Combinatorics 2018 Fall

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Key words: Inversion Formula on Poset, Group Action, Burnside Lemma

<u>**Theorem**</u> **1.** ***(Inverse Formula I)*** *If P has a minimum element m(i.e. $m \leq x$, $\forall x \in P$). Let $f,\ g$ be functions $P \to \mathbb{R}$,then*

$$f(y) = \sum_{z\ \leq\ y} g(z),\ \forall y \in P \iff g(y) = \sum_{z\ \leq\ y} f(z)\mu_P(z,y),\ \forall y \in P$$

<u>**Theorem**</u> **2.** ***(Inverse Formula II)*** *If P has a maximum element M(i.e. $x \leq M$, $\forall x \in P$). Let $f,\ g$ be functions $P \to \mathbb{R}$,then*

$$f(x) = \sum_{x\ \leq\ z} g(z),\ \forall x \in P \iff g(x) = \sum_{x\ \leq\ z} \mu_P(x,z)f(z),\ \forall x \in P$$

E.g. $P = (2^X, \subseteq)$, then $\mu_P(A,B) = (-1)^{|B|-|A|}$

<u>***proof:***</u> Let $X = [n], \varphi : 2^X \longrightarrow \{0,1\}^n, A \longmapsto \varphi(A)$, where $\varphi(A)$ is the indicator vector.
Let $S = \{0,1\}, S^n = \{0,1\}^n = S \times S \times \cdots \times S$, define $(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) \Leftrightarrow x_i \leq y_i, \forall i \in [n]$, then $A \subseteq B \Leftrightarrow \varphi(A) \leq \varphi(B)$, φ is isomorphism.
$\mu_P(A,B) = \mu_{S^n}(\varphi(A), \varphi(B)) = \mu_{S^n}((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \prod_{i=1}^{n} \mu_S(x_i, y_i) = \prod_{i=1}^{n} (-1)^{y_i - x_i} = (-1)^{|B|-|A|}$

$\square$

Theorem 3. ***(IEP)*** $|A_1^c \cap \cdots \cap A_n^c| = \sum\limits_{I \subseteq [n]} (-1)^{|I|}|A_I|.$

proof: Let $X = [n], P = (2^X, \subseteq)$.
Define

$$f: \ P \to \mathbb{R} \text{ as } f(|I|) = |A_I| = |\bigcap_{i \in I} A_i|,$$
$$g: \ P \to \mathbb{R} \text{ as } g(|I|) = |A_I \cap (\bigcap_{j \notin I} A_j^c)|.$$

Then $f(I) = \sum\limits_{I \subseteq J \subseteq [n]} g(J)$. Since P has a maximal element $[n]$, by *Inverse Formula II*

$$g(I) = \sum_{I \subseteq J \subseteq [n]} \mu_P(I,J)f(J) = \sum_{I \subseteq J \subseteq [n]} (-1)^{|J \setminus I|}|A_J|.$$

Particularly, $g(\emptyset) = |A_1^c \cap \cdots \cap A_n^c| = \sum\limits_{I \subseteq [n]} (-1)^{|I|}|A_I|.$

$\square$

Polya Counting Theorem

Def: Group action: a finite group $(G, \cdot)$, finite set X, a group action (G, X) is a map $* : G \times X \to X$,s.t.

(1) $e * x = x$,where e is id of $G, \forall x \in X$.

(2) $\forall g, h \in G, \forall x \in X, g * (h * x) = (g \cdot h) * x$.

E.g.

(1) $(S_n, \circ), X = [n]$, define a group action by $\sigma * (i) = \sigma(i)$.

(2) $(G, \cdot), X = G$, define a group action by $g * h = g \cdot h, \forall g, h \in G$.

Def:

(1) Orbit of $x, Orb(x) = \{g * x : \forall g \in G\}$.

(2) Stabilizer of $x, Stab(x) = \{g \in G : g * x = x\}$.

(3) Fixed points of $g \in G, Fix(g) = \{x \in X : g * x = x\}$.

Theorem 1. ***(Orbit-Stabilizer Thm)*** $|G| = |Orb(x)| \cdot |Stab(x)|$

proof: Let $G_x = Stab(x)$, and $G = g_1G_x \cup g_2G_x \cup \cdots \cup g_nG_x$, denote $G/G_x = \{g_iG_x : i \in [n]\}$. we have $|G| = n|G_x| = |G/G_x| \cdot |G_x|$.
Define a function $f : G/G_x \to Orb(x)$ by $f(g_iG_x) = g_i * x$.
Ex: check f is a bijection.

$\square$

Theorem 2. ***(Burnside Lemma)***
Finite group action (G, X). Let $N(G)$ be number of distinct orbits of X, then $N(G) = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$.

proof: (Double Counting)
Consider the set $S = \{(g, x) : g * x = x\}$.
group (g, x) by the first element

$$|S| = \sum_{g \in G} |\{x : g * x = x\}| = \sum_{g \in G} |Fix(g)|.$$

group (g, x) by the second element

$$\begin{aligned}
|S| &= \sum_{x \in X} |\{g : g * x = x\}| = \sum_{x \in X} |Stab(x)| \\
&= \sum_{i=1}^{N(G)} \sum_{x \in Orb(x_i)} |Stab(x_i)| = \sum_{i=1}^{N(G)} |Orb(x_i)| \cdot |Stab(x_i)| \\
&= N(G) \cdot |G|.
\end{aligned}$$

$\square$

E.g. $\tau = (123)(45)$, $G =< \tau >$, $X = [5]$, then $Orb(1) = Orb(2) = Orb(3) = \{1, 2, 3\}$, $Orb(4) = Orb(5) = \{4, 5\}$ and $2 = N(G) = \frac{1}{6}(5 + 0 + 2 + 3 + 2) = 2$

Theorem 3. ***(Weighted Burnside Lemma)***
Finite group action (G, X). Let $\Omega_i = Orb(x_i), i \in [N(G)]$ be all distinct orbits. Let $\omega : X \to R$ be a weighted function, where $\omega(x) = \omega(y)$ if $x, y \in \Omega_i$, for some $i \in [N(G)]$. Define $\omega(\Omega_i) = \omega(x_i)$, then

$$\sum_{i=1}^{N(G)} \omega(\Omega_i) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in Fix(g)} \omega(x).$$

proof: Let $S = \{(g, x) : g * x = x\}$.
compute $M = \sum_{(g,x) \in S} \omega(x)$, Then

(1)

$$M = \sum_{g \in G} \sum_{x \in Fix(g)} \omega(x).$$

(2)

$$\begin{aligned} M &= \sum_{x \in X} \sum_{g \in Stab(x)} \omega(x) = \sum_{x \in X} \omega(x)|Stab(x)| \\ &= \sum_{i=1}^{N(G)} \sum_{x \in \Omega_i} \omega(x)|Stab(x)| = \sum_{i=1}^{N(G)} \omega(x_i)|\Omega_i||Stab(x_i)| \\ &= |G| \sum_{i=1}^{N(G)} \omega(\Omega_i). \end{aligned}$$

□

Def: A coloring of X in m colors is a function $f : X \to C$, where C is a set of m colors. Let C^X ={all coloring of X in C}.
Define (G, X) induces a group action (G, C^X):

$$(g * f)(x) = f(g^{-1} * x)$$