theta)$$

- In other words, the Fourier Transform of a discrete-time signal is obtained by evaluating the z-Transform for $z = e^{j\theta}$.
- Put another way, the Fourier Transform is equal to the z-Transform evaluated on the unit circle in the z-plane

Relationship between the z-Transform and Fourier Transform

- $z = 1$ corresponds to $\theta = 0$ radians, $z = j$ corresponds to $\theta = \pi/2$, $z = -1$ corresponds to $\theta = \pi$ (or $-\pi$), while $z = -j$ corresponds to $\theta = 3\pi/2$ (or $-\pi/2$).
- Also, note that as we arrive back at $z = 1$ after completing our circuit, we're back where we started, i.e. $\theta = 2\pi$ is the same as $\theta = 0$
- As we increase θ beyond 2π, we go around the unit circle again – periodicity again.
- No point in evaluating the Transform for values of θ beyond 2π

Relationship between the z-Transform and Fourier Transform

- Thus, the spectrum from $\theta = 0$ to $\theta = 2\pi$ fully defines the spectrum. Alternatively, we can say that the spectrum from $\theta = -\pi$ to $\theta = \pi$ defines the spectrum, because of "symmetry" in the spectrum – the information contained in the spectrum between $\theta = \pi$ and $\theta = 2\pi$ is the same as that contained in the spectrum from $\theta = -\pi$ to $\theta = 0$.
- Same thing applies for systems – in particular, the frequency response of a system can be obtained from the transfer function of the system, by setting $z = e^{j\theta}$

Relationship between the z-Transform and Fourier Transform

- Alternatively, the frequency response may be obtained directly from the impulse response, using the equation defining the Fourier Transform above.
- Fourier Transform is generally a complex quantity.
- Generally interested in the magnitude spectrum or phase spectrum – if we represent the spectrum at a given frequency θ by $X(\theta) = a + jb$, then:

$$|X(\theta)| = \sqrt{a^2 + b^2}$$

$$\phi(\theta) = \tan^{-1}\left(\frac{b}{a}\right)$$

Relationship between the z-Transform and Fourier Transform

- For real functions of time it is sufficient to calculate the spectrum over the range of frequencies from $\theta = 0$ to $\theta = \pi$, because of conjugate symmetry.

Unit Impulse and Delayed Unit Impulse

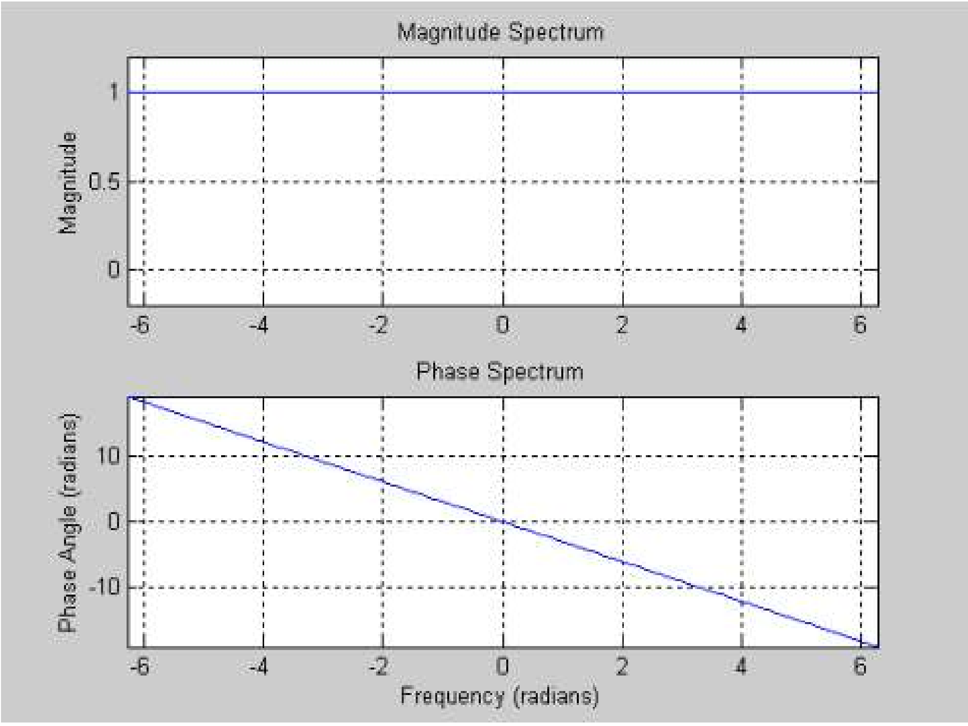
$$Z[\delta(n)] = X(z) = 1$$

- The spectrum of the unit impulse can be found by setting $z = e^{j\theta}$ in $X(z)$ to obtain $X(\theta) = 1$
- In other words, the unit impulse contains all frequencies with amplitude of 1 and a phase angle of 0 degrees
- Delayed unit impulse (delay of k samples):

$$X(\theta) = X(z)\Big|_{z=e^{j\theta}} = z^{-k}\Big|_{z=e^{j\theta}} = e^{-jk\theta} = \cos(k\theta) - j\sin(k\theta)$$

