3. Frequency-Domain Analysis of Discrete-Time Signals and Systems

3.1 Introduction

Section 1 covered the analysis of discrete-time signals and systems from the perspective of the time-domain, while Section 2 covered this analysis in the transform domain through the use of the z-Transform. In this Section, we turn our attention to the frequency domain, and how Fourier analysis can be applied to discrete-time signals and systems. This is extremely important in signal and systems analysis, as the view of a signal (or system) in the frequency domain can reveal insights that are not apparent in the time domain.

You will already be familiar with Fourier analysis from previous studies. For example, in *EE221/222 Electrical Circuits and Systems*, you will have studied the frequency response of continuous-time systems, and how this can be obtained from the system transfer function. In *EE308 Signals and Communications*, you will have examined Fourier analysis in more depth, and used it to analyse signals and systems in the time domain. It is important to note that many of the concepts that you learned in these earlier courses also apply in the case of discrete-time signals and systems – the mathematics are slightly different, that's all. However, a "new" topic of great importance is what happens to the spectrum of a continuous-time signal when it is sampled, and this is what we will start with.

3.2 Fourier Transform of Sampled Signals

From previous studies, you will be aware that the Fourier Transform of a signal $x(t)$ is defined by the following equation:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

while the Inverse Fourier Transform is defined by:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

[Recall that there is also the Fourier Series for periodic signals, however, the Fourier transform as defined above is the more general form, and that is what we will use].

For discrete-time signals, the definition of the Fourier Transform is:

$$X(\theta) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\theta}$$

where $X(\theta)$ is the Fourier Transform, and θ is digital frequency (in radians), and is equal to $2\pi fT$, where f is "analogue" frequency in Hz and T, as usual, is the sampling period. The Inverse Fourier Transform is:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)e^{jn\theta}\,d\theta$$

[Note that in the case of a causal signal, the limit of n in the summation in the equation for the Fourier Transform would, of course, start at 0 instead of $-\infty$].

For notational simplicity, we will denote the Fourier Transform and Inverse Fourier Transform as follows:

$$X(\theta) = \Im[x(n)],\; x(n) = \Im^{-1}[X(\theta)]$$

It can be shown that $X(\theta)$ is periodic, with period 2π, as follows. Let's assume that we have a continuous-time signal $x(t)$, which is band limited to some frequency. We will sample this signal at a particular sampling rate, and further assume that the sampling rate is at least twice the bandwidth of $x(t)$, so that the sampling theorem is satisfied and aliasing does not occur.

We represent the sampled signal in terms of the Fourier Transform of the continuous-time signal as follows:

$$x(n) = x(t)\big|_{t=nT} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega\bigg|_{t=nT} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega nT}\,d\omega$$

If we "divide" the frequency axis into segments of length 2π, we can rewrite $x(n)$ as:

$$x(n) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{k2\pi-\pi}^{k2\pi+\pi} X(\omega)e^{j\omega nT}\,\frac{d\omega T}{T}$$

(note also that the variable of integration has been multiplied by T/T).

If we now change the argument, we obtain:

$$x(n) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\frac{1}{T}\int_{-\pi}^{\pi} X(\omega + k\frac{2\pi}{T})e^{j(\omega+k\frac{2\pi}{T})nT}\,d\omega T$$

The function $e^{j\omega nT}$ is periodic, with period 2π, so we can write:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T})\right]e^{j\omega nT}\,d\omega T$$

(with some additional re-ordering of terms). Finally, we note that $\theta=\omega T$, and therefore:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T})\right]e^{jn\theta}\,d\theta$$

If we compare this with the earlier expression for the Inverse Fourier Transform of a sampled signal, we see that the expression inside the square brackets is equal to $X(\theta)$, i.e.:

$$X(\theta) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega + k\frac{2\pi}{T})$$

In other words, the Fourier Transform of a discrete-time signal, obtained by sampling a continuous-time signal, is equal to an infinite set of "copies" of the original continuous-time spectrum, repeating at multiples of $2\pi/T$ (which happens to be equal to the sampling frequency in radians/sec).

Recall that we assumed that the sampling frequency was greater than twice the bandwidth of the underlying continuous-time signal, so that aliasing would not occur. This situation can be illustrated in the frequency domain by the plot in Figure 3.1 (for simplicity, the scaling factor of $1/T$ has been omitted):

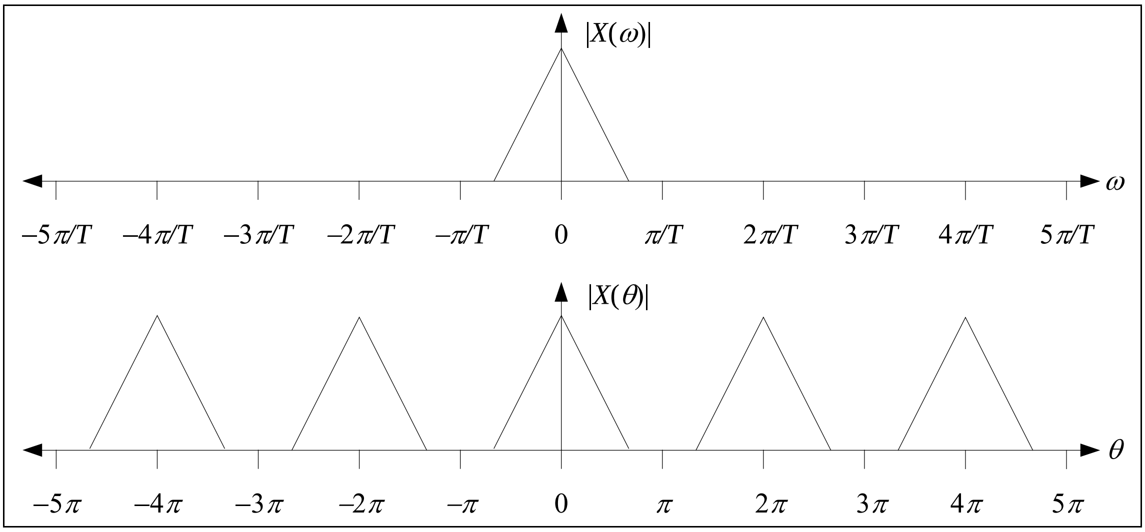

Figure 3.1. Magnitude spectra of continuous-time signal and its sampled version, where no aliasing has occurred.

