SEMESTER 1 EXAMINATIONS 2014-15

EE445 DIGITAL SIGNAL PROCESSING

SOLUTIONS

1. (a) $y(n) = x(n) - 0.4x(n-1) - 0.35y(n-1) + 0.6y(n-2)$

$$H(z) = \frac{1 - 0.4z^{-1}}{1 + 0.35z^{-1} - 0.6z^{-2}}$$ 1

$$H(\theta) = H(z)\big|_{z=e^{j\theta}}$$

$$= \frac{1 - 0.4e^{-j\theta}}{1 + 0.35e^{-j\theta} - 0.6e^{-j2\theta}}$$ 1

$$= \frac{1 - 0.4\cos\theta + j0.4\sin\theta}{1 + 0.35\cos\theta - j0.35\sin\theta - 0.6\cos 2\theta + j0.6\sin 2\theta}$$ 1

$$|H(\theta)| = \frac{\sqrt{(1-0.4\cos\theta)^2 + (0.4\sin\theta)^2}}{\sqrt{(1+0.35\cos\theta - 0.6\cos 2\theta)^2 + (-0.35\sin\theta + 0.6\sin 2\theta)^2}}$$ 1

$$\underline{/H(\theta)} = \tan^{-1}\left[\frac{0.4\sin\theta}{1-0.4\cos\theta}\right] - \tan^{-1}\left[\frac{-0.35\sin\theta + 0.6\sin 2\theta}{1+0.35\cos\theta - 0.6\cos 2\theta}\right]$$ 1

$$\frac{f_s}{5} = \frac{2\pi}{5}$$ 1

$$\underline{/H(\theta)}\Big|_{\theta=\frac{2\pi}{5}} = \tan^{-1}\left[\frac{0.4\sin\frac{2\pi}{5}}{1-0.4\cos\frac{2\pi}{5}}\right] - \tan^{-1}\left[\frac{-0.35\sin\frac{2\pi}{5} + 0.6\sin\frac{4\pi}{5}}{1+0.35\cos\frac{2\pi}{5} - 0.6\cos\frac{4\pi}{5}}\right]$$

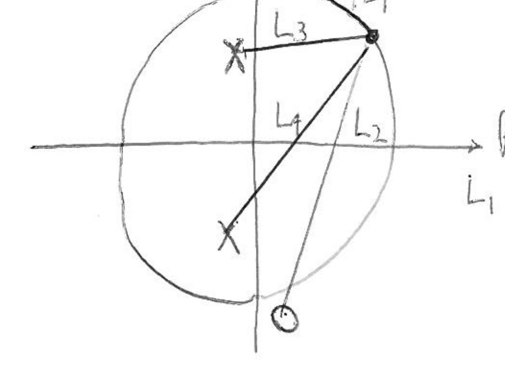
$$= \tan^{-1}\left[\frac{0.3804}{0.8764}\right] - \tan^{-1}\left[\frac{0.0198}{1.5936}\right]$$ 1

$$= \tan^{-1}[0.4340] - \tan^{-1}[0.0124]$$

$$= 0.4095 - 0.0124$$

$$= 0.3971 \text{ radians}$$ 1

(b)



$$\theta_x = 2\pi\frac{200}{1600} = \frac{\pi}{4}$$ 1

$$= \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$ 1

$$|H(\theta_x)| = \frac{L_1 L_2}{L_3 L_4}$$ 1

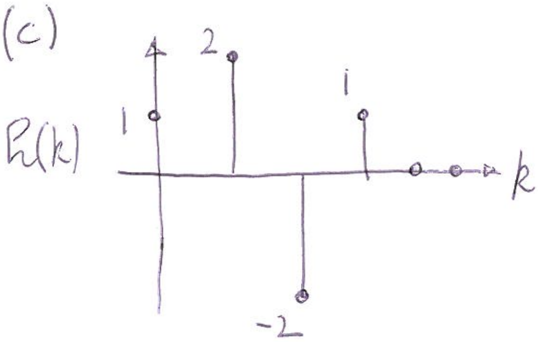
$$L_1 = \sqrt{(0.7071-0.2)^2 + (0.7071-1.2)^2}$$ 1

