Combinatorics, 2018 Fall, USTC
Homework 11

- **The due is on Thursday, Dec. 13.**
- **Please sign your name and student number.**
- **Please solve as many problems as you can.**

1. Let $\{A_1, \cdots, A_m\}$ be an L-intersecting family of subsets of $[n]$, where each A_i is of a constant size, say, k. Prove that $m \leqslant \binom{n}{|L|}$.
   Hint: Besides f_i defined in Frankl-Wilson Theorem, define more polynomials. For each $I \subset [n]$, $|I| \leqslant |L| - 1$, define $g_I(x) = [(\sum_{j=1}^{n} x_j) - k] \prod_{i \in I} x_i$. Show $f_i, i \in [m]$ and $g_I, I \subset [n]$ & $|I| \leqslant |L| - 1$ are linearly independent.

2. Let $A_1, \cdots A_m$ and $B_1, \cdots B_m$ be subsets of $[n]$ such that $|A_i \cap B_i|$ is odd for all $i \in [m]$ and $|A_i \cap B_j|$ is even for all $1 \leqslant i < j \leqslant m$. Show that $m \leqslant n$.
   Hint: Use the fact that $rank(AB) \leqslant \min\{rank(A), rank(B)\}$ for any matrices A, B over any field.

3. Let $\mathcal{F} \subset 2^{[n]}$ be such that $|A|$ is even for all $A \in \mathcal{F}$ and $|A \cap B|$ is even for all distinct $A, B \in \mathcal{F}$. Show that $|\mathcal{F}| \leqslant 2^{n/2}$.

4. Prove that there are at most $n+1$ points in $\mathbb{R}^n$ such that the distance between any 2 points is 1.
Hint: Assume $0, v_1, \cdots, v_{m-1}$ are such points (*i.e.* column vectors) in $\mathbb{R}^n$. Let $A = (v_1 \ \cdots \ v_{m-1})^T$. Show $rank(AA^T) = m-1$ and use the Hint of Question 2.