If the sampling frequency is exactly equal to twice the bandwidth of $x(t)$, then the "copies" of the original spectrum (i.e. each term in the summation in the equation for $X(\theta)$ above) will move closer together so that they effectively "touch". Note, however, that the sampling theorem is still satisfied, and aliasing does not occur.

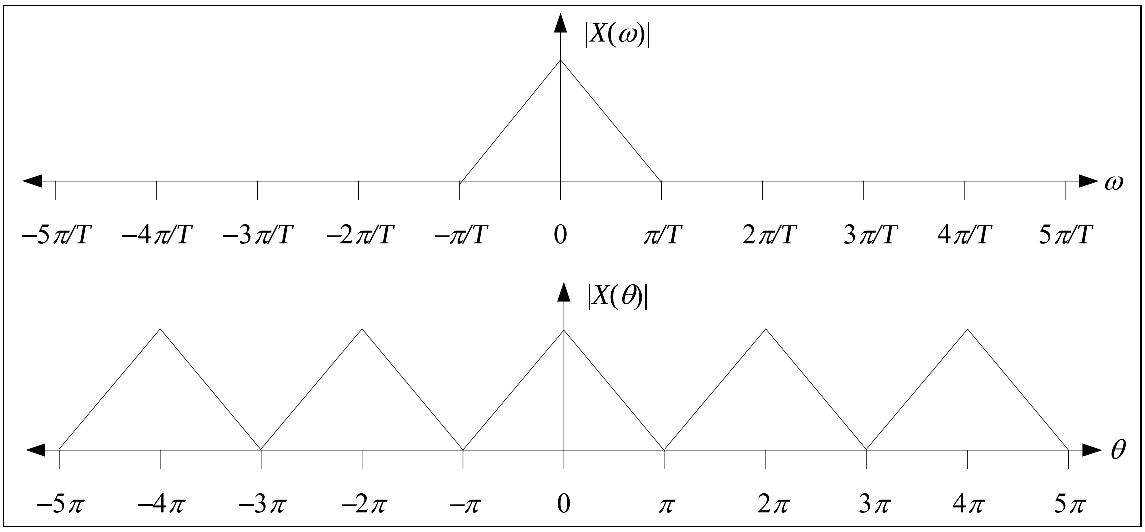

Figure 3.2. Magnitude spectra when signal is sampled at the Nyquist Rate, i.e. exactly twice the bandwidth of $x(t)$.

However, if the sampling frequency is less than twice the bandwidth of $x(t)$, then the "copies" of the original spectrum in the sampled signal will move closer together and will in fact overlap and sum, as is illustrated in Figure 3.3:

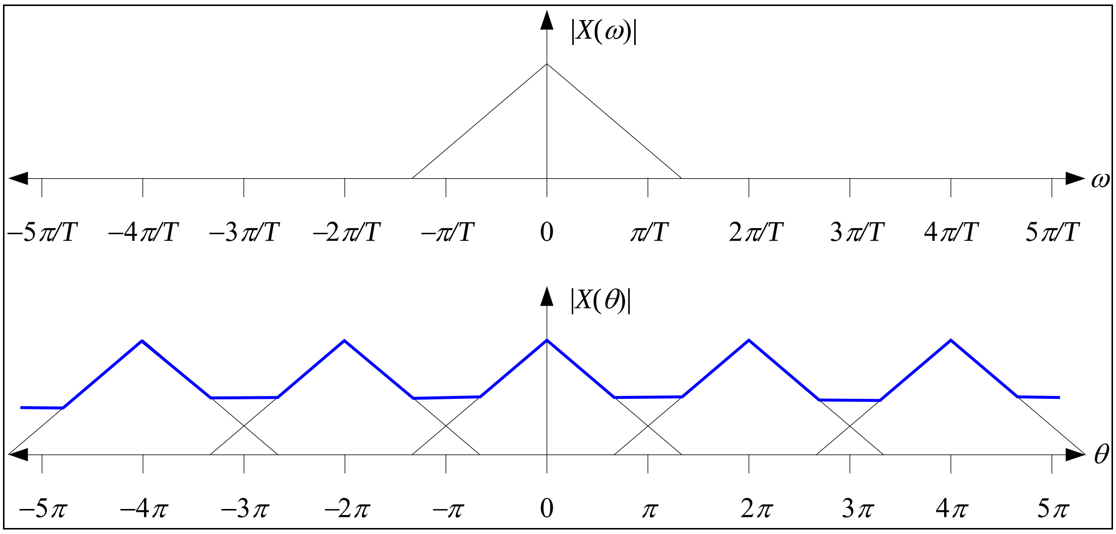

Figure 3.3. Illustration of situation where sampling frequency is too low (the "bold" line indicates the summation of the overlapping copies).

