Combinatorics 2018 Fall

Taught by: Professor Xiande Zhang

2018.12.27

Key words: Deletion Method, Markov's Inequality, Random Graph

Recall. $cc(G) = \min \#$ cliques in a clique covering of G

Theorem. Let G be a graph on n vertices $s.t.$ every vertex has at least one neighbor and at most d non-neighbors. Then

$$cc(G) \leqslant O(d^2 \ln n).$$

<u>proof</u>: Consider the following way of choosing cliques of $G = (V, E)$. First, pick $v \in V$ independently with probability $p = \frac{1}{1+d}$ to get a set $W \subset V$. Then remove from W all vertices having at least one non-neighbor in W. Then we get a clique of G. Apply this way independently t times to get t cliques $H_1, \cdots, H_t$.
Let X be the number of edges not covered by any H_i. Let X_{uw} be the indicator random variable of the event that $u \sim w$ is not covered by any H_i. Then $X = \sum\limits_{u \sim w} X_{uw}$.
Note that H_i covers $u \sim w$ if both u and w are chosen and none of their $\leqslant 2d$ non-neighbors are chosen. So

$$\begin{aligned}\Pr[u \sim w \text{ is covered by } H_i] \geqslant p^2(1-p)^{2d} &= \frac{1}{(1+d)^2}(\frac{d}{1+d})^{2d} \\ &= \frac{1}{(1+d)^2}\frac{1}{(1+\frac{1}{d})^{2d}} \geqslant \frac{1}{(1+d)^2 e^2}\end{aligned}$$

So

$$
\begin{aligned}
\Pr[u \sim w \text{ is not covered by any } H_i] &= \Pr[\bigcap_{i=1}^{t} \{u \sim w \text{ is not covered by } H_i\}] \\
&= \prod_{i=1}^{t} \Pr[u \sim w \text{ is not covered by } H_i] \\
&\leqslant (1 - \frac{1}{(1+d)^2 e^2})^t \leqslant e^{-\frac{t}{(1+d)^2 e^2}}
\end{aligned}
$$

Then, by linearity of expectation,

$$
\begin{aligned}
\mathrm{E}[X] &= \sum_{u \sim w} \mathrm{E}[X_{uw}] = \sum_{u \sim w} \Pr[u \sim w \text{ is not covered by any } H_i] \\
&\leqslant \binom{n}{2} e^{-\frac{t}{(1+d)^2 e^2}} \leqslant \frac{e^{2\ln n}}{2} e^{-\frac{t}{(1+d)^2 e^2}}
\end{aligned}
$$

Choose $t \geqslant 2 \ln n (1+d)^2 e^2$, $t = O(d^2 \ln n)$, we have $\mathrm{E}[X] < 1$. Hence, there is at least one choice of $t = O(d^2 \ln n)$ cliques that form a clique covering of G. $\Longrightarrow cc(G) \leqslant O(d^2 \ln n)$. $\square$

Markov's Inequality. Let $X \geqslant 0$ be a random variable, $a > 0$, then $\Pr[X \geqslant a] \leqslant \dfrac{\mathrm{E}[X]}{a}$.

proof: Let $I_{\{X \geqslant a\}}$ be the indicator random variable.
Then $aI_{\{X \geqslant a\}} \leqslant X$. $\Longrightarrow \mathrm{E}[aI_{\{X \geqslant a\}}] = a\Pr[X \geqslant a] \leqslant \mathrm{E}[X]$. $\square$

Corollary. Let $X_n \geqslant 0$ be integer-valued random variables in $(\Omega_n, \Pr_n)$, $n \in \mathbb{Z}_{>0}$. If $\mathrm{E}[X_n] \to 0$ as $n \to \infty$, then $\Pr[X_n = 0] \to 1$ as $n \to \infty$.

proof: $\Pr[X_n \geqslant 1] \leqslant \mathrm{E}[X_n] \to 0$ as $n \to \infty$. $\square$

Definition. Let $X_n \geqslant 0$ be integer-valued random variables in $(\Omega_n, \Pr_n)$, $n \in \mathbb{Z}_{>0}$. If $\Pr[X_n = 0] \to 1$ as $n \to \infty$, we say $X_n = 0$ almost surely occur.

Definition. The random graph $G(n,p)$ for $0 \leqslant p \leqslant 1$ is a graph with vertex set $[n]$, where each of the potential $\binom{n}{2}$ edges appears with probability p independently.

