# Combinatorics 2018 Fall

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**Key words:** Odd/Even-town, Linear Algebra Method (about polynomials)

**Recall.**

(1) (Frankl-Wilson Theorem) $\mathcal{F} \subseteq 2^{[n]}$ is an $L$-intersecting family, then $|\mathcal{F}| \leqslant \sum_{i=0}^{|L|} \binom{n}{i}$.

(2) Let $p$ be a prime and $L \subset \mathbb{Z}_p = \{0, 1, \cdots, p-1\}$. Assume $\mathcal{F} = \{A_1, \cdots A_m\} \subseteq 2^{[n]}$ such that

(a) $|A_i| \pmod p \notin L, \forall\ i \in [m]$;

(b) $|A_i \cap A_j| \pmod p \in L, \forall\ i \neq j$.

Then $|\mathcal{F}| \leqslant \sum_{i=0}^{|L|} \binom{n}{i}$

(3) (Application) $\exists$ graph on $\binom{p^3}{p^2-1}$ vertices such that the size of maximum clique or maximum independent set is $\leqslant \sum_{i=0}^{p-1} \binom{p^3}{i}$.
$\Longrightarrow R(t+1, t+1) \geqslant t^{\Omega(\ln t / \ln \ln t)}$.

**Odd/Even-town.** Let $\mathcal{F} \subseteq 2^{[n]}$ *s.t.* $|A|$ is odd for all $A \in \mathcal{F}$, and $|A \cap B|$ is even for $\forall\ A \neq B \in \mathcal{F}$. Then $|\mathcal{F}| \leqslant n$.

proof: $\forall\ A \in \mathcal{F}$, let $\mathbf{e_A}$ be the indictor vector of $A$. View $\mathbf{e_A} \in \mathbb{F}_2^n$, then

$$< \mathbf{e_A}, \mathbf{e_A} >= 1, \quad \forall\ A \in \mathcal{F}$$
$$< \mathbf{e_A}, \mathbf{e_B} >= 0, \quad \forall\ A \neq B \in \mathcal{F}$$

Assume $\exists\ \alpha_A \in \mathbb{F}_2 = \{0, 1\}$ $s.t.$ $\sum_{A \in \mathcal{F}} \alpha_A \mathbf{e_A} = \mathbf{0}$, then

$$0 =< \sum_{A \in \mathcal{F}} \alpha_A \mathbf{e_A}, \mathbf{e_B} >= \alpha_B < \mathbf{e_B}, \mathbf{e_B} >= \alpha_B, \quad \forall\ B \in \mathcal{F}$$

So $\mathbf{e_A},\ A \in \mathcal{F}$ are linearly independent, which implies $|\mathcal{F}| \leqslant n$. $\square$

**Corollary.** $R(t+1, t+1) > \binom{t}{3} \sim t^3$.

proof: Define $G = (V, E)$ as follows:

$$V = \binom{[t]}{3}, \quad A \sim B \text{ iff } |A \cap B| = 1.$$

Consider a clique $A_1, \cdots, A_m$, then $|A_i \cap A_j| = 1$, $\forall\ i \neq j$. By Fisher's Inequality, $m \leqslant t$.
Condsider an independent set $B_1, \cdots B_s$, then $|B_i| = 3$ $(Odd)$, $|B_i \cap B_j| = 0$ *or* $2$ $(Even)$. By Odd/Even-town, $s \leqslant t$. $\square$

**Even/Odd-town.** Let $\mathcal{F} \subseteq 2^{[n]}$ $s.t.$ $|A|$ is even for all $A \in \mathcal{F}$, and $|A \cap B|$ is odd for $\forall\ A \neq B \in \mathcal{F}$. Then $|\mathcal{F}| \leqslant n$.

