A Crusonia-Plant Model of Capital: Consideration and Criticism

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Abstract. The Crusonia plant is an imaginary perennial plant that some agent can use as a permanent reservoir of utility.

Keywords: Capital; Intertemporal Exchange

J.B. Clark (1907: 354): "What we mean by capital is a mass of things belonging to a certain genus."

F. Machlup (1935: 580): "There was and is always the choice between maintaining, increasing, or consuming capital... Capital is not necessarily perpetual."

Friday's choice of how to harvest Crusonia is a sequence of dovetailing actions. It is a policy!

The Crusonia plant is a tool for vividly and clearly thinking about how human beings make the best use of the capital at their disposal through the analogy of capital as some kind of perennial plant. The stocks of durable and intermediate goods that constitute an economy's capital embody alternatively possible streams of useful services across time. The economic problem that is peculiar to the use of capital is thus the problem of choosing the most useful of those alternative streams. In addition, one of the essential attributes of capital is that it can permanently yield a stream of services at a specific level depending on the size and composition of the underlying stock of goods. In the real world, people use capital accounting, which entirely separates their capital's yield from its underlying substance, to think about the throng of possibilities that their capital's alternative streams present and thereby choose the most useful stream. The Crusonia plant models this financial view of capital by representing somebody's capital as a homogeneous plant that that person draws useful services from by harvesting its mass. The Crusonia plant is therefore an analogy for capital as a self-perpetuating fund that can be readily allocated to satisfy any demand.

1 A Literature Review of Fanciful Economic Botany

The Crusonia plant is an analogy that has a long history and it has been applied to the study of capital for both pedagogical and polemical motivations.

The Crusonia-plant model probably has a direct progenitor in John Bates Clark's (1908: 313-4) use of the analogy of a managed forest when discussing his argument that capital synchronized consumption and production. Clark (1894: 65; 1895: 257; 1907: 354-5) famously argued that capital proper, or pure capital as he referred to it, abides within an abstract fund that can maintain its productive capabilities while persistently changing its material embodiment. He asked his readers to imagine a forest whose owner managed so that it could provide a constant stream of firewood. Across each period of time, that owner plants and harvests the same number of trees, such that, although individual trees grow and are eventually harvested, the forest can perpetually maintain a constant size.[1] Clark argued that the owner's maintenance of his forest at that constant size

[1] J.B. Clark (1908: 313): "Again, let a forest twenty acres in extent suffice to furnish fire-wood for a family. A tree will mature in twenty years; and the forest must be kept intact, in point of size and maturity, or the supply of wood will fail. Each year we plant a row of trees along one side of the forest, and cut a row from the other. The planting and the cutting are, in a way, simultaneous. We do not burn to-day the tree that we plant to-day; but we do burn a tree, the consuming of which is made practicable by to-day's planting."

expunged any temporal concerns from its management, such as the period of production. For the owner, the amount of time that a tree takes to grow is an irrelevant technical datum because trees of the desired size are ready at hand each period (Cohen 2008: 159-60; Skousen 1990: 27-9).[2] In this manner, the forest, as a fund that yields a permanent flow of wood, synchronizes its owner's production and consumption such that the period of production becomes a technical datum irrelevant to the owner's economic problem. For the sake of providing a complete account of Clark's analogy, it must be noted that it is only valid in a stationary state and Clark's contemporary Frank Taussig (1908), in particular, provides an insightful critique of the theory behind the analogy.

It was Frank Knight who invented the analogy of the Crusonia plant as we know it today, and he deployed that analogy in order to make the essential conditions that he thought characterized sound capital theory more evident. Knight (1933: 335-5; 1936: 444-5; 1944: 31) argued that an economy's capital emerges from the capitalization of various productive instruments in terms of perpetual rates of return and that those values made otherwise dissimilar investment opportunities commensurate to each other. In his analysis of capital, Knight (1936: 460) explicitly followed in Clark's intellectual footsteps and characterized capital as essentially a permanent fund of value: "a continuous organize whole, a fund measured in value units, though at any moment it is largely (not entirely) embodied in things of a sort which more or less regularly wear out, are used up or become obsolete, and are replaced by other items of the same or different disposition."[3] Knight (1935b: 11) first used a botanical analogy in his discussion of capital when he compared "the total social stock of owned economic agencies" to a plant, and Knight (1935c: 86) also mentions a "social plant" that presumably also represents its society's economic agencies. Knight (1944: 30) later made that analogy more explicit when described a society "living on the natural growth of some perennial which grows at a constant (geometric) rate, except as new tissue is cut away for consumption."[4] That society then manages its Crusonia plant by choosing its rate of consumption. Knight (1935b: 11 15, and 1944: 30-1) assumed

[2] Clark (1893: 308): "Goods have periods of production. Capital has no periods, but acts incessantly."

[3] In article about Knight's career teaching economics courses at Chicago University, D. Patinkin noted that Knight's analysis of capital built on Clark's prior theorizing: "In his more formal treatment of capital theory, Knight—like J. B. Clark before him (1899)—emphasized the basic distinction between specific capital goods in the concrete which are periodically worn out and reproduced, and the stock of capital in the abstract—which is permanent, and which with the exception of a few historical periods (for example, periods of prolonged warfare) has been continuously growing over the course of human history."

[4] Unlike in subsequent iterations on the analogy, Knight titled the society living off of the plant's growth, rather than the plant itself, 'Crusonia.' To my knowledge the term 'Crusonia' was first invented by the English radical Thomas Spence (1782) to describe the island that Robinson Crusoe was marooned on. Within Spence's fictional history, Crusonia was given its name in honor of Crusoe later inhabitants of the island.

that the chosen rate would only exceed the plant's growth rate in the conditions of a decadent society that was beyond the purview of economics proper.

Knight wielded the Crusonia plant as a sharp tool to cut to the heart of the economic problems of capital by abstracting away from the technical problem of combining heterogeneous resources into finished products and focusing on the problem of managing capital. The analogy captures two key propositions that Knight argued must inform sound capital theory: 1) That production and consumption, at least from an economic point of view, necessarily occurred simultaneously; 2) That the only relevant magnitudes for economic choice is a flow of pure services, or consumptive income; and 3) That, at least from a purely economic point of view, all productive agencies could be treated as capital. For the sake of brevity in my main text, I discuss Knight's views on those propositions in footnotes.[5] [6] [7] The Crusonia plant captures Knight's first proposition because, as a fund, the Crusonia plant yields its services in the same period of time in

[5] According to Knight, the simultaneity of production and consumption followed from a sound economic point of view, which he argued could only analyze an economy in stationary conditions. Knight (1933: 340) argued that their simultaneity was therefore a matter of sound definition: "Defining production and consumption correctly, from the stationary economy standpoint, as respectively the enjoyment and the rendering of enjoyable service, the two are equal and simultaneous." Knight (1934: 276) even argued that their simultaneity was self evident: "Viewed in economic terms, production means the rendering of services, and it is self-evident that a service can only be produced when it is rendered, and only enjoyed or consumed at that same instant." Knight (1936: 456) then reiterated his argument that economic analysis was restricted to a stationary state: "In a stationary economy what is produced and consumed is *services*, and all production of things constitutes *replacement* and is a part of the production of the services *simultaneously* consumed. To go back to Clark's managed forest, Knight argued that production only occurs when wood is harvested for use, with that wood being the service that the forest yields. The trees' growth over time should not be considered production, according to Knight, because their growth only maintains the forest's size, it does not yield a service.

[6] F. Knight (1934: 260) asserts the flow of pure services as being the only relevant magnitude for economic choice: "The only primary value magnitude possible for economic thought is consumptive income, a pure service, a pure intangible, a flow, at some intensity, for some interval." Knight (1935b: 12-3) restates this claim: "Perpetual service income is the primary magnitude in economic analysis."

[7] Knight argued that all productive agencies, or factors of production as we might call them today, were economically equivalent and were all subsumed within the abstract fund that embodied a society's productive capabilities. Classical economics had taught that all agencies could be classified as either land, labor, or capital. Knight condemned that schema as entirely incoherent or as at least irrelevant to economic analysis. Knight (1934: 263-4f) wrote: "The point to be noted here is simply that as regards economic—not technical—role in production and distribution, no classification of productive factors has any validity." Later, Knight (1935b: 22) makes the even stronger argument: "'Capital' is the ideal 'factor of production' and is 'theoretically' (meaning in a society completely economic) all-inclusive." Don Paktinkin (1973: 794) noted that "Knight rallied against the classical 'trinity of factors of production'" in the classroom and that Knight's tendency to treat all productive

which they are consumed, thereby synchronizing its society's production and consuming. The plant captures his second proposition because the services that it yields are homogeneous and it also captures his third proposition because it represents all of a society's factors of production as a single homogeneous mass. These three aspects of the Crusonia plant enabled Knight to entirely purge time from his analysis of capital. For one thing, the period of production entirely vanishes because the Crusonia plant yields its useful services in the same increment of time that they are used. In addition, Knight used the Crusonia plant to argue, contra time-preference theories, that the rate of interest was entirely determined by capital's technical productivity, which manifests itself here as the plant's exogenous rate of growth. Overall, Knight invented the Crusonia plant as an analogy to make evident key propositions that he argued informed sound capital theory and how those propositions necessarily framed capital as a perpetual fund of value.

After Knight's invention of the Crusonia plant, Donald Dewey (1963 and 1965) employed the Crusonia plant as a pedagogical device to efficiently capture the essential properties of capital through the language of geometry. The two aspects of the Crusonia-plant model that Dewey (1965: 60) identified as contributing to that goal were that the model enabled changes in the quantity of capital to be easily measured and that it abstracted from real goods' periods of production. Jack Hirshleifer (1970: 158-64) also employed the Crusonia plant towards pedagogical ends. He viewed the Crusonia plant as one of many alternative lenses that capital theory could be viewed through and, like Dewey, employed geometry in elucidating that lens.[8] Following Dewey's and Hirscheifer's geometric presentation of the Crusonia plant. Bo Sandelin (1989) suggested the Crusonia plant as an improvement over Knut Wicksell's own analogy of aging wine as a model for demonstrating the Wicksell effect.

More recently, Tyler Cowen (2018) has made use of the Crusonia plant for didactic purposes to argue that people should consider future consequences more when thinking about how they ought to act. Cowen uses the Crusonia plant's compounding growth as an analogy for those businesses and institutions that yield consistent benefits over time. When growth compounds, tiny immediate differences can have very large distant consequences. In a world of scarce resources, Cowen argues that humanity in general would be much better off if people directed their resources towards investing in those businesses and institutions that resemble Crusonia plants, rather than consuming them for one-off benefits.

agencies as equivalent to capital was "the most distinctive part of Knight's theory course."

[8] J. Hirshleifer (1970: 159): "We may perhaps think of [the Crusonia plant] as an undifferentiated mass of consumables (a vast, edible fungus, perhaps) that physically grows... at some over-all proportionate rate ρ over time—except as portions are excised for current consumption."

2 The Crusonia Plant

2.1 Defining the Crusonia Plant

Returning to the subject of fanciful botany *per se*, let us imagine the Crusonia as some species of perennial vegetation whose mass can be put to any imaginable use and that thereby functions as a source of universal utility. That plant grows over time at some rate that is proportionate to its mass. The Crusonia plant might be imagined to resemble a tree, such as an ash or oak tree, or a bush, such as a hydrangea or beech hedge. It might even be imagined to better resemble a fungus, such as a polypore or a slime mold. One might even go in a more Lovecraftian direction and *attempt* to imagine the Crusonia plant as some inchoate, non-Euclidean mass that defies any organized shape. Regardless of its physical properties, the Crusonia plant's flesh and foliage can be harvested and then employed in any imaginable use.

