Question 1

Compute eigenvalues and eigenvectors for the matrix A and its power of 2

$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}$

When A has eigenvalues λ1 and λ2, then A^2 has eigenvalues $(\lambda 1)^2$ and $(\lambda 2)^2$?

- [x] True
- [ ] False

Conclusion

- Both A and A^2 share the same eigenvectors but they don't share the same eigenvalues. The eigenvalues of A^2 are squares of those of A.
- Both matrices apply their transformations along the same line/plane of the of the eigenvectors that are defined. The matrices stretch or compress the eigenvectors by different amounts.

In this problem we are trying to compute the eigenvalues and the corresponding eigenvectors by hand.

Eigenvalues and eigenvectors arise from linear transformations represented by matrices. The eigenvector **only** stretches or compresses the matrix. It does not rotate or change the direction of the matrix.

Note: A vector in linear algebra is a arrow that starts at the origin and goes through the vectors point that is how a vector has direction

<u>**Systems of linear equations vs. Eigenvectors**</u>

The eigenvalue being applied to a matrix is not the same as a system of linear equations. Eigenvalues arise from studying matrix transformations (aka linear transformations), specifically those that scale a vector without changing its direction. A system of linear equations does not focus on matrix transformations since it is solving for a vector x that satisfies Ax=b, for a given right hand side b.

Eigenvalues tell us how a matrix stretches space in special directions. A system of equations tells us how a matrix transforms one specific input to a output.

The matrix form of a system of linear equations.

$$\boxed{A\vec{x} = \vec{b}}$$

Called the System of Linear Equations because:

- Matrix A only contains the coefficients of the variables

The defining equation for eigenvalues and eigenvectors

$$\boxed{A\vec{v} = \lambda\vec{v}}$$

Important points to note

- Direction stays the same
- Length changes by a factor of lambda
- When we look at A times vector v alone (without the equal sign) then in most vectors the formula A times vector v would change the length and direction of vector v
- We are trying to find a non-zero solution for vector v which can only be solved if the determinant is zero

$$det(A - \lambda I) = 0$$

- $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

Note: A vector in linear algebra is a arrow that starts at the origin and goes through the vectors point. That is how a vector has direction.

How to solve for Eigenvalues and Eigenvectors:

1. Compute eigenvalues of A by solving $det(A - \lambda I) = 0$
2. Find eigenvectors for each eigenvalue
3. Use the rule: eigenvalues of A squared are lambda squared

Step 1: Solving for eigenvalues

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad A - \lambda I = \begin{bmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{bmatrix}$$

Determinant formula for 2 by 2 matrix

$$det(A) = AD - BC$$

$$A = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$det(A - \lambda I) = (-1 - \lambda)(-\lambda) - 3 \cdot 2 = \lambda + \lambda^2 - 6 = 0$$

The equation should equal this value

$$\lambda^2 + \lambda - 6 = 0$$
$$(\lambda + 3)(\lambda - 2) = 0$$
$$\lambda = -3$$
$$\lambda = 2$$

Step 2: Find eigenvectors for each eigenvalue

we are solving for a homogeneous system of linear equations

$$(A - \lambda I)\vec{v} = \vec{0}$$

Remember, A homogeneous system of linear equations is a system where all the constant terms (from the right side of the equation) are zero

$$\lambda = -3$$

$$A + 3I = \begin{bmatrix} -1+3 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

$$A\vec{v} + 3I\vec{v} = \vec{0}$$

$$\begin{array}{cc} x & y \end{array}$$
$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 3 & 0 \end{array}\right]$$

$$2x + 3y = 0$$
$$2x = -3y$$
$$x = -\frac{3}{2}y$$

Since both rows are the same I can reduce the system to one equation without the need to reduce the matrix to RREF

General Solution for eigenvector

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}y \\ y \end{bmatrix} = y\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} = 2\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Replace y with 2 to remove fraction

The eigenvector is any scalar multiple (only multiplication) of:

$$\vec{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\lambda = 2$$

Formula: $A\vec{v} - 2I\vec{v} = 0$

$$\vec{v} = \begin{bmatrix} -1-2 & 3 \\ 2 & -2 \end{bmatrix} = \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{R_1: \frac{R_1}{-3}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right] \xrightarrow{R_2: R_2 - 2R_1} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$$x - y = 0 \qquad x = 1$$
$$x = y \qquad y = 1$$

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

When a matrix transforms an eigenvector, the vector only gets stretched or shrunk. It does not change direction. A eigenvalue can tell you how much a matrix is stretched, flipped, shrunk, or it could stay the same.

