Lecture 02 – Linear Programming

Optimisation CT5141

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Linear programming

In this first section of the module, we'll discuss **linear programming** and its close relative **integer programming**.

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These methods work for real-valued and integer-valued search spaces respectively, and even mixed ones. They require strong assumptions:

- the search space is composed of **decision variables**, either real or integer-value;
- **linearity** of the objective in the decision variables;
- there are **constraints** which mean that large parts of the search space are **infeasible**. Constraints are expressed as **linear** equations and inequalities in the decision variables.

Overview

1. **What is linear programming? A motivating problem**
2. Graphical solution in 2D
3. More applications
4. A little theory

Terminology

An LP is a **mathematical program**, a specific type of optimisation problem:

- a set of decision variables
- a set of **constraints**
- an objective function to be minimised (or maximised).

Terminology

Terms such as **dynamic programming**, **mathematical programming** and **linear programming** are misleading to modern ears. They **predate** the use of "programming" to mean "writing computer code".

From Wikipedia:

> The term "linear programming" for certain optimization cases was due to George B. Dantzig, although much of the theory had been introduced by Leonid Kantorovich in 1939. (**Programming** in this context does not refer to computer programming, but comes from the use of **program** by the United States military to refer to proposed training and logistics schedules, which were the problems Dantzig studied at that time.) Dantzig published the Simplex algorithm in 1947, and John von Neumann developed the theory of duality in the same year.

Motivating problem: product mix

Manufacturing hand sanitizer

Suppose we have a small manufacturing unit for two hand sanitizer products.

- Products have different requirements of a key raw material, of labour, give different profits, etc
- Constraints on labour (time), raw materials, etc
- Want to maximise profits
- How much of each product should we manufacture, ie what **product mix**?

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Decision variables

- x_1 is the quantity of Product 1 to make (in L)
- x_2 is the quantity of Product 2 to make (in L)

Formalising the objective

- Product 1 gives profit €15/L
- Product 2 gives profit €10/L
- Total profit = profit from Product 1 + profit from Product 2
- Maximise $15x_1 + 10x_2$

Constraints

If some values for decision variables are not allowed, that is a **constraint**

- Maximum demand for Product 1 is 18L
- Maximum demand for Product 2 is 30L
- Maximum supply of active ingredient is 2.2L
- Each unit of either product requires 0.1L of active ingredient
- Product 1 requires 1.5 hours machine time
- Product 2 requires 2.5 hours machine time
- The machine operator cannot work more than 45 hours per week
- Cannot make a negative amount of either Product

Formalising the constraints

- Demand for Product 1: $x_1 \leq 18$
- Demand for Product 2: $x_2 \leq 30$
- Labour: $1.5x_1 + 2.5x_2 \leq 45$
- Active ingredient: $0.1x_1 + 0.1x_2 \leq 2.2$
- Non-negativity: $x_1 \geq 0$ and $x_2 \geq 0$

Formalising the problem

$$
\begin{aligned}
\text{Maximise profit: } & 15x_1 + 10x_2 \\
\text{Subject to: } 0.1x_1 + 0.1x_2 &\leq 2.2 \text{ (raw materials)} \\
1.5x_1 + 2.5x_2 &\leq 45 \text{ (labour)} \\
x_1 &\leq 18 \text{ (demand for Product 1)} \\
x_2 &\leq 30 \text{ (demand for Product 2)} \\
x_1 &\geq 0 \text{ (non-negativity)} \\
x_2 &\geq 0 \text{ (non-negativity)}
\end{aligned}
$$

Example

- A potential solution: 2 units of Product 1 and 2 units of Product 2
- Put the point $(2, 2)$ into the labour constraint:

$$\begin{aligned} & 1.5x_1 + 2.5x_2 \\ = & 1.5 \times 2 + 2.5 \times 2 \\ = & 3 + 5 \qquad = 8 \leq 45 \end{aligned}$$

- This solution satisfies this constraint…
- … in fact it satisfies all constraints…
- … but is not optimal.

Feasible solutions

- A potential solution which satisfies all constraints is called **feasible**
- The **best** feasible solution is the optimum.