proof: For $\forall\ A \in \mathcal{F}$, define $A' = A \cup \{n+1\}$. Define $\mathcal{F}' = \{A' : A \in \mathcal{F}\} \subseteq 2^{[n+1]}$. Then, by Odd/Even-town, $|\mathcal{F}| = |\mathcal{F}'| \leqslant n+1$.
Assume $|\mathcal{F}| = n+1$. For $\forall\ A \in \mathcal{F}$, let $\mathbf{e_A} \in \mathbb{F}_2^n$ be the indictor vector of $A$. Then $\mathbf{e_A},\ A \in \mathcal{F}$ are linearly dependent, $i.e.$ $\exists$ not all zero $\alpha_A \in \mathbb{F}_2 = \{0, 1\}$ $s.t.$ $\sum_{A \in \mathcal{F}} \alpha_A \mathbf{e_A} = \mathbf{0}$.
Note that

$$< \mathbf{e_A}, \mathbf{e_A} >= 0, \quad \forall\ A \in \mathcal{F}$$
$$< \mathbf{e_A}, \mathbf{e_B} >= 1, \quad \forall\ A \neq B \in \mathcal{F}$$

So

$$0 =< \sum_{A \in \mathcal{F}} \alpha_A \mathbf{e_A}, \mathbf{e_B} >= \sum_{A \in \mathcal{F}: A \neq B} \alpha_A = \sum_{A \in \mathcal{F}} \alpha_A - \alpha_B, \quad \forall\ B \in \mathcal{F}$$

$\Longrightarrow \alpha_B = \sum\limits_{A \in \mathcal{F}} \alpha_A = 1, \forall\ B \in \mathcal{F}$ (Since $\alpha_B$ are not all zero.)
$\Longrightarrow 1 = n+1$ in $\mathbb{F}_2$. $\quad \Longrightarrow n$ is even.
Consider $\mathcal{F}^c = \{A^c = [n] \backslash A : A \in \mathcal{F}\}$, then

- $|A^c|$ is even, $\forall\ A \in \mathcal{F}$
- $|A^c \cap B^c| = n - |A \cup B| = n - (|A| + |B| - |A \cap B|)$ is odd, $\forall\ A \neq B \in \mathcal{F}$

So $\mathcal{F}^c$ is also Even/Odd-town & $|\mathcal{F}^c| = |\mathcal{F}| = n+1$.
Repeat the previous proof, we have

$$\sum_{A \in \mathcal{F}} \mathbf{e}_{\mathbf{A}^\mathbf{c}} = \mathbf{0}.$$

Combine with $\sum\limits_{A \in \mathcal{F}} \mathbf{e}_\mathbf{A} = \mathbf{0}$,
$\Longrightarrow \mathbf{0} = \sum\limits_{A \in \mathcal{F}} (\mathbf{e}_\mathbf{A} + \mathbf{e}_{\mathbf{A}^\mathbf{c}}) = \sum\limits_{A \in \mathcal{F}} \mathbf{1} = (n+1)\mathbf{1} = \mathbf{1}$, a contradiction.
Therefore, $|\mathcal{F}| \leqslant n$. $\square$

$F$ : field. $F[x_1, \cdots x_n] = \{$polynomial $f : F^n \to F\}$.

**Definition.** Say polynomial $f$ vanishes on $E \subset F^n$ if $f(x_1, \cdots, x_n) = 0, \forall\ (x_1, \cdots, x_n) \in E$.

$f$ is zero polynomial if and only if all coefficients are 0.

**Note.** If $n = 1$, univariate $f(x) \neq 0$, $deg(f) \leqslant d$, then $f$ has at most $d$ roots.

(1) $f \neq 0$ vanishes on $S$, then $|S| \leqslant deg(f)$.

(2) If $\nexists\ f \neq 0$ that vanishes on $S$, then $|S| > deg(f)$.

**Lemma 1.** Given $E \subset F^n$, $|E| < \binom{n+d}{d}$, then $\exists\ 0 \neq f \in F[x_1, \cdots, x_n]$ with $deg(f) \leqslant d$ that vanishes on $E$.

proof: Let $V_d = \{f \in F[x_1, \cdots, x_n] : deg(f) \leqslant d\}$, then $dim\ (V_d) = \begin{pmatrix} n+d \\ d \end{pmatrix}$.

Let $F^E = \{$all functions from $E$ to $F\} = \{$all vectors over $F$ of length $|E|\}$. *i.e.* $u \in F^E$, $u = (u_a)_{a \in E}$, $u_a \in F$.

Define a map: $V_d \to F^E,\ f \mapsto (f(a))_{a \in E}$.

