SEMESTER I EXAMINATIONS 2016-17

EE445 DIGITAL SIGNAL PROCESSING

SOLUTIONS

1 (a) $$H(z) = \frac{1-0.5z^{-1}}{1+0.7z^{-1}+0.4z^{-2}}$$ 1

$$H(\theta) = \frac{1-0.5e^{-j\theta}}{1+0.7e^{-j\theta}+0.4e^{-j2\theta}}$$

$$= \frac{1-0.5\cos\theta - j0.5\sin\theta}{1+0.7\cos\theta - j0.7\sin\theta + 0.4\cos2\theta - j0.4\sin2\theta}$$ 1

$$|H(\theta)| = \frac{\left[(1-0.5\cos\theta)^2+(0.5\sin\theta)^2\right]^{1/2}}{\left[(1+0.7\cos\theta+0.4\cos2\theta)^2+(-0.7\sin\theta-0.4\sin2\theta)^2\right]^{1/2}}$$ 1

$$\underline{/H(\theta)} = \tan^{-1}\left[\frac{0.5\sin\theta}{1-0.5\cos\theta}\right]-\tan^{-1}\left[\frac{-(0.7\sin\theta+0.4\sin2\theta)}{1+0.7\cos\theta+0.4\cos2\theta}\right]$$ 1

DC gain : $$H(\theta)\big|_{\theta=0} = \frac{1-0.5}{1+0.7+0.4} = 0.2381$$ 1

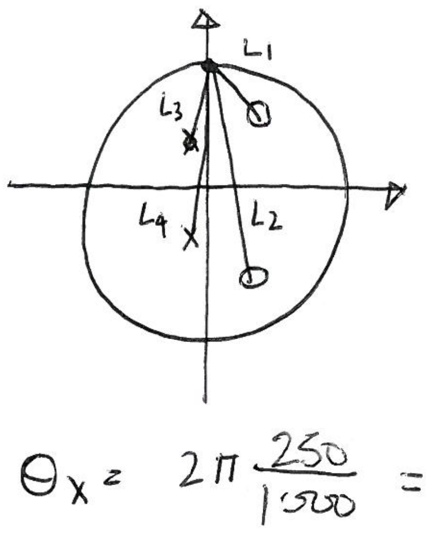
$$\frac{f_s}{4}=\frac{\pi}{2} \quad \underline{/H(\theta)}\big|_{\theta=\frac{\pi}{2}} = \tan^{-1}\left[\frac{0.5}{1-0}\right]-\tan^{-1}\left[\frac{-0.7}{1+0-0.4}\right]$$

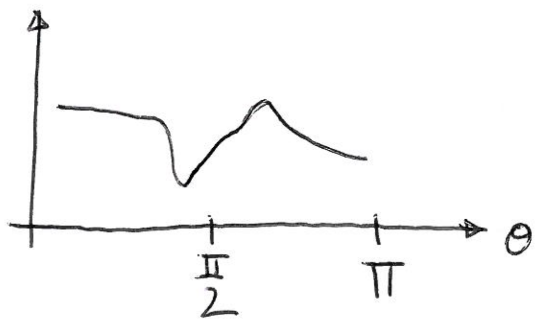
$$= \tan^{-1}[0.5]-\tan^{-1}[-1.167]$$

$$= 0.464 + 0.8623$$

$$= \cancel{-0.398} \quad 1.3263$$ 1

(b)





