Combinatorics 2018 Fall

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Given $\{a_n\}_{n\geqslant 0}$, GF$=\sum\limits_{n\geqslant 0} a_n x^n$, EGF$=\sum\limits_{n\geqslant 0} \frac{a_n}{n!}x^n$.

Definition. Dirichlet series of $\{a_n\}_{n=1}^{\infty}$ is $a(x)=\sum\limits_{n\geqslant 1}\frac{a_n}{n^x}$.

Let $b(x)=\sum\limits_{n\geqslant 1}\frac{b_n}{n^x}$, $c(x)=a(x)b(x)=\sum\limits_{n\geqslant 1}\frac{c_n}{n^x}$. Then $c_n=\sum\limits_{rs=n} a_r b_s = \sum\limits_{d|n} a_d b_{\frac{n}{d}} = \sum\limits_{d|n} a_{\frac{n}{d}} b_d$ where $\sum\limits_{d|n}$ means d runs over all positive divisors of n.

Definition. The Dirichlet Convolution of $f=\{f(n)\}_1^{\infty}$ and $g=\{g(n)\}_1^{\infty}$ is a sequence $f\odot g$, where $(f\odot g)(n)=\sum\limits_{d|n} f(d)g(\frac{n}{d}) = \sum\limits_{d|n} f(\frac{n}{d})g(d)$.

Fact.

(1) $\odot$ is commutative, associative and distributive.

(2) All real sequences form a ring under $\odot$ and $+$.

(3) $I=\{I(n)\}=\begin{cases} 1, & n=1 \\ 0, & n>1 \end{cases}$.

$$(I\odot f)(n)=\sum_{d|n} I(d)f(\frac{n}{d})=f(n).$$

(4) If $f\odot g=I$, then we say f is the D-inverse of g.

(5) $e = \{e(n)\}_{n\geqslant 1} = \{1, 1, 1, \cdots\}$.

$(e \odot f)(n) = \sum_{d|n} e(d) f(\frac{n}{d}) = \sum_{d|n} f(d)$.

Definition. Möbius Function $\mu = \{\mu(n)\}_1^\infty$, where

$$\mu(n) = \begin{cases} 1 & n = 1 \\ (-1)^r & n = p_1 \cdots p_r \text{ where } p_1, \cdots, p_r \text{ are distinct primes} \\ 0 & n \text{ is not square-free} \end{cases}.$$

Theorem 1. $\mu \odot e = I$, that is,

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}.$$

proof: $n = 1$: true.
$n > 1$: Write $n = p_1^{a_1} \cdots p_r^{a_r}$, where $p_1, \cdots, p_r$ are distinct primes and $a_i \geqslant 1$. $d \mid n \Rightarrow d = p_1^{b_1} \cdots p_r^{b_r}$ where $0 \leqslant b_i \leqslant a_i$. If d is not square-free, then $\mu(d) = 0$; If d has no square factors, then $d \mid p_1 \cdots p_r$, which implies $0 \leqslant b_i \leqslant 1$. So

$$\sum_{d|n} \mu(d) = \sum_{d|\prod_{i=1}^r p_i} \mu(d) = \sum_{I \subset [r]} \mu(\prod_{i \in I} p_i) = \sum_{I \subset [r]} (-1)^{|I|} = \sum_{i=0}^r \binom{r}{i} (-1)^i = 0$$

$\square$

Lemma 2. $f = \{f(n)\}$ is D-invertible if and only if $f(1) \neq 0$.

proof: $\Rightarrow$: f is D-invertible means $\exists\ g = \{g(n)\}$ such that $f \odot g = I$. So $1 = I(1) = (f \odot g)(1) = f(1)g(1)$, which implies $f(1) \neq 0$.
$\Leftarrow$: We want to prove $\exists\ g = \{g(n)\}$ such that $f \odot g = I$.

$n = 1:\ I(1) = f(1)g(1) = 1 \Rightarrow g(1) = \frac{1}{f(1)}$.

$n > 1:\ 0 = I(n) = \sum_{d|n, d\neq 1} f(d) g(\frac{n}{d}) + f(1)g(n)$

$\Rightarrow g(n) = -\frac{1}{f(1)} \sum_{d|n, d\neq 1} f(d) g(\frac{n}{d})$. $\square$

Möbius Inversion Formula. For any two sequences $\{f(n)\}$ and $\{g(n)\}$, we have

$$f(n) \equiv \sum_{d|n} g(d) \Longleftrightarrow g(n) \equiv \sum_{d|n} \mu(\frac{n}{d}) f(d)$$

Remark: i.e. $f = e \odot g \Longleftrightarrow g = \mu \odot f$ (Note that $\mu \odot e = I$.)

Recall. Euler function $\varphi(n) = \#\ m \in [n]$ such that $\gcd(m, n) = 1$. Write $n = p_1^{k_1} \cdots p_r^{k_r}$, then $\varphi(n) = \sum\limits_{I \subset [r]} (-1)^{|I|} \frac{n}{\prod\limits_{i \in I} p_i}$. Let $\varphi = \{\varphi(n)\}$.

Theorem 3. Let $N = \{N(n)\} = \{1, 2, 3, \cdots\}$. Then

(1) $\varphi = N \odot \mu$, i.e. $\varphi(n) = \sum\limits_{d|n} \frac{n}{d} \mu(d)$.

(2) $N = \varphi \odot e$, i.e. $n = \sum\limits_{d|n} \varphi(d)$.

<u>proof:</u>

(1)

$$n \cdot \sum_{d|n} \frac{1}{d} \mu(d) = n \cdot \sum_{d | \prod_{i=1}^{r} p_i} \frac{1}{d} \mu(d) = n \cdot \sum_{I \subset [r]} \frac{1}{\prod\limits_{i \in I} p_i} (-1)^{|I|} = \varphi(n)$$

(2) By (1) and Möbius Inversion Formula. $\square$

Arrangements in a cycle (without seat number)

If no repetition, then $\frac{n!}{n} = (n-1)!$.
Let $C_m(n) = \#$ cycles of length n over $[m]$ with repetition.
To find $C_m(n)$, we consider how many lines of length n correspond to the same n-cycle.
Suppose we have an n-cycle of the smallest period p(here $p \mid n$):

$$a_1 a_2 \ldots a_p a_1 a_2 \ldots a_p \ldots a_1 a_2 \ldots a_p.$$

Cut it into lines, then we have p different lines:

$$\begin{gathered} a_1a_2\dots a_pa_1a_2\dots a_p\dots a_1a_2\dots a_p \\ (a_2\dots a_pa_1)(a_2\dots a_pa_1)\dots(a_2\dots a_pa_1) \\ \vdots \\ (a_pa_1\dots a_{p-1})(a_pa_1\dots a_{p-1})\dots(a_pa_1\dots a_{p-1}) \end{gathered}$$

Let $L(p) = \#$ lines with period p, $M(p) = \#$ cycles with period p. Then

$$L(p) = pM(p).$$

So we have

$$C_m(n) = \sum_{p|n} M(p) = \sum_{p|n} \frac{1}{p} L(p).$$

Note that $m^n = \sum\limits_{p|n} L(p)$ where m is fixed.

Let $f = \{f(n)\}$ where $f(n) = m^n$, then by Möbius Inversion Formula,

$$L(n) = \sum_{d|n} \mu(\frac{n}{d}) f(d) = \sum_{d|n} \mu(\frac{n}{d}) m^d$$

So

$$C_m(n) = \sum_{p|n} \frac{1}{p} \sum_{d|p} \mu(\frac{p}{d}) m^d$$

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Exercise. Prove that $C_m(n) = \dfrac{1}{n} \sum\limits_{d|n} \varphi(\dfrac{n}{d}) m^d$.