We've already discussed aliasing in the time domain for the simple case of a sinusoidal waveform – this is a further illustration of aliasing in the frequency domain for a more complex waveform. Further, recall that this type of distortion is quite serious, and once it occurs, it cannot generally be removed.

3.3 Relationship between z-Transform and Fourier Transform

A simple relationship exists between the z-Transform of a signal $x(n)$, and its Fourier Transform. Recall:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Since z is a complex variable, we can express it in polar form as $z = re^{j\theta}$. Thus:

$$X(z)\Big|_{z=re^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-jn\theta}$$

Suppose we let $r = 1$. Then:

$$X(z)\Big|_{z=e^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n) e^{-jn\theta} = X(\theta)$$

In other words, the Fourier Transform of a discrete-time signal is obtained by evaluating the z-Transform for $z = e^{j\theta}$. Put another way, the Fourier Transform is equal to the z-Transform evaluated on the unit circle in the z-plane (analogous to the manner in which the Fourier Transform of a continuous-time signal is obtained by evaluating its Laplace Transform along the imaginary axis). For a particular point on the unit circle, θ is the angle that this point makes with the positive real axis. For example, as we travel around the unit circle in a counter-clockwise direction, $z = 1$ corresponds to $\theta = 0$ radians, $z = j$ corresponds to $\theta = \pi/2$, $z = -1$ corresponds to $\theta = \pi$ (or $-\pi$), while z = -j corresponds to $\theta = 3\pi/2$ (or $-\pi/2$). Also, note that as we arrive back at z = 1 after completing our circuit, we're back where we started, i.e. $\theta = 2\pi$ is the same as $\theta = 0$. As we increase θ beyond 2π, we go around the unit circle again. This is simply another illustration of the periodicity that exists in the spectrum of a sampled signal.

From a practical point of view, the periodicity in the Fourier Transform of a sampled signal means that there's nothing to be gained by evaluating the Transform for values of θ beyond 2π (because we're only calculating the same numbers over and over again). Thus, the spectrum from $\theta = 0$ to $\theta = 2\pi$ fully defines the spectrum. Alternatively, we can say that the spectrum from $\theta = -\pi$ to $\theta = \pi$ defines the spectrum, because the information contained in the spectrum between $\theta = \pi$ and $\theta = 2\pi$ is the same as that contained in the spectrum from $\theta = -\pi$ to $\theta = 0$ (this can readily be seen in Figure 3.1). This is also consistent with the usual definition of two-sided spectra, which contain both positive and negative frequencies. From the perspective of the z-domain, calculation of the spectrum is like moving around the unit circle in the counter-clockwise direction from $z = -1$ to $z = 1$ for negative frequencies from $-\pi$ to 0, and from $z = 1$ to $z = -1$ for positive frequencies from 0 to $+\pi$.

While the relationship between the z-Transform and the Fourier Transform has been discussed in the context of a signal, clearly the same principle applies in the case of systems. In particular, the frequency response of a system can be obtained from the transfer function of the system, by setting $z = e^{j\theta}$. Alternatively, the frequency response may be obtained directly from the impulse response, using the equation defining the Fourier Transform above. So, we can obtain the frequency response of a system, or the spectrum of a signal, in one of two ways, starting either in the transform domain or in the time domain.

The Fourier Transform (whether it represents the spectrum of a signal, or the frequency response of a system) is generally a complex quantity, and can be represented in either polar or Cartesian form. Furthermore, we're generally interested in the magnitude spectrum or phase spectrum, which can be readily obtained from the value of $X(\theta)$ at any particular frequency - after all, it's just a complex number. If we represent the spectrum at a given frequency θ by $X(\theta) = a + jb$, then we can write the magnitude spectrum as

$$|X(\theta)| = \sqrt{a^2 + b^2}$$

and the phase spectrum as:

$$\phi(\theta) = \tan^{-1}\left(\frac{b}{a}\right)$$

[**Note**: When we talk about a signal, we generally refer to the magnitude and phase spectra of the signal, while when we refer to a system, we usually refer to frequency response, and magnitude and phase responses (which are, in essence, the magnitude and phase spectra of the system impulse response).]

Also, it was observed above that, because of the periodicity of the spectrum of a sampled signal (or the frequency response of a digital filter), the signal is completely characterised by the spectrum from $\theta = -\pi$ to $\theta = \pi$. However, in the case of real functions of time (normally the situation of most interest), it is sufficient to calculate the spectrum over the range of frequencies from $\theta = 0$ to $\theta = \pi$. This is because the

Fourier Transform of real-valued signals (or impulse responses) exhibits conjugate symmetry, so that the magnitude spectrum is an even function of frequency, while the phase spectrum is an odd function of frequency (recall that the same principle holds for continuous-time systems).

3.4 Examples

3.4.1 Unit Impulse and Delayed Unit Impulse

We have previously shown that the z-Transform of the unit impulse function, $\delta(n)$, is given by

$$Z[\delta(n)] = X(z) = 1$$

Therefore, the spectrum of the unit impulse can be found by setting $z = e^{j\theta}$ in $X(z)$ to obtain

$$X(\theta) = 1.$$

In other words, the unit impulse contains all frequencies with amplitude of 1 and a phase angle of 0^o. For a delayed unit impulse (delay of k samples), carrying out the same operation yields

$$X(\theta) = X(z)\big|_{z=e^{j\theta}} = z^{-k}\Big|_{z=e^{j\theta}} = e^{-jk\theta} = \cos(k\theta) - j\sin(k\theta)$$

