2. The z-Transform

2.1 Introduction

In *EE221/222 Electrical Circuits and Systems*, you studied the Laplace Transform, and learned how it could be used in the representation and analysis of continuous-time signals and systems. The Laplace transform can be viewed as a "transform domain" representation of the signal (or system). There are a number of benefits associated with this. For example, you will recall that continuous-time system behaviour in the time-domain can be represented by means of linear differential equations. The solution of such equations in the time-domain is perfectly feasible, if a little tedious. However, when the equation is re-written in the transform domain (by means of the Laplace Transform), differential equations become simpler algebraic equations, and it becomes much easier to obtain the transform domain representation of the system output (which can then be transformed back to the time-domain by means of the Inverse Laplace Transform). As another example, if you want to obtain the overall impulse response of a cascade of two linear systems in the time domain, you have to carry out convolution of the individual impulse responses. However, if the individual impulse responses are represented by their Laplace transforms, the multiplication of these two Laplace transforms gives you the transform of the overall impulse response (inverse Laplace transformation then gives the overall impulse response back in the time domain).

The transform domain representation of a continuous-time system also provides an easy route for obtaining the frequency response of a system – you simply set $s = \mathrm{j}\omega$ in the Laplace transform of the system's impulse response (which is the system transfer function).

For discrete-time signals and systems, the analogue of the Laplace Transform is the z-Transform, and many of the same concepts, properties, and techniques that are associated with the Laplace Transform are equally applicable to the z-Transform.

2.2 Definition of the z-Transform

The z-Transform of a sequence $x(n)$ is defined by the following equation:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is a complex variable, and plays a role similar to that played by the complex variable s in the Laplace Transform. Because z is a complex number, it may be represented in the complex "z-plane" either in Cartesian form as $a+jb$, where a is the real part and b is the imaginary part, or in polar form as $r\mathrm{e}^{\mathrm{j}\theta}$, where r is the radius of a point in the z-plane, and θ is the corresponding angle that the point makes with the positive direction of the real axis. The above equation defines the two-sided z-transform; the more common case of a causal signal (or system impulse response) gives the following "one-sided" z-Transform:

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

It can be seen that the z-Transform of a sequence is, effectively, an infinite sum, with the sample values of the sequence as the coefficients of the power series. As was the case for the Laplace Transform, the z-Transform may not converge to a finite value for all values of z. The range of values of z for which the z-Transform converges is known as the Region of Convergence (more below).

Formally, the Inverse z-Transform is defined by the following equation:

$$x(n) = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

i.e. the signal $x(n)$ is expressed as a weighted superposition of complex exponentials of the form z^n, with weighting factors $(1/2\pi j)X(z)z^{-1}$. The inverse z-Transform is found by means of contour integration over a closed contour in the z-plane. This is generally a difficult talk to carry out, so we will normally rely on simpler methods to move from the transform domain back to the time domain. In particular, we will tend to use a subset of “common” signals with well-defined z-Transforms, and simply rely on the use of tables to transform between time sequences and z-Transforms. In other cases, if z-Transform we wish to invert is not included in our Table, we may be able to rewrite it in terms of transforms that do appear in the Table. For example, the Method of Partial Fractions may be used for this (see Exercise 2.2 below).

2.3 Z-transforms of Some Useful Sequences

2.3.1 Unit Impulse and Delayed Unit Impulse

The z-transform of the unit impulse is calculated as follows:

$$X(z) = \sum_{n=-\infty}^{\infty}\delta(n)z^{-n} = 1 + 0.z^{-1} + 0.z^{-2} + ... = 1$$

The z-transform of the delayed unit impulse is found as follows:

$$X(z) = \sum_{n=-\infty}^{\infty}\delta(n-k)z^{-n} = 0 + 0.z^{-1} + 0.z^{-2} + ... + z^{-k} + 0.z^{-(k+1)} + ... = z^{-k}$$

This neatly illustrates an important way of viewing the z variable – essentially, as a time-shift variable. In other words, delaying a sequence by k samples in the time domain is equivalent to multiplying its z-transform by z^{-k}. Alternatively, multiplication by z corresponds to a time advance by one sample period in the time domain. This is an important property that we will frequently use.

2.3.2 Arbitrary, Finite-Duration Sequence

Consider the following sequence:

$x(n)$ = {1, -3, 0, 2, 4, 2} commencing at n = 3.

The z-Transform of this sequence can be obtained as follows:

$$X(z) = \sum_{n=-\infty}^{\infty}x(n)z^{-n} = \sum_{n=3}^{8}x(n)z^{-n} = z^{-3} - 3z^{-4} + 2z^{-6} + 4z^{-7} + 2z^{-8}$$

i.e. the z-Transform can be directly written by inspection of the original sequence.

2.3.3 Unit Step Function

The z-Transform of the unit step function is:

$$X(z)=\sum_{n=-\infty}^{\infty}u(n)z^{-n}=\sum_{n=0}^{\infty}z^{-n}=1+z^{-1}+z^{-2}+...=\frac{1}{1-z^{-1}}=\frac{z}{z-1}$$

if $|z^{-1}|< 1$ (see Exercise 2.1 below).