One can then imagine a solitary human being, whom I shall name Friday, whose economic life entirely revolves around harvesting and consuming quantities of a Crusonia plant. All of Friday's productive activities are directed towards cultivating that plant; he produces no other goods, nor does he need to. Harvesting Crusonia occurs practically instantaneously and requires a negligible amount of labor. Friday is thus something of a rentier whose income entirely derives from resources that he has ownership over rather than any exertion on his part. The Crusonia plant's environment does not have any carrying capacity that might constraint its growth, nor does that growth decelerate with age. So long as an infinitesimally small particle is left unharvested, the Crusonia plant continues to grow at a constant rate, irrespective of environmental circumstances.

At any period of time, the Crusonia plant can be understood as a reservoir of services that are universally useful. Friday draws services from that reservoir by harvesting an appropriate quantity of that plant's mass. As he does so, the Crusonia plant's mass, which I shall call *Crusonia*, and thus its reservoir of useful services, is gradually exhausted. If he draws enough services from it, that reservoir can run draw, rendering the Crusonia plant, or at least what is left of it (if anything), entirely useless. The Crusonia plant is therefore a non-permanent resource that is gradually exhausted in the very process of yielding its services. Of course, Friday can choose to harvest Crusonia at varying speeds such that the Crusonia plant's mass represents a myriad number of alternatively possible streams of services of varying shapes and magnitudes.[9] Furthermore, the plant grows over time so its reservoir of services is consistently being refreshed over time with the plant's natural growth. As a result, the Crusonia plant can yield many alternative streams of useful services over an indefinite period of time.

[9] F. Hayek (2007: 154): "A given stock of capital goods does not represent one single stream of potential output of definite size and shape; it represents a great number of alternatively possible streams of different time shapes and magnitudes.

2.2 The Crusonia Plant and Capital Theory

Considered as an economic means, the Crusonia plant's mass is a good that can jointly satisfy Friday's present and future demands. Friday uses the Crusonia plant to directly satisfy his prevent demands by harvesting and then consuming some of that plant's flesh or foliage. *Consumption* is that class of action that uses the means at its disposal to directly realize immediate demands. Once harvested, Crusonia becomes a consumption good that can satisfy any imaginable demand, at least in sufficient quantities. The more Crusonia he harvests, the more demands Friday can satisfy and the happier he is. Friday always prefers more Crusonia to less; Crusonia is therefore a *good*. Crusonia is also perfectly homogeneous. It follows that Friday's consumption at any period of time can be measured in units of the weight. Each period, Friday might consume, say, 50 kgs, or around 110 lbs, or Crusonia. Because consumption is the sole end of Friday's economic life, any interaction he has with the Crusonia plant is for the self-regarding end of improving his capability to consume Crusonia, either now or in the future. When planning how to use his Crusonia plant across time, Friday must plan for how much Crusonia he should harvest each period within the relevant time frame. Following the jargon of dynamic programming, I call that plan a *harvesting policy*, or just a *policy* for short.

Friday uses the Crusonia plant to indirectly satisfy his future demands by presently abstaining from some portion of that plant so that he can harvest that portion, as well as its natural growth, at some future date. *Investment* is that class of action that, by forgoing the immediate consumption of its means, enables those means to be directed towards more distant ends.[10] Investments indirectly satisfy future demands by putting into place the conditions that enable their future satisfaction, generally through the production of more goods. By directing means towards distant ends, they enable agents to better realize those ends by making use of more productive, though time consuming, techniques.

Friday invests Crusonia simply by abstaining from its consumption: By abstaining from harvesting Crusonia today, Friday enables his future self to later harvest that Crusonia in order to satisfy some urgent demand at that time; he saves that Crusonia for future use. Friday's investment pays off because the Crusonia plant grows at some constant rate, r. Whatever Crusonia that he abstains from now grows into even more Crusonia in the future.[11] Friday's abstinence therefore substitutes the distant consumption of a portion of his Crusonia plant, plus its natural growth, in the future for the immediate consumption of that portion today. By postponing the date at which he consumes a given mass of the Crusonia plant, he can draw more useful services from it at some future date than had he immediately consumed it. All-in-all, Friday can indirectly realize a

[10] L. Mises (1998: 487): "The postponement of consumption makes it possible to direct action towards temporally remoter ends."

[11] Nassau William Senior (1938: 58): "Abstinence: A term by which we express the conduct of a person who either abstains from the unproductive use of what he can command, or designedly prefers the production of remote to that of immediate results."

distant end using a quantity of Crusonia that would be incapable of immediately realizing that same end by investing that Crusonia and allowing it to grow into a sufficient quantity that can realize that end sometime in the future.

The Crusonia plant functions as Friday's capital because it can yield a stream of useful services that Friday can use to permanently maintain his consumption at at a specific level. In general, the things that we consider capital emerge from commensurate stocks of goods that can jointly satisfy present and future demands. Those stocks can yield many alternatively possible streams of services of varying time profiles. Capital, as well as the associated techniques of capital accounting, are concepts that help agents choose between those alternative streams.[12] By definition, capital can yield a unique stream that permanently maintains that capital's magnitude across time.[13] So long as he abstains from its total consumption, an agent's capital embodies a potentially permanent reservoir of services. An aspect of real-world capital that the Crusonia plant entirely abstracts from is that, if capital is to yield a permanent income, the real goods that underlie it must be consistently maintained and replaced as they depreciate in the process of yielding their useful services. The use of capital thus gives rise to the need for investment in order to maintain that capital intact.[14] Although the stock of real goods is in perpetual flux, the capital itself can yield a permanent stream of services. That capital can therefore be thought of as a *mobile perpetuum* that drives its owner's consumption.[15]

The notion that capital can be used to provide a constant stream of services over a given time frame enables a precise definition of income, as well as the associated phenomena of saving and dissaving. Here, for the sake of conceptual clarity, I differentiate between an agent's income within a given time frame and the permanent income that his capital, by definition, can yield. I define an agent's *income* as the highest constant stream of services that he can draw from his capital across a given time frame. I then define a capital's *permanent income* as the highest constant stream of services that the capital can yield without

[12] F. Hayek (2007: 278): "Capital accounting in this sense is simply a shorthand device for preventing involuntary encroachments upon future income."

[13] F. Hayek (2007: 75): "The term capital, in so far as it is required to describe a particular part of the productive resources, will accordingly be used here to designate the aggregate of those non-permanent resources which can be used only in this indirect manner to contribute to the permanent maintenance of the income at a particular level."

[14] F. Hayek (2007: 102): "The essential characteristic of capital, and the one which affects the use of current output, is that it needs replacement and in consequence leads to investment... The important thing is not that the capital has been produced, but that it (or some equivalent) has to be reproduced."

[15] K. Wicksell (1954: 99): "The seemingly paradoxical phenomenon, that consumable goods—that is to say, goods which exhaust or seem to exhaust their whole content of their usefulness in a limited series of acts of uses—can nevertheless be employed 'capitalistically,' so that their entire value remains stored up for the owner, and yet provides him with an income—this *perpetuum mobile* of the economic mechanism forms, as we said previously, the real pith of the theory of capital."

diminishing its magnitude. Equivalently, a capital's permanent income is the highest stream of services that it can perpetually yield. So long as he is planning across a sufficiently small time frame, as is the case is the next two sections, an agent can raise his income above his capital's permanent income by gradually consuming his capital. However, as he plans across an ever longer time frame, an agent's income converges to his capital's permanent income, which I shall demonstrate in the fourth section.

That agent then *saves* when he consumes fewer services than his capital's permanent income can provide, and he *dissaves* when he consumes more services than his capital's permanent income can provide.

An agent's income is a natural baseline for framing the characteristic trade-off between using up his capital either relatively slowly or relatively quickly. When an agent uses up his capital slowly, he consumes fewer services than his income can provide. He *saves* and, by investing that portion of his income that he abstains from, can thereby increase his future income. When an agent uses up his capital quickly, he consumes more services than his income can provide. He *dissaves* and thereby decreases his future income. Here, the choice between saving and dissaving is not, as it is in ordinary language, a choice between increasing and decreasing one's capital. Instead, it is a choice between alternative streams of services of varying time profiles, specifically time profiles that increase or decrease over time.[16] Capital is a means to an end. An agent's problem of making the best use of his capital is a problem of choosing a stream of services that best jointly satisfies his present and future demands. The choice between larger or small quantities of capital is ultimately a financial shorthand for thinking about those alternative streams.

If terms are needed to describe actions that either increase or decrease the quantity of capital at an agent's disposal, D.H. Robertson's (1949: 41) terms lacking and dis-lacking would probably be more appropriate. Lacking and dis-lacking must both be defined with respect to a capital's permanent income. [17] An agent *lacks* when he consumers fewer services than his capital's permanent income can provide, and *vice versa* for dis-lacking. Lacking enables an agent to invest a portion of the output that his capital yields and thereby increase the quantity of capital at his disposal.

Turning to the Crusonia plant, the economic problem that Friday faces when using his Crusonia plant is the problem of making the best use of that plant's scarce mass to jointly satisfy his present and future demands. Although he can choose to consume his income and thereby enjoy a constant stream of services, that stream might not be jointly satisfy his present and future demands. In

[16] F. Hayek (2007: 280): "What is relevant is whether a person maintains a stock of non-permanent resources which will secure him an increasing, constant, or decreasing income stream, not whether this stock itself increases, remains constant, or decreases in any of its directly measurable dimensions."

[17] D.H. Robertson (1949: 41): "A man is lacking if during a given period he consumes less than the value of his current economic output... If during a given period a man consumes more than the value of his current economic output, he may be said to be dis-lacking."

choosing his harvesting policy, Friday faces the trade-off, which characterizes the use of all capital, of using the Crusonia plant either relatively quickly or relatively slowly: On the one hand, Friday can use up the Crusonia plant relatively quickly. By doing so, he can consume more Crusonia now to satisfy relatively more of his immediate demands, but he does so at the cost of having less Crusonia later to satisfy his distant demands. On the other hand, Friday can use up the Crusonia plant relatively slowly. With that course of action, he can consume more Crusonia later to satisfy relatively more of his distant demands, but at the cost of having less Crusonia now to satisfy his immediate demands. There is also the possibility that Friday might choose to harvest a stream of services from the Crusonia plant that maintains his consumption at a constant level across time.

Friday solves his economic problem, and thereby makes the best possible use of his Crusonia plant, when he chooses the harvesting policy that maximizes the plant's utility to him or, equivalently, when he chooses the policy that he prefers to all alternative policies. There are two factors that affect the shape and magnitudes of Friday's optimal policy: 1) The Crusonia plant's productivity, and 2) Friday's time preference. Taking the factors in sequential order, the Crusonia plant's *productivity* determines how quickly the plant grows over time and thus grows more useful to Friday.[18] The Crusonia plant's productivity determines Friday's income because it specifies how much Crusonia Friday can consume each period without reducing the plant's mass and thereby trenching on future consumption. Friday's *time preference* then encompasses any reason why he might prefer the realization of immediate to distant ends. For one reason or another, future demands might strike Friday's imagination with less force and vivacity than present demands, thereby inducing him to impute more urgency to the satisfaction of present demands to future ones. As a result, Friday might choose to use up his Crusonia plant relatively quickly to satisfy them. However, if the Crusonia plant grows at a fast rate, then the relatively small portion of the Crusonia plant that Friday abstains from rapidly grows larger, and might even entirely cancel out Friday's high time preference. Overall, the Crusonia plant's productivity and Friday's time preference therefore have duelling influences on the time profile of his optimal harvesting policy.

3 Intertemporal Choice Over a Couple of Periods: The Two-Period Model

J. Hirshleifer (1970: 32): "The objective of the individual economic agent is to achieve a preferred time-pattern of consumption, an optimal balance among consumption claims of differing dates."