This has a magnitude of 1 for all frequencies; however, the phase spectrum is given by $\phi(\theta) = -k\theta$. This is plotted in Figure 3.4:

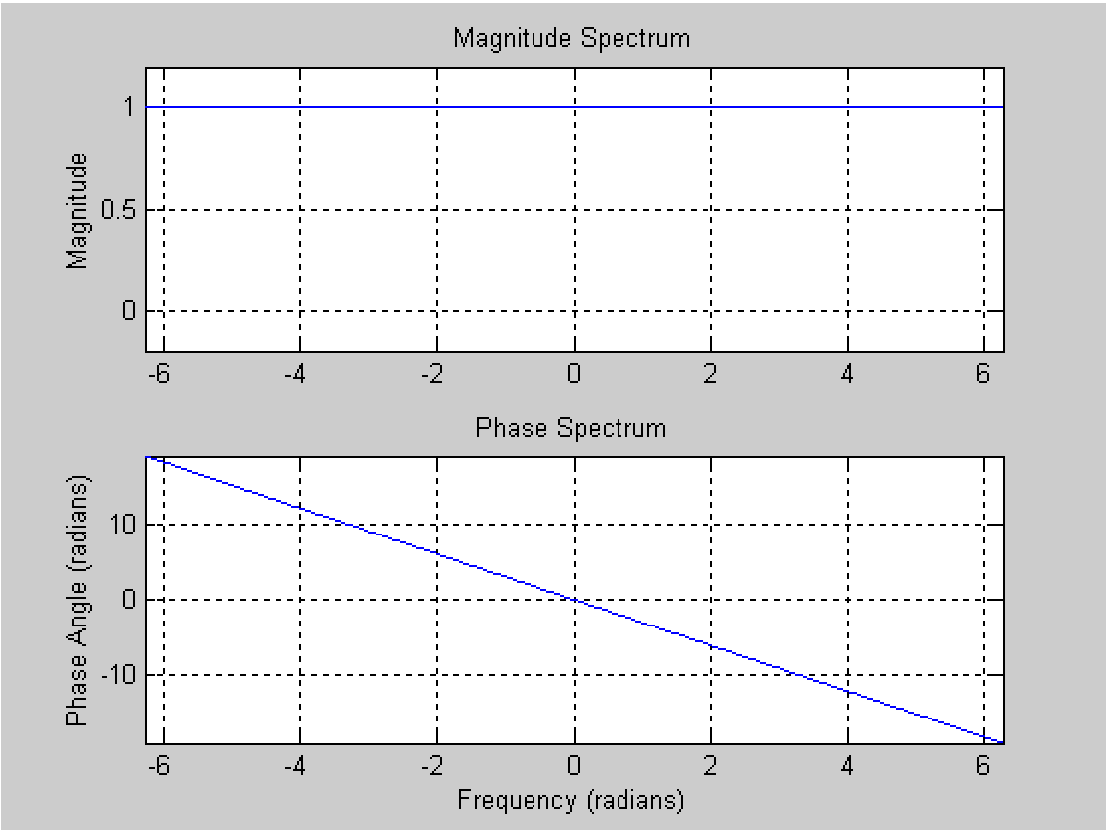

Figure 3.4. Magnitude and phase spectra of delayed unit impulse (delay of 3 samples).

3.4.2 Sampled Exponential

The spectrum of the sampled exponential waveform is obtained as follows:

$$x(n) = a^n u(n)$$

$$X(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

$$\begin{aligned} X(\theta) &= X(z)\big|_{z=e^{j\theta}} = \frac{1}{1-ae^{-j\theta}} \\ &= \frac{1}{1-[a\cos(\theta) - ja\sin(\theta)]} \end{aligned}$$

We obtain the magnitude and phase spectra as indicated above, i.e. treat the spectrum as a complex number:

$$\begin{aligned} X(\theta) &= \frac{1}{1-[a\cos(\theta) - ja\sin(\theta)]} \\ &= \frac{1}{[1-a\cos(\theta)] + ja\sin(\theta)} \end{aligned}$$

To obtain the magnitude spectrum, simply take the magnitude of this complex number:

$$\begin{aligned} |X(\theta)| &= \frac{1}{\sqrt{[1-a\cos(\theta)]^2 + [a\sin(\theta)]^2}} \\ &= \frac{1}{\sqrt{1-2a\cos(\theta) + a^2\cos^2(\theta) + a^2\sin^2(\theta)}} \\ &= \frac{1}{\sqrt{1-2a\cos(\theta) + a^2}} \end{aligned}$$

The phase spectrum is given by:

$$\begin{aligned} \phi(\theta) &= \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) \\ &= -\tan^{-1}\left(\frac{a\sin(\theta)}{1-a\cos(\theta)}\right) \end{aligned}$$

[the negative sign is required because the real and imaginary parts are in the denominator of the expression for the Fourier Transform].

