SEMESTER I EXAMINATIONS 2013/14 V11

EE445 DIGITAL SIGNAL PROCESSING

SOLUTIONS

1 (a) $y(n) = x(n) - 0.6x(n-1) + 0.4y(n-1) - 0.5y(n-2)$

$$H(z) = \frac{1 - 0.6z^{-1}}{1 - 0.4z^{-1} + 0.5z^{-2}} \qquad 1$$

$$H(\theta) = H(z)\big|_{z=e^{j\theta}}$$

$$= \frac{1 - 0.6e^{-j\theta}}{1 - 0.4e^{-j\theta} + 0.5e^{-j2\theta}} \qquad 1$$

$$= \frac{1 - 0.6\cos\theta + j0.6\sin\theta}{1 - 0.4\cos\theta + j0.4\sin\theta + 0.5\cos 2\theta - j0.5\sin 2\theta} \qquad 1$$

$$|H(\theta)| = \frac{\sqrt{(1-0.6\cos\theta)^2 + (0.6\sin\theta)^2}}{\sqrt{(1-0.4\cos\theta + 0.5\cos 2\theta)^2 + (+0.4\sin\theta - 0.5\sin 2\theta)^2}} \qquad 1$$

$$\angle H(\theta) = \tan^{-1}\left[\frac{0.6\sin\theta}{1-0.6\cos\theta}\right]$$

$$- \tan^{-1}\left[\frac{0.4\sin\theta - 0.5\sin 2\theta}{1 - 0.4\cos\theta + 0.5\cos 2\theta}\right] \qquad 1$$

$$\frac{f_s}{4} = \frac{\pi}{2} \qquad 1$$

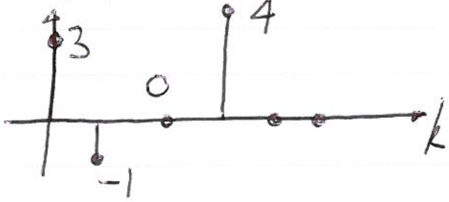
$$\left|H\left(\tfrac{\pi}{2}\right)\right| = \frac{\sqrt{(1-0.6\cos\frac{\pi}{2})^2 + (0.6\sin\frac{\pi}{2})^2}}{\sqrt{(1-0.4\cos\frac{\pi}{2} + 0.5\cos\pi)^2 + (0.4\sin\frac{\pi}{2} - 0.5\sin\pi)^2}} \qquad 1$$

$$= \frac{\sqrt{(1-0)^2 + (0.6)^2}}{\sqrt{(1-0-0.5)^2 + (0.4-0)^2}} \qquad 1$$

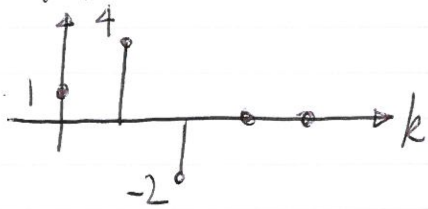
$$= \frac{\sqrt{1.36}}{\sqrt{0.25+0.16}} = \frac{\sqrt{1.36}}{\sqrt{0.41}}$$

$$= 1.8213 \qquad 1$$

(b) $x(k)$

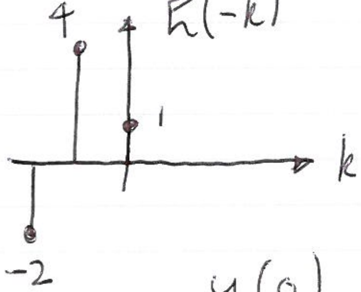


$h(k)$



1

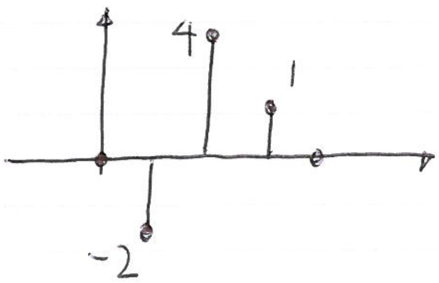
$h(-k)$



$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$ 1

$$y(0) = 3(1) = 3$$
$$y(1) = 4(3) - 1(1) = 11$$
$$y(2) = 3(-2) - 1(4) + 0(1) = -10$$

