SEMESTER I EXAMINATIONS 2015-16
EE445 DIGITAL SIGNAL PROCESSING
SOLUTIONS

1. (a) $$H(z) = \frac{0.3 - 0.75z^{-2}}{1 + 0.65z^{-1} - 0.5z^{-2}}$$ 1

$$H(\theta) = H(z)\big|_{z=e^{j\theta}}$$

$$= \frac{0.3 - 0.75e^{-j2\theta}}{1 + 0.6e^{-j\theta} - 0.5e^{-j2\theta}}$$ 1

$$= \frac{0.3 - 0.75\cos 2\theta + j0.75\sin 2\theta}{1 + 0.6\cos\theta - j0.6\sin\theta - 0.5\cos 2\theta + j0.5\sin 2\theta}$$

$$|H(\theta)| = \frac{\sqrt{(0.3 - 0.75\cos 2\theta)^2 + (0.75\sin 2\theta)^2}}{\sqrt{(1 + 0.6\cos\theta - 0.5\cos 2\theta)^2 + (-0.6\sin\theta + 0.5\sin 2\theta)^2}}$$ 1

$$\underline{/H(\theta)} = \tan^{-1}\left[\frac{0.75\sin 2\theta}{0.3 - 0.75\cos 2\theta}\right] - \tan^{-1}\left[\frac{-0.6\sin\theta + 0.5\sin 2\theta}{1 + 0.6\cos\theta - 0.5\cos 2\theta}\right]$$ 1

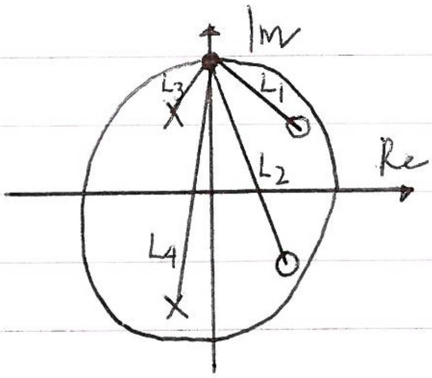
$$\frac{f_s}{8} = \frac{2\pi}{8} = \frac{\pi}{4}$$ 1

$$\underline{/H(\theta)}\Big|_{\theta=\frac{\pi}{4}} = \tan^{-1}\left(\frac{0.75}{0.3}\right) - \tan^{-1}\left(\frac{\cancel{0.0757}\ 0.04038}{\cancel{0.9243}\ 1.4596}\right)$$

$$= 1.1903 - \cancel{0.0817}\ 0.0277$$

$$= \cancel{1.1086}\ 1.1626 \text{ rads}$$ 2

(b)



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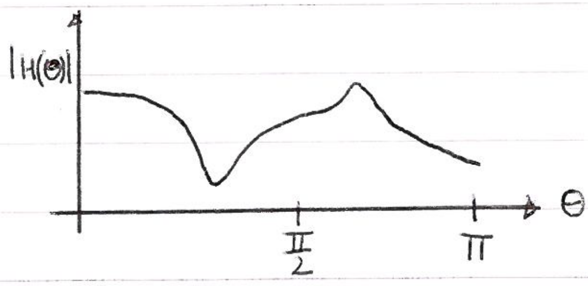


1

$$f_x = 500\text{ Hz} \Rightarrow \theta_x = \frac{2\pi 500}{2000} = \frac{\pi}{2} = 0 + j1$$ 1

$$|H(\theta_x)| = \frac{L_1 L_2}{L_3 L_4}$$ 1

$$L_1 = \sqrt{(0 - 0.7)^2 + (1 - 0.5)^2} = 0.8602$$

$$L_2 = \sqrt{(0 - 0.7)^2 + (1 + 0.5)^2} = 1.6553$$

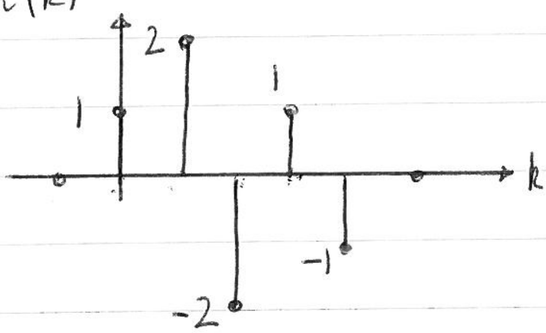
$$L_3 = \sqrt{(0+0.3)^2 + (1-0.7)^2} = 0.4243$$

$$L_4 = \sqrt{(0+0.3)^2 + (1+0.7)^2} = 1.7263$$ 1

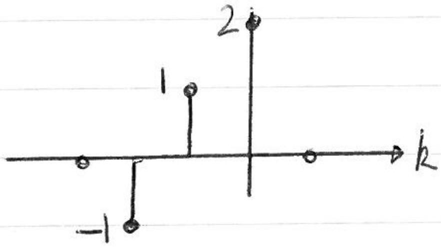
$$|H(\theta_x)| = \frac{(0.8602)(1.6553)}{(0.4243)(1.7263)} = 1.944$$ 2