The magnitude and phase spectra of the sampled exponential are plotted in Figure 3.5:

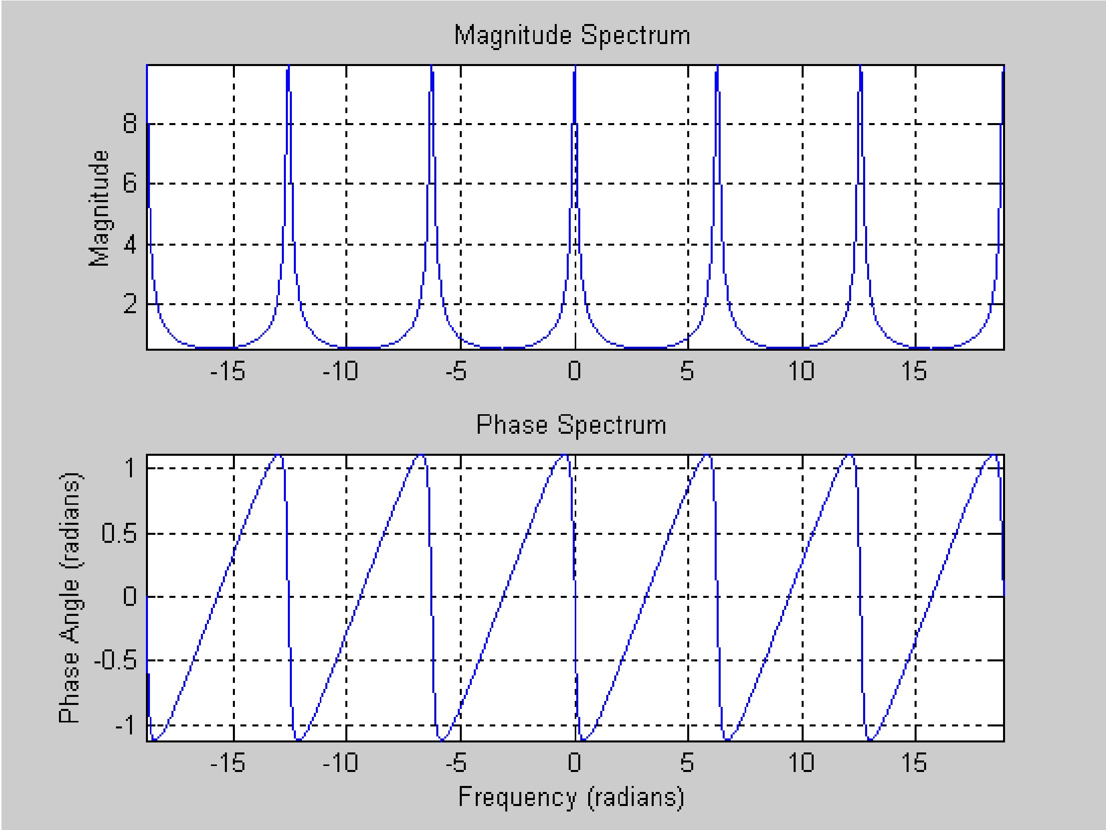

Figure 3.5. Magnitude and phase spectra of sampled exponential (*a*=0.9).

Exercise 3.1

Obtain expressions for the spectra (magnitude and phase) of the following signals:

(a) The unit step function.

(b) $x(n) = 0.5^n u(n) + 0.9^n u(n)$

(c) The sequence consisting of the following samples {1, 0, 1} starting at n = 0.

3.5 System Frequency Response

We have already observed that the frequency response of a system may be obtained either from the system transfer function, or directly by means of Fourier transformation of the system impulse response. Also, recall that convolution in the time domain was replaced by multiplication in the z-domain (through the convolution property). The convolution property also holds for the Fourier Transform, in other words, the Fourier Transform of the output of a digital filter is equal to the product of the Fourier Transform of the filter input, and the frequency response of the filter. Mathematically:

$$\text{Fourier Transform of input}: X(\theta) = \Im[x(n)]$$

$$\text{Filter Frequency Response}: H(\theta) = \Im[h(n)]$$

$$\text{Then, } Y(\theta) = H(\theta)X(\theta)$$

If we express $Y(\theta)$ in polar form as $|Y(\theta)|e^{j\phi(\theta)}$, i.e. we explicitly indicate the gain and phase of the filter, then we can write:

$$|Y(\theta)| = |X(\theta)||H(\theta)|$$
$$\phi_y(\theta) = \phi_x(\theta) + \phi_h(\theta)$$

i.e. the magnitude spectrum of the filter output is equal to the magnitude spectrum of the filter input, multiplied by the magnitude response (gain) of the filter. This is exactly the same result as obtained for the continuous-time Fourier Transform. The phase response of the filter output at a given frequency is equal to the sum of the phase angle of the input and the phase response of the filter. Normally, the magnitude response of a filter is expressed in dB on a log scale, while the phase response is often shown "modulo-2π", i.e. the phase angle is plotted in the range $-\pi$ to $+\pi$, and if the angle exceeds $+\pi$ or $-\pi$ at any frequency, it "wraps" around to the opposite end of the range.

3.6 Examples

3.6.1 FIR Filter

A system is described by the following difference equation:

$$y(n) = x(n) + x(n-1) + x(n-2)$$

Calculate and plot the frequency response of the system.

Solution

As noted above, the two general methods of obtaining the frequency response of a system are either through the transfer function, or by Fourier transformation of the impulse response. In the case of an FIR filter, we have seen in Section 1 that the samples of the impulse response are the same as the digital filter coefficients, so we could easily use either method; for completeness, we will look at both.

Starting with the difference equation, we obtain the transfer function, and thence the frequency response. The transfer function can be obtained by inspection of the difference equation:

$$H(z) = 1 + z^{-1} + z^{-2}$$

The frequency response is obtained by substituting $e^{j\theta}$ for z, to obtain:

$$\begin{aligned} H(\theta) &= 1 + e^{-j\theta} + e^{-j2\theta} \\ &= e^{-j\theta}[e^{j\theta} + 1 + e^{-j\theta}] \end{aligned}$$

By Euler's equations, the term in square brackets can be seen to be equal to $1+2\cos(\theta)$, so we obtain:

$$H(\theta) = e^{-j\theta}[1 + 2\cos(\theta)]$$

The magnitude and phase responses of the filter are given by:

$$\begin{aligned} |H(\theta)| &= 1 + 2\cos(\theta) \\ \phi(\theta) &= -\theta \end{aligned}$$

These are plotted in Figure 3.6.

