Combinatorics, 2018 Fall, USTC
Homework4

- **The due is on Thursday, Oct. 18 .**
- **Please sign your name and student number.**
- **Please solve as many problems as you can.**

1. Let $P(n) = \prod_{\substack{k \in [n] \\ (k,n)=1}} k$. Prove $P(n) = n^{\varphi(n)} \prod_{d \mid n} (\frac{d!}{d^d})^{\mu(\frac{n}{d})}$

2. Let $\alpha \in \mathbb{R}$, define $\varphi_\alpha(n) = \sum_{\substack{k \in [n] \\ (k,n)=1}} k^\alpha$. Prove

   (1) $\sum_{d \mid n} \frac{\varphi_\alpha(d)}{d^\alpha} = \frac{1}{n^\alpha} \sum_{k=1}^{n} k^\alpha$.

   (2) when $n \geq 2$, let $n = p_1^{a_1} \cdots p_m^{a_m}$ be the standard decomposition of n. Show

      (i) $\varphi_1(n) = \frac{n\varphi(n)}{2}$

      (ii) $\varphi_2(n) = \frac{1}{3}n^2\varphi(n) + \frac{n}{6}\prod_{i=1}^{m}(1-p_i)$.

3. Given a poset P, a chain in P is any subset $C \subseteq P$ such that any two elements in C are comparable, that is, for any $x, y \in C$, we have either $x \leq y$ or $y \leq x$.

   For any $a, b \in P$, define a function η by:

$$\eta(a,b) = \zeta(a,b) - \delta(a,b).$$

Show that $\eta^k(a,b)$ is equal to the number of chains of length k with the smallest element a and the largest element b.

[say the length of a chain $a < a_1 < \cdots < a_{k-1} < b$ is k.]

[**Hint:** use the matric representation of functions]

4. Show that for any invertible function $f \in \mathbb{A}(P)$, the inverse f^{-1} can be written as $f^{-1} = a_0\delta + a_1 f + \cdots + a_k f^k$ for some integer k and $a_i \in \mathbb{R},\ i \in [k]$. Here f^i means the Dedekind convolution of i functions f

   [**Hint:** use the matric representation of functions]

5. A simple poset consists of a top element a and a bottom element b and n incomparable elements $z_1, \cdots, z_n$ between. i.e. $b \leq z_i \leq a, i \in [n]$. Calculate its *Möbius function*.

6. Show that $N \odot (N\mu) = I$, and $\varphi \odot (N\mu \odot e) = I$