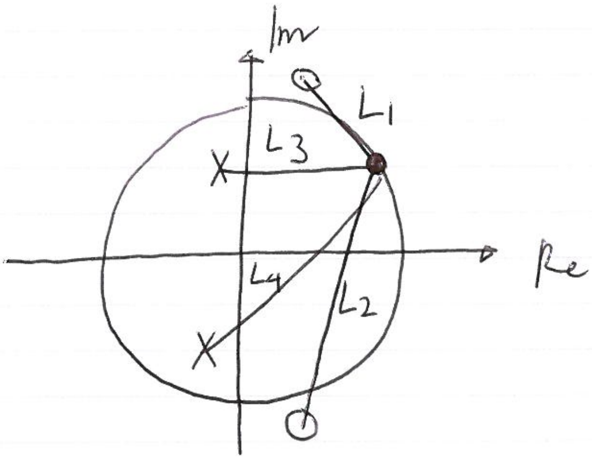
$h(3-k)$



$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k)$$ 1
$$= \sum_{k=1}^{3} x(k)h(3-k)$$
$$= -1(-2) + 0(4) + 4(1) = 6$$ 1

$$y(4) = 0(-2) + 4(4) = 16$$
$$y(5) = 4(-2) = -8$$
$$y(n) = \{3, 11, -10, 6, 16, -8\}$$ 2

(c)



$$\theta_x = 2\pi \frac{100}{800} = \frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$ 1

1

$$|H(\theta_x)| = \frac{L_1 L_2}{L_3 L_4}$$ 1

$$L_1 = \sqrt{(0.707-0.3)^2+(0.707-1.1)^2} = 0.565$$

$$L_2 = \sqrt{(0.707-0.3)^2+(0.707-(-1.1))^2} = 1.8523$$

1

$$L_3 = \sqrt{(0.707-(-0.2))^2+(0.707-0.7)^2} = 0.9070$$

$$L_4 = \sqrt{(0.707-(-0.2))^2+(0.707-(-0.7))^2} = 1.674$$

$$\Rightarrow |H(\theta x)| = \frac{(0.565)(1.8523)}{(0.907)(1.674)} = 0.6893$$

1

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2. (a) $H(z) = \dfrac{0.15}{1-0.85z^{-1}} = \dfrac{1-a}{1-az^{-1}}$ where $a = 0.85$

$$H(\theta) = \frac{1-a}{1-ae^{-j\theta}}$$ 1

$$|H(\theta)|^2 = \frac{(1-a)^2}{(1-a\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \frac{(1-a)^2}{1-2a\cos\theta + a^2}$$ 1

At $\theta = \theta_c$, $|H(\theta)|^2 = \frac{1}{2}$

$$\Rightarrow \frac{(1-a)^2}{1-2a\cos\theta_c + a^2} = 0.5$$ 1

$$\Rightarrow (1-a)^2 = 0.5 - a\cos\theta_c + 0.5a^2$$

$$\Rightarrow \cos\theta_c = \frac{0.5-(1-a)^2+0.5a^2}{a}$$

$$= \frac{0.5-1+2a-a^2+0.5a^2}{a}$$

$$= \frac{-0.5+2a-0.5a^2}{a}$$

$a = 0.85 \Rightarrow \cos\theta_c = 0.9868$

$$\Rightarrow \theta_c = 0.1627$$ 1

$$= 2\pi\frac{f_c}{f_s}$$

$$\Rightarrow f_c = \frac{\theta_c f_s}{2\pi}$$

$$= 258.9\text{ Hz}$$ 1

(b) $X(z) = z^{-1} + 2z^{-2} + 3z^{-3} - z^{-4}$ NB: starts @ $n=1$ 1

$H(z) = 1 - z^{-1} + z^{-2}$ 1

$$\begin{array}{rrrrrrr}
X(z)H(z) = & z^{-1} & +2z^{-2} & +3z^{-3} & -z^{-4} & & \\
 & & -z^{-2} & -2z^{-3} & -3z^{-4} & +z^{-5} & \\
 & & & z^{-3} & +2z^{-4} & +3z^{-5} & -z^{-6} \\
\hline
 & z^{-1} & +z^{-2} & +2z^{-3} & -2z^{-4} & +4z^{-5} & -z^{-6}
\end{array}$$ 1

