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S PARAMETER THEORY OF LOSSLESS BLOCK NETWORK

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Abstract—The energy conservation of lossless network reflects a series of novel symmetry in S parameter. This paper presents the generalized modulus symmetry, spurious reciprocity, constant characteristic phase and determinant of the lossless block network. The perfect matching condition of block load network $[\Gamma_l]$ and the invariable lossless property of S parameter of generalized block network are developed. Application examples are given to illustrate the application and validity of the proposed theory.

1. INTRODUCTION

The lossless network is a very important basis for microwave synthesis in microwave engineering. In fact, with the assumption of the lossless condition synthetic models of a considerable amount of microwave components can reasonably approximate a number of practical engineering problems, such as microwave filters [1], power dividers [2] and directional couplers [3].

The most essential property of the lossless network is the energy conservation. In other words, all energy entering into the network can be expressed in terms of reflection or scattering. As Emmy Noether, a known mathematician, revealed, any sort of the conservation must correspond to some kind of symmetry. In the lossless network, it is the Hermite symmetry. Specifically, scattering matrix $[S]$ satisfies

$$[S]^{+}[S] = [I] \tag{1}$$

where $[\ ]^{+} = [^{*}]^{T} = ([\ ]^{T})^{*}$. Here superscript $*$ represents conjugation. $[\ ]^{T}$ denotes matrix transpose, and $[I]$ is a unity matrix. Eq. (1)

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is referred to the lossless unitarity. For a two-port lossless network, the unitary condition (1) can be expressed as three independent equations [4]

$$\begin{cases} |S_{11}|^2 + |S_{21}|^2 = 1 \\ |S_{12}|^2 + |S_{22}|^2 = 1 \\ S_{11}^* S_{12} + S_{21}^* S_{22} = 0 \end{cases} \tag{2}$$

By some derivations of (2), the S parameter properties of the two-port lossless network can be summarized as follows [5]:

1. Modulus symmetry
$$|S_{11}| = |S_{22}| \tag{3}$$
That is to say, there is an electromagnetic asymmetry at port 1 and port 2, i.e., $S_{11} \neq S_{22}$. But with the lossless condition, we can obtain the same moduli.
2. Spurious reciprocity
$$|S_{12}| = |S_{21}| \tag{4}$$
Similarly, the two-port lossless network can be nonreciprocal, viz. $S_{12} \neq S_{21}$. However, the lossless condition guarantees the same moduli.
3. Characteristic phase Φ
$$\Phi = (\varphi_{12} + \varphi_{21}) - (\varphi_{11} + \varphi_{22}) = \pm\pi \tag{5}$$
in which $S_{ij} = |S_{ij}|e^{\varphi_{ij}}$ $(i, j = 1, 2)$. There is a constant characteristic phase or standing property in the lossless network. The determinant of S parameter of the lossless network $\det[S]$ can be expressed as
$$\det[S] = e^{j(\varphi_{11}+\varphi_{22})} = -e^{j(\varphi_{12}+\varphi_{21})} \tag{6}$$
and the modulus of $\det[S]$ is equal to 1.

One of the applications of the two-port lossless network is the perfect matching problem. Specifically, the network is connected by the lossy load Γ_L through $[S]$ in order to get $\Gamma_{in} = 0$. In this case, the perfect matching condition becomes

$$S_{22}^* = \Gamma_L \tag{7}$$

In recent years, a lot of researches on the multi-port lossless network have been done on the basis of S parameter properties of the two-port lossless networks. Liang and Qiu [6] first extended the modulus symmetry of the two-port lossless networks to a multi-port lossless reciprocal network. The phase relation of the two-port lossless network has been generalized to a lossless n-port network [7]. Characteristic phase Φ of the three- and four-port even symmetry networks has been discussed [8]. Liang et al. further developed the characteristic phase Φ of a lossless n-port network in the case of a non-square sub-matrix S parameter [9]. Heiber and Vernon derived the matching conditions for a reciprocal lossless five-port network [10]. In [11], a condition to achieve a kind of completely matching problem, i.e., $\min|S_{ii}|$ $(i = 1, 2, \ldots, n)$, has been derived. Some bounding conditions for S parameter of a three-port reciprocal lossless network have been discussed by Butterweck [12]. A lot of works on the modulus symmetry, spurious reciprocity and characteristic phase of the lossless n-port network have been carried out, but to our knowledge the perfect matching problem and invariable lossless property of the multi-port lossless network have not been studied yet.

In this paper, the block network method is first utilized to develop the modulus symmetry, spurious reciprocity, constant characteristic phase and determinant of the multi-port lossless network in analogy with those of the two-port lossless networks. In the following, the perfect matching condition of the multi-port lossless network is proposed when the block load network $[\Gamma_l]$ is considered. Furthermore, the invariable lossless property of S parameter of generalized block network is studied. Finally, some practical multi-port lossless networks are given to validate the proposed theory.

2. THE LOSSLESS BLOCK NETWORK

With development of the large complex systems, the multi-port network theory [6–12] becomes increasingly important. Especially the block network in essence is the function division of the complex system. Fig. 1 shows a general n-port network. According to the function of the network, the ports can be divided into two parts: Part I denotes the input port or generalized source port consisting of p ports; Part II denotes the output port or generalized load port consisting of q ports. Here $n = p + q$.

