Combinatorics 2018 Fall

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Recall.

$\mathbb{A}(P) = \{f : P \times P \to \mathbb{R} | f(x, y) = 0 \text{ whenever } x \nleqslant y\}$

$f * g :\ (f * g)(x, y) = \sum\limits_{z: x \leqslant z \leqslant y} f(x, z) g(z, y)$ for $x \leqslant y$

identity: $\delta(x, y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}.$

Zeta function: $\zeta(x, y) = \begin{cases} 1, & x \leqslant y \\ 0, & x \nleqslant y \end{cases}.$

Möbius function: $\mu_P = \zeta^{-1}$ where

$$\mu_P(x, y) = \begin{cases} 1, & x = y \\ -\sum\limits_{x < z \leqslant y} \mu_P(z, y) = -\sum\limits_{x \leqslant z < y} \mu_P(x, z), & x < y \\ 0, & else \end{cases}$$

Theorem (Inversion Formula I). Suppose P has a minimum element m (i.e $m \leqslant x,\ \forall\, x \in P$). Let f, g be functions $P \to \mathbb{R}$, then

$$f(y) = \sum_{z \leqslant y} g(z),\ \forall\, y \in P \Longleftrightarrow g(y) = \sum_{z \leqslant y} f(z) \mu_P(z, y),\ \forall\, y \in P.$$

<u>proof:</u> Define $\overline{f}(x, y) = \begin{cases} f(y), & x = m \\ 0, & x \neq m \end{cases}$. $\overline{g}(x, y) = \begin{cases} g(y), & x = m \\ 0, & x \neq m \end{cases}.$

Then $\overline{f}, \overline{g} \in \mathbb{A}(P)$.
Claim: $\overline{f} = \overline{g} * \zeta$
Proof of Claim: If $x = m$, then $\overline{f}(m, y) = f(y) = \sum\limits_{z \leqslant y} g(z) = \sum\limits_{z: m \leqslant z \leqslant y} \overline{g}(m, z)\zeta(z, y) = (\overline{g} * \zeta)(m, y)$. If $x > m$, then $\overline{f}(x, y) = 0 = \sum\limits_{z: x \leqslant z \leqslant y} \overline{g}(x, z)\zeta(z, y) = (\overline{g} * \zeta)(x, y)$.
So $\overline{g} = \overline{f} * \mu_P$, which implies

$$g(y) = \overline{g}(m, y) = \sum_{z: m \leqslant z \leqslant y} \overline{f}(m, z)\mu_P(z, y) = \sum_{z \leqslant y} f(z)\mu_P(z, y).$$

□

Theorem (Inversion Formula II). Suppose P has a maximal element M (i.e $x \leqslant M,\ \forall\ x \in P$). Let f, g be functions $P \to \mathbb{R}$, then

$$f(x) = \sum_{x \leqslant z} g(z),\ \forall\ x \in P \Longleftrightarrow g(x) = \sum_{x \leqslant z} \mu_P(x, z) f(z),\ \forall\ x \in P.$$

proof: Similar to the proof above !
Define $\overline{f}(x, y) = \begin{cases} f(x), & y = M \\ 0, & y \neq M \end{cases}$. $\overline{g}(x, y) = \begin{cases} g(x), & y = M \\ 0, & y \neq M \end{cases}$.
Claim: $\overline{f} = \zeta * \overline{g}$.
So $\overline{g} = \mu_P * \overline{f}$, which implies

$$g(x) = \overline{g}(x, M) = \sum_{z: x \leqslant z \leqslant M} \mu_P(x, z)\overline{f}(z, M) = \sum_{x \leqslant z} \mu_P(x, z) f(z).$$

□

Example. $P = C_n = \{x_1 < \cdots < x_n\}$ is a totally ordered set of size n.
$\forall\ f \in \mathbb{A}(P)$, define A_f by $A_f(i, j) = f(x_i, x_j)$, then A_f is an upper triangle matrix.
Claim: $A_{f*g} = A_f \cdot A_g$.

Pf of Claim:

$$\begin{aligned}
A_{f*g}(i,j) &= (f*g)(x_i,x_j) = \sum_{x_i \leqslant z \leqslant x_j} f(x_i,z)g(z,x_j) \\
&= \sum_{z\in P} f(x_i,z)g(z,x_j) = \sum_{k=1}^{n} f(x_i,x_k)g(x_k,x_j) \\
&= \sum_{k=1}^{n} A_f(i,k)A_g(k,j)
\end{aligned}$$

Definition (linear extension). $P=(X,\leqslant)$, $|X|=n$, linear extension of P is permutation $x_1x_2\cdots x_n$ such that $x_i<x_j$ implies $i<j$.

