Combinatorics, 2018 Fall, USTC
Homework 9

- **The due is on Thursday, Nov. 29.**
- **Please sign your name and student number.**
- **Please solve as many problems as you can.**

1. Let $A_1, \cdots, A_m$ be a collection of sets of size a and $B_1, \cdots, B_m$ be a collection of sets of size b such that $|A_i \cap B_i| = t$ for all $i \in [n]$ and $|A_i \cap B_j| > t$ for all $i \neq j$. Show that $m \leqslant \binom{a+b-t}{a-t}$.

2. Let $\mathcal{F}$ be a k-uniform family, and suppose that it is intersection-free, i.e. $A \cap B \nsubseteq C$ for any three sets A, B, C of $\mathcal{F}$. Prove that $|\mathcal{F}| \leqslant 1 + \binom{k}{\lfloor \frac{k}{2} \rfloor}$.

3. Let $n = 2k$. Characterize (i.e. list) all intersecting families $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\mathcal{F}| = \binom{n-1}{k-1}$.

4. Let $n \leqslant 2k$, and let $A_1, \cdots, A_m$ be a family of k-element subsets of $[n]$ such that $A_i \cup A_j \neq [n]$ for all i, j.
Show that $m \leqslant (1 - \frac{k}{n})\binom{n}{k} = \binom{n-1}{k}$.

5. Use Fisher's Inequality to show that $|D| \geqslant |X|$, where (X, D) is a (v, k, λ) design.

6. Let (X, D) be a (v, k, λ) design with b blocks and replication number r. Prove that its complement $(X, \overline{D})$, where $\overline{D} = \{X \setminus B : B \in D\}$, is a $(v, v-k, b-2r+\lambda)$ design provided that $b - 2r + \lambda > 0$.

7. Let A be the adjacency matrix of a finite d-regular graph G with vertex set $[n]$. Let $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$ be a column vector of length n. Show that

   (1) If $Ax = y$ and $y = (y_1, \cdots, y_n)^T$, then $y_i = \sum\limits_{j: i\sim j} x_j$.

   (2) Consider any orientation of G, that is, any assignment of direction to each edge, then each edge $u \sim v$ becomes an arc $u \to v$ or $v \to u$. Let D be the incidence matrix of the orientation of G, whose rows and columns are indexed by vertices and edges, and for $\forall\ v \in V$, $e \in E$,

   $$D_{v,e} = \begin{cases} 1, & \text{if } v \text{ is the head of } e \\ -1, & \text{if } v \text{ is the tail of } e \\ 0, & \text{otherwise.} \end{cases}$$

   Show that $DD^T = dI - A$. Here, $L = DD^T$ is called *Laplacian matrix* of G.

   (3) If λ is an eigenvalue of A, show that $d \geqslant \lambda$.