Combinatorics 2018 Fall

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Theorem 1. *(**Hall's marriage Theorem**) The sets of $S_1, S_2, \cdots, S_m$ has a SDR $\Longleftrightarrow |\cup_{i\in I} S_i| \geq |I|$ for $I \subset [m]$(Hall's condition).*

Corollary 1. *$|X| = n$, $|S_i| = r, S_i \subset X, i \in [m].s.t.|\{i : x \in S_i\}| = d$ for all $x \in X$. If $m \leq n$, then $S_1, \cdots, S_m$ have a SDR.*

Theorem 2. *Suppose elements in X are colored by either red or blue. $S_i \subset X, i \in [m]$, then $S_1, \cdots, S_m$ have a SDR with $\leq t$ red elements iff $S_1, \cdots, S_m$ have a SDR and $\forall I \subset [m], \cup_{i\in I} S_i$ has $\geq |I| - t$ blue elements.*

proof:
"$\Longrightarrow$" Let $x_1, \cdots, x_m$ be a SDR of $S_1, \cdots, S_m$ with $\leq t$ red elements. then $\forall I \subset [m]$, $\{x_i, i \in I\}$ has at least $|I|-t$ blue elements $\Rightarrow \cup_{i\in I} S_i$ has at least $|I| - t$ blue elements.
"$\Longleftarrow$" Let R be the set of red elements in X. If $|R| \leq t$, trivial. Assume $|R| > t$, let $S_{m+1} = S_{m+2} = \cdots = S_{m+r} = R$, where $r = |R| - t$. then $S_1, \cdots, S_m$ have a SDR with $\leq t$ red elements $\Longleftrightarrow S_1, \cdots, S_m, S_{m+1}, \cdots, S_{m+r}$ have a SDR. So we need to check Hall's conditio n for $S_1, \cdots, S_{m+r}$, let $Y = \cup_{i\in I} S_i$, if $I \subset [m]$, then $|Y| \geq |I|$ since $S_1, \cdots, S_m$ have a SDR. if $I = J_1 \cup J_2$, where $J_1 \subset [m], J_2 \subset [m+1, m+r]$, then $|J_2| \leq |R| - t, |Y| = |\cup_{i\in J_1} (S_i \backslash R)| + |R| \geq |J_1| - t + |R| = |J_1| + (|R| - t) \geq |J_1| + |J_2| = |I|$.

□

Application

Def: A $r \times n (r \leq n)$ Latin rectangle is $r \times n$ matrix over $[n]$ s.t.numbers $1, 2, \cdots, n$ occurs once in each row and $\leq$once in each column. A Latin square is an $n \times n$ Latin rectangle.

Theorem 3. *(**Evans conjecture**) If fewer than n cells of an $n \times n$ matrix are filled, then one can always complete it into a Latin square.*

Theorem 4. *If $r < n$, then any given $r \times n$ Latin rectangle can be extended to an $(r+1) \times n$ Latin rectangle.*

proof:
Let R be $r \times n$ LR, For $j \in [n]$, let S_j be the set of integers in $[n]$ which don't occur in the j-th column. Then it suffices to prove $S_1, \cdots, S_n$ have a SDR. Since $|S_j| = n - r$, and each $i \in [n], i$ occurs in $n - r$ sets S_j, by Corollary1, $S_1, \cdots, S_n$ have a SDR.

□

Def: An $n \times n$ matrix $A = \{A_{ij}\}$ with $a_{ij} \geq 0$ is called doubly stochastic if $\sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ij} = 1$ for $\forall i, j \in [n]$. If $a_{ij} = 0$ or 1, then it is a permutation matrix.

Theorem 5. *(**Birkhoff**) Every doubly stochastic matrix A is a convex combination of permutation matrixes, that is, $\exists$ permutation matrixes $P_1, \cdots, P_s$ and non-negative reals $\lambda_1, \cdots, \lambda_s$ s.t.$A = \sum_{i=1}^{s} \lambda_i P_i$ and $\sum_{i=1}^{s} \lambda_i = 1$.*