2.3.4 Sampled Exponential

The z-Transform of the exponential sequence:

$$x(n)=a^n u(n)$$

is found as follows:

$$X(z)=\sum_{n=0}^{\infty}a^n z^{-n}=1+az^{-1}+a^2z^{-2}+a^3z^{-3}...=\frac{1}{1-az^{-1}}=\frac{z}{z-a}$$

if $\left|az^{-1}\right|<1$.

The exponential sequence is a good example to use to illustrate the issue of convergence of the z-Transform, so we will look some more at this topic in the next sub-section.

Table 2.1 summarises the z-Transforms of some common sequences.

	Sequence	**z-Transform**
1. Unit impulse	$\delta(n)$	1
	$\delta(n-k)$	z^{-k}
2. Unit step	$u(n)$	$z/(z-1)$
3. Exponential	$a^n\, u(n)$	$z/(z-a)$
4. Sinusoidal	$\sin(\theta_O n)\, u(n)$	$\dfrac{z\sin\theta_O}{z^2-2z\cos\theta_O+1}$
	$\cos(\theta_O n)\, u(n)$	$\dfrac{z^2-z\cos\theta_O}{z^2-2z\cos\theta_O+1}$
5. Unit ramp	$nu(n)$	$\dfrac{z}{(z-1)^2}$

6. Product of ramp and signal $nx(n)$	$-z\dfrac{dX(z)}{dz}$
7. Sum of Series: $1+z^{-1}+z^{-2}+z^{-3}+..........+z^{-(N-1)} = \dfrac{1-z^{-N}}{1-z^{-1}}$	

Table 2.1. Table of some useful z-Transform pairs.

2.4 Convergence of the Z-transform

As noted above, the Region of Convergence (ROC) of the z-Transform represents the region in the z-plane within which the z-Transform converges. For the most part, we will not be too concerned about the ROC when dealing with the sampled signals and digital filters that are of most interest. However, we will look at one or two specific examples.

First, let us re-examine the sampled exponential sequence, and evaluate the z-Transform for the specific value of $z = 2a$:

$$X(z)\Big|_{z=2a} = \sum_{n=0}^{\infty} a^n (2a)^{-n} = \sum_{n=0}^{\infty} a^n 2^{-n} a^{-n} = \sum_{n=0}^{\infty} 2^{-n} = 1 + \frac{1}{2} + \frac{1}{4} + ... = 2$$

i.e. the transform converges. On the other hand, if we evaluate the z-Transform for $z = a/2$, we get the following:

$$X(z)\Big|_{z=a/2} = \sum_{n=0}^{\infty} a^n \left(\frac{a}{2}\right)^{-n} = \sum_{n=0}^{\infty} a^n \left(\frac{1}{2}\right)^{-n} a^{-n} = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + ... = \infty$$

i.e. the transform does not converge. In this particular example, the ROC is defined by

$$|z| > a \Rightarrow |az^{-1}| < 1$$

In other words, the ROC is defined by a circle in the z-plane of radius a (the ROC is outside this circle). This is illustrated in Figure 2.1.

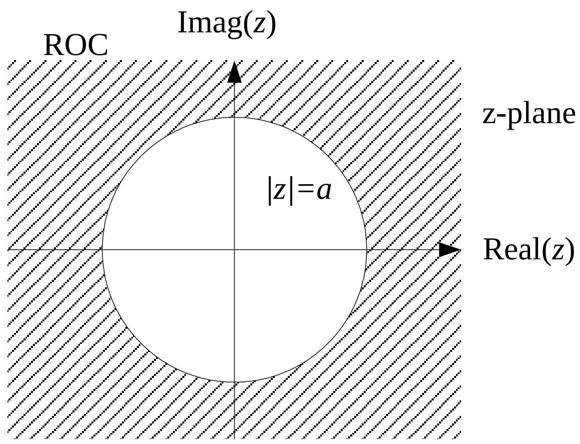

Figure 2.1. Region of Convergence for z-Transform of sampled exponential sequence.

The above example is of a signal that has infinite duration. For a finite-duration signal, it can be shown that the ROC consists of the entire z-plane, except perhaps for z=0 and/or $|z| = \infty$. For example, suppose $x(n)$ is non-zero only for the range of sample indices $N_1 \le n \le N_2$. Then, the z-Transform is given by:

$$X(z) = \sum_{n=N_1}^{N_2} x(n) z^{-n}$$

This sum will converge if each term in the sum is finite. If N_2>0, then the sum will involve terms of the form z^{-1}, z^{-2} etc., which means that it will not converge for z = 0. On the other hand, if N_1<0 (i.e. the signal is non-causal), then the sum will include terms involving z, z^2 etc., so it will not converge for $|z| = \infty$.