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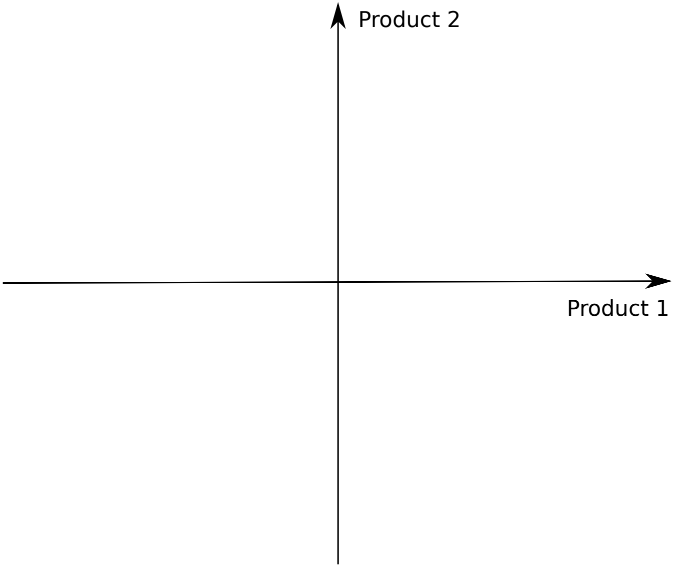
Graphical solution in 2D

Of course, this works only with 2 decision variables:

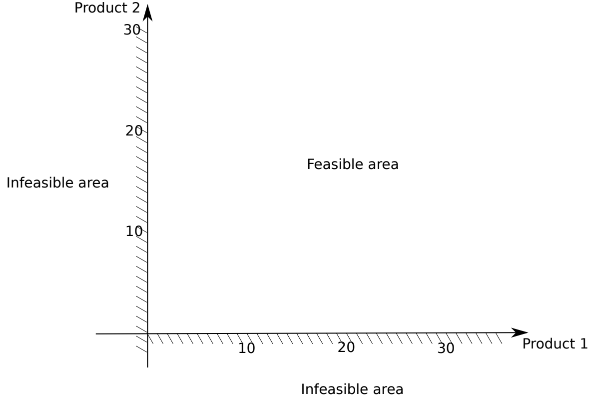
1. Draw a 2D graph
2. For each constraint, draw it as a line **cutting out half the plane**
3. Find the **feasible area**
4. Draw some contour lines: **lines of equal profit**
5. Identify the best corner point and find value of objective function there

