Combinatorics 2018 Fall

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Probabilistic Methods

- **Ideas:**

  ① Image we need to find some combinatorial object satisfying certain property, call them "good" objects. We consider a random object. If the probability that the random object is "good" is positive (i.e. Pr[the object is good]>0), then we say there must exist "good" objects.

  ② To compute Pr[good object]>0, we often prove Pr[bad object]<1.

Def: A finite probability space is $(\Omega, \Pr)$, where Ω is a finite set and $\Pr: \Omega \to [0,1]$ such that $\sum\limits_{x\in\Omega} \Pr[x] = 1$.

• $A \subseteq \Omega$ is called an event, $\Pr[A] = \sum\limits_{x\in A} \Pr[x]$.

• $\Pr[\emptyset] = 0$, $\Pr[\Omega] = 1$.

• $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B] \le \Pr[A] + \Pr[B]$(union bound).

"=" holds iff $\Pr[A \cap B] = 0$.

<u>Thm1:</u> Let n, s be integers satisfying $\binom{n}{s}2^{1-\binom{s}{2}} < 1$, then $R(s,s) > n$.

<u>proof :</u>.

We need to find a 2-edge-coloring of K_n such that it has NO monochromatic clique K_s. Consider a random 2-edge-coloring of K_n: each

edge is colored blue or red, each with probability $\frac{1}{2}$, independent of other edges.
Let A be the event that the defined K_n has a monochromatic K_s (need to show $\Pr[A] < 1$).
For each $B \in \binom{[n]}{s}$, let A_B be the event that K_n has a monochromatic K_s with vertex set B. Then $A = \bigcup_{B \in \binom{[n]}{s}} A_B$.
By union bound, $\Pr[A] = \Pr[\bigcup_{B \in \binom{[n]}{s}} A_B] \leq \sum_{B \in \binom{[n]}{s}} \Pr[A_B] = \binom{n}{s} 2^{1-\binom{s}{2}} < 1$. Thus $\Pr[\bar{A}] = 1 - \Pr[A] > 0$, i.e. the probability that K_n has NO monochromatic K_s is positive. So there must exist a 2-edge-coloring of K_n such that it has NO monochromatic K_s.
$\Longrightarrow R(s,s) > n$

□

<u>Cor2:</u> $R(k,k) \geq \frac{1}{\mathrm{e}\sqrt{2}} k 2^{\frac{k}{2}}$.

<u>proof :</u>.
Let $n = \frac{1}{\mathrm{e}\sqrt{2}} k 2^{\frac{k}{2}} \left(\frac{\mathrm{e}}{2}\right)^{\frac{1}{k}}$. Recall $\binom{n}{k} < \frac{n^k}{k!}$ and $k! > \mathrm{e}\left(\frac{k}{\mathrm{e}}\right)^k$. We have

$$\binom{n}{k} 2^{1-\binom{k}{2}} < \frac{n^k}{\mathrm{e}\left(\frac{k}{\mathrm{e}}\right)^k} 2^{1-\binom{k}{2}} = \left(\frac{\mathrm{e}n}{k}\right)^k \left(\frac{2}{\mathrm{e}}\right) 2^{-\binom{k}{2}}.$$

Since $n = \frac{1}{\mathrm{e}\sqrt{2}} k 2^{\frac{k}{2}} \left(\frac{\mathrm{e}}{2}\right)^{\frac{1}{k}}$, we have $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$. Then $R(k,k) > n \geq \frac{1}{\mathrm{e}\sqrt{2}} k 2^{\frac{k}{2}}$.

□

Def: A **tournament** of n vertices is an orientation of K_n. Say vertex x beats vertex y if $x \to y$ or $(x,y) \in E$. **Property $\mathbf{P_k}$** : $\forall$ k-subset $A \subset V$, there is a vertex who beats all vertices in A.

Question: For $\forall k \geq 2$, does there exist a tournament T with property P_k?