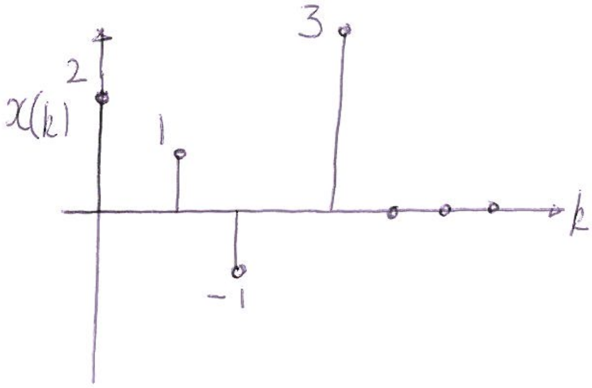
$$= 0.7072$$

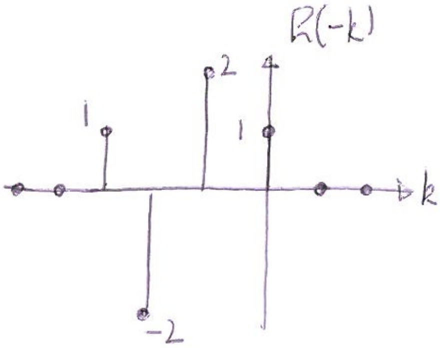
$$L_2 = \sqrt{(0.7071-0.2)^2 + (0.7071+1.2)^2} = 1.9734$$
$$L_3 = \sqrt{(0.7071+0.1)^2 + (0.7071-0.6)^2} = 0.8142$$
$$L_4 = \sqrt{(0.7071+0.1)^2 + (0.7071+0.6)^2} = 1.5362 \quad 1$$

$$\Rightarrow |H(\theta_x)| = \frac{(0.7072)(1.9734)}{(0.8142)(1.5362)} = 1.1158 \quad 1$$

(c)





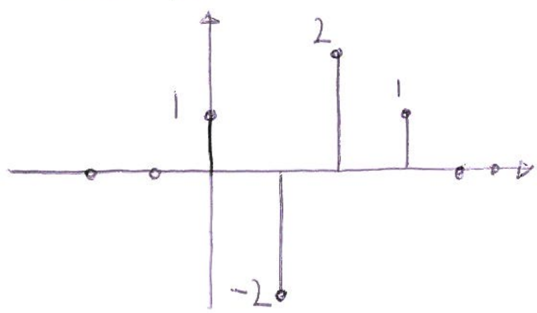


$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad 2$$

$$y(0) = 2(1) = 2$$
$$y(1) = 2(2)+1(1) = 5$$
$$y(2) = 2(-2)+1(2)-1(1) = -3$$

Calculation of $y(3)$:

$h(3-k)$



$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k)$$
$$= \sum_{k=0}^{3} x(k)h(3-k)$$
$$= 2(1)+1(-2)-1(2)+3(1)$$
$$= 1 \quad 2$$

$$y(4) = 1(1)-1(-2)+3(2) = 9$$
$$y(5) = -1(1)+3(-2) = -7$$
$$y(6) = 3(1) = 3$$

$\therefore y(n) = \{2, 5, -3, 1, 9, -7, 3\}$ starting at $n=0$ 2

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2. (a) $H(z) = \frac{0.1}{1-0.9z^{-1}} = \frac{1-a}{1-az^{-1}}$ where $a = 0.9$

$$H(\theta) = \frac{1-a}{1-ae^{-j\theta}} = \frac{1-a}{1-a\cos\theta + ja\sin\theta}$$

$$|H(\theta)|^2 = \frac{(1-a)^2}{(1-a\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \frac{(1-a)^2}{1-2a\cos\theta + a^2}$$

