Combinatorics 2018 Fall

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Reference:

Extremal Combinatorics with applications in Computer Science. 2nd Edition.Stasys Jukna,Springer.

《组合数学》潘永亮，徐俊明，科学出版社2006

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Key words: bijection, binomial coefficient, double counting

Notation:

(1) A set X (a collection of distinct elements)
$|X| :=\#$ (the number of) elements of X

(2) for an integer $n > 0$ $[n] := \{1, 2, \cdots, n\}$

(3) A vector(or string) of length n over X
$(x_1, \cdots, x_n)$. $x_i \in X, i \in [n]$

(4) i.e. that is.

$\exists$ exist.

$\Rightarrow$ imply.

s.t. such that.

! unique.

Counting:

Function: For sets A,B. a function $A \to B$ is a relation between A and B. $(a, f(a))$ s.t. for each $a \in A$, there is exactly one $b \in B$ satisfying $b = f(a)$

Injection(one to one function): if $a \neq a'$,then $f(a) \neq f(a')$.

Surjection(onto): $\forall b \in B, \exists a \in A, s.t. f(a) = b$.

Bijection(one to one correspondence): Injection & Surjection.

Porpostion1. For sets A,B, $f : A \to B$ is a function

(1) if f is an injection, then $|A| \leq |B|$.

(2) if f is a surjection, then $|A| \geq |B|$.

(3) if f is a bijection, then $|A| = |B|$.

proof:

(1) Proof by contradiction. Assume $|B| < |A|$ since f is a function, by definition, $\forall a \in A, \exists! b \in B$ s.t.$f(a) = b$. since $|B| < |A|$ then $\exists a \neq a\prime$ s.t. $f(a) = f(a\prime)$ (by pigeonhole principal) This is contradiction with the fact that f is an injection.

(2) Proof by contradiction. Assume $|B| > |A|$ since f is a function, by definition, $\forall a \in A, \exists! b \in B$ s.t.$f(a) = b$. $\exists b \in B$ s.t.there does not exist $a \in A$ satisfying $f(a) = b$ This is contradiction with the fact that f is a surjection.

(3) (1)&(2) $\Rightarrow$ (3). $\square$

Porpostion2:
For sets X,Y. $|X| = n, |Y| = r$. Let $X^Y := \{all\ functions\ f : Y \to X\}$.Then $|X^Y| = n^r$.

proof: Let B=$\{$all vectors of length r over $X\}$. $|B|$=n^r. Assume $Y = \{y_1, y_2, \cdots, y_r\}$ Define a function

$$g : X^Y \to B.$$

$$f \mapsto (f(y_1), \cdots, f(y_r))$$

First, g is well-defined.
g is a injection since $f \neq f' \Rightarrow (f(y_1), \cdots, f(y_r)) \neq (f'(y_1), \cdots, f'(y_r))$.
g is a surjection since $\forall (b_1, \cdots, b_r) \in B$, define $f : Y \to X, y_i \mapsto b_i (i.e. f(y_i) = b_i)$ That is easy to get $g(f) = (b_1, \cdots, b_r)$.
$\Rightarrow g$ is a bijection. $\Rightarrow |X^Y| = |B| = n^r$. □

Porpostion3:
Let $Y = [r], 2^Y$:={all subsets of Y}.then $|2^Y| = 2^r$

proof: Let B ={all vectors of length r over {0,1}}. define function

$$f : 2^Y \to B, A \to f(A) = (b_1, \cdots, b_r)$$

$$\text{where } \mathrm{b}_i = \begin{cases} 0, if\ i \notin A; \\ 1, if\ i \in A. \end{cases}$$

Ex: f is a bijection. $\Rightarrow |Y| = |B| = 2^r$.

Remark: f is called indicator(characteristic) function, $g(A)$ is called the indicator vector of A, A is called the support of $f(A) = (b_1, \cdots, b_r)$.
□

Porpostion4:
Let $|X| = n, |Y| = r, n \geq r$, then # {injection $f : Y \to X$} $=n(n-1) \cdot (n-r+1) := (n)_r$ (r-th factorial of n)

proof: Let A={injection $f : Y \to X$}, B ={all vectors of length r over X with r different entries}
Ex: define a bijection $g : A \to B$ and complete the proof. □

Binomial Coefficient:

- $\binom{n}{k} :=$ #$k-$subsets of an $n-$elements set.
- $|X| = n, \binom{X}{k}$:={$A \subset X : |A| = k$}. $\left|\binom{X}{k}\right| = \binom{n}{k}$.

- $n! = n(n-1)\cdots 2\cdot 1.$

  $(n)_r := n(n-1)\cdots(n-r-1).$

  $0! = 1.$

  $\binom{n}{0} = 1.$

  $\binom{n}{k} = 0$ if $k > n$.

Porposition5: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

proof: Let B =$\{$all vectors of length k over $[n]$ consisting k different elements.$\}$
(Double Counting)
Way1: Just directly count the number of vectors. $|B| = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$.
Way2: There are $\binom{n}{k}$ ways to choose $k-$subset of X. For each $k-$subset ,there are $k!$ ways to order it to a vector.
$\Rightarrow |B| = \binom{n}{k}k!$
$\Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$. □