3. Frequency-Domain Analysis of Discrete-Time Signals and Systems

3.1 Introduction

Section 1 covered the analysis of discrete-time signals and systems from the perspective of the time-domain, while Section 2 covered this analysis in the transform domain through the use of the z-Transform. In this section, we turn our attention to the frequency domain, and how Fourier analysis can be applied to discrete-time signals and systems.

3.2 Fourier Transform of Sampled Signals.

The Fourier Transform of a signal $x(t)$ is defined by:

$$X(w) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-jwt} dt$$

while the Inverse Fourier Transform is defined by:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(w) e^{jwt} dw$$

For discrete-time signals, the definition of the Fourier Transform is:

$$X(\theta) = \sum_{-\infty}^{+\infty} x(n) e^{-jn\theta} = \sum_{n=0}^{\infty} x(n) \cdot e^{-jn\theta}$$

where $X(\theta)$ is the Fourier Transform, and θ is digital frequency (in radians), and is equal to $2\pi T f_a$ ($\theta = 2\pi f_a$) where f_a is the analogue frequency in Hz, and T is the sampling period.

$T = \frac{1}{f_s}$ $\therefore \theta = 2\pi f_a / f_s$

The Inverse Fourier Transform is

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\theta) e^{jn\theta} d\theta$$

$X(\theta) = \mathcal{F}[x(n)]$, $x(n) = \mathcal{F}^{-1}[X(\theta)]$

It can be shown that $X(\theta)$ is periodic, with period 2π.

We assume that we have a continuous-time signal $x(t)$.

We represent the sampled signal in terms of the Fourier Transform of the continuous-time signal as follows:

$$x(n) = x(t)\Big|_{t=nT} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(w)e^{jwt}\,dw\Big|_{t=nT}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(w)e^{jwnT}\,dw$$

If we devide the frequency axis into segments of length 2π, we can rewrite $x(n)$ as:

$$x(n) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\int_{2k\pi-\pi}^{2k\pi+\pi} X(w)e^{jwnT}\,\frac{dwT}{T}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\frac{1}{T}\int_{2k\pi-\pi}^{2k\pi+\pi} X(w)e^{jwnT}\,dwT$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\frac{1}{T}\int_{2k\pi-\pi}^{2k\pi+\pi} X(w\cdot\frac{T}{T})e^{jwnT}\,dwT$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\frac{1}{T}\int_{-\pi}^{\pi} X\left(\frac{wT+2k\pi}{T}\right)\cdot e^{jn(wT+2k\pi)\cdot\frac{T}{T}}\,d(wT+2k\pi)$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\frac{1}{T}\int_{-\pi}^{\pi} X\left(w+\frac{2k\pi}{T}\right)\cdot e^{j\cancel{\pi}(w+\frac{2k\pi}{T})nT}\,dwT$$

The function e^{jwnT} is periodic, with period 2π. So we can write

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T}\sum_{k=-\infty}^{+\infty} X\left(w+k\cdot\frac{2\pi}{T}\right)\right]e^{jwnT}\,dwT$$

$\because \boxed{\theta = wT}$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\boxed{\left[\frac{1}{T}\sum_{k=-\infty}^{+\infty} X\left(w+k\cdot\frac{2\pi}{T}\right)\right]}e^{jn\theta}\,d\theta$$

The Inverse Fourier Transform for discrete-time signal: $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\boxed{X(\theta)}e^{jn\theta}\,d\theta$

$\therefore X(\theta) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X\left(w+k\cdot\frac{2\pi}{T}\right)$

$$X(\theta) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(w + \frac{2\pi}{T}k)$$

In other words, the Fourier Transform of a discrete-time signal, obtained by sampling a continuous-time signal, is equal to an infinite set of "copies" of the original continuous-time spectrum, repeating at multiples of $2\pi/T$ (which happens to be equal to ~~the be~~ the sampling frequency in radians/sec).