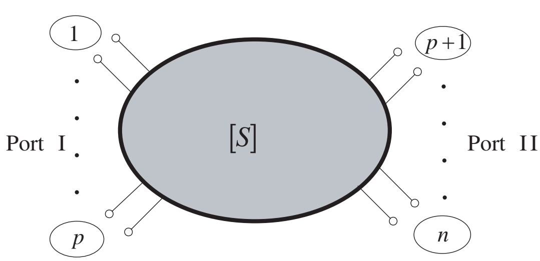

Figure 1. A n-port lossless block network.

According to the lossless unitarity (1), we can obtain the determinant of (1) as

$$|\det[S]| = 1 \tag{8}$$

Using the function division above, we have in the form of the block matrix:

$$\begin{bmatrix} b_{\mathrm{I}} \\ b_{\Pi} \end{bmatrix} = \begin{bmatrix} S_{\mathrm{I\ I}} & S_{\mathrm{I}\ \Pi} \\ S_{\Pi\ \mathrm{I}} & S_{\Pi\ \Pi} \end{bmatrix} \begin{bmatrix} a_{\mathrm{I}} \\ a_{\Pi} \end{bmatrix} \tag{9}$$

where $S_{\mathrm{I\ I}}$ and $S_{\Pi\ \Pi}$ are the $p \times p$ and $q \times q$ square matrixes, and $S_{\mathrm{I}\ \Pi}$ and $S_{\Pi\ \mathrm{I}}$ are the $p \times q$ and $q \times p$ arbitrary matrixes. Using (1) we can obtain three independent equations similar to (2) as

$$\begin{cases} S^{+}_{\mathrm{I\ I}} S_{\mathrm{I\ I}} + S^{+}_{\Pi\ \mathrm{I}} S_{\Pi\ \mathrm{I}} = I_p \\ S^{+}_{\mathrm{I}\ \Pi} S_{\mathrm{I}\ \Pi} + S^{+}_{\Pi\ \Pi} S_{\Pi\ \Pi} = I_q \\ S^{+}_{\mathrm{I\ I}} S_{\mathrm{I}\ \Pi} + S^{+}_{\Pi\ \mathrm{I}} S_{\Pi\ \Pi} = 0_{p\ q} \end{cases} \tag{10}$$

2.1. Generalized Modulus Symmetry

Similar to the derivation process in [9], introducing matrix $[U]$

$$[U] = \begin{bmatrix} S_{\mathrm{I\ I}} & 0_{p\ q} \\ 0_{q\ p} & I_q \end{bmatrix} \tag{11}$$

and left multiplying $[S]$ by $[U]^{+}$, we can get

$$[U]^{+}[S] = \begin{bmatrix} S^{+}_{\mathrm{I\ I}} S_{\mathrm{I\ I}} & S^{+}_{\mathrm{I\ I}} S_{\mathrm{I}\ \Pi} \\ S_{\Pi\ \mathrm{I}} & S_{\Pi\ \Pi} \end{bmatrix} \tag{12}$$

By calculating the determinant of (12), we can obtain

$$\begin{aligned} \det S^{+}_{\mathrm{I\ I}} \det[S] &= \det \begin{bmatrix} S^{+}_{\mathrm{I\ I}} S_{\mathrm{I\ I}} + S^{+}_{\Pi\ \mathrm{I}} S_{\Pi\ \mathrm{I}} & S^{+}_{\mathrm{I\ I}} S_{\mathrm{I}\ \Pi} + S^{+}_{\Pi\ \mathrm{I}} S_{\Pi\ \Pi} \\ S_{\Pi\ \mathrm{I}} & S_{\Pi\ \Pi} \end{bmatrix} \\ &= \det \begin{bmatrix} I_p & 0_{p\ q} \\ S_{\Pi\ \mathrm{I}} & S_{\Pi\ \Pi} \end{bmatrix} = \det[S_{\Pi\ \Pi}] \end{aligned} \tag{13}$$

Considering (8), we can get [9]

$$|\det S_{\mathrm{I\ I}}| = |\det S_{\Pi\ \Pi}| \tag{14}$$

Eq. (14) is the generalized modulus symmetry. Note that in general cases $S_{\mathrm{I\ I}}$ and $S_{\Pi\ \Pi}$ are the square matrices of different orders.

2.2. Generalized Spurious Reciprocity

The definitions of the inverse matrixes have two kinds of forms, i.e., the left inverse and right inverse. Due to the definition of the unitarity matrix similar to that of the inverse matrix, we have the other form of the lossless unitarity except (1) [9]

$$[S]\,[S]^{+} = [I] \tag{15}$$

Eq. (15) can be expanded as

$$\begin{cases} S_{\mathrm{I}\ \mathrm{I}} S^{+}_{\mathrm{I}\ \mathrm{I}} + S_{\mathrm{I}\ \Pi} S^{+}_{\mathrm{I}\ \Pi} \ = I_p \\ S_{\Pi\ \mathrm{I}} S^{+}_{\Pi\ \mathrm{I}} + S_{\Pi\ \Pi} S^{+}_{\Pi\ \Pi} = I_q \\ S_{\Pi\ \mathrm{I}} S^{+}_{\mathrm{I}\ \mathrm{I}} + S_{\Pi\ \Pi} S^{+}_{\mathrm{I}\ \Pi} \ = 0_{q\ p} \end{cases} \tag{16}$$

