Combinatorics 2018 Fall

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Key words: Polya Theorem

Recall (Polya Theorem) $|X| = n$, $|C| = m$, $G \leqslant S_n$. Then,

$$\# \text{ orbits in } C^X \text{ under } G = P_G(m, \cdots, m)$$

where $P_G(x_1, x_2, \cdots, x_n) = \dfrac{1}{|G|} \sum_{g \in G} x_1^{\lambda_1(g)} x_2^{\lambda_2(g)} \cdots x_n^{\lambda_n(g)}$.

Problem 1. Consider coloring of a regular cube in m colors under rotations in 3^{rd} space.

Solution. Note that rotations of the cube induce different automorphisms of vertices, edges and faces.
Since there are $3 \times 3 = 9$ centers of opposite faces, $1 \times 6 = 6$ centers of opposite edges, and $2 \times 4 = 8$ centers of opposite vertices, $|G| = 24$ (id is also an automorphism).

(1) Consider colorings of vertices $X = \{x_1, \cdots, x_8\}$, $G_1 = G$ restricted on vertices.
So, $P_{G_1}(x_1, \cdots, x_8) = \dfrac{1}{24}({x_1}^8 + 3(2{x_4}^2 + {x_2}^4) + 6{x_2}^4 + 2{x_1}^2{x_3}^2 \times 4)$,
which implies $P_{G_1}(m, \cdots, m) = \dfrac{1}{24}(m^8 + 17m^4 + 6m^2)$.

(2) Consider colorings of faces $X = \{x_1, \cdots, x_6\}$, $G_2 = G$ restricted on faces.
So, $P_{G_2}(x_1, \cdots, x_6) = \dfrac{1}{24}({x_1}^6 + 3(2{x_1}^2 x_4 + {x_1}^2{x_2}^2) + 6{x_2}^3 + 8{x_3}^2)$,
which implies $P_{G_2}(m, \cdots, m) = \dfrac{1}{24}(m^6 + 3m^4 + 12m^3 + 8m^2)$.

(3) **(Exercise.)** Consider colorings of edges $X = \{x_1, \cdots, x_{12}\}$

Problem 2. $G = Aut(H),\ X = \{A, B, C, D, E, F\},\ |C| = m.$

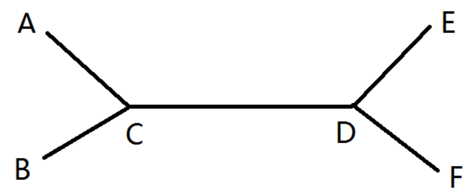

Figure 1: H

Consider C^X.

Solution.

(1) By considering the automorphisms of the graph H, we have $|Aut(H)| \geqslant 8$. (enumeration)

(2) $|Stab(A)| = 2$, $|Orbit(A)| = 4$, so $|G| = 8$.

Now it's easy to calculate $P_G(x_1, \cdots, x_6)$ by Polya Theorem.

Recall (Weighted Burnside Lemma) Finite group action (G, X). Let $\Omega_i = Orb(x_i),\ i \in [N(G)]$ be all distinct orbits. Let $\omega : X \to \mathbb{R}$ be a weighted function, where $\omega(x) = \omega(y)$ if $x, y \in \Omega_i$ for some $i \in [N(G)]$. Define $\omega(\Omega_i) = \omega(x_i)$, then

$$\sum_{i=1}^{N(G)} \omega(\Omega_i) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in Fig(x)} \omega(x).$$

Problem 3. The number of cycles of length n over $[m]$ such that i occurs k_i times, $i \in [m]$.

Definition. (G, X), C^X : colorings. Let $\omega : C \longrightarrow \mathbb{R}$ be any function, then ω induces a weight function of C^X as follows:

$$\forall\ f \in C^X, \text{define } \omega(f) = \prod_{x \in X} \omega(f(x)).$$

If $f_1,\ f_2 \in C^X$ are in the same orbit, i.e. $f_2 = g * f_1$, i.e. $f_2(x) = f_1(g^{-1} * x)$, then $\omega(f_2) = \prod_{x \in X} \omega(f_2(x)) = \prod_{x \in X} \omega(f_1(g^{-1} * x)) = \prod_{x \in X} \omega(f_1(x)) = \omega(f_1)$. i.e. ω has the same value on each orbit.

Theorem 1. (G, C^X), $\omega : C^X \longrightarrow \mathbb{R}$ is defined as above. $\mathcal{F} = \{$all orbits in $C^X\}$. Then

$$\sum_{F \in \mathcal{F}} \omega(F) = P_G(\sum_{c \in C} \omega(c), \sum_{c \in C} \omega(c)^2, \cdots, \sum_{c \in C} \omega(c)^n).$$

Corollary 1. (G, C^X), $|x| = n$, $C = \{c_1, c_2, \cdots, c_m\}$, $k_1 + k_2 + \cdots + k_m = n$, $k_i \geq 0$. Let $N(k_1, \cdots k_m)$ be the number of colorings in Problem 3. Then

$$P_G(\sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \cdots, \sum_{i=1}^{m} y_i^n) = \sum_{k_1 + \cdots + k_m = n} N(k_1, \cdots, k_m) y_1^{k_1} \cdots y_m^{k_m}.$$

proof: Let $\omega(c_i) = y_i$, $i \in [m]$. Let f be a coloring with c_i occuring k_i times, $i \in [m]$. Then

$$\omega(f) = \prod_{x \in X} \omega(f(x)) = y_1^{k_1} \cdots y_m^{k_m}$$

$$\begin{aligned} N(k_1, \cdots, k_m) &= \#\text{ orbits } F \text{ such that } \omega(F) = y_1^{k_1} \cdots y_m^{k_m} \\ &= \text{coefficient of } y_1^{k_1} \cdots y_m^{k_m} \text{ in } \sum_{F \in \mathcal{F}} \omega(F) \\ &= \text{coefficient of } y_1^{k_1} \cdots y_m^{k_m} \text{ in } P_G(\sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \cdots, \sum_{i=1}^{m} y_i^n) \end{aligned}$$

$\square$

Solution to Problem 3.
$G = \sigma_n$. $P_{\sigma_n}(x_1, \cdots, x_n) = \dfrac{1}{n}\sum_{d|n} \varphi(d)(x_d)^{\frac{n}{d}}$. Then

$$P_{\sigma_n}(\sum_{i=1}^{m} y_i, \sum_{i=1}^{m} y_i^2, \cdots, \sum_{i=1}^{m} y_i^n) = \frac{1}{n}\sum_{d|n} \varphi(d)(\sum_{i=1}^{m} y_i^d)^{\frac{n}{d}}.$$

Let $T(k_1, \cdots, k_m)$ be the coefficient of $y_1^{k_1} \cdots y_m^{k_m}$.
If $d \nmid k_i$, then no monomial $y_1^{k_1} \cdots y_m^{k_m}$.
So
$$T(k_1, \cdots, k_m) = \frac{1}{n} \sum_{d|(k_1, \cdots, k_m)} \varphi(d) \frac{(\frac{n}{d})!}{(\frac{k_1}{d})! \cdots (\frac{k_m}{d})!}$$

$\square$