Combinatorics 2018 Fall

Taught by: Professor Xiande Zhang

2018.11.8

Key words: Poset, chain, antichain, Pigeonhole Principle

Pigeonhole Principle and Poset

Definition. We say y covers x if $x < y$ and $\nexists\, t$ such that $x < t < y$, denoted by $x \lhd y$.

Infinite ?

Fact. $\forall\ x, y \in P = (X, \leqslant)$, $x < y$ iff $\exists\ x_1, \cdots, x_k \in X$ where $k \geqslant 0$ s.t. $x \lhd x_1 \lhd \cdots \lhd x_k \lhd y$.

proof: " $\Leftarrow$ " : trivial.
" $\Rightarrow$ " : $\forall\ x < y$, let $M_{xy} = \{t \in X : x < t < y\} = [x, y] \backslash \{x, y\}$.
Prove by induction on $|M_{xy}|$.
If $|M_{xy}| = 0$, then $x \lhd y$ holds.
Suppose the claim holds for any $x < y$ with $|M_{xy}| < n$. Consider $x < y$ with $|M_{xy}| = n \geqslant 1$.
Pick $t \in M_{xy}$. Consider M_{xt} and M_{ty}.
Since $M_{xt} \subsetneqq M_{xy}$ and $M_{ty} \subsetneqq M_{xy}$, we have $|M_{xt}| < n$ and $|M_{ty}| < n$.
By induction, $\exists\ x_1, \cdots, x_k \in X$ and $y_1, \cdots, y_l \in X$ such that $x \lhd x_1 \lhd \cdots \lhd x_k \lhd t$ and $t \lhd y_1 \lhd \cdots \lhd y_l \lhd y$.
So $x \lhd x_1 \lhd \cdots \lhd x_k \lhd t \lhd y_1 \lhd \cdots \lhd y_l \lhd y$. $\square$

Definition. Hasse diagram of $P = (X, \leqslant)$ in the plane:

每层由独立的点组成
层之间有继关系.

- $\forall\, x \in X$ is drawn as a node.
- $\forall\, x, y \in X$ with $x \lhd y$ is connected by a line segment.
- If $x \lhd y$, then x appears lower than y.

Definition.

(1) $A \subset P = (X, \leqslant)$ is a chain if any two elements of A are comparable, i.e. A is totally ordered. Let $\omega(P)$ be the maximum size of a chain of P.

(2) $A \subset P = (X, \leqslant)$ is an antichain if any two elements of A are incomparable. Let $\alpha(P)$ be the maximum size of an antichain of P.

Remark. In Hasse diagram, $\omega(P)$ is called the height of P, and $\alpha(P)$ is called the width of P. w(P)最长链

Fact. The set of minimal (or maximal) elements of $P = (X, \leqslant)$ forms an antichain.

Theorem 1. $\forall$ finite poset $P = (X, \leqslant)$, we have $\alpha(P) \cdot \omega(P) \geqslant |X|$.

proof: Let $P_1 = P$, $X_1 = X$, $M_1 =$ {minimal elements of P_1}.
Let $X_2 = X_1 - M_1$, $P_2 = (X_2, \leqslant)$, $M_2 =$ {minimal elements of P_2}.
Inductively, we can define a sequence of posets $P_i = (X_i, \leqslant)$, and $M_i =$ {minimal elements of P_i}. $X_i = X - \bigcup\limits_{j=1}^{i-1} M_j$ for $i \leqslant l$, and $X_{l+1} = \emptyset$.
Each M_i is an antichain of P_i, thus it is also an antichain of P. So $|M_i| \leqslant \alpha(P)$.
Claim: $\forall\, x \in M_{i+1}$, $\exists\, y \in M_i$ such that $y < x$. (Since otherwise $x \in M_i$.)
So, we can find a chain $x_1 < x_2 < \cdots < x_l$ in P with $x_i \in M_i$, which implies $l \leqslant \omega(P)$.
Since $X = M_1 \sqcup M_2 \sqcup \cdots \sqcup M_l$, we have:

$$|X| = \sum_{i=1}^{l} |M_i| \leqslant \alpha(P) \cdot l \leqslant \alpha(P) \cdot \omega(P). \qquad \square$$

Theorem 2 (Dilworth). For any $P = (X, \leqslant)$ with $|X| \geqslant sr + 1$, there exists a chain of size $s + 1$ or an antichain of size $r + 1$.

proof: Suppose not, i.e. $\alpha(P) \leqslant r$ and $\omega(P) \leqslant s$.
Then by Theorem 1, $|X| \leqslant \alpha(P) \cdot \omega(P) \leqslant sr$, a contradiction. $\square$

Definition. Let $A = (a_1, \cdots, a_n)$ be a sequence of n different real numbers, and $B = (a_{i_1}, \cdots, a_{i_k})$ with $i_1 < \cdots < i_k$ be a subsequence. Call B an increasing subsequence (or decreasing subsequence) if $a_{i_1} \leqslant \cdots \leqslant a_{i_k}$ (or $a_{i_1} \geqslant \cdots \geqslant a_{i_k}$).

