SEMESTER I EXAMINATIONS 2018-2019

EE445 DIGITAL SIGNAL PROCESSING

SOLUTIONS

1. (a) $$H(z) = \frac{0.3 - 0.25z^{-1}}{1 - 0.5z^{-1} + 0.4z^{-2}}$$

$$H(\theta) = H(z)\big|_{z=e^{j\theta}} = \frac{0.3 - 0.25e^{-j\theta}}{1 - 0.5e^{-j\theta} + 0.4e^{-j2\theta}}$$ 1

$$= \frac{0.3 - 0.25\cos\theta + j0.25\sin\theta}{1 - 0.5\cos\theta + j0.5\sin\theta + 0.4\cos 2\theta - j0.4\sin 2\theta}$$

$$|H(\theta)| = \frac{\sqrt{(0.3 - 0.25\cos\theta)^2 + (0.25\sin\theta)^2}}{\sqrt{(1 - 0.5\cos\theta + 0.4\cos 2\theta)^2 + (0.5\sin\theta - 0.4\sin 2\theta)^2}}$$ 1

$$\angle H(\theta) = \tan^{-1}\left[\frac{0.25\sin\theta}{\cancel{1 - 0.5\cos\theta}\ 0.3 - 0.25\cos\theta}\right] - \tan^{-1}\left[\frac{0.5\sin\theta - 0.4\sin 2\theta}{1 - 0.5\cos\theta + 0.4\cos 2\theta}\right]$$ 1

$$\frac{f_s}{4} \Rightarrow \theta = \frac{2\pi}{4} = \frac{\pi}{2}$$ 1

$$|H(\theta)|_{\theta=\frac{\pi}{2}} = \frac{\sqrt{(0.3 - 0.25\times 0)^2 + (0.25\times 1)^2}}{\sqrt{(1 - 0.5.\phi + 0.4(-1))^2 + (0.5(1) - 0.4(0))^2}}$$

$$= \frac{\sqrt{0.09 + 0.0625}}{\sqrt{(0.6)^2 + (0.5)^2}}$$

$$= \frac{0.3905}{0.7810} = 0.5$$ 1

$$\angle H(\theta)\big|_{\theta=\frac{\pi}{2}} = \tan^{-1}\left[\frac{0.25\sin(\frac{\pi}{2})}{\cancel{1 - 0.5\cos(\frac{\pi}{2})}\ 0.3 - 0.25\cos(\frac{\pi}{2})}\right] - \tan^{-1}\left[\frac{0.5\sin(\frac{\pi}{2}) - 0.4\sin(\pi)}{1 - 0.5\cos(\frac{\pi}{2}) + 0.4\cos(\pi)}\right]$$

$$= \tan^{-1}\left[\frac{0.25}{\cancel{1}\ 0.3}\right] - \tan^{-1}\left[\frac{0.5}{0.6}\right]$$ 1

$$= \cancel{0.245}\ \cancel{0.8333}\ 0.6947 - 0.6947 = \cancel{-0.4497}\ \cancel{0.1386}\ 0 \text{ rads}$$ 1

