Combinatorics 2018 Fall

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Recall (König's Theorem) Let A be an $m \times n$ 0-1 matrix. Let r be the maximum number of independent 1's, R be the minimum number of rows and columns required to cover all 1's in A. Then, $r = R$. ($\Longleftrightarrow$ A doesn't have a 0-submatrix of size $c \times d$ such that $c + d = m + n - r + 1$.)

Definition. A system of common representatives (SCR) of two sequences of sets $A_1, \cdots, A_m$ and $B_1, \cdots, B_m$ is a sequence $x_1, \cdots, x_m$ (not necessarily distinct) such that $x_i \in A_i \cap B_{\pi(i)},\ i \in [m]$ for some $\pi \in S_m$.

Theorem 1. Suppose X has two partitions: $X = A_1 \cup \cdots \cup A_m = B_1 \cup \cdots \cup B_m$. Then they have an SCR if and only if for $\forall\ I \subset [m]$, $\bigcup\limits_{i \in I} A_i$ contains at most $|I|$ sets of $B_j,\ j \in [m]$.

proof 1: Let $Y = \{B_1, \cdots, B_m\}$. For $\forall\ i \in [m]$, let $S_i = \{B_j : B_j \cap A_i \neq \emptyset\}$. Then $\exists$ SCR $\Longleftrightarrow$ $S_1, \cdots, S_m$ have an SDR.
So it suffices to check that $S_1, \cdots, S_m$ satisfy the Hall Condition, i.e. $\forall\ I \subset [m]$, $|\cup_{i \in I} S_i| \geqslant |I|$.
In fact, this is equivalent to: for $\forall\ I \subset [m]$, we have at least $|I|$ sets B_j such that each B_j intersect some A_i, $i \in I$. Since $A_1, \cdots, A_m$ and $B_1, \cdots, B_m$ are partitions, this is equivalent to say that $\cup_{j \in [m] \setminus I} A_j$ contains at most $m - |I|$ sets B_j, which is just the given condition.$\square$

proof 2: Consider the intersection matrix of the two partitions $C = (c_{ij})$, where

$$c_{ij} = \begin{cases} 1 & \text{if } A_i \cap B_j \neq \emptyset \\ 0 & \text{else} \end{cases} .$$

Then $\exists$ SCR $\iff C$ has m independent 1's $\iff C$ has no $(m-k) \times (k+1)$ 0-submatrix for any $0 \leqslant k \leqslant m-1 \iff \nexists\ m-k$ sets A_i such that their union is disjoint with $k+1$ sets $B_j \iff \nexists\ k$ sets A_i such that their union contains $k+1$ sets $B_j \iff$ The union of any k sets A_i contains at most k sets B_j. $\square$

Definition.

- $G = (V, E)$ is called a bipartite graph, if $V = A \sqcup B$, and $i \sim j$ only if i, j are in different sets. G is called a complete bipartite graph, if G is bipartite, and every possible edge belongs to E.
- Two edges are disjoint, if they have no common vertex.
- A matching is a set of pairwise disjoint edges.
- A vertex is called matched (or saturated) if it is an endpoint of one of the edges in the matching. Otherwise the vertex is called free (or unmatched).
- If a matching matches all vertices in A, then we say it's a matching of A into B. A perfect matching is a matching of A into B when $|A| = |B|$.
- A vertex $x \in A$ is a neighbor of a vertex $y \in B$, if $\{x, y\} \in E$. Let S_x be the set of all neighbors of x, and call $\deg(x) := |S_x|$ the degree of x.

Fact. $\exists$ a matching of A into B if and only if $S_x,\ x \in A$ has an SDR.

Theorem 2. If G is a bipartite graph with bipartitions A, B, then G has a matching of A into B iff $\forall\ k$ vertices in A have at least k neighbors.

Definition. A vertex cover in $G = (A \cup B, E)$ is a set of vertices

$S \subseteq A \cup B$ such that every edge is incident to at least one vertex in S.

Theorem 3. For bipartite graphs, the maximum size of a matching is equal to the minimum size of a vertex cover.

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Proposition 4. $|X| = n$. For any $k \leqslant \frac{n-1}{2}$, it is possible to extend every k-element subset of X to a $(k+1)$-subset such that the extensions of no two sets coincide.

proof: Consider the bipartite graph $G = (A \cup B, E)$, where $A = \{$all k-subsets of $X\}$, $B = \{$all $(k+1)$-subsets of $X\}$, and for any $x \in A$, $y \in B$, $x \sim y$ iff $x \subset y$. Then we only need to prove G has a matching of A into B.
For each $x \in A$, $\deg(x) = n - k$. For each $y \in B$, $\deg(y) = k + 1$. For $\forall\ I \subseteq A$, let $S(I) = \underset{x \in I}{\cup} S_x$, then

$$|I|(n-k) \leqslant |S(I)|(k+1) \Longrightarrow |S(I)| \geqslant |I|\frac{n-k}{k+1} \geqslant |I|.$$

By Theorem 2, we're done. $\square$

Pigeonhole Principle and Graphs

Remark. Here, we only consider finite simple graphs: no loops, no multiple edges.

Theorem 1 (Handshaking Lemma) In any finite graph, the number of vertices which have odd degrees is even.

proof: Double counting.
For $G = (V, E)$, count # ordered pairs $(x, y) \in E$,

$$2|E| = \sum_{x \in V} \deg(x) = \sum_{x \in V,\ \deg(x) \text{ is odd}} \deg(x) + \sum_{x \in V,\ \deg(x) \text{ is even}} \deg(x).$$

If we have odd number of vertices with odd degree, then the first item is odd and the second item is even, a contradiction. $\square$

Definition. $G = (V, E)$ is a graph.

- A set of pairwise nonadjacent vertices is called an independent set. Independent number $\alpha(G)$ is the maximum size of such a set, i.e. $\alpha(G)$ is the maximum number of pairwise nonadjacent vertices of G.
- A coloring of V is called proper, if no two adjacent vertices have the same color. Chromatic number $\chi(G)$ is the minimum number of colors in a proper coloring.

Proposition 2. In any graph G with n vertices, $n \leqslant \alpha(G)\chi(G)$.

proof: Given a proper coloring of G with $\chi(G)$ colors and partition $V(G)$ into $\chi(G)$ color classes. By Pigeonhole Principle, one of the color classes has size $\geqslant \dfrac{n}{\chi(G)}$. Note that these vertices are pairwise nonadjacent, so $\alpha(G) \geqslant \dfrac{n}{\chi(G)}$. $\square$