$X(\theta)$ $x(t)$ $\sum_{-\infty}^{+\infty}$ $X(w)$ $X(w + k \cdot \frac{2\pi}{T})$

$$\boxed{\frac{1}{T_s} \cdot 2\pi = f_s \Rightarrow \text{radians/sec}}$$

The discrete-time sinusoidal signal is derived from the underlying analogue sinusoidal signal $x(t) = A\sin(w_a t)$

where A is the amplitude of the sinusoid, and w_a is the frequency in radians per second.

By sampling this signal every T seconds, we obtain:

$$x(n) = A\sin(w_a n T) = A\sin(\boxed{2\pi \cdot f_a}\, n \cdot 1/f_s)$$

$$T = 1/f_s$$

$$= A\sin(2\pi n\, f_a/f_s)$$

where f_a is the frequency of the sinusoid (in Hz), f_s is the sampling frequency (in Hz) and n is the sample index.

The quantity $2\pi \cdot f_a/f_s$ has a special meaning in the context of DSP, and is often referred to as the digital frequency or relative frequency with symbol θ : $x(n) = A\sin(n\theta)$ $\quad (\theta = 2\pi \cdot f_a/f_s)$

the units of θ are radians. $\theta = 2\pi \cdot f_a/f_s = \underbrace{2\pi f_a}_{w_a\ (\text{radian/sec})} \cdot \underbrace{T_s}_{\text{sec}} = \underset{(\text{radians})}{\text{unit}}$

f_a should $\le \frac{1}{2} f_s$.

if f_a increase from 0 to f_s $\theta \in [0, 2\pi]$

A digital sequence is said to be periodic, with period N, if N is the smallest integer for which $x(n+N) = x(n)$.

For example, the sampled sinusoid is periodic with $\boxed{\text{period } N}$ if

$$A\sin([n+N]\theta) = A\sin(n\theta)$$

This can only be satisfied if $N\theta$ is an integer of 2π, i.e. $N\theta = 2\pi k$

where k is an integer. $N = \frac{2\pi k}{\theta} = \frac{2\pi k}{2\pi \frac{f_a}{f_s}} = f_s/f_a \cdot k$

sampling number per period. (取点数)

Since N must be an integer, this means f_s/f_a must be an ~~int~~ integer.

In other words, in the case of a sinusoid, there will be an exact integer number of digital samples in every period of the underlying analogue waveform. f_s $T_a \cdot = 1/f_a$.

$$f_s \cdot T_a = f_s/f_a \text{ must be an exact integer}$$

For example, a 100 Hz analogue sinewave sampled at 8 kHz
$f_a = 100\,Hz$ $f_s = \cancel{80,000\,Hz}$ 8000 Hz

will have $N = f_s/f_a \cdot k = 8000/100 \cdot k = 80 \cdot k$

80 samples per period of the sinewave,

and this set of 80 samples ~~per period~~ will repeat it itself over and over again.

3.3. Relationship between Z-Transform and Fourier Transform.

Z-Transform: $\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}}$

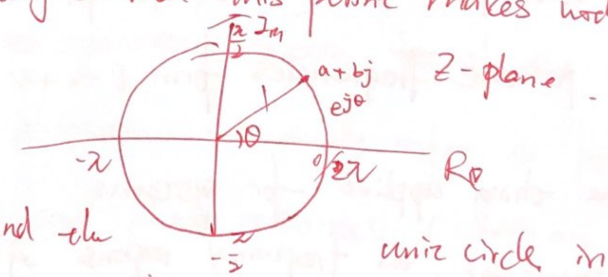
Since z is a complex variable, we can express it in polar form as $\boxed{z = re^{j\theta}}$. Thus: $X(z)\big|_{z=r\cdot e^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n)\cdot(r\cdot e^{j\theta})^{-n}$

$$= \sum_{n=-\infty}^{\infty} x(n)\cdot r^{-n}\cdot e^{-j\theta n}$$

Suppose we let r=1. Then: $X(z)\big|_{z=e^{j\theta}} = \sum_{n=-\infty}^{\infty} x(n)\cdot e^{-jn\theta} = X(\theta)$

In other words, the Fourier Transform of a discrete-time signal is obtained by evaluating the z-Transform for $\boxed{z = e^{j\theta}}$.