<u>Thm3:</u> If $n \geq k^2 2^{k+1}$, then $\exists$ a tournament T of n players with property P_k.

proof :.
Consider a random tournament of K_n over n. For any $i < j$, the arc $i \to j$ occurs with probability $\frac{1}{2}$ independently. Let B be the event that T doesn't have P_k. For $A \in \binom{[n]}{k}$, let B_A be the event that $\forall x \in [n] \backslash A$, x can not beat every vertices in A. Then $B = \bigcup_{A \in \binom{[n]}{k}} B_A$.
Let $B_{A,x}$ be the event that x can not beat all vertices in A. Then $B_A = \bigcap_{x \in [n] \setminus A} B_{A,x}$. Clearly, $\Pr[B_{A,x}] = 1 - \left(\frac{1}{2}\right)^k$.
Note that arcs between x and A are disjoint, so all events $B_{A,x}$ are independent for $x \in [n] \setminus A$. $\Longrightarrow \Pr[B_A] = \prod_{x \in [n] \setminus A} \left(1 - \frac{1}{2^k}\right) = \left(1 - \frac{1}{2^k}\right)^{n-k}$.
By union bound, $\Pr[B] \leq \sum_{A \in \binom{[n]}{k}} \Pr[B_A] = \binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k}$.
Check $\binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k} < 1$ by following steps:

(i) $n > 2^k \cdot k + 1$, $\binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k}$ is decreasing.

(ii) $\binom{n}{k} \left(1 - \frac{1}{2^k}\right)^{n-k} < \frac{n^k}{k!} e^{-\left(\frac{1}{2}\right)^k (n-k)} < n^k e^{-\frac{n}{2^k}}$

(iii) $n = k^2 2^{k+1}$, $n^k e^{-\frac{n}{2^k}} < 1$

$\square$

Def: $\mathcal{F} \subseteq 2^{[n]}$, say $\mathcal{F}$ is **2-colorable** if $\exists$ a function $f : [n] \to \{\text{blue, red}\}$, s.t. every set $A \in \mathcal{F}$ is not monochromatic.

Note: $\mathcal{F} \subseteq \binom{[n]}{2}$, $\mathcal{F}$ is 2-colorable iff $G = ([n], \mathcal{F})$ is bipartite.

Thm4: Every k-uniform family $\mathcal{F}$ with $|\mathcal{F}| < 2^{k-1}$ is 2-colorable.

proof :.
Let $\mathcal{F}$ be any fixed k-uniform family over finite set $X = [n]$. Consider a random function $f : X \to \{\text{blue, red}\}$, s.t.$\forall x \in X$ is colored by blue or red with probability $\frac{1}{2}$, and the coloring of different elements are independent. Let B be the event that f is "bad", i.e. $\exists\ A \in \mathcal{F}$ is monochromatic.
For $A \in \mathcal{F}$, let B_A be the event that A is monochromatic. So $B = \bigcup_{A \in \mathcal{F}} B_A$. $\forall A \in \mathcal{F}$, $\Pr[B_A] = 2^{1-k}$. Hence $\Pr[B] \leq \sum_{A \in \mathcal{F}} \Pr[B_A] = |\mathcal{F}| 2^{1-k} < 1$. So $\Pr[\bar{B}] > 0$, i.e. $\exists$ a 2-coloring s.t. no $A \in \mathcal{F}$ is monochromatic.

$\square$

Note:

(i) $\mathcal{F} = \binom{[2k]}{k}$ is not 2-colorable

(ii) Let $B(k)$ be the minimum possible number of sets in a k-uniform family which is not 2-colorable, then $2^{k-1} \leq B(k) \leq \binom{2k}{k} < 2^{2k}$. But 2^{2k} is not good enough and we can prove the following Thm.

Thm5: If k is sufficiently large, then there exists a k-uniform family $\mathcal{F}$ such that $|\mathcal{F}| \leq k^2 2^k$ and $\mathcal{F}$ is not 2-colorable.

proof :.
To be continued. □