Step 1: 2D graph



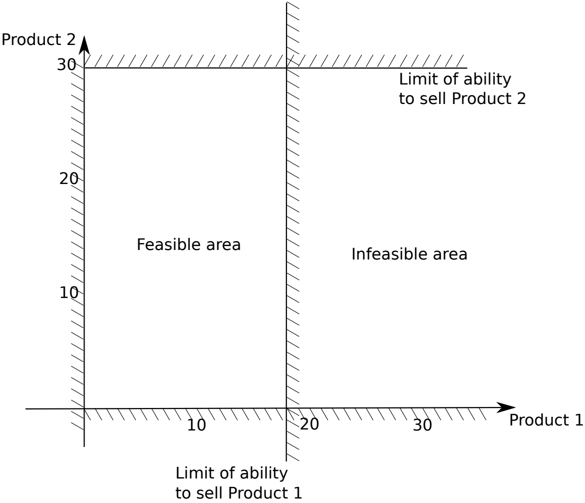
Step 2: Draw constraints

Non-negativity: $x_1 \geq 0$, $x_2 \geq 0$



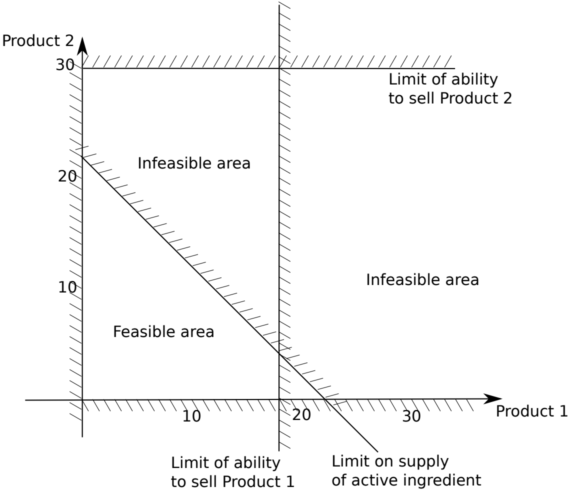
Step 2: Draw constraints and label them

Demand: $x_1 \leq 18$, $x_2 \leq 30$



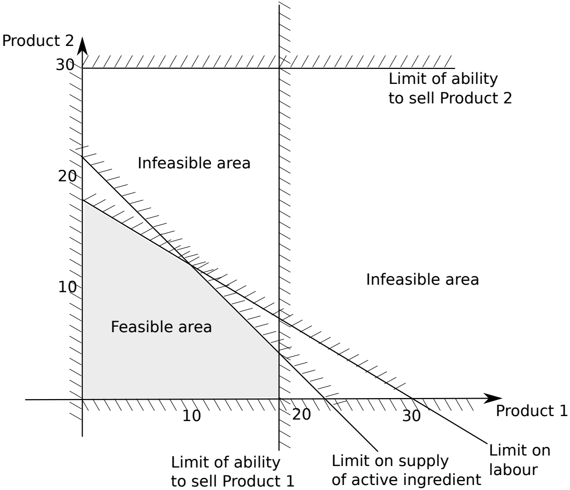
Step 2: Draw constraints and label them

Supply of active ingredient: $0.1x_1 + 0.1x_2 \leq 2.2$



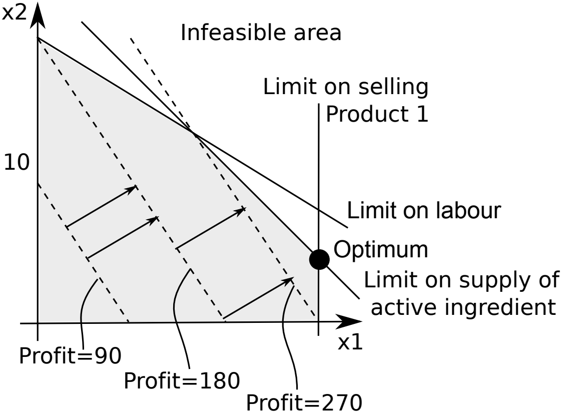
Step 3: find the feasible area

Labour: $1.5x_1 + 2.5x_2 \leq 45$



Step 4: lines of equal profit

- Lines of equal profit (LOEPs) are **contour lines**
- Choose an arbitrary value for profit, e.g. 90
- Objective function: $15x_1 + 10x_2 = 90$. **Draw this line**
- Repeat for other values, e.g. profit=180, profit=270

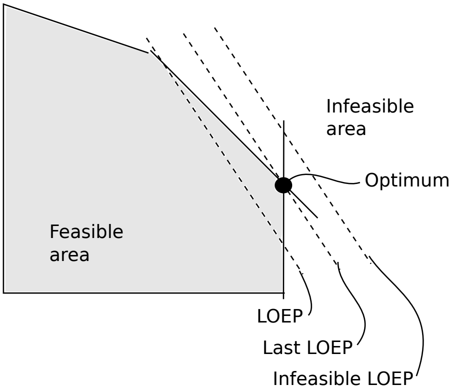


Step 4: lines of equal profit

- LOEPs are parallel to each other
- Largest profit: rightward and upward
- Each LOEP may have some points in the feasible area, some outside.