Put another way, the Fourier Transform is equal to the z-Transform evaluated on the unit circle in the z-plane. For a particular point on the unit circle, θ is the angle that this point makes with the ~~positive~~ real axis.



For example, as we travel around the unit circle in a counter-clockwise direction, $z = 1 = e^{j\cdot 0}$, corresponds to $\theta = 0$ radians

$z = j = 0 + 1\cdot j$, corresponds to $\theta = \frac{\pi}{2}$ radians.

$z = -j = 0 + (-1)\cdot j$. correspods to $\theta = \frac{3}{2}\pi$ radians or $(-\frac{\pi}{2})$

$Z = 1 = 1 + 0\cdot j$ Corresponds to $\theta = 2\pi$ or $\theta = 0$

As we increase θ beyond 2π, we go around the unit circle again.
This is simply another illustration of the periodicity that exists in the spectrum of a sampled signal

From a practical point of view, the periodicity in the Fourier Transform of a sampled signal means that there's nothing to be gained by evaluating the Transform for values of θ beyond 2π.

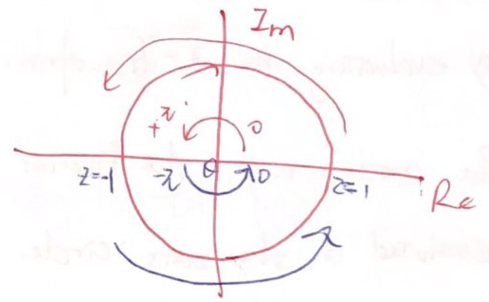
Thus, the spectrum from $\theta = 0$ to $\theta = 2\pi$ fully defines the spectrum.

Alternatively, we can say that the spectrum from $\theta = -\pi$ to $\theta = \pi$ defines the spectrum. because of "symmetry" in the spectrum

This is also consistent with the usual definition of two-sided spectra which contain both positive and negative frequencies.

From the perspective of the z-domain, calculation of the spectrum is like moving around the unit circle in the counter-clockwise direction from $z = -1$ to $z = 1$ for negative frequencies from $-\pi$ to 0.

and from $z = 1$ to $z = -1$ for positive frequencies from 0 to $+\pi$



Same thing applies for systems.

In particular, the frequency response of a system can be obtained from the transfer function of the system, by setting $\boxed{z = e^{j\theta}}$.

Alternatively, the frequency response may be obtained directly from the impulse response, using the equation defining the Fourier Transform above.

So we can obtain frequency response of a system, or the spectrum of a signal, in one or two ways, starting either in the transform domain, or in the time domain.

The Fourier Transform (whether it represents the spectrum of a signal, or the frequency response of a system) is generally a complex quantity, and can be represented in either polar or Cartesian form.

~~Furthermore~~ Furthermore, we're generally interested in the magnitude spectrum or phase spectrum, which can be readily obtained from the value of $X(\theta)$ at any particular frequency - after all, it's just a complex number.

If we represent the spectrum at a given frequency θ by $X(\theta) = a + jb$, then we can write the

magnitude spectrum as $|X(\theta)| = \sqrt{a^2 + b^2}$

phase ~~respons~~ spectrum as $\phi(\theta) = \tan^{-1}\left(\frac{b}{a}\right)$

Note: When we talk about a signal, we generally refer to the magnitude and phase ~~res spectrum~~ spectra ~~for~~ of the signal. while when we refer to a system, we usually refer to frequency response, and magnitude and phase responses (which are, in essence, the magnitude and phase spectra of the system impulse response.)

Also, it was observed above that, because of the periodicity of the spectrum of a sampled signal (or the frequency response of a digital filter) the signal is completely characterised by the spectrum from $\theta=-\pi$ to $\theta=\pi$.

However, in the case of real functions of time, it is sufficient to calculate the spectrum over the range of frequencies from $\theta=0$ to $\theta=\pi$.

