BAYES' THEOREM:

$$Pr(A|B) = \frac{Pr(B|A)Pr(A)}{Pr(B)} = \frac{Pr(B|A)Pr(A)}{Pr(B|A)Pr(A) + Pr(B|A^C)Pr(A^C)}$$

Note: A^C = ~A in the applet

EQUAL PROBABILITIES FOR Pr(B|A) and Pr(B|~A)
First, we will look at how the probability of A given B, Pr(A|B), changes in relation to the probability of A, Pr(A), when Pr(B|A) = Pr(B|~A).

1. Start by setting "Pr(B|A)" to 0.50, as well as "Pr(B|~A)" to 0.50, if they are not already set to those values. Observe the plot to the right, what sort of relationship do you notice between Pr(A) and P(A|B).

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2. Now change "Pr(B|A)" and "Pr(B|~A)" both to 0.73. Did the relationship on the graph change at all after both were changed? If yes, explain.

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   Big Idea: When the probability of B is the same whether or not A has occurred, Pr(B|A) = Pr(B|~A), then there is a <u>perfect linear relationship</u> between Pr(A) and Pr(A|B).

3. In addition to the the perfect linear relationship, can you find the equation that would relate Pr(A) and Pr(A|B) when Pr(B|A) = Pr(B|~A)?

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4. What does the equation you found in exercise 3 imply about A and B?

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   Big Idea: When Pr(B|A) = Pr(B|~A), there is not only a perfect linear relationship between Pr(A|B) and Pr(A), but rather Pr(A|B) = Pr(A). This means that the probability of A occurring is the same regardless of whether or not B occurred, just as the probability of B occurring was the same regardless of whether or not A occurred. Thus, in this case, the events A and B are <u>independent</u>.

Pr(B|A) IS HIGHER THAN Pr(B|~A)
Now we will look at how Pr(A) and Pr(A|B) are related when the probability of B given A occurred is higher than the probability of B given A did not occur.

5. Can you think of two events (A and B) in which this would be true? Write down two events whereas event B has a higher probability of occurring given that A has occurred compared to when A did not occur.

6. Change both "Pr(B|A)" and "Pr(B|~A)" to 0.20 and once again notice the linear relationship on the plot to the right.

7. Change "Pr(B|A)" to 0.70. What sort of relationship do Pr(A) and Pr(A|B) now have? Is this what you expected?

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8. Now update the probability of B given A, Pr(B|A), to many values, slowly increasing it to 1. (Plug-in 0.30, 0.40, 0.50, 0.60, 0.70, 0.80, 0.90, and 1) What happens to the relationship between Pr(A) and Pr(A|B) as Pr(B|A) increases to be far greater than Pr(B|~A)?

Big Idea: When Pr(B|A) is greater than Pr(B|~A), Pr(A) and Pr(A|B) have a logarithmic relationship. The relationship becomes more concave downward as the difference between Pr(B|A) and Pr(B|~A) grows. The Pr(A|B) grows at a faster rate for changes in Pr(A) when Pr(A) is smaller as compared to when it is closer to 1.

Pr(B|A) IS LOWER THAN Pr(B|~A)
The next step will be to look at the relationship between Pr(A) and Pr(A|B) when the probability of B given that A occurred is less than the probability of B given that A did not occur.

9. Can you think of two events (A and B) in which this would be true? Write down two events whereas event B has a lower probability of occurring given that A has occurred compared to when A did not occur.

10. What type of relationship between Pr(A) and Pr(A|B) do you expect that we will find when change Pr(B|A) to be less than Pr(B|~A)?

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11. Proceed to change both "Pr(B|A)" and "Pr(B|~A)" to 0.80 and verify that the relationship between Pr(A) and Pr(A|B) is linear.

12. Change "Pr(B|A)" to 0.30. What sort of relationship do Pr(A) and Pr(A|B) now have? Is this what you predicted above?

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13. Now update the probability of B given A, Pr(B|A), to many values, slowly decreasing it toward 0. (Plug-in 0.70, 0.60, 0.50, 0.40, 0.30, 0.20, 0.10 and 0.01) What happens to the relationship between Pr(A) and Pr(A|B) as Pr(B|A) decreases to be far less than Pr(B|~A)?