$\Rightarrow y(n) = \{1, 1, 2, -2, 4, -1\}$ commencing @ $n = 1$ 2

(c) $$H(z) = \frac{1 + 3z^{-1}}{1 + 0.4z^{-1} - 0.7z^{-2}}$$

$$H(\theta) = \frac{1 + 3e^{-j\theta}}{1 + 0.4e^{-j\theta} - 0.7e^{-j2\theta}}$$ 1

$$= \frac{1 + 3\cos\theta - j3\sin\theta}{1 + 0.4\cos\theta - j0.4\sin\theta - 0.7\cos 2\theta + j0.7\sin 2\theta}$$ 1

$$\underline{\angle H(\theta)} = \tan^{-1}\left[\frac{-3\sin\theta}{1 + 3\cos\theta}\right] - \tan^{-1}\left[\frac{-0.4\sin\theta + 0.7\sin 2\theta}{1 + 0.4\cos\theta - 0.7\cos 2\theta}\right]$$ 1

$$\frac{f_s}{5} = \frac{2\pi}{5}$$

$$\Rightarrow \underline{\angle H(\theta)}\Big|_{\theta = \frac{2\pi}{5}} = \tan^{-1}\left[\frac{-3\sin\frac{2\pi}{5}}{1 + 3\cos\frac{2\pi}{5}}\right] - \tan^{-1}\left[\frac{-0.4\sin\frac{2\pi}{5} + 0.7\sin\frac{4\pi}{5}}{1 + 0.4\cos\frac{2\pi}{5} - 0.7\cos\frac{4\pi}{5}}\right]$$

$$= \tan^{-1}\left[\frac{-2.85}{1.927}\right] - \tan^{-1}\left[\frac{0.031}{1.6899}\right]$$ 1

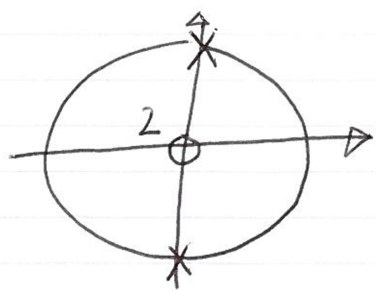
$$= -0.9763 - 0.0183$$

$$= -0.9946 \text{ rads}$$ 1

(d) $$\theta_0 = 2\pi\frac{4}{16} = \frac{\pi}{2}$$ 1

$$r = 1 - \frac{\Delta f}{f_s}\pi = 1 - \frac{40}{16000}\pi = 0.9921$$ 1

$$b_1 = -2r\cos\theta_0 = 0$$
$$b_2 = r^2 = 0.9843$$ 1



$$H(z) = \frac{1.9843}{1 + 0.9843z^{-2}}$$ 2

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3. (a) Notch filter

$$\theta_0 = 2\pi \frac{60}{1000} = 0.12\pi$$ 1

$$r = 1 - \frac{\Delta f}{f_s}\pi = 1 - \frac{25}{1000}\pi = 0.9215$$ 1

Poles:

$$b_1 = -2r\cos\theta_0 = -1.7136$$

$$b_2 = r^2 = 0.8492$$ 1

Zeros:

$$a_1 = -2\cos\theta_0 = -1.8596$$ 1

$$a_2 = 1$$

$$\therefore H(z) = \frac{1 - 1.8596z^{-1} + z^{-2}}{1 - 1.7136z^{-1} + 0.8492z^{-2}}$$ 1