This is because the Fourier-Transform of real-valued signals (or impulse responses) exhibits $\boxed{\text{conjugate symmetry}}$, so that the $\boxed{\text{magnitude spectrum}}$ is an <u>even function of frequency</u> (偶函数), while the $\boxed{\text{phase spectrum}}$ is an <u>odd function of frequency</u> (奇函数).

3.4. Examples.

3.4.1 Unit Impulse and Delayed Unit Impulse.

The Z-transform of the unit impulse function $\delta(n)$.

$$\boxed{Z[\delta(n)] = X(z) = \sum_{n=-\infty}^{\infty} \delta(n)\cdot z^{-n} = 1}$$

Therefore, the spectrum of the unit impulse can be found by setting $\boxed{z=e^{j\theta}}$ in $X(z)$ to obtain: $X(\theta)=1$

$$\underbrace{X(z)\Big|_{z=e^{j\theta}}}_{=1} = \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n} = \sum_{n=-\infty}^{+\infty} x(n)\cdot e^{-j\theta n} = \underbrace{X(\theta)}_{=1}.$$

$$Z[\delta(n)] = X(z) = 1$$

$$X(\theta) = 1$$

In other words, the unit impulse contains all frequencies with amplitude of 1 and a phase angle of 0°

The Fourier Transform is a complex quantity $X(\theta) = a + jb$.

In this case $X(\theta) = a + jb = 1 = 1 + 0 \cdot j$

$$\boxed{|X(\theta)| = \sqrt{a^2+b^2} = 1}$$

$$\boxed{\phi(\theta) = \tan^{-1}\left(\frac{b}{a}\right) = 0°}$$

or ~~$z = e^{j\theta} = 1 \rightarrow \theta = 0$~~

For a delayed unit impulse (delay of k samples), carrying out the

$$X(\theta) = X(z)\Big|_{z=e^{j\theta}} = \sum_{n=-\infty}^{+\infty} \delta(n-k) z^{-n} = z^{-k} = e^{-j\theta k}$$

$$= \cos(k\theta) - j \cdot \sin(k\theta).$$

This has a $\boxed{\text{magnitude of 1 for all frequencies}}$; $\left(\sqrt{\cos^2(k\theta) + \sin^2(k\theta)} = 1\right)$

However, the $\boxed{\text{phase spectrum}}$ is given by $\phi(\theta) = -k\theta$.

3.4.2 Sampled Exponential.

The spectrum of the sampled exponential waveform is obtained as follows:

$$x(n) = a^n u(n)$$

$$X(z) = \sum_{n=0}^{\infty} a^n u(n) \cdot z^{-n} = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

$$X(\theta) = X(z)\Big|_{z=e^{j\theta}} = \frac{1}{1-a \cdot e^{-j\theta}} = \frac{1}{1-[a\cos\theta - ja\sin\theta]}$$

$$= \frac{1}{[1-a\cos(\theta)] - j \cdot a\sin(\theta)}$$

$$|X(\theta)| = \frac{1}{\left([1-a\cos(\theta)]^2 + [a\sin(\theta)]^2\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\left(1-2a\cos(\theta) + a^2\cos^2(\theta) + a^2\sin^2(\theta)\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\left(1-2a\cos(\theta) + a^2\right)^{\frac{1}{2}}}$$

$$\phi(\theta) = \tan^{-1}\left(\frac{Im}{Re}\right)$$

$$= -\tan^{-1}\left(\frac{a\sin(\theta)}{1-a\cos(\theta)}\right)$$

the negative sign is required because the real and imaginary parts are in the denominator of the expression for the Fourier Transform?

3.5 System Frequency Response

We have observed that the frequency response of a system may be obtained either from the system transfer function, or directly by means of Fourier Transformation of the system impulse response.

