Question 1

Determine which vectors are solutions to a linear system:

$2x_1 - 4x_2 - x_3 = 1$
$x_1 - 3x_2 + x_3 = 1$
$3x_1 - 5x_2 - 3x_3 = 1$

Hint: linear system can have multiple solutions, this problem has multiple correct answers

- ☐ $(3,1,1)$
- ☐ $(3,-1,1)$
- ☐ $(13,5,2)$
- ☐ $(\frac{13}{2},\frac{5}{2},2)$
- ☐ $(17,7,5)$

Method 1: Using Gauss Jordan Elimination by reducing the matrix to reduced row echelon form (RREF)

In Gauss Jordan Elimination you use the elementary row operations (ERO):
1. Swap 2 rows
2. Multiply rows by a non-zero constant
3. Add or subtract the multiple of one row to another

$$\left[\begin{array}{ccc|c} 2 & -4 & -1 & 1 \\ 1 & -3 & 1 & 1 \\ 3 & -5 & -3 & 1 \end{array}\right] \xrightarrow{R_1:\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -2 & -\frac{1}{2} & \frac{1}{2} \\ 1 & -3 & 1 & 1 \\ 3 & -5 & -3 & 1 \end{array}\right] \xrightarrow{R_2:R_2-R_1}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & \frac{3}{2} & \frac{1}{2} \\ 3 & -5 & -3 & 1 \end{array}\right] \xrightarrow{R_1:R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & -\frac{1}{2} \\ 0 & -1 & \frac{3}{2} & \frac{1}{2} \\ 3 & -5 & -3 & 1 \end{array}\right] \xrightarrow{R_3:R_3-3R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & -\frac{1}{2} \\ 0 & -1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -5 & \frac{15}{2} & \frac{5}{2} \end{array}\right] \xrightarrow{R_3:R_3-5R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & -\frac{1}{2} \\ 0 & -1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2:-R_2}$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \end{array}$$

$$x_1 - \frac{7}{2}x_3 = -\frac{1}{2} \qquad x_2 - \frac{3}{2}x_3 = -\frac{1}{2}$$
$$x_1 = -\frac{1}{2} + \frac{7}{2}x_3 \qquad x_2 = -\frac{1}{2} + \frac{3}{2}x_3$$

$x_3 \in R$ ← This means that X_3 is any real number

Method #2: Inverse Matrix Method or Matrix inversion Method

$$[A]\{x\} = \{b\}$$
$$\{x\} = [A]^{-1}\{b\}$$

$[A] =$ matrix

$\{b\} =$ vector (can be a matrix if you are solving for multiple systems of linear equations simultaneously)

Plugging in values for X_3

$x_3 = 1$
$$x_1 = -\frac{1}{2} + \frac{7}{2} = \frac{6}{2} = 3$$
$$x_2 = -\frac{1}{2} + \frac{3}{2} = \frac{2}{2} = 1$$
$(3,1,1)$

$x_3 = 2$
$$x_1 = -\frac{1}{2} + \frac{14}{2} = \frac{13}{2}$$
$$x_2 = -\frac{1}{2} + \frac{6}{2} = \frac{5}{2}$$
$\left(\frac{13}{2}, \frac{5}{2}, 2\right)$

$x_3 = 5$
$$x_1 = -\frac{1}{2} + \frac{35}{2} = \frac{34}{2} = 17$$
$$x_2 = -\frac{1}{2} + \frac{15}{2} = 7$$
$(17,7,5)$

Note: Gauss-Jordan Elimination is not the same as Gaussian Elimination since Gaussian Elimination transforms/reduces the augmented matrix to row echelon form (REF)

Question 2

Which option is equal to the DOT product $u \cdot v$ of vectors

$$u = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$$

- ○ 26
- ○ -8
- ○ 24

$$(3 \cdot 2) + (1 \cdot 1) - (4 \cdot 4) = u \cdot v$$
$$6 + 2 + \; - 16 = u \cdot v$$
$$-8 = u \cdot v$$

Question 3

Is the matrix A invertible

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

- ○ True
- ○ False

A square matrix is invertible if and only if:
1. its determinant is non-zero
2. it row-reduces to the identity matrix
3. it has full rank (all rows & columns are linearly independent)

Method #1: calculating the determinant

$$det(A) = ad - bc \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$det(A) = 6 - 6$$
$$det(A) = 0$$

Method 2: Using Gauss Jordan Elimination by reducing the matrix to reduced row echelon form (RREF)

$$\left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 3 & -6 & 0 & 1 \end{array}\right] \xrightarrow{R_1 : -R_1}$$

$$\left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 3 & -6 & 0 & 1 \end{array}\right] \xrightarrow{R_2 : R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

Question 4

Let A be the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

Which answer is correct for the A^3 ?

- ○ $\begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix}$
- ○ $\begin{bmatrix} 11 & -15 \\ -30 & 41 \end{bmatrix}$
- ✗ $\begin{bmatrix} 41 & 15 \\ 30 & 11 \end{bmatrix}$

$$\underbrace{\underset{2 \times 2}{\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}} \cdot \underset{2 \times 2}{\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}}}_{2 \times 2} = \underset{2 \times 2}{\begin{bmatrix} 3 \cdot 3 + 1 \cdot 2 = 11 & 3 \cdot 1 + 1 \cdot 1 = 4 \\ 2 \cdot 3 + 1 \cdot 2 = 8 & 2 \cdot 1 + 1 \cdot 1 = 3 \end{bmatrix}} = A^2$$

final matrix Shape

$$\begin{bmatrix} 11 & 4 \\ 8 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 \cdot 3 + 4 \cdot 2 = 41 & 11 \cdot 1 + 4 \cdot 1 = 15 \\ 8 \cdot 3 + 3 \cdot 2 = 30 & 8 \cdot 1 + 3 \cdot 1 = 11 \end{bmatrix} = A^3$$

If A and B are matrices such that AB is defined, then it is true that $(AB)^T = A^T B^T$

- ○ True
- ✗ False

$$(AB)^T = B^T \cdot A^T \longleftarrow$$ order in linear algebra matters