Combinatorics 2018 Fall

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Key words: Decomposition in chains/antichains, Sperner's Thm

Recall:

– Hassa diagram of a poset P $\longrightarrow$ Graph G

– $\alpha(P)$ = max size of antichain, $\alpha(P) = \alpha(G)$

– $w(P)$ = max size of chain, $w(P) = \chi(G)$

– $\alpha(P)w(P) \geq |X|$, $\alpha(G)\chi(G) \geq |X|$

Fact:

(1) Any finite poset P can be partitioned into $w(P)$ antichains

(2) If a poset P has a chain(antichan) of size r, then it can not be partitioned into fewer than r antichains(chains).

proof of Fact:

(1) Let Let $A_i = \{x \in P :$ the longest chain with greatest element x has size $i\}$.Then since $w(P) = r$, we have $A_i = \phi$ if $i \geq r + 1$. so $P = A_1 \cup A_2 \cup \cdots \cup A_r$ is a partition (some of them may be empty). Show each A_i is an antichain. If not, say $x, y \in A_i, x < y$, then the largest chain with greatest element $x, x_1 < x_2 < \cdots < x_i = x$ can be prolonged to a longer chain $x_1 < x_2 < \cdots < x < y$, then the largest chain with greatest element y has size $\geq i + 1$, i.e. $y \notin A_i$. Contradiction!

(2) Use pigeonhole principle.

□

Theorem 1 (Dilworth's Decomposition Thm). *If $\alpha(P) = r$, then P can be partitions into r chains.*

proof:
Induction on $|P|$.
Assume the claim holds with any poset of size less than $|P|$.
Let a be a maximal element of P. Let $P' = P \setminus a$ and assume $\alpha(P') = s$, then P' can be partitioned into s chains $c_1, \cdots, c_s$

(1) $\alpha(P) = r = s + 1$, then $c_1, \cdots, c_s, \{a\}$ form a partition of P

(2) $\alpha(P) = r = s$, let a_i be the maximal element in $\{x \in C_i : x$ belongs to some antichain of size $s\} \neq \phi$ and assume $a_i \in B_i$, where B_i is an antichain in P' of size s.
Then $A = \{a_1, \cdots a_s\}$ is an antichain of P'. (If not, $\exists a_i < a_j, i \neq j$, $\because \exists x \in B_j \cap c_i, \therefore x \leq a_i < a_j$, then $x \leq a_j \in B_j$, a contradiction.)
$\because \alpha(P) = s$, $\therefore a > a_i$ for some $i \in [s]$. Let $K = \{a\} \cup \{x \in c_i : x \leq a_i\}$, then K is a chain. $\alpha(P \setminus K) = s - 1$ since K breaks all antichains of size of s. By assumption, $P \setminus K$ is a union of $s - 1$ chains. Then P is a union of s chains.

□

Def: A set system (i.e. a family of subsets) $\mathscr{F}$ is an antichain with inclusion order (or sperner system) if $A \neq B \in \mathscr{F}$, neither $A \subset B$ nor $B \subset A$.
Sperner's Thm: Let $\mathscr{F}$ be a set system over $[n]$, i.e. $\mathscr{F} \subset 2^[n]$. If $\mathscr{F}$ is an antichain,then $|\mathscr{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.
Note:

(1) $P = (2^{[n]}, \subset)$, $\alpha(P) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$

(2) P can be partitioned into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ symmetric chains.

LYM Inequality:

$\mathscr{F}$ is an antichain over $[n]$, then $\sum\limits_{A \in \mathscr{F}} \binom{n}{|A|}^{-1} \leq 1$.

Note: LYM Inequality $\Rightarrow$ Sperner's Thm: since $\binom{n}{|A|} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

proof of LYM Inequality::

A permutation$(x_1, x_2, \cdots, x_n)$, consider a maximal symmetric chain $\{\phi\}$ - $\{x_1\}$ - $\{x_1, x_2\}$ - $\cdots$ - $\{x_1, \cdots, x_n\}$

Count $S = \{(A, C) : A \in \mathscr{F}, C$ is a a maximal symmetric chain, $A \in C\}$, fix A, $\exists$ $|A|!(n-|A|)!$ permutations $(x_1, \cdots, x_n)$ s.t. $A = (x_1, \cdots, x_{|A|})$, then $\exists$ $|A|!(n-|A|)!$ maximal symmetric chains. Since $\mathscr{F}$ is an antichain, then each maximal symmetric chain contains at most one $A \in \mathscr{F}$. Then

$\sum\limits_{A \in \mathscr{F}} |A|!(n-|A|)! = |S| \leq |\{\text{maximal symmetric chains}\}| = n!$

$\Rightarrow \sum\limits_{A \in \mathscr{F}} \binom{n}{|A|}^{-1} \leq 1.$

□

Bollobás Thm: Let $A_1, \cdots, A_m$ and $B_1 \cdots, B_m$ be two sequence of sets,such that $A_i \cap B_j = \phi$ iff $i = j$. Then $\sum\limits_{i=1}^{m} \binom{a_i + b_i}{a_i}^{-1} \leq 1$, where $|A_i| = a_i, |B_i| = b_i, i \in [m]$. In particular,if $a_i = a$ and $b_i = b$ for all $i \in [m]$, then $m \leq \binom{a+b}{a}$.

Note: Bollobás Thm $\Rightarrow$ LYM Inequality $\Rightarrow$ Sperner's Thm.

proof of Note:

Let $\mathscr{F} = \{A_1, \cdots, A_n\} \subset 2^{[n]}$, $B_i = [n] \backslash A_i$, easily check $A_i \cap B_j = \phi$ iff $i = j$. By Bollobás Thm, $\sum\limits_{i=1}^{m} \binom{n}{a_i}^{-1} \leq 1$

□

proof of Bollobás Thm:
Let $X = (\cup_{i=1}^m A_i) \cup (\cup_{i=1}^m B_i)$, suppose $|X| = n$. $\forall A, B \subset X$, and $A \cap B = \phi$, say a permutation $(x_1, \cdots, x_n)$ of X separates the pair (A, B) if no element of B precedes an element of A, i.e. if $x_k \in A,\ x_l \in B$, then $k < l$.

We claim that each permutation separates at most one pair $(A_i, B_i), i \in [m]$. Indeed, suppose $(x_1, \cdots, x_n)$ separates $(A_i, B_i), (A_j, B_j), i \neq j$ and assume $\max\{k : x_k \in A_i\} \leq \max\{k : x_k \in A_j\}$. Since $(x_1, \cdots, x_n)$ separates (A_j, B_j),we have
$\min\{k : x_k \in B_j\} > \max\{k : x_k \in A_j\} \geq \max\{k : x_k \in A_i\}$
$\Rightarrow A_i \cap B_j = \phi$, a contradiction.

Now count the number N_i of permutations separated one fixed pair say (A_i, B_i). First choose $a_i + b_i$ positions for $A \cup B$, there are $\binom{n}{a_i + b_i}$ choices. Then A occupy the first a_i positions, giving $a_i!$ choices for the order of A, and $b_i!$ choices for the order of B, the remaining elements can be arranged in $(n - a_i - b_i)!$ ways. So $N_i = \binom{n}{a_i + b_i} a_i! b_i! (n - a_i - b_i)! = n! \binom{a_i + b_i}{a_i}^{-1}$.
Summing over all pairs, we have

$$\sum_{i=1}^{m} n! \binom{a_i + b_i}{a_i}^{-1} \leq n! \Rightarrow \sum_{i=1}^{m} \binom{a_i + b_i}{a_i}^{-1} \leq 1.$$

$\square$