Fourier Transform of input: $X(\theta) = \mathcal{F}[x(n)]$

Filter Frequency Response $H(\theta) = \mathcal{F}[h(n)]$

Then, $Y(\theta) = X(\theta)\, H(\theta)$

$$X(\theta) = \mathfrak{F}[x(n)]$$

$$H(\theta) = \mathfrak{F}[h(n)]$$

$$Y(\theta) = H(\theta)\,X(\theta)$$

If we express $Y(\theta)$ in polar form as $\boxed{|Y(\theta)| \cdot e^{j\phi(\theta)}}$

i.e. we explicitly indicate the gain and phase of the filter, then we can write:

$$\boxed{\begin{aligned} |Y(\theta)| &= |X(\theta)|\,|H(\theta)| \\ \phi_y(\theta) &= \phi_x(\theta) + \phi_h(\theta) \end{aligned}}$$

ie. the magnitude spectrum of the filter output $|Y(\theta)|$ is equal to the magnitude spectrum of the filter input $|X(\theta)|$, multiplied by the magnitude response (gain) of the filter $|H(\theta)|$.

This is exactly the same result as obtained for the continuous-time Fourier Transform.

The phase response of the filter output at a given frequency $\phi_y(\theta)$ is equal to the sum of the phase angle of the input $\phi_x(\theta)$ and the phase response of the filter. $\phi_h(\theta)$

Normally, the magnitude response of a filter is expressed in $\boxed{dB}$ on $\boxed{\text{a log scale}}$, while the phase response is often shown "modulo-2π", i.e. the phase angle is plotted in the range $-\pi$ to $+\pi$, and if the angle exceeds $+\pi$ or $-\pi$ at any frequency, it wraps around to the opposite end of the range.

3.6 Examples.

3.6.1 FIR Filter

A system: $y(n) = x(n) + x(n-1) + x(n-2)$.

Calculate and plot the frequency response of the system.

Solution:

The two general methods of obtaining the frequency response of a system are either through the transfer function, or by Fourier transformation of the impulse response.

In the case of an FIR filter, we have seen in Section 1 that the samples of the impulse response are the same as the digital filter coefficients, so we could easily use either method

We look at both here.

S1: starting with the differences equation, we obtain the transfer function, and thence the frequency response.

$$y(n) = x(n) + x(n-1) + x(n-2)$$

$$Y(z) = X(z)\,(1 + z^{-1} + z^{-2})$$

Transfer function: $H(z) = 1 + z^{-1} + z^{-2}$

Frequency response: let $z = e^{j\theta}$

$$H(\theta) = H(z)\big|_{z=e^{j\theta}} = \boxed{1 + e^{-j\theta} + e^{-j2\theta}}$$

$$= e^{-j\theta}\,[e^{j\theta} + 1 + e^{-j\theta}]$$

By Euler's equations, the term in square brackets can be seen to be equal to $1 + 2\cos(\theta)$

$$H(\theta) = e^{-j\theta}\,[e^{j\theta} + 1 + e^{-j\theta}] = e^{-j\theta}\,[\cos\theta + j\sin\theta + 1 + \cos\theta - j\sin\theta$$

$$= e^{-j\theta}\,[1 + 2\cos(\theta)]$$

The magnitude and phase response of the filter are given by:

$$|H(\theta)| = 1 + 2\cos(\theta)$$

$$\phi(\theta) = -\theta$$

S2: The impulse response of this filter: $y(n) = x(n) + x(n-1) + x(n-2)$

can be written as $\{1, 1, 1\}$, (starting @ $n=0$).

Hence, the Fourier Transform of the impulse response is.

$$H(\theta) = \sum_{n=-\infty}^{\infty} h(n) \cdot e^{-jn\theta} = \sum_{n=0}^{2} h(n) \cdot e^{-jn\theta}$$

$$= \sum_{n=0}^{2} e^{-jn\theta}$$

$$= e^{-j0\cdot\theta} + e^{-j\cdot 1\cdot\theta} + e^{-j\cdot 2\cdot\theta}$$

$$\boxed{= 1 + e^{-j\theta} + e^{-j2\theta}}$$

$$= e^{-j\theta}\left[e^{j\theta} + 1 + e^{-j\theta}\right]$$

$$= e^{-j\theta}\left[1 + 2\cos(\theta)\right]$$

$$|H(\theta)| = 1 + 2\cos(\theta)$$

$$\phi(\theta) = -\theta$$

3.6.2. IIR Filter

A filter: $y(n) = x(n) + 0.6\,x(n-1) - 0.1\,x(n-2) + 0.3\,y(n-1) - 0.3\,y(n-2)$

Because the impulse response of this filter is of infinite length, it is not practical to evaluate the frequency response by means of Fourier transformation of the impulse response.