- This has a magnitude of 1 for all frequencies; however, the phase spectrum is given by $\varphi(\theta) = -k\theta$

Unit Impulse and Delayed Unit Impulse



Sampled Exponential

$$x(n) = a^n u(n)$$

$$X(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

$$X(\theta) = X(z)\big|_{z=e^{j\theta}} = \frac{1}{1-ae^{-j\theta}}$$

$$= \frac{1}{1-[a\cos(\theta) - ja\sin(\theta)]}$$

- We obtain the magnitude and phase spectra as indicated above, i.e. treat the spectrum as a complex number

$$X(\theta) = \frac{1}{1-[a\cos(\theta) - ja\sin(\theta)]}$$

$$= \frac{1}{[1-a\cos(\theta)] + ja\sin(\theta)}$$

Sampled Exponential

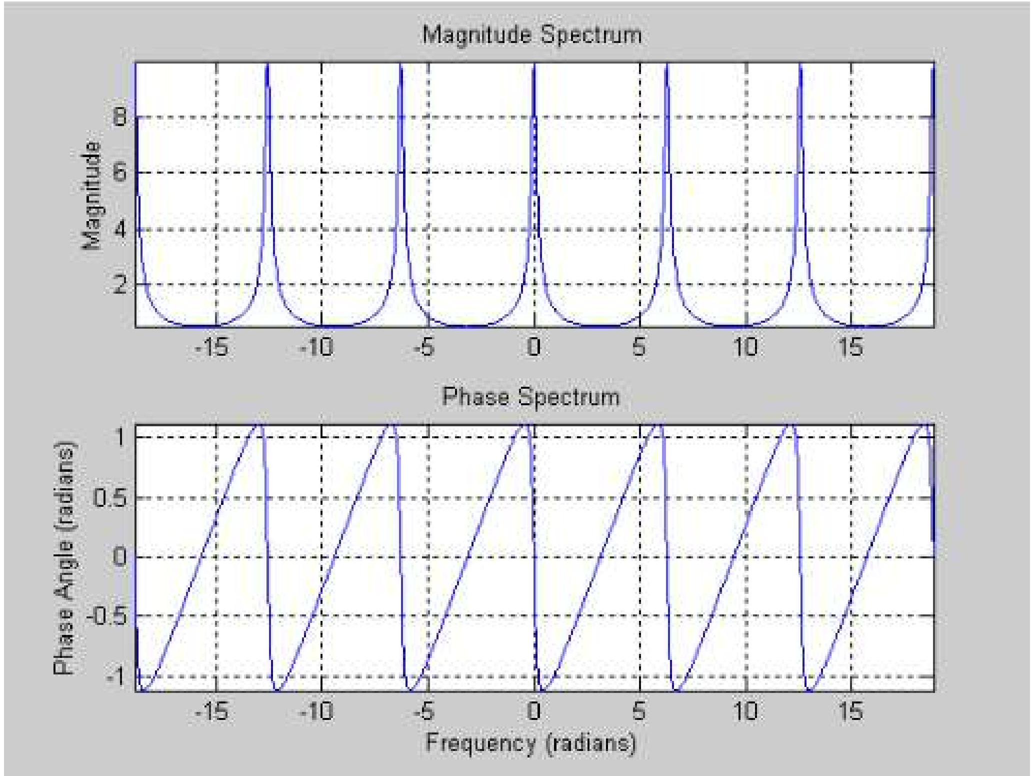
- To obtain the magnitude spectrum, simply take the magnitude of this complex number

$$|X(\theta)| = \frac{1}{\sqrt{[1-a\cos(\theta)]^2+[a\sin(\theta)]^2}}$$
$$= \frac{1}{\sqrt{1-2a\cos(\theta)+a^2\cos^2(\theta)+a^2\sin^2(\theta)}}$$
$$= \frac{1}{\sqrt{1-2a\cos(\theta)+a^2}}$$

- The phase spectrum is given by

$$\phi(\theta) = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)$$
$$= -\tan^{-1}\left(\frac{a\sin(\theta)}{1-a\cos(\theta)}\right)$$

Sampled exponential



Exercise

- Exercise 3.1
- Obtain expressions for the spectra (magnitude and phase) of the following signals:
- (a) The unit step function.
- (b) $x(n) = 0.5^n u(n) + 0.9^n u(n)$
- (c) The sequence consisting of the following samples {1, 0, 1} starting at n = 0.