By left multiplying the third equation in (10) by $S_{\Pi\ \mathrm{I}}$ and considering the third equation in (16), we can get

$$S_{\Pi\ \Pi} S^{+}_{\mathrm{I}\ \Pi} S_{\mathrm{I}\ \Pi} = S_{\Pi\ \mathrm{I}} S^{+}_{\Pi\ \mathrm{I}} S_{\Pi\ \Pi} \tag{17}$$

Taking the determinant of (17), we have

$$\det\left(S^{+}_{\mathrm{I}\ \Pi} S_{\mathrm{I}\ \Pi}\right) = \det\left(S_{\Pi\ \mathrm{I}} S^{+}_{\Pi\ \mathrm{I}}\right) \tag{18}$$

Similarly, we also obtain

$$\det\left(S_{\mathrm{I}\ \Pi} S^{+}_{\mathrm{I}\ \Pi}\right) = \det\left(S^{+}_{\Pi\ \mathrm{I}} S_{\Pi\ \mathrm{I}}\right) \tag{19}$$

Here (18) and (19) denote the generalized spurious reciprocity. Note that (18) is not equivalent to (19), because $S_{\mathrm{I}\ \Pi} S^{+}_{\mathrm{I}\ \Pi}$ is a p-order square matrix, whereas $S^{+}_{\mathrm{I}\ \Pi} S_{\mathrm{I}\ \Pi}$ is a q-order square matrix. When p is equal to q, we further have from (18) and (19) [7]

$$|\det S_{\mathrm{I}\ \Pi}| = |\det S_{\Pi\ \mathrm{I}}| \tag{20}$$

In addition, according to (10) and (16) we also get

$$S_{\mathrm{I}\ \Pi} S^{+}_{\mathrm{I}\ \Pi} - S^{+}_{\Pi\ \mathrm{I}} S_{\Pi\ \mathrm{I}} \ = \ S^{+}_{\mathrm{I}\ \mathrm{I}} S_{\mathrm{I}\ \mathrm{I}} - S_{\mathrm{I}\ \mathrm{I}} S^{+}_{\mathrm{I}\ \mathrm{I}} \tag{21}$$

$$S^{+}_{\mathrm{I}\ \Pi} S_{\mathrm{I}\ \Pi} - S_{\Pi\ \mathrm{I}} S^{+}_{\Pi\ \mathrm{I}} \ = \ S_{\Pi\ \Pi} S^{+}_{\Pi\ \Pi} - S^{+}_{\Pi\ \Pi} S_{\Pi\ \Pi} \tag{22}$$

Therefore, only if

$$S^{+}_{\mathrm{I}\ \mathrm{I}} S_{\mathrm{I}\ \mathrm{I}} \ = \ S_{\mathrm{I}\ \mathrm{I}} S^{+}_{\mathrm{I}\ \mathrm{I}} \tag{23}$$

$$S_{\Pi\ \Pi} S^{+}_{\Pi\ \Pi} \ = \ S^{+}_{\Pi\ \Pi} S_{\Pi\ \Pi} \tag{24}$$

we can get

$$S_{\mathrm{I}\ \Pi} S^{+}_{\mathrm{I}\ \Pi} \ = \ S^{+}_{\Pi\ \mathrm{I}} S_{\Pi\ \mathrm{I}} \tag{25}$$

$$S^{+}_{\mathrm{I}\ \Pi} S_{\mathrm{I}\ \Pi} \ = \ S_{\Pi\ \mathrm{I}} S^{+}_{\Pi\ \mathrm{I}} \tag{26}$$

In this scenario, we can clearly see the reciprocity in (25) and (26).

2.3. Determinant of Lossless Network $\det[S]$

Assume

$$\det S_{\mathrm{I\ I}} = \left|\det S_{\mathrm{I\ I}}\right| e^{j\Phi_{\mathrm{I\ I}}} \quad \det S_{\Pi\ \Pi} = \left|\det S_{\Pi\ \Pi}\right| e^{j\Phi_{\Pi\ \Pi}} \tag{27}$$

Substituting (27) and (14) into (13), we can obtain

$$\det[S] = e^{j(\Phi_{\mathrm{I\ I}}+\Phi_{\Pi\ \Pi})} \tag{28}$$

Eq. (28) indeed is the generalization of the determinant of the two-port lossless network. In order to further express $\det[S]$ using the block matrix, we introduce the block inverse matrix of S

$$[S]^{-1} = \begin{bmatrix} R_{\mathrm{I\ I}} & R_{\mathrm{I}\ \Pi} \\ R_{\Pi\ \mathrm{I}} & R_{\Pi\ \Pi} \end{bmatrix} \tag{29}$$

in which

$$R_{\mathrm{I\ I}} = \left(S_{\mathrm{I\ I}} - S_{\mathrm{I}\ \Pi} S_{\Pi\ \Pi}^{-1} S_{\Pi\ \mathrm{I}}\right)^{-1} \tag{30}$$

$$R_{\Pi\ \Pi} = \left(S_{\Pi\ \Pi} - S_{\Pi\ \mathrm{I}} S_{\mathrm{I\ I}}^{-1} S_{\mathrm{I}\ \Pi}\right)^{-1} \tag{31}$$

