Combinatorics 2018 Fall

Taught by: Professor Xiande Zhang

2018.10.25

Key words: Polya Thm, SDR

Recall:
$N(k_1, \cdots k_m) = \sharp$ colorings of C^X under G s.t. there are k_i elements of X have color c_i, $\sum\limits_{i\in[m]} k_i = |X| = n$
Weight Function: define $\omega : C \longrightarrow \mathbb{R}, \forall f \in C^X,\ \omega(f) = \prod\limits_{x\in X} \omega(f(x))$
Weighted Burnside Lemma: $\sum\limits_{i\in[N(G)]} \omega(\Omega_i) = \frac{1}{|G|} \sum\limits_{g\in G} \sum\limits_{x\in Fix(g)} \omega(x)$

Theorem 1. *(G, C^X), $\omega : C^X \longrightarrow \mathbb{R}$ as defined above. $\mathcal{F} = \{$all orbits in $C^X\}$, then*

$$\sum_{F\in\mathcal{F}} \omega(F) = P_G(\sum_{c\in C} \omega(c), \sum_{c\in C} \omega(c)^2, \cdots, \sum_{c\in C} \omega(c)^n).$$

Corollary 1.

$$P_G(\sum_{i=1}^m y_i, \sum_{i=1}^m y_i^2, \cdots, \sum_{i=1}^m y_i^n) = \sum_{k_1+\cdots+k_m=n} N(k_1, \cdots, k_m) y_1^{k_1} \cdots y_m^{k_m}.$$

proof of thm1:
$\forall g \in G$, let $k = k(g)$ be the number of cycles of g and $A_1, \cdots, A_k$ be the set of elements of each cycle. For $f \in Fix(g)$, suppose $f(x) = c_i$,

if $x \in A_i, i \in [k]$. Then $\omega(f) = \prod_{i=1}^{k}(\omega(c_i)^{|A_i|})$. So

$$\sum_{f \in Fix(g)} \omega(f) = \sum_{(c_1,\cdots,c_k)\in C^k} (\prod_{i=1}^{k}(\omega(c_i)^{|A_i|})) = \prod_{i=1}^{k}(\sum_{c\in C}\omega(c)^{|A_i|}).$$

Denote $type(g) = (\lambda_1(g), \cdots, \lambda_n(g))$, then

$$\sum_{f \in Fix(g)} \omega(f) = \prod_{i=1}^{n}(\sum_{c\in C}\omega(c)^i)^{\lambda_i(g)}$$

By weighted Burnside Lemma,

$$\begin{aligned}\sum_{F\in\mathcal{F}}\omega(F) &= \frac{1}{|G|}\sum_{g\in G}\sum_{f\in Fix(g)}\omega(f)\\ &= \frac{1}{|G|}\sum_{g\in G}(\prod_{i=1}^{n}(\sum_{c\in C}\omega(c)^i)^{\lambda_i(g)})\\ &= P_G(\sum_{c\in C}\omega(c), \sum_{c\in C}\omega(c)^2, \cdots, \sum_{c\in C}\omega(c)^n).\end{aligned}$$

□

Def: $(G, X), (H, Y)$, where $X \cap Y = \emptyset$. Let $G \times H = \{(g, h) : g \in G, h \in H\}$, define $(g, h) : X \cup Y \longrightarrow X \cup Y$ by

$$(g,h)*(a) = \begin{cases} g*a, & if\ a \in X \\ h*a, & if\ a \in Y. \end{cases}$$

Then $G \times H$ is permutation group of $X \cup Y$.

__**Lemma**__ **1.** *$(G \times H, X \cup Y), |X| = n, |Y| = m$. Then*

$$P_{G\times H}(x_1, \cdots, x_n, y_1, \cdots, y_m) = P_G(x_1, \cdots, x_n)P_H(y_1, \cdots, y_m)$$

Ex: prove the lemma.

__**Distribution Problem**__:
n balls with r colors, n_i balls with color i, $\sum_{i=1}^{r} n_i = n$, put them into m boxes: $B_1, \cdots, B_m$, s.t. there are k_i balls in B_j, $\sum_{i=1}^{m} k_i = n$

(1) If $n_i = 1, i \in [r]$ and $r = n$,then it is an ordered partition problem, s.t. the $i-$th part is of size $k_i, i \in [m]$, so $\frac{n!}{k_1!\cdots k_m!}$

(2) In general, it is a coloring problem as follows. $X_1, \cdots, X_r$, where $|X_i| = n_i$ with permutation group G_i, since each ball in X_i are considered to be the same, $G_i = S_{n_i}$. So we have a group action $S_{n_1} \times \cdots \times S_{n_r}$ over $X_1 \cup \cdots X_r$. By lemma 1, it has cycle index $\prod_{i=1}^{r} P_{s_i}(x_1, \cdots, x_{n_i})$. then

total number of distributions$=\prod_{i=1}^{r} P_{s_{n_i}}(m, \cdots, m)$

$N(k_1, \cdots, k_n) = [y_1^{k_1} \cdots y_m^{k_m}] \prod_{i=1}^{n} P_{S_{n_i}}(\sum_{l=1}^{m} y_l, \cdots, \sum_{l=1}^{m} y_l^{n_i})$

(3) If $r = 1$, all balls have same color, $n_1 = n$. then total the number of distributions is the number of sols to $k_1 + \cdots + k_m = n, k_i \geq 0.i.e.\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$. Since cycle index is $P_{s_n}(x_1, \cdots, x_n)$, so $P_{s_n}(m, \cdots, m) = \binom{n+m-1}{n}$. so $\prod_{i=1}^{r} P_{s_{n_i}}(m, \cdots, m) = \prod_{i=1}^{r} \binom{n_i+m-1}{n_i}$.

