Combinatorics 2018 Fall

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Recall.

- $\mathcal{F} \subset \binom{[n]}{k}$ is 2-colorable if $\exists\ f : [n] \to \{\text{blue, red}\}$ $s.t.$ NO $A \in \mathcal{F}$ is monochromatic.
- $B(k) = \min |\mathcal{F}|$ $s.t.$ $\mathcal{F}$ is not 2-colorable
- $|\mathcal{F}| < 2^{k-1}$ is 2-colorable, and

$$2^{k-1} \leqslant B(k) \leqslant \binom{2k}{k}.$$

Theorem. If k is sufficiently large, then there exists a k-uniform $\mathcal{F}$ $s.t.$ $|\mathcal{F}| \leqslant k^2 \cdot 2^k$ and $\mathcal{F}$ is not 2-colorable.

<u>proof:</u> Let $X = [r]$. Choose b k-subsets from $\binom{[r]}{k}$ independently at random. Specifically, pick a k-subset A_1 from $\binom{[r]}{k}$ with probability $\frac{1}{\binom{r}{k}}$, and repeat this process b times to get $\mathcal{F} = \{A_1, \cdots, A_b\}$ (allow repetition).
Let B be the event that $\mathcal{F}$ is 2-colorable. It suffices to prove $\Pr[B] < 1$.
Let B_χ be the event that $\mathcal{F}$ is 2-colorable under coloring χ, then

$B = \cup_{\chi \in 2^{[r]}} B_\chi$.

$$\begin{aligned}
\Pr[B_\chi] &= \Pr[\bigcap_{i=1}^{b} \{A_i \text{ is not monochromatic under } \chi\}\ ] \\
&= \prod_{i=1}^{b} \Pr[A_i \text{ is not monochromatic under } \chi] \\
&= \prod_{i=1}^{b} (1 - \Pr[A_i \text{ is monochromatic under } \chi])
\end{aligned}$$

Suppose χ has s red points and $r-s$ blue points, then

$$\begin{aligned}
\Pr[A_i \text{ is monochromatic under } \chi] &= \frac{\binom{s}{k} + \binom{r-s}{k}}{\binom{r}{k}} \\
&\overset{\text{Jensen's Inequality}}{\geqslant} \frac{2\binom{r/2}{k}}{\binom{r}{k}} \triangleq p
\end{aligned}$$

So, $\Pr[B_\chi] \leqslant (1-p)^b$ and

$$Pr[B] \leqslant \sum_{\chi \in 2^{[r]}} (1-p)^b = 2^r (1-p)^b \leqslant 2^r \cdot e^{-pb} = e^{r \ln 2 - pb}$$

Note that $p = \frac{1}{2^{k-1}} \prod_{i=0}^{k-1} (1 - \frac{i}{r} \frac{1}{1-\frac{i}{r}})$ and $1 - \frac{i}{r}\frac{1}{1-\frac{i}{r}} \sim 1 - \frac{i}{r} + O(\frac{i^2}{r^2})$, so $p \leqslant \frac{1}{2^{k-1}} \prod_{i=0}^{k-1} e^{-\frac{i}{r} + O(\frac{i^2}{r^2})} \leqslant \frac{1}{2^{k-1}} e^{-\frac{k(k-1)}{2r} + O(\frac{k^3}{r^2})}$. Let $b = \frac{r \ln 2}{p}$, then $b \geqslant r \ln 2 \cdot 2^{k-1} e^{\frac{k(k-1)}{2r} + O(\frac{k^3}{r^2})}$. If we choose $r = \frac{k^2}{2}$ for k sufficiently large, then $b \geqslant \frac{k^2 \ln 2}{2} \cdot 2^{k-1} e^{1+O(1)}$, which implies that we can choose $b = k^2 \cdot 2^k$, and thus get $\Pr[B] < 1$. $\square$

Linearity of Expectation

- $A \subset \Omega$, $X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$
  $\mathrm{E}[X_A] = \sum_{\omega \in \Omega} \Pr[\omega] X_A(\omega) = \sum_{\omega \in A} \Pr[\omega] = \Pr[A]$
- $X : \Omega \to \mathbb{R}$ is a random variable, $\mathrm{E}[X] = \sum_{\omega \in \Omega} \Pr[\omega] X(\omega)$
- $\Pr[X \geqslant \mathrm{E}[X]] > 0$, $\Pr[X \leqslant \mathrm{E}[X]] > 0$.
- $\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$

Definition. Let A be a subset of an additive group. A is sum-free if $\forall\ x, y \in A$, $x + y \notin A$.

Example.