Considering the lossless condition, i.e., $[S]^{-1} = [S]^{+}$, we have

$$R_{\mathrm{I\ I}} = S_{\mathrm{I\ I}}^{+} \tag{32}$$

$$R_{\Pi\ \Pi} = S_{\Pi\ \Pi}^{+} \tag{33}$$

When $p = q$, we can obtain

$$S_{\mathrm{I\ I}} \left(S_{\Pi\ \Pi}^{+}\right)^{-1} = S_{\mathrm{I\ I}} S_{\Pi\ \Pi} - S_{\mathrm{I\ I}} S_{\Pi\ \mathrm{I}} S_{\mathrm{I\ I}}^{-1} S_{\mathrm{I}\ \Pi} \tag{34}$$

By taking the determinant of (34) and considering

$$\det\left[S_{\mathrm{I\ I}} \left(S_{\Pi\ \Pi}^{+}\right)^{-1}\right] = e^{j[\Phi_{\mathrm{I\ I}}+\Phi_{\Pi\ \Pi}]} \tag{35}$$

we can derive $\det[S]$ of the lossless network in the case of $p = q$, namely

$$\det[S] = \det\left[S_{\mathrm{I\ I}} S_{\Pi\ \Pi} - S_{\mathrm{I\ I}} S_{\Pi\ \mathrm{I}} S_{\mathrm{I\ I}}^{-1} S_{\mathrm{I}\ \Pi}\right] \tag{36}$$

Similarly, we can also obtain

$$\det[S] = \det\left[S_{\mathrm{I\ I}} S_{\Pi\ \Pi} - S_{\mathrm{I}\ \Pi} S_{\Pi\ \Pi}^{-1} S_{\Pi\ \mathrm{I}} S_{\Pi\ \Pi}\right] \tag{37}$$

Eqs. (36) and (37) are the block matrix expressions for $\det[S]$ of the lossless network when $p = q$.

2.4. Generalized Characteristic Phase Φ

The generalized characteristic phase of the lossless network is meaningful only when p is equal to q. In other words, when $S_{\mathrm{I\ II}}$ and $S_{\mathrm{II\ I}}$ are square matrices, there is a generalized characteristic phase. Assuming

$$\det S_{\mathrm{I\ II}} = |\det S_{\mathrm{I\ II}}|\, e^{j\Phi_{\mathrm{I\ II}}} \quad \det S_{\mathrm{II\ I}} = |\det S_{\mathrm{II\ I}}|\, e^{j\Phi_{\mathrm{II\ I}}} \tag{38}$$

the determinant of the third equation in (16) can be expressed as

$$|\det S_{\mathrm{II\ I}}|\, |\det S_{\mathrm{I\ I}}|\, e^{j(\Phi_{\mathrm{II\ I}}-\Phi_{\mathrm{I\ I}})} = -\,|\det S_{\mathrm{I\ II}}|\, |\det S_{\mathrm{II\ II}}|\, e^{j(\Phi_{\mathrm{II\ II}}-\Phi_{\mathrm{I\ II}})} \tag{39}$$

Noticing the phase term in (39), we can define the generalized characteristic phase Φ as [9]

$$\Phi = (\Phi_{\mathrm{I\ II}} + \Phi_{\mathrm{II\ I}}) - (\Phi_{\mathrm{I\ I}} + \Phi_{\mathrm{II\ II}}) = \pm\pi \tag{40}$$

It can be seen from (40) that the n-port lossless network has the constant generalized characteristic phase.

If we take into further consideration the magnitude of (39), we can get the generalized spurious reciprocity (20) in the case of $p = q$. Note that it is an inevitable consequence of the lossless phase property in the case of the generalized modulus symmetry. According to (40), we further have

$$\det[S] = -e^{j(\Phi_{\mathrm{I\ II}}+\Phi_{\mathrm{II\ I}})} \tag{41}$$

Moreover, $\det[S]$ can be expressed as in terms of the block matrix

$$\det[S] = \frac{1}{|\det S_{\mathrm{I\ I}}|^2 + |\det S_{\mathrm{I\ II}}|^2} \det \begin{bmatrix} \det S_{\mathrm{I\ I}} & \det S_{\mathrm{I\ II}} \\ \det S_{\mathrm{II\ I}} & \det S_{\mathrm{II\ II}} \end{bmatrix} \tag{42}$$

Note that the valid condition for (42) is $p = q$.

3. PERFECT MATCHING CONDITION

For an n-port lossless network, the load network $[\Gamma_L]$ is connected with the Port II, as shown in Fig. 2. According to (9), we have

$$a_{\mathrm{II}} = \Gamma_L b_{\mathrm{II}} \tag{43}$$

Denoting

$$b_{\mathrm{I}} = S_p a_{\mathrm{I}} \tag{44}$$

where S_p is a p-order matrix and can be considered as the generalized reflection at the Port I of the block network. We can easily get

$$S_p = S_{\mathrm{I\ I}} + S_{\mathrm{I\ II}} \left(\Gamma_L^{-1} - S_{\mathrm{II\ II}}\right)^{-1} S_{\mathrm{II\ I}} \tag{45}$$

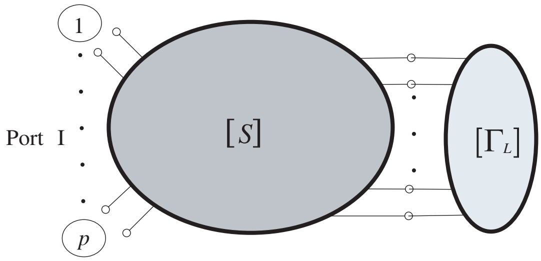

Figure 2. Perfect matching problem for an n-port lossless block network.

