Combinatorics 2018 Fall

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Proposition 1.

(1) $\binom{n}{k} = \binom{n}{n-k}$.

(2) (Pascal Triangle) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

proof:

(1) $|X| = n$. Define $f : \binom{X}{k} \to \binom{X}{n-k}$, $A \mapsto X \backslash A$. It's easy to check that f is a bijection.

(2) $|X| = n$. Separate $\binom{X}{k}$ into two parts, and fix an element $t \in X$.

- $\#\{$all $k-$subsets containing t$\}=\binom{n-1}{k-1}$.
- $\#\{$all $k-$subsets avoiding t$\}=\binom{n-1}{k}$.

Combine these two situations, and we prove (2). □

Selections with repetition

Proposition 2. # integer solutions to $x_1 + \cdots + x_n = k$ where $x_i > 0$ is $\binom{k-1}{n-1}$.

proof: This question is equivalent to *How many ways are there of distributing k sweets to n children such that each child has at least one sweet.*
Lay out the sweets in a single row of length k, and cut it into n pieces. Then give the sweets of the i-th piece to child i, which means that we need $n-1$ cuts from $k-1$ possibles.

$$\Rightarrow \begin{pmatrix} k-1 \\ n-1 \end{pmatrix}.$$

$\square$

Proposition 3. # integer solutions to $x_1 + \cdots + x_n = k$ where $x_i \geqslant 0$ is $= \binom{n+k-1}{n-1}$.

proof: Let $A = \{\text{integer solutions to } x_1 + \cdots + x_n = k, x_i \geq 0\}$.
$B = \{\text{integer solutions to } y_1 + \cdots + y_n = n+k, y_i > 0\}$.
Define $f : A \to B$, $(x_1, \cdots, x_n) \mapsto (y_1, \cdots, y_n)$ by $y_i = x_i + 1$, $i \in [n]$.
CHECK: f is a bijection.
(1) f is well-defined: If $(x_1, \cdots, x_n) \in A$, then $(y_1, \cdots, y_n) \in B$.
(2) f is injective.
(3) f is surjective.
$\Rightarrow |A| = |B| = \binom{n+k-1}{n-1}$ $\square$

Proposition 4. $X = [n]$, $A = \{(a_1, \cdots, a_r) : \ a_i \in X, \ 1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \leqslant n, \ a_{i+1} - a_i \geqslant k+1, \ i \in [r-1]\}$. Then $|A| = \binom{n-k(r-1)}{r}$.

proof: Let $B = \{(b_1, \cdots, b_{r+1}) : b_1 + \cdots + b_{r+1} = n-1, \ b_1 \geqslant 0, \ b_i \geqslant k+1, \ i = 2, \cdots, r, \ b_{r+1} \geqslant 0\}$.
Define $f : A \to B$, $(a_1, \cdots, a_r) \mapsto (b_1, \cdots, b_{r+1})$ by

$$b_1 = a_1 - 1 \geqslant 0.$$

$$b_i = a_i - a_{i-1} \geqslant k+1, i = 2, \cdots, r.$$

$$b_{r+1} = n - a_r \geqslant 0.$$

It's easy to check that f is a bijection.
Now, let $C = \{(c_1, \cdots, c_{r+1}) : \ c_1 + \cdots + c_{r+1} = n - 1 - (k+1)(r-1), \ c_1, \cdots, c_{r+1} \geqslant 0\}$.
Define $g : B \to C$, $(b_1, \cdots, b_{r+1}) \mapsto (c_1, \cdots, c_{r+1})$ by

$$c_1 = b_1 \geqslant 0.$$

$$c_i = b_i - (k+1) \geqslant 0, i = 2, \cdots, r.$$

$$c_{r+1} = b_{r+1} \geqslant 0.$$

Also, it's easy to check that g is a bijection. Hence,

$$|A| = |B| = |C| = \binom{n-1-(k+1)(r-1)+(r+1)-1}{(r+1)-1} = \binom{n-k(r-1)}{r}. \square$$

Arrangements with repetition

Proposition 5. $X = \{x_1, \cdots, x_n\}$, $B =$\{all vectors of length r over X such that x_i occurs a_i times in each vector, $i \in [n]$, $\sum\limits_{i=1}^{n} a_i = r$\}. Then,

$$|B| = \frac{r!}{a_1! a_2! \cdots a_n!}.$$

proof: Trivial. $\square$

Corollary. (Multinomial Theorem)

$$(x_1 + x_2 + \cdots + x_n)^r = \sum_{a_1 + a_2 + \cdots + a_n = r} \frac{r!}{a_1! a_2! \cdots a_n!} {x_1}^{a_1} \cdots {x_n}^{a_n}.$$

proof:

$$(x_1 + x_2 + \cdots + x_n)^r = \sum_{1 \leqslant i_1, i_2, \cdots, i_r \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_r}$$

The coefficient of ${x_1}^{a_1}\cdots{x_n}^{a_n}$ is equal to the number of vectors $(i_1,\cdots,i_n)$ over $[n]$ such that i occurs a_i times. By Proposition 5, we get this theorem. $\square$

Remark:

(1) **(Binomial Theorem)** If $n=2$, $x_1=a$, $x_2=b$, then we get $(a+b)^r=\sum\limits_{i=0}^{r}\binom{r}{i}a^ib^{r-i}$.