Step 5: find the optimum

- Optimum will always lie at a corner, where two constraints meet
- An LOEP through that point will have no other points in the feasible area
- Any larger LOEP will have no points in the feasible area



Step 5: find the optimum

- Optimum is at **the corner of two constraint lines**
- Treat them as **equations** (not inequalities)
- Solve them simultaneously

Step 5: find the optimum

$$
\begin{aligned}
0.1x_1 + 0.1x_2 &= 2.2 \text{ (raw materials)} \\
x_1 &= 18 \text{ (demand for P1)} \\
\implies 0.1x_2 &= 2.2 - 0.1 \times 18 \\
\implies x_2 &= 0.4/0.1 = 4 \\
&\implies \text{Optimum is } (18, 4)
\end{aligned}
$$

Step 5: find the optimum profit

$$\text{Optimum: } (18, 4)$$

$$\begin{aligned} \text{Optimum profit: } f(x_1, x_2) &= 15x_1 + 10x_2 \\ &= 15 \times 18 + 10 \times 4 \\ &= \text{EUR}310 \end{aligned}$$

To be careful: test the optimum

- Is it plausible?
- Does it make sense in context of original (verbal) problem?
- Use common sense.

Common misunderstanding

- Our goal is **not** to **solve the equation(s)**
- We solve the **problem** by following the graphical method
- The optimum is at a corner point which we find by solving some specific equations simultaneously.

Overview

Exercise: diet problem

In LP we can also **minimize**, e.g. minimize weight:

Diet problem: our astronauts need minimum amounts of several nutrients per day: 12mg of calcium, 10mg of zinc, 25mg of iron.

- **Space-biscuits** provide 2mg of calcium, 3mg of zinc, and 5mg of iron per serving. They weigh 100g per serving.
- **Astro-smoothies** provide 9mg of calcium, 5mg of zinc, and 12mg of iron per serving. They weigh 300g per serving.

How many servings of each should be supplied per astronaut per day, to minimize the weight?

Formulate this problem and solve it graphically.

Diet problem: solution

Decision variables

- x_1 servings of space-biscuits
- x_2 servings of astro-smoothies

Constraints

- $2x_1 + 9x_2 \geq 12$ calcium
- $3x_1 + 5x_2 \geq 10$ zinc
- $5x_1 + 12x_2 \geq 25$ iron

Objective

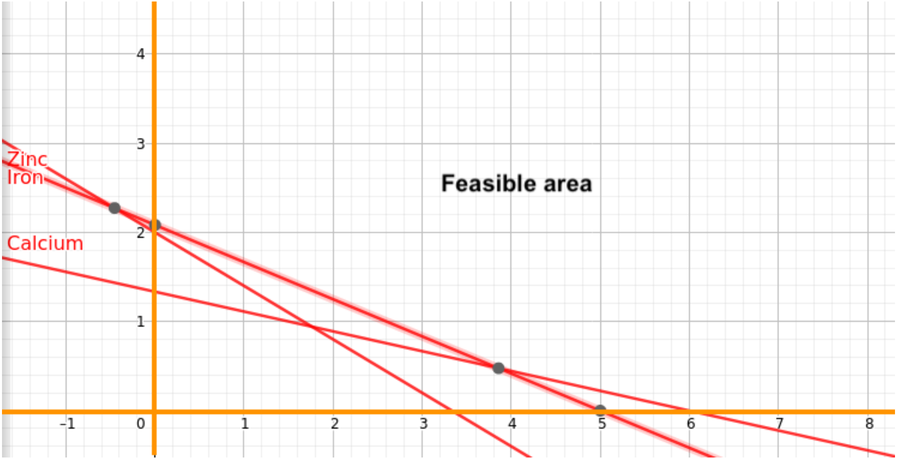
- Minimize $100x_1 + 300x_2$ weight

Diet problem: solution

Constraints:

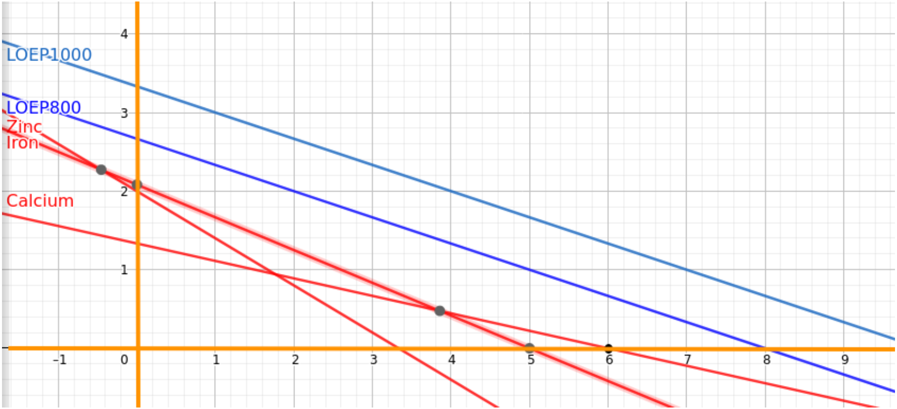
Diet problem: solution

Feasible area:



Diet problem: solution

Since we are now minimising **weight**, the “LOEPs” are now “LOEWs”:



Diet problem: solution

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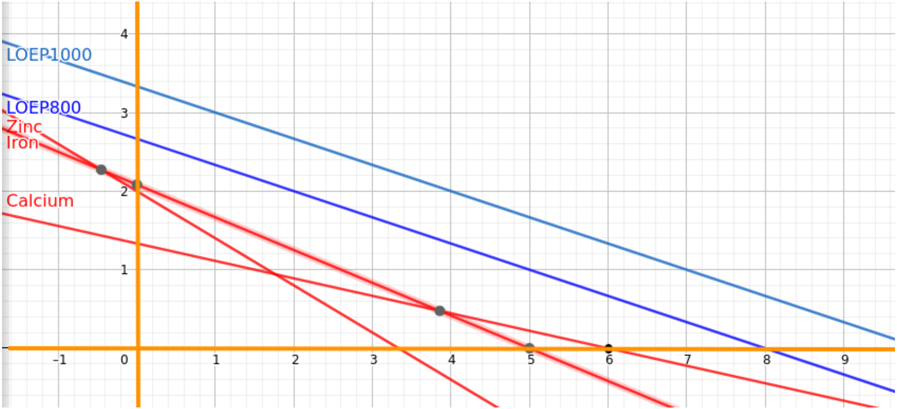

Best solution is the point approx (3.8, 0.5) (confirm by calculation)

Geograba graphing calculator

I’ve been using this to make the diagrams:

https://www.geogebra.org/graphing/tad337zk

Portfolio allocation

A pension fund has a total amount of money to invest, say EUR1M, to be allocated among many possible assets. It wants to maximise the total expected return.

It is obliged by regulations to avoid certain types and combinations of risks, e.g.:

- Diversification: no more than EUR700K in any one asset.
- No more than EUR800K in assets in the same sector, to avoid risk (e.g. over-exposure to tech sector).
- No more than EUR600K in assets in the same geographical region, to avoid risk (e.g. over-exposure to European stocks).

Portfolio allocation: problem formulation

- Suppose there are one thousand possible assets.
- Decision is represented by 1000 decision variables $x_i \in \mathbb{R}$.
- A solution $x \in \mathbb{R}^{1000}$
- Maximise the total expected return: objective function $f(x) = \sum_i r_i x_i$, where r_i is the expected return for investment i.
- Constraint on total investment: $\sum_i x_i \leq 1000000$
- Can't invest less than 0 in any asset: $x_i \geq 0 \ \forall \ i$
- Can't invest more than 700K in any asset: $x_i \leq 700000 \ \forall \ i$
- Sectoral diversification: $\sum_{i \in S_j} x_i \leq 800000$ where S_j is set of indices e.g. $S_0 = \{0, 17, 23, 190, 255\}$ for stocks in sector j.
- Regional diversification: $\sum_{i \in R_j} x_i \leq 600000$ where R_j is a set of indices representing stocks in region j.

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Too many variables to solve by hand: postpone til we have covered suitable LP software.

Double-subscript decision variables

Sometimes we have a **2D matrix** of decision variables, i.e. x_{ij}

Two examples to follow:

- Transport problem
- Blend problem

Canning Company (a transport problem)

A canning company operates two canning plants, P_1 and P_2. Three growers can supply fresh fruit:

- Grower G_1: up to 200 tonnes at €11/tonne
- Grower G_2: up to 310 tonnes at €10/tonne
- Grower G_3: up to 420 tonnes at €9/tonne

Plant capacities and labour costs are:

	Plant P_1	Plant P_2
Capacity:	460 tonnes	560 tonnes
Labour cost:	€26/tonne	€21/tonne

Shipping cost €3/tonne from any supplier to any plant.