Big Idea: When Pr(B|A) is lower than the Pr(B|~A), Pr(A) and Pr(A|B) are exponentially related. The relationship becomes more concave upward as the difference between Pr(B|A) and Pr(B|~A) increases. Pr(A|B) grows at a slower rate as Pr(A) increases when Pr(A) is small, compared to when Pr(A) is close to 1.

DISEASE TESTING EXAMPLE

Let's look at a practical example where Bayes' Theorem can be used in the real world. Suppose someone is walking down the street and the Department of Public Health is giving free tests for AIDS. The test being administered is 95.3% accurate, wherein one will test positive 95.3% of the time if one has the disease. Also, the test will only give a positive result if one does not have AIDS 8.9% of the time.

14. The person decides to get tested and receives positive test results for AIDS, but what is the actual probability that he or she has AIDS? Write your prediction below.

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Let A = the event that the person has AIDS; B = the event that the person tests positive for AIDS

15. Using the defined events above, how would you write the probability of testing positive given that one has AIDS? Also, use the paragraph description above to determine the numeric value of this probability statement. (*Ex. Pr(B) = 0.65*)

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16. Using the defined events above, how would you write the probability of testing positive given that one does not have AIDS, also known as the false positive rate? Also, use the paragraph description above to determine the numeric value of this probability statement. (*Ex. Pr(B) = 0.65*)

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17. Now using the values you found in exercises 10 and 11, change the values of "Pr(B|A)" and "Pr(B|~A)" to the appropriate values.

18. Look at the plot to the right. What is the form of the relationship? Is this what you expected from what you learned above?

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19. What would Pr(A) represent in the context of this problem?

20. What would Pr(A|B) represent in the context of this problem?

21. Now, assume the person being tested had no prior knowledge that he or she would be any more likely to have AIDS than anyone else. This would make the chances of this person having AIDS equal to the proportion of Americans that have AIDS, 0.0040. With this prior probability, Pr(A) = 0.0040, use the plot to roughly estimate what the updated probability of this person having AIDS is, given a positive test result (Pr(A|B)).

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22. Perform the exact calculation using the Bayes' Theorem equation to find the actual value of Pr(A|B) in this example. Compare this value to the one you obtained in exercise 21, it should be fairly close.

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23. Is the values you found in exercise 22 what you predicted in exercise 14? If you did not predict the value correctly, can you explain any differences between your prediction and the actual result? Why do you think the difference is?

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Big Idea: The probability of an event A, given the occurrence of another event B, is largely dependent on the inverse probabilities of B given A and B given not A. Also, Pr(A|B) depends on the value of Pr(A). Thus, even with a very accurate test, the actual probability of having AIDS given that you test positive can be very low if the prior probability of having AIDS, Pr(A), was low to begin with.

24. Now, go to 0.80 on the y-axis, what would this value represent in the context of this problem?

25. What prior probability of having AIDS, Pr(A), would the person have needed to have in order to have a 80% chance of having AIDS given a positive test? (*Hint: Use the plot to estimate*)

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26. Say the test for AIDS became more accurate, increasing its accuracy rate to 100%. Change "Pr(B|A)" to 1, and note how the curve changed. Is a higher or lower prior probability of having AIDS now needed to obtain an updated probability of having AIDS equalling 0.80, Pr(A|B) = 0.80?

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27. What if the test was less accurate? Change "Pr(B|A)" to 0.80, representing 80% accuracy. How does this change the prior probability of having AIDS needed to obtain an updated probability of having AIDS equalling 0.80?

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28. Now play around with different values of Pr(B|~A), higher and lower than 0.089 and note how the plot changes. Be sure to understand what those values would mean in context.

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Big Idea: Bayes' Theorem is useful in finding inverse probabilities. If we know the probability of B given that A occurred, then we can find the probability of A given that B occurred. When using Bayes' Theorem, we must have some prior knowledge of the probability of A, which allows us to find the updated probability of A, with the addition of more knowledge of B's occurrence. The use of additional information to update previous knowledge is the basis for Bayesian Inference, and is why Bayes' Theorem is so critical in statistics.