Theorem 3 (Erdös–Szekeres). Let $A = (a_1, \cdots, a_n)$ be a sequence of n different real numbers. If $n \geqslant sr + 1$, then there exists an increasing subsequence of length $s + 1$ or a decreasing subsequence of length $t + 1$.

proof: Let $X = \{a_1, \cdots, a_n\}$. Define a poset $P = (X, \preccurlyeq)$, where $a_i \preccurlyeq a_j$ iff $a_i \leqslant a_j$ and $i \leqslant j$ (i.e. a_j appears after a_i in A).
Verify that P is indeed a poset:

注意
双重序关系.

(1) $a_i \preccurlyeq a_i$, $\forall\, i \in [n]$.

(2) If $a_i \preccurlyeq a_j$ and $a_j \preccurlyeq a_i$, then $a_i = a_j$.

(3) If $a_i \preccurlyeq a_j$ and $a_j \preccurlyeq a_l$, then $a_i \preccurlyeq a_l$.

In $P = (X, \preccurlyeq)$, a chain $a_{i_1} \prec \cdots \prec a_{i_k}$ is an increasing subsequence, since $a_{i_j} < a_{i_{j+1}}$ and $i_j < i_{j+1}$ for $j \in [k-1]$; an antichain $\{b_{i_1}, \cdots, b_{i_k}\}$ with $i_1 < \cdots < i_k$ is a decreasing subsequence, since b_{i_j} and $b_{i_{j+1}}$ are incomparable, which means $b_{i_j} < b_{i_{j+1}}$, $\forall\, j \in [k-1]$.
By Theorem 2, we can get the desired result. $\square$

Rational approximation (Dirichlet, 1879)

Theorem 4. Given any integer $n > 0$, $\forall\, x \in \mathbb{R}$, $\exists$ a rational number $\dfrac{p}{q}$ with $1 \leqslant q \leqslant n$ and $|x - \dfrac{p}{q}| < \dfrac{1}{nq} \leqslant \dfrac{1}{n}$.

proof: Let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x.
Consider $\{ax\}$ for $a = 1, \cdots, n+1$, which are $n+1$ real numbers in $[0, 1)$.
Put these numbers into n pigeonholes: $[0, \frac{1}{n})$, $[\frac{1}{n}, \frac{2}{n}), \cdots, [\frac{n-1}{n}, 1)$.
Then by P-P, there are 2 numbers, say $\{ax\}$ and $\{bx\}$ with $a > b$, are in the same interval. So $|\{ax\} - \{bx\}| < \frac{1}{n}$.
Since $ax - bx = (\lfloor ax \rfloor - \lfloor bx \rfloor) + \{ax\} - \{bx\}$, we have $\{ax\} - \{bx\} = (a-b)x - (\lfloor ax \rfloor - \lfloor bx \rfloor)$.
Let $q = a - b$, so $1 \leqslant q \leqslant n$, and let $p = \lfloor ax \rfloor - \lfloor bx \rfloor$.
So we have $|qx - p| < \frac{1}{n}$, i.e. $|x - p/q| < \frac{1}{nq}$. $\square$

Subset without divisors

Question. How large a subset $S \subset [2n]$ can be such that for $\forall\ i, j \in S$, we have $i \nmid j$ and $j \nmid i$?

Example. $S = \{n+1, n+2, \cdots, 2n\}$ with $|S| = n$.

Theorem 5. For any $S \subset [2n]$ with $|S| \geqslant n+1$, $\exists\ i, j \in S$ such that $i \mid j$.

proof: For each $k \in [n]$, let $S_k = \{x \in S : x$ can be written as $x = 2^i(2k-1)$ for some $i\}$. Then S can be partitioned into n sets: $S_1, S_2, \cdots, S_n$.
Since $|S| \geqslant n+1$, by P-P, $\exists\ k$ such that $|S_k| \geqslant 2$.
Assume $i, j \in S_k \subset S$, then either $i \mid j$ or $j \mid i$. $\square$