Exercise 2.1

Find the z-Transforms of the following sequences, and determine their ROCs:

(a) {1, 3, 5, 3, 1} starting at n=-5 (i.e. non-causal)

(b) {1, 3, 5, 3, 1} starting at n=-2 (i.e. non-causal)

(c) {1, 3, 5, 3, 1} starting at n=0 (i.e. causal)

(d) The unit step function (causal, but of infinite duration)

Exercise 2.2

Determine the inverse z-Transforms of the following functions:

(a) $X(z) = 3 + 2z^{-1} + 4z^{-2} - 3z^{-4}$

(b) $X(z) = \dfrac{1}{z(z-1)(2z-1)}$ (use partial fractions)

2.5 Properties of the z-Transform

As noted above, many of the properties of the z-Transform are the same as those of the Laplace Transform. For convenience, we denote the z-Transform of a signal $x(n)$ by:

$$X(z) = Z[x(n)]$$

and the inverse z-Transform of a function $X(z)$ by:

$$x(n) = Z^{-1}[X(z)]$$

2.5.1 Linearity

If $Z[x(n)] = X(z)$ and $Z[y(n)]=Y(z)$, then the linearity property states that:

$$Z[ax(n) + by(n)] = aX(z) + bY(z)$$

2.5.2 Time shift

The time-shift (or "translation") property of the z-Transform may be written as:

$Z[x(n\text{-}k)] = z^{-k}X(z)$

Proof

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z[x(n-k)] = \sum_{n=-\infty}^{\infty} x(n-k) z^{-n}$$

Let $i = n\text{-}k$, so that $n=i+k$. Then, the equation for $Z[x(n\text{-}k)]$ becomes:

$$Z[x(n-k)] = \sum_{i=-\infty}^{\infty} x(i) z^{-(i+k)} = \sum_{i=-\infty}^{\infty} x(i) z^{-i} z^{-k} = z^{-k} X(z)$$

2.5.3 Convolution

This is an extremely important property of the z-Transform. It states that:

$Z[x(n)*y(n)] = X(z)Y(z)$

i.e. the z-Transform of the convolution of two sequences is equal to the product of the z-Transforms of the two sequences.

Exercise 2.3

Prove the convolution property of the Z-Transform.

Exercise 2.4

Using the z-Transform convolution property, determine the sequences resulting from the convolution of the following pairs of sequences:

(a) $x(n) = \{1, -2, 1\}$ and $y(n) = \{1, 1, 1, 1, 1, 1\}$

(b) $x(n) = \{1, 2, 3, 1, -1, 1\}$ and $h(n) = \{1, 1, 1\}$

All sequences commence at $n = 0$. Verify your result by calculating the convolution in the time domain in the usual manner.

2.6 Relationship between the Laplace and z-Transforms

We now look a little more at the relationship between the Laplace Transform and the z-Transform. To illustrate this, let us return to an arbitrary continuous-time signal $x(t)$, and sample it at a rate of one sample every T seconds. Mathematically, what we end up with here is a sequence of delta functions, with each function weighted by value of $x(t)$ at the time of occurrence:

$$x^*(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

(of course, for DSP purposes, the sampled signal is simply a sequence of numbers denoted by $x(n)$).

If we take the Laplace Transform of this signal (and note that the Laplace Transform of a delayed delta function $\delta(t-\tau)$ is $e^{-s\tau}$), we obtain:

$$X^*(s) = \sum_{n=-\infty}^{\infty} x(nT)e^{-snT}$$

If we let

$$z = e^{sT}$$

then we can write

$$X^*(s)\Big|_{z=e^{sT}} = \sum_{n=-\infty}^{\infty} x(nT)z^{-n} = X(z)$$

In other words, the z-Transform of the sequence $x(nT)$ is the Laplace transform of the sampled signal $x^*(t)$, with the variable z substituted for e^{sT}. The equation $z = e^{sT}$ provides a mapping between the s-plane and the z-plane. Furthermore, you will recall that shifting a continuous-time function by an amount T corresponds to multiplication of its Laplace transform by e^{sT}. By the same token, shifting a discrete-time sequence by one sample period corresponds to multiplying its z-Transform by z, which is consistent with the mapping.

If we let $s = \sigma + j\omega$, we can gain further insights into the mapping. In this case:

$$z = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T}$$

If we let σ = 0, the imaginary axis of the s-plane maps into the unit circle in the z-plane through the complex exponential function. Furthermore, because $e^{j\omega T}$ is periodic, as we move along the imaginary axis of the s-plane from 0 to $2\pi/T$, we move once around the unit circle in the z-plane; as ω increases beyond $2\pi/T$, we start another circuit of the unit circle. The result is that the entire left half of the s-plane maps to the interior of the unit circle in the z-plane, while the right-half of the s-plane maps to the exterior of the unit circle. Another way of looking at this is to observe that each "strip" of the left half of the s-plane of width $2\pi/T$ maps into the interior of the unit circle. The mapping from s-plane to z-plane is illustrated in Figure 2.2.