At $\theta = \theta_c$, $|H(\theta)| = \frac{1}{\sqrt{2}} \Rightarrow |H(\theta)|^2 = \frac{1}{2}$

$$\Rightarrow \frac{(1-a)^2}{1-2a\cos\theta_c + a^2} = 0.5$$

$$\Rightarrow (1-a)^2 = 0.5 - a\cos\theta_c + 0.5a^2$$

$$\Rightarrow \cos\theta_c = \frac{1}{a}\left[0.5 - (1-a)^2 + 0.5a^2\right]$$

$$= \frac{1}{a}\left[0.5 - 1 + 2a - a^2 + 0.5a^2\right]$$

$$= \frac{1}{a}\left[-0.5 + 2a - 0.5a^2\right]$$

Substitute $a = 0.9$ to obtain

$$\cos\theta_c = \frac{1}{0.9}\left[-0.5 + 2(0.9) - 0.5(0.9)^2\right]$$

$$= 0.9944$$

$$\Rightarrow \theta_c = 0.1059 \text{ rads}$$

$$= 2\pi \frac{f_c}{f_s}$$

$$\Rightarrow f_c = \frac{\theta_c f_s}{2\pi}$$

$$= 134.83 \text{ Hz}$$

(b) $h(n) = \{1, -1, 0, 3\}$ starting @ $n = 0$

$$\Rightarrow H(z) = 1 - z^{-1} + 3z^{-3}$$

$x(n) = \{1, 3, -1, 2\}$ starting @ $n = 2$

$$\Rightarrow X(z) = z^{-2} + 3z^{-3} - z^{-4} + 2z^{-5}$$

$X(z)H(z)$:

$$\begin{array}{r} z^{-2} + 3z^{-3} - z^{-4} + 2z^{-5} \\ - z^{-3} - 3z^{-4} + z^{-5} - 2z^{-6} \\ 3z^{-5} + 9z^{-6} - 3z^{-7} + 6z^{-8} \\ \hline \end{array}$$

$$Y(z) = z^{-2} + 2z^{-3} - 4z^{-4} + 6z^{-5} + 7z^{-6} - 3z^{-7} + 6z^{-8}$$

$\therefore y(n) = \{1, 2, -4, 6, 7, -3, 6\}$

starting @ $n = 2$

(c) $H(z) = \dfrac{1-2z^{-1}}{1+0.3z^{-1}-0.55z^{-2}}$

$|H(\theta)| = \dfrac{1-2e^{-j\theta}}{1+0.3e^{-j\theta}-0.55e^{-j2\theta}}$

$= \dfrac{1-2\cos\theta + j2\sin\theta}{1+0.3\cos\theta - j0.3\sin\theta - 0.55\cos 2\theta + j0.55\sin 2\theta}$

$|H(\theta)| = \dfrac{\sqrt{(1-2\cos\theta)^2+(2\sin\theta)^2}}{\sqrt{(1+0.3\cos\theta-0.55\cos 2\theta)^2+(-0.3\sin\theta+0.55\sin 2\theta)^2}}$

$f = \dfrac{f_s}{4} \Rightarrow \theta = \dfrac{\pi}{2}$

$\Rightarrow |H(\theta)| = \dfrac{\sqrt{(1-2(0))^2+(2)^2}}{\sqrt{(1+0.3(0)-0.55(-1))^2+(-0.3(1)+0.55(0))^2}}$

$= \dfrac{\sqrt{1+4}}{\sqrt{2.4025+0.09}}$ $\quad 1.107 + 0.191 = 1.298$

$= 1.4163$

(d) $\theta_0 = 2\pi \dfrac{f_0}{f_s} = 2\pi\dfrac{2}{10} = 0.4\pi$

$r = 1 - \dfrac{\Delta f}{f_s}\pi = 1 - \dfrac{30}{10000}\pi = 0.9906$

Coefficients:

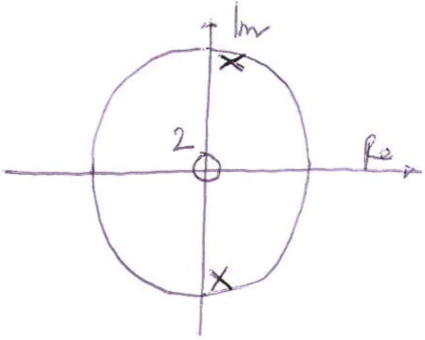
$b_1 = -2r\cos\theta_0 = -2(0.9906)\cos(0.4\pi) = -0.6122$

$b_2 = r^2 = 0.9813$

DC gain of 1 $\Rightarrow$ numerator $= 1+b_1+b_2$
$= 1.3691$

DC gain of 0.8 $\Rightarrow$ numerator $= 0.8(1.3691)$
$= 1.0953$

$\therefore H(z) = \dfrac{1.0953}{1-0.6122z^{-1}+0.9813z^{-2}}$



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3. (a) Notch filter with two notches required (cascade)