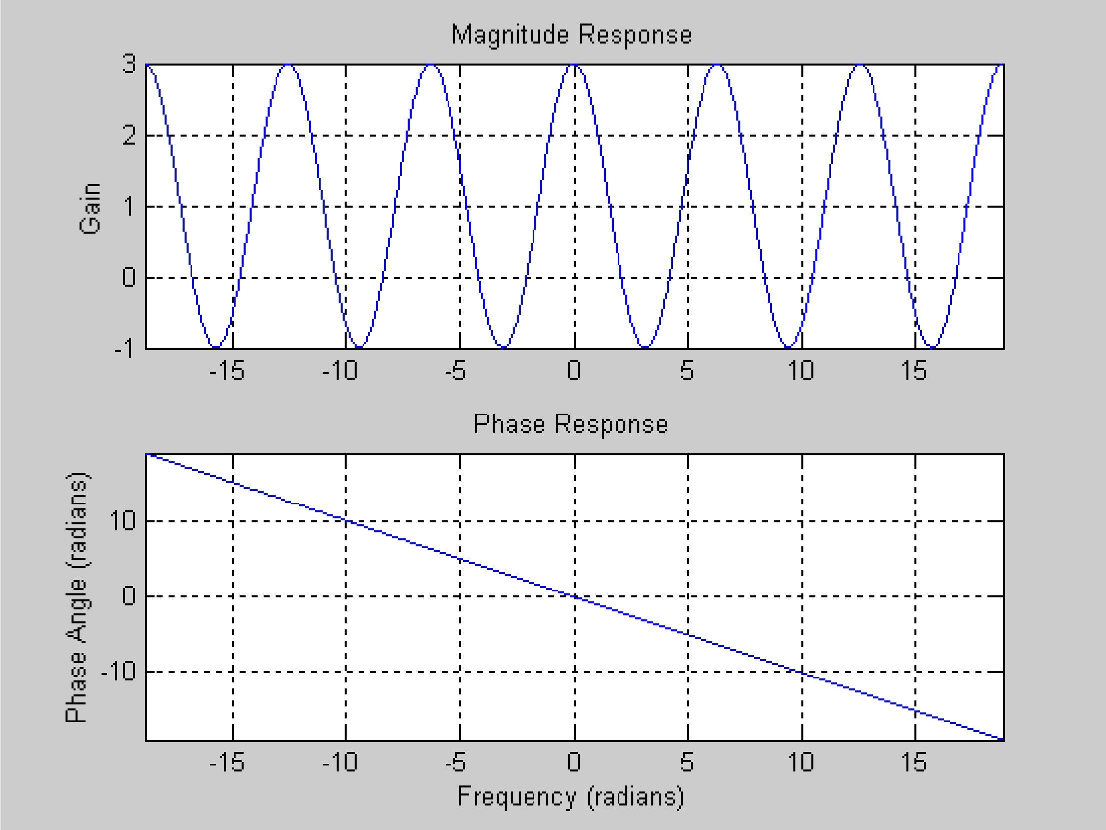

Figure 3.6. Frequency response of simple FIR filter.

The impulse response of this filter can be written as {1, 1, 1} (starting at n = 0). Hence, the Fourier Transform of the impulse response is:

$$\begin{aligned} H(\theta) &= \sum_{n=-\infty}^{\infty} h(n)e^{-jn\theta} \\ &= \sum_{n=0}^{2} e^{-jn\theta} \\ &= 1 + e^{-j\theta} + e^{-j2\theta} \end{aligned}$$

This is clearly the same expression as we obtained using the transfer function, so the frequency response is the same by this method (as expected).

3.6.2 IIR Filter

Suppose a filter is described by the following expression:

$$y(n) = x(n) + 0.6x(n-1) - 0.1x(n-2) + 0.3y(n-1) - 0.3y(n-2)$$

[From Exercise 1.1]

Because the impulse response of this filter is of infinite length, it is not practical to evaluate the frequency response by means of Fourier transformation of the impulse response. Therefore, we will use the transfer function, which we know from Section 2 is given by

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1+0.6z^{-1} - 0.1z^{-2}}{1-0.3z^{-1} + 0.3z^{-2}}$$

Hence, the frequency response is given by:

$$\begin{aligned} H(\theta) &= \frac{1+0.6e^{-j\theta} - 0.1e^{-j2\theta}}{1-0.3e^{-j\theta} + 0.3e^{-j2\theta}} \\ &= \frac{1+0.6\cos(\theta) - j0.6\sin(\theta) - 0.1\cos(2\theta) + j0.1\sin(2\theta)}{1-0.3\cos(\theta) + j0.3\sin(\theta) + 0.3\cos(2\theta) - j0.3\sin(2\theta)} \end{aligned}$$

The magnitude and phase responses are obtained in the usual way by calculating the magnitude and phase:

$$|H(\theta)| = \frac{\sqrt{[1+0.6\cos(\theta) - 0.1\cos(2\theta)]^2 + [-0.6\sin(\theta) + 0.1\sin(2\theta)]^2}}{\sqrt{[1-0.3\cos(\theta) + 0.3\cos(2\theta)]^2 + [0.3\sin(\theta) - 0.3\sin(2\theta)]^2}}$$

$$\phi(\theta) = \tan^{-1}\left(\frac{-0.6\sin(\theta) + 0.1\sin(2\theta)}{1+0.6\cos(\theta) - 0.1\cos(2\theta)}\right) - \tan^{-1}\left(\frac{[0.3\sin(\theta) - 0.3\sin(2\theta)}{1-0.3\cos(\theta) + 0.3\cos(2\theta)}\right)$$

[note how the numerator and denominator each contribute to the phase response]

At this point, things are getting a little tedious, so it's best to calculate the actual magnitude and phase responses using Matlab, or some other software package.