Therefore, we will use the transfer function,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.6z^{-1} - 0.1z^{-2}}{1 - 0.3z^{-1} + 0.3z^{-2}}$$

Hence, the frequency response is given by

$$H(\theta)\Big|_{z=e^{j\theta}} = \frac{1 + 0.6\,e^{-j\theta} - 0.1\,e^{-2\theta j}}{1 - 0.3\,e^{-j\theta} + 0.3\,e^{-2\theta j}}$$

$$= \frac{1 + 0.6\cos\theta - j0.6\sin\theta - 0.1\cos(2\theta) + j0.1\sin(2\theta)}{1 - 0.3\cos\theta + j0.3\sin\theta + 0.3\cos(2\theta) - j0.3\sin(2\theta)}$$

$$|H(\theta)| = \frac{\left[(1+0.6\cos\theta - 0.1\cos2\theta)^2 + (-0.6\sin\theta + 0.1\sin2\theta)^2\right]^{\frac{1}{2}}}{\left[(1-0.3\cos\theta + 0.3\cos2\theta)^2 + (0.3\sin\theta - 0.3\sin2\theta)^2\right]^{\frac{1}{2}}}$$

$$\phi(\theta) = \tan^{-1}\left(\frac{-0.6\sin\theta + 0.1\sin2\theta}{1+0.6\cos\theta - 0.1\cos2\theta}\right) - \tan^{-1}\left(\frac{0.3\sin\theta - 0.3\sin2\theta}{1-0.3\cos\theta + 0.3\cos2\theta}\right)$$

3.7 Phase Response.

In Magnitude response of the filter $|H(\theta)|$: tells us how much of each frequency component in the input the filter will pass or attenuate

Phase response $\phi(\theta)$ ~~is generally~~ of the filter indicates the amount of delay (in radians) which a sinusoid at a particular frequency suffers as it passes through the filter (even if the magnitude response of the filter is 1 at this frequency, the phase response will still affect the filter).

This is often expressed in terms of the filter phase delay, which is the time delay (~~the delay in unitsof time~~). suffered by each frequency component as it passes through the filter:

$$\boxed{\tau_p = -\frac{\phi(\theta)}{\theta}}$$

Another measure that is often used to describe the "delay" through a filter is the $\boxed{\text{group delay}}$. This is the "average" time delay that a composite signal suffers at each frequency when passing through the filter:

$$\tau_g = -\frac{d\phi(\theta)}{d\theta}$$

If the group delay is not the same for all frequency components in the signal, it means that different frequencies suffer different amounts of delay through the filter (even if their magnitudes remain unaffected).

This results in phase distortion in the signal.

~~In order to have constant group delay at all frequencies.~~

In order to have constant group delay at all frequencies, it is necessary for the filter, to have linear ~~space~~ phase.

It can be shown that this can be achieved only with certain types of non-recursive filters (FIRs)

A filter is said to have linear phase if the phase response satisfies one of the following relationships:

$$\begin{cases} \phi(\theta) = -a\theta \\ \phi(\theta) = b - a\theta \end{cases}$$

where a and be are constants (and a is the slope of the phase response). Clearly, the derivative of the phase response in these cases will be equal to the constant a.

If the filter has a non-linear phase response, then the group delay will not be constant with frequency.

$$\tau_g = -\frac{d\phi(\theta)}{\theta}$$

Effect of non-linear and linear phase response.

The linear phase filter does not cause any ~~disortion~~ distortion of the input signal, the only effect is a delay.

However, the non-linear phase filter does introduce significant distortion even though the magnitude response is 1 at all of the frequencies contained in the input.