In this section, I use a two-period model to begin my analysis of Friday's economic problem of best using his Crusonia plant to jointly satisfy his most urgent present and future demands. The Crusonia plant constitutes the capital that Friday uses to jointly satisfy his present and future demands. As his

[18] A. Alchian and W. Allen (1964: 40): "*Production* means an act that increases utility."

capital, Friday can draw useful services from his Crusonia plant at a variety of time schedules of different shapes and magnitudes. Economizing that plant's use requires that Friday choose to harvest Crusonia on the schedule that best jointly satisfies his most urgent present and future demands. Solving for that optimal harvesting policy across a long time space can be mathematically daunting, so I begin my inquiry here by analyzing the simplest possible case: A two-period model in which the present and future are defined by two periods, today and tomorrow respectively. This two-period model distills Friday's choice of policy into a choice between two dated goods, *Crusonia* today and *Crusonia tomorrow*.

The principal advantage of this two-period model is that it focuses attention on the abstract rule that optimizes Friday's stream of Crusonia rather than on the mathematical apparatus that formalizes that rule. Another advantage of this simple model is that it presents Friday's economic problem in a way that readily enables the application of the familiar tools of two-dimensional indifference analysis to that rule. Within the context of such analysis, Friday's optimal harvesting policy occurs at the point at which one of his indifference curves is tangent to the plant's budget constraint. In other words, Friday makes the best possible use of his Crusonia plant when he harvests Crusonia today up to the point at which his intertemporal marginal rate of substitution coincides with the plant's marginal rate of transformation. In the following sub-sections, I derive both of those rates of equivalence and explain how they model Friday's economic problem.

3.1 The Marginal Rate of Substitution

I first derive the marginal rate of substitution. Friday strives to harvest Crusonia today and tomorrow in order to best jointly satisfy his most urgent present and future demands. He consumes the Crusonia that he harvests today to satisfy present demands, and he then consumes the Crusonia that he harvests tomorrow to satisfy future demands. As a result, the time profile of Friday's consumption of Crusonia can be expressed as an intertemporal bundle of goods: (c_1, c_2), where the variables c_1 and c_2 measure how much Crusonia he consumes today and tomorrow respectively. The myriad alternatively possible bundles of Crusonia today and tomorrow can be represented as points along a goods plane.

The utility function $\mathrm{U}(c_1, c_2)$ then describes how well any given bundle can jointly satisfy his most urgent present and future demands. Friday prefers more useful bundles of dated Crusonia to less useful ones. The utility function assigns a numerical value to each bundle that ranks those bundles relative to alternative bundles according to Friday's preferences. Here, I make the behavioral assumption that Friday's consumption satisfies his demands in order of their relative urgency. Each additional increment of Crusonia that he consumes satisfies a less urgent demand than the previous increment. Friday's utility function, $\mathrm{U}(c_1, c_2)$, thus exhibits diminishing marginal utility and, as a result, is a concave, twice-differentiable function, $U''(c_1, c_2) < 0$. At least since Frank Ramsey (1928: 546), economists have also assumed that any intertemporal utility function, $\mathrm{U}(c_i)$, can be decomposed into a discounted sum of utility flows across i independent periods. I adopt that approach to describe the total utility of Friday's harvesting policy the sum of the utility that Friday consumes today and the discounted utility of the Crusonia that he consumes tomorrow:

$$U(c_1, c_2) = U(c_1) + (1+\rho)U(c_2) \tag{1}$$

where ρ is a parameter that measures Friday's time preference. I assume that Friday never prefers future to present consumption such that $\rho \leq 0$. For ease of expression, I then define Friday's *discount factor* as the parameter $\beta = (1+\rho)$:

$$U(c_1, c_2) = U(c_1) + \beta U(c_2) \tag{2}$$

So long as he considers present demands as more urgent than future ones, Crusonia today is more useful to him than Crusonia tomorrow to Friday. Equation describes how the utility that Friday imputes to a given bundle of dated Crusonia, $U(c_1, c_2)$, is determined by the sum of the utility of Crusonia today and the discounted utility of Crusonia tomorrow.

The intertemporal marginal rate of substitution is the rate of equivalence that is germane to Friday's utility function, and it relates those bundles of dated Crusonia that are equally useful to Friday. Although his utility function is a three-dimensional object, it is convenient to summarize Friday's preferences between various bundles of dated Crusonia as a map of indifference curves by projecting his utility function's level curves on the goods plane. Figure 1 provides a sample utility surface and its associated indifference map. The marginal rate of substitution describes the contours of the utility function by linearly approximating the slopes of those indifference curves. I can then derive an equation for Friday's marginal rate of substitution by totally differentiating Friday's utility function and setting any change in total utility to 0:

$$\begin{aligned} \partial U(c_1, c_2) &= U'(c_1)dc_1 + \beta U'(c_2)dc_2 \\ 0 &= \\ dc_2/dc_1 &= -U'(c_1)/\beta U'(c_2) \end{aligned} \tag{3}$$

Friday is thus indifferent between any two alternative bundles of dated Crusonia only when the rate at which he substitutes Crusonia today for Crusonia tomorrow equals the negative ratio of the marginal utility of his consumption today and the discounted marginal utility of his consumption tomorrow. In other words, the marginal rate if substitution measures how much Crusonia tomorrow is required to perfectly compensate Friday for the loss of another increment of Crusonia today, and *vice versa*. Equation (4) specifically describes an intertemporal marginal rate of substitution because the marginal utility of Friday's consumption tomorrow is discounted by his discount factor.

Equation (4) also identifies two factors that influence Friday's choice between present and future Crusonia: 1) Diminishing marginal utility, and 2) time preference. Those two factors must be strictly differentiated because whereas diminishing marginal utility ubiquitously affects economic choice, time preference uniquely affects intertemporal choice. They also affect Friday's choice in different manners: Whereas diminishing marginal utility induces him to allocate his Crusonia plant equally between present and future consumption, time preference induces Friday to prefer the former to the latter.

Those distinct effects are made evident in the shapes of the indifference curves that describe Friday's preferences between present and future consumption: Diminishing marginal utility occasions those curves to assume a convex shape, and time preference occasions them to approach their asymptote along the future axis slower than their asymptote along the present axis. On the one hand, the concave shapes of Friday's indifference curves illustrate how diminishing marginal utility induces him to smooth his consumption between today and tomorrow. *Ceteris paribus*, Friday chooses the optimal harvesting policy when he harvests Crusonia today up until the point at which its marginal utility is equal to the marginal utility of the Crusonia that he shall harvest tomorrow. In this manner, Friday's optimal policy is a manifestation of Gosset's Second Law, also known as the equimarginal principle, that a good with competing uses is economized when it is allocated between those uses up until the point that those uses' marginal utilities are equal.[19] On the other hand, that Friday's indifference curves might approach their asymptotes along the future axis slower than their asymptotes along the present axis, thereby illustrating Friday's preference of present to future consumption. As β increases in Equation (4), Friday's intertemporal marginal rate of substitution grows ever larger in the negative direction and his indifference curve approach their future asymptote slower. The reason is that, as Friday attributes less and less urgency to future demands relative to present ones, it requires ever larger quantities of Crusonia tomorrow to induce him to forgo a unit of Crusonia today. As originally demonstrated by Irving Fisher, Friday's time preference can be isolated by measuring his indifference curves' marginal rates of substitution along the diagonal, where $c_1 = c_2$, because an indifference curve's slope is equal to the negative reciprocal of his discount factor at that point and that point only.[20] Figure II below graphically depicts the affect that various rates of time preference have on an intertemporal indifference curve.

[19] J. Schumpeter (1954: 910-1):"Unlike the first [Gosset's second law] is not a postulate but a theorem: in order to secure a maximum satisfaction from a good that is capable of satisfying different wants (including money and labor), and individual (or household) must allocate it to these different uses in such a way as to equalize its marginal utilities in all of them."

[20] Where $c_1 = c_2$, Friday's marginal rate of substitution can be written as follows:

$$\begin{aligned} MRS &= -U'(c_1)/\beta U'(c_2) \\ &= -U'(c_1)/\beta U'(c_1) \\ &= -1/\beta \end{aligned}$$

Friday's marginal rate of substitution is therefore equal to his discount factor's negative reciprocal.

3.2 The Marginal Rate of Transformation

I now derive the marginal rate of transformation. Although he would always prefer to substitute a bundle of dated Crusonia along a higher indifference curve for one along a lower curve, Friday's choice is constrained by the capital at his disposal, which is embodied within his Crusonia plant's mass. As a result, at the margin of choice, Friday faces the trade-off between harvesting another increment of Crusonia today or a relatively larger increment of Crusonia tomorrow. The marginal rate of transformation describes that trade-off.

The budget constraint $B(c_1, c_2)$ enumerates the alternative bundles of present and future Crusonia that Friday must choose between when making an efficient use of his Crusonia plant. It is the frontier of the opportunity set that is the subset of the goods plane that specifies the alternative bundles of Crusonia today and Crusonia tomorrow that Friday's capital enables him to choose between. Its extent is determined by the initial size of the Crusonia plant and that plant's rate of growth. The budget constraint is then the frontier of that opportunity set and its equation can be written implicitly as follows:

$$B(c_1, c_2) = c_2 - (1 + r)(c_0 - c_1) \tag{4}$$

However, I can also write the budget constraint as a function of the quantity of Crusonia tomorrow that Friday harvests:

$$c_2 = (1 + r)(c_0 - c_1) \tag{5}$$

where c_0 identifies the Crusonia plant's initial size and and r identifies the plant's growth rate; I assume that $r > 0$. Once again for ease of expression, I define the Crusonia plant's productivity as the parameter $\pi = 1 + r$ so that I can write the budget constraint more succinctly:

$$c_2 = \pi(c_0 - c_1) \tag{6}$$

Equation (6) describes how any quantity of Crusonia that Friday abstains from today, $c_0 - c_1$, grows into a larger mass of Crusonia, c_2, that Friday then consumes tomorrow.[21] The budget constraint thus enumerates those *efficient* harvesting policies because there are no feasible policies that can produce as much future Crusonia using less present Crusonia as its bundles. A description of his budget constraint suffices for modeling Friday's economic problem because he is always eager to substitute a bundle along the budget constraint for a bundle within his opportunity set's interior.

A complementary way of thinking about Friday's budget constraint is in terms of the Crusonia plant's present value and how that present value funds Friday's harvesting policy. As discussed above, capital can be thought of as a

[21] A given quantity of Crusonia today is therefore more useful to Friday than that same quantity tomorrow because that quantity today can grow into an ever larger quantity tomorrow.

reservoir that can yield myriad alternative streams of useful services of varying time profiles. Present value is an accounting method for thinking about the practical volume of such a reservoir. A capital's present volume is equal to the most valuable streams of services that it can yield, discounted by the relevant rate of return. The Crusonia plant's present value is equal to the value of one of Friday's efficient harvesting policies discounted by the plant's productivity. Like his consumption each period, the value of Friday's Crusonia plant can be reckoned in units of weight. It follows that I can express Friday's budget constraint as a present discounted stream of Crusonia:

$$c_0 \geq c_1 + \pi^{-1} c_2 \tag{7}$$

Any portion of the Crusonia plant that is left over tomorrow is wasted because it can no longer be used to satisfy any of Friday's urgent demands. Because consumption is the sole end of his economic life, I assume that Friday's harvesting policy tends to exhaust the Crusonia plant's present value so that I can write the budget constraint as follows:

$$c_0 = c_1 + \pi^{-1} c_2 \tag{8}$$

Equation (8) describes how the Crusonia plant's initial mass, or present value, is equal to the present discounted value of Friday's consumption today and tomorrow. The Crusonia plant's capital value can be efficiently transformed into a variety of alternative streams of services that are all represented as points along the budget constraint. Over time, that value is transformed into the consumers' goods that satisfy Friday's most urgent present and future demands. Equation (8) is described as a *present discounted stream* because the size of Friday's future consumption is discounted by his Crusonia plant's productivity. Although he consumes c_2 units of Crusonia tomorrow, Friday only needs to abstain from $\pi^{-1}c_2$ units of Crusonia today because $\pi^{-1}c_2$ units of Crusonia today grows into c_2 units tomorrow. The Crusonia plant's present value can also be thought of as its *capital value* because that fund *capitalizes* Friday's harvesting policy.