The perfect matching condition for the n-port lossless network becomes

$$S_p = 0_p \tag{46}$$

In this case, (45) can be rewritten as

$$S_{\mathrm{I\ I}} = -S_{\mathrm{I\ II}}\left(\Gamma_L^{-1} - S_{\mathrm{II\ II}}\right)^{-1} S_{\mathrm{II\ I}} \tag{47}$$

By solving (47), we can obtain the perfect matching condition

$$\Gamma_L = \left\{S_{\mathrm{II\ II}} - \left(S_{\mathrm{II\ I}}S_{\mathrm{II\ I}}^{+}\right)\left(S_{\mathrm{I\ II}}^{+}S_{\mathrm{I\ I}}S_{\mathrm{II\ I}}^{+}\right)\left(S_{\mathrm{I\ II}}^{+}S_{\mathrm{I\ II}}\right)\right\}^{-1} \tag{48}$$

Especially when $p = q$, (48) becomes

$$\Gamma_L = \left\{S_{\mathrm{II\ II}} - S_{\mathrm{II\ I}}S_{\mathrm{I\ I}}^{-1}S_{\mathrm{I\ II}}\right\}^{-1} \tag{49}$$

According to (31) and (33), we can finally get

$$\Gamma_L = S_{\mathrm{II\ II}}^{+} \tag{50}$$

Eq. (50) is the generalization of the perfect matching condition of the two-port lossless networks. Note that for the square matrix of $p = q$, if independent load Γ_L is a diagonal matrix, $S_{\mathrm{II\ II}}$ must also be a diagonal matrix so that the perfect matching problem can be achieved; on the contrary, when there are some non-diagonal elements in $S_{\mathrm{II\ II}}$, the perfect matching condition requires that Γ_L should include some non-diagonal elements. However, the problem becomes very complex for the case of $p \neq q$. Detailed examples will be discussed in Section 5.

4. CONSTANT LOSSLESS PROPERTY OF GENERALIZED BLOCK PARAMETER [S]

In order to easily study the large complex system, the origin parameter $\left[S^0\right]$ is generally extended to the generalized parameter $[S]$. The origin parameter $\left[S^0\right]$ is defined as

$$\left[b^0\right] = \left[S^0\right]\left[a^0\right] \tag{51}$$

where $\left[a^0\right]$ and $\left[b^0\right]$ at Ports I and II are normalized according to impedance $[I_p]$ and $[I_q]$, respectively. Specifically, we can get

$$a_{\mathrm{I}}^0 = \frac{1}{2}\left(V_{\mathrm{I}} + I_{\mathrm{I}}\right) \qquad b_{\mathrm{I}}^0 = \frac{1}{2}\left(V_{\mathrm{I}} - I_{\mathrm{I}}\right) \tag{52}$$

$$a_{\Pi}^0 = \frac{1}{2}\left(V_{\Pi} + I_{\Pi}\right) \qquad b_{\Pi}^0 = \frac{1}{2}\left(V_{\Pi} - I_{\Pi}\right) \tag{53}$$

Note that the voltages and currents at Ports I and II have been normalized according to the characteristic impedance of the system. The extension of $[S^0]$ to $[S]$ can be made using

$$[b] = [S]\,[a] \tag{54}$$

in which $[a]$ and $[b]$ at Ports I and II are normalized according to arbitrary complex diagonal impedance matrices $[Z_g]$ and $[Z_L]$, respectively. They can be expressed as in terms of the voltages and currents

$$\begin{cases} a_{\mathrm{I}} = \frac{1}{2}\left(V_{\mathrm{I}} + Z_g I_{\mathrm{I}}\right)\left[\sqrt{\mathrm{Re}\left(Z_g\right)}\right]^{-1} \\ b_{\mathrm{I}} = \frac{1}{2}\left(V_{\mathrm{I}} - Z_g^* I_{\mathrm{I}}\right)\left[\sqrt{\mathrm{Re}\left(Z_g\right)}\right]^{-1} \end{cases} \tag{55}$$

$$\begin{cases} a_{\Pi} = \frac{1}{2}\left(V_{\Pi} + Z_L I_{\Pi}\right)\left[\sqrt{\mathrm{Re}\left(Z_L\right)}\right]^{-1} \\ b_{\Pi} = \frac{1}{2}\left(V_{\Pi} - Z_L^* I_{\Pi}\right)\left[\sqrt{\mathrm{Re}\left(Z_L\right)}\right]^{-1} \end{cases} \tag{56}$$

Considering (51)$\sim$(56) and assuming

$$\left[S^0\right] = \begin{bmatrix} S_{\mathrm{I\,I}}^0 & S_{\mathrm{I}\,\Pi}^0 \\ S_{\Pi\,\mathrm{I}}^0 & S_{\Pi\,\Pi}^0 \end{bmatrix} \qquad [S] = \begin{bmatrix} S_{\mathrm{I\,I}} & S_{\mathrm{I}\,\Pi} \\ S_{\Pi\,\mathrm{I}} & S_{\Pi\,\Pi} \end{bmatrix} \tag{57}$$