proof:
Let A be an $n \times n$ doubly stochastic matrix, let m be the number of non-zero entries in A, then $m \geq n$. prove by induction on m. If $m = n$, then each non-zero entry is 1, so A itself is a permutation matrix. If $m > n$ and the results holds for matrices with $< m$ non-zero entries. Define $S_i = \{j : a_{ij} > 0\}, i \in [n]$. If for some of the sets $S_{i_1}, S_{i_2}, \cdots, S_{i_k}, |\cup_{i=1}^{k} S_{i_k}| \leq k-1$. that is all non-zero entries in rows $i_1, \cdots, i_k$ occupy at most $k-1$ columns, say columns $j_1, \cdots, j_{k-1}$, if count by rows, we have the sum is k, but if count by columns, the sum is at most $k-1$, a contradiction. By Hall's Theorem, there is a SDR $j_1 \in S_1, j_2 \in S_2, \cdots, j_n \in S_n$. Take a permutation matrix $P_1 =$

(P_{ij}) with entries $p_{ij} = 1$ iff $j = j_i$. Let $\lambda_1 = min\{a_{1j_1}, \cdots, a_{nj_n}\}$. and consider $B_1 = A - \lambda_1 P_1$. By definition of S_i, we have $\lambda_1 > 0$, matrix B_1 has at most $m-1$ non-zero entries, and the row sum and column sum of B_1 is $1 - \lambda_1$. Let $A_1 = \frac{1}{1-\lambda_1} B_1$, then A_1 is a doubly stochastic matrix with less than m non-zero entries. By assumption $A_1 = \mu_2 P_2 + \cdots + \mu_s P_s$ a convex combination. Hence, $A = \lambda_1 P_1 + (1-\lambda_1)A_1 = \lambda_1 P_1 + (1-\lambda_1)\mu_2 P_2 + \cdots + (1-\lambda_1)\mu_s P_s$. Since $\sum_{i=2}^{s} \mu_i = 1$, we have $\lambda_1 + (1-\lambda_1)(\sum \mu_i) = 1$.

□

Def: Let $S_1, \cdots, S_m \subseteq X = \{x_1, \cdots, x_n\}$, and $M = (a_{ij})$ be the corresponding incidence matrix. The **permanent** of M is

$$Per(M) = \sum_{(i_1, \cdots, i_m) \in S_n(m)} a_{i_1 1} a_{i_2 2} \cdots a_{i_m m},$$

where $S_n(m)$ is the set of all vectors of length m over $[n]$ without repetition.
Fact: $Per(M) = \sharp$different SDR's of $S_1, \cdots, S_m$.

Def: Let A be a 0-1 matrix. Two 1's are dependent if they are in the same row or the same column, otherwise, they are independent.

<u>Theorem</u> 6. *(**König**) Let A be an $m \times n$ 0-1 matrix, then max $\sharp$ independent 1's r = the min $\sharp$ rows and columns R required to cover all 1's in A.*

<u>proof:</u> Clearly, $R \geq r$, since we can find r independent 1's and every row or column covers at most one of them.
Now we show $r \geq R$. Assume that some a rows and b columns cover all 1's and $a + b = R$. We may assume the first a rows and the first b columns cover all the 1's. Write A as the form

$$A = \begin{pmatrix} B_{a \times b} & C_{a \times (n-b)} \\ D_{(m-a) \times b} & E_{(m-a) \times (n-b)} \end{pmatrix},$$

with no 1 in $E_{(m-a)\times(n-b)}$. If we can show that there are a independent 1's in C and b independent 1's in D, then we find at least $a + b$ independent 1's, so we have $r \geq a + b = R$.
For each $1 \leq i \leq a$, let $S_i = \{j : \ c_{ij} = 1\} \subseteq [n-b]$. If $S_1, \cdots, S_a$ have an SDR, then we find a independent 1's in C If not, by Hall's

theorem, there are some $k \in [a]$ sets, say $S_{i_1}, \cdots, S_{i_k}$, such that $\left| \bigcup\limits_{j=1}^{k} S_{i_j} \right| < k$, i.e. the 1's in these k rows occupy at most $k-1$ columns of C, say $j_1, \cdots, j_{k-1}$. Then the first b columns of A, the columns $j_1, \cdots, j_{k-1}$ of C and the first a rows of A deleting the rows $i_1, \cdots, i_k$ will cover all 1's in A. So we find $b+(k-1)+a-k = a+b-1$ rows and columns cover all 1's in A, contradiction!

$\square$