System Frequency Response

- Convolution property of z-Transform (and continuous-time Fourier Transform) also holds for Fourier Transform of discrete-time signals:
- Fourier Transform of input: $X(\theta) = \mathfrak{J}[x(n)]$
- Filter Frequency Response: $H(\theta) = \mathfrak{J}[h(n)]$
- Then, $Y(\theta) = H(\theta)X(\theta)$
- Also:

$$|Y(\theta)| = |X(\theta)||H(\theta)|$$
$$\phi_y(\theta) = \phi_x(\theta) + \phi_h(\theta)$$

System Frequency Response

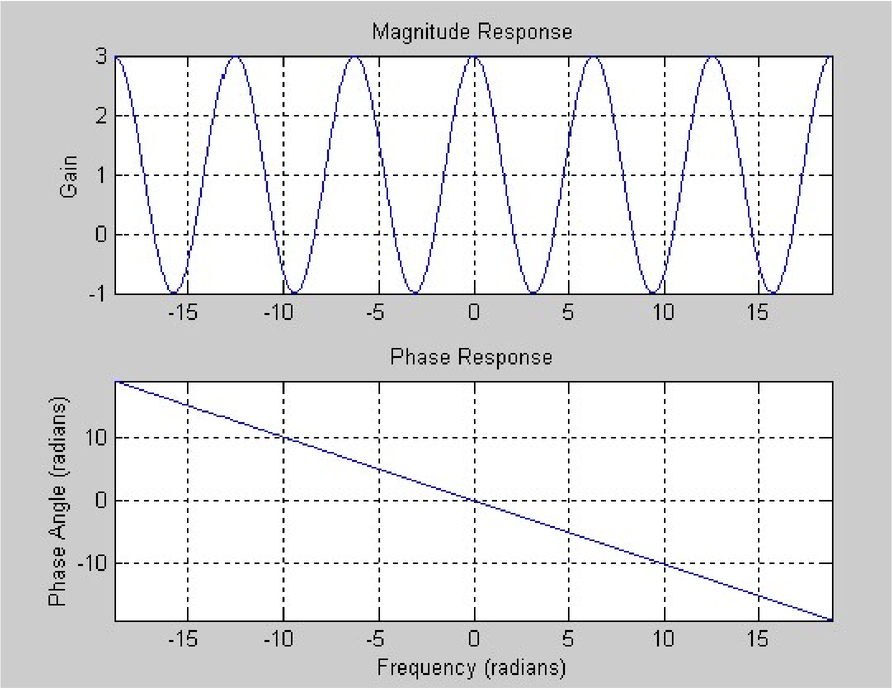
- Normally, the magnitude response of a filter is expressed in dB on a log scale
- Phase response is often shown "modulo-2π", i.e. the phase angle is plotted in the range $-\pi$ to $+\pi$

Examples – FIR filter

- FIR filter
  - Calculate and plot the frequency response of the following system:
    $$y(n) = x(n) + x(n-1) + x(n-2)$$
- Solution
  - Use both approaches – from transfer function, and from impulse response…
  - Transform function can be easily obtained: $H(z) = 1 + z^{-1} + z^{-2}$
  - The frequency response is obtained by substituting $e^{j\Theta}$ for z, to obtain $H(\Theta) = 1 + e^{-j\Theta} + e^{-j2\Theta} = e^{-j\Theta}\left[e^{j\Theta} + 1 + e^{-j\Theta}\right]$
- Cont'd…

Examples – FIR filter

- By Euler's equations, the term in the square brackets can be seen to be equal to $1 + 2\cos(\Theta)$, so we obtain
$$H(\Theta) = e^{-j\Theta}[1 + 2\cos(\Theta)]$$
- The magnitude and phase response of the filter are given by:
$$|H(\Theta)| = 1 + 2\cos(\Theta)$$
$$\phi(\Theta) = -\Theta$$



