Combinatorics 2018 Fall

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Key words: Intersecting Family, EKR Thm

Recall:

LYM Inequality:

$\mathscr{F}$ is an antichain in $(2^{[n]}, \subseteq)$, then $\sum_{A\in\mathscr{F}} \binom{n}{|A|}^{-1} \leq 1$.

proof:.
A maximal symmetric chain $\{\phi\}$ - $\{x_1\}$ - $\{x_1, x_2\}$ - $\cdots$ - $\{x_1, \cdots, x_n\}$, consider a permutation $\pi = (x_1, x_2, \cdots, x_n)$
Double counting $S = \{(A, \pi) : A \in \mathscr{F}, \pi \in S_n, \pi \text{ contains } A\}$, where 'contain' means $A = \{x_1, \cdots, x_{|A|}\}$
Fix A, $\sharp\pi$ contains $A = |A|!(n - |A|)!$
Fix π, since $\mathscr{F}$ is an antichain, π contains at most one $A \in \mathscr{F}$. Then
$\sum_{A\in\mathscr{F}} |A|!(n - |A|)! = |S| = \sum_{\pi\in S_n} |\{A \in \mathscr{F} : \pi \text{ contains } A\}| \leq n!$
$\Rightarrow \sum_{A\in\mathscr{F}} \binom{n}{|A|}^{-1} \leq 1$.

$\square$

Intersecting Family: A family of sets $\mathscr{F} \subset 2^{[n]}$ is intersecting if $\forall A, B \in \mathscr{F},\ |A \cap B| \geq 1$
e.g.

(1) $\mathscr{F} = \{A \in [n] :\ 1 \in a\},\ |\mathscr{F}| = 2^{n-1}$

(2) for n odd, $\mathscr{F} = \{A \in [n] : \ |A| \geq \frac{n+1}{2}\}, \ |\mathscr{F}| = \sum_{k=\frac{n+1}{2}}^{n} \binom{n}{k} = 2^{n-1}$

Fact: For any intersecting family $\mathscr{F} \subset 2^{[n]} \implies |\mathscr{F}| \leq 2^{n-1}$, because $\mathscr{F}$ can not contain both A and $A^c = [n] \setminus A$

Uniform case: $\mathscr{F} \subset \binom{[n]}{k}$, $\mathscr{F}$ is called $k - uniform$.
How large is $|\mathscr{F}|$ if $\mathscr{F}$ is intersecting?
e.g.

(1) $\mathscr{F} = \{A \in \binom{[n]}{k} : \ 1 \in A\}, \ |\mathscr{F}| = \binom{n-1}{k-1}$

(2) $k \geq \frac{n+1}{2}$, $\mathscr{F} = \binom{[n]}{k}, \ |\mathscr{F}| = \binom{n}{k}$

Theorem 1 (**Erdős-Ko-Rado Thm, *EKR***). *If $n \geq 2k, \mathcal{F} \subset \binom{[n]}{k}$ is intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.*

proof:.
doubling count (A, π), where $A \in \mathcal{F}$, π is a cyclic permutation vector and $A \in \mathcal{F}_\pi$ (which means that A appears as consecutive elements in π)
Fix A, $\sharp\pi = |A|!(n - |A|)! = k!(n-k)!$
Fix π, $\forall A \in \mathcal{F} \cap \mathcal{F}_\pi, \exists 2k-2$ sets $B \in \mathcal{F}_\pi$ s.t. $B \cap A \neq \emptyset, \ B \neq A$. But they can be partitioned into k-1 pairs of disjoint subsets. Since $\mathcal{F}$ is intersecting, $\mathcal{F}$ contains at most one set of each pair.
$\implies |\mathcal{F} \cap \mathcal{F}_\pi| \leq k$
$\implies |\mathcal{F}| \cdot k!(n-k)! = \sum_{A \in \mathcal{F}} k!(n-k)! = \sum_{\pi} |\mathcal{F} \cap \mathcal{F}_\pi| \leq \sum_{\pi} k = (n-1)!k$
$\implies |\mathcal{F}| \leq \binom{n-1}{k-1}$

