Combinatorics 2018 Fall

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2018.11.29

Key words: Projective Plane, Resolvable Design, Affine Plane

Recall.

(1) A (v, k, λ) design $(X, \mathcal{D})$: $|X| = v$, $\mathcal{D} \subset \binom{X}{k}$ such that $\forall \{x, y\} \subset X$, $\{x, y\}$ appears in exactly λ blocks.
r: replication number.
$\Rightarrow r(k-1) = \lambda(v-1)$, $bk = vr$, $b = |\mathcal{D}| = \dfrac{\lambda v(v-1)}{k(k-1)} \geqslant v$.

(2) A (v, k, λ) difference set D.
$(G, \{a + D : a \in G\})$ is a (v, k, λ) design.

(3) Finite linear space $(X, \mathcal{L})$: $\mathcal{L} \subseteq 2^X$; $\forall\, L \in \mathcal{L}, |L| \geqslant 2$ such that $\forall \{x, y\} \subset X$ determine exactly one line.
$|\mathcal{L}| \geqslant 2 \Rightarrow |\mathcal{L}| \geqslant |X|$. Equality holds $\Longleftrightarrow$ Every two lines intersect in exactly one point.

<u>Projective Plane ($PG(q)$)</u>

Definition. A projective plane of order $q \geqslant 1$ is a finite linear space with $q^2 + q + 1$ points, and each line has $q + 1$ points.

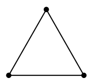
Remark. A projective plane of order q, denoted by $PG(q)$, is a $(q^2 + q + 1, q + 1, 1)$ symmetric design.

$$b = \frac{C^2_{q^2+q+1}}{C^2_{q+1}} = q^2 + q + 1.$$

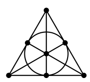
事实上，线和点地位相同，可以互换.

Example.

(1) $q = 1$:



(2) $q = 2$: Fano plane, $(7, 3, 1)$ design:



Proposition. In $PG(q)$:

(1) Any point lies on $q + 1$ lines.

(2) There are in total $q^2 + q + 1$ lines.

(3) Any two lines meet in a unique point.

Construction of $PG(q)$ for prime power $q \geqslant 2$.

Consider $\mathbb{F}_q^3$: 3-dim vector space over $\mathbb{F}_q$.
$V = \{(x_0, x_1, x_2) \in \mathbb{F}_q^3 \ \&\ (x_0, x_1, x_2) \neq (0, 0, 0)\}$, then $|V| = q^3 - 1$.

(1) **points:** $[x_0, x_1, x_2] := \{(cx_0, cx_1, cx_2) : c \in \mathbb{F}_q \backslash \{0\}\}$. So there are $\dfrac{|V|}{q-1} = \dfrac{q^3 - 1}{q - 1} = q^2 + q + 1$ points.

(2) **lines:** $L(a_0, a_1, a_2)$, where $(a_0, a_1, a_2) \in V$, is defined to be the set of points $[x_0, x_1, x_2]$ for which $a_0x_0 + a_1x_1 + a_2x_2 = 0$. There are $q^2 - 1$ solutions to this equation, so there are $\dfrac{q^2 - 1}{q - 1} = q + 1$ points in line $L(a_0, a_1, a_2)$.

(3) **CHECK any two points lie on a unique line:**
i.e. $\forall\ [x_0, x_1, x_2] \neq [y_0, y_1, y_2]$, $\exists!\ L(a_0, a_1, a_2)$ such that

$$\begin{cases} a_0x_0 + a_1x_1 + a_2x_2 = 0 \\ a_0y_0 + a_1y_1 + a_2y_2 = 0 \end{cases}.$$

Since $\begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{bmatrix}$ has rank 2, the solution space has dimension 1, i.e. $\exists!$ line $L(a_0, a_1, a_2)$ containing both $[x_0, x_1, x_2]$ and $[y_0, y_1, y_2]$.

Remark. $\forall\ q \geqslant 2$: prime power, $\exists\ (q^2+q+1, q+1, 1)$ design: $PG(q)$.

Conjecture (open). If q is not a prime power, $\exists\ PG(q)$?

Resolvable Design

Definition. $(X, \mathcal{D})$ is a (v, k, λ) design, r : replication number. A parallel class is a set of blocks from $\mathcal{D}$ such that they partition X. ($\Rightarrow$ Each parallel class has $\frac{v}{k}$ blocks.)
A partition of $\mathcal{D}$ into r parallel classes is called a resolution.
A design is said to be resolvable if it has a resolution.

Problem 1 (Kirkman's schoolgirl problem).
15 girls in a school walk out 3 abreast for 7 days in succession. Is it possible to arrange them daily so that no two shall walk twice abreast?

