Combinatorics, 2018 Fall, USTC
Homework 3

- **The due is on Thursday, Oct. 11.**
- **Please sign your name and student number.**
- **Please solve as many problems as you can.**

1. Show that the number $\frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]$ is an integer for all $n \geqslant 1$.

2. Let $f_n(x) = x(x-1)\cdots(x-n+1)$. Prove that
$$f_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} f_k(x) f_{n-k}(y).$$
Hint: Similar to the proof of Vandermonde's Identity.

3. Let a_n denote the number of mappings $f : [n] \to [n]$ such that if f takes a value i, then it also takes every value j for $1 \leqslant j \leqslant i$. Let $a_0 = 1$. Find the closed form of the exponential generating function $f(x)$ of $\{a_n\}_{n\geqslant 0}$.

Hint: Find the recurrence first.

4. (Bell numbers) Denote by B_n the number of unordered partitions of n distinct elements into disjoint subsets. Let $B_0 = 1$. Find a formula for B_n.

   Hint: Find the recurrence first.

5. Let $f(x)$ be a real function with $0 \leqslant x \leqslant 1$. If for any positive integer n, we have $F(n) = \sum_{k=1}^{n} f(\frac{k}{n})$ and $F^*(n) = \sum_{\substack{k=1 \\ \gcd(k,n)=1}}^{n} f(\frac{k}{n})$, then show that $F^*(n) = \sum_{d|n} \mu(d) F(\frac{n}{d})$.

6. Suppose that we are given infinitely many tickets, each with one natural number on it. For any $n \in \mathbb{N}$, the number of tickets on which divisors of n are written are exactly n. For example, the divisors of 6, $\{1, 2, 3, 6\}$, are written in some variation on 6 tickets, and no other ticket has these numbers written on it. Prove that any number $n \in \mathbb{N}$ is written on at least one ticket.