$[S]$ can be expressed as in terms of $\left[S^0\right]$

$$\begin{aligned} S_{\mathrm{I\,I}} = &\left[\left(I - S_{\mathrm{I\,I}}^0\Gamma_g\right) - S_{\mathrm{I}\,\Pi}^0\Gamma_L\left(I - S_{\Pi\,\Pi}^0\Gamma_L\right)^{-1}\Gamma_g S_{\Pi\,\mathrm{I}}^0\right]^{-1} \\ &\cdot\left[\left(S_{\mathrm{I\,I}}^0 - \Gamma_g^*\right) + S_{\mathrm{I}\,\Pi}^0\Gamma_L\left(I - S_{\Pi\,\Pi}^0\Gamma_L\right)^{-1} S_{\Pi\,\mathrm{I}}^0\right]\left[e^{-j2\varphi_g}\right] \end{aligned} \tag{58}$$

$$\begin{aligned} S_{\Pi\,\Pi} = &\left[\left(I - S_{\Pi\,\Pi}^0\Gamma_L\right) - S_{\Pi\,\mathrm{I}}^0\Gamma_g\left(I - S_{\mathrm{I\,I}}^0\Gamma_g\right)^{-1}\Gamma_L S_{\mathrm{I}\,\Pi}^0\right]^{-1} \\ &\cdot\left[\left(S_{\Pi\,\Pi}^0 - \Gamma_L^*\right) + S_{\Pi\,\mathrm{I}}^0\Gamma_g\left(I - S_{\mathrm{I\,I}}^0\Gamma_g\right)^{-1} S_{\mathrm{I}\,\Pi}^0\right]\left[e^{-j2\varphi_L}\right] \end{aligned} \tag{59}$$

$$\begin{aligned} S_{\mathrm{I}\,\Pi} = &\left[\sqrt{1 - \left|\Gamma_g\right|^2}\right]\left[\left(I - S_{\mathrm{I\,I}}^0\Gamma_g\right) - S_{\mathrm{I}\,\Pi}^0\Gamma_L\left(I - S_{\Pi\,\Pi}^0\Gamma_L\right)^{-1}\Gamma_g S_{\Pi\,\mathrm{I}}^0\right]^{-1} \\ &\cdot S_{\mathrm{I}\,\Pi}^0\left(I - S_{\Pi\,\Pi}^0\Gamma_L\right)^{-1}\left[\sqrt{1 - \left|\Gamma_L\right|^2}\right]\left[e^{-j(\varphi_g + \varphi_L)}\right] \end{aligned} \tag{60}$$

$$S_{\mathrm{II\ I}} = \left[\sqrt{1-|\Gamma_L|^2}\right]\left[\left(I - S^0_{\mathrm{II\ II}}\Gamma_L\right) - S^0_{\mathrm{II\ I}}\Gamma_g\left(I - S^0_{\mathrm{I\ I}}\Gamma_g\right)^{-1}\Gamma_L S^0_{\mathrm{I\ II}}\right]^{-1}$$
$$\cdot S^0_{\mathrm{II\ I}}\left(I - S^0_{\mathrm{I\ I}}\Gamma_g\right)^{-1}\left[\sqrt{1-|\Gamma_g|^2}\right]\left[e^{-j(\varphi_g+\varphi_L)}\right] \tag{61}$$

where the source reflection matrix of p-order Γ_g and load reflection matrix of q-order Γ_L are

$$\Gamma_g = (Z_g - I)(Z_g + I)^{-1} = (Z_g + I)^{-1}(Z_g - I) \tag{62}$$
$$\Gamma_L = (Z_L - I)(Z_L + I)^{-1} = (Z_L + I)^{-1}(Z_L - I) \tag{63}$$

and

$$\left[\sqrt{\mathrm{Re}\,(Z_g)}\right][Z_g + I]^{-1} = \left[\sqrt{1-|\Gamma_g|^2}\right]\left[e^{-j\varphi_g}\right] \tag{64}$$
$$\left[\sqrt{\mathrm{Re}\,(Z_L)}\right][Z_L + I]^{-1} = \left[\sqrt{1-|\Gamma_L|^2}\right]\left[e^{-j\varphi_L}\right] \tag{65}$$
$$(Z_g + I)^{-1}\left(Z_g^* + I\right) = \left[e^{-j2\varphi_g}\right] \tag{66}$$
$$(Z_L + I)^{-1}\left(Z_L^* + I\right) = \left[e^{-j2\varphi_L}\right] \tag{67}$$