Solution. $\Leftrightarrow$ Find $(15, 3, 1)$ resolvable design.
Known results:

(1) $\exists\ (v, 3, 1)$ design for $v \equiv 1, 3 \pmod 6$: Steiner Triple System (STS)

(2) $\exists\ (v, 3, 1)$ resolvable design for $v \equiv 3 \pmod 6$: Kirkman Triple System (KTS)

Example. KTS (9,3,1)

123	147	159	168
456	258	267	249
789	369	348	357

Problem 2. A football league of $2n$ teams. Is it possible to arrange a schedule such that they play in $2n-1$ days, and on each day every team plays one match?

Solution. The answer is a resolvable $(2n, 2, 1)$ design $(X, \mathcal{D})$.
Let $X = \{*\} \cup [2n-1]$, $\mathcal{D} = \binom{X}{2}$, $\mathcal{D} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_{2n-1}$ where $\mathcal{D}_i := \{\{i, *\}\} \cup \{\{a, b\} : \ a+b \equiv 2i \pmod{2n-1}\}$ for $i \in [2n-1]$.

(1) $\forall\ a \neq b \in X$: If $a = *$, then $\{a, b\} \in \mathcal{D}_b$. If $a \neq *$, $b \neq *$, then $\exists!\ i \in [2n-1]$ s.t. $a + b \equiv 2i \pmod{2n-1}$, i.e. $\exists!\ \mathcal{D}_i$ s.t. $\{a, b\} \in \mathcal{D}_i$. $\implies \mathcal{D} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_{2n-1}$ is a partition.

(2) Given $i \in [2n-1]$, CHECK $\mathcal{D}_i$ is a parallel class. $\forall\ a \in X$: If $a = *$, then the unique block in D_i containing a is $\{i, *\}$. If $a \neq *$, then $\exists!\ b$ s.t. $a + b \equiv 2i \pmod{2n-1}$, i.e. the unique block in D_i containing a is $\{a, b\}$.

<u>Affine Plane $AG(q)$</u>

Definition. Affine plane $AG(q)$ is a finite linear space such that all lines form a $(q^2, q, 1)$ design.

Remark. For $AG(q)$, we have: $r = \dfrac{q^2-1}{q-1} = q+1$, $b = \dfrac{vr}{k} = q^2+q$.

Construction of $AG(q)$ from $PG(q)$.

Construction 1.
$(X, \mathcal{L})$: $PG(q)$. Then $|X| = q^2+q+1$, $|\mathcal{L}| = q^2+q+1$, $|L| = q+1$. Fix $L_0 \in \mathcal{L}$. Let $X' = X \setminus \{L_0\}$, $\mathcal{L}' = \{L \setminus L_0 : L \in \mathcal{L}, L \neq L_0\}$. Then it's easy to see that $(X', \mathcal{L}')$ is $AG(q)$.
Next, we show resolvability. $\forall\ x \in L_0$, let $\mathcal{L}_x = \{L \setminus \{x\} : L_0 \neq L \in \mathcal{L}, x \in L\}$. Then $\mathcal{L}_x$ is a parallel class. Since in $PG(q)$, every $L \neq L_0$ intersects L_0 in exactly one point, we know that $\bigcup_{x \in L_0} \mathcal{L}_x$ is a partition of $\mathcal{L}'$.

Construction 2.
$q \geqslant 2$: prime power. Consider $\mathbb{F}_q$.
Let $X = \mathbb{F}_q \times \mathbb{F}_q$. Let $\mathcal{L}$ be the set of all lines of the form

$$L(a,b) = \{(x,y) \in X : y = ax + b\}, \ a, b \in \mathbb{F}_q,$$

and

$$L(c) = \{(c,y) \in X : y \in \mathbb{F}_q\}, \ c \in \mathbb{F}_q.$$

Then $(X, \mathcal{L})$ is $AG(q)$.
<u>proof:</u> $|X| = q^2$, $|L| = q$. For $\forall\ (x_1, y_1) \neq (x_2, y_2) \in X$,

(1) If $x_1 = x_2$, then the unique line containing them is $L(x_1)$.

(2) If $x_1 \neq x_2$, then

$$\begin{cases} y_1 = ax_1 + b \\ y_2 = ax_2 + b \end{cases}$$

has a unique solution (a, b), i.e. the unique line containing them is $L(a, b)$.

Next, show resolvability. $\{L(a,b) : b \in \mathbb{F}_q\}$ $(a \in \mathbb{F}_q)$, $\{L(c) : c \in \mathbb{F}_q\}$ is a resolution. (Note that $r = q + 1$.)

Fact. Any $AG(q)$ is resolvable.