The canned fruits are sold at €50/tonne, with no upper limit on sales.

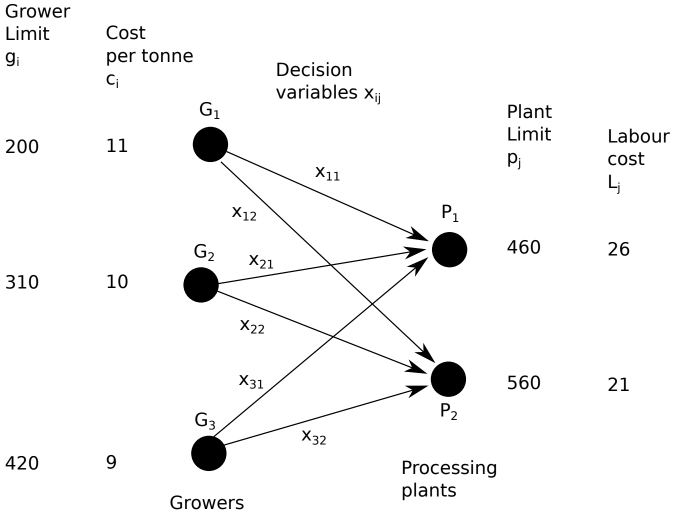
Canning Company (a transport problem)

The objective is to maximise profits. The decision is how much each grower should supply to each plant.

Formulate the problem by identifying the decision variables and formulating the constraints and objective.

Canning Company (a transport problem)

Decision variables: let x_{ij} be the number of tonnes supplied from grower i to plant j where $x_{ij} \geq 0$, $i = 1, 2, 3$; $j = 1, 2$



Canning Company (a transport problem)

The objective function is to maximise profit.

Let x_{ij} be the number of tonnes supplied from grower i to plant j where $x_{ij} \geq 0$, $i = 1, 2, 3$; $j = 1, 2$

Profit for goods shipped from Grower 1 to Plant 1:

- Sale price - Labour cost - fruit cost - shipping cost = $50 - 26 - 11 - 3 = 10$

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Repeat to find the profit coefficient a_{ij} for every DV.

This gives the objective **maximise** $\sum_{i,j} a_{ij}x_{ij}$.

Canning Company (a transport problem)

Constraint on supply for each grower:

- $x_{11} + x_{12} \leq 200$
- $x_{21} + x_{22} \leq 310$
- $x_{31} + x_{32} \leq 420$
- ie $\forall i, \sum_j x_{ij} \leq g_i$

Constraint on processing by each canning plant:

- $x_{11} + x_{21} + x_{31} \leq 460$
- $x_{12} + x_{22} + x_{32} \leq 560$
- ie $\forall j, \sum_i x_{ij} \leq p_j$

More complex transport problems

Above, we had this information:

- Shipping cost €3/tonne from any supplier to any plant.

In a more complex problem, shipping costs could vary:

- Shipping cost from supply point i to demand point j is given by a constant s_{ij}.

Then we would have a table of shipping costs (e.g. shape 3×2 for the Canning Company). This would change the calculation of the DVs' objective function coefficients c_{ij}.

Blend problem

Suppose we work for an oil company. We have three components (ingredients), and we produce three products, each a blend of the ingredients. No processing, just blending.

We have contracts to produce at least 3,000 barrels of each grade of motor oil per day.

Determine the optimal mix of the three components in each grade of motor oil to maximize profit.

Blend problem

Component	Maximum barrels available/day	Cost/barrel
1	4500	12
2	2700	10
3	3500	14

Grade	Component specification	Selling price/barrel
Super	At least 50% of C1	23
	No more than 30% of C2	
Premium	At least 40% of C1	20
	No more than 25% of C3	
Extra	At least 60% of C1	18
	At least 10% of C2	

Blend problem

Decision variables

The quantity of each of the three components used in each grade of gasoline (9 decision variables):
x_{ij} = barrels of component i used in motor oil grade j per day, where i = 1, 2, 3 and j = s (super), p (premium), and e (extra).

