SML - ASSIGNMENT 1

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SML - Assignment - 1

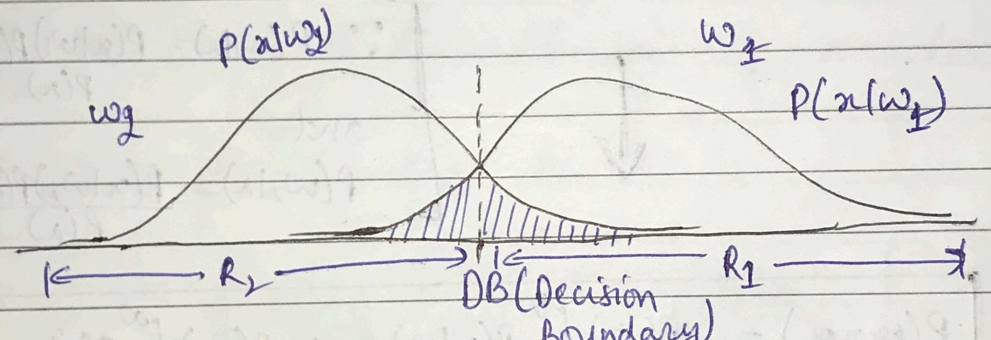
Q.1 $$P(error) = P(\omega_1)\int_{-\inf}^{\theta} p(x|\omega_1)dx + P(\omega_2)\int_{\theta}^{\inf} p(x|\omega_2)dx$$

a)

We have to decide based on the following decision rule :-

Decide ω_1 if $x>\theta$, otherwise decide ω_2

Since, there are only 2 decision classes, we can assume 2 gaussian curves :-



$DB = \theta$

Shaded part in above graph is the error region.

We have 2 regions :- R_1 and R_2

If $x \in \omega_2$ and lies in R_1, then choose ω_1 as $P(\omega_1|x)$ has higher probability. So,

Case 1 :- R_1 : $x \in \omega_2 \rightarrow$ Decide $\omega_1 \rightarrow$ error

Similarly Case 2 :- R_2 : $x \in \omega_1 \rightarrow$ Decide $\omega_2 \rightarrow$ error

Probability of error in case 1 $= \int_{DB=\theta}^{\infty} P(\omega_2|x) P(x) dx$ — ①

Probability of error in case 2 $= \int_{-\infty}^{\theta} P(\omega_1|x) P(x) dx$ — ②

Total error = ① + ②

$$P(\text{error}) = \int_{-\infty}^{\theta} P(\omega_1|x) P(x) dx + \int_{\theta}^{\infty} P(\omega_2|x) P(x) dx$$

$$= \int_{-\infty}^{\theta} \frac{P(x|\omega_1)\, P(\omega_1) \times \cancel{P(x)}}{\cancel{P(x)}} dx + \int_{\theta}^{\infty} \frac{P(x|\omega_2)\, P(\omega_2) \times \cancel{P(x)}}{\cancel{P(x)}} dx$$

$$\left[\because P(\omega_1|x) = \frac{P(x|\omega_1)P(\omega_1)}{P(x)} \text{ and } P(\omega_2|x) = \frac{P(x|\omega_2)P(\omega_2)}{P(x)} \right]$$

$$\downarrow$$

$$\boxed{P(\text{error}) = P(\omega_1)\int_{-\infty}^{\theta} P(x|\omega_1)dx + P(\omega_2)\int_{\theta}^{\infty} P(x|\omega_2)dx}$$

b) To minimize P(error), we'll just differenciate wrt x the above expression and equate it equal to 0.