(c) $$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$ 1

$h(k)$



$x(-k)$



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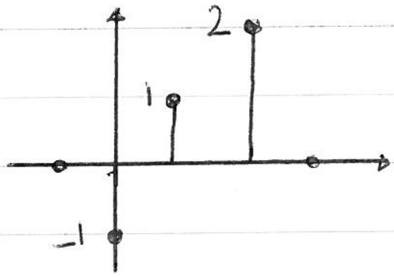
$x(2-k)$



$y(0) = 1(2) = 2$

$y(1) = 1(1) + 2(2) = 5$

$y(2) = 1(-1) + 1(2) + 2(-2)$
$= -3$

$y(3) = 2(-1) + (-2)1 + 1(2)$
$= -2$

$$y(2) = \sum_{k=0}^{2} h(k)x(2-k)$$ 1

$y(4) = (-2)(-1) + 1(1) + (-1)2$
$= 1$

$y(5) = 1(-1) + (-1)(1) = -2$ 3

$y(6) = (-1)(-1) = 1$

$\therefore y(n) = \{2, 5, -3, -2, 1, -2, 1\}$

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2. (a) $H(z) = 1 - z^{-1} + 3z^{-3}$

$X(z) = z^{-2} + 3z^{-3} - z^{-4} + 2z^{-5}$ 1

$$
\begin{array}{r}
z^{-2}+3z^{-3}-z^{-4}+2z^{-5} \\
-z^{-3}-3z^{-4}+z^{-5}-2z^{-6} \\
3z^{-5}+9z^{-6}-3z^{-7}+6z^{-8} \\
\hline
\end{array}
$$
3

$Y(z) = z^{-2} + 2z^{-3} - 4z^{-4} + 6z^{-5} + 7z^{-6} - 3z^{-7} + 6z^{-8}$

$\therefore y(n) = \{1, 2, -4, 6, 7, -3, 6\}$ starting @ $n=2$ 2

(b) $H(\theta) = \dfrac{1 - 0.6e^{-j2\theta}}{1 + 0.2e^{-j\theta} - 0.7e^{-j2\theta}}$ 1

$= \dfrac{1 - 0.6\cos 2\theta + j0.6\sin 2\theta}{1 + 0.2\cos\theta - j0.2\sin\theta - 0.7\cos 2\theta + j0.7\sin 2\theta}$ 1

$\underline{/H(\theta)} = \tan^{-1}\left[\dfrac{0.6\sin 2\theta}{1 - 0.6\cos 2\theta}\right] - \tan^{-1}\left[\dfrac{-0.2\sin\theta + 0.7\sin 2\theta}{1 + 0.2\cos\theta - 0.7\cos 2\theta}\right]$

$f = \dfrac{f_s}{4} \Rightarrow \theta = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$ 1

$\Rightarrow \underline{/H(\tfrac{\pi}{2})} = \tan^{-1}\left[\dfrac{0}{1.6}\right] - \tan^{-1}\left[\dfrac{-0.2}{1.7}\right]$ 1

$= +0.1171$ rads

(c) $H(z) = \dfrac{0.2}{1 - 0.8z^{-1}} = \dfrac{1-a}{1 - az^{-1}}, a = 0.8$ 1

$H(\theta) = \dfrac{1-a}{1 - ae^{-j\theta}} = \dfrac{1-a}{1 - a\cos\theta + ja\sin\theta}$