Blend problem

Decision variables

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Objective function

Maximise:

$$11x_{1s} + 13x_{2s} + 9x_{3s} + 8x_{1p} + 10x_{2p} + 6x_{3p} + 6x_{1e} + 8x_{2e} + 4x_{3e}$$

Blend problem constraints

What are the constraints? This is tricky! We will attempt this exercise in the lab.

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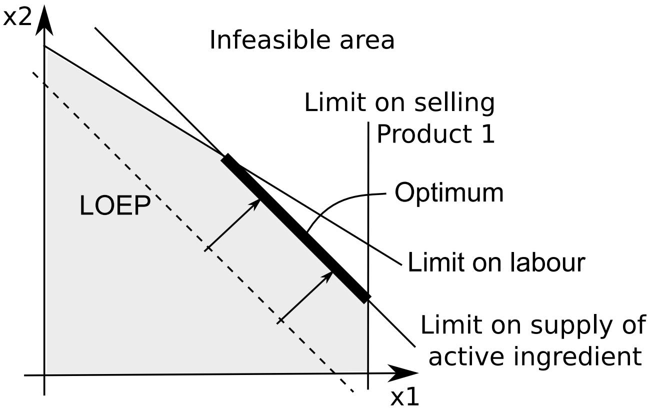
Possible outcomes of LP

- Normal outcome (we find the optimum)
- Multiple equal optima
- Problem is infeasible
- Problem is unbounded
- Degeneracy (one constraint is redundant)

Multiple equal optima



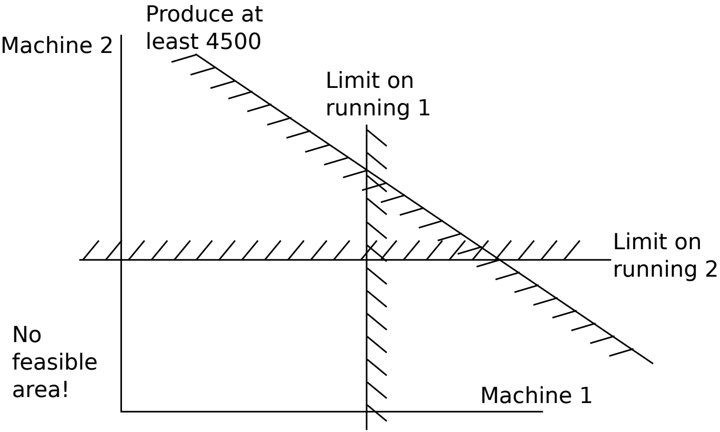
- (A variant on hand sanitizer problem)
- Slope of LOEP equals slope of “top-right” edge
- Then **all points on that edge** are equal optima.

Infeasibility

- Definition: a problem is **infeasible** if there is no feasible area
- Every point violates some constraint
- Obvious example: $x_1 \geq 3$, and $x_1 < 2$

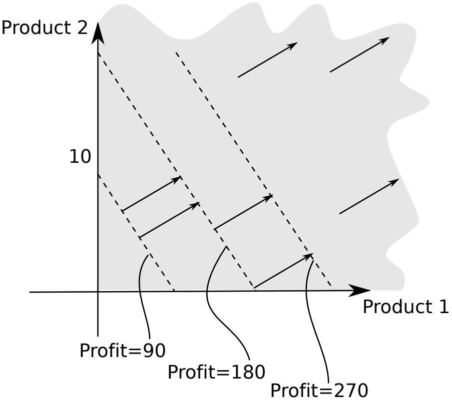
Infeasibility

More interesting example: a manufacturer has limits on how long they can run their machines, but they also have a contract to provide a certain number of products. Can they do it?



Unboundedness

- No limit on profits? Unbounded.
- Maybe we forgot a constraint, or tried to maximise a cost function!



Degeneracy

E.g. we have two constraints:

$$3x_1 + 12x_2 \leq 100$$
$$x_1 + 4x_2 \leq 80$$

One is redundant and can be deleted. (Exercise: which? Draw a picture if needed.)

After deleting, we can re-run.