The marginal rate of transformation is the rate of equivalence that is germane to the budget constraint and it relates those bundles of dated Crusonia that require equivalent quantities of capital to finance. As capital, the Crusonia plant can be used up quickly or slowly. The marginal rate of transformation specifies the trade-off that Friday faces at the margin of choice between using up his Crusonia plant either relatively more quickly or relatively more slowly. It measures the rate at which Friday can substitute Crusonia tomorrow for Crusonia today, or *vice versa*. I can derive that rate of equivalence by differentiating either Equation (6) or Equation (8), though the former is definitely an easier operation:

$$dc_2/dc_1 = -\pi \tag{9}$$

Equation (9) makes evident that the marginal rate of transformation is determined by the Crusonia plant's productivity. That rate specifies that in order to consume an additional unit of Crusonia today, Friday must forgo π units of

Crusonia tomorrow. In other words, the cost of Friday's consumption of an additional unit of Crusonia today is the forgone opportunity to consume π units tomorrow. Overall, the trade-off between present and future consumption that the marginal rate of transformation measures characterizes the trade-off that Friday faces using up his Crusonia either relatively more slowly or relatively more quickly.

3.3 Constrained Maximization

Having defined the utility function and the budget constraint, I can model Friday's economic problem as a constrained-optimization problem. Friday solves his economic problem when he chooses that bundle of dated Crusonia that best jointly satisfies his most urgent demands today and tomorrow. The budget constraint specifies the bundles of dated Crusonia that Friday can transform his Crusonia plant into. The utility function then measures how well a given bundle of Crusonia jointly satisfies his most urgent present and future demands relative to other bundles. Friday solves his economic problem when maximizes his utility function against his budget constraint and chooses the most useful bundle where those two equations' rates of equivalence are equal. That optimal bundle occurs on the goods plane where an indifference curve is just tangent to the budget constraint.

I can solve for that optimal bundle using the logic of constrained optimization and the associated method of Lagrangian multipliers where the utility function is the objective function and the budget constraint is the constraint. By combining the objective function and the constraint, I can set up a Lagrangian function and then derive the Euler-Lagrange equations whose solution specifies the optimal bundle of dated Crusonia. The Lagrangian function is as follows:

$$\mathcal{L}(c_1, c_2, \lambda) = U(c_1) + \beta U(c_2) + \lambda[c_0 - c_1 - \pi^{-1} c_2] \tag{10}$$

I then take the Lagrangian function's partial derivatives to derive the following Euler-Lagrange equations:

$$\partial\mathcal{L}/\partial c_1 = U'(c_1) - \lambda = 0 \tag{11a}$$
$$\partial\mathcal{L}/\partial c_2 = \beta U'(c_2) - \pi^{-1}\lambda = 0 \tag{11b}$$
$$\partial\mathcal{L}/\partial\lambda = c_0 - c_1 - \pi^{-1} C_2 = 0 \tag{11c}$$

In general, Euler-Lagrange equations are fundamental tool for solving dynamical problems because they describe a system's stationary points where the associated function has its extrema. In this case, I can solve that system of equations for the stationary point, (c_1^*, c_2^*), that represents the bundle of dated Crusonia along the budget constraint that solves Friday's economic problem.

I can find the bundle of dated Crusonia that solves Friday's economic problem by solving the System of Equations (11) above. To do so, I can first use equations (11a) and (11b) to derive two different expressions of the Lagrange multiplier,

λ, and I then set those expressions equal to each other to arrive at the following equation:

$$U'(c_1^*) = \beta\pi U'(c_2^*)$$

I then rearrange that equation so that its left side resembles Equation (4) above:

$$U'(c_1^*)/\beta U'(c_2^*) = \pi$$

Friday thus solves his economic problem when he chooses that bundle of dated Crusonia at which the ratio of his marginal utility of consumption today to his discounted marginal utility of consumption tomorrow is equal to the Crusonia plant's productivity. To put the point in fewer words, Friday's optimal bundle occurs where his intertemporal marginal rate of substitution equals the plant's marginal rate of transformation. I can also rearrange the above equation to arrive at another expression of Friday's optimality condition:

$$U'(c_1^*)/U'(c_2^*) = \beta\pi \tag{12}$$

Equation (12) specifies that Friday's optimal bundle of Crusonia occurs where his marginal rate of substitution is equal to the product of his discount factor and the Crusonia plant's productivity. Below, I shall rely on Equation (12) as my principal expression of the optimality condition that solves Friday's economic problem,

Equation (12) makes it evident that productivity and time preference are dueling forces in the determination of the shape of Friday's optimal harvesting policy. Although the Crusonia plant's productivity causes whatever part of the Crusonia plant that Friday abstains from to grow larger, Friday's time preference motivates him to value present consumption more urgently than future consumption. Equation (12) condenses those dueling influences into the product of the discount factor and the plant's productivity, $\pi\beta$. The time profile of Friday's policy is determined by that product. If he discounts future consumption at a faster rate than his Crusonia plant grows, $\beta > \pi$, then Friday's optimal harvesting policy uses that plant up relatively quickly, such that his consumption declines over time, $U'(c_1) < U'(c_2)$.[22] If he discounts future consumption at a slower rate than his Crusonia plant grows, $\beta < \pi$, then Friday's optimal harvesting policy uses that plant up relatively slowly, such that his consumption increases over time, $U'(c_1) > U'(c_2)$. Of course, *tertium datur*: Friday might discount the future at such a rate that it entirely cancels out the Crusonia plant's growth, $\beta = \pi$. In that case, Friday's consumption remains constant over time, $U'(c_1) = U'(c_2)$. That permanent stream of consumption can serve as a baseline

[22] The marginal utility of his consumption today is lower than the marginal utility than that of his consumption tomorrow when Friday uses up the Crusonia plant relatively quickly due to the law of diminishing marginal utility. Each increment of Friday consumes in any time period is less useful than his previous increment. As a result, because he consumes more Crusonia today than Crusonia tomorrow, the marginal utility of Crusonia today to Friday is less than that of Crusonia tomorrow. Figure III below provides a graphical presentation of this result.

from which whether Friday is using up his capital relatively slowly or quickly can be measured. Figure IV portrays the three possible general shapes of Friday's harvesting policy.

Taking a step back, the two-period model demonstrates how capital only emerge as a means of human action when it is considered in the passing of time. As discussed above, capital, as well as the underlying stocks of durable goods, are means that are not necessarily immediately used up upon yielding a useful service. Instead, it can yield a steam of useful services that can jointly satisfy an agent's present and future demands. Capital is thus characterized by the trade-off between using them up quickly of slowly, with that trade-off determining the shape of its stream of services across time. When a good is not useful for satisfying any future demands, there is no trade-off between using that good up quickly or slowly, so that good is a consumption good, rather than a durable good.

Friday values the Crusonia plant, qua capital, for its capability to jointly satisfy his present and future demands. He imputes the discounted urgency of his future demands back to the present Crusonia plant. Today, the Crusonia plant is useful to Friday as an indirect means for satisfying his urgent demands tomorrow. He thus imputes the discounted urgency of his future demands back to his present Crusonia plant. Tomorrow, though, the Crusonia plant has no utility as an indirect utility to Friday has no future demands to provide for. Once he arrives at the terminal period, the Crusonia plant is only useful to him for the useful services that it can immediately yield. Friday therefore faces no trade-off when he harvests the whole Crusonia plant in one go tomorrow to provide for his demands. In that period, the Crusonia plant ceases to be capital to Friday. As a result, the Crusonia plant becomes a mere consumption good to Friday tomorrow that is only valued for the one-off services that it can yield.

Overall, the two-period model deploys the familiar tools of indifference analysis to describe Friday's economic problem, and it captures the insight that the time profile of his optimal harvesting policy depends on the duelling influences of his discount factor and the Crusonia plant's productivity. The time profile of Friday's optimal harvesting policy is determined by the relation between those two magnitudes. If his time preference is greater than the plant's productivity, then Friday uses up the Crusonia plant relatively quickly, and *vice versa.* If those two magnitudes are equal, then Friday uses up the Crusonia plant at a constant rate.

4 Intertemporal Choice over Multiple Periods: A Four-Period Model

Although a two-period model can describe Friday's economic problem of how to best use his Crusonia plant with the tools of two-dimensional indifference analysis, the model unnaturally restricts Friday's time horizon to two periods. In this section, I expand that time horizon and analyze Friday's choice over four periods. This four-period model demonstrates that Friday solves his economic problem when he applies the decision rule of equating his intertemporal marginal rate of substitution with the Crusonia plant's marginal rate of transformation across adjacent periods until he reaches the fourth period, at which point consumes the plant entirely. Friday's optimal harvesting policy across four periods is therefore a sequence of three dovetailing choices until the fourth period, at which point the plant ceases to be capital and he thus consumes it entirely in one go.

In this section, I assume that Friday must economize his Crusonia plant to best jointly satisfy his most urgent demands across four periods of time. By planning how to make use of his plant, Friday chooses a policy that governs how he harvests Crusonia across those periods. Once again, Friday's choice of harvesting policy can be represented as a choice of a bundle of dated Crusonia, though here that bundle is a vector of four, rather than two, variables, (c_1, c_2, c_3, c_4). A utility function then measures how well a given policy jointly satisfies Friday's present and future demands. Following the method of Equation (2), I can write out that function as a sum of discounted utility flows across four periods:

$$u(c_1, c_2, c_3, c_4) = u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + \beta^3 u(c_4)$$

where $\beta \leq 1$ and $U''(c_t) < 0$. I can also use sigma notation to express the utility function more succinctly:

$$u(c_1, ..., c_4) = \sum_{t=1}^{4} \beta^{t-1} U(c_t) \tag{13}$$

Equation (13) describes how, due to his time preference, the utility of future consumption might strike Friday's mind with exponentially decaying vivacity. Friday therefore normally places less and less weight to the satiation of future demands as those demands recede into the future. That exponential discounting might be understood as a mathematical description of Arthur Pigou's (1920: 25) assertion that "our telescopic faculty is defective, and we, therefore, see future pleasures, as it were, on a diminished scale. " Present demands seem strong and lively whereas future ones seem weak and dull.

The budget constraint then describes how the scarcity of the Crusonia plant's mass checks Friday's capability to jointly satisfy his most urgent distant and immediate demands. Following Equation above, I describe the budget constraint in terms of the Crusonia plant's net-present value:

$$c_0 = c_1 + \pi^{-1} c_2 + \pi^{-2} c_3 + \pi^{-3} c_4$$

where $\pi \geq 1$. Once again, I can use sigma notation to express my equation more succinctly:

$$c_0 = \sum_{t=1}^{4} \pi^{1-t} c_t \tag{14}$$

Equation (**??**) describes how the Crusonia that Friday continues to abstain from grows larger and larger, at an exponential rate, across time periods. Friday must therefore allocate an exponentially diminishing portion of his Crusonia plant's initial mass to provide for his consumption in ever more distant periods. For example, although he must allocate $\pi^{-1}c_2$ units of present Crusonia to provide for c_2 units of consumption in Period 2, Friday must only allocate $\pi^{-3}c_4$ units of present Crusonia to provide for c_4 units of consumption in Period 4.