Fact. Any finite poset has a linear extension.

Lemma. μ_P is Möbius function of P.

(1) If $\varphi: P\to Q$ is an isomorphism (i.e. $x\leqslant y \Leftrightarrow \varphi(x)\leqslant\varphi(y)$), then $\mu_P(x,y)=\mu_Q(\varphi(x),\varphi(y))$.

(2) $u,v\in P$, $P_1=[u,v]$, then $\mu_{P_1}(x,y)=\mu_P(x,y)$ for $x,y\in[u,v]$.

(3) If P is a direct product of $(P_i,\leqslant_i)$ for $i\in[k]$, i.e. $P=P_1\times P_2\times\cdots\times P_k$ and $(x_1,x_2,\cdots,x_k)\leqslant(y_1,y_2,\cdots,y_k)\Leftrightarrow x_i\leqslant_i y_i$ for $i\in[k]$, then $\mu_P(x,y)=\prod_{i=1}^{k}\mu_{P_i}(x_i,y_i)$.

Example. Still consider the Example above.
By Claim, we have $A_{\mu_{C_n}}=A_{\zeta^{-1}}=(A_\zeta)^{-1}$.
So $\mu_{C_n}(x,y)=\begin{cases}1, & x=y\\ -1, & y\ cover\ x,\ i.e.[x,y]=\{x,y\}for\ x<y\\ 0, & else\end{cases}$
Let $P=(\mathbb{Z}_{\geqslant 0},\leqslant)$: usual order.
Lemma $\Rightarrow \mu_P(x,y)=\mu_{[1,n]}(x,y)=\mu_{C_n}(\varphi(x),\varphi(y))$.
So $\mu_P(x,y)=\begin{cases}1, & x=y\\ -1, & y=x+1\\ 0, & else\end{cases}$
$f(y)=\sum_{z\leqslant y} g(z) \Longleftrightarrow g(y)=\sum_{z\leqslant y} f(z)\mu_P(z,y)=f(y)-f(y-1)$

Example. $\Omega = (\mathbb{Z}_{>0}, \leqslant)$ is the divisor poset, i.e. $a \leqslant b \Leftrightarrow a \mid b$, then $\mu_\Omega(x, y) = \mu(\dfrac{y}{x})$ if $x \mid y$.
<u>proof</u>: If $x \mid y$, then $[x, y] \approx [1, \dfrac{y}{x}]$ by considering $\varphi : z \to \dfrac{z}{x}$.
By Lemma, we have $\mu_\Omega(x, y) = \mu_{[x,y]}(x, y) = \mu_{[1,\frac{y}{x}]}(1, \dfrac{y}{x}) \triangleq f(\dfrac{y}{x})$
Next, we show: $f(n) = \mu(n)$ for $\forall\ n > 0$.
If $n = 1$, then $\mu(1) = 1 = f(1)$.
If $n > 1$, let $n = p_1^{e_1} \cdots p_k^{e_k}$. Then $[1, n] = \{p_1^{h_1} \cdots p_k^{h_k} : 0 \leqslant h_i \leqslant e_i\}$. So $[1, n] \approx \{(h_1, \cdots, h_k) :\ 0 \leqslant h_i \leqslant e_i\}$. Define $\overline{\Omega} = \{(h_1, \cdots, h_k) : 0 \leqslant h_i \leqslant e_i\}$, where $(h_1, \cdots, h_k) \leqslant (t_1, \cdots, t_k) \Leftrightarrow p_1^{h_1} \cdots p_k^{h_k} \mid p_1^{t_1} \cdots p_k^{t_k} \Leftrightarrow h_i \leqslant t_i$ for $\forall\ i \in [k]$. Let $\Omega_i = \{0, 1, \cdots, e_i\}$: usual order. Then $\overline{\Omega} = \Omega_1 \times \cdots \times \Omega_k$. By Lemma, we have

$$
\begin{aligned}
f(n) &= \mu_{[1,n]}(1, n) = \mu_{\overline{\Omega}}((0, \cdots 0), (e_1, \cdots, e_k)) = \prod_{i=1}^{k} \mu_{\Omega_i}(0, e_i) \\
&= \begin{cases} (-1)^k & e_1 = \cdots = e_k = 1 \\ 0, & e_i \geqslant 2\ for\ some\ i \end{cases}
\end{aligned}
$$

which implies $f(n) = \mu(n)$.