

Question 1

Determine which vectors are solutions to a linear system:

$$\begin{aligned} 2x_1 - 4x_2 - x_3 &= 1 \\ x_1 - 3x_2 + x_3 &= 1 \\ 3x_1 - 5x_2 - 3x_3 &= 1 \end{aligned}$$

Hint: linear system can have multiple solutions, this problem has multiple correct answers

- (3, 1, 1)
- (3, -1, 1)
- (13, 5, 2)
- $(\frac{13}{2}, \frac{5}{2}, 2)$
- (17, 7, 5)

Method 1: Using Gauss Jordan Elimination by reducing the matrix to reduced row echelon form (RREF)

In Gauss Jordan Elimination you can only:

1. Swap 2 rows
2. Multiply rows by a non-zero constant
3. Add or subtract the multiple of one row to another

$$\left[\begin{array}{ccc|c} 2 & -4 & -1 & 1 \\ 1 & -3 & 1 & 1 \\ 3 & -5 & -3 & 1 \end{array} \right] \xrightarrow{R_1: \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -2 & -\frac{1}{2} & \frac{1}{2} \\ 1 & -3 & 1 & 1 \\ 3 & -5 & -3 & 1 \end{array} \right] \xrightarrow{R_2: R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -2 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & \frac{3}{2} & \frac{1}{2} \\ 3 & -5 & -3 & 1 \end{array} \right]$$

Method #2: Inverse Matrix Method or Matrix inversion Method

$$\begin{aligned} [A]\{x\} &= \{b\} \\ \{x\} &= [A]^{-1}\{b\} \end{aligned}$$

$[A]$ = matrix

$\{b\}$ = vector (can be a matrix if you are solving for multiple systems of linear equations simultaneously)

$$\left[\begin{array}{ccc|c} 1 & -2 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & \frac{3}{2} & \frac{1}{2} \\ 3 & -5 & -3 & 1 \end{array} \right] \xrightarrow{R_1: R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & -\frac{1}{2} \\ 0 & -1 & \frac{3}{2} & \frac{1}{2} \\ 3 & -5 & -3 & 1 \end{array} \right] \xrightarrow{R_3: R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & -\frac{1}{2} \\ 0 & -1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -5 & \frac{15}{2} & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & -\frac{1}{2} \\ 0 & -1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -5 & \frac{15}{2} & 1 \end{array} \right] \xrightarrow{R_3: R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & -\frac{1}{2} \\ 0 & -1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_2: -R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{aligned} x_1 - \frac{7}{2}x_3 &= -\frac{1}{2} & x_2 - \frac{3}{2}x_3 &= -\frac{1}{2} \\ x_1 &= -\frac{1}{2} + \frac{7}{2}x_3 & x_2 &= -\frac{1}{2} + \frac{3}{2}x_3 \end{aligned}$$

$x_3 \in \mathbb{R}$ ← This means that x_3 is any real number

Plugging in values for X_3

$$\begin{aligned} x_3 &= 1 & x_3 &= 2 & x_3 &= 5 \\ x_1 &= -\frac{1}{2} + \frac{7}{2} = \frac{6}{2} = 3 & x_1 &= -\frac{1}{2} + \frac{14}{2} = \frac{13}{2} & x_1 &= -\frac{1}{2} + \frac{35}{2} = \frac{34}{2} = 17 \\ x_2 &= -\frac{1}{2} + \frac{3}{2} = \frac{1}{2} = 1 & x_2 &= -\frac{1}{2} + \frac{6}{2} = \frac{5}{2} & x_2 &= -\frac{1}{2} + \frac{15}{2} = 7 \end{aligned}$$

$$(3, 1, 1) \quad \left(\frac{13}{2}, \frac{5}{2}, 2\right) \quad (17, 7, 5)$$

Question 2

Which option is equal to the DOT product $u \cdot v$ of vectors

$$u = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$$

- 26
- 8
- 24

$$(3 \cdot 2) + (1 \cdot 1) - (4 \cdot 4) = u \cdot v$$

$$6 + 2 + -16 = u \cdot v$$

$$-8 = u \cdot v$$

Question 3

Is the matrix A invertible

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

- True
- False

A square matrix is invertible if and only if:

1. its determinant is non-zero
2. it row-reduces to the identity matrix
3. it has full rank (all rows & columns are linearly independent)

Method #1: calculating the determinant

$$\det(A) = ad - bc$$

$$\det(A) = 6 - 6$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = 0$$

Method 2: Using Gauss Jordan Elimination by reducing the matrix to reduced row echelon form (RREF)

$$\left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 3 & -6 & 0 & 1 \end{array} \right] \xrightarrow{R_1: -R_1}$$

$$\left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 3 & -6 & 0 & 1 \end{array} \right] \xrightarrow{R_2: R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{array} \right]$$

Question 4

Let A be the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

Which answer is correct for the A^3 ?

$\begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix}$

$\begin{bmatrix} 11 & -15 \\ -30 & 41 \end{bmatrix}$

$\begin{bmatrix} 41 & 15 \\ 30 & 11 \end{bmatrix}$

$$\left[\begin{array}{cc} 3 & 1 \\ 2 & 1 \end{array} \right] \cdot \left[\begin{array}{cc} 3 & 1 \\ 2 & 1 \end{array} \right] = \left[\begin{array}{cc} 3 \cdot 3 + 1 \cdot 2 = 11 & 3 \cdot 1 + 1 \cdot 1 = 4 \\ 2 \cdot 3 + 1 \cdot 2 = 8 & 2 \cdot 1 + 1 \cdot 1 = 3 \end{array} \right] = A^2$$

2×2 2×2 2×2

2×2 2×2 2×2

final matrix shape

$$\left[\begin{array}{cc} 11 & 4 \\ 8 & 3 \end{array} \right] \cdot \left[\begin{array}{cc} 3 & 1 \\ 2 & 1 \end{array} \right] = \left[\begin{array}{cc} 11 \cdot 3 + 4 \cdot 2 = 41 & 11 \cdot 1 + 4 \cdot 1 = 15 \\ 8 \cdot 3 + 3 \cdot 2 = 30 & 8 \cdot 1 + 3 \cdot 1 = 11 \end{array} \right] = A^3$$

If A and B are matrices such that AB is defined, then it is true that

$$(AB)^T = A^T B^T$$

- True
- False

$$(AB)^T = B^T \cdot A^T \leftarrow \text{order in linear algebra matters}$$