(b) Poles @ $z = 0.5e^{\pm j0.7}$

$= 0.5\cos(0.7) \pm j0.5\sin(0.7)$

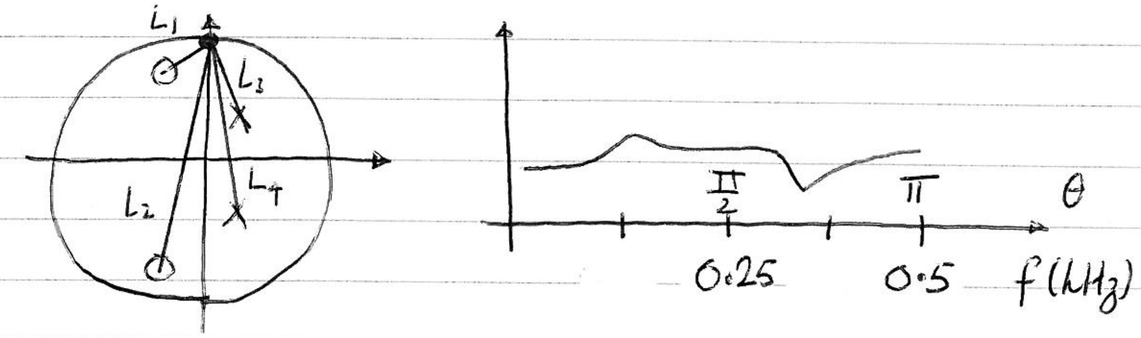
$= 0.38 \pm j0.32$ 1

zero @ $z = 0.8e^{\pm j2}$

$= 0.8\cos(2) \pm j\sin(2)$

$= -0.33 \pm j0.73$ 1

Pole-zero map



1

1

$$\theta_x = 2\pi \frac{250}{1000} = \frac{\pi}{2} = 0 + j1$$

$L_1 = \sqrt{(-0.33-0)^2 + (0.73-1)^2} = 0.4264$

$L_2 = \sqrt{(-0.33-0)^2 + (-0.73-1)^2} = 1.7612$

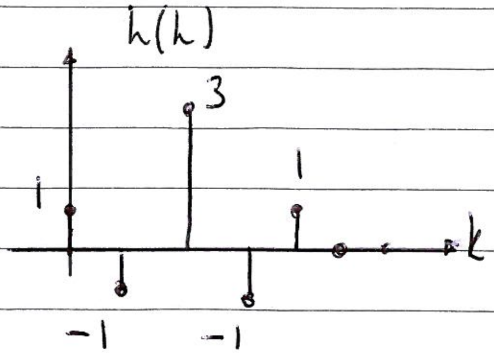
$L_3 = \sqrt{(0.38-0)^2 + (0.32-1)^2} = 0.7790$

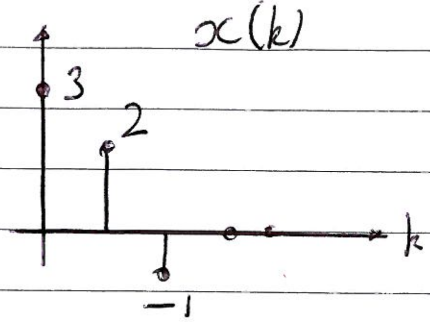
$L_4 = \sqrt{(0.38-0)^2 + (-0.32-1)^2} = 1.3736$ 1

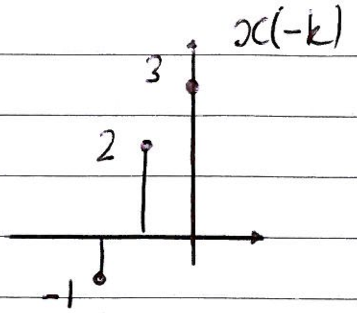
$$|H(\theta_x)| = \frac{L_1 L_2}{L_3 L_4} = \frac{(0.4264)(1.7612)}{(0.7790)(1.3736)} = 0.7078$$ 1

Q1(c) solution corrected Dec 2019

(c)





