Combinatorics 2018 Fall

Teaching by: Professor Xiande Zhang

2018.09.20

Key words: IEP, Generating Function

IEP: $|A_1^c \cap A_2^c \cap \cdots \cap A_n^c| = \sum\limits_{I \subseteq [n]} (-1)^{|I|} |A_I|$, where $A_I := \cap_{i \in I} A_i$

Porpostion2:
If $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$, where p_i are distinct primes, then $\varphi(n) = n \prod\limits_{i=1}^{t} (1 - \frac{1}{p_i})$.

<u>proof:</u> $\Omega = [n]$, let $A_i = \{m \in [n] :\ p_i | m\}$, then $\varphi(n) = |A_1^c \cap \cdots \cap A_t^c|$
$|A_i| = \frac{n}{p_i}$ and $|A_I| = \frac{n}{\prod\limits_{i \in I} p_i}$
By IEP, $\varphi(n) = \sum\limits_{I \subseteq [t]} (-1)^{|I|} \frac{n}{\prod\limits_{i \in I} p_i} = n \sum\limits_{I \subseteq [t]} (-1)^{|I|} \frac{1}{\prod\limits_{i \in I} p_i}$
Notice that $\prod\limits_{i=1}^{t} (1 + x_i) = (1 + x_1) \cdots (1 + x_t) = \sum\limits_{I \subseteq [t]} \prod\limits_{i \in I} x_i$,
then $\varphi(n) = n \sum\limits_{I \subseteq [t]} (-1)^{|I|} \frac{1}{\prod\limits_{i \in I} p_i} = n \prod\limits_{i=1}^{t} (1 - \frac{1}{p_i})$. $\square$

Def: A permutation $\sigma :\ X \longrightarrow X$ is a bijection. It is called a derangement of X if $\sigma(i) \neq i,\ \forall i \in X$.

Porpostion3:
Let $D_n = \sharp$ derangement of $[n]$, then $D_n = n! \sum\limits_{k=0}^{n} \frac{(-1)^k}{k!}$.

<u>Proof:</u> Ω=\{all permutations\}, let A_i=\{all permutations satisfying $\sigma(i) = i$\}, $i \in [n]$. Then $|A_i| = (n-1)!$, $|A_I| = (n - |I|)!$.

$D_n = |A_1^c \cap \cdots \cap A_n^c| = \sum\limits_{I \subseteq [n]} (-1)^{|I|}|A_I| = \sum\limits_{I \subseteq [n]} (-1)^{|I|}(n-|I|)! = \sum\limits_{i=0}^{n} \binom{n}{i}(-1)^i(n-i)! = n! \sum\limits_{i=0}^{n} \frac{(-1)^i}{i!}$ $\square$

Generating function(GF)

Problem1: There are 30 red balls, 40 blue balls and 50 white balls in the box, # ways of selecting 70 balls?

<u>Sol:</u> Consider the coefficient of x^{70} in $(1+x+\cdots+x^{30})(1+x+\cdots+x^{40})(1+x+\cdots+x^{50})$

Def: The (ordinary) GF of an infinity sequence $a_0, a_1, a_2, \cdots$ is a power series $f(x) = \sum\limits_{n \geq 0} a_n x^n$.

Consequences:

Let $a(x) = \sum\limits_{n \geq 0} a_n x^n$ and $b(x) = \sum\limits_{n \geq 0} b_n x^n$, then

$$a(x) + b(x) = \sum_{n \geq 0} (a_n + b_n)x^n,$$

$$a(x)b(x) = \sum_{n \geq 0} c_n x^n,$$

where $c_n = \sum\limits_{i+j=n} a_i b_j = \sum\limits_{i=0}^{n} a_i b_{n-i}$.

1. $\frac{1}{1-x} = \sum\limits_{n \geq 0} x^n$ is the GF of $\{1, 1, 1, \cdots\}$
2. $\frac{1}{1-ax} = \sum\limits_{n \geq 0} (ax)^n = \sum\limits_{n \geq 0} a^n x^n$ is the GF of $\{a^0, a^1, \cdots\}$
3. $\frac{1}{1-x^2} = \sum\limits_{n \geq 0} x^{2n}$ is the GF of $\{1, 0, 1, 0, \cdots\}$

4. $f(x)=\sum\limits_{n\geq 0} a_n x^n$ is the GF of $\{a_0, a_1, a_2, \cdots\}$

5. $cf(x)=\sum\limits_{n\geq 0} ca_n x^n$ is the GF of $\{ca_0, ca_1, ca_2, \cdots\}$

6. $f(cx)=\sum\limits_{n\geq 0} c^n a_n x^n$ is the GF of $\{c^0a_0, c^1a_1, c^2a_2, \cdots\}$