I can then use the method of Lagrangian multipliers to solve for Friday's optimal choice of how to harvest his plant to best jointly satisfy his present and future demands. Following that method, I set up the Lagrangian function:

$$\begin{aligned}\mathcal{L}(c_1, c_2, c_3, c_4, \lambda) = U(c_1) + \beta U(c_2) + \beta^2 U(c_3) + \beta^3 U(c_4) \\ +\lambda[c_0 - c_1 - \pi^{-1}c_2 - \pi^{-2}c_3 - \pi^{-3}c_4]\end{aligned}$$

This equation is an extended variation on Equation 10 above. Once again, I can simply the equation's expression by using sigma notation:

$$\mathcal{L}(c_1, c_2, c_3, c_4, \lambda) = \sum_{t=1}^{4} \beta^{t-1} U(c_t) - \sum_{t=1}^{4} \lambda_t \pi^{1-t} c_t \tag{15}$$

Although it is equal across all four time periods, it is useful to apply t-subscripts to the Lagrangian multiplier in order to keep track of how its expression, though not its ultimate value, changes across time periods. Rinsing and repeating the method of Lagrangian multipliers in Section II above, I can take the partial derivatives of my Lagrangian function to derive the Euler-Lagrange equations:

$$\partial\mathcal{L}/\partial c_1 = U'(c_1) - \lambda_1 = 0 \tag{16a}$$
$$\partial\mathcal{L}/\partial c_2 = \beta U'(c_2) - \pi^{-1}\lambda_2 = 0 \tag{16b}$$
$$\partial\mathcal{L}/\partial c_3 = \beta U'(c_3) - \pi^{-2}\lambda_3 = 0 \tag{16c}$$
$$\partial\mathcal{L}/\partial c_4 = \beta U'(c_4) - \pi^{-3}\lambda_4 = 0 \tag{16d}$$
$$\partial\mathcal{L}/\partial \lambda = c_0 - c_1 - \pi^{-1}c_2 - \pi^{-3}c_3 - \pi^{-3}c_4 = 0 \tag{16e}$$

That system of equations can solve to yield the bundle of dated Crusonia, $(c_1^*, c_2^*, c_3^*, c_4^*)$, that best jointly satisfies Friday's most urgent demands across those four periods. Because System of Equations has five variables, as opposed to System (11)'s three, the computation is more complicated, but it can still be executed without excessive difficulty. (I still have yet to expand the number of time periods to the point that such a computation founders on the reef of dimensionality.) I can then solve for various ways of expressing the Lagrangian multiplier, λ_t, and express them as a vector:

$$\begin{aligned}\lambda_i &= [U'(c_1), \pi\beta U'(c_2), \pi^2\beta^2 U'(c_3), \pi^3\beta^3 U'(c_4)] \\ &= [\pi^{1-t}\beta^{t-1}U'(c_t)]\end{aligned} \tag{17}$$

Each period's Lagrange multiplier is thus expressed as the marginal utility of that period's consumption discounted by Friday's discount factor and compounded by the Crusonia plant's productivity across each preceding period. The Lagrange multiplier therefore relates Friday's marginal utility from consumption across all four periods. I can then use those varying expressions of the Lagrange multiplier to solve for the optimality condition that governs Friday's best use of his Crusonia plant. I first do so for the multipliers associated with the first and second period:

$$\begin{aligned} \lambda_1 &= \lambda_2 \\ U'(c_1) &= \pi\beta U'(c_2) \\ U'(c_1)/\beta U'(c_2) &= \pi \end{aligned} \tag{18}$$

Equation (**??**) is identical to Equation (12) above. I now find the optimality condition implied by the multipliers associated with the second and third periods, and then the third and fourth periods. The optimality condition associated with the second and third period is as follows:

$$\begin{aligned} \lambda_2 &= \lambda_3 \\ \pi\beta U'(c_2) &= \pi^2\beta^2 U'(c_3) \\ U'(c_2)/\beta U'(c_3) &= \pi \end{aligned} \tag{19}$$

The condition for the third and fourth periods is then as follows:

$$\begin{aligned} \lambda_3 &= \lambda_4 \\ \pi^2\beta^2 U'(c_3) &= \pi^3\beta^3 U'(c_4) \\ U'(c_3)/\beta U'(c_4) &= \pi \end{aligned} \tag{20}$$

Equations (**??**), (**??**), and (20) are identical optimality conditions. I can even use mathematical induction to arrive at a more abstract description of the optimality condition:

$$U'(c_t)/\beta U'(c_{t-1}) = \pi \tag{21}$$

Equation 30 specifies that Friday makes the best use of his Crusonia plant between any two adjacent periods when he harvests Crusonia up to the point that his marginal rate of substitution equals the plant's marginal rate of transformation.

The four-period model demonstrates that, at least up until the terminal period, Friday's optimal harvesting policy is determined by an abstract rule and thus consistent across time. The same abstract rule that governs Friday's best use of his Crusonia plant in Period 1 also governs his best use of that means in Periods 2 and 3 as well. Friday's optimal choices in Periods 1, 2, and 3 therefore dovetail into a dynamically consistent policy over those three periods.

The same rule that governs Friday's best use of his Crusonia plant across two periods also governs his best use of that plant across four periods. The time shape of Friday's consumption then depends on the relative values of his time preference and the Crusonia plant's productivity. In addition, just like in the two-period model, Friday's choice in Period 4, the terminal period, is an exception to Equation 30's general rule because there is no future period and the Crusonia plant thus ceases to have any future utility for Friday. The four-period model is therefore time variant at the terminal period because the nature of Friday's economic problem changes at that point. Friday's choice of how to use his Crusonia plant in Period 4 is based on the fact that Period 4 is the terminal period, not on his prior choice in Period 3. From an economic point of view, the Crusonia plant ceases to be capital and Friday consumes the entire plant for the immediate utility it can confer.

This four-period model also mathematically represents how Friday's optimal harvesting policy can be analyzed as a sequence of dovetailing choices across four periods. That sequence begins in Period 1 when Friday must choose how to allocate his present Crusonia so that he best jointly satisfies his most urgent demands across four time periods. He makes that choice based on the relative urgency that he ascribes to each period's demands and the present Crusonia that he allocates to future consumption then grows between Period 1 and 2. Once he gets to Period 2, Friday faces almost the exact same choice that he faced in Period 1: He must choose how to best allocate his present Crusonia between multiple periods. The difference between those two choices is that Friday must allocate his present Crusonia over three periods from the perspective of Period 2 whereas Friday had to allocate his present Crusonia over four periods from the perspective of Period 1. Once again, the present Crusonia that Friday abstains from grows between periods. In a similar manner, in Period 3, Friday must allocate his present Crusonia over two periods, and, in Period 4, he must only allocate it over a single period, the present. All those sequential choices dovetail into a single coherent plan for the economic use of the Crusonia plant across four periods because they are decided by the same rule and identical parameters; neither Friday's discount rate, nor the plant's productivity change over time. The dovetailing nature of Friday's choices over time enables the deployment of dynamic programming to intertemporal choice that I explore in the next section.

The four-period model emphasizes how the time profile of Friday's optimal harvesting policy is sequentially determined by Friday allocating his Crusonia plant to various time periods and the plant then growing in the meantime. Here, examining the baseline case of Friday choosing to harvest a constant stream of Crusonia is a particularly illuminating case. At least without Equation (21), or even (12) in the previous section, one might think that Friday chooses to harvest a constant stream of Crusonia when he ascribes equal urgency to present and future demands. After all, that policy would provide Friday an equal amount of Crusonia each period to satisfy his demands therein. However, this supposition neglects to consider the impact that the Crusonia plant's growth has on the time profile of Friday's optimal policy. Friday's discount factor alone does not determine that time profile. It only determines how much of his Crusonia plant in Period 1 that he allocates for present and future harvesting. The Crusonia that he allocates for future harvesting can then into even larger quantities of Crusonia in the meantime. So long as $\pi \geq 1$, it follows that Friday only chooses to harvest a constant stream of Crusonia when he discounts future demands relative to present ones. More specifically, he only chooses to do so when his discount factor is the reciprocal of his Crusonia plant's productivity, $\beta = \pi^{-1}$, as follows from Equation (21).

A numerical example might make how the time profile of Friday's optimal harvesting policy is determined more vivid and clear. For example, if $\pi = 1.05$, $\beta = 1.05^{-1}$ and $c_0 = 100$kgs, in Period 1, Friday initially allocates 26.86 kgs to harvesting in Period 1, 25.58 kgs in Period 2, 24.31 kgs in Period 3, and 23.2 kgs in Period 4. I can express that initial allocation more succinctly using vector notation:

$$w = \begin{bmatrix} 26.86 \\ 25.58 \\ 24.31 \\ 23.2 \end{bmatrix} \tag{22}$$

I use the variable 'W' because Friday's in Period 1 can be measured by the size of his Crusonia plant. Vector W_1 then makes evident how, as future demands become ever less lively within his mind, Friday allocates ever diminishing portions of his Crusonia plant in Period 1 towards satisfying those demands. Although the Crusonia that is allocated for consumption in Period 1 is used up before it can grow, the Crusonia that is allocated for future consumption does grow in the meantime. The 25.58 kgs that Friday allocates towards Period 2's demands grows into 26.86 kgs over one period, the 24.31 kgs that he allocates towards Period 3's demands grow into 26.86 kgs over two periods, and the 23.2 kgs that he allocates towards Period 4's demands grow into 26.86 kgs over three periods. Friday's initial wealth vector is thereby transformed into his consumption vector:

$$c_i = \begin{bmatrix} 26.86 \\ 26.86 \\ 26.86 \\ 26.86 \end{bmatrix} \tag{23}$$

Friday therefore chooses to harvest a constant stream of Crusonia. He only chooses that specific stream, though, because the Crusonia plant's productivity perfectly compensates for his discount factor. With this example, the four-period model allows us to vividly and clearly perceive how the time profile of Friday's optimal harvesting policy is determined sequentially, first by Friday choosing how to allocated his Crusonia between alternative time periods and then the Crusonia allocated to future periods growing over time.

Another peculiar case that the four-period model can help us better understand in a wider context is the case in which Friday chooses a policy that does not discount the utility of future Crusonia. In this case, Friday allocates his Crusonia plant equally between all four periods because he considers demands across all four periods to be equally urgent. So long as his Crusonia plant grows over time, he then harvests a stream of Crusonia that increases over time. Moreover, this zero-discount policy maximizes the quantity of Crusonia that Friday can harvest over the four periods. The reason is that whatever portion of his Crusonia plant that he allocates towards future consumption grows over time. Friday therefore maximizes the quantity of Crusonia that he harvests over time when he chooses the zero-discount policy because that policy, relative to all alternative policies, allocates the greatest faction of the Crusonia plant to future consumption. The zero-discount policy best exploits the Crusonia plant's productivity. I prove this result numerically in Figure XXX. Of course, that policy might not be Friday's optimal policy if future demands strike him with less urgency than present ones. However, from a wider, more impartial perspective, we can perceive that a zero-discounting policy would maximize the Crusonia plant's utility to Friday. Although future demands might seem weak and dull right now, when the future arrives, as it inevitably does, those demands are as strong and lively to him as any other demands. When he chooses a zero-discount policy, Friday chooses the policy that can best satisfy all his demands when they become strong and lively. From that wider, more impartial perspective, there is a moral impetus for Friday to view his future demands as counting just as much as his present ones. His patience shall be rewarded.

Overall, the four-period model demonstrates how Friday's solution to his economic problem assumes the form of a sequence of choices across adjacent time periods that are governed by a common rule until he reaches the terminal period. That rule is that Friday should consume Crusonia up until the point that his intertemporal marginal rate of substitution is equal to the plant's productivity. However, at the terminal period, the Crusonia period ceases to be capital because it has no future utility, so Friday harvests the entire plant for present consumption.

