Combinatorics 2018 Fall

Teaching by: Professor Xiande Zhang

2018.09.17

Key words: estimate, IEP

Estimates

Definition. $f(n) = O(g(n))$ means $\exists$ constants n_0 and C such that for $\forall\ n \geqslant n_0$, the inequality $|f(n)| \leqslant C \cdot g(n)$ holds.

Fact. Let $C,\ \alpha,\ \beta,\ a > 0$ be fixed real numbers. Then

(1) $n^\alpha = O(n^\beta)$ if $\beta \geqslant \alpha$

(2) $n^C = O(a^n)$ if $a > 1$

(3) $(\ln n)^C = O(n^\alpha)$ if $\alpha > 0$

Definition.

(1) $f(n) = o(g(n)) \Leftrightarrow \lim\limits_{n\to\infty} \dfrac{f(n)}{g(n)} = 0$

(2) $f(n) = \Omega(g(n)) \Leftrightarrow g(n) = O(f(n))$

(3) $f(n) = \Theta(g(n)) \Leftrightarrow f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

(4) $f(n) \sim g(n) \Leftrightarrow \lim\limits_{n\to\infty} \dfrac{f(n)}{g(n)} = 1$

Theorem 1. For $\forall\ n \geqslant 1$, we have

$$e(\frac{n}{e})^n \leqslant n! \leqslant en(\frac{n}{e})^n$$

.

proof: Consider $\int_1^n \ln x dx$, then

$$\ln(n-1)! = \sum_{i=1}^{n-1} \ln i \leqslant \int_1^n \ln x dx \leqslant \sum_{i=1}^{n} \ln i = \ln n!$$

$$\Longrightarrow \ln(n-1)! \leqslant n \ln n - n + 1 \leqslant \ln n!$$

Thus,

$$(n-1)! \leqslant e^{n \ln n - n + 1} \leqslant n!$$

where $e^{n \ln n - n+1} = (e^{\ln n})^n e^{-n} e = (\frac{n}{e})^n e$.

Therefore,

$$e(\frac{n}{e})^n \leqslant n! \leqslant en(\frac{n}{e})^n$$

$\square$

Exercise.

(1) Prove Theorem 1 by induction using the fact: $1 + x \leqslant e^x$.

(2) Prove $n! \leqslant e\sqrt{n}(\frac{n}{e})^n$ by definite integral.

Stirling formula. $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$

Fact. $\max\{\binom{n}{k} : k = 0, 1, 2, \cdots, n\} = \begin{cases} \binom{n}{\frac{n}{2}}, if\ n\ is\ even; \\ \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}, if\ n\ is\ odd. \end{cases}$

Corollary. $\dfrac{2^n}{n+1} \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor} \leqslant 2^n$.

Stirling approximation. $\binom{n}{\lfloor \frac{n}{2} \rfloor} \sim \dfrac{2^n}{\sqrt{n}}\sqrt{\dfrac{2}{\pi}}$.

Theorem 2. For $1 \leqslant k \leqslant n$, we have

$$(\frac{n}{k})^k \leqslant \binom{n}{k} \leqslant (\frac{en}{k})^k$$

proof: For lower bound, note that $\dfrac{n}{k} \leqslant \dfrac{n-i}{k-i}$ for $\forall\ i < k$, then

$$(\frac{n}{k})^k \leqslant \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \dots \cdot \frac{n-k+1}{1} = \binom{n}{k}$$

For upper bound, note that for $0 < t < 1$, we have

$$\binom{n}{k} \leqslant \sum_{i=0}^{k} \binom{n}{i} \leqslant \sum_{i=0}^{k} \binom{n}{i} \frac{t^i}{t^k} \leqslant \sum_{i=0}^{n} \binom{n}{i} \frac{t^i}{t^k} = \frac{(1+t)^n}{t^k}$$

