Combinatorics 2018 Fall

Taught by: Professor Xiande Zhang

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Recall (EKR Theorem). If $n \geqslant 2k$, $\mathcal{F} \subset \binom{[n]}{k}$ is an intersecting family, then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$.

Definition. Kneser graph $KG(n,k)$ for $n \geqslant 2k$ is a graph with vertex set $\binom{[n]}{k}$ such that $A, B \in \binom{[n]}{k}$, $A \sim B$ iff $A \cap B = \emptyset$.

Fact.

(1) $\mathcal{F} \subset \binom{[n]}{k}$ is an intersecting family. $\Longleftrightarrow$ $\mathcal{F}$ is an independent set in $KG(n,k)$.

(2) EKR Theorem $\Longleftrightarrow$ $\alpha(KG(n,k)) \leqslant \binom{n-1}{k-1}$

Definition. $G = (V, E)$, $V = [n]$, the adjacency matrix of G $A_G = (a_{ij})_{n \times n}$ is defined by $a_{ij} = \begin{cases} 1, & \text{if } i \sim j \\ 0, & \text{otherwise.} \end{cases}$

Definition. The real eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ of A_G is called the eigenvalues of G. The eigenvectors $v_1, v_2, \cdots, v_n$ of A_G

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such that $\begin{cases} A_G v_i = \lambda_i v_i, \|v_i\| = 1 \text{ for all } i \in [n] \\ v_i \perp v_j \text{ for all } i \neq j. \end{cases}$ are called the orthogonal eigenvectors of G.

Definition. A graph G is called d-regular if each vertex has degree d.

Theorem 1 (Hoffman's Theorem). If an n-vertex graph G is d-regular with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, then

$$\alpha(G) \leqslant n \frac{-\lambda_n}{\lambda_1 - \lambda_n}.$$

proof: Let $v_1, \cdots, v_n$ be the corresponding orthogonal eigenvectors of $\lambda_1, \cdots, \lambda_n$. Let I be any independent set of G. Let e_I be the column indicator vector of I, and write $e_I = \sum_{i=1}^{n} \alpha_i v_i$, where $\alpha_i = \langle e_I, v_i \rangle$. Since G is d-regular, we have $\lambda_1 = d$ (*prove this as an exercise!*) and $v_1 = (\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}})^T$. So $\alpha_1 = \langle e_I, v_1 \rangle = \frac{|I|}{\sqrt{n}}$.
Since I is an independent set of G, $e_I^T A_G e_I = \sum_{i,j} x_i a_{ij} x_j = 0$, where we assume $e_I = (x_1, \cdots, x_n)^T$.
So we have

dI−A 行和为0
v 为 λ_1 e-vec
x 为绝对值最大分量. $v_x > 0$.
$\lambda_1 v_x = (Av)_x$
$= \sum_y A_{xy} v_y$
$\leq \sum_y A_{xy} v_x$
$\leq d v_x$.

$$\begin{aligned} 0 = e_I^T A_G e_I = e_I^T \sum_{i=1}^{n} \alpha_i A_G v_i &= \sum_{i=1}^{n} \alpha_i \lambda_i \langle e_I, v_i \rangle \\ &= \sum_{i=1}^{n} \alpha_i^2 \lambda_i \geqslant \alpha_1^2 \lambda_1 + (\alpha_2^2 + \cdots + \alpha_n^2) \lambda_n \\ &= (\frac{|I|}{\sqrt{n}})^2 \lambda_1 + (|I| - (\frac{|I|}{\sqrt{n}})^2) \lambda_n \end{aligned}$$

$$\Longrightarrow 0 \geqslant \frac{|I|^2}{n} \lambda_1 + (|I| - \frac{|I|^2}{n}) \lambda_n = |I| (\frac{|I|}{n} \lambda_1 + \lambda_n - \frac{|I|}{n} \lambda_n)$$

$$\Longrightarrow \frac{|I|}{n} \lambda_1 + \lambda_n - \frac{|I|}{n} \lambda_n \leqslant 0 \qquad \Longrightarrow \frac{|I|}{n} (\lambda_1 - \lambda_n) \leqslant -\lambda_n$$

$$\Longrightarrow |I| \leqslant n\frac{-\lambda_n}{\lambda_1-\lambda_n}$$

$\square$

Lemma 2. The eigenvalues of $KG(n,k)$ are $u_j = (-1)^j \binom{n-k-j}{k-j}$ of multiplicity $\binom{n}{j} - \binom{n}{j-1}$ for every $0 \leqslant j \leqslant k$.

proof: Omitted. (Refer to Chapter 9 of *GTM 207* Algebraic Graph Theory.) $\square$