$\square$

Theorem 2 (**EKR Extremal Case**). *If $n > 2k.$, then the intersecting family $\mathcal{F} \subset \binom{X}{k}$ with $|\mathcal{F}| = \binom{n-1}{k-1}$ must be a star. i.e. $|\mathcal{F}| = \{A \in \binom{[n]}{k} : \ i \in A\}$ for some $i \in [n]$*

Note: $n = 2k$, e.g. $\mathcal{F}$ contains exactly one set of (A, A^c), $|A| = k$, $|\mathcal{F}| = \dbinom{2k}{k}/2 = \binom{2k-1}{k-1}$

proof:.

See from the proof of EKR, if $|\mathcal{F}| = \dbinom{n-1}{k-1}$

(1) $\forall$ cyclic π, $|\mathcal{F} \cap \mathcal{F}_\pi| = k$

(2) if $\pi = (a_1, a_2, \cdots, a_n)$ cyclic vector, then $\mathcal{F} \cap \mathcal{F}_\pi = \{A_1, A_2, \cdots, A_k\}$, $A_j = \{a_j, \cdots, a_{j+k-1}\}$, $j = 1, \cdots, k$.

Fix π, let $\{a_k = A_1 \cap \cdots \cap A_k\}$. If all $A \in \mathcal{F}$, $a_k \in A$, then $\mathcal{F}$ is a star, done. If not, $\exists A_0 \in \mathcal{F}, a_k \notin A_0$, we will show that $\mathcal{F} = \dbinom{A_1 \cup A_k}{k}$, then $|\mathcal{F}| = \dbinom{2k-1}{k} = \dbinom{2k-1}{k-1} \leq \dbinom{n-1}{k-1}$ contradiction.

<u>**Claim1**</u>: $\forall B \in \binom{A_1 \cup A_k \setminus \{a_k\}}{k-1}$, $B \cup \{a_k\} \in \mathcal{F}$.
Proof: $B = B_1 \cup B_2$, where $B_1 \subset A_1$ and $B_2 \subset A_k$. Construct $\pi' = (A_1 \setminus (B_1 \cup \{a_k\}), B_1, a_k, B_2, A_k \setminus (B_2 \cup a_k))$. Since $A_1, A_k \in \mathcal{F}$, by 2, $B \cup \{a_k\} = B_1 \cup B_2 \cup \{a_k\} \in \mathcal{F}$

<u>**Claim2**</u>: $A_0 \subseteq (A_1 \cup A_k) \setminus \{a_k\}$.
Proof: If $A_0 \nsubseteq A_1 \cup A_k \setminus \{a_k\}, |A_0 \cap (A_1 \cup A_k \setminus \{a_k\})| \leq k-1$. $\exists B \subset (A_1 \cup A_k \setminus \{a_k\}) \setminus A_0, |B| = k-1$. By Claim1, $B \cup \{a_k\} \in \mathcal{F}$. But $A_0 \cap (B \cup \{a_k\}) = \emptyset$. Contradiction.

<u>**Claim3**</u>: $\binom{A_1 \cup A_k}{k} \subset \mathcal{F}$.
Proof: $\forall C \in \binom{A_1 \cup A_k}{k}, a_k \notin C, C \cap A_0 = \{b_1, b_2, \cdots, b_s\}$. $\pi = (\cdots b_1, \cdots, b_s \cdots), C \in \mathcal{F}$.

<u>**Claim4**</u>: $\mathcal{F} \subset \binom{A_1 \cup A_k}{k}$.
Proof: If $\exists B \in \mathcal{F}. B \notin A_1 \cup A_k$, then $|B \cap (A_1 \cup A_k)| \leq k-1$. Then $\exists C \in \binom{A_1 \cup A_k}{k}, C \cap B = \emptyset$. Therefore, $C \notin \mathcal{F}$. But by Claim3, contradiction.

$\square$