Let $t = \dfrac{k}{n} < 1$, then

$$\binom{n}{k} \leqslant \sum_{i=0}^{k} \binom{n}{i} \leqslant \frac{(1+t)^n}{t^k} = \frac{(1+\frac{k}{n})^n}{(\frac{k}{n})^k} \leqslant \frac{(e^{\frac{k}{n}})^n}{(\frac{k}{n})^k} = (\frac{en}{k})^k$$

$\square$

The Inclusion-exclusion Principle (IEP)

Let $A_1,\ A_2,\ \cdots, A_n$ be subsets of Ω (general set).
For $I \subseteq [n],\ A_I := \cap_{i \in I} A_i.\ A_\emptyset := \Omega.$

Theorem 3 (IEP).

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{I \in \binom{[n]}{k}} |A_I| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} |A_I|.$$

proof: Rewrite the right hand side

$$\sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} |A_I| = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \sum_{x \in A_I} 1 = \sum_{x \in \Omega} \sum_{\emptyset \neq I \subseteq [n] : x \in A_I} (-1)^{|I|+1}$$

Consider the contribution of each $x \in \Omega$ to both sides.
For the left hand side, $|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{x \in \Omega} \delta_x$, where $\delta_x = 1$ if $x \in A_1 \cup A_2 \cup \cdots \cup A_n$ and 0 otherwise.

For the right hand side, when $x \notin A_1 \cup A_2 \cup \cdots \cup A_n$, we have $\sum\limits_{\emptyset \neq I : x \in A_I} (-1)^{|I|+1} = 0$.
When $x \in A_1 \cup A_2 \cup \cdots \cup A_n$, let $J = \{j : \ x \in A_j\}$, then

$$\begin{aligned}\sum_{\emptyset \neq I : x \in A_I} (-1)^{|I|+1} = \sum_{\emptyset \neq I \subseteq J} (-1)^{|I|+1} &= \sum_{i=1}^{|J|} \binom{|J|}{i} (-1)^{i+1} \\ &= (-1) \sum_{i=1}^{|J|} \binom{|J|}{i} (-1)^i = (-1)[(1-1)^{|J|} - 1] = 1\end{aligned}$$

by the Binomial Theorem. $\square$

Exercise. Prove Theorem 3 by induction on n.

Theorem 4. $|A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = \sum\limits_{I \subseteq [n]} (-1)^{|I|} |A_I|$.

proof: $|A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = |\Omega| - |A_1 \cup A_2 \cup \cdots \cup A_n| = |A_\emptyset| - \sum\limits_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} |A_I| = |A_\emptyset| + \sum\limits_{\emptyset \neq I \subseteq [n]} (-1)^{|I|} |A_I| = \sum\limits_{I \subseteq [n]} (-1)^{|I|} |A_I|$. $\square$

Applications of IEP

Recall.

(1) $S(n,k) = \#$ partitions of $[n]$ into k non-empty parts.

(2) $S(n,k) \cdot k! = \#$ surjections from $[n]$ to $[k]$.

Proposition 1. $S(n,k) = \dfrac{1}{k!} \sum\limits_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$

proof: It suffices to show that the number of surjections from $[n]$ to $[k]$ is $\sum\limits_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$.
Let $X = [k]$, $Y = [n]$, $\Omega = X^Y$.
Define $A_i = \{f : \ Y \to X \backslash \{i\}\}$ for $i \in [k]$, then $|A_i| = (k-1)^n$, and $A_I = \cap_{i \in I} A_i = \{f : \ Y \to X \backslash I\}$ for $I \subseteq [k]$ with $|A_I| = (k - |I|)^n$.
Note that $\#$ surjections from $[n]$ to $[k]$ is $A_1^c \cap A_2^c \cap \cdots \cap A_k^c$. Using

IEP, we can easily get what we want. $\square$

Definition. $\varphi(n) = \#$ integers $m \in [n]$ $s.t.$ $gcd(m, n) = 1$.

Proposition 2. If $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ where p_i are distinct primes in $[n]$, then $\varphi(n) = n \prod_{i=1}^{t} (1 - \frac{1}{p_i})$.