The second proof of EKR Theorem.
Consider the eigenvalues of $KG(n,k)$, say $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{\binom{n}{k}}$, where $\lambda_1 = \binom{n-k}{k} = u_0$ and $\lambda_{\binom{n}{k}} = -\binom{n-k-1}{k-1} = u_1$.
Then by Hoffman's bound, we have

$$\alpha(KG(n,k)) \leqslant \binom{n}{k}\frac{-\lambda_{\binom{n}{k}}}{\lambda_1-\lambda_{\binom{n}{k}}} = \binom{n}{k}\frac{\binom{n-k-1}{k-1}}{\binom{n-k}{k}+\binom{n-k-1}{k-1}} = \binom{n-1}{k-1}.$$

By Fact(2), we are done. $\square$

Theorem 3 (Fisher's Inequality). Let $A_1, \cdots, A_m$ be distinct subsets of $[n]$ such that $|A_i \cap A_j| = k$ for some fixed $k \in [n]$, $\forall\, i \neq j$, then $m \leqslant n$.

proof: Consider the incidence matrix of $A_1, \cdots, A_m$, and let $v_1, \cdots, v_m$ be the m column vectors. Then it suffices to show that $v_1, \cdots, v_m$ are linearly independent in $\mathbb{R}$.
Assume, for the sake of contradiction, that $\exists\, \lambda_i \neq 0$ s.t. $\sum\limits_{i=1}^{m} \lambda_i v_i = 0$.

Since $\langle v_i, v_j \rangle = |A_i \cap A_j| = \begin{cases} |A_i| \geqslant k, & i = j \\ k, & i \neq j. \end{cases}$ we have

$$0 = \langle \sum_{i=1}^{m} \lambda_i v_i, \sum_{j=1}^{m} \lambda_j v_j \rangle = \sum_{i=1}^{m} \lambda_i^2 \langle v_i, v_i \rangle + \sum_{1 \leqslant i \neq j \leqslant m} \lambda_i \lambda_j \langle v_i, v_j \rangle$$
$$= \sum_{i=1}^{m} \lambda_i^2 |A_i| + k \sum_{1 \leqslant i \neq j \leqslant m} \lambda_i \lambda_j = \sum_{i=1}^{m} \lambda_i^2 (|A_i| - k) + k (\sum_{i=1}^{m} \lambda_i)^2$$

Note that $|A_i| = k$ for at most one $i \in [m]$. (Otherwise $|A_i \cap A_j| < k$ for $|A_i|, |A_j| = k$ with $i \neq j$.) But there are at least two $\lambda_i \neq 0$. So $\sum_{i=1}^{m} \lambda_i^2 (|A_i| - k) > 0$, a contradiction. $\square$

Designs

Definition. A (v, k, λ) design over X is a pair (X, D), where

most important

- X is a set of points. $|X| = v$.
- $D \subset \binom{X}{k}$ is called a set of blocks. $|D| = b$.
- $\forall$ pair of distinct points of X is contained in exactly λ blocks.

Example. $X = [7]$, $v = 7$, $k = 3$, $\lambda = 1$, $D = \{123, 147, 156, 345, 367, 257, 246\}$.

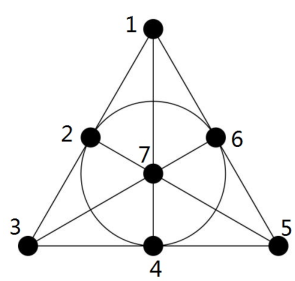

Figure 1: the Fano plane

Fact. $b \geqslant v$.

proof: $\forall\ x \in X$, define $A_x = \{B \in D : \ x \in B\}$.
Since each pair of distinct points of X is contained in exactly λ blocks, $|A_x \cap A_y| = \lambda$ for $x \neq y$.
By Fisher's Inequality, $v \leqslant b$. $\square$

Definition. If $b = v$, a (v, k, λ) design is called a symmetric design.

Definition. D is called r-regular if every point appears in exactly r blocks, and r is called replication number.

即：同时是
$1-(v,k,r)$
$2-(v,k,\lambda)$

Theorem 4. (X, D) is a (v, k, λ) design with b blocks, then D is r-regular satisfying $r(k-1) = \lambda(v-1)$ and $bk = vr$.

所有 design 都是 r-regular

proof: $\forall\ a \in X$, assume a occurs in r_a blocks.
Consider $S = \{(x, B) : B \in D;\ a, x \in B;\ x \neq a\}$.

- Since a is fixed, there are $v-1$ possibilities for x. For each chosen x, there are exactly λ blocks containing both x and a. Hence $|S| = (v-1)\lambda$.
- For each of the r_a blocks containing a, there are $k-1$ ways to choose $x \in B \setminus \{a\}$. Hence $|S| = r_a(k-1)$.

So, by double counting, we have $r_a(k-1) = (v-1)\lambda$ for $\forall\ a \in X$, which implies that r_a is independent of a, i.e. D is regular.
To prove $bk = vr$, consider $T = \{(x, B) : B \in D, x \in B\}$.

- $\forall\ x \in X$, we can choose B in r ways. Hence $|T| = vr$.
- $\forall\ B \in D$, we can choose x in k ways. Hence $|T| = bk$.

$\square$