Combinatorics 2018 Fall

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Key words: Catalan Number, GF, EGF

1. The Catalan number

Recall: $\binom{\frac{1}{2}}{n} = \frac{(-1)^{n-1}\cdot 2}{4^n n}\binom{2n-2}{n-1}$

Def: A triangulation of an n-gon ia a way to add lines between corners to make triangles such that these lines do not cross insides of the n-gon.

Catalan number:
Let b_{n-1} be $\sharp$ triangulations of n-gon, compute b_{n-1}

Sol: $b_2 = 1, b_3 = 2$ and $b_4 = 5$
For $4 \leq i \leq n-1$, $\Delta_{1,2,i}$ can split the n-gon into two parts: (i-1)-gon and (n-i+2)-gon, then:
$b_{n-1} = b_{n-2} + \sum_{i=4}^{n-1} b_{i-2}b_{n-i+1} + b_{n-2}$
Let $b_0 = 0, b_1 = 1$,then
$b_{n-1} = b_0 b_{n-1} + b_1 b_{n-2} + \sum_{i=4}^{n-1} b_{i-2}b_{n-i+1} + b_{n-2}b_1 + b_{n-1}b_0 = \sum_{j=0}^{n-1} b_j b_{n-1-j}$
that is $b_n = \sum_{j=0}^{n} b_j b_{n-j}$ for $n \geq 2$
Let $f(x) = \sum_{n\geq 0} b_n x^n$, then $f^2(x) = \sum_{n\geq 0}(\sum_{j=0}^{n} b_j b_{n-j})x^n = \sum_{n\geq 2} b_n x^n = f(x) - x$
$\Longrightarrow f^2(x) = f(x) - x$

$\Longrightarrow f(x) = \frac{1+\sqrt{1-4x}}{2}$ or $f(x) = \frac{1-\sqrt{1-4x}}{2}$
$\because f(0) = 0$
$\therefore f(x) = f(x) = \frac{1-\sqrt{1-4x}}{2} = \frac{1}{2} - \frac{1}{2}(1-4x)^{\frac{1}{2}} = \frac{1}{2} - \frac{1}{2}\sum\limits_{n\geq 0}\binom{\frac{1}{2}}{n}(-4x)^n = -\sum\limits_{n\geq 1}\frac{1}{2}\cdot\frac{(-1)^{n+1}2}{n4^n}\binom{2n-2}{n-1}(-4)^n x^n = \sum\limits_{n\geq 1}\frac{1}{n}\binom{2n-2}{n-1}x^n$
$\Longrightarrow b_n = \frac{1}{n}\binom{2n-2}{n-1}$ $\square$

2. GF & Selection

Recall: $\frac{1}{(1-x)^k} = \sum\limits_{n\geq 0}\binom{n+k-1}{k-1}x^n = (1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots) = \sum\limits_{n\geq 0}(\sum\limits_{n_1+n_2+\cdots+n_k=n, n_i\geq 0} 1)x^n$

Problem1:
Compute a_{20}= #ways to pay 20-yuan given unlimited numbers of three kinds of bills: 1-yuan, 2-yuan, 5-yuan

Sol:

$$\begin{aligned} f(x) &= (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^5+x^{10}+\cdots) \\ &= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^5} \end{aligned}$$

$[x^{20}]f(x)$ is the answer. $\square$

Integer Partition: write n as a sum of positive integers with no order

Problem2:
Let P_n be $\sharp$ of the integer partition of n, find the GF of $\{P_n\}$

Sol: Let n_j be $\sharp$ of the j's in such a partition of n, then

$$\sum_{j\geq 1} j\cdot n_j = n.$$

Let $i_j = j \cdot n_j$, which is the contribution of the addends j in a partition of n, then $i_j \in \{0, j, 2j, 3j, \cdots\}$. Let $f_j(x) = 1 + x^j + x^{2j} + x^{3j} + \cdots = \frac{1}{1-x^j}$, then the GF of $\{P_n\}$ is

$$P(x) = \prod_{j\geq 1} f_j(x) = \prod_{j\geq 1} \frac{1}{1-x^j}.$$

$\square$

Remark:

1. Compute $P_4 = 5$
2. $P_n = e^{\theta(\sqrt{n})}$, i.e. $\exists c_2 \geq c_1 \geq 0$ such that $e^{c_1\sqrt{n}} \leq P_n \leq e^{c_2\sqrt{n}}$

3. Exponential GF(EGF) and Arrangements

Problem3:
$T_n = \sharp$ ways of picking n balls in order from unlimited number of red, blue and white balls s.t. the $\sharp$ of red and blue balls both even.

Sol: A selection of e_1 red balls, e_2 blue balls and e_3 white balls with $e_1 + e_2 + e_3 = n$ contributes $\frac{n!}{e_1!e_2!e_3!}$ to T_n. Therefore

$$T_n = \sum_{\substack{e_1 + e_2 + e_3 = n \\ e_1, e_2 \in 2\mathbb{Z}_{\geq 0}, e_3 \in \mathbb{Z}_{\geq 0}}} \frac{n!}{e_1!e_2!e_3!}$$

What's the GF of T_n ?

Def: The EGF of the sequence $\{a_n\}$ is $f(x) = \sum\limits_{n\geq 0} \frac{a_n}{n!} x^n$.

Eg:

1. the EGF of $\{1, 1, \cdots\}$ is $\sum\limits_{n\geq 0} \frac{x^n}{n!} = e^x$
2. $f_j(x) = \sum_{n\geq 0} \frac{f_j(n)}{n!} x^n$, $j \in [k]$.

$$f(x) = \prod_{j=1}^{k} f_j(x) = \prod_{j=1}^{k} (\sum_{n\geq 0} \tfrac{f_j(n)}{n!} x^n) = \sum_{n\geq 0} \left(\sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\cdots n_k \geq 0}} \frac{\Pi_{j=1}^k f_j(n_j)}{n_1! n_2! \cdots n_k!} \right) x^n,$$

i.e. $f(x)$ is the EGF of $a_n = \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\cdots n_k \geq 0}} \frac{n! \Pi_{j=1}^k f_j(n_j)}{n_1! n_2! \cdots n_k!}$.

Problem3:

Find EGF of $\{T_n\}$ and compute T_n

Sol: Let

$$T(x) = \sum_{n=0}^{\infty} \frac{T_n}{n!} x^n,$$

$$f_1(x) = f_2(x) = \sum_{i\in 2\mathbb{Z}_{\geq 0}} \frac{x^i}{i!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{e^x + e^{-x}}{2},$$

$$f_3(x) = \sum_{i\in \mathbb{Z}_{\geq 0}} \frac{x^i}{i!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

Therefore,

$$T(x) = f_1 f_2 f_3 = \frac{e^{3x} + e^{-x} + 2e^x}{4} = \sum_{n\geq 0} \left(\frac{3^n + 2 + (-1)^n}{4} \right) \frac{x^n}{n!}$$

$\Rightarrow \; T_n = \frac{3^n+2+(-1)^n}{4}$.

□

Problem4:

Let a_n be the number of ways to send n students to 4 classes R_1, R_2, R_3, R_4, s.t. each class has at least one student.

Sol:

$$a_n = \sum_{\substack{i_1+i_2+i_3+n_4=n \\ i_1,\cdots n_4 \geq 1}} \frac{n!}{i_1! i_2! i_3! i_4!}.$$

Let $f_i(x) = \sum_{n\geq 1} \frac{x^n}{n!} = e^x - 1$, $i \in [4]$. Then the EGF of $\{a_n\}$ is

$$f = f_1 f_2 f_3 f_4 = \sum_{n\geq 0} (4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4) \frac{x^n}{n!} + 1.$$

$\Rightarrow a_n = 4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4,\ n \geq 1$, and $a_0 = 0$. $\square$

Exercise: Use EGF to find $a_n = \sharp$ vectors of length n over $[k]$.