Combinatorics 2018 Fall

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Key words: Graphs, Ramsey Theorem

Recall:

– $\alpha(G)$: Independent number = max $\sharp$ pairwise nonadjacent vertices of G

– $\chi(G)$: Chromatic number = min $\sharp$ colors s.t. $\exists$ a coloring of $V(G)$ is a proper coloring

– $n \leq \alpha(G)\chi(G)$

Def: path $v_1 v_2 \cdots v_s$, where $v_i \sim v_{i+1}$ and $v_i \neq v_j, \forall i \neq j \in [s]$. If $v_1 = v_s$, call it a cycle. A graph G is **connected** if there is a path between any two vertices.

Theorem 1. *$|V(G)| = n$. If for any $x \in V(G)$, $\deg(x) \geq \frac{n-1}{2}$, then G is connected.*

proof: Take any different $x, y \in V(G)$. If $x \sim y$, then done.
If $x \nsim y$, since $\deg(x), \deg(y) \geq \frac{n-1}{2}$, there are at least $n-1$ edges joining x, y to $V(G) \setminus \{x, y\}$. Since $|V(G) \setminus \{x, y\}| = n - 2$, by P-P, $\exists z \in V(G) \setminus \{x, y\}$, $z \sim x, z \sim y$. $\square$

Remark:

(1) The condition above is best possible: e.g. n even, G is the union of two vertex disjoint complete graphs of $\frac{n}{2}$ vertices, each vertex has degree $\frac{n-2}{2}$, but G is disconnected.

(2) Define the **diameter** of G is the smallest number k, s.t. every two vertices are connected by a path with at most k edges. Then Theorem 1 says G has diameter at most two.

Fact(A party of six): Suppose a party has 6 participants. Participants may know each other or not. Then there must be 3 people such that any 2 know each other or any 2 don't know each other.

proof: Construct a graph with vertices $[6]$, where $i \sim j$ iff i and j know each other. Then we need to show that there are 3 vertices in G which form a triangle or an independent set of size 3.
Consider vertex 1, by P-P, 1 is either adjacent to ≥ 3 vertices or nonadjacent to ≥ 3 vertices.

① Suppose 1 is adjacent to $2, 3, 4$. If one of the pairs $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$ is adjacent, then we have a K_3. If not, $\{2, 3, 4\}$ is an independent set of size 3.

② Suppose 1 is nonadjacent to $2, 3, 4$. Similar arguments.

□

Def: $\forall s, t \geq 1$, let $R(s, t)$ denote the smallest integer n, s.t. in any graph with n or more vertices, there exists either a clique(a complete subgraph) with s vertices K_s or an independent set with t vertices I_t.

Remark:

① $R(s, t) \leq L \iff$ any graph with L vertices has either a K_s or an I_t.

② $R(s, t) > M \iff \exists$ a graph with M vertices has neither K_s nor I_t.

Fact:

① $R(s, t) = R(t, s)$.

② $R(2, t) = R(t, 2) = t$.

③ $R(3, 3) = 6$.

Theorem 2. *For $s \geq 2,\ t \geq 2,\ R(s,t) \leq R(s,t-1) + R(s-1,t)$.*

proof: Let G be a graph on $n = R(s,t-1)+R(s-1,t)$ vertices. We need to prove any graph on n vertices has either a K_s or an I_t. Take an arbitrary vertex $x \in V(G)$. Let $S_x = \{y \in V(G) :\ x \sim y\}$ and $T_x = (V \backslash \{x\}) \backslash S_x$, then $|S_x|+|T_x| = n-1 = R(s,t-1)+R(s-1,t)-1$. By P-P, we have either $|T_x| \geq R(s,t-1)$ or $|S_x| \geq R(s-1,t)$.

① $|S_x| \geq R(s-1,t)$. Consider the induced subgraph $G[S]$: a graph on S, in which $v \sim w$ iff $v \sim w$ in G. Since $G[S_x]$ has at least $R(s-1,t)$ vertices, $G[S]$ has either a K_{s-1} or an I_t. Therefore $G[S_x \cup \{x\}]$ has either a K_s or an I_t.

② $|T_x| \geq R(s,t-1)$. Similar.

$\square$

Theorem 3. $R(s,t) \leq \binom{s+t-2}{s-1} = \binom{s+t-2}{t-1}$.

proof: By induction on $s+t$. $R(2,t) = t$, $R(s,2) = s$, true. Assume the claim holds for $R(k,l)$ with $k+l < s+t$. Then $R(s,t) \leq R(s,t-1) + R(s-1,t) \leq \binom{s+t-3}{s-1} + \binom{s+t-3}{s-2} = \binom{s+t-2}{s-1}$.

$\square$

Note: $2^{\frac{t}{2}} \leq R(t,t) \leq 2^{2t}$ $(Erdös1947)$

Theorem 4. *If $R(s,t-1), R(s-1,t)$ are even, then $R(s,t) \leq R(s,t-1) + R(s-1,t) - 1$*

proof: $n = R(s,t-1) + R(s-1,t) - 1$, odd. We need to show any G on n vertices has a K_s or an I_t
$\forall x \in V$, let $S_x = \{y \in V(G) : x \sim y\}$ and $T_x = (V \setminus \{x\}) \setminus S_x$.

① If $\exists x$ s.t. $|S_x| \geq R(s-1,t)$ or $|T_x| \geq R(s,t-1)$, done

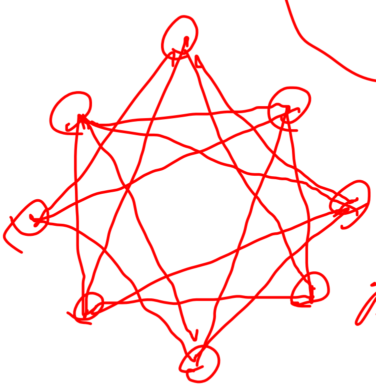
② $\forall x,\ |S_x| \leq R(s-1,t) - 1$ and $|T_x| \leq R(s,t-1) - 1$. $\because |S_x| + |T_x| = n-1 = R(s-1,t)+R(s,t-1)-2$, $\therefore |S_x| = R(s-1,t)-1$, odd. Contradiction to the Handshaking Lemma.

$\square$

Ex: Compute $R(3,4)$

$6 < R(3,4) \leq \binom{5}{2} = 10$.

$R(3,3) = 6$, $R(4,2) = 4$ } even.



无K_4

补无K_3

$R(3,4) \leq 9$. 构造8不行.

2-coloring version of Ramsey's theorem. Define a r-edge-coloring of K_n to be a coloring of edges of K_n by r colors. Then $R(s,t)$ denotes the smallest integer N s.t. any 2-edge-coloring of K_N has either a blue K_s or a red K_t.

Generalized Ramsey number $R_k(s_1, s_2, \ldots, s_k)$ is the smallest integer N such that any k-edge-coloring of K_N has a K_{s_i} in color i for some $i \in [k]$.

Ramsey Thm: $R_k(S_1, \cdots, S_k) \leq +\infty$