$|H(\theta)|^2 = \dfrac{(1-a)^2}{(1 - a\cos\theta)^2 + (a\sin\theta)^2}$ 1

$= \dfrac{(1-a)^2}{1 - 2a\cos\theta + a^2}$

At $\theta = \theta_c$, $|H(\theta)| = \dfrac{1}{\sqrt{2}} \Rightarrow |H(\theta)|^2 = \dfrac{1}{2}$ 1

$$\Rightarrow \quad \frac{(1-a)^2}{1-2a\cos\theta_c + a^2} = 0.5$$

$$\Rightarrow \quad (1-a)^2 = 0.5 - a\cos\theta_c + 0.5a^2$$

$$\Rightarrow \quad \cos\theta_c = \left[0.5-(1-a)^2+0.5a^2\right]/a$$

$$= \frac{1}{a}\left[-0.5+2a-0.5a^2\right]$$ 1

Set $a = 0.8$ to get

$$\cos\theta_c = 0.975$$

$$\Rightarrow \quad \theta_c = 0.2241$$

$$= 2\pi\frac{f_c}{f_s}$$ 1

$$\Rightarrow \quad f_c = \frac{\theta_c f_s}{2\pi}$$

$$= 285.33\ \text{Hz}$$ 1

(d) $$H(z) = \frac{a}{1-bz^{-1}}$$ 1

$$H(\theta) = \frac{a}{1-be^{-j\theta}}$$ 1

$$H(\theta)\big|_{\theta=0} = \frac{a}{1-b}$$ 1

For $|H(\theta)|_{\theta=0} = 0.8$, we require

$$\frac{a}{1-b} = 0.8$$

$$\Rightarrow \quad a = 0.8(1-b)$$ 1

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3. (a) $\theta_0 = 2\pi \frac{1.5}{20} = \frac{3\pi}{20}$ 1

$r = 1 - \frac{\Delta f}{f_s}\pi = 1 - \frac{40}{20kHz}\pi$ 1

$= 0.9937$

$b_1 = -2r\cos\theta_0 = -1.7708$

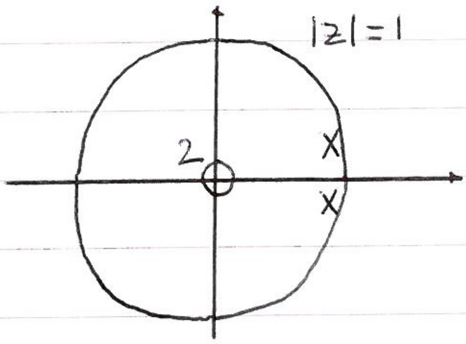
$b_2 = r^2 = 0.9874$ 1

Numerator $= 0.6(1 + b_1 + b_2)$

$= 0.13$

$\Rightarrow H(z) = \dfrac{0.13}{1 - 1.7708z^{-1} + 0.9874z^{-2}}$

1



$y(n) = 0.13x(n) + 1.7708y(n-1)$
$- 0.9874y(n-2)$ 1

(b) $f_s = 32\,kHz$

$T_{win} = \cancel{25}\ 40\ ms \Rightarrow N = (25\,ms)(32\,kHz)$

$= \cancel{800}\ 1280$ samples 1

We require $\Delta f \leq 20\,Hz$

$\Rightarrow \frac{32000}{N_{FFT}} \leq 20\,Hz$ 1

$\Rightarrow N_{FFT} \geq \frac{32000}{20}$

≥ 1600 1

N_{FFT} must be a power of 2, therefore

$N_{FFT} = 2048$

$\Rightarrow$ no. of samples for zero-padding

$= 2048 - \cancel{800}\ 1280 = \cancel{1,248}\ 768$ 1

(c) FIR filter, $N = 512$

linear phase $\Rightarrow$ 256 unique coefficients 1

Each output calculation requires

256 MPY

512 ADD 1

30 seconds of data requires

$(30 \times 96\,000) \times 256 = 737{,}280{,}000$ MPY

$(30 \times 96\,000) \times 256 = 1{,}474{,}560{,}000$ ADD 1

For FFT, each frame of data requires:

Windowing	N	512
FFT	$2N\log_2(N)$	9216
$H(\theta) \times X(\theta)$	$4N$	2048
IFFT	$2N\log_2(N)$	9216
		$20{,}99\cancel{8}2$

2

30 seconds of data @ $f_s = 96\,kHz$

$\Rightarrow$ 5,625 frames

50% overlap $\Rightarrow$ 11,250 "equivalent" frames 1

$\Rightarrow$ FFT approach requires

236,160,000

$11{,}250 \times 20{,}99\cancel{8}2 = \cancel{236{,}227{,}500}$ MPY

Saving = ~~67%~~ 67.9% 1

(d) Oscillator : $b_1 = 2\cos\theta_0$

$\theta_0 = 2\pi \frac{1}{30} = \frac{2\pi}{30} \Rightarrow b_1 = 1.9563 \; ; \; b_2 = 1$ 1

Initial condition: we require phase shift of $\frac{\pi}{3}$, which is $1/6$ of a period