Linear and non-linear functions

- Linear programming only works with **linear** objective functions and constraints
- (Otherwise the LOEPs or constraints are not straight lines!)

LP assumptions

Linear objective: The objective function is a linear function of the decision variables.

Linear constraints: Each constraint of the form LHS OP RHS:

- LHS (left-hand side) is a linear function of the decision variables
- A **linear function** is of the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n$
- OP is an operator $=, \geq, >, \leq,$ or $<$.
- RHS (right-hand side) is a constant.

Deterministic: All the parameters (objective function coefficients, LHS coefficients, RHS values) are known with certainty.

Divisibility: Decision variables can take on fractional values.

Rewriting for linearity

Sometimes simple algebra is needed to see that an objective function or a constraint LHS is indeed linear.

Suppose we are a bank, and we're going to make many loans of different types: normal mortgages, subprime mortgages, personal loans, small business loans, credit card debt, etc. Each loan type i carries a different risk of default, d_i. We want to limit our exposure to default, with a rule such as: "the expected amount lost to default must not exceed 15% of the total lending".

Model the amount we lend in category i as x_i, the decision variable.

$$\frac{d_1 x_1 + d_2 x_2 + d_3 x_3}{x_1 + x_2 + x_3} \leq 0.15$$

Is this a linear constraint?

Constraints

- Each constraint corresponds to a straight line (2D) or hyperplane (in general) in the plot, which makes **half** of the space **infeasible** (i.e. disallowed)
- If a constraint involves only one variable, it is **vertical** or **horizontal**. If it involves two or more, it is **diagonal**.

Example

Why don't we just use LP on problems like fitting linear regression?

- In simple LR, the decision variables are a and b
- There are **no constraints** on a and b
- The objective is $\sum_i (a + bx_i - y_i)^2$

Example

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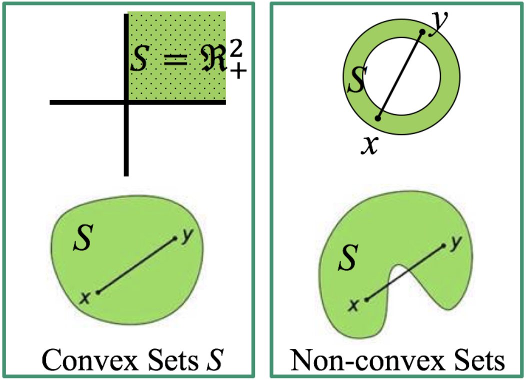
The objective is **not linear**.

Convexity

Intuitively, a convex set is **one** component, with **no dents**

A set $S \subset \mathbb{R}^n$ is **convex** if the line segment between any pair of points x, y in S is itself in S. That is: $\lambda x + (1 - \lambda)y \in S, \forall x, y \in S, \lambda \in [0, 1]$.



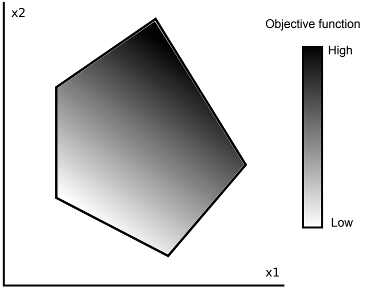
Convex sets (left); non-convex sets (right)

Convexity

Given **linear constraints**…

- The feasible set is a **polytope**: a convex, connected set with flat, polygonal faces
- In 2D, a polytope is a **convex polygon**.

Fundamental theorem of LP



- The extremum (min or max) of a linear function on a polytope is at one of the corner points.
- (Unless there are multiple equally-good optima: then they're at corners **and** all along the line segment or face between these corners.)
- (We won't prove this, but a picture is enough to be convincing.)

Reflection

- What would happen with an LP with no constraints?
- Does LP (with real-valued decision variables) suffer from the curse of dimensionality?

Next week

- **Integer programming**: the decision variables are integer-valued, not real.

Homework

- Optional reading: "Basic OR Concepts" from Beasley's OR-notes: http://people.brunel.ac.uk/~mastjjb/jeb/or/basicor.html
- Lab: see `lab02.pdf` and `sol02.pdf`.