1

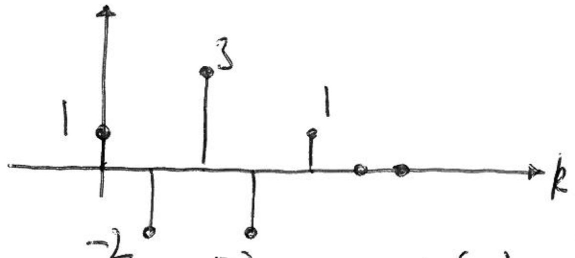
2

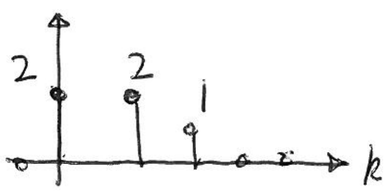
$$\theta_x = 2\pi\frac{250}{1000} = \frac{\pi}{2} = 0+j1$$ 1

$$L_1 = \sqrt{(0-0.3)^2+(1-0.6)^2} = \sqrt{0.09+0.16} = 0.5$$
$$L_2 = \sqrt{(0-0.3)+(1+0.6)^2} = \sqrt{0.09+2.56} = 1.627$$
$$L_3 = \sqrt{(0+0.1)^2+(1-0.4)^2} = \sqrt{0.01+0.36} = 0.608$$
$$L_4 = \sqrt{(0+0.1)^2+(1+0.4)^2} = \sqrt{0.01+1.96} = 1.403$$ 1

$$|H(\theta_x)| = \frac{L_1 L_2}{L_3 L_4} = 0.954$$ 1

(c) $h(k)$ $x(k)$

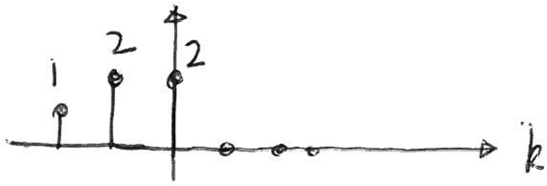




1

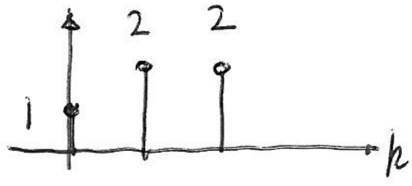
$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$ 1

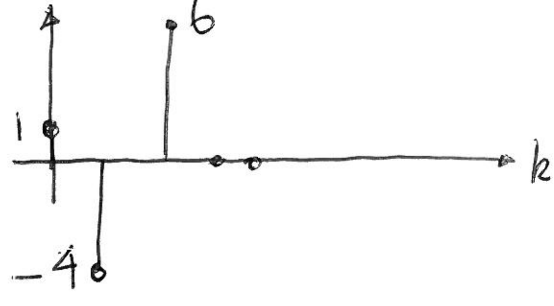
$x(-k)$



$$y(0) = 1\times 2 = 2$$
$$y(1) = 1\times 2 + (-2)\times 2 = -2$$ 1

$x(2-h)$ $h(h)x(2-h)$





1

$$y(2) = \sum_{k=0}^{2} h(k)\,x(2-k)$$
$$= 1 - 4 + 6 = 3$$

$$y(3) = 1(-2) + 2(3) + 2(-2) = 0$$
$$y(4) = 1(3) + 2(-2) + 2(1) = 1$$
$$y(5) = 1(-2) + 2(1) = 0$$
$$y(6) = 1(1) = 1$$
$$y(n) = \{2, -2, 3, 0, 1, 0, 1\}$$ 1

Check no. of output samples
= 5 + 3 − 1 = 7 correct 1

Filter has symmetric $h(n)$
⇒ linear phase 1

Group delay = $\frac{N-1}{2}$ = 2 samples 1

20

2. (a) $H(z) = 1 - 3z^{-3}$
$X(z) = z^{-2} - z^{-3} + 2z^{-4}$ 1

$$
\begin{array}{rl}
X(z)H(z) = & z^{-2} - z^{-3} + 2z^{-4} \\
 & \qquad\qquad\quad -3z^{-5} + 3z^{-6} - 6z^{-7} \\
\hline
 & z^{-2} - z^{-3} + 2z^{-4} - 3z^{-5} + 3z^{-6} - 6z^{-7}
\end{array}
$$
2