(b) $f_s = 16\text{ kHz}$

$$T_{win} = 30\text{ msec} \Rightarrow N = (30\text{ msec})(16\text{ kHz})$$

$$= 480$$ 1

$$\Delta f \leq 20\text{ Hz}$$

$$\Rightarrow N_{FFT} \geq \frac{16000}{20} \geq 800$$ 1

N_{FFT} must be a power of 2, therefore

$$N_{FFT} = 1024$$ 1

$\Rightarrow$ no. of samples for zero-padding

$$= 1{,}024 - 480 = 544$$ 1

(c) FIR filter, $N = 512$ coefficients

linear phase $\Rightarrow$ coefficients are symmetric
$\Rightarrow$ 256 unique coefficients 1

Each coefficient multiplies two data values added together, in each sampling interval

$\Rightarrow$ each output sample requires
512 ADD
256 MPY 1

10 seconds of data at $f_s = 16$ kHz requires

$(10 \times 16000) \times 512 = 81{,}920{,}000$ ADD
$(10 \times 16000) \times 256 = 40{,}960{,}000$ MPY 1

For FFT, each frame requires the following number of MPY

Windowing	N	512
512-point FFT	$2N\log_2(N)$	9,216
Multiplication of $X(\theta)$ by $H(\theta)$	$4N$	2,048
Inverse FFT	$2N\log_2(N)$	9,216
		20,992

1

1

10 seconds of data = 312.5 frames
50% overlap $\Rightarrow$ 625 "equivalent" frames

$\Rightarrow$ total MPY $= 625 \times 20{,}992$
$= 13{,}120{,}000$

$\Rightarrow$ saving $= 83.98\%$ 1

(d) Oscillator

$b_1 = 2\cos\theta_0$

$b_2 = -1$ 1

$\theta_0 = 2\pi \frac{2}{48} = \frac{\pi}{12} \Rightarrow b_1 = 1.9318$ 1

Initial conditions

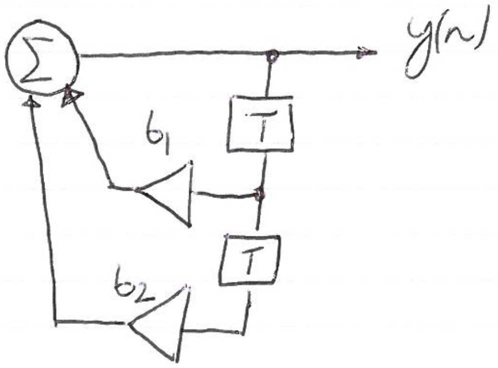
We require a phase shift of $\frac{\pi}{3}$, which is one-sixth of a period

Each cycle contains $\frac{48}{2} = 24$ samples 1

$\Rightarrow \frac{\pi}{3}$ corresponds to 4 samples

$\Rightarrow y(n-1) = \cos(3\theta_0) = 0.7071$

$y(n-2) = \cos(2\theta_0) = 0.8660$ 1



1

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4. (a) $\theta_c = 2\pi \frac{800}{4000} = \frac{2\pi}{5}$ 1

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\theta) e^{jn\theta} d\theta$$ 1

$$= \frac{1}{2\pi}\int_{-\pi}^{-\frac{2\pi}{5}} e^{jn\theta} d\theta + \frac{1}{2\pi}\int_{\frac{2\pi}{5}}^{\pi} e^{jn\theta} d\theta$$ 1

$$= \frac{1}{2\pi}\left[\frac{e^{jn\theta}}{jn}\right]_{-\pi}^{-\frac{2\pi}{5}} + \frac{1}{2\pi}\left[\frac{e^{jn\theta}}{jn}\right]_{\frac{2\pi}{5}}^{\pi}$$

$$= \frac{1}{jn2\pi}\left[e^{-jn\frac{2\pi}{5}} - e^{-jn\pi} + e^{jn\pi} - e^{jn\frac{2\pi}{5}}\right]$$

$$= \frac{1}{jn2\pi}\left[(e^{jn\pi} - e^{-jn\pi}) - (e^{jn\frac{2\pi}{5}} - e^{-jn\frac{2\pi}{5}})\right]$$ 1