In conclusion, the four-period model describes Friday's choice of an optimal policy for harvesting Crusonia as a choice between alternative bundles of two goods, Crusonia today and Crusonia tomorrow. It treats the choice of an optimal harvesting policy as a constrained-maximization problem of maximizing a utility function against a budget constraint. The model specifies that Friday chooses his optimal policy when he harvests Crusonia today up until the point that his marginal rate of substitution equals the plant's marginal rate of transformation and then harvests the rest of the plant tomorrow.

5 Intertemporal Choice over Many Periods: An N-Period Model

The Lagrangian function models how Friday solves his economic problem by allocating his scarce Crusonia between competing uses by maximizing Friday's utility function against his budget constraint.

I now expand Friday's time horizon from two or four periods to an infinite number of periods or at least, as I shall explain below, to a number of periods that Friday cannot foresee. The last two sections have served as a sort of propaedeutic. Both sections have built up a mathematical apparatus for analyzing intertemporal choice while keeping an economic interpretation of that apparatus, principally in terms of relevant rates of equivalence, in the foreground. In this section, I expand that mathematical apparatus by using dynamic programming to solve for that stream of consumption that optimizes Friday's use of his Crusonia plant across an infinite number of periods and I shall discuss how that result points to an economic interpretation that I first mentioned in

Section II above. Overall, the case of an infinite number of periods demonstrates how, in the absence of a terminal period in which capital loses its value, Friday economizes his Crusonia plant by persistently equating his intertemporal marginal rate of substitution with the plant's marginal rate of transformation across adjacent periods *ad infinitum*.

Dynamic programming is a mathematical technique for analyzing problems of many dimensions that might otherwise founder of the reef of dimensionality if one were to try to solve them enumeratively. The crux of dynamic programming is that it frames a problem in a way that manages the growth of its dimensions before the stifle mathematical analysis and obstruct computation. Problems that have a large number of dimensions, or a large dimensionality, are those problems that require large volumes of information to describe and those large volumes of information, in turn, entail large numbers of dimensions. For example, describing Friday's consumption across two periods requires a 1×2 vector, describing his consumption across four periods requires a 1×4 vector, and describing his consumption across, say, a thousand periods requires a $1 \times 1,000$ vector. As his time horizon expands, so does the volume of information that is required to describe the time path of Friday's consumption and thus the dimensionality of the underlying problem.

Dynamic programming works by dividing a problem of many dimensions into a sequence of problems of fewer dimensions that can be computed recursively. Rather than searching out the optimal plan from the set of all feasible plans, dynamic programming analyzes a plan as a sequence of choices and computes each optimal choice as a function of some system's current state. Along those lines, it analyzes the structure of an optimal policy (here, the shape of Friday's consumption across time) through the relations between recursive choices. Richard Bellman (1954: 4) established the principle that governs an optimal plan: "An optimal policy has the property that whatever the initial state and the initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." Bellman uses the term 'policy' where I use the term 'plan,' that is to describe a sequence of choices. Bellman's principle of optimality is a mathematical expression of the idea that, at least in equilibrium, all plans must dovetail and that the optimal plan can be computed from the perspective of any arbitrary period with the previous period's choice as the input. Friday's choice of how much Crusonia he consumes is therefore time invariant; that choice is not a function of time, but of the current state of the system, that is the size of his Crusonia plant. Overall, dynamic programming analyzes an optimal plan's structure by substituting a system of recursive choices of relatively small dimensionality for a many-dimensional maximization problem.

It follows that dynamic programming can only be applied to problems that admit of their division into smaller problems. (optimal substructure)

It follows that dynamic programming can only be applied to problems with optimal sub-structure. Problems with optimal sub-structure are problems whose optimal solution can be inferred from the optimal solutions to its sub-problems.

The four-period model above has already demonstrated that, at least until the terminal period, Friday solves his economic problem by applying the same rule to choices between adjacent periods, with each of those choices representing sub-problems whose solution dovetails with Friday's optimal plan. Unlike the four-period model, the infinite-period model lacks a terminal period such that Friday's choice of his optimal path of consumption can be exhaustively dissected into sub-problems entailing the choice of how to allocate his scarce Crusonia between adjacent periods. Because his optimal stream is a stationary process across time, the shape of Friday's choice of consumption across two periods provides information about the shape of his stream of consumption across all time.

Just like in the two previous sections, I begin my analysis of Friday's economic problem by constructing a Lagrangian function, if only to demonstrate how the same-old methods I used above cannot work here. I hope that I shall not wear on a reader's patience if I take the following two paragraphs as an opportunity to review the Lagrangian function's place within the logic of optimization. That Lagrangian function models how Friday solves his economic problem by allocating his scarce Crusonia between competing uses by maximizing Friday's utility function against his budget constraint. All optimization problems can be framed in terms of an objective function being maximized against some constraint.

In this section, I expand my analysis of Friday's economic problem to n periods. Just like in my previous two sections, I begin my analysis by constructing a Lagrangian function that specifies the harvesting policy that makes the best of Friday's Crusonia plant. The utility function describes the utility of a given harvesting policy to Friday in terms of a discounted flow of utility across n periods:

$$U(c_1, u_2, ..., u_\infty) = \sum_{t=1}^{\infty} \beta^{t-1} U(c_t) \tag{24}$$

The budget constraint is the Crusonia plant's value that funds Friday's choice of policy:

$$c_0 = \sum_{t=1}^{\infty} \pi^{1-t} c_t \tag{25}$$

I can then combine those equations to construct a Lagrangian function that models how Friday tends to choose the most useful policy that the Crusonia plant enables him to:

$$\mathcal{L}(c_1, c_2, ..., c_\infty) = \sum_{t=1}^{\infty} \beta^{t-1} U(c_t) - \sum_{t=1}^{\infty} \lambda_t \pi^{1-t} c_t \tag{26}$$

In the previous two sections, I have proceeded to take the Lagrangian function's partial derivatives that constituted a system of equations whose solution specified the optimal policy that solved Friday's economic problem. For a horizon of two or four periods, that system is of a manageable size such that it can readily yield an analytic solution. However, the size of the space of possible policies grows exponentially in the number of periods. As a result, it becomes increasingly impractical to solve Friday's optimal bundle with a system of equations as Friday's economic horizon extends across a broader and broader horizon. Eventually, my problem becomes intractable by classical methods and my computation thus founders on the reef of dimensionality.

I can use dynamic programming to unravel Friday's complex problem of choosing, all at once, how to best economize the use of his Crusonia plant over a large number of periods into a simpler problem of choosing how to best economize its use over a sequence of many periods. Dynamic programming can be applied to this problem because, for any sequence of choices, there exists a value function that optimizes the payoff to a policy, or sequence of choices, over that sequence. Now, a *value function* specifies the policy that maximizes the discounted utility of the services that Friday draws from his Crusonia plant:

$$V(a_i) = Max\{\sum_{t=i}^{n} \beta^{t-1} U(c_t)\} \tag{27}$$

where a is a state variable that describes the Crusonia plant's capital value in a given period. The operator $V(\cdot)$ is an operator that returns the optimal policy that the plant's capital can be transformed into.

Friday's utility function is an objective function that measures how well a given allocation of Crusonia can jointly satisfy Friday's most urgent relatively immediate as well as his relatively distant demands. Through his actions, Friday strives to maximize that utility function by purposefully substituting more useful allocations for less useful ones. As I did before, I assume that the utility of the Crusonia that Friday consumes in one period has no impact on the utility of the Crusonia that he consumes in any other period. I therefore express Friday's utility function as a sum of his instantaneous utilities across many, many independent periods:

$$U(c_1, u_2, ..., u_n) = \sum_{t=1}^{n} \beta^{t-1} U(c_t) \tag{28}$$

where β is a parameter that measures the degree to which Friday considers relatively immediate demands more urgent than relatively distant ones.

The budget constraint is then the constraint that limits the quantity of Crusonia that Friday can consume and thus the highest value that the utility function may assume. I assume that Friday begins with a Crusonia plant of a given mass, c_0 and that his consumption across time cannot exceed that mass, plus the plant's natural growth. I can thus write Friday's budget constraint in terms of his Crusonia plant's net present value:

$$c_0 = \sum_{t=1}^{n} \pi^{1-t} c_t \tag{29}$$

where π is a parameter that measures the plant's rate of growth.

I can then combine the utility function and the budget constraint to arrive at the Lagrangian function that models Friday's economic problem:

$$\mathcal{L}(c_1, c_2, ..., c_n) = \sum_{t=1}^{n} \beta^{t-1} U(c_t) - \sum_{t=1}^{n} \lambda_t \pi^{1-t} c_t \tag{30}$$

In the previous two sections, I have derived a system of equations whose optimal values characterize the solution to Friday's economic problem by taking the Lagrangian function's partial derivatives. However, when I take the partial derivatives of my Lagrangian function here, I end up with an infinite number of partial derivatives and, as foreshadowed by the four-period model above, my computation founders on the reef of dimensionality.

The *value function* specifies the best possible value of some objective function as a function of some state. In this case, the value function specifies the maximum value of the utility that Friday can accrue from his consumption of Crusonia across time. It is a function of the quantity of Crusonia that Friday chose to consume in some period. I can write the value function as follows:

$$V(c_i) = Max\{\sum_{t=i}^{n} \beta^{t-1} U(c_t)\} \tag{31}$$

where the value function V(·) maximizes the utility that Friday accrues over time from consuming Crusonia. I can then use Bellman's principle of optimality to break up my value function and transform Friday's choice of an optimal stream of consumption into the sum of an instantaneous level of consumption and the Crusonia plant's optimal value from the next period on:

$$\begin{aligned} V(c_i, c_{i+1}) &= Max\{U(c_i) + V(c_{i+1})\} \\ &= Max\{U(c_i) + \sum_{t=i+1}^{n} \beta^{t-1} U(c_t)\} \end{aligned} \tag{32}$$

Equation 35 above is known as the *Bellman equation*, and its principal explanatory virtue is that it breaks the computation of a complex value function into the recursive computation of simpler sub-problems. Equation 36 is recursive because the Crusonia plant's mass in Period i + 1, and hence how much Crusonia Friday can possibly consume, depends on how much Crusonia he chose to consume in Period i.

Friday faces a dynamic constraint that changes over time as a function of the controls because the Crusonia plant is a dynamical system whose state changes based on Friday's choices. I therefore need an *equation of motion* that keeps track of how the Crusonia plant's mass changes over time and I can write that equation as follows:

$$a(t) = \begin{cases} a_0 & \text{for } t = 1 \\ \pi a_{t-1} & \text{for } t > 1 \end{cases} \tag{33}$$

where a_{t-1} is the quantity of Crusonia that Friday abstained from in Period t-1. Although it is constrained by the plant's initial mass in Period 1, in every subsequent period, Friday's choice of consumption is constrained by that portion of the plant's mass that he abstained from in the previous period. (For example, in Period 7, Friday's choice would be constrained by the quantity of Crusonia that he abstained from in Period 6, plus its natural growth.) The Crusonia plant's equation of motion, Equation 36, is therefore a piecewise function. Now, I can use that equation of motion to express my Bellman equation $V(c_i, c_{i+t}$ as a function of a single variable, a_t. First, I note that:

$$c_t = \pi a_{t-1} - a_t \tag{34}$$

$$c_{t+1} = \pi a_t - a_{t+1} \tag{35}$$

I can insert the right-hand sides of Equations 37 and 38 into my Bellman equation as follows:

$$\begin{aligned} V(c_i) &= Max\{U(\pi a_{t-1} - a_t) + V(\pi a_t - a_{t+1})\} \\ &= Max\{U(\pi a_{t-1} - a_t) + \sum_{t=i+1}^{\infty} \beta^{t-1} U(\pi a_{t-1} - a_t)\} \end{aligned} \tag{36}$$

This version of my Bellman equation expresses the value function in a way that I can apply the calculus to solve for the stream of consumption that optimizes the Crusonia plant's utility to Friday.

$$\frac{\partial V}{a_t} = -U'(\pi a_{t-1} - a_t) + \pi\beta U'(\pi a_{t-1} - a_t) = 0 \tag{37}$$

I can then use that first-order condition to solve for Friday's optimal policy:

$$\begin{aligned} -U'(\pi a_{t-1} - a_t) + \pi\beta U'(\pi a_{t-1} - a_t) &= 0 \\ \frac{U'(\pi a_{t-1} - a_t)}{\beta U'(\pi a_{t-1} - a_t)} &= \pi \end{aligned} \tag{38}$$

I can then substitute the left-hand sides of Equations 37 and 38 to arrive at a familiar optimality condition:

$$\frac{U'(c_t)}{\beta U'(c_{t+1}} = \pi \tag{39}$$

Once again, I find that Friday makes the best possible use of his Crusonia plant when, in every period, he consumes Crusonia up until the point that his intertemporal marginal rate of substitution equals the Crusonia plant's marginal rate of transformation. Even in the case in which he faces an infinite time horizon, Friday's economic problem is solved by persistently applying the rule that the two-period model made evident with Figure II.

