# Combinatorics 2018 Fall

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**Key words:** Polya Theorem

**Recall. (Burnside Lemma)** Finite group action $(G, X)$. Let $N(G)$ be the number of distinct orbits of $X$, then $N(G) = \frac{1}{|G|}\sum_{g\in G}|Fix(g)|$.

**Definition.**

(1) A coloring of $X$ in $m$ colors is a function $f : X \to C$, where $C$ is a set of $m$ colors.

(2) $C^X$ ={all colorings of $X$ in $C$}.

(3) $(G, X)$ induces a group action $(G, C^X)$ as $\forall\ g \in G,\ \forall\ f \in C^X$, define $g * f$ by $(g * f)(x) = f(g^{-1} * x)$, $\forall\ x \in X$.

**Fact.** Any finite group can be viewed as a permutation group.

**Example.**

(1) $X = [6]$, $g = (1\ 2\ 3\ 4\ 5\ 6)$, $C = \{g, r, b\}$
$f_1 : 1 \to r \quad 2 \to b \quad 3 \to b \quad 4 \to g \quad 5 \to r \quad 6 \to b$
$f_2 = g * f_1 : 1 \to b \quad 2 \to r \quad 3 \to b \quad 4 \to b \quad 5 \to g \quad 6 \to r$

(2) $X = [6]$, $g = (1\ 3\ 5)(2\ 4\ 6)$, $C = \{g, r, b\}$
$f_1 : 1 \to r \quad 2 \to b \quad 3 \to b \quad 4 \to g \quad 5 \to r \quad 6 \to b$
$f_2 = g * f_1 : 1 \to r \quad 2 \to b \quad 3 \to r \quad 4 \to b \quad 5 \to b \quad 6 \to g$

**Note.** $f \in C^X$ is fixed by $g$ if and only if $f$ has the same color along each cycle of $g$, so $|Fix(g)| = |C|^{c(g)}$ where $c(g)$ is the number of cycles of $g$.

**Theorem (Polya Theorem).** $|X| = n$, $|C| = m$, $G \leqslant S_n$. Then,

$$\#\text{different colorings in } C^X \text{ under } G = \#\text{orbits of } C^X = \frac{1}{|G|}\sum_{g\in G} m^{c(g)}.$$

**Definition (cycle type).** $g \in S_n$, let $\lambda_i(g)$ be the number of cycles of length $i$ in $G$. $type(g) = (\lambda_1(g), \lambda_2(g), \cdots, \lambda_n(g))$.

**Example.** $g = (135)(246)$, $type(g) = (0, 0, 2, 0, 0, 0)$.

**Definition (cycle index).** $G \leqslant S_n$, the cycle index of $G$ is a polynomial

$$P_G(x_1, x_2, \cdots, x_n) = \frac{1}{|G|}\sum_{g\in G} x_1^{\lambda_1(g)} x_2^{\lambda_2(g)} \cdots x_n^{\lambda_n(g)}.$$

**Corollary (Polya Theorem).**

$$\frac{1}{|G|}\sum_{g\in G} m^{c(g)} = P_G(m, m, \cdots, m).$$

**Problem 1.** $\sigma_n =< (1, 2, \cdots, n) >$. Compute $P_{\sigma_n}(x_1, x_2, \cdots, x_n)$.

**Solution.** $\tau = (1, 2, \cdots, n)$, $\sigma_n = \{\tau, \tau^2, \cdots, \tau^n\}$.
For $\forall\ k \in [n]$, consider $\tau^k$. What is $type(\tau^k)$?
$\forall\ i \in [n], \tau^k(i) \equiv i + k \pmod{n}$. Suppose the cycle containing $i$ has length $l$, then the cycle is $(i, i+k, \cdots, i+(l-1)k)$. So $l = \frac{n}{(k, n)}$.
So, all cycles of $\tau^k$ have length $\frac{n}{(k, n)}$, i.e.

$$\lambda_l(\tau^k) = \begin{cases} 0 & l \neq \frac{n}{(k, n)}; \\ (k, n) & l = \frac{n}{(k, n)}. \end{cases}$$

So,

$$P_{\sigma_n}(x_1, x_2, \cdots, x_n) = \frac{1}{n}\sum_{i=1}^{n}(x_{\frac{n}{(k,n)}})^{(k,n)}$$
$$= \frac{1}{n}\sum_{j|n}\sum_{k:(k,n)=j}(x_{\frac{n}{j}})^j$$

Let

$$r(j) = \#\{k \in [n]: \ (k,n) = j\}$$
$$= \#\{\frac{k}{j} \in [\frac{n}{j}] : (\frac{k}{j}, \frac{n}{j}) = 1\}$$
$$= \varphi(\frac{n}{j}).$$

So,

$$P_{\sigma_n}(x_1, x_2, \cdots, x_n) = \frac{1}{n}\sum_{j|n}\varphi(\frac{n}{j})(x_{\frac{n}{j}})^j$$
$$= \frac{1}{n}\sum_{d|n}\varphi(d)(x_d)^{\frac{n}{d}}.$$

**Example.** $C_m(n) = \#$ orbits of $[m]^{[n]}$ under $\sigma_n = P_{\sigma_n}(m, m, \cdots, m) = \frac{1}{n}\sum_{d|n}\varphi(d)m^{\frac{n}{d}}$.

