Combinatorics, 2018 Fall, USTC
Homework 1

- **The due is on Thursday, Sep. 20.**
- **Please sign your name and student number.**
- **Please solve as many problems as you can.**

1. Prove: Let X be a set of size n, $r \geqslant n$, then the number of surjections $f : [r] \to X$ is $S(r, n) \cdot n!$

2. Prove:
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k}{m} = (-1)^m \delta_{m,n}.$$

$$\text{where } \delta_{m,n} = \begin{cases} 0 & if\ m \neq n \\ 1 & if\ m = n \end{cases}$$

3. Let $n \geqslant m$. Give a combinatorial proof of the following identity:
$$\sum_{k=0}^{m} \binom{m}{k} \binom{n+k}{m} = \sum_{k=0}^{m} \binom{m}{k} \binom{n}{k} 2^k.$$

4. How many functions $f : [n] \to [n]$ are there that are monotone; that is, for $i < j$, we have $f(i) \leqslant f(j)$?

5. Show that

$$\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}.$$

6. Let p be a permutation of the set $[n]$. Let us write it in the one-line notation, and let us mark the *increasing segments* in the resulting sequence of numbers. For example, in (4 5 7 2 6 8 3 1), there are 4 increasing segments: (4 5 7), (2 6 8), (3), and (1). Let $f(n, k)$ denote the number of permutations over $[n]$ with exactly k increasing segments. Show that:

   (1) $f(n, k) = f(n, n + 1 - k)$

   (2) $f(n, k) = k \cdot f(n - 1, k) + (n + 1 - k) \cdot f(n - 1, k - 1)$