I need to end my discussion of the Bellman equation by reinforcing the lesson that the value function specifies the highest value that a function can take.

6

I exaggerate Friday's time preference to vividly display the discount factor's effect on the curve's shape

7 Figures

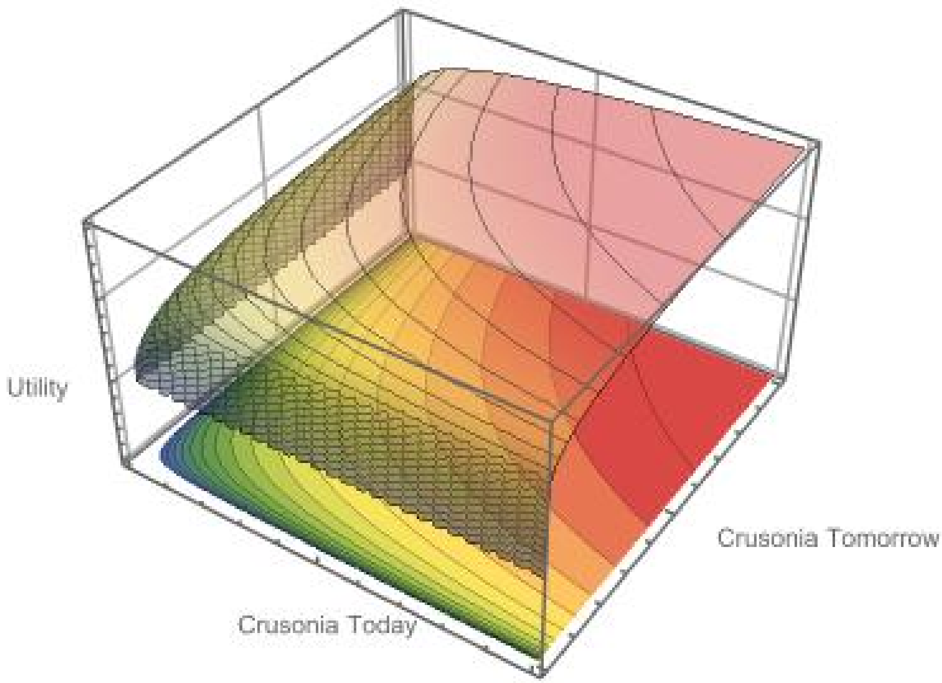

Fig. 1: This figure demonstrates how an indifference curve map that describes Friday's preferences between bundles of Crusonia today and Crusonia can be derived by projecting the associated utility surface's level curves on the goods plane.

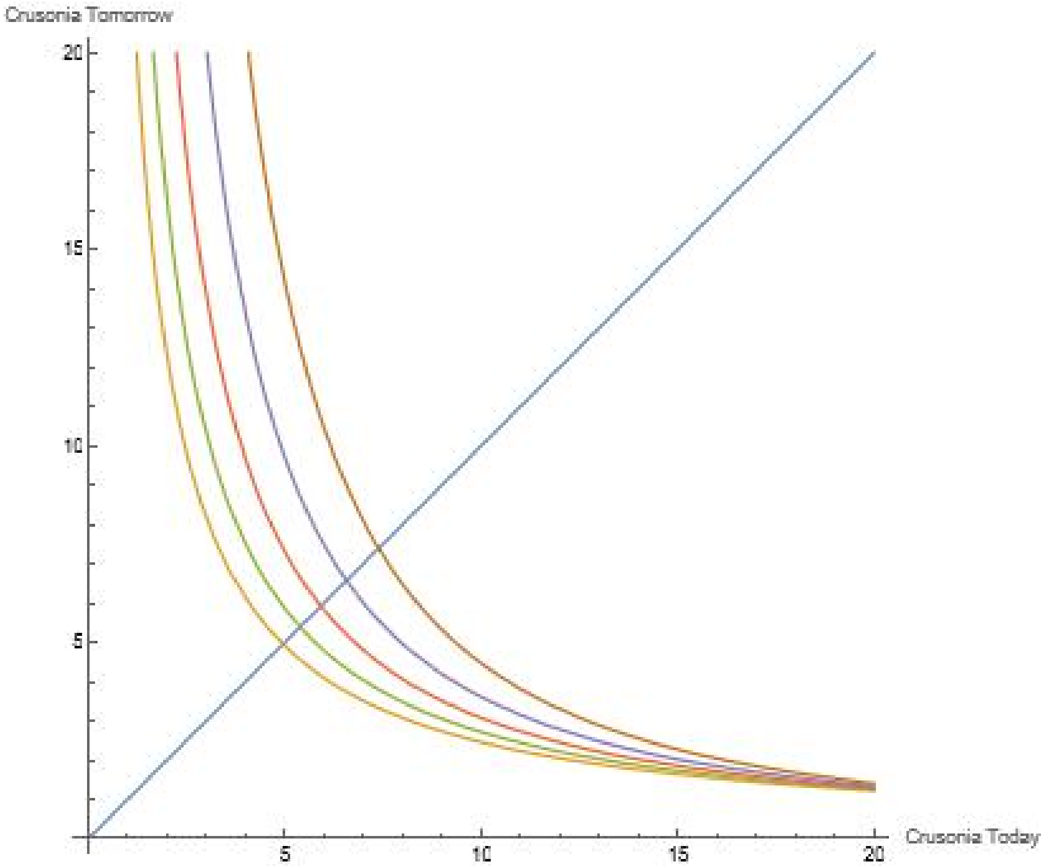

Fig. 2: The figure above illustrates the effect of varying time preference on a given indifference curve, $\bar{c} = ln(c_1) + \beta ln(c_2)$. The straight line is a 45° line. In the furthest curve to the left, Friday weights present and future consumption equally, $\beta = 1.00$. However, in each following curve, he discounts future consumption's utility by an ever greater amount, with β decreasing by increments of one tenth each curve. The greater his time preference, the more future Crusonia it requires to induce Friday to substitute that future Crusonia for some increment of present Crusonia, hence the increasing slopes of the curves from left to right. In addition, the more he discounts future consumption, the slower Friday's indifference curve approaches its asymptote along the future axis.

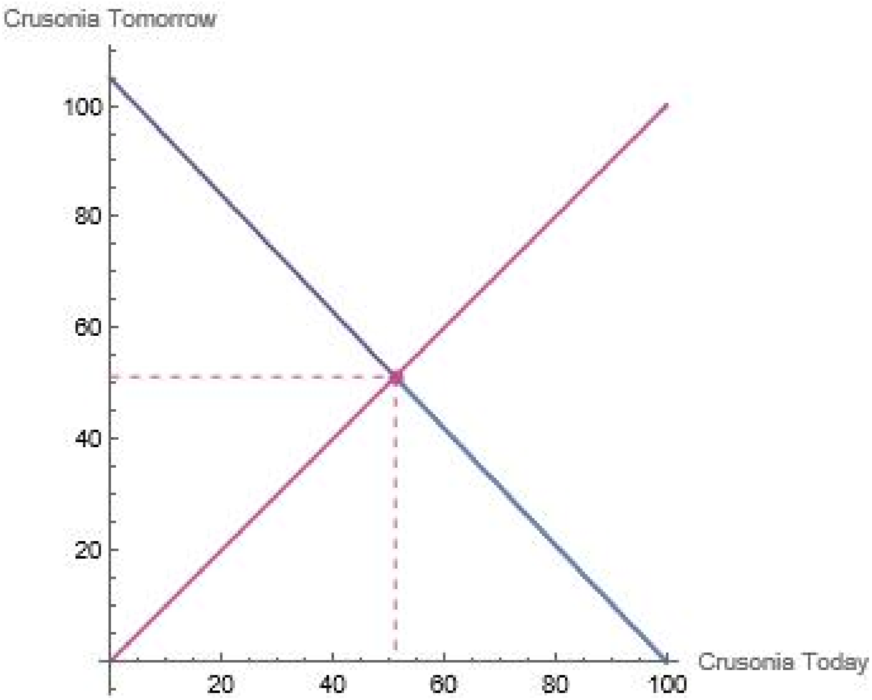

Fig. 3: Friday's *income* can be graphically illustrated as the bundle of dated Crusonia at which a 45° line intersects his budget constraint. Friday *saves* when he chooses a bundle to the left of that intersection and he *dissaves* when he chooses a bundle to its right. By saving, Friday increases his future consumption relative to his present consumption, and *vice versa* for dissaving.

$$\begin{aligned} V(c_i, c_{i+1}) &= Max\{U(c_i) + V(c_{i+1})\} \\ &= Max\{U(c_i) + \sum_{t=i+1}^{\infty} \beta^{t-1} U(c_t)\} \end{aligned} \tag{40}$$

$$a(t) = \begin{cases} a_0 & \text{for } t = 1 \\ \pi a_{t-1} & \text{for } t > 1 \end{cases} \tag{41}$$

where a(t) is the equation of motion that describes the evolution of the Crusonia plant's mass across periods of time.

$$V(a_i) = Max\{U(\pi a_{i-1} - a_i) + V(\pi a_t - a_{t+1})\} \tag{42}$$

$$a_{t-1} = \begin{cases} c_0 & \text{for } t = 1 \\ \pi a_{t-1} & \text{for } t > 1 \end{cases} \tag{43}$$

$$\begin{cases} c_0 & t = 1 \\ \pi a_{t-1} & t > 1 \end{cases} \tag{44}$$

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$$\psi(u) = \int_o^T \left[\frac{1}{2} \left(\Lambda_o^{-1} u, u \right) + N^*(-u) \right] dt \; . \tag{45}$$

$$\begin{multline*} p(x) = 3x^6 + 14x^5y + 590x^4y^2 + 19x^3y^3 \\ - 12x^2y^4 - 12xy^5 + 2y^6 - a^3b^3 \end{multline*}$$