$D_n \leqslant S_n$: dihedral group.
$D_n = \{\tau, \tau^2, \cdots, \tau^n\} \cup \{\pi_1, \pi_2, \cdots, \pi_n\}$.
$\pi_k(i) \equiv -i + k \pmod{n}$.

**Problem 2.** Show that

$$P_{D_n}(x_1, x_2, \cdots, x_n) = \begin{cases} \frac{1}{2}P_{\sigma_n}(x_1, x_2, \cdots, x_n) + \frac{1}{2}x_1 x_2^{\frac{n-1}{2}} & n \ odd; \\ \frac{1}{2}P_{\sigma_n}(x_1, x_2, \cdots, x_n) + \frac{1}{4}(x_2^{\frac{n}{2}} + x_1^2 x_2^{\frac{n-2}{2}}) & n \ even. \end{cases}$$

**Solution.**

$$P_{D_n}(x_1, x_2, \cdots, x_n) = \frac{1}{2n}(\sum_{i=1}^{n} x_1^{\lambda_1(\tau^i)} \cdots x_n^{\lambda_n(\tau^i)} + \sum_{i=1}^{n} x_1^{\lambda_1(\pi_i)} \cdots x_n^{\lambda_n(\pi_i)})$$
$$= \frac{1}{2}P_{\sigma_n}(x_1, \cdots, x_n) + \frac{1}{2n}\sum_{i=1}^{n} x_1^{\lambda_1(\pi_i)} \cdots x_n^{\lambda_n(\pi_i)}$$

Since $\pi_k$ is a reflection, i.e. $(\pi_k)^2 = id$, $\pi_k$ is a product of $m_k$ cycles of length 1 and $\frac{n-m_i}{2}$ cycles of length 2.
So
$$\frac{1}{2n}\sum_{i=1}^{n} x_1^{\lambda_1(\pi_i)}\cdots x_n^{\lambda_n(\pi_i)} = \frac{1}{2n}\sum_{i=1}^{n} x_1^{m_i} x_2^{\frac{n-m_i}{2}}.$$
$m_k = |Fix(\pi_k)| = \#\{i : -i+k \equiv i \pmod{n}\} = \#\{i : 2i \equiv k \pmod{n}\}$.
If $n$ is odd, $(2,n)=1$, 2 has inverse in $\mathbb{Z}_n$, so $m_1 = \cdots = m_n = 1$.
So,
$$\frac{1}{2n}\sum_{i=1}^{n} x_1^1 x_2^{\frac{n-1}{2}} = \frac{1}{2} x_1 x_2^{\frac{n-1}{2}}.$$
If $n$ is even, then $2i \equiv k \pmod{n}$ has no solutions if $k$ is odd, and two solutions $i = \frac{k}{2}$ and $i = \frac{k}{2} + \frac{n}{2}$ if $k$ is even.
So
$$m_k = \begin{cases} 0 & k \textit{ odd}; \\ 2 & k \textit{ even}. \end{cases}$$
Then,
$$\begin{aligned}\frac{1}{2n}\sum_{i=1}^{n} x_1^{m_k} x_2^{\frac{n-m_k}{2}} &= \frac{1}{2n}\left(\frac{n}{2}x_1^0 x_2^{\frac{n}{2}} + \frac{n}{2}x_1^2 x_2^{\frac{n-2}{2}}\right) \\ &= \frac{1}{4}\left(x_2^{\frac{n}{2}} + x_1^2 x_2^{\frac{n-2}{2}}\right).\end{aligned}$$

**Corollary.** # cycles of length $[n]$ over $[m]$ under rotations and reflections $= P_{D_n}(m, \cdots m)$
$$= \frac{1}{2}C_m(n) + \begin{cases} \frac{1}{2}m^{\frac{n+1}{2}} & n \textit{ odd}; \\ \frac{1}{4}\left(m^{\frac{n}{2}} + m^{\frac{n}{2}+1}\right) & n \textit{ even}. \end{cases}$$

**Definition.**

(1) Simple graph $H = (V, E)$. $V$: vertex set, $E \subseteq \binom{V}{2}$: edge set.

(2) If $\{i, j\} \in E$, then we say $\{i, j\}$ is incident with $i$ and $j$, and $i$ is adjacent to $j$.

(3) $H = (V, E)$, $\varphi : V \to V$. If $\forall\ u, v \in V$, $\{u, v\} \in E \Leftrightarrow \{\varphi(u), \varphi(v) \in E\}$, then we say $\varphi$ is an automorphism of $H$.

(4) $Aut(H) =$\{all automorphisms of $H$\}$\leqslant Sym(V)$.

**Example.**

(