Combinatorics 2018 Fall

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Key words: Linear Algebra method, Frankl-Wilson Thm

Recall:

(1) intersecting family $\mathcal{F} \subset 2^{[n]}$: $\forall A, B \in \mathcal{F}, |A \cap B| \neq 0$

(2) Fisher inequality: let $A_1, \cdots, A_m$ be subsets of $[n]$, $|A_i \cap A_j| = k$, for some fixed $k \in [n]$, then $m \leq n$

<u>Def:</u> $\mathcal{F} \subset 2^{[n]}$, $L \subset \{0, 1, \cdots\}$ be a finite set of integers, say $\mathcal{F}$ is $L-$intersecting if $|A \cap B| \in L$ for $\forall A \neq B \in \mathcal{F}$

<u>Theorem</u> 1 (Frankl-Wilson)**.** *If $\mathcal{F}$ is an $L-$intersecting family over $[n]$, then* $|\mathcal{F}| \leq \sum\limits_{i=0}^{|L|} \binom{n}{i}$.

Note: The bound is best possible: Let $L = \{0, 1, \cdots, k\}$ $\mathcal{F} = \{\emptyset\} \cup \binom{[n]}{1} \cup \cdots \binom{[n]}{k+1}$

Note:

(1) Let F is a field $(\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{F}_q)$, Ω is a set. $F^{\Omega} = \{\text{function:}\Omega \to F\}$ is a linear space over F. A set of functions $f_1, \cdots, f_m$ is linearly independent if $\forall$ combination $\lambda_1 f_1 + \cdots + \lambda_m f_m = 0, \lambda_i \in F$, then $\lambda_i = 0$, $i \in [m]$

(2) Consider $\{f(x_1, \cdots, x_n)$ polynomials with degree$\leq d\}$, then each of f is combination of $x_1^{t_1} \cdots x_n^{t_n}$, with $t_1 + \cdots + t_n \leq d, t_i \geq 0$. The dimension is $\sum\limits_{i=0}^{d} \binom{i+n-1}{i} = \binom{n+d}{d}$

Lemma 1 (Independence criterion)**.** *If $i \in [m]$, let $f_i : \Omega \to F$ (where F is a field) be functions and $v_i \in \Omega$ such that*

(1) $f_i(v_i) \neq 0, \forall i \in [m]$.

(2) $f_i(v_j) = 0, \forall 1 \leq j < i \leq m$.

不是所有j！因为可以重排以满足要求.

Then $f_1, \cdots, f_m$ are linearly independent in the function space F^Ω.

proof :.
Assume $\exists$ not all zeros λ_i, $i \in [m]$, s.t.

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m = 0$$

Suppose j is the smallest index such that $\lambda_j \neq 0$. then

$$0 = \lambda_{j+1} f_{j+1}(v_j) + \lambda_{j+2} f_{j+2}(v_j) + \cdots + \lambda_n f_n(v_j) = -\lambda_i f_j(v_j) \neq 0$$

$\square$

proof of Thm1:.
Suppose $\mathcal{F} = \{A_1, \cdots, A_m\}$,with $|A_1| \leq \cdots \leq |A_m|$, and $L = \{l_1, \cdots l_s\}$. Let v_i be the indicator vector of A_i, $i \in [m]$, then $< v_i, v_j >= |A_i \cap A_j| = l_k$ for some $k \in [s]$. For $i = 1, \cdots, m$, define f_i with n variables $\vec{x} = (x_1, \cdots, x_n)$ by

$$f_i : \mathbb{R}^n \to \mathbb{R}.$$

$$f_i(\vec{x}) = \prod_{k: l_k < |A_i|} (< v_i, \vec{x} > - l_k).$$

$f_i(v_i) \neq 0, \forall i \in [m]$
If $1 \leq j < i \leq m$, $< v_i, v_j >= |A_i \cap A_j| < |A_i| \implies f_i(v_j) = 0$
By **Lemma1**, $f_1, \cdots, f_m$ are linearly independent over $\mathbb{R}$.
$\because f_i$ are polynomials of degree at most s, $\therefore m \leq \sum\limits_{i=0}^{s} \binom{i+n-1}{i}$, but we can do it better!
Define new polynomials $\bar{f}_i$ form f_i by replacing all term $x_i^{r_i}$ $(r_i \geq 1)$ by x_i. Since v_i are $0-1$ vectors, we have $\bar{f}_i(v_j) = f_i(v_j), \forall i, j$, so $\bar{f}_i, \cdots, \bar{f}_m$ are linearly independent, who lie in a space with basis $x_1^{r_1} \cdots x_n^{r_n}$ with $r_1 + \cdots + r_n \leq s$ and $r_i \in \{0, 1\} \implies m \leq \sum\limits_{i=0}^{s} \binom{n}{i}$.

