SEMESTER I EXAMINATIONS 2017-2018

EE445 DIGITAL SIGNAL PROCESSING

SOLUTIONS

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1. (a) $$H(z) = \frac{0.3 - 0.25z^{-1}}{1 - 0.5z^{-1} + 0.4z^{-2}}$$

$$H(\theta) = H(z)\big|_{z=e^{j\theta}} = \frac{0.3 - 0.25e^{-j\theta}}{1 - 0.5e^{-j\theta} + 0.4e^{-j2\theta}}$$
1

$$= \frac{0.3 - 0.25\cos\theta + j0.25\sin\theta}{1 - 0.5\cos\theta + j0.5\sin\theta + 0.4\cos 2\theta - j0.4\sin 2\theta}$$

$$|H(\theta)| = \frac{\left[(0.3 - 0.25\cos\theta)^2 + (0.25\sin\theta)^2\right]^{1/2}}{\left[(1 - 0.5\cos\theta + 0.4\cos 2\theta)^2 + (0.5\sin\theta - 0.4\sin 2\theta)^2\right]^{1/2}}$$
1

$$\angle H(\theta) = \tan^{-1}\left[\frac{0.25\sin\theta}{0.3 - 0.25\cos\theta}\right] - \tan^{-1}\left[\frac{0.5\sin\theta - 0.4\sin 2\theta}{1 - 0.5\cos\theta + 0.4\cos 2\theta}\right]$$
1

$$\frac{f_s}{3} \Rightarrow \theta = \frac{2\pi}{3}$$
1

$$|H(\theta)|_{\theta=\frac{2\pi}{3}} = \frac{\left[(0.3 - 0.25\times(-0.5))^2 + (0.25\times 0.866)^2\right]^{1/2}}{\left[(1 - 0.5(-0.5) + 0.4(-0.5))^2 + (0.5(0.866) - 0.4(-0.866))^2\right]^{1/2}}$$

$$= \frac{\left[0.425^2 + 0.2165^2\right]^{1/2}}{\left[1.05^2 + 0.7794^2\right]^{1/2}} = \frac{0.477}{1.3077} = 0.3648$$
1

$$\angle H(\theta)|_{\theta=\frac{2\pi}{3}} = \tan^{-1}\left[\frac{0.25(0.866)}{0.3 - 0.25(-0.5)}\right] - \tan^{-1}\left[\frac{0.5(0.866) - 0.4(-0.866)}{1 - 0.5(-0.5) + 0.4(-0.5)}\right]$$
1

$$= \tan^{-1}\left[\frac{0.2165}{0.425}\right] - \tan^{-1}\left[\frac{0.7794}{1.05}\right]$$