Flipped	Shrunk	Stays the same	Stretched
$\lambda < 0$	$0 < \lambda < 1$	$1 = \lambda$	$\lambda > 1$

Finding eigenvalues for A^2

$$A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix} \quad -\lambda I = \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix} \quad A^2 - \lambda I = \begin{bmatrix} 7-\lambda & -3 \\ -2 & 6-\lambda \end{bmatrix}$$

$$det\left(A^2 - \lambda I\right) = (7-\lambda)(6-\lambda) - (-3)(-2) = 42 - 7\lambda - 6\lambda + \lambda^2 - 6 = 36 - 13\lambda + \lambda^2$$

$$\lambda^2 - 13\lambda + 36 = 0$$
$$(\lambda - 9)(\lambda - 4) = 0$$

$$\lambda = 9$$
$$\lambda = 4$$

← Eigenvalues

Finding eigenvector when eigenvalue is $\lambda = 9$ Formula: $A\vec{v} - 9I\vec{v} = 0$

$$\begin{bmatrix} 7-9 & -3 \\ -2 & 6-9 \end{bmatrix} = \begin{array}{c} \begin{matrix} x & y & \end{matrix} \\ \left[\begin{array}{cc|c} -2 & -3 & 0 \\ -2 & -3 & 0 \end{array}\right] \end{array}$$

← Since both rows are the same I can reduce the system to one equation without the need to reduce the matrix to RREF

$$-2x - 3y = 0$$
$$-2x = 3y$$
$$x = -\frac{3}{2}y$$

← To eliminate the denominator make $y=2$

$$x = -3$$
$$y = 2$$

$$\vec{v} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Finding eigenvector when eigenvalue is $\lambda = 4$ Formula: $A\vec{v} - 4I\vec{v} = 0$

$$\begin{bmatrix} 7-4 & -3 \\ -2 & 6-4 \end{bmatrix} = \begin{array}{c} \begin{matrix} x & y & \end{matrix} \\ \left[\begin{array}{cc|c} 3 & -3 & 0 \\ -2 & 2 & 0 \end{array}\right] \end{array}$$

$$\left[\begin{array}{cc|c} 3 & -3 & 0 \\ -2 & 2 & 0 \end{array}\right] \xrightarrow{R_1: \frac{R_1}{3}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right] \xrightarrow{R_2: R_2 + 2R_1} \begin{array}{c} \begin{matrix} x & y & \end{matrix} \\ \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$$x - y = 0$$
$$x = y$$
$$x = 1$$
$$y = 1$$

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Question 2

Compute the eigenvalues of matrix A and its inverse

$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{-3}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$

If A has eigenvalues λ1 and λ2, then its inverse A^{-1} has eigenvalues :

○ λ1 and λ

✓ $\frac{1}{\lambda 1}$ and $\frac{1}{\lambda 2}$

○ (-λ1) and (-λ2)

$$A - I\lambda = \begin{bmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

$$det(A - I\lambda) = \lambda(3-\lambda) - 2 \cdot 2 = -3\lambda + \lambda^2 - 4$$
$$= \lambda^2 - 3\lambda - 4$$

$$\lambda^2 - 3\lambda - 4 = 0$$
$$(x-4)(x+1) = 0$$
$$x = 4$$
$$x = -1$$

$$A^{-1} - I\lambda = \begin{bmatrix} -\frac{3}{4} - \lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{bmatrix}$$

$$det(A^{-1} - I\lambda) = 0$$
$$\left(-\frac{3}{4} - \lambda\right)(-\lambda) - \frac{1}{2} \cdot \frac{1}{2} = 0$$
$$\frac{3}{4}\lambda + \lambda^2 - \frac{1}{4} = 0$$
$$\lambda^2 + \frac{3}{4}\lambda - \frac{1}{4} = 0$$

$$\lambda^2 + 3\lambda - 1 = 0$$
$$4. -1 = -4$$
$$4 - 1 = 3$$
$$4\lambda^2 + 4\lambda - \lambda - 1 = 0$$
$$4\lambda(\lambda+1) - 1(\lambda+1) = 0$$
$$(4\lambda - 1)(\lambda + 1) = 0$$
$$\lambda = -1$$
$$\lambda = \frac{1}{4}$$

Question 3

1 pts

Let B be an ordered basis for R^2 and linear transformation to be "identity transformation" $L(v) = v$ from old basis B to new basis C. Which of the following answers can be new basis C if B is given as a set below?

$B = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

○ $C = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

✗ $C = \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$

○ $C = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$

A ordered basis is just a basis with a specified order (the order of the vectors).

A basis of a vector space:

1. Spans the vector space (R^n)
2. Is linearly independent
3. Has exactly n elements where n is the dimension of the space.

The identity transformation maps every vector to itself.

Question 4

1 pts

Let A be 5x5 matrix. Which of the following criteria will ensure that A is diagonalizable over real numbers?

○ The determinant d of A is not zero d≠0

✓ A has 5 distinct eigenvalues

○ The rows of A are linearly independent

A square matrix is diagonalizable if it is similar to the diagonal matrix (that is, having numbers only on the diagonal of the matrix and no where else) This means:

$$A = PDP^{-1}$$

- D is the diagonal matrix
- P is a matrix made up of the eigenvectors of A (as columns)

Question 5

1 pts

If matrix of transformation A is both invertible and orthogonally diagonalizable, then A^{-1} is orthogonally diagonalizable

✓ True

○ False

Two vectors are orthogonal if they are at a right angle (90 degrees) to each other. Another way to word this statement is "two vectors are orthogonal if their dot product is zero". Both statements are not two separate conditions, they are two ways of saying the same thing.