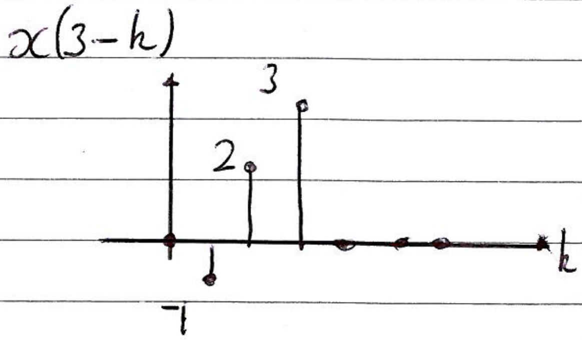


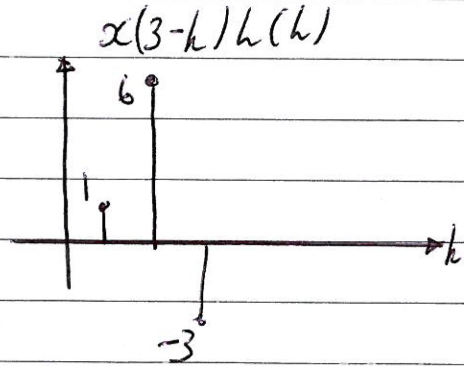
$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\,h(k)$$ 1

$y(0) = 3(1) = 3$

$y(1) = 2(1) + 3(-1) = -1$

$y(2) = -1(1) + 2(-1) + 3(3) = 6$





1

1

$y(3) = 1 + 6 - 3 = 4$

$y(4) = -1(3) + 2(-1) + 3(1) = -2$

$y(5) = -1(-1) + 2(1) = 3$

$y(6) = -1(1) = -1$

$y(n) = \{3, -1, 6, 4, -2, 3, -1\}$ 2

Expect 5+3-1 samples out – correct 1

~~Coeffi~~ Samples of h(n) are symmetric

$\Rightarrow$ linear phase 1

group delay = 2 samples 20

2. (a) $H(z) = 1 + 2z^{-2} - 2z^{-3}$

$X(z) = 1 + z^{-1} - z^{-2} + 3z^{-3}$ 1

$$
\begin{array}{lrrrrrrr}
H(z)\,X(z) & 1 & +z^{-1} & -z^{-2} & +3z^{-3} & & & \\
 & & & 2z^{-2} & +2z^{-3} & -2z^{-4} & +6z^{-5} & \\
 & & & & -2z^{-3} & -2z^{-4} & +2z^{-5} & -6z^{-6} \\
\hline
 & 1 & +z^{-1} & +z^{-2} & +3z^{-3} & -4z^{-4} & +8z^{-5} & -6z^{-6}
\end{array}
$$ 2

$y(n) = \{1, 1, 1, 3, -4, 8, -6\}$ 1

If $x(n)$ is delayed by 3 samples, then $y(n)$ is simply delayed by the same number of samples. 1

(b) $H(\theta) = H(z)\big|_{z=e^{j\theta}}$

$$= \frac{1 - 0.6e^{-j2\theta}}{1 + 0.2e^{-j\theta} - 0.7e^{-j2\theta}}$$ 1

$$= \frac{1 - 0.6\cos 2\theta + j0.6\sin 2\theta}{1 + 0.2\cos\theta - j0.2\sin\theta - 0.7\cos 2\theta + j0.7\sin 2\theta}$$ 1

$$\angle H(\theta) = \tan^{-1}\left[\frac{0.6\sin 2\theta}{1 - 0.6\cos 2\theta}\right] - \tan^{-1}\left[\frac{-0.2\sin\theta + 0.7\sin 2\theta}{1 + 0.2\cos\theta - 0.7\cos 2\theta}\right]$$ 1

$f = \frac{f_s}{4} \Rightarrow \theta = \frac{\pi}{2}$

$$\angle H(\theta)\big|_{\theta = \frac{\pi}{2}} = \tan^{-1}\left[\frac{0}{1.6}\right] - \tan^{-1}\left[\frac{-0.2}{1.7}\right]$$ 1

$$= +0.1171 \text{ rads}$$

(c) $H(z) = \frac{0.15}{1 - 0.85z^{-1}} = \frac{1-a}{1-az^{-1}}$ with $a = 0.85$

$$H(\theta) = \frac{1-a}{1-ae^{-j\theta}}$$ 1

$$|H(\theta)|^2 = \frac{(1-a)^2}{(1-a\cos\theta)^2 + (a\sin\theta)^2}$$

$$= \frac{(1-a)^2}{1-2a\cos\theta + a^2}$$

At $\theta = \theta_c$, $|H(\theta)| = \frac{1}{\sqrt{2}} \Rightarrow |H(\theta)|^2 = \frac{1}{2}$ 1