$$= \tan^{-1}[0.5094] - \tan^{-1}[0.7423]$$

$$= 0.4711 - 0.6386$$

$$= -0.1675 \text{ rads}$$
1

(b) poles @ $z = 0.5e^{\pm j0.7}$

$= 0.5\cos 0.7 \pm j0.5\sin 0.7$

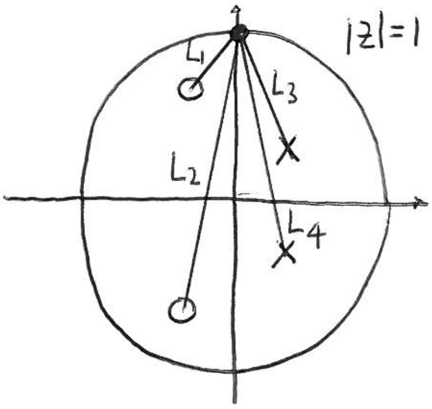
$= 0.38 \pm j0.32$ 1

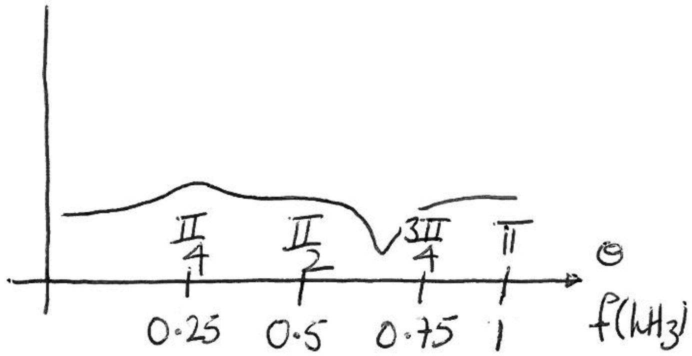
zeros @ $z = 0.8e^{\pm j2}$

$= 0.8\cos(2) \pm j0.8\sin(2)$

$= -0.33 \pm j0.73$ 1

Pole-zero map





1

1

$$\theta_x = 2\pi \frac{500}{2000} = \frac{\pi}{2} = 0 + j1$$

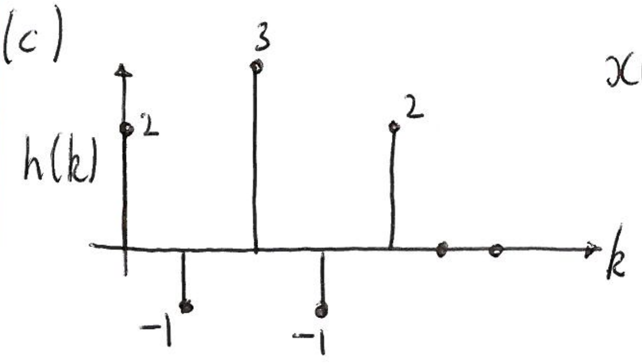
$$L_1 = \sqrt{(-0.33-0)^2 + (0.73-1)^2} = 0.4264$$
$$L_2 = \sqrt{(-0.33-0)^2 + (-0.73-1)^2} = 1.7612$$

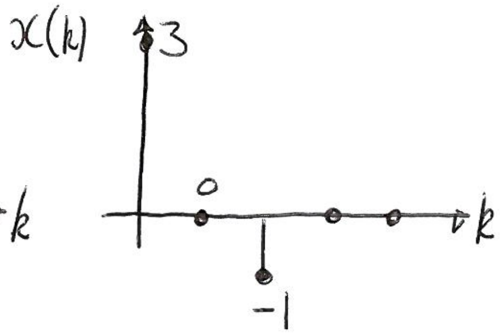
$$L_3 = \sqrt{(0.38-0)^2 + (0.32-1)^2} = 0.7790$$ 1
$$L_4 = \sqrt{(0.38-0)^2 + (-0.32-1)^2} = 1.3736$$

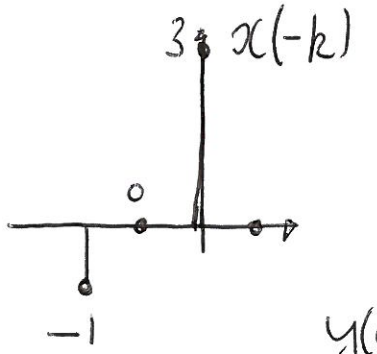
$$|H(\theta_x)| = \frac{L_1 L_2}{L_3 L_4} = \frac{(0.4264)(1.7612)}{(0.7790)(1.3736)}$$

$$= 0.7018$$ 1

(c)





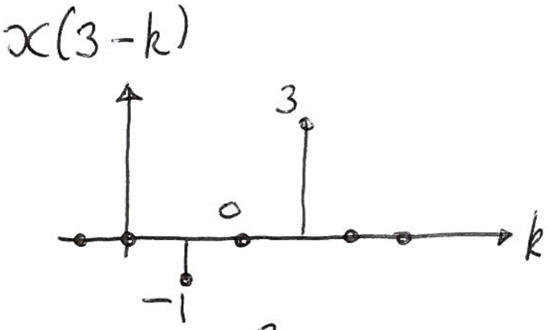


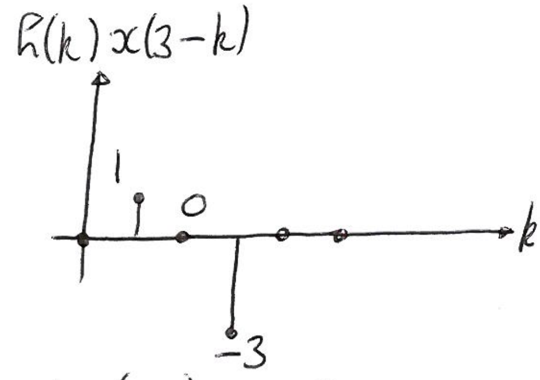
$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$ 1