Theorem. $G(n, \frac{1}{2})$ almost surely is NOT bipartite.

proof: Let A_n be the event that $G(n, \frac{1}{2})$ is bipartite. Then it suffices to show $\Pr[A_n] \to 0$ as $n \to \infty$.
For $U \in 2^{[n]}$, let A_U be the event that all edges of G are between U and $[n]\backslash U$. Then $A_n = \cup_{U \in 2^{[n]}} A_U$.

$$\Pr[A_U] = (\frac{1}{2})^{\binom{|U|}{2}} (\frac{1}{2})^{\binom{n-|U|}{2}} = \frac{1}{2^{\binom{|U|}{2} + \binom{n-|U|}{2}}} \leqslant \frac{1}{2^{2\binom{n/2}{2}}} = 2^{-\frac{n^2}{4} + \frac{n}{2}}$$

So

$$\Pr[A_n] \leqslant 2^n \cdot 2^{-\frac{n^2}{4} + \frac{n}{2}} = 2^{-\frac{n^2}{4} + \frac{3n}{2}} \to 0 \text{ as } n \to \infty$$

$\square$

Theorem. $G(n,p)$ for fixed $p \in (0,1)$, then

$$\Pr[\alpha(G) \leqslant \left\lceil \frac{2 \ln n}{p} \right\rceil] \to 1 \text{ as } n \to \infty.$$

proof: Let $k = \left\lceil \frac{2 \ln n}{p} \right\rceil$. Let X_n be the number of independent sets of size $k+1$. For $\forall\, S \in \binom{[n]}{k+1}$, let $I_{\{S \text{ is independent}\}}$ be the indicator random variable, then $X_n = \sum_{S \in \binom{[n]}{k+1}} I_{\{S \text{ is independent}\}}$.

$$\mathrm{E}[I_{\{S \text{ is independent}\}}] = \Pr[S \text{ is independent}] = (1-p)^{\binom{k+1}{2}}$$

So,

$$\mathrm{E}[X_n] = \sum_{S \in \binom{[n]}{k+1}} \mathrm{E}[I_{\{S \text{ is independent}\}}] = \binom{n}{k+1} (1-p)^{\binom{k+1}{2}}$$

$$\leqslant \frac{n^{k+1}}{(k+1)!} e^{-p\binom{k+1}{2}} = \frac{(ne^{-p\frac{k}{2}})^{k+1}}{(k+1)!} < \frac{1}{(k+1)!} \to 0 \text{ as } n \to \infty$$

By Corollary, $\Pr[X_n = 0] \to 1$ as $n \to \infty$.
$\Longrightarrow \Pr[\alpha(G) \leqslant k] \to 1$ as $n \to \infty$. $\square$

Definition. The girth of G, denoted by $g(G)$, is the length of the shortest cycle in G.

Theorem. For $\forall\ k \in \mathbb{N}^+$, there exists a graph G with $\chi(G) \geqslant k$ and $g(G) \geqslant k$.

proof: Consider $G = G(n,p)$ where p will be determined later.
Let X be the number of cycles of length less than k. Let X_i be the number of cycles of length i. Then $X = \sum_{i=3}^{k-1} X_i$.

$$\begin{aligned}\mathrm{E}[X_i] &= \sum_{(x_1\cdots x_i)} \mathrm{E}[I_{\{(x_1\cdots x_i)\text{ is a cycle}\}}] = \sum_{(x_1\cdots x_i)} \Pr[(x_1\cdots x_i)\text{ is a cycle}]\\ &= \frac{n(n-1)\cdots(n-i+1)}{2i}p^i\end{aligned}$$

So

$$\mathrm{E}[X] = \sum_{i=3}^{k-1} \frac{n(n-1)\cdots(n-i+1)}{2i}p^i \leqslant \sum_{i=0}^{k-1}(np)^i = \frac{(np)^k - 1}{np-1}.$$

By Markov's Inequality,

$$\Pr[X \geqslant \frac{n}{2}] \leqslant \frac{\mathrm{E}[X]}{\frac{n}{2}} \leqslant \frac{2[(np)^k-1]}{n(np-1)}.$$

Let $p = n^{-\frac{k-1}{k}}$, then $\Pr[X \geqslant \frac{n}{2}] \leqslant \dfrac{2(n-1)}{n(n^{\frac{1}{k}}-1)} \to 0$ as $n \to \infty$.
Delete a vertex from each cycle of length less than k, we have a graph $G' \subset G$ with no cycle of length less than k.
Note that $|G'| \geqslant n - \dfrac{n}{2} = \dfrac{n}{2}$, $g(G') \geqslant k$.
Recall that $\chi(G) \geqslant \dfrac{|G|}{\alpha(G)}$, and $\Pr[\alpha(G) \leqslant \left\lceil \dfrac{2\ln n}{p} \right\rceil] \to 1$ as $n \to \infty$.
So $\alpha(G') \leqslant \alpha(G) \leqslant \left\lceil \dfrac{2\ln n}{p} \right\rceil \leqslant 3\ln n \cdot n^{\frac{k-1}{k}}$.

$$\Longrightarrow \chi(G') \geqslant \frac{|G'|}{\alpha(G')} \geqslant \frac{\frac{n}{2}}{3\ln n \cdot n^{\frac{k-1}{k}}} = \frac{n^{\frac{1}{k}}}{6\ln n} \gg k \text{ as } n \to \infty. \quad \square$$