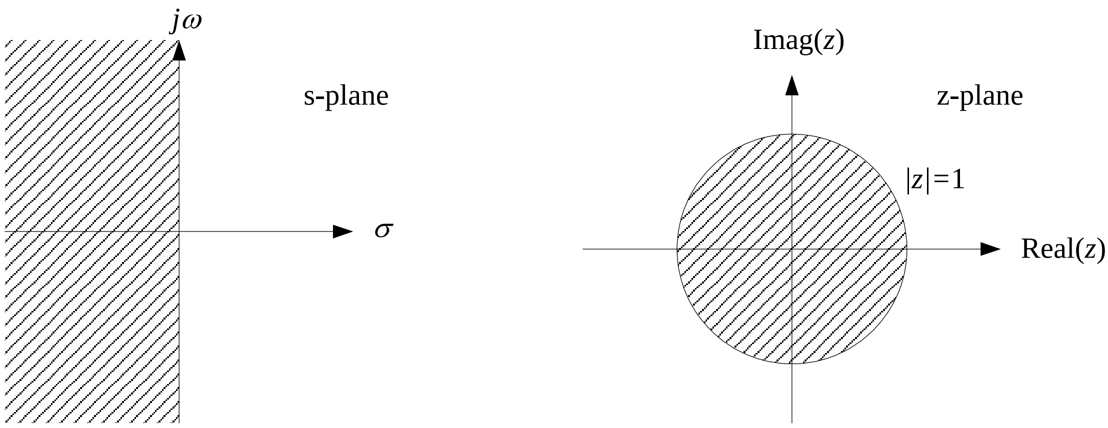

Figure 2.2. Mapping between s-plane and z-plane.

We will be returning to this basic mapping again and again in our studies in DSP.

2.7 Transfer Functions

So far, we have looked at the z-Transform of arbitrary sequences, without regard to what those sequences represent. For example, they could have represented impulse responses of systems, or they could simply have represented input signals. In this sub-section, we will look in more detail at the representation of discrete-time systems (digital filters) by means of the z-Transform. In particular, the linearity, time shift, and convolution properties of the transform are very important in systems analysis.

In Section 1, we represented the behaviour of digital filters by means of the difference equation, which explicitly gave the output of the filter in terms of the input, previous outputs, and the digital filter coefficients. We also described the behaviour of a system in terms of its impulse response (and the response to a given input was then obtained by means of convolution with the impulse response). It is straightforward to obtain a "transform domain" representation of a digital filter, starting with the difference equation. For example, suppose we take the following difference equation (from Exercise 1.1, Part (c)):

$$y(n) = x(n) + 0.6x(n-1) - 0.1x(n-2) + 0.3y(n-1) - 0.3y(n-2)$$

Taking the z-Transform of this equation, and in particular, making use of the linearity and time shift properties, we obtain:

$$Y(z) = X(z) + 0.6X(z)z^{-1} - 0.1X(z)z^{-2} + 0.3Y(z)z^{-1} - 0.3Y(z)z^{-2}$$

Collecting terms in $Y(z)$ on the left-hand side and terms in $X(z)$ on the right-hand side, we obtain:

$$Y(z) - 0.3Y(z)z^{-1} + 0.3Y(z)z^{-2} = X(z) + 0.6X(z)z^{-1} - 0.1X(z)z^{-2}$$

Factorise to obtain:

$$Y(z)[1 - 0.3z^{-1} + 0.3z^{-2}] = X(z)[1 + 0.6z^{-1} - 0.1z^{-2}]$$

Then, the ratio of $Y(z)$ to $X(z)$ is given by:

$$\frac{Y(z)}{X(z)} = \frac{1 + 0.6z^{-1} - 0.1z^{-2}}{[1 - 0.3z^{-1} + 0.3z^{-2}]}$$

This is the ratio of the z-Transform of the output to the z-Transform of the input, and is the transfer function, denoted by $H(z)$.

We can use the convolution property of the z-Transform to arrive at the same result by a slightly different route. We know that the response of a filter to a given input $x(n)$ is given by the convolution of the input and the filter's impulse response, $h(n)$, i.e.

$$y(n) = x(n) * h(n)$$

The convolution property of the z-Transform tells us that:

$$Z[y(n)] = Z[x(n)]Z[h(n)]$$

Since $Z[y(n)] = Y(z)$ and $Z[x(n)] = X(z)$, we then have:

$$Y(z) = X(z)\, Z[h(n)]$$

Therefore, the transfer function $H(z) = Y(z)/X(z)$ is equal to $Z[h(n)]$, i.e. the z-Transform of the filter impulse response.

In Section1, we saw how a digital filter could be described by a block diagram, in the “time domain”. A digital filter can also be represented by a block diagram in the “transform domain”. For example, Figure 2.3 shows z-domain block diagrams of the two simple digital filters that were originally given in Figure 1.8.