$y(n) = \{1, -1, 2, -3, 3, -6\}$ commencing @ $n=2$ 1

(b) Inverse z-transforms

(i) $x(n) = \{1, 2, 0, -3, 0, -2\}$ commencing @ $n=0$ 1

(ii) $X(z) = \dfrac{1}{z(z-1)(2z-1)} = \dfrac{A}{z} + \dfrac{B}{z-1} + \dfrac{C}{2z-1}$ 1

$A = X(z)z\big|_{z=0} = 1$
$B = X(z)(z-1)\big|_{z=1} = 1$
$C = X(z)(2z-1)\big|_{z=\frac{1}{2}} = -4$ 1

$$X(z) = \frac{1}{z} + \frac{1}{z-1} - \frac{4}{2z-1}$$
$$= z^{-1}\left[1 + \frac{z}{z-1} - \frac{4z}{2z-1}\right]$$ 1
$$= z^{-1}\left[1 + \frac{z}{z-1} - \frac{2z}{z-0.5}\right]$$
$\delta(n)$ $u(n)$ $a^n u(n)$ 1

Take Inverse z-transform, and add delay of 1 sample, to get

$x(n) = \delta(n-1) + u(n-1) - 2(0.5)^{n-1}u(n-1)$ 1

(c) $h(n) = \alpha^n u(n)$

$$H(z) = \frac{1}{1-\alpha z^{-1}}$$ 1

$$H(\theta) = \frac{1}{1-\alpha e^{-j\theta}} = \frac{1}{(1-\alpha\cos\theta) + j\alpha\sin\theta}$$ 1

Phase response : $\phi(\theta) = -\tan^{-1}\left[\frac{\alpha\sin\theta}{1-\alpha\cos\theta}\right]$ 1

group delay : $\tau(\theta) = -\frac{d\phi(\theta)}{d\theta}$ 1

$$= \frac{\alpha^2 - \alpha\cos\theta}{1+\alpha^2 - 2\alpha\cos\theta}$$

(d) gain of $-20\,dB = 0.1$

$$H(z) = \frac{0.1}{1-0.9z^{-1}} = \frac{1-a}{1-az^{-1}} \text{ with } a = 0.9$$

$$H(\theta) = \frac{1-a}{1-ae^{-j\theta}} = \frac{1-a}{1-a\cos\theta + ja\sin\theta}$$ 1

$$|H(\theta)|^2 = \frac{(1-a)^2}{(1-a\cos\theta)^2 + a^2\sin^2\theta}$$

$$= \frac{(1-a^2)}{1-2a\cos\theta + a^2}$$

At $\theta = \theta_x$, $|H(\theta)| = 0.1 \Rightarrow |H(\theta)|^2 = 0.01$ 2

$$\Rightarrow \frac{(1-a)^2}{1-2a\cos\theta_x + a^2} = 0.01$$

$$\Rightarrow (1-a)^2 = 0.01[1-2a\cos\theta_x + a^2]$$

$$2a\cos\theta_x = 1 + a^2 - \frac{(1-a)^2}{0.01}$$

$$\cos\theta_x = \frac{1}{2a}\left[1 + a^2 - \frac{(1-a)^2}{0.01}\right]$$ 1

$$= 0.555[1 + 0.81 - 1]$$

$$= 0.4496$$

$$\Rightarrow \theta_x = 1.1045 \text{ rads}$$

$\theta_x = 2\pi\frac{f_x}{f_s} \Rightarrow f_x = \frac{\theta_x f_s}{2\pi} = 175.8\ Hz$ 2