7. $f(x^3)=\sum\limits_{n\geq 0} a_n x^{3n}$ is the GF of $\{a_0, 0, 0, a_1, 0, 0, \cdots\}$

8. $x^3 f(x)=\sum\limits_{n\geq 0} a_n x^{n+3}$ is the GF of $\{0, 0, 0, a_0, a_1, \cdots\}$

9. $\frac{f(x)-a_0x^0-a_1x^1-a_2x^2}{x^3}$ is the GF of $\{a_3, a_4, \cdots\}$

10. $f'(x)=\sum\limits_{n\geq 1} na_n x^{n-1}$ is GF of $\{a_1, 2a_2, 3a_3, \cdots\}$

11. $\sum\limits_{n\geq 1} nx^{n-1}=(\frac{1}{1-x})'=\frac{1}{(1-x)^2}$ is GF of $\{1, 2, 3, \cdots\}$

12. Take the (k-1)-th derivative of $\sum\limits_{n\geq 0} x^n=\frac{1}{1-x}$, we get $\sum\limits_{n\geq k-1} n(n-1)\cdots(n-(k-2))x^{n-(k-1)}=\frac{(k-1)!}{(1-x)^k}$, then $\frac{1}{(1-x)^k}=\sum\limits_{n\geq k-1}\binom{n}{k-1}x^{n-(k-1)}=\sum\limits_{n\geq 0}\binom{n+k-1}{k-1}x^n$

    $\frac{1}{(1-x)^k}$ is the GF of $\{a_n\}_{n=0}^{\infty}$, where a_n is the $\sharp$integer solutions to $x_1+x_2+\cdots+x_k=n,\ x_i\geq 0$.

Generalized binomial theorem: r any real number, $(1+x)^r=\sum\limits_{n\geq 0}\binom{r}{n}x^n$, where $\binom{r}{n}=\frac{(r)(r-1)\cdots(r-n+1)}{n!}$, we call $\binom{r}{n}$ generalized binomial coefficient.

Remark: $(1-x)^{-k}=\sum\limits_{n\geq 0}\binom{-k}{n}(-x)^n$, $\binom{-k}{n}=\frac{(-k)(-k-1)\cdots(-k-n+1)}{n!}=(-1)^n\frac{(n+k-1)\cdots(k+1)k}{n!}=(-1)^n\binom{n+k-1}{n}\Longrightarrow(1-x)^{-k}=\sum\limits_{n\geq 0}\binom{n+k-1}{n}x^n$

Recall Problem1: There are 30 red balls, 40 blue balls and 50 white balls in the box, # ways of selecting 70 balls?

<u>Sol:</u>

$$\begin{aligned} f(x) &= (1+x+\cdots+x^{30})(1+x+\cdots+x^{40})(1+x+\cdots+x^{50}) \\ &= (\frac{1}{1-x}-\frac{x^{31}}{1-x})(\frac{1}{1-x}-\frac{x^{41}}{1-x})(\frac{1}{1-x}-\frac{x^{51}}{1-x}) \\ &= \frac{(1-x^{31})(1-x^{41})(1-x^{51})}{(1-x)^3} \\ &= (\sum_{n\geq 0}\binom{n+2}{2}x^n)(1-x^{31})(1-x^{41})(1-x^{51}) \end{aligned}$$

$\Longrightarrow [x^{70}]f(x) = \binom{72}{2} - \binom{41}{2} - \binom{31}{2} - \binom{21}{2}$ $\square$

Problem2:

Fibonacci number $F_0 = 0, F_1 = 1. F_{n+2} = F_{n+1} + F_n, n \geq 0$, compute $\{F_n\}$

<u>Sol:</u> Let $f(x) = \sum\limits_{n\geq 0} F_n x^n$, then $\sum\limits_{n\geq 0} F_{n+2}x^n = \sum\limits_{n\geq 0} F_{n+1}x^n + \sum\limits_{n\geq 0} F_n x^n$

$\Longrightarrow \frac{f(x)-F_0-F_1x}{x^2} = \frac{f(x)-F_0}{x} + f(x)$

$\Longrightarrow f(x) - x = xf(x) + x^2 f(x) \Longrightarrow f(x) = \frac{x}{1-x-x^2} = \frac{a}{1-\lambda_1 x} + \frac{b}{1-\lambda_2 x}$,

where $\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}, a = \frac{1}{\sqrt{5}}, b = -\frac{1}{\sqrt{5}}$

$\Longrightarrow f(x) = \sum\limits_{n\geq 0} a(\lambda_1 x)^n + \sum\limits_{n\geq 0} b(\lambda_2 x)^n = \sum\limits_{n\geq 0}(a\lambda_1^n + b\lambda_2^n)x^n$

$\Longrightarrow F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$ $\square$