(2) If $a=b=1$, then we get $2^r=\sum\limits_{i=0}^{r}\binom{r}{i}$

Partition: $X=R_1\cup R_2\cup\cdots\cup R_n$. There are two cases:

- unordered partition $\{R_1,R_2,\cdots,R_n\}=\{R_2,R_1,\cdots,R_n\}$.
- ordered partition $(R_1,R_2,\cdots,R_n)\neq(R_2,R_1,\cdots,R_n)$.

Proposition 6. $|X|=r$, $A=${ordered partitions of X into n parts such that the i-th part has size a_i, $i\in[n]$, $\sum\limits_{i=1}^{n}a_i=r$}. Then

$$|A|=\frac{r!}{a_1!a_2!\cdots a_n!}.$$

proof: This question is equivalent to *How many ways are there of partitioning r students into class $1,\cdots,n$ such that class i has a_i students*. There are $\binom{r}{a_1}$ ways of choosing the students in class 1. Then, there are $\binom{r-a_1}{a_2}$ ways of choosing the students in class 2. $\cdots$. So, there are $\binom{r}{a_1}\binom{r-a_1}{a_2}\binom{r-a_1-a_2}{a_3}\cdots\binom{r-a_1-\cdots-a_{n-1}}{a_n}$ ways to partition the students as required, which is exactly $\frac{r!}{a_1!a_2!\cdots a_n!}$.

Exercise. Find the connection between Proposition 5 and Proposition 6.

Proposition 7. $|X| = r$, $S = \{$unordered partitions of X such that there are k_i blocks of size i, $i \in [n]$, $\sum\limits_{i=1}^{n} ik_i = r\}$. Then,

$$|S| = \frac{r!}{(1!)^{k_1}(2!)^{k_2}\cdots(r!)^{k_r}k_1!k_2!\cdots k_r!}.$$

<u>proof:</u> This question is equivalent to *How many ways are there of partitioning r students into different unordered groups such that there are k_i groups having i students.*

First, there are $\binom{r}{k_1 \cdot 1}$ ways of choosing the students in groups of size 1. To partion these students into these k_1 unordered groups, there are $\frac{(k_1 \cdot 1)!}{(1!)^{k_1}(k_1!)}$ ways.

Similarly, for $i = 2, \cdots, n$, there are $\binom{r - k_1 \cdot 1 - \cdots - k_{i-1}\cdot(i-1)}{k_i \cdot i}$ ways of choosing the students in groups of size i. To partion these students into these k_i unordered groups, there are $\frac{(k_i \cdot i)!}{(i!)^{k_i}(k_i!)}$ ways.

So, after simple calculations, we get the desired result.

Stirling Number of the 2nd kind

Def: $S(r,n)$ is the number of partitions of $[r]$ into n unordered non-empty subsets.

$$S(r,n) = \sum_{\sum\limits_{i=1}^{m} k_i = n,\ \sum\limits_{i=1}^{m} ik_i = r} \frac{r!}{(1!)^{k_1}(2!)^{k_2}\cdots(m!)^{k_m}k_1!k_2!\cdots k_m!}.$$

Remark: $S(r,1) = 1 \qquad S(r,r) = 1$

Exercise. $S(r,2) = ?$

Theorem (Vandermonde's Identity)

(1) $\binom{m+n}{r} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i} = \sum_{i+j=r} \binom{n}{i}\binom{m}{j}$.

(2) $\binom{m+n}{r+m} = \sum_{i-j=r} \binom{n}{i}\binom{m}{j}$.

proof:

(1)

$$(1+x)^{m+n} = (1+x)^m(1+x)^n$$
$$\Rightarrow \sum_{r=0}^{m+n} \binom{m+n}{r} x^r = \sum_{i=0}^{n} \binom{n}{i} x^i \cdot \sum_{j=0}^{m} \binom{m}{j} x^j$$

Compute the coefficient of x^r for both sides, and we get

$$\binom{m+n}{r} = \sum_{i+j=r} \binom{n}{i}\binom{m}{j} = \sum_{i=0}^{r} \binom{n}{i}\binom{m}{r-i}.$$

(2)

$$\sum_{i-j=r} \binom{n}{i}\binom{m}{j} = \sum_{i-j=r} \binom{n}{i}\binom{m}{m-j} = \sum_{i+(m-j)=m+r} \binom{n}{i}\binom{m}{m-j} = \binom{m+n}{r+m}. \square$$

Exercise.

(1) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

(2) $\sum_{k=0}^{n} \binom{m}{k}\binom{n}{p+k} = \binom{n+m}{p+m}$.

(3) $\sum_{k=1}^{m} \binom{m}{k}\binom{n-1}{k-1} = \binom{n+m-1}{n}$.

(4) $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} \quad (n \geq k \geq m)$.

(5) $\sum_{k=0}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m} 2^{n-m}$.