$$= \frac{1}{jn2\pi} 2j\left[\sin(n\pi) - \sin(n\tfrac{2\pi}{5})\right]$$

$$= \frac{1}{n\pi}\sin(n\pi) - \frac{1}{n\pi}\sin(n\tfrac{2\pi}{5})$$

$$= \delta(n) - \frac{1}{n\pi}\sin(n\tfrac{2\pi}{5})$$ 1

We require group delay of 5 msec
$= 20$ samples @ $f_s = 4$ kHz

$\alpha = \frac{N-1}{2} \Rightarrow N = 2\alpha + 1 = 41$ coefficients 1

$$\therefore h(n) = \delta(n-20) - \frac{1}{(n-20)\pi}\sin\left[(n-20)\tfrac{2\pi}{5}\right]$$
$n = 0, 1, \dots 40$ 1

(b) $H(s) = \frac{\omega_c}{s+\omega_c}$

Desired digital cut-off frequency
$\theta_c = 2\pi \frac{1.8}{16} = 0.225\pi$ 1

Pre-warp: $\omega_c = \frac{2}{T}\tan\left(\frac{\theta_c}{2}\right)$

$= 32000 \tan\left(\frac{0.225\pi}{2}\right)$ 1

$= 11{,}805.4 \text{ rad/s}$

Bilinear transform

$$s = \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}} \qquad 1$$

$$H(z) = \frac{w_c}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + w_c} = \frac{w_c(1+z^{-1})}{(2f_s + w_c) + (w_c - 2f_s)z^{-1}} \qquad 1$$

Substitute for w_c and f_s to get:

$$\frac{11805.4(1+z^{-1})}{(32000 + 11805.4) + (11805.4 - 32000)z^{-1}}$$

$$= \frac{11805.4(1+z^{-1})}{43805.4 - 20194.6z^{-1}}$$

$$= \frac{0.2695(1+z^{-1})}{1 - 0.461z^{-1}} \qquad 1$$

If pre-warping was not carried out:

$$\theta_d = 2\tan^{-1}\left(\frac{w_c T}{2}\right)$$

$$= 2\tan^{-1}\left(\frac{3600\pi}{32000}\right) = 0.6794 \text{ rad} \qquad 1$$

$$\theta_d = 2\pi \frac{f_d}{f_s}$$

$$\Rightarrow f_d = \frac{\theta_d f_s}{2\pi} = 1{,}730.1 \text{ Hz } (< 1.8 \text{ kHz}) \qquad 1$$

(c) $$H(s) = \frac{3}{(s+4)(s+5)}$$

$$H(s) = \frac{A}{s+4} + \frac{B}{s+5} \qquad 1$$

$$A = H(s)(s+4)\Big|_{s=-4} = \frac{3}{s+5}\Big|_{s=-4} = 3$$

$$B = H(s)(s+5)\Big|_{s=-5} = \frac{3}{s+4}\Big|_{s=-5} = -3 \qquad 1$$

$$\Rightarrow H(s) = \frac{3}{s+4} - \frac{3}{s+5}$$

Impulse Invariant Transformation:

$$\frac{K}{s+a} \longrightarrow \frac{K}{1-e^{-aT}z^{-1}}$$ 1

$$\Rightarrow H(z) = \frac{3}{1-e^{-4T}z^{-1}} - \frac{3}{1-e^{-5T}z^{-1}}$$

$$= \frac{3(1-e^{-5T}z^{-1}) - 3(1-e^{-4T}z^{-1})}{(1-e^{-4T}z^{-1})(1-e^{-5T}z^{-1})}$$

$$= \frac{3(e^{-4T}-e^{-5T})z^{-1}}{1-(e^{-4T}+e^{-5T})z^{-1}+e^{-9T}z^{-2}}$$ 1

Choice of sampling frequency:

Highest pole frequency = 5 rad/s

$= 0.796\,Hz$

$f_s = 7.96\,Hz$

$\Rightarrow T = 0.126\,s$ 1

$$H(z) = \frac{3(0.604-0.533)z^{-1}}{1-(0.604+0.533)z^{-1}+0.322z^{-2}}$$

$$= \frac{0.213z^{-1}}{1-1.137z^{-1}+0.322z^{-2}}$$ 1

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