$\square$

Theorem 2. *Let p be a prime and $L \subset \mathbb{Z}_p = \{0, 1, \cdots, p-1\}$. Assume $\mathcal{F} = \{A_1, \cdots A_m\} \subset 2^{[n]}$ such that*

(a) $|A_i| \notin L \pmod{p}, \forall i \in [m]$.

(b) $|A_i \cap A_j| \in L \pmod{p}, \forall i \neq j$.

Then $|\mathcal{F}| \leq \sum_{i=0}^{|L|} \binom{n}{i}$

Hint: $f_i : \mathbb{R}^n \to \mathbb{F}_p,\ f_i(x) = \prod_{l \in L} (< v_i, x > -l) \pmod{p}, i \in [m]$.

Note: Consider $p = 2, L = \{0\}$, then $|A_i|$ is odd and $|A_i \cap A_j|$ is even($\forall i \neq j$), $|\mathcal{F}| \leq n + 1$ (Actually $|\mathcal{F}| \leq n$, which will be proved in **Odd/Even Town**)

Ramsey number $R(s, t)$ = least integer N s.t. any graph on N vertices has either a K_s or an I_t

(1) $2^{\frac{t}{2}} < R(t, t) < \binom{2t-2}{t-1} < 2^{2t}$

(2) $R(t, t) > (t - 1)^2$. (Homework 11.8)

(3) $R(s, t) > \Omega(t^3)$ 1972,Zsigmond Nagy

(4) $R(t, t) > t^{\Omega(\ln t / \ln \ln t)}$. 1977,Frankl, 1981 F&Wlison.

Theorem 3. *For any prime p, there is a graph G on $n = \binom{p^3}{p^2-1}$ vertices s.t. the size of maximum clique or maximum independent set is* $\leq \sum_{i=0}^{p-1} \binom{p^3}{i}$.

proof :.
Let $G = (V, E)$ be as follows

- $V = \binom{[p^3]}{p^2-1}$
- for $A, B \in V, A \sim B$ iff $|A \cap B| \neq p - 1 \pmod{p}$.

(i) consider the cliques $A_1, \cdots, A_m \in V, |A_i| = p^2 - 1 = p - 1 \pmod{p}$,
$|A \cap B| \neq p - 1 \pmod{p}$ means $|A \cap B| \in L \pmod{p}$, where $L = \{0, 1, \cdots p - 2\} \subset \mathbb{Z}_p$.
By **Thm2**, we have $m \leq \sum_{i=0}^{p-1} \binom{p^3}{i}$.

(ii) consider the independent sets $B_1, \cdots, B_s$, $|B_i \cap B_j| = p - 1 \pmod{p}$,
so $|B_i \cap B_j| \in \{p-1, 2p-1, \cdots, p(p-1)-1\} = L^* \subset \mathbb{Z}_{\geq 0}$.
and $|L^*| = p - 1$, By **Thm1**, we have $s \leq \sum_{i=0}^{p-1} \binom{p^3}{i}$.

$\square$

Corollary 1. $R(t+1, t+1) \geq t^{\Omega(\ln t / \ln \ln t)}$

proof(sketch) :.

Let $t = \sum_{i=0}^{p-1} \binom{p^3}{i}$, $n = \binom{p^3}{p^2-1} \Longrightarrow R(t+1, t+1) > n$
Recall $(\frac{n}{k})^k \leq \binom{n}{k} \leq (\frac{en}{k})^k \Longrightarrow$
$t \sim \binom{p^3}{p} \sim (\frac{p^3}{p})^p = p^{2p}$
$n \sim \binom{p^3}{p^2} \sim p^{p^2}$
$\ln t \sim p \ln p$, $\ln \ln t \sim \ln(p \ln p) = \ln p + \ln \ln p \sim \ln p$
$p \sim \frac{\ln t}{\ln \ln t}$, $n \sim (p^{2p})^{\frac{p}{2}} = t^{\Omega(\ln t / \ln \ln t)}$

$\square$