Examples – FIR filter

- Impulse response of this filter can be written as {1, 1, 1} (starting at n=0). Hence the Fourier Transform is:

$$H(\Theta) = \sum_{n=-\infty}^{\infty} h(n)e^{-jn\Theta}$$

$$= \sum_{n=0}^{2} e^{-jn\Theta}$$

$$= 1 + e^{-j\Theta} + e^{-j2\Theta}$$

- This is clearly the same as we obtained with the transfer function

Examples – IIR filter

- From previous exercise:

$$y(n) = x(n) + 0.6x(n-1) - 0.1x(n-2) + 0.3y(n-1) - 0.3y(n-2)$$

- IIR filter=> impractical to calculate Fourier Transform of impulse response. So, we use the transfer function:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1 + 0.6z^{-1} - 0.1z^{-2}}{1 - 0.3z^{-1} + 0.3z^{-2}}$$

- Hence, the frequency response is given by:

$$H(\Theta) = \frac{1 + 0.6e^{-j\Theta} - 0.1e^{-2j\Theta}}{1 - 0.3e^{-j\Theta} + 0.3e^{-2j\Theta}}$$

$$= \frac{1 + 0.6\cos(\Theta) - j0.6\sin(\Theta) - 0.1cos(2\Theta) + j0.1\sin(2\Theta)}{1 - 0.3\cos(\Theta) + \mathrm{j}0.3\sin(\Theta) + 0.3\cos(2\Theta) - \mathrm{j}0.3\sin(2\Theta)}$$

Examples – IIR filter

- Get the magnitude and phase responses:

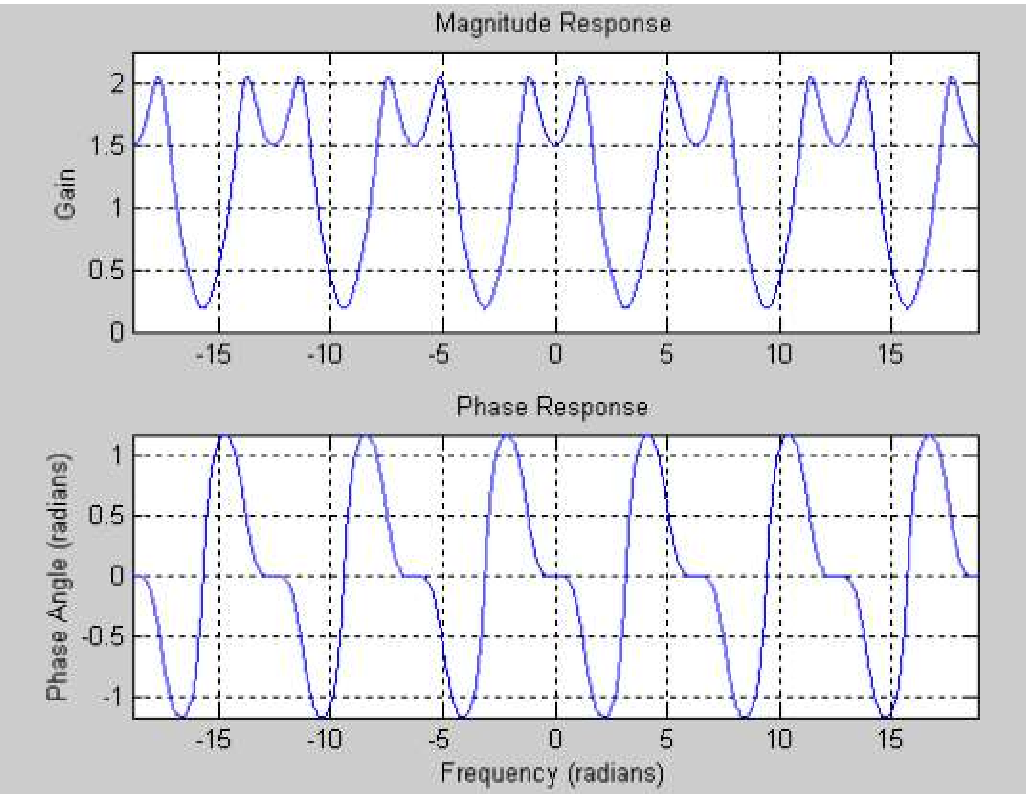
$$|H(\Theta)| = \frac{\sqrt{[1 + 0.6cos(\Theta) - 0.1\cos(2\Theta)]^2 + [-0.6\sin(\Theta) + 0.1\sin(2\Theta)]^2}}{\sqrt{[1 - 0.3\cos(\Theta) + 0.3\cos(2\Theta)]^2 + [0.3\sin(\Theta) - 0.3\sin(2\Theta)]^2}}$$

$$\phi(\Theta) = \tan^{-1}\frac{-0.6\sin(\Theta) + 0.1\sin(2\Theta)}{1 + 0.6cos(\Theta) - 0.1\cos(2\Theta)} - \tan^{-1}\frac{0.3\sin(\Theta) - 0.3\sin(2\Theta)}{1 - 0.3\cos(\Theta) + 0.3\cos(2\Theta)}$$