<u>Exercise 3.2</u>

Write Matlab code to evaluate the magnitude and phase response of the above filter, over the frequency range from -3π to $+3\pi$.

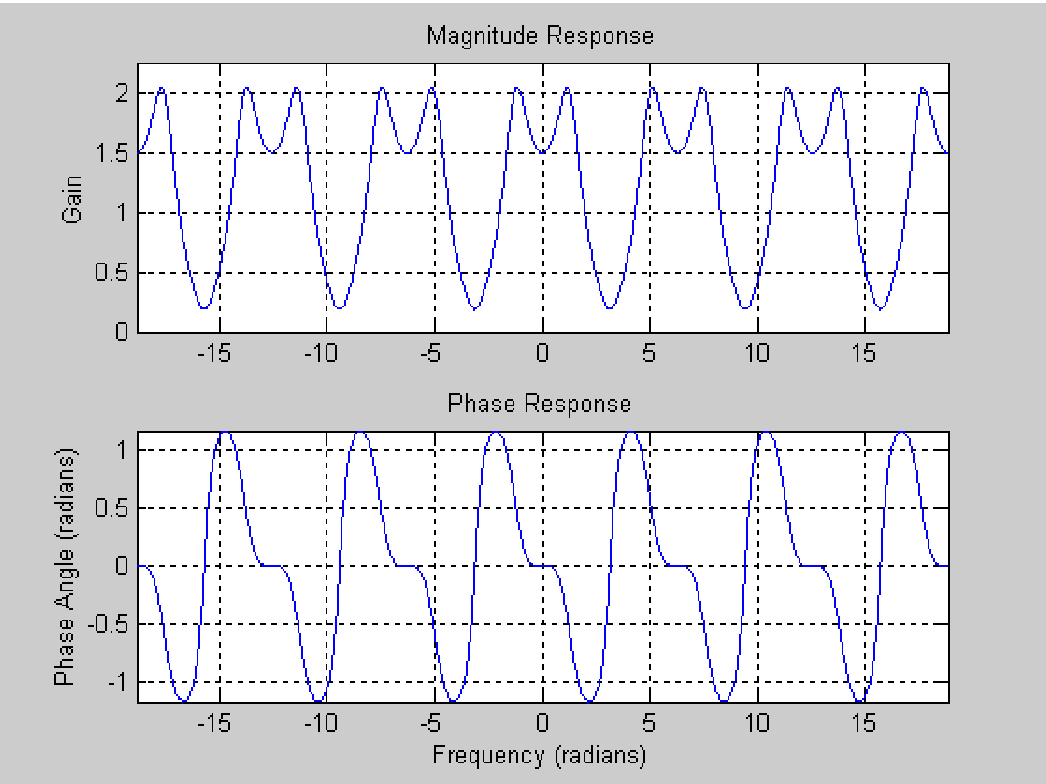

Figure 3.7. Frequency response of IIR filter.

As it happens, Matlab provides an easier method for calculating the frequency response, involving the use of the "freqz" function.

Exercise 3.3

Evaluate the frequency response of the IIR filter above using "freqz", and verify that the magnitude and phase responses obtained in this way are the same as those obtained in Exercise 3.2.

Note that by default, "freqz" calculates the frequency response over the frequency range from 0 to π (corresponding to 0 Hz to half the sampling frequency).

3.7 Phase Response

In most applications, we are more interested in the magnitude response of the filter, since this tells us how much of each frequency component in the input the filter will pass (or attenuate); the phase response is generally of less importance, so we generally do not mind what form it takes. However, there are cases where the phase response of the filter becomes important, especially when the signal being filtered consists of many different frequency components.

In essence, the phase response of the filter indicates the amount of delay (in radians) which a sinusoid at a particular frequency suffers as it passes through the filter (even if the magnitude response of the filter is 1 at this frequency, the phase response will still affect the filter). This is often expressed in terms of the filter phase delay, which is the time delay suffered by each frequency component as it passes through the filter:

$$\tau_p = -\frac{\phi(\theta)}{\theta}$$

Another measure that is often used to describe the "delay" through a filter is the group delay. This is the "average" time delay that a composite signal suffers at each frequency when passing through the filter:

$$\tau_g = -\frac{d\phi(\theta)}{d\theta}$$

If the group delay is not the same for all frequency components in the signal, it means that different frequencies suffer different amounts of delay through the filter (even if their magnitudes remain unaffected). This results in phase distortion in the signal. In order to have constant group delay at all frequencies, it is necessary for the filter to have linear phase. It can be shown that this can be achieved only with certain types of non-recursive (FIR) filters. A filter is said to have linear phase if the phase response satisfies one of the following relationships:

$$\phi(\theta) = -a\theta$$
$$\phi(\theta) = b - a\theta$$

where a and b are constants (and a is the slope of the phase response). Clearly, the derivative of the phase response in these cases will be equal to the constant a. If the filter has a non-linear phase response, then the group delay will not be constant with frequency.