3. (a) $\theta_0 = 2\pi \frac{2.5}{10} = \frac{\pi}{2}$ 1

$r = 1 - \frac{\Delta f}{f_s}\pi = 1 - \frac{30}{10000}\pi = 0.9906$

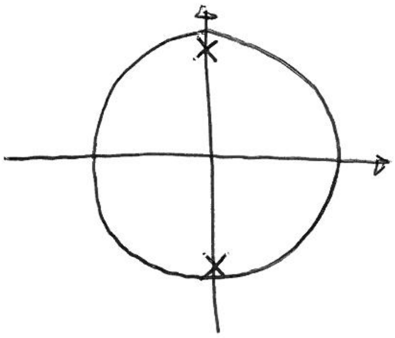
$b_1 = -2r\cos(\theta_0) = 0$ 1

$b_2 = r^2 = 0.9813$

DC gain of 0.75 ⇒ numerator $= 0.75(1+b_1+b_2)$

$= 1.486$ 1

$$H(z) = \frac{1.486}{1 + 0.9813 z^{-2}}$$



1

$y(n) = 1.486x(n) - 0.9813y(n-2)$ 1

(b) FIR filter with 256 coefficients

linear phase ⇒ 128 unique coefficients

Each output sample requires:

128 MPY

256 ADD 1

20 seconds requires:

$(20 \times 32000) \times 128 = 81{,}920{,}000$ MPY

$(20 \times 32000) \times 256 = 163{,}840{,}000$ ADD 1

FFT: each frame requires:

Windowing	N	256
FFT	$2N\log_2(N)$	4096
$H(\theta) \times X(\theta)$	$4N$	1024
Inverse FFT	$2N\log_2(N)$	4096
		9472

1

1

20 seconds of data @ $f_s = 32$ kHz
$\Rightarrow$ 2500 frames
50% overlap $\Rightarrow$ 5000 frames 1

$\Rightarrow$ total MPY $= 5{,}000 \times 9{,}472$
$= 47{,}360{,}000$ MPY
saving $= 42\%$ 1

Further savings:
1. $H(\theta), X(\theta)$ have conjugate symmetry
$\Rightarrow$ only 128 unique complex values each 1
~~2. Window is also symmetric~~ 1
~~$\Rightarrow$ only 128 unique values~~ 1

1. $H(\theta) \times X(\theta)$ would only need 512 MPY 1
~~2. Windowing would only require 128 MPY~~
$\Rightarrow$ ~~8832~~ 8960 MPY / frame instead 1

c(~~d~~) $f_s = 50$ kHz
$T_{WIN} = 20$ msec $\Rightarrow N_{WIN} = 1{,}000$ samples 1

We require $\Delta f \leq 10$ Hz
$\Rightarrow \frac{f_s}{N_{FFT}} \leq 10$ Hz 1
$\Rightarrow N_{FFT} \geq \frac{f_s}{10} \geq 5000$ 1

N_{FFT} must be a power of 2,
$\Rightarrow N_{FFT} = 8{,}192$ 1

$\therefore$ no. of samples for zero-padding
$= 8{,}192 - 1{,}000 = 7{,}192$ 1

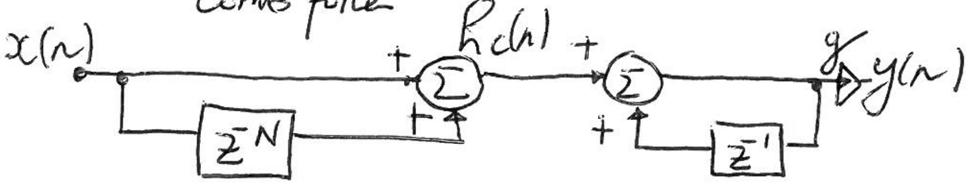
20

8/10

4. (a) $H(z) = g[1+z^{-1}+z^{-2}+ \quad + z^{-(N-1)}] = g\sum_{k=0}^{N-1} z^{-k} = g\frac{1-z^{-N}}{1-z^{-1}}$ 1

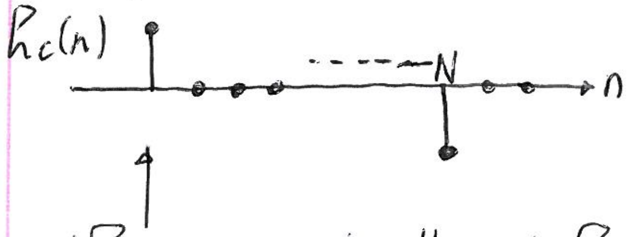
$= (1-z^{-N})\,\frac{g}{1-z^{-1}}$ 1

comb filter



2

Impulse Response:

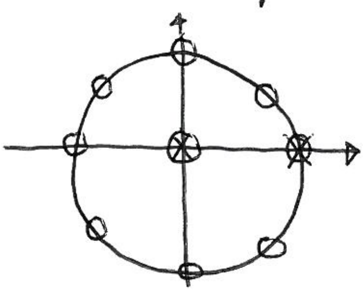


Impulse response of comb filter 1

- this sample "switches on" first order pole
- second sample of hc(n) "switches off" the first order recursive filter 1

Pole @ z = 1 is cancelled out by zero @ z = 1

pole-zero map



1

(b) $$H(s) = \frac{1}{(s+3)(s+12)} = \frac{A}{s+3} + \frac{B}{s+12}$$

$$A = H(s)(s+3)\big|_{s=-3} = \frac{1}{9}$$
$$B = H(s)(s+12)\big|_{s=-12} = -\frac{1}{9}$$ 1

$$\Rightarrow H(s) = \frac{1/9}{s+3} - \frac{1/9}{s+12}$$

IIT: $$\frac{K}{s+a} \longrightarrow \frac{K}{1-e^{-aT}z^{-1}}$$ 1

$$\Rightarrow H(z) = \frac{1/9}{1-e^{-3T}z^{-1}} - \frac{1/9}{1-e^{-12T}z^{-1}}$$ 1

$$= \frac{1/9(1-e^{-12T}z^{-1}) - 1/9(1-e^{-3T}z^{-1})}{(1-e^{-3T}z^{-1})(1-e^{-12T}z^{-1})}$$
$$= \frac{1/9(e^{-3T}-e^{-12T})z^{-1}}{1-(e^{-3T}+e^{-12T})z^{-1}+e^{-15T}z^{-2}}$$ 1

Sampling rate:
10 × highest pole frequency $= 10 \times \frac{12}{2\pi}$
$= 19.1$ Hz
$\Rightarrow T = 0.052$ s 1

$$H(z) = \frac{1/9(0.8555 - 0.5358)z^{-1}}{1-(0.8555+0.5358)z^{-1}+0.4584z^{-2}}$$
$$= \frac{0.0355z^{-1}}{1-1.3913z^{-1}+0.4584z^{-2}}$$ 1

(c) $H(\theta) = e^{-j7\theta}, \quad |\theta| \le \frac{\pi}{3}$

$\quad\quad\;\; = 0 \quad$ otherwise

$$h(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\theta)\, e^{jn\theta}\, d\theta \quad 1$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} e^{-j7\theta} e^{jn\theta}\, d\theta \quad 1$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} e^{j(n-7)\theta}\, d\theta$$

$$= \frac{1}{j2\pi(n-7)} e^{j(n-7)\theta}\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \quad 1$$

$$= \frac{1}{j2\pi(n-7)}\left[e^{j(n-7)\frac{\pi}{3}} - e^{-j(n-7)\frac{\pi}{3}}\right]$$

$$= \frac{1}{j2\pi(n-7)}\, 2j \sin(n-7)\frac{\pi}{3}$$

$$= \frac{1}{\pi(n-7)} \sin(n-7)\frac{\pi}{3}$$

$$= \frac{1}{3}\,\frac{\sin\left([n-7]\frac{\pi}{3}\right)}{[n-7]\frac{\pi}{3}} \quad 1$$

Group delay $= 7$ samples $= \alpha \quad 1$

$$\alpha = \frac{N-1}{2}$$

$$\Rightarrow N = 2\alpha + 1 = 15 \quad 1$$

$$\therefore h(n) = \frac{1}{3}\,\frac{\sin\left([n-7]\frac{\pi}{3}\right)}{[n-7]\frac{\pi}{3}} \quad n = 0, \ldots, 14 \quad 1$$

20