$$\Rightarrow \frac{(1-a)^2}{1-2a\cos\theta_c + a^2} = \frac{1}{2}$$

$$\Rightarrow (1-a)^2 = 0.5 - a\cos\theta_c + 0.5a^2$$

$$\cos\theta_c = \frac{1}{a}[0.5 - (1-a)^2 + 0.5a^2]$$

$$= \frac{1}{a}[0.5 - 1 + 2a \quad a^2 + 0.5a^2]$$ 1

$$= \frac{1}{a}[-0.5 + 2a - 0.5a^2]$$

$a = 0.85 \Rightarrow \cos\theta_c = 0.9868$

$$\Rightarrow \theta_c = 0.1627 \text{ rad}$$

$$= 2\pi\frac{f_c}{f_s}$$ 1

$f_s = 10\text{kHz} \Rightarrow f_c = \frac{\theta_c f_s}{2\pi}$

$$= 258.9\text{Hz}$$ 1

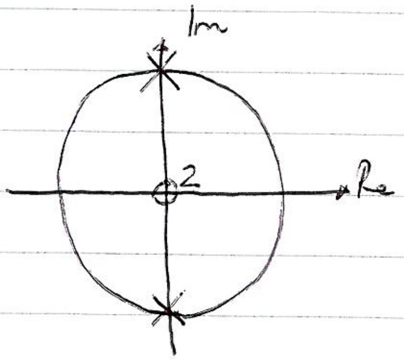
(d) $\theta_0 = 2\pi\frac{4}{16} = \frac{\pi}{2}$ 1

$$r = 1 - \frac{\Delta f}{f_s}\pi = 1 - \frac{40}{16000}\pi = 0.9921$$ 1

$b_1 = -2r\cos\theta_0 = 0$

$b_2 = r^2 = 0.9843$ 1

$1 + b_1 + b_2 = 1.9843$ 1



$$H(z) = \frac{1.9843}{1 + 0.9843z^{-2}}$$ 1

$$y(n) = 1.9843x(n) - 0.9843y(n-2)$$ 1

20

3. (a) Notch filter

$$\theta_0 = 2\pi \frac{60}{1000} = 0.12\pi \qquad 1$$

$$r = 1 - \frac{\Delta f}{f_s}\pi = 1 - \frac{25}{1000}\pi = 0.9215 \qquad 1$$

Poles:

$$b_1 = -2r\cos\theta_0 = -1.7136$$
$$b_2 = r^2 = 0.8492 \qquad 1$$

Zeros:

$$a_1 = -2\cos\theta_0 = -1.8596$$
$$a_2 = 1 \qquad 1$$

$$H(z) = \frac{1 - 1.8596z^{-1} + z^{-2}}{1 - 1.7136z^{-1} + 0.8492z^{-2}} \qquad 1$$

(b) FIR filter with 512 coefficients
Each output sample requires 512 MPY
512 ADD 1

20 seconds of signal ⇒ 1.92 million samples 1

⇒ total of 983,040,000 MPY
983,040,000 ADD 1

FFT: each frame requires

Windowing	$\cancel{N/2}$ 2N	512 ~~256~~	~~symmetric window~~ 1
FFT	$2N\log_2(N)$	9216	
$H(\theta) \times X(\theta)$	$2N$	1024	exploit conjugate symmetry 1
IFFT	$2N\log_2(N)$	9216	
		~~19,712~~ 19,968	MPY 1

1,920,000 samples ⇒ 3750 frames
50% overlap ⇒ 7,500 effective frames 1

FFT requires 7500 x ~~19,712~~ 19,968 = ~~147.84 million~~ MPY 149.76 million MPY 1