(i) poles: $\theta_0 = 2\pi \frac{100}{1000} = 0.2\pi$ 1

radius $r = 1 - \frac{10}{1000}\pi = 0.9686$ 1

Denominator: $b_{11} = -2(0.9686)\cos(0.2\pi)$

$= -1.5672$

$b_{12} = r^2$

$= 0.9382$ 1

zeros: $\theta_0 = 0.2\pi$, $r = 1$

Numerator: $a_{11} = -2\cos(0.2\pi)$

$= -1.6180$

$a_{12} = 1$ 1

(ii) poles: $\theta_0 = 2\pi \frac{150}{1000} = 0.3\pi$

radius $r = 0.9686$ 1

Denominator: $b_{21} = -2(0.9686)\cos(0.3\pi)$

$= -1.1387$

$b_{22} = 0.9382$ 1

Numerator: $a_{21} = -2\cos(0.3\pi)$

$= -1.1756$

$a_{22} = 1$ 1

$$\therefore H(z) = \frac{1 - 1.6180z^{-1} + z^{-2}}{1 - 1.5672z^{-1} + 0.9382z^{-2}} \cdot \frac{1 - 1.1756z^{-1} + z^{-2}}{1 - 1.1387z^{-1} + 0.9382z^{-2}}$$ 1

(b) FIR filter, $N = 512$

linear phase $\Rightarrow$ coefficients are symmetric

$\Rightarrow$ 256 "unique" coefficients 1

Each coefficient multiplies two digital filter memory values added together, in each sample period

$\Rightarrow$ each $y(n)$ calculation requires

256 + 256 = 512 ADD

256 MPY 1

10 seconds of data at $f_s = 16{,}000\,Hz$ requires:

$(10 \times 16000) \times 512 = 81{,}920{,}000$ ADD

$(10 \times 16{,}000) \times 256 = 40{,}960{,}000$ MPY 1

FFT: each frame requires the following number of multiplies:

Windowing	N	512
512-point FFT	$2N\log_2(N)$	9,216
Multiply $H(\theta) \times X(\theta)$	$4N$	2,048
Inverse FFT	$2N\log_2(N)$	9,216
		20,992 MPY/frame

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10 seconds of data = 312.5 frames ⇒ 313

50% overlap ⇒ "effective" number of frames = 626 1

$\Rightarrow$ total MPY $= 626 \times 20{,}992 = 13{,}140{,}992$

Saving $= 1 - \dfrac{13{,}140{,}992}{40{,}960{,}000} = 67.92\%$ 1

(c) Oscillator: $b_1 = 2\cos\theta_0$

$b_2 = -1$ 1

$\theta_0 = 2\pi \frac{2}{48} = \frac{\pi}{12} \Rightarrow b_1 = 2\cos\left(\frac{\pi}{12}\right) = 1.9318$ 1

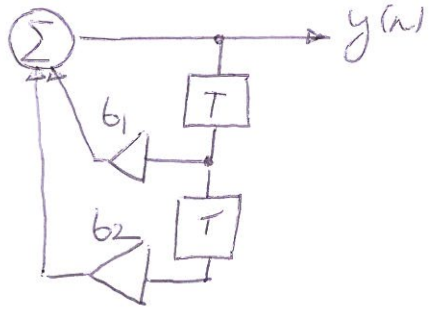
Initial conditions: we require a phase shift of $\frac{\pi}{4}$ which is $\frac{1}{8}$ of a period 1