One period = 30 samples 1

$\Rightarrow \frac{\pi}{3} = 5$ samples 1

$y(n-1) = \cos(4\theta_0) = 0.6691$ 20

$y(n-2) = \cos(3\theta_0) = 0.8090$ 1

4. (a) Notch filter

$\theta_0 = 2\pi \frac{50}{500} = \frac{\pi}{5}$ 1

$r = 1 - \frac{10}{500}\pi = 0.9372$ 1

Poles: $b_1 = -2r\cos\theta_0 = -1.5164$

$b_2 = r^2 = 0.8783$ 1

Zeros: $a_1 = -2\cos\theta_0 = -1.6180$ 1

$a_2 = 1$ 1

$H(z) = \dfrac{1 - 1.6180 z^{-1} + z^{-2}}{1 - 1.5164 z^{-1} + 0.8783 z^{-2}}$ 1

(b) $\theta_c = 2\pi \frac{1}{8} = \frac{\pi}{4}$

$|H(\theta)| = 1, \quad |\theta| \geq \frac{\pi}{4}$

$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\theta) e^{jn\theta}\, d\theta$ 1

$= \frac{1}{2\pi}\int_{-\pi}^{-\frac{\pi}{4}} e^{jn\theta}\, d\theta + \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\pi} e^{jn\theta}\, d\theta$

$= \frac{1}{2\pi}\left[\frac{e^{jn\theta}}{jn}\right]_{-\pi}^{-\frac{\pi}{4}} + \frac{1}{2\pi}\left[\frac{e^{jn\theta}}{jn}\right]_{\frac{\pi}{4}}^{\pi}$ 1

$= \frac{1}{j2n\pi}\left[e^{-jn\frac{\pi}{4}} - e^{-jn\pi} + e^{jn\pi} - e^{jn\frac{\pi}{4}}\right]$ 1

$= \frac{1}{j2n\pi}\left[\left(e^{jn\pi} - e^{-jn\pi}\right) - \left(e^{jn\frac{\pi}{4}} - e^{-jn\frac{\pi}{4}}\right)\right]$

$= \frac{1}{j2n\pi} 2j\left[\sin(n\pi) - \sin\left(n\frac{\pi}{4}\right)\right]$

$= \frac{1}{n\pi}\sin(n\pi) - \frac{1}{n\pi}\sin\left(\frac{n\pi}{4}\right)$ 1

$= \delta(n) - \frac{1}{n\pi}\sin\left(\frac{n\pi}{4}\right)$

Group delay of 5 msec = 40 samples 1

$\alpha = \frac{N-1}{2} \Rightarrow N = 2\alpha + 1$

$= 81$ coefficients 1

$\Rightarrow h(n) = \delta(n-40) - \frac{1}{(n-40)\pi}\sin\left[\frac{(n-40)\pi}{4}\right]$

$n = 0, \ldots, 80$ 1

(c) $H(s) = \dfrac{1}{(s+4)(s+9)} = \dfrac{A}{s+4} + \dfrac{B}{s+9}$ 1

$A = H(s)(s+4)\big|_{s=-4} = \frac{1}{5}$

$B = H(s)(s+9)\big|_{s=-9} = -\frac{1}{5}$ 1

$\Rightarrow H(s) = \dfrac{1/5}{s+4} - \dfrac{1/5}{s+9}$

IIT: $\dfrac{1}{s+a} \longrightarrow \dfrac{1}{1-e^{-aT}z^{-1}}$ 1

$\Rightarrow H(z) = \dfrac{1/5}{1-e^{-4T}z^{-1}} - \dfrac{1/5}{1-e^{-9T}z^{-1}}$ 1

$= \dfrac{\frac{1}{5}(1-e^{-9T}z^{-1}) - \frac{1}{5}(1-e^{-4T}z^{-1})}{(1-e^{-4T}z^{-1})(1-e^{-9T}z^{-1})}$

$= \dfrac{\frac{1}{5}(e^{-4T}-e^{-9T})z^{-1}}{1-(e^{-4T}+e^{-9T})z^{-1}+e^{-13T}z^{-2}}$ 1

Sampling rate:

Highest pole frequency = 9 rad/s = 1.432 Hz

$\Rightarrow f_s = 8 \times 1.432 = 11.456$ Hz

$\Rightarrow T = 0.0873$ s 1

$\Rightarrow H(z) = \dfrac{\frac{1}{5}(0.7053 - 0.4558)z^{-1}}{1-(0.7053+0.4558)z^{-1}+0.3215z^{-2}}$ 1

$= \dfrac{0.05z^{-1}}{1-1.1611z^{-1}+0.3215z^{-2}}$ 1

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