$\Rightarrow$ saving $= 1 - \dfrac{\cancel{147.84}\ 149.76}{983.07}$

$= \cancel{84.96\%}$ 84.76% 1

(c) Oscillator

$H(z) = \dfrac{1}{1 - b_1 z^{-1} - b_2 z^{-2}}$

$b_1 = 2\cos\theta_0$

$b_2 = -1$ 1

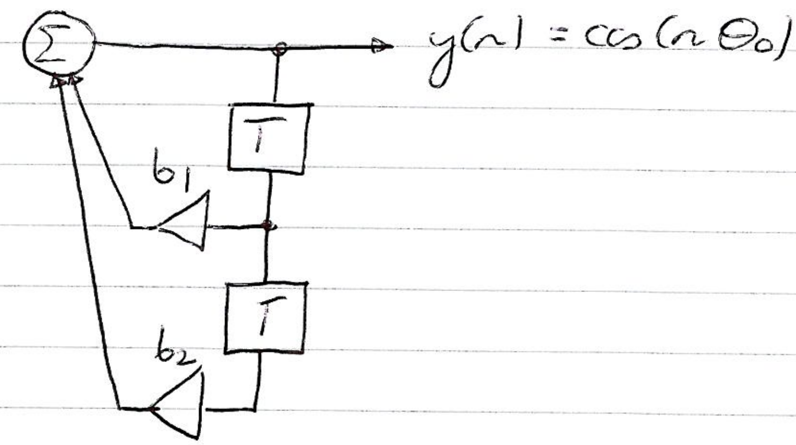
$\theta_0 = 2\pi \dfrac{2}{48} = \dfrac{\pi}{12}$

$\Rightarrow b_1 = 2\cos\left(\frac{\pi}{12}\right) = 1.9318$ 1

Initial conditions: start at $n=0$

$y(n-1) = y(-1) = \cos(-\theta_0) = 0.9659$

$y(n-2) = y(-2) = \cos(-2\theta_0) = 0.8660$ 1



1

Sinewave phase: require phase shift of

$-90° = +\frac{3\pi}{2} = \frac{3}{4}$ of one period

Each cycle contains 24 samples 1

$\Rightarrow$ need starting phase of 18 samples

$y(n-1) = y(17) = -0.2588$

$y(n-2) = y(16) = -0.5$ 1

20

4. (a) $\theta_c = 2\pi \frac{800}{4000} = \frac{2\pi}{5}$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\theta) e^{jn\theta} d\theta$$ 1

$$= \frac{1}{2\pi}\int_{-\pi}^{-\frac{2\pi}{5}} e^{jn\theta} d\theta + \frac{1}{2\pi}\int_{\frac{2\pi}{5}}^{\pi} e^{jn\theta} d\theta$$

$$= \frac{1}{2\pi}\left[\frac{e^{jn\theta}}{jn}\right]_{-\pi}^{-\frac{2\pi}{5}} + \frac{1}{2\pi}\left[\frac{e^{jn\theta}}{jn}\right]_{\frac{2\pi}{5}}^{\pi}$$ 1

$$= \frac{1}{jn2\pi}\left[e^{-jn\frac{2\pi}{5}} - e^{-jn\pi} + e^{jn\pi} - e^{jn\frac{2\pi}{5}}\right]$$

$$= \frac{1}{jn2\pi}\left[\left(e^{jn\pi} - e^{-jn\pi}\right) - \left(e^{jn\frac{2\pi}{5}} - e^{-jn\frac{2\pi}{5}}\right)\right]$$

$$= \frac{1}{jn2\pi} 2j\left[\sin(n\pi) - \sin\left(n\frac{2\pi}{5}\right)\right]$$