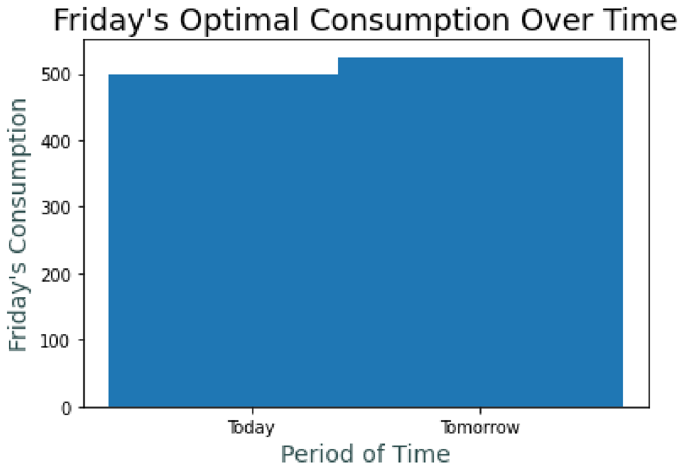
$$\begin{array}{rl} a+b+c &= d \\ e+f &= g \\ h &= i \end{array} \tag{46}$$

$$\begin{aligned} a+b+c &= d \\ e+f &= g \\ h &= i \end{aligned} \tag{47}$$

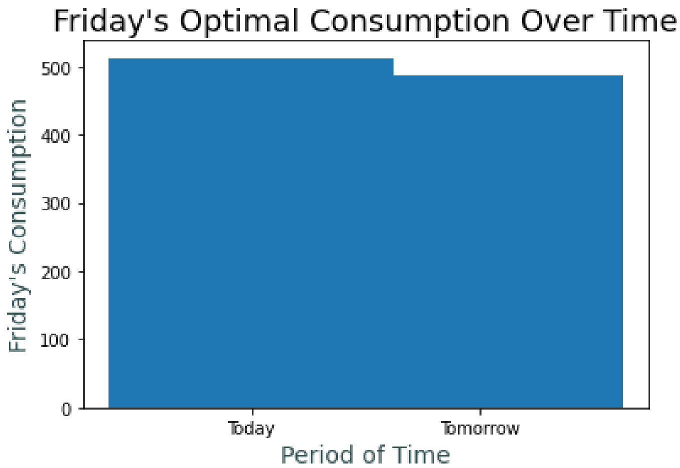
$$\partial \mathcal{L} / \partial c_1 = \qquad U'(c_1) - \lambda = 0 \tag{48a}$$

$$\partial \mathcal{L} / \partial c_2 = \qquad \beta U'(c2) - \pi^{-1} \lambda = 0 \tag{48b}$$

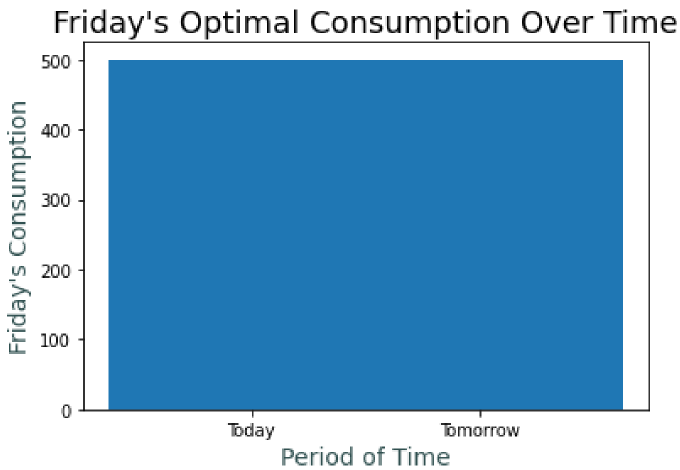
$$\partial \mathcal{L} / \partial \lambda = \qquad c_0 - c_1 - \pi^{-1} C_2 = 0 \tag{48c}$$



(a) $\pi > \beta$



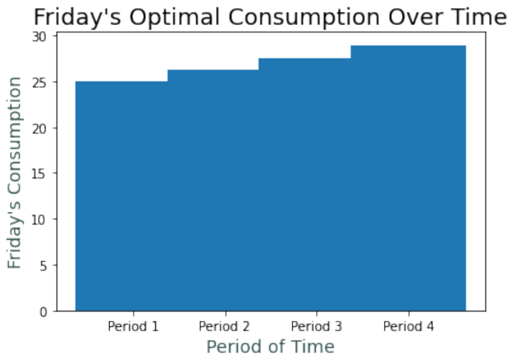
(b) $\pi < \beta$



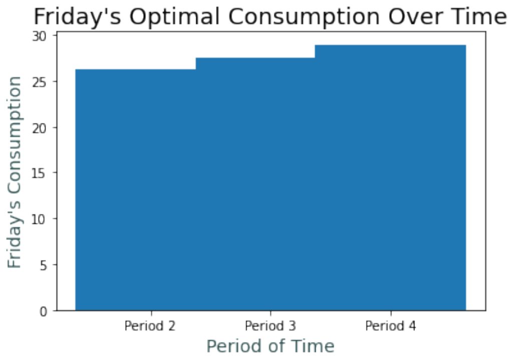
(c) $\pi = \beta$

Fig. 4: Three possible shapes of Friday's optimal harvesting policy given $c_0 = 1,000$

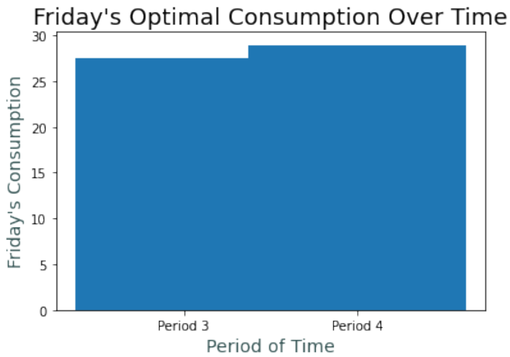
.



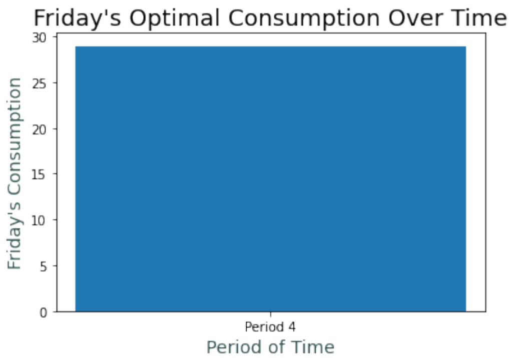
(a) Friday's economic problem from the perspective of Period 1.



(b) Friday's economic problem from the perspective of Period 2.



(c) Friday's economic problem from the perspective of Period 3.



(d) Friday's economic problem from the perspective of Period 4.

Fig. 5: Friday's dovetailing policies chosen across four periods.

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Period	First Policy	Second Policy	Third Policy	Fourth Policy
1	25 kgs	0	0	0
2	26.25 kgs	26.25 kgs	0	0
3	27.56 kgs	27.56 kgs	27.56 kgs	0
4	28.941 kgs	28.941 kgs	28.941 kgs	28.941 kgs

Fig. 6: Friday's optimal harvesting policies over four periods considered numerically. Here, I assume that $c_0 = 100$, $\beta = 0$, and $\pi = 1.05$.

For example, see (48). Or see equations (48a) and (48b).

$$\partial\mathcal{L}/\partial c_1 = U'(c_1) - \lambda = 0 \tag{49a}$$
$$\partial\mathcal{L}/\partial\lambda = \beta U'(c2) - \pi^{-1}\lambda = 0 \tag{49b}$$
$$\partial\mathcal{L}/\partial\lambda = c_0 - c_1 - \pi^{-1}C_2 = 0 \tag{49c}$$

$$m'_j = 0 \quad \Longrightarrow \quad m_{ij} = -m_j \tag{50}$$

8 Editing-Room Floor

Those distinct effects can be perceived in an indifference curve's shape as depicted in Figure II: Diminishing marginal utility occasions the indifference curve to bow out, but time preference occasions the curve to approach its asymptote along the future axis relatively slowly compared to its asymptote along the present axis. The convex shape of his indifference curves illustrates that, given the fact that his consumption is characterized by diminishing marginal utility, Friday can best jointly satisfy his present and future demands by allocating his scarce between present and future consumption so that their marginal utilities are equal. However, Friday's time preference might also induce him to discount the utility of future relative to present consumption and to therefore allocate more Crusonia to the latter than the former. The higher his time preference is, the steeper are his indifference curves and the more Crusonia tomorrow it takes to induce Friday to forgo a unit of Crusonia today.

Literature Review The analogy probably has its genesis in John Bates Clark's (1908: 313-4) use of a botanical metaphor when arguing against the notion of the average period of production as practically relevant. Clark asked his readers to imagine that, within that managed forest, an equal number of trees are planted and harvested in the same period of time so that the forest permanently maintains its size, as measured in number of trees. The forest is thus a permanent fund that yields a constant supply of wood over time; it is irrelevant to the people harvesting that wood whether the associated trees took 10, 20, or even more years to grow. Clark used that metaphor to argue that a permanent fund of capital served to synchronize production and consumption.

It was Frank Knight who invented the analogy of the Crusonia plant as we know it today, and, much like his predecessor Clark, Knight deployed this analogy for polemical purposes, in order to argue that the use of capital enabled consumption and production to occur simultaneously. Knight (1944: 30) described a society "living on the natural growth of some perennial which grows at a constant (geometric) rate, except as new tissue is cut away for consumption."[23] Knight used that model to entirely abstract capital from time and thereby argue, contra time-preference theory, that the rate of interest derived entirely from the technical marginal productivity of capital. Capital, Knight argued, is thus a permanent stock that yielded a flow of income in the form of interest.

At least in a stationary state, he argued that all factors shared the same economic form as instruments that cooperate in the production of services. Those factors yield services over time and their value can thus be computed on the basis of present value of those services. Through their capitalized present values, all factors of production can then be summed up in the wider abstract fund of productive capabilities.

Simultaneity of Production and Consumption: Although the claim might seem puzzling (even on a second, third, or even fourth view), it makes more sense when one considers the fact that Knight argued that production only concluded in the rendering of services. F. Knight (1935b: 4) criticized Classical theory for supposing that production consisted in the production of wealth: "Production was defined as production of *wealth*. But in fact, primary production consists in the rendering of *services*. Wealth is an agency by which services are rendered, not a product in the primary sense." Patkinson (1973: 792) recounted how this point influenced Knight's classroom teaching at Chicago University: "Knight also emphasized that essentially what was produced (and, even more so, consumed) was not goods, but services though (he admitted) the distinction tended to vanish in the case of perishable goods which were immediately destroyed in the process of consumption."

Equivalence of All Factors of Production: At least in the stationary state that concern economics proper, Knight argued that almost all productive agencies, or factors of production as we might call them today, shared the same economic form as productive instruments and whose values can be computed in terms of the income that they can yield in perpetuity. As productive instruments, all factors cooperate in the production of services, as well as the maintenance and replacement of other used-up factors. As Knight (1935c: 196f) wrote: "All the productive agencies use each other (or are used jointly by the enterprise), and

[23] Unlike in subsequent iterations of the model, Knight titled the society living off of the plant's growth, rather than the plant itself, 'Crusonia.' To my knowledge the term 'Crusonia' was first invented by the English radical Thomas Spence (1782) to describe the island that Robinson Crusoe was marooned on. Within Spence's fictional history, Crusonia was given its name in honor of Crusoe later inhabitants of the island.

all are jointly maintained and reproduced by all, in a continuing process." The value of each factor of production can then be computed in terms of the the net returns that can yield in perpetuity after incurring costs for maintaining and replacing that factor; those net returns can then either be consumed or invested in order to obtain more returns in the future from one's factors. The economic problem of best using any factor of production by maximizing its net returns leads to those factors becoming functionally equivalent to capital. As Knight (1944: 31) argues: "What is essential for the theory of the return on capital in any situation is merely that some complex of 'productive agents,' used alone or in co-operation with others, maintains itself and yields an excess for consumption or reinvestment at the will of the owner." Knight (1944: 33f) then emphasizes that: "In connection with any social change involving reallocation of production, most productive agents become more or less assimilated to capital goods and have to be considered by the theorist as quantities of capital." The only exception to Knight's argument are those original and indestructible forces of nature that the Classical schema classified as land, but Knight argued that there were exceedingly few cases of such forces.