$y(0) = 2 \times 3 = 6$

$y(1) = \cdot \quad \cdot + (-1)3 = -3$

$y(2) = 2(-1) + (-1)0 + 3(3) = 7$





1

1

$$y(3) = \sum_{k=1}^{3} h(k)x(3-k) = 1 + (-3) = -2$$

$y(4) = 3(-1) + (-1)0 + 2(3) = 3$

$y(5) = (-1)(-1) + 2(0) = 1$

$y(6) = 2(-1) = -2$

$y(n) = \{6, -3, 7, -2, 3, 1, -2\}$ 2

Coefficients/$h(n)$ of filter are symmetric $\Rightarrow$ linear phase 1

Group delay = $\dfrac{\text{"filter duration"}}{2}$ = 2 samples 1

20

2. (a) $H(z) = 1 + z^{-2} - 2z^{-3}$ | 1

$X(z) = 1 + 2z^{-1} - z^{-2} + 3z^{-3}$ | 1

$H(z) \times X(z)$

$$\begin{array}{rrrrrrr}
1 & +2z^{-1} & -z^{-2} & +3z^{-3} & & & \\
 & & z^{-2} & +2z^{-3} & -z^{-4} & +3z^{-5} & \\
 & & & -2z^{-3} & -4z^{-4} & +2z^{-5} & -6z^{-6} \\
\hline
1 & +2z^{-1} & +0 & +3z^{-3} & -5z^{-4} & +5z^{-5} & -6z^{-6}
\end{array}$$ | 2

$y(n) = \{1, 2, 0, 3, -5, 5, -6\}$ starting at $n = 0$ | 1

If $x(n)$ is delayed to $n=3$, then $y(n)$ is simply delayed by the same number of samples. | 2

(b) $H(\theta) = H(z)|_{z=e^{j\theta}}$

$$= \frac{1 - 0.4e^{-j3\theta}}{1 + 0.7e^{-j\theta} - 0.6e^{-j2\theta}}$$ | 1

$$= \frac{1 - 0.4\cos 3\theta + j0.4\sin 3\theta}{1 + 0.7\cos\theta - j0.7\sin\theta - 0.6\cos 2\theta + j0.6\sin 2\theta}$$ | 1

$$\underline{/H(\theta)} = \tan^{-1}\left[\frac{0.4\sin 3\theta}{1 - 0.4\cos 3\theta}\right] - \tan^{-1}\left[\frac{-0.7\sin\theta + 0.6\sin 2\theta}{1 + 0.7\cos\theta - 0.6\cos 2\theta}\right]$$ | 1

$\frac{f_s}{6} \Rightarrow \theta = \frac{2\pi}{6} = \frac{\pi}{3}$ | 1

$$\underline{/H(\theta)}\Big|_{\theta=\frac{\pi}{3}} = \tan^{-1}\left[\frac{0}{1 - 0.4(-1)}\right] - \tan^{-1}\left[\frac{(-0.7)(0.866) + 0.6(0.866)}{1 + 0.7(0.5) - 0.6(-0.5)}\right]$$

$$= \tan^{-1}(0) - \tan^{-1}\left(\frac{-0.0866}{1.65}\right)$$ | 1

$$= -\tan^{-1}(-0.0525) = +0.0525$$ | 1

(c) $H(\theta) = \dfrac{0.1}{1-0.9e^{-j\theta}} = \dfrac{0.1}{1-0.9\cos\theta+j0.9\sin\theta}$ 1