The effect of linear or non-linear phase is illustrated in Figure 3.8. Figure 3.8(a) shows an input signal that consists of a number of different frequency components. Figure 3.8(b) shows the output of an FIR filter with linear phase, and whose magnitude response is such that it does not attenuate any of the input frequency components. Figure 3.8(c) shows the effect of a filter with the same magnitude response, but with non-linear phase. It can be clearly seen that the linear phase filter does not cause any distortion of the input signal (the only effect is a delay), however, the non-linear phase filter does introduce significant distortion – even though the magnitude response is 1 at all of the frequencies contained in the input. One way of understanding this is to observe that the different frequency components (actually harmonics of the fundamental because it's a periodic signal) are delayed by different amounts through the filter, so they do not add "coherently" at the output.

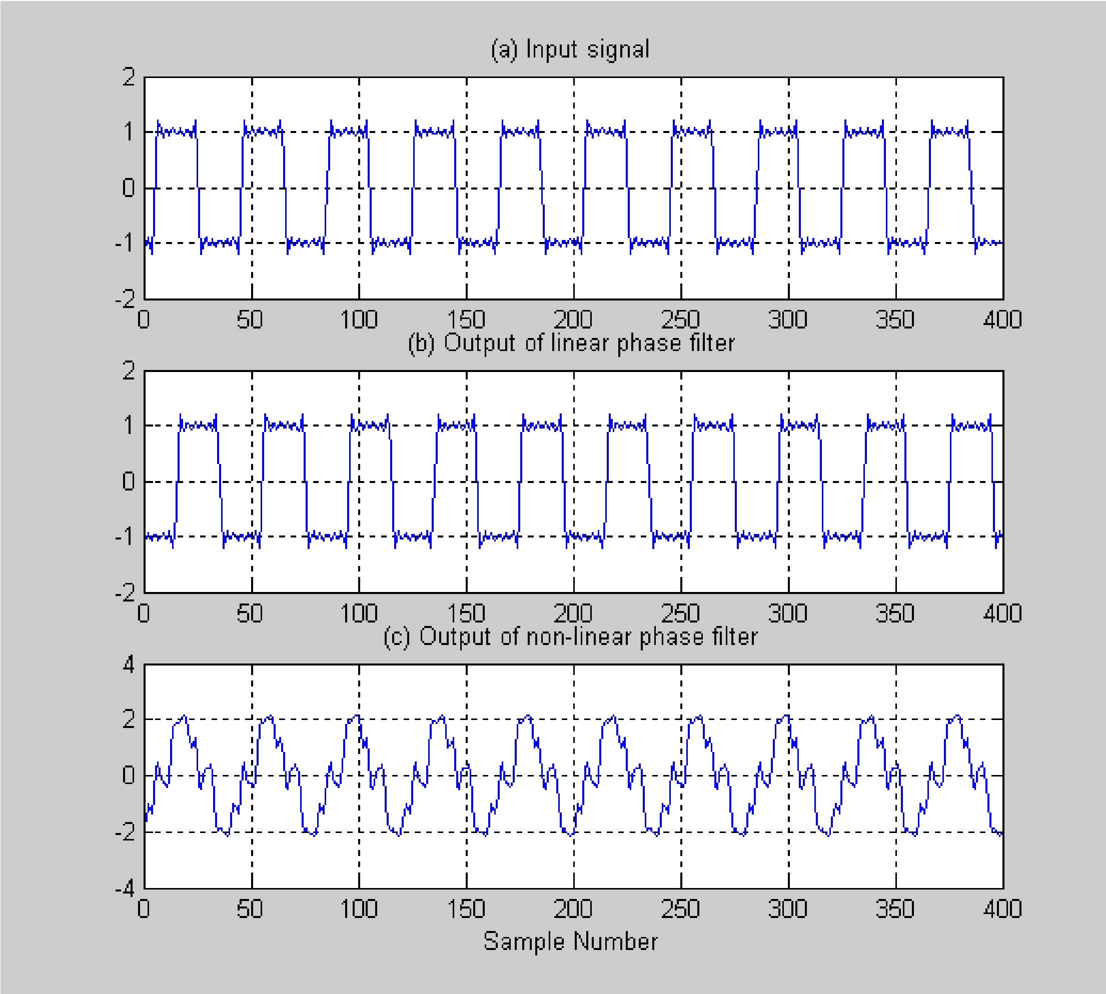

Figure 3.8. Effect of linear and non-linear phase response.

<u>Exercise 3.4</u>

Using Matlab, calculate and plot the frequency responses of the systems described by the difference equations in Exercise 2.7.

3.8 Summary

In this Section, we have looked at Fourier analysis for discrete-time signals and systems. We have seen how the Fourier Transform of a signal can be obtained either from the time-domain representation of the signal, or from the z-Transform of the signal. Furthermore, we have seen how the frequency response of a digital filter may be obtained by the same methods.

At this point, we have looked at three ways of describing or analysing signals and systems – time domain, z-domain and frequency domain. We have looked at techniques for obtaining the z-Transform of a signal, or system impulse response, and have also looked at ways of getting the frequency response, as well as “inverse” operations to get back to where we started. With the z-Transform, we also have the related technique of pole-zero analysis to gain further insights into system behaviour. Which technique is used largely depends on where we’re starting from, and what we want to do. However, it is important to bear in mind the relationships between the different “domains” – after all, they’re just different ways of describing the same “entity” (i.e. a signal or a system). Furthermore, what we’re doing for discrete-time systems has an exact analogy in the theory of continuous-time systems.

The next Section will continue the analysis of discrete-time systems, including some more “intuitive” observations on system behaviour that may be gleaned from the transfer function or the pole-zero map, as well as looking at techniques for implementing more complex digital filters.