$$= \frac{1}{n\pi}\left[\sin(n\pi) - \sin\left(n\frac{2\pi}{5}\right)\right]$$ 1

$$= \frac{\sin(n\pi)}{n\pi} - \frac{1}{n\pi}\sin\left(n\frac{2\pi}{5}\right)$$

$$= \delta(n) - \frac{1}{n\pi}\sin\left(n\frac{2\pi}{5}\right)$$ 1

We require group delay of 5 msec

$= 20$ samples @ $f_s = 4$ kHz 1

$$\alpha = \frac{N-1}{2} \Rightarrow N = 2\alpha + 1$$ 1

$$= 41$$

$$\therefore h(n) = \delta(n-20) - \frac{1}{(n-20)\pi}\sin\left[(n-20)\frac{2\pi}{5}\right]$$ 1

$$n = 0, 1, \ldots, 40$$

(b) $H(s) = \frac{w_c}{s + w_c}$

Desired digital cutoff frequency

$$\theta_c = 2\pi\frac{1.8}{16} = 0.225\pi$$ 1

Pre-warp: $w_c = \frac{2}{T}\tan\left(\frac{\theta_c}{2}\right)$

$$= 32000\tan\left(\frac{0.225\pi}{2}\right)$$

$= 11,805.4$ rad/s 1

Bilinea transform:

$$s = \frac{2}{T} \cdot \frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = \frac{\omega_c}{\frac{2}{T}\cdot\frac{1-z^{-1}}{1+z^{-1}} + \omega_c} = \frac{\omega_c(1+z^{-1})}{(2f_s+\omega_c) + (\omega_c - 2f_s)z^{-1}}$$ 1

Substitute for fs and wc to get

$$H(z) = \frac{11805.4(1+z^{-1})}{(32000+11805.4) + (11805.4-32000)z^{-1}}$$ 1

$$= \frac{11805.4(1+z^{-1})}{43805.4 - 20194.6z^{-1}}$$

$$= \frac{0.2695(1+z^{-1})}{1-0.461z^{-1}}$$ 1

If pre-warping was not carried out:

$$\theta_d = 2\tan^{-1}\left(\frac{\omega_c T}{2}\right)$$ 1

$$= 2\tan^{-1}\left(\frac{3600\pi}{32000}\right) = 0.6794 \text{ rads}$$

$$\theta_d = \frac{2\pi f_d}{f_s}$$

$$\Rightarrow f_d = 1,730.1 \text{ Hz } (\angle 1.8 \text{ kHz})$$ 1

(c) $$H(s) = \frac{3}{(s+4)(s+5)}$$

$$= \frac{A}{s+4} + \frac{B}{s+5}$$ 1

$$B = H(s)(s+5)\big|_{s=-5} = \frac{3}{s+4}\Big|_{s=-5} = -3$$

$$A = H(s)(s+4)\big|_{s=-4} = \frac{3}{s+5}\Big|_{s=-4} = 3$$

$$\Rightarrow H(s) = \frac{3}{s+4} - \frac{3}{s+5}$$ 1

Impulse Invariant Transformation:

$$\frac{K}{s+a} \longrightarrow \frac{K}{1-e^{-aT}z^{-1}}$$ 1

$$H(z) = \frac{3}{1-e^{-4T}z^{-1}} - \frac{3}{1-e^{-5T}z^{-1}}$$

$$= \frac{3(1-e^{-5T}z^{-1})-3(1-e^{-4T}z^{-1})}{(1-e^{-4T}z^{-1})(1-e^{-5T}z^{-1})}$$

$$= \frac{3(e^{-4T}-e^{-5T})z^{-1}}{1-(e^{-4T}+e^{-5T})z^{-1}+e^{-9T}z^{-2}}$$ 1

Choice of fsamp

Highest pole frequency = 5 rad/s

= 0.796 Hz

$f_s = 8 \times 0.796$

$= 6.368\ Hz$

$\Rightarrow T = 0.1575$ 1

$$H(z) = \frac{3(0.5337 - 0.4561)z^{-1}}{1-(0.5337+0.4561)z^{-1}+0.2434z^{-2}}$$

$$= \frac{0.2328z^{-1}}{1-0.9898z^{-1}+0.2434z^{-2}}$$ 1

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