(4) $r = n, \ \ n_i = 1$, the number of n different balls into m membered boxes.

$$\prod_{i=1}^{n} P_{s_1}(m) = \prod_{i=1}^{n} m = m^n.$$

$N(k_1, \cdots, k_m) = \frac{n!}{k_1!\cdots k_m!} = [y_1^{k_1} \cdots y_m^{k_m}](\prod_{i=1}^{n} P_{s_1}(\sum_{i=1}^{m} y_i))$
$= [y_1^{k_1} \cdots y_m^{k_m}](\sum_{i=1}^{m} y_i)^n$

(5) The number of putting a, a, b, b into boxes A, B. $r = 2, n_1 = n_2 = 2$.

$$\prod_{i=1}^{2} P_{s_{n_i}}(2, 2) = \binom{2+2-1}{2}^2 = 9.$$

(6) The number of putting a, a, b, b, c, c into boxes A, B, C. s.t. 3 balls in A, 2 balls in B, 1 ball in C.

$$P_{s_2}(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2).$$

$$P_{s_2}(y_1+y_2+y_3, y_1^2+y_2^2+y_3^2) = \frac{1}{2}[(y_1+y_2+y_3)^2+y_1^2+y_2^2+y_3^2].$$

$$[y_1^3y_2^2y_3][P_{s_2}(y_1+y_2+y_3, y_1^2+y_2^2+y_3^2)]^3 = 15.$$

Note:

(1) numbered balls and numbered boxes: vectors

(2) unnumbered balls and numbered boxes: #solutions to $\sum_{i=1}^{m} k_i = n$

(3) numbered balls and unnumbered boxes: stirling number

(4) unnumbered balls and unnumbered boxes: integer partition

SDR

Def: A system of distinct representatives(SDR) for a sequence of set $S_1, S_2, \cdots, S_m$(are not necessarily distinct) is a sequence of distinct elements $x_1, \cdots, x_m$, s.t. $x_i \in S_i, i \in [m]$.

<u>**Theorem**</u>**1.**(**Hall's marriage Theorem**) The sets of $S_1, S_2, \cdots, S_m$ has a SDR iff $|\cup_{i\in I} S_i| \geq |I|$ for $I \subset [m]$(Hall's condition) i.e. for $\forall k \in [m]$, the union of any k sets has at least k elements.

<u>***proof:***</u>
"$\Longrightarrow$" If $S_1, S_2, \cdots, S_m$ have a SDR $x_1, \cdots, x_m$, then $\forall I \subset [m], |\cup_{i\in I} S_i| \geq |\{x_i, i \in I\}| = |I|$.
$\Longleftarrow$ Prove by induction on m. The case $m = 1$ is clear. Assume the claim holds for any collection with $< m$ sets.

- if for all $I \subsetneq [m], |\cup_{i\in I} S_i| > |I|$, take $x \in S_1$ as its representative, let $S_i' = S_i \backslash \{x\}, i = 2, \cdots, m$. then for all $I \subset [2, m], |\cup_{i\in I} S_i'| \geq |I|$. by assumption $S_2', \cdots, S_m'$ have a SDR $x_2, \cdots, x_m$. then $x_1, \cdots, x_m$ is a SDR of $S_1, \cdots, S_m$.
- if for some $I \subsetneq [m], |\cup_{i\in I} S_i| = |I| = k$ for some k. by recording $S_i, i \in [m]$, we may assume $|\cup_{i=1}^{k} S_i| = k$. by assumption, $S_1, \cdots, S_k$ have a SDR $x_1, \cdots, x_k$. Let $S_i' = S_i \backslash \{x_1, \cdots, x_k\}, i \in [k+1, m]$. then for all $I \subset [k+1, m], |\cup_{i\in I} S_i'| \geq |I|$, if not,

$|(\cup_{i\in I}S_i) \cup (\cup_{j=1}^{k} S_j)| < |I| + k$, a contradiction. So by assumption $S'_{k+1}, S'_{k+2}, \cdots, S'_m$ have a SDR $x_{k+1}, \cdots, x_m$, then $x_1, x_2, \cdots, x_m$ is a SDR of $S_1, S_2, \cdots, S_m$.

$\square$

Corollary1.$|X| = n$, $|S_i| = r, S_i \subset X, i \in [m].s.t.|\{i : x \in S_i\}| = d$ for all $x \in X$. If $m \leq n$, then $S_1, \cdots, S_m$ have a SDR.

proof:
Consider the incidence matrix $M = (m_{x,i})$ of $S_i, i \in [m]$. That is M is a 0-1 matrix with $|X|$ rows labeled by elements $x \in X$, and with m column labeled by $i \in [m]$, such that $m_{x,i} = 1$ iff $x \in S_i$, count the number of 1 in M. we have $dn = mr$, then $m \leq n$ implies $d \leq r$. Suppose $S_1, \cdots, S_m$ don't have a SDR, by Hall's Theorem, $\exists k$ sets $S_{i_1}, \cdots, S_{i_k}$ for some $k \in [m]$,

$$|Y| = |S_{i_1} \cup \cdots \cup S_{i_k}| < k$$

$\forall x \in Y$, let $d_x = |\{j \in [k] : x \in S_{i_j}\}| \leq d$. count the number of 1 in rows labeled by $x \in Y$ and columns $i_j, j \in [k]$.

$$rk = \sum_{j=1}^{k} |S_{i_j}| = \sum_{x\in Y} d_x \leq d|Y| < dk.$$

a contradiction.

$\square$