in which

$$\left[\sqrt{1-|\Gamma_g|^2}\right] = \mathrm{diag}\left[\sqrt{1-|\Gamma_{g1}|^2} \quad \sqrt{1-|\Gamma_{g2}|^2} \quad \dots \quad \sqrt{1-|\Gamma_{gp}|^2}\right] \tag{68}$$
$$\left[\sqrt{1-|\Gamma_L|^2}\right] = \mathrm{diag}\left[\sqrt{1-|\Gamma_{L1}|^2} \quad \sqrt{1-|\Gamma_{L2}|^2} \quad \dots \quad \sqrt{1-|\Gamma_{Lq}|^2}\right] \tag{69}$$
$$\left[e^{-j\varphi_g}\right] = \mathrm{diag}\left[\ e^{-j\varphi_{g1}} \quad e^{-j\varphi_{g2}} \quad \dots \quad e^{-j\varphi_{gp}}\ \right] \tag{70}$$
$$\left[e^{-j\varphi_L}\right] = \mathrm{diag}\left[\ e^{-j\varphi_{L1}} \quad e^{-j\varphi_{L2}} \quad \dots \quad e^{-j\varphi_{Lq}}\ \right] \tag{71}$$
$$\begin{cases} Z_{gi} = R_{gi} + jX_{gi} \\ \varphi_{gi} = \tan^{-1}\left(\frac{X_{gi}}{1+R_{gi}}\right) \end{cases} \qquad i = 1, 2, \dots, p \tag{72}$$
$$\begin{cases} Z_{Lm} = R_{Lm} + jX_{Lm} \\ \varphi_{Lm} = \tan^{-1}\left(\frac{X_{Lm}}{1+R_{Lm}}\right) \end{cases} \qquad m = 1, 2, \dots, q \tag{73}$$

Note that both the source reflection matrix and load reflection matrix are diagonal matrices. When origin parameter $[S^0]$ is a lossless network, parameter $[S]$ must also be a lossless network, which can be proved by considering the energy conservation relation (the input power at Port I equal to the output power at Port II) and (55) and (56). This is referred to the invariable lossless property in the generalized block matrix.

5. APPLICATION EXAMPLE

In this section, the generalized modulus symmetry and the perfect matching condition for a completely symmetric reciprocal three-port network are discussed, as shown in Fig. 3.

The S parameter of the three-port lossless network can be written as

$$[S] = \begin{bmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{bmatrix} \tag{74}$$

in which $\alpha = |\alpha|\, e^{j\varphi_\alpha}$ and $\beta = |\beta|\, e^{j\varphi_\beta}$. Here we let

$$S_{\mathrm{I\ I}} = \alpha \quad S_{\Pi\ \Pi} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \quad S_{\mathrm{I}\ \Pi} = [\beta \quad \beta] \quad S_{\Pi\ \mathrm{I}} = [\,\beta \quad \beta\,]^T \tag{75}$$

In this case, the unitarity becomes

$$\begin{cases} |\alpha|^2 + 2\,|\beta|^2 = 1 \\ \alpha\beta^* + \alpha^*\beta + |\beta|^2 = 0 \end{cases} \tag{76}$$

According to the second equation of (76), we can obtain

$$\cos\theta = \cos(\varphi_\alpha - \varphi_\beta) = -\frac{|\beta|}{2\,|\alpha|} \tag{77}$$

It is easily seen from (77) that $|\beta|/2\,|\alpha| \leq 1$.

5.1. Case 1. Generalized Modulus Symmetry

According to (75), we have

$$|\det S_{\mathrm{I\ I}}| = |\alpha| \tag{78}$$

$$|\det S_{\Pi\ \Pi}| = \left|\det \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}\right| = \sqrt{|\alpha|^4 + |\beta|^4 - 2\,|\alpha|^2\,|\beta|^2 \cos 2\theta} \tag{79}$$

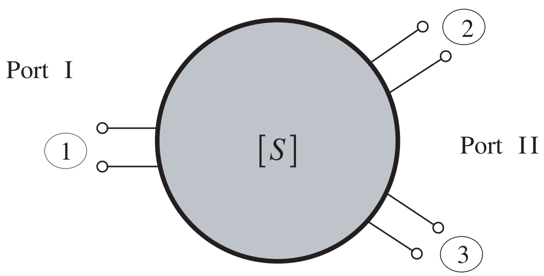

Figure 3. A three-port lossless completely symmetric reciprocal network.

Considering $\cos 2\theta = 2\cos^2\theta - 1$ and (77), we can get

$$|\det S_{\text{II II}}| = \sqrt{|\alpha|^4 + 2|\alpha|^2|\beta|^2} \tag{80}$$

Substituting the first equation of (76) into (80), we can obtain

$$|\det S_{\text{II II}}| = |\alpha| \tag{81}$$

Hence, we get the generalized modulus symmetry, i.e.,

$$|\det[S_{\text{I I}}]| = |\det[S_{\text{II II}}]| \tag{82}$$

5.2. Case 2. Perfect Matching Condition

The perfect matching problem for the completely symmetric reciprocal three-port network is shown in Fig. 4. Assume that two independent loads Γ_{L2} and Γ_{L3} are connected with Ports 2 and 3, respectively.

According to (45), we know

$$S_m = S_{\text{I I}} + S_{\text{I II}}\left(\Gamma_L^{-1} - S_{\text{II II}}\right)^{-1} S_{\text{II I}} \tag{83}$$

where

$$\Gamma_L = \begin{bmatrix} \Gamma_{L2} & 0 \\ 0 & \Gamma_{L3} \end{bmatrix} \tag{84}$$

Substituting (84) into (83), we can obtain

$$S_m = \alpha + \frac{\beta^2(\Gamma_{L2} + \Gamma_{L3}) + 2\beta^2(\beta - \alpha)\Gamma_{L2}\Gamma_{L3}}{(1 - \alpha\Gamma_{L2})(1 - \alpha\Gamma_{L3}) - \beta^2\Gamma_{L2}\Gamma_{L3}} \tag{85}$$