- Note how the numerator and denominator each contribute to the phase response

- Next step – implement in Matlab…

Examples – IIR filter



Note: can also use Matlab "freqz" function

See example

Phase response

- In general, we are not too concerned about the filter phase response – but there are exceptions. Let’s examine this a bit more
- Phase response of the filter indicates the amount of delay (in radians) which a sinusoid at a particular frequency suffers as it passes through a filter
- Often expressed in terms of a filter phase delay, which is the delay (in units of time) suffered by each frequency component as it passes through the filter

$$\tau_p = -\frac{\phi(\Theta)}{\Theta}$$

Phase response

- We also use *Group Delay*. This is the “average” time delay that a composite signal suffers at each frequency when passing through the filter:

$$\tau_g = -\frac{d\phi(\Theta)}{d\Theta}$$

- If the group delay is not the same for all frequency components in the signal, it means that different frequencies suffer different amounts of delay through the filter - this results in phase distortion in the signal.

Phase response

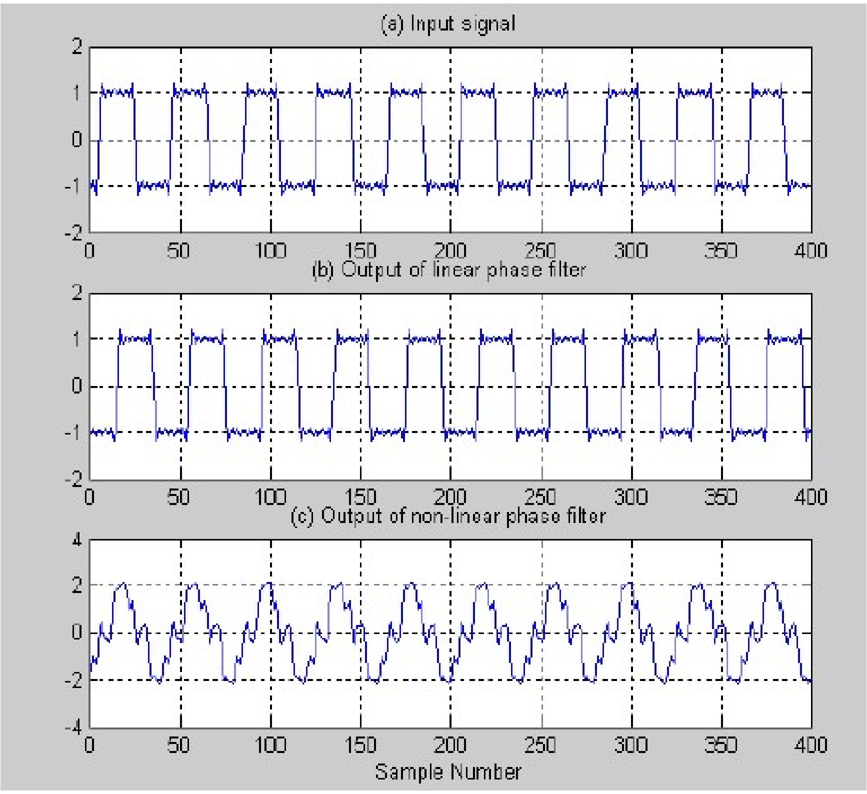
- To have constant group delay at all frequencies, it is necessary for the filter to have linear phase.
- This can be achieved only with certain types of non-recursive (FIR) filters
- A filter is said to have linear phase if the phase response satisfies one of the following relationships

$$\phi(\Theta) = -a\Theta$$

$$\phi(\Theta) = b - a\Theta)$$

- Where a and b are constants (a is the slope of the phase response)
- If the filter has a non-linear phase response, then the group delay will not be constant with frequency.
- Effect of linear and non-linear phase response illustrated in the next slide…

Effect of non-linear phase response



Matlab examples…

Summary

- We have looked at Fourier analysis for discrete-time signals and systems.
- Fourier Transform can be obtained either from the time-domain representation of the signal, or from the z-Transform of the signal.
- Frequency Response of a digital filter may be obtained by the same methods.
- Three ways of describing or analysing signals and systems – time domain, z-domain and frequency domain.
- We can readily move between the three “domains”.
- Important to note the various inter-relationships between the domains …