$$|H(\theta)|^2 = \frac{(0.1)^2}{(1-0.9\cos\theta)^2+(0.9\sin\theta)^2}$$

$$= \frac{(0.1)^2}{1-1.8\cos\theta+0.81}$$ 1

At $\theta=\theta_x$, we have $|H(\theta)| = -20\text{dB} = 0.1$ 1

$$\Rightarrow |H(\theta)|^2 = 0.01$$

$$\frac{0.1^2}{1-1.8\cos\theta_x+0.81} = 0.01$$

$$\Rightarrow 0.1^2 = 0.01[1-1.8\cos\theta_x+0.81]$$

$$= 0.01 - 0.018\cos\theta_x + 0.0081$$ 1

$$\Rightarrow 0.018\cos\theta_x = 0.01+0.0081-0.01$$

$$= 0.0081$$

$$\Rightarrow \cos\theta_x = 0.45$$

$$\Rightarrow \theta_x = 1.104 \text{ rads}$$

$\theta_x = 2\pi\dfrac{f_x}{f_s} \Rightarrow f_x = \dfrac{\theta_x f_s}{2\pi} = 175.7\text{ Hz}$ 1

$\underline{|H(\theta)} = -\tan^{-1}\left[\dfrac{0.9\sin\theta}{1-0.9\cos\theta}\right]$ 1

$$\underline{|H(\theta)}\Big|_{\theta=1.104} = -\tan^{-1}\left[\frac{(0.9)(0.893)}{1-0.9(0.45)}\right] = -\tan^{-1}[1.351]$$

$$= -0.9336 \text{ rads}$$ 1

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3. (a) Notch filter

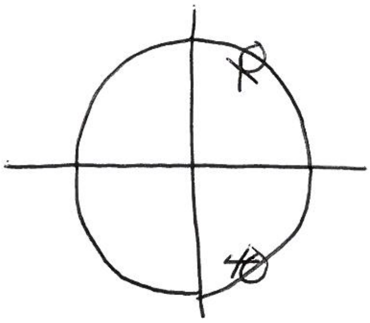
$$\theta_0 = 2\pi \frac{50}{300} = \frac{\pi}{3}$$

$$r = 1 - \frac{\Delta f}{f_s}\pi = 1 - \frac{10}{300}\pi = 0.8953$$

Numerator : $a_1 = -2r\cos(\theta_0)$
$= -2\cos(\theta_0)$
$= -1$
$a_2 = r^2 = 1$

Denominator : $b_1 = -2r\cos(\theta_0)$
$= -2(0.8953)\cos\theta_0$
$= -0.8953$
$b_2 = r^2 = 0.8016$

$$H(z) = \frac{1 - z^{-1} + z^{-2}}{1 - 0.8953z^{-1} + 0.8016z^{-2}}$$



$$y(n) = x(n) - x(n-1) + x(n-2) + 0.8953y(n-1) - 0.8016y(n-2)$$

(b) FIR filter with 512 coefficients
Each output sample requires 512 MPY
512 ADD
10 seconds of signal $\Rightarrow$ 10 X 96,000 = 960,000 samples

$\Rightarrow$ Total of 491,520,000 MPY
491,520,000 ADD

FFT: each frame requires

Windowing	~~$N/2$~~ $2N$	~~256~~ 512	~~Symmetric window~~	1
FFT	$2N\log_2(N)$	9216		
$H(\Theta) \times X(\Theta)$	$2N$	1024	conjugate symmetry	1
IFFT	$2N\log_2(N)$	9216		
		~~19,712~~ 19,968		2

960,000 samples ⇒ 1875 frames

50% overlap ⇒ effective number of frames is $2 \times 1875 = 3750$ 1

Total MPY for FFT-based approach

$3{,}750 \times \cancel{19{,}712}$ 19,968 $=$ ~~73,920,000~~ 74,880,000 1

$$\text{Saving} = 1 - \frac{73.92}{491.52} = \cancel{85\%}\ 84.7\%$$ 1

(c) $f_s = 30\,\text{kHz}$

$T_{win} = 16\,\text{msec} \Rightarrow N_{win} = 480$ samples 1

$$\Delta f = \frac{f_s}{N_{FFT}} \leq 10\,\text{Hz}$$

$$\Rightarrow N_{FFT} \geq \frac{f_s}{10} \geq 3000$$ 1

Next highest power of 2 is 4096 1

⇒ no. of samples for zero-padding $= 4096 - 480 = 3{,}616$ 2

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4. (a) $H(s) = \dfrac{W_c}{s+W_c}$