Each cycle contains 24 samples

$\Rightarrow \frac{\pi}{4}$ corresponds to 3 samples

$\Rightarrow y(n-1) = \cos(2\theta_0) = 0.8660$

$y(n-2) = \cos(\theta_0) = 0.9659$ 1



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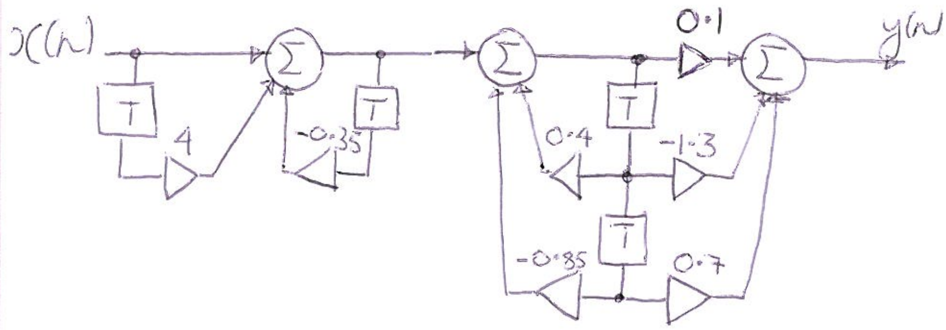
(20)

4. (a) Cascade implementation:



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(b) $H(s) = \frac{\omega_c}{s + \omega_c}$

Desired digital cut-off frequency
$$\theta_c = 2\pi \frac{2}{20} = 0.2\pi$$ 1

Pre-warp: $\omega_c = \frac{2}{T} \tan\left(\frac{\theta_c}{2}\right)$
$$= 40{,}000 \tan(0.1\pi)$$
$$= 12{,}997 \text{ rad/s}$$ 1

Bilinear Transform:
$$s = \frac{2}{T} \cdot \frac{1-z^{-1}}{1+z^{-1}}$$ 1

$$H(z) = \frac{\omega_c}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + \omega_c} = \frac{\omega_c(1+z^{-1})}{(2f_s+\omega_c) + (\omega_c - 2f_s)z^{-1}}$$ 1

Substitute for ω_c and f_s to obtain:
$$\frac{12997(1+z^{-1})}{(40000+12997) + (12997-40000)z^{-1}}$$
$$= \frac{12997(1+z^{-1})}{52997 - 27003 z^{-1}}$$ 1
$$= \frac{0.2452(1+z^{-1})}{1 - 0.5095 z^{-1}}$$ 1

If pre-warping is not carried out:
$$\theta_d = 2\tan^{-1}\left(\frac{\omega_c T}{2}\right)$$
$$= 2\tan^{-1}\left(\frac{4000\pi}{40000}\right) = 0.6058$$ 1
$$\theta_d = 2\pi \frac{f_d}{f_s}$$
$$\Rightarrow f_d = \frac{\theta_d f_s}{2\pi} = 1937.9 \text{ Hz} \quad (< 2 \text{ kHz})$$ 1

(c) $H(s) = \dfrac{2}{(s+3)(s+8)}$

$$= \frac{A}{s+3} + \frac{B}{s+8}$$ 1

$$A = H(s)(s+3)\Big|_{s=-3} = \frac{2}{s+8}\Big|_{s=-3} = 2/5$$

$$B = H(s)(s+8)\Big|_{s=-8} = \frac{2}{s+3}\Big|_{s=-8} = -2/5$$ 1

$$\Rightarrow H(s) = \frac{0.4}{s+3} - \frac{0.4}{s+8}$$

Impulse Invariant Transformation:

$$\frac{K}{s+a} \longrightarrow \frac{K}{1-e^{-aT}z^{-1}}$$ 1

$$\Rightarrow H(z) = \frac{0.4}{1-e^{-3T}z^{-1}} - \frac{0.4}{1-e^{-8T}z^{-1}}$$ 1

$$= \frac{0.4(1-e^{-8T}z^{-1}) - 0.4(1-e^{-3T}z^{-1})}{(1-e^{-3T}z^{-1})(1-e^{-8T}z^{-1})}$$

$$= \frac{0.4(e^{-3T}-e^{-8T})z^{-1}}{1-(e^{-3T}+e^{-8T})z^{-1}+e^{-11T}z^{-2}}$$ 1

Choice of sampling frequency:

Highest pole frequency = 8 rad/s = 1.2732 Hz 1

$\Rightarrow f_s = 10 \times 1.2732 = 12.732$ Hz

$\Rightarrow T = 0.0785$ s 1

$$\Rightarrow H(z) = \frac{0.4(0.7902 - 0.5337)z^{-1}}{1-(0.7902+0.5337)z^{-1}+0.4217z^{-2}}$$

$$= \frac{0.1026z^{-1}}{1-1.3239z^{-1}+0.4217z^{-2}}$$ 1

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