Combinatorics, 2018 Fall, USTC
Homework 13

- **The due is on Thursday, Dec. 27.**
- **Please sign your name and student number.**
- **Please solve as many problems as you can.**

1. (1) A fixed point of a permutation π is an integer i such that $\pi(i) = i$. Let X_π be the number of fixed points of permutation π. What is the expectation $\mathrm{E}[X_\pi]$ for a random permutation π uniformly chosen from the set S_n?

   (2) We toss a fair 0-1 coin n times. What is the expected number of *runs*? Here, a *run* is a maximal set of consecutive tosses with the same result. For example, the sequence 0001111100 has 3 *runs*.

2. Prove that if there is a real $p \in [0, 1]$ such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$

   then Ramsey number $R(k, t) > n$. Using this to show that $R(4, t) \geqslant C \cdot (\frac{t}{\ln t})^{3/2}$ for some constant $C > 0$.

3. Let $k \geqslant 4$ be an integer and $\mathcal{F} \subset \binom{X}{k}$ be a k-uniform family. Prove that if $|\mathcal{F}| < \frac{4^{k-1}}{3^k}$, then there is a coloring of elements

of X with four colors such that in any set of $\mathcal{F}$, all four colors are presented.

4. A family of pairs of sets $\mathcal{F} = \{(A_i, B_i)\}_{i=1}^{h}$ is called a (k,l)-system if $|A_i| = k$, $|B_i| = l$ for $i \in [h]$, and $A_i \cap B_j = \emptyset$ iff $i = j$. Give a probabilistic proof of $|\mathcal{F}| \leqslant \binom{k+l}{k}$.

5. Use probabilistic method to prove EKR Theorem, *i.e.* if $\mathcal{F}$ is a k-uniform intersecting family over $[n]$, $n \geqslant 2k$, then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$.