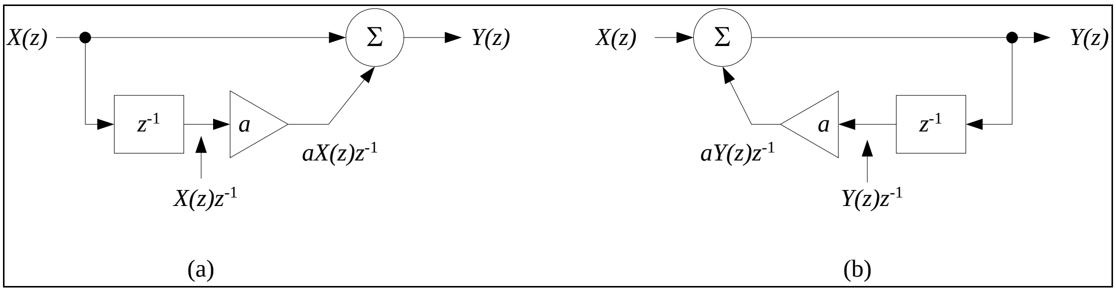

Figure 2.3. Z-domain block diagrams of filters from Figures 1.8(a) and (b).

<u>Exercise 2.5</u>

A digital filter is described by the following equation:

$$y(n) - y(n-1) + 0.9y(n-2) = x(n) + 0.5x(n-1)$$

(a) Draw block diagrams of this system in the time domain.

(b) Determine the system transfer function, and draw a block diagram of the system in the z-domain.

(c) Write Matlab code to calculate and plot the impulse response for the first 100 samples.

<u>Exercise 2.6</u>

Write the difference equations and draw block diagrams for the following transfer functions:

(a) $$H(z) = \frac{1 + 0.3z^{-1}}{1 - 0.4z^{-1}}$$

(b) $$H(z) = \frac{1 + 0.2z^{-1} + 4z^{-2}}{1 - 4z^{-1} + 2z^{-2}}$$

(c) $$H(z) = \frac{1 + 0.2z^{-1} + 4z^{-2}}{2 - 4z^{-1} + 2z^{-2}}$$

(d) $$H(z) = \frac{2 + 2z^{-2}}{1 - 2.2^{-1} + 1.7z^{-2}}$$

Note: When obtaining the difference equation from the transfer function, care must be taken to use the correct sign (positive or negative) with the coefficients of terms in the *denominator* of the transfer function. In particular, the sign of denominator coefficients may change when transforming back to the difference equation (when terms are moved to the right-hand side of the equation).

2.8 Poles and Zeros

2.8.1 Introduction

We now have an efficient means of representing digital filters by means of the z-domain transfer function. The transfer function can be obtained either from the difference equation, or from the z-Transform of the impulse response. The z-domain transfer function for discrete-time systems means exactly the same thing as the s-domain transfer function does for continuous systems. In particular, it can be used to obtain the frequency response of the system (more in Section 3). It also gives great insights into aspects of system behaviour, such as the form of the impulse response, and whether or not the system is stable (this can also be determined from the impulse response). In particular, the transfer functions of digital filters can be written in terms of poles and zeros, in exactly the same way as the transfer functions of continuous-time systems can, and poles and zeros means the same thing here as they do there. The zeros of a system are the values of z for which $H(z) = 0$, while the poles are the values of z for which $H(z) = \infty$.

2.8.2 Representing Systems in terms of Poles and Zeros

Thus far, we have written transfer functions as ratios of polynomials in negative powers of z; the general form of such a transfer function is:

$$H(z) = \frac{\sum_{k=0}^{K} a_k z^{-k}}{\sum_{n=0}^{N} b_n z^{-n}} = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + ... + a_K z^{-K}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + ... + b_N z^{-N}}$$

The order of the numerator is K and the order of the denominator is N; the constants a_k and b_n are the digital filter coefficients. Frequently, the transfer function is scaled such that coefficient $b_0 = 1$, so that the transfer function can be re-written as:

$$H(z) = \frac{\sum_{k=0}^{K} a_k z^{-k}}{1 + \sum_{n=1}^{N} b_n z^{-n}} = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + ... + a_K z^{-K}}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + ... + b_N z^{-N}}$$

Determination of the poles and zeros of the system is more readily done if the numerator and denominator are expressed in terms of positive powers of z (because it's easier to factorise them). Therefore, we need to multiply the numerator by z^K and the denominator by z^N to achieve this (of course, we also need to multiply the entire transfer function by z^N/z^K to compensate for this):

$$\begin{aligned} H(z) &= \frac{z^N}{z^K}\frac{z^K}{z^N}\frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + ... + a_K z^{-K}}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + ... + b_N z^{-N}} \\ &= z^{N-K}\frac{a_0 z^K + a_1 z^{K-1} + a_2 z^{K-2} + a_3 z^{K-3} + ... + a_K}{z^N + b_1 z^{N-1} + b_2 z^{N-2} + b_3 z^{N-3} + ... + b_N} \\ &= z^{N-K}\frac{A(z)}{B(z)} \end{aligned}$$

The polynomials A(z) and B(z) can then be factorised as follows:

$$H(z) = z^{N-K}\frac{a_0(z - z_1)(z - z_2)(z - z_3)...(z - z_K)}{(z - p_1)(z - p_2)(z - p_3)...(z - p_N)}$$

Clearly, z_1, z_2 etc. are the values of z for which $H(z) = 0$, while p_1, p_2 etc. are the values of z for which $H(z) = \infty$. The term a_0 in the numerator is simply a scaling factor. The leading term z^{N-K} will result in multiple zeros at $z = 0$ if $N>K$, or multiple poles at $z = 0$ if $N<K$.