Desired digital cutoff frequency
$$\theta_c = 2\pi \frac{4}{24} = \frac{\pi}{3}$$ 1

Prewarp $W_c = \dfrac{2}{T} \tan\left(\dfrac{\theta_c}{2}\right)$
$$= 48000 \tan\left(\frac{\pi}{6}\right)$$
$$= 27{,}712.8 \text{ rad/s}$$ 1

Bilinear Transform
$$s = \frac{2}{T} \cdot \frac{1-z^{-1}}{1+z^{-1}}$$ 1

$$H(z) = \frac{W_c}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + W_c} = \frac{W_c(1+z^{-1})}{(2f_s+W_c)+(W_c-2f_s)z^{-1}}$$ 1

Substitute for W_c (pre-warped value) and f_s:
$$H(z) = \frac{27{,}713\,(1+z^{-1})}{(48000+27713)+(27713-48000)z^{-1}}$$
$$= \frac{27713(1+z^{-1})}{75713 - 20287z^{-1}}$$ 1
$$= \frac{0.366(1+z^{-1})}{1 - 0.2679z^{-1}}$$

If pre-warping was not carried out:
$$\theta_d = 2\tan^{-1}\left(\frac{W_cT}{2}\right)$$
$$= 2\tan^{-1}\left(\frac{8000\pi}{48000}\right) = 0.9647$$ 1

$$\theta_d = \frac{2\pi f_d}{f_s} \Rightarrow f_d = \frac{\theta_d f_s}{2\pi}$$
$$= 3684.9\text{ Hz } (< 4\text{ kHz})$$ 1

(b) $$H(s) = \frac{4}{(s+3)(s+12)} = \frac{A}{s+3} + \frac{B}{s+12}$$ 1

$$A = H(s)(s+3)\big|_{s=-3} = \frac{4}{(-3+12)} = 4/9$$

$$B = H(s)(s+12)\big|_{s=-12} = \frac{4}{(-12+3)} = -4/9$$ 1

$$\Rightarrow H(s) = \frac{4/9}{s+3} - \frac{4/9}{s+12}$$

IIT: $$\frac{K}{s+a} \longrightarrow \frac{K}{1-e^{-aT}z^{-1}}$$ 1

$$\Rightarrow H(z) = \frac{4/9}{1-e^{-3T}z^{-1}} - \frac{4/9}{1-e^{-12T}z^{-1}}$$

$$= \frac{4/9(1-e^{-12T}z^{-1}) - 4/9(1-e^{-3T}z^{-1})}{(1-e^{-3T}z^{-1})(1-e^{-12T}z^{-1})}$$ 1

$$= \frac{4/9(e^{-3T}-e^{-12T})z^{-1}}{1-(e^{-3T}+e^{-12T})z^{-1}+e^{-15T}z^{-2}}$$ 1

Sampling frequency:

$$f_s = 8 \times \frac{12}{2\pi} = 15.28\,Hz \Rightarrow T = 0.065\,s$$ 1

$$\Rightarrow H(z) = \frac{4/9(0.8228-0.4584)z^{-1}}{1-(0.8228+0.4584)z^{-1}+0.3772z^{-2}}$$

$$= \frac{0.1619z^{-1}}{1-1.2812z^{-1}+0.3772z^{-2}}$$ 1

(c) Window method
- description (brief) as per notes
- windowing IIR response
- relationship between group delay and length
- can design linear phase filters 4

Group delay: 5 ms @ 8 kHz = 40 samples

$N = 2(\text{group delay}) + 1 = 81$ coefficients 2

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