- $A = \{n+1, n+2, \cdots, 2n\} \subset \mathbb{Z}$
- $A = \{\text{odd integers}\} \subset \mathbb{Z}$

Theorem. For any set A of non-zero integers, there is a sum-free set $B \subset A$ with $|B| \geqslant |A|/3$.

proof: Let $p = 3k + 2$ be a prime $s.t.$ $p > 2 \max_{a \in A} |a|$. (Such a prime does exist because of Dirichlet's Prime Number Theorem: $\forall\ a, b$ with $(a, b) = 1$, $\exists$ infinitely many primes of the form $a + nb$.)
Let $S = \{k+1, k+2, \cdots 2k+1\} \subset \mathbb{Z}_{3k+2} = \mathbb{Z}_p$, then S is sum-free in $\mathbb{Z}_p$.
For $x \in \mathbb{Z}_p \backslash \{0\}$, define $A_x = \{a \in A : (ax \mod p) \in S\} \subset A$.
Claim: A_x is sum-free in $\mathbb{Z}$.
proof of Claim: $\forall\ a, b \in A_x$, $(ax \mod p) \in S$ and $(bx \mod p) \in S$. If $a + b \in A_x$, then $((a+b)x \mod p) \in S$, a contradiction to the fact that S is sum-free.
Choose $x \in \mathbb{Z}_p \backslash \{0\}$ uniformly at random, and compute $\mathrm{E}[|A_x|]$.
Define a random variable $X_{a,S} : x \to \begin{cases} 1 & \text{if } (ax \mod p) \in S \\ 0 & \text{otherwise} \end{cases}$. Then

$|A_x| = \sum_{a\in A} X_{a,S}$.
So, by linearity of expectation,

$$\mathrm{E}[|A_x|] = \sum_{a\in A} \mathrm{E}[X_{a,S}] = \sum_{a\in A} \Pr[(ax \mod p) \in S]$$

Note that for a fixed $a \in A$, $\{(ax \mod p) : x \in \mathbb{Z}_p\backslash\{0\}\} = \mathbb{Z}_p\backslash\{0\}$, so running over $x \in \mathbb{Z}_p\backslash\{0\}$ for a fixed $a \in A$, there are exactly $|S|$ many $x \in \mathbb{Z}_p\backslash\{0\}$ satisfying $(ax \mod p) \in S$.
So, $E[|A_x|] = \sum_{a\in A} \frac{|S|}{p-1} = \sum_{a\in A} \frac{k+1}{3k+1} > \sum_{a\in A} \frac{1}{3} = \frac{|A|}{3}$.
Therefore, $\Pr[|A_x| > \frac{|A|}{3}] > 0. \implies \exists\, x \in \mathbb{Z}_p\backslash\{0\}$ *s.t.* A_x is sum-free and $|A_x| \geqslant |A|/3$. $\square$

Definition. A dominating set of $G = (V, E)$ is a set $A \subset V$ *s.t.* every $v \in V\backslash A$ has a neighbor in A.

Theorem. Let $G = (V, E)$ be a graph on n vertices with minimum degree $\delta > 1$. Then G contains a dominating set of at most $\frac{1+\ln(1+\delta)}{1+\delta} n$ vertices.

proof: For $p \in (0, 1)$ which will be determined later, we pick each vertex in V with probability p independently at random.
Let X be the random set of vertices picked. Let Y be the random set of vertices $y \in V\backslash X$ which has no neighbor in X. That is, $y \in Y$ iff y is not picked and all neighbors of y are not picked. We consider $X \cup Y$.
$\mathrm{E}[|X \cup Y|] = \mathrm{E}[|X|] + \mathrm{E}[|Y|]$.
$\mathrm{E}[|X|] = \sum_{x\in V} \Pr[x \in X] = np$.
$\Pr[y \in Y] = (1-p) \cdot (1-p)^{deg(y)} \leqslant (1-p)^{1+\delta}$.
$\implies \mathrm{E}[|Y|] = \sum_{y\in V} \Pr[y \in Y] \leqslant n(1-p)^{1+\delta}$.
So, $\mathrm{E}[|X \cup Y|] \leqslant np + n(1-p)^{1+\delta} \leqslant np + ne^{-p(1+\delta)}$.
Choose $p = \frac{\ln(1+\delta)}{1+\delta}$ such that $p + e^{-p(1+\delta)}$ is minimized, then

$\mathrm{E}[|X \cup Y|] \leqslant \dfrac{1 + \ln(1+\delta)}{1+\delta} n.$

$\implies \exists$ a dominating set of size $\dfrac{1 + \ln(1+\delta)}{1+\delta} n.$ $\square$