So, $$\frac{P(\text{error})}{dx} = \frac{d}{dx}\left[P(\omega_1)\int_{-\infty}^{\theta} P(x|\omega_1)dx\right] + \frac{d}{dx}\left[P(\omega_2)\int_{\theta}^{\infty} P(x|\omega_2)dx\right] = 0$$

$$\Rightarrow \underset{\downarrow}{-}P(\omega_1) \times [P(x|\omega_1)] + P(\omega_2) \times [P(x|\omega_2)] = 0$$

−ve sign because of $(-\infty, \theta]$ integrating limits

$\Rightarrow$

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At $x=0$

$$-P(\omega_1)P(0|\omega_1) + P(\omega_2)P(0|\omega_2) = 0$$

$$\boxed{P(0|\omega_1)P(\omega_1) = P(0|\omega_2)P(\omega_2)}$$

Q.2 Cauchy distribution :-

$$P(x|\omega_i) = \frac{1}{\pi b} \times \frac{1}{1+\left(\frac{x-a_i}{b}\right)^2} \qquad i=1,2$$

$$P(\omega_1) = P(\omega_2)$$

To show: $P(\omega_1|x) = P(\omega_2|x)$ if $x = \frac{(a_1+a_2)}{2}$

Consider : $P(\omega_1|x) = P(\omega_2|x)$

$$\Rightarrow \frac{P(x|\omega_1)\cancel{P(\omega_1)}}{\cancel{P(x)}} = \frac{P(x|\omega_2)\cancel{P(\omega_2)}}{\cancel{P(x)}} \qquad \left[\because P(\omega_1) = P(\omega_2) \text{ and } P(a|b) = \frac{P(b|a) \times P(a)}{P(b)}\right]$$

$$\Rightarrow P(x|\omega_1) = P(x|\omega_2)$$

from above given distribution, expand $P(x|\omega_1)$ and $P(x|\omega_2)$

$$\frac{\cancel{1}}{\cancel{\pi} b} \cdot \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2} = \frac{\cancel{1}}{\cancel{\pi} b} \cdot \frac{1}{1+\left(\frac{x-a_2}{b}\right)^2}$$

$$\cancel{1}+\left(\frac{x-a_1}{b}\right)^2 = \cancel{1}+\left(\frac{x-a_2}{b}\right)^2$$

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$$\left(\frac{x-a_1}{b}\right)^2 = \left(\frac{x-a_2}{b}\right)^2 \Rightarrow \frac{x-a_1}{\cancel{b}} = \pm \frac{(x-a_2)}{\cancel{b}}$$

$+$ $-$

$x - a_1 = x - a_2$ ✗
Not possible

$x - a_1 = -x + a_2$

$x = \left[\frac{a_1 + a_2}{2}\right]$ ✓

So $\boxed{x = \frac{a_1 + a_2}{2}}$

The minimum error decision boundary is a point midway $\left[x = \frac{a_1+a_2}{2}\right]$ b/w the peaks of the two distributions, regardless of b.

Q.3

$$P(x|\omega_1) \sim N(0, I)$$

$$P(x|\omega_2) \sim N\left(\begin{bmatrix}1\\1\end{bmatrix}, I\right)$$

$$P(x|\omega_3) \sim 0.5N\left(\begin{bmatrix}0.5\\0.5\end{bmatrix}, I\right) + 0.5N\left(\begin{bmatrix}-0.5\\0.5\end{bmatrix}, I\right)$$

The given ~~proba~~ catagories are equi-probable
So, $P(\omega_1) = P(\omega_2) = P(\omega_3)$

$P(x|\omega)$ can be reduced further $\Rightarrow$

$P(x|\omega_3) \sim 0.5\, N\left(\begin{bmatrix}0\\0.5\end{bmatrix}, 2I\right)$ $\left[\because \text{If } A \sim N(\mu_1,\Sigma_1)+N(\mu_2,\Sigma_2) \text{ then } A \sim N(\mu_1+\mu_2, \Sigma_1+\Sigma_2)\right]$

$P(x|\omega_3) \sim N\left(\begin{bmatrix}0\\0.5\end{bmatrix}, I\right)$ $\left[\because \text{If } A \sim aN(\mu_1,\Sigma_1) \text{ then } A \sim N(\mu_1, a\Sigma_1)\right]$

From question 1, we know

$$P(error) = \sum P(\omega_i)\int P(x|\omega_i)dx$$

So, $P(error) = P(\omega_1)\int P(x|\omega_1)dx + P(\omega_2)\int P(x|\omega_2) + P(\omega_3)\int P(x|\omega_3)$