For simplicity, we let

$$\Gamma_l = \Gamma_{L2} = \Gamma_{L3} \tag{86}$$

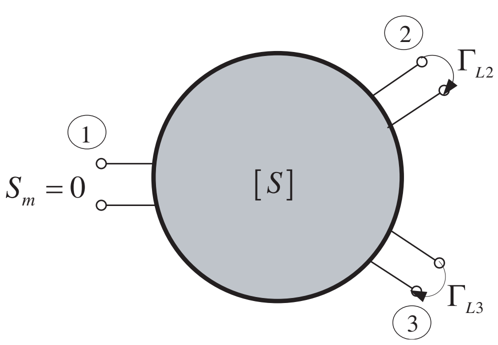

Figure 4. Perfect matching problem for a three-port lossless completely symmetric reciprocal network.

Therefore, the perfect matching condition $\Gamma_{in} = S_m = 0$ becomes

$$\left[\alpha\left(\alpha^2-\beta^2\right)+2\beta^2\left(\beta-\alpha\right)\right]\Gamma_l^2+2\left(\beta^2-\alpha^2\right)\Gamma_l+\alpha=0 \qquad (87)$$

This is a quadratic equation of one variable. For example, we choose

$$\begin{cases}\alpha=\frac{1}{3}\\ \beta=-\frac{2}{3}\end{cases} \qquad (88)$$

It is worthwhile pointing out that the assumption in (88) must satisfy the unitarity of the S matrix. Substituting (88) into (87), we have

$$\begin{cases}\Gamma_{l1}=-\frac{1}{3}\\ \Gamma_{l2}=1\end{cases} \qquad (89)$$

Here only Γ_{l1} is kept, because Γ_{l2} makes the denominator of (85) equal to zero. So when the perfect matching condition is achieved, we have

$$\Gamma_{L2}=\Gamma_{L3}=\frac{1}{3} \qquad (90)$$

Note that in this example p is not equal to q. Therefore, $[\Gamma_L]$ is a diagonal matrix although there are non-diagonal elements in $S_{\Pi\ \Pi}$.

6. CONCLUSION

It is a problem worthy of much research effort to reflect the Hermite symmetry of the lossless network in the block form. The inherent lossless property lies in the energy conservation. Reflecting on the network, this kind of conservation includes not only the magnitude conservation but also the phase condition, which reveals zero interaction term of the energy in depth.

The block network is a very important tool in the analysis of the large complex systems. In reality, it groups the ports of the same function as a whole, which is called as the function division. In this scenario, various properties of the two-port networks can be generalized including the generalized modulus symmetry, generalized spurious reciprocity and constant generalized characteristic phases.

It is worthwhile pointing out that this paper not only concentrates on the theory development, but emphasizes the practical application background. For example, the problems about maximum power output in the large complex systems can be converted into the perfect matching problems or best matching problems. In this way, the origin problems become clearer and more concise. Further results will be reported in other papers.

REFERENCES

1. Abu-Hudrouss, A. M. and M. J. Lancaster, "Design of multiple-band microwave filters using cascaded filter elements," *Journal of Electromagnetic Waves and Applications*, Vol. 23, No. 16, 2109–2118, 2009.
2. Monti, G., F. Congedo, and L. Tarricone, "On the use of a rat-race coupler in the design of a 180° phase shifter," *Journal of Electromagnetic Waves and Applications*, Vol. 23, No. 8, 1201–1210, 2009.
3. Wu, Y., Y. Liu, and S. Li, "A new dual-frequency Wilkinson power divider," *Journal of Electromagnetic Waves and Applications*, Vol. 23, No. 4, 483–492, 2009.
4. Liang, C., *Computational Microwave*, Northwest Electronics Academy Press, Xi'an, 1985.
5. Liang, C., Y. Xie, and B. Guan, *Concise Microwave*, Higher Education Press, Beijing, 2006.
6. Liang, C. and C. Qiu, "Some theorems for the lossless reciprocal network," *Acta Electronica Sinica*, Vol. 19, No. 3, 101–109, Jun. 1991 (in Chinese).
7. Shi, X., C. Liang, and Y. Han, "A useful theorem for a lossless multiport network," *IEEE Trans. Microwave Theory Tech.*, Vol. 43, No. 3, 697–699, Mar. 1995.
8. Liang, C. and L. Li, "Study of characteristic phases of a kind of lossless even symmetry networks," *Acta Electronica Sinica*, Vol. 13, No. 2, 271–274, 2004.
9. Liang, C., L. Li, and X. Shi, "Generalized constraint for a lossless multiport network," *Journal of Electromagnetic Waves and Applications*, Vol. 18, No. 12, 1591–1604, 2004.
10. Heiber, A. L. and R. J. Vernon, "Matching consideration of lossless reciprocal 5-port waveguide junctions," *IEEE Trans. Microwave Theory Tech.*, Vol. 21, No. 4, 547–552, Aug. 1973.
11. Liang, C. and X. Shi, "Generalized matching theory for multi-port network," *Acta Electronica Sinica*, Vol. 12, No. 21, 17–22, 1993 (in Chinese).
12. Butterweck, H. J., "Scattering bounds for the general lossless reciprocal three-port," *IEEE Trans. Circuit Theory*, Vol. 13, No. 2, 290–293, Sep. 1966.