Note that $F^E$ is a linear space, $dim\ (F^E) = |E| < \begin{pmatrix} n+d \\ d \end{pmatrix} = dim\ (V_d)$. So $\exists\ f_1 \neq f_2 \in V_d$ such that $(f_1(a))_{a \in E} = (f_2(a))_{a \in E}$.

$\Longrightarrow ((f_1 - f_2)(a))_{a \in E} = 0$, *i.e.* $0 \neq f_1 - f_2 \in V_d$ vanishes on $E$. $\square$

**Lemma 2.** $\forall\ 0 \neq f \in \mathbb{F}_q[x_1, \cdots, x_n]$ with $deg(f) = d < q$ has at most $dq^{n-1}$ roots.

proof: $n = 1$ : univariate case, OK.

Assume $n \geqslant 2$. Write $f = g + h$, where $g$ is homogenous of degree $d$ and $deg(h) \leqslant d - 1$. Then $g \neq 0$, $\exists\ \omega \in \mathbb{F}_q^n \backslash \{0\}$ such that $g(\omega) \neq 0$.
$\forall\ u \in \mathbb{F}_q^n$, let $L_u = \{u + t\omega : t \in \mathbb{F}_q\}$ (line). $\Longrightarrow \quad |L_u| = q$.
If $v \notin L_u$, then $L_u \cap L_v = \emptyset$. (Since otherwise, $u + t\omega = v + t'\omega \Longrightarrow v = u + (t - t')\omega \in L_u$.) Hence $\mathbb{F}_q^n$ is partitioned into $q^n/q = q^{n-1}$ disjoint lines.
Now, it remains to show that $f$ has at most $d$ roots in each line.
Let $P_u(t) = f(u + t\omega)$, then $deg(P_u) \leqslant d$ and a root of $P_u$ corresponds to a root of $f$ in line $L_u$.

$$[t^d]P_u(t) = [t^d]f(u + t\omega) = [t^d]g(u + t\omega) = [t^d]g(t\omega) = g(\omega) \neq 0$$

$\Longrightarrow P_u \neq 0$.
$\Longrightarrow P_u$ has at most $d$ roots.
$\Longrightarrow f$ has at most $d$ roots in line $L_u$. $\square$

**Lemma 3.** $\forall\ S \subset F$, $|S| \geqslant d$, $\forall\ 0 \neq f \in F[x_1, \cdots, x_n]$ of degree $d$. Then $f$ has at most $d|S|^{n-1}$ roots in $S^n = S \times \cdots \times S$.

proof: Prove by induction on $n$.
$n = 1$ : univariate case, OK.
Assume $n \geqslant 2$. Write $f = f_0 + f_1 x_n + \cdots + f_t x_n^t$, $t \leqslant d$, $f_t \neq 0$,

$f_i \in F[x_1, \cdots, x_{n-1}]$, $deg(f_i) \leqslant d - i$.
Now estimate the number of points $(a, b) \in S^{n-1} \times S$ $s.t.$ $f(a, b) = 0$. Fix $t$

(1) $f_t(a) = 0$. Since $deg(f_t) \leqslant d - t$, by assumption, $f_t$ has at most $(d-t)|S|^{n-2}$ roots in $S^{n-1}$. So there are at most $(d-t)|S|^{n-2} \cdot |S| = (d-t)|S|^{n-1}$ points $(a, b) \in S^{n-1} \times S$ $s.t.$ $f(a, b) = 0$ and $f_t(a) = 0$.

(2) $f_t(a) \neq 0$. Given such $a \in S^{n-1}$, let $g_a(x_n) = f(a, x_n)$, then $deg(g_a) \leqslant t$. So $g_a \neq 0$ has at most $t$ roots in $S$. $i.e.$ Given $a \in S^{n-1}$ satisfying $f_t(a) \neq 0$, $\exists$ at most $t$ elements $b \in S$ $s.t.$ $f(a, b) = g_a(b) = 0$. Since there are at most $|S|^{n-1}$ such $a$, the number of points $(a, b) \in S^{n-1} \times S$ $s.t.$ $f(a, b) = 0$ and $f_t(a) \neq 0$ is at most $t|S|^{n-1}$.

Together, there are at most $d|S|^{n-1}$ points $(a, b) \in S^{n-1} \times S$ such that $f(a, b) = 0$. $\square$

思路：

将定义域分块.