For minimum probability of error :-

$$\frac{dP(error)}{dx} = 0$$

So,

$$\frac{1}{(2\pi)|\Sigma_1|^{1/2}} e^{-\frac{1}{2}x^T I^{-1} x} + \frac{1}{(2\pi)|\Sigma_2|^{1/2}} e^{-\frac{1}{2}\left(x-\begin{bmatrix}1\\1\end{bmatrix}\right)^T I^{-1}\left(x-\begin{bmatrix}1\\1\end{bmatrix}\right)}$$

$$+ \frac{1}{2\pi|\Sigma_3|^{1/2}} e^{-\frac{1}{2}\left(x-\begin{bmatrix}0\\1/2\end{bmatrix}\right)^T I^{-1}\left(x-\begin{bmatrix}0\\1/2\end{bmatrix}\right)}$$

Simplifying :-

$$x^T x = \left(x-\begin{bmatrix}1\\1\end{bmatrix}\right)^T\left(x-\begin{bmatrix}1\\1\end{bmatrix}\right) + \left(x-\begin{bmatrix}0\\1/2\end{bmatrix}\right)^T\left(x-\begin{bmatrix}0\\1/2\end{bmatrix}\right)^T$$

Let $x = \begin{bmatrix}a\\b\end{bmatrix}$ → Simplifying above equation gives $a = 0.3$ $b = 0.3$

So $\boxed{x = \begin{bmatrix}0.3\\0.3\end{bmatrix}}$

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Q.4 c) $N\left(\mu_1 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \Sigma_1 = \sigma^2 I\right)$ $d=2$

$N\left(\mu_2 = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}, \Sigma_2 = \sigma^2 I\right)$ $\Sigma_1^{-1} = \Sigma_2^{-1} = \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix}$

$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $P(\omega_1) = 1/3$ $P(\omega_2) = 2/3$

Discriminate function equation is given as :-

$$g_i(x) = \ln\left(\frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}}\right) - \frac{1}{2}(x-\mu_i)^T \Sigma_i^{-1}(x-\mu_i) + \ln(P(\omega_i))$$

This term is same, so it can be dropped

$$g_i(x) = -\frac{1}{2}\left[x^T \Sigma_i^{-1} x - \mu_i^T \Sigma_i^{-1} x - x^T \Sigma_i^{-1} \mu_i + \mu_i^T \Sigma_i^{-1} \mu_i \right] + \ln(P(\omega_i))$$

Same term, so drop it

$$g_i(x) = -\frac{1}{2}\left[-2\mu_i^T \Sigma_i^{-1} x + \mu_i^T \Sigma_i^{-1} \mu_i \right] + \ln P(\omega_i)$$

$$\boxed{g_i(x) = \mu_i^T \Sigma_i^{-1} x - \frac{1}{2} \mu_i^T \Sigma_i^{-1} \mu_i + \ln P(\omega_i)}$$

Computing $g_i(x)$ for both the normal distribution:

$$g_1(x) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}^T \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$+ \ln\left(\frac{1}{3}\right)$$

$$g_1(x) = \frac{1}{\sigma^2} - \frac{1}{4\sigma^2} - 1.09 = \frac{3}{4\sigma^2} - 1.09$$

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Similarly, $g_2(x) = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix} \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}^T \begin{bmatrix} 1/\sigma^2 & 0 \\ 0 & 1/\sigma^2 \end{bmatrix} \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$

$+ \ln(2/3)$

$g_2(x) = 0 - \frac{1}{4\sigma^2} \cancel{+} - 0.4$

So, $g_2(x) = -\frac{1}{4\sigma^2} - 0.4$ —(1) $g_1(x) = \frac{3}{4\sigma^2} - 1.09$ —(2)

$\downarrow$

$\cancel{g_2(x)}$ $\cancel{\frac{1}{g_2(x)} =}$

Eliminating σ^2 from (1) and (2)

$-(g_2(x) + 0.4) = \frac{1}{4\sigma^2}$ $\frac{1}{3}(g_1(x) + 1.09) = \frac{1}{4\sigma^2}$

$-3g_2(x) - 1.2 = g_1(x) + 1.09$

$$\boxed{g_1(x) = -3g_2(x) - 2.29}$$

The above equation clearly compares discriminant function of 2 normally distributed curves at $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.