2.8.3 Pole-Zero Maps

A pole-zero map (diagram) is a plot of the positions of the poles and zeros in the z-plane (similar in many ways to the pole-zero diagrams used to described continuous-time systems). In the following exercise, we will consider some simple examples to illustrate how poles and zeros may be calculated from a difference equation.

Exercise 2.7

Determine the poles and zeros, and plot the pole-zero map, for each of the following difference equations:

(a) $y(n) = x(n) - x(n-1) - 0.5y(n-1)$

(b) $y(n) = x(n) - 0.9x(n-1) - 0.5x(n-2)$

(c) $y(n) = 2x(n) - x(n-1) - 0.3y(n-1)$

(d) $y(n) = x(n) - 0.8y(n-1) - 0.8y(n-2)$

(e) $y(n) = x(n) + 0.45x(n-1) - 0.77x(n-2) - 0.86y(n-1) - 0.95y(n-2)$

By way of example, pole-zero maps for the systems in Exercise 2.7, Parts (c) and (d), are given in Figure 2.4 (note the double pole at the origin for Part (d)).

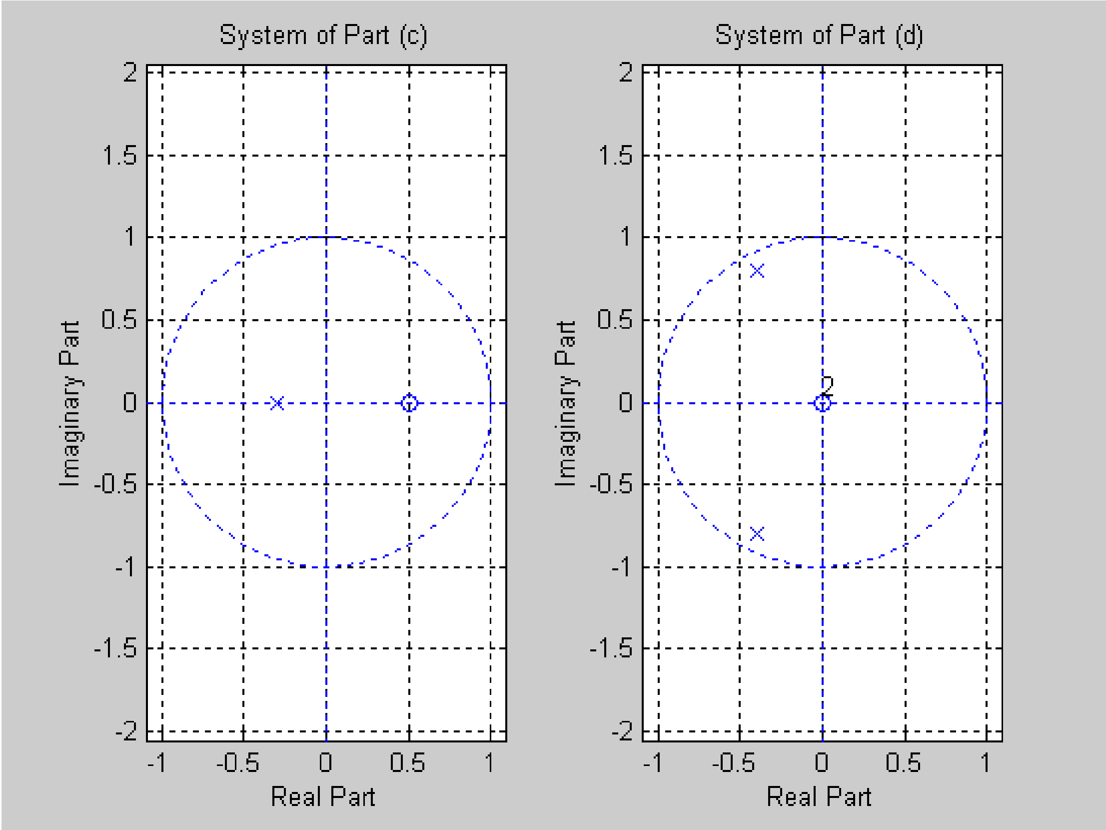

Figure 2.4. Pole-zero maps for systems in Exercise 2.7, Parts (c) and (d).

Note: Generally-speaking, we are primarily interested in systems that have real coefficients only. In this case, the poles and zeros of such systems are either:

- Purely real;
- Complex, but occurring in complex conjugate pairs, i.e. one pole/zero will be of the form *a+jb*, while another pole/zero will be of the form *a-jb* (for example, see Exercise 2.7, Part (d)). In this case, two complex conjugate terms will combine to produce a second-order term with only real coefficients. Alternatively, complex conjugate poles/zeros may be represented in polar form as $re^{j\phi}$ and $re^{-j\phi}$, where ϕ is the angle of the pole/zero with respect to the positive direction of the real axis.

There are some applications (for example, in data communications) where the use of complex digital filter coefficients is very helpful in describing the behaviour of a system, but even the implementation of a filter with complex coefficients has to be decomposed into a series of "real" operations (e.g. the multiplication of a complex coefficient by a complex data value results in a set of four "real" multiplies, plus a couple of additions and subtractions).

2.8.4 Stability in terms of the Pole-Zero Map

The locations of poles can be used to infer the stability of a system. In particular, for a system to be stable, all of its poles must be located inside the unit circle (this fact follows intuitively from the earlier observation that the left-half of the s-plane maps to the interior of the unit circle in the z-plane). This implies that a non-recursive filter is always stable, because it will only have poles that are located at the origin (z = 0). A recursive filter, on the other hand, may or may not be stable, depending on the positions of its poles. If a system has poles sitting on the unit circle, then that system may continually oscillate – such a system is said to be marginally stable (a similar situation holds for continuous-time systems with poles on the imaginary axis).

For example, the system with the following difference equation is unstable:

$$y(n) = x(n) - 0.56y(n-1) - 1.5y(n-2)$$

It can be seen from the pole-zero map in Figure 2.5 that the poles are outside the unit circle.

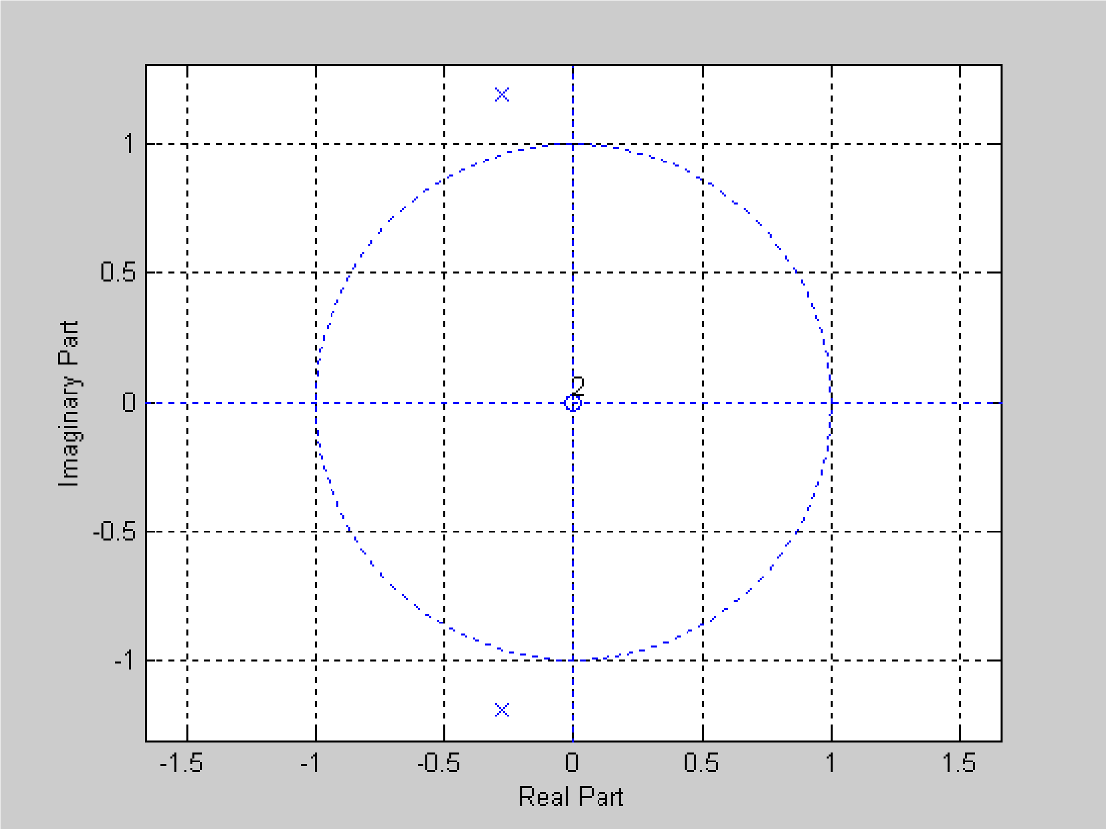

Figure 2.5. Pole-zero map for system $y(n) = x(n) - 0.56y(n-1) - 1.5y(n-2)$

2.8.5 Determination of Impulse Response from Pole-Zero Map

Apart from stability, a great deal can be inferred about the impulse response from the pole-zero map, in particular from the positions of the poles. For example, if a digital filter has a purely real pole located at $z = a$, then this corresponds to an impulse response of the form:

$$h(n) = a^n$$

If $a < 1$, then $h(n)$ is decaying, while if $a > 1$, then $h(n)$ grows larger and larger as n increases, i.e. the filter is unstable. Also, if $a > 0$, then the exponential is "unipolar", i.e. all of the sample values are positive, whereas if $a < 0$, then $h(n)$ "oscillates" (see Table 1.1 and Figure 1.5).

For the case of a complex conjugate pole pair, the situation is a little more complicated. In this case, it is more instructive to express the poles in polar form, i.e, we have poles at z = $re^{j\phi}$ and z = $re^{-j\phi}$. This gives an impulse response of the form:

$$h(n) = r^n \sin(n\phi)$$

i.e. a sinusoid that oscillates at a "digital" frequency of ϕ radians, with a decaying exponential term (note that r, the pole radius, is positive). If $r < 1$, then the term in r^n causes the oscillation to decay, while if $r > 1$, then the oscillation grows without limit, i.e. the filter is unstable (as expected, because the poles are outside the unit circle). If $r = 1$, then the poles are sitting on the unit circle, and the digital filter is an oscillator.

For example, the pole-zero plot and the impulse response of the filter with the following difference equation are plotted in Figure 2.6:

$$y(n) = x(n) + 1.8y(n-1) - 0.95y(n-2)$$

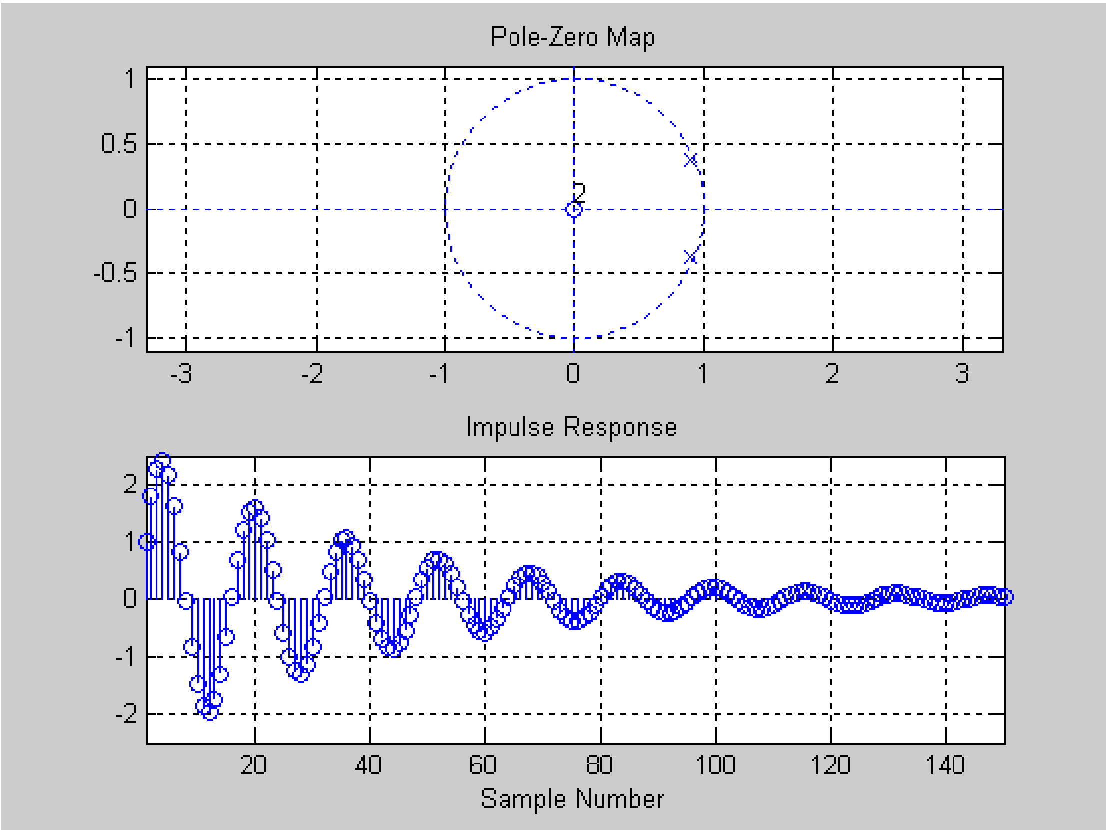

Figure 2.6. Plot of pole-zero map and impulse response for filter with difference equation $y(n) = x(n) + 1.8y(n-1) - 0.95y(n-2)$.

Exercise 2.8

For the system shown in Figure 2.6, estimate the value of ϕ from the impulse response, and verify this value using the pole-zero map.

2.9 Conclusion

Section 1 treated discrete-time signals and systems from the point of view of the time domain, in particular, dealing with such topics as processing signals using operations like convolution, and describing systems by means of the impulse response. This Section has discussed the description of signals and systems by means of the z-Transform, and has displayed the relationship between the time-domain and the z-domain. The important topic of system transfer functions has also been covered. The next Section will examine Fourier analysis for discrete-time signals and systems, and will also discuss in more detail the inter-relationships between the time-domain, the z-domain and the frequency domain.