Combinatorics, 2018 Fall, USTC
Homework 7

- **The due is on Thursday, Nov. 8.**
- **Please sign your name and student number.**
- **Please solve as many problems as you can.**

1. Let $S_1, \cdots, S_m$ be a sequence of sets of size at least r, and assume that it has an SDR.

   Prove that it has at least $f(r, m) = \prod_{i=1}^{min\{r,m\}} (r + 1 - i)$ SDRs.

2. Show that there are at least $(n - r)!$ ways to add a new row to every $r \times n$ Latin rectangle such that the resulting $(r + 1) \times n$ matrix is still a Latin rectangle.

3. Let G be a bipartite graph with bipartition A, B. Let a be the minimum degree of a vertex in A, and b be the maximum degree of a vertex in B.
   Prove: If $a \geqslant b$, then there exists a matching of A into B.

4. Prove that every bipartite graph G with l edges has a matching of size at least $\frac{l}{\Delta(G)}$, where $\Delta(G)$ is the maximum degree of a vertex in G.

5. Let $\mathcal{F}$ be a family of sets, each of size at least 2. Let A, B be two sets such that $|A| = |B|$, both A and B intersects all the members of $\mathcal{F}$, and no set of fewer than $|A|$ elements does this. Consider a bipartite graph G with parts A and B, where $a \in A$

is connected to $b \in B$ if there is an $F \in \mathcal{F}$ containing both a and b. Show that this graph has a perfect matching.

6. Let $t < \frac{n}{2}$, and let $\mathcal{F}$ be a family of subsets of an n-element set X. Suppose that: (i) each member of $\mathcal{F}$ has size at most t, and (ii) $\mathcal{F}$ is an antichain, i.e. no member of $\mathcal{F}$ is a subset of another one. Let $\mathcal{F}_t$ be the family of all those t-element subsets of X, which contain at least one member of $\mathcal{F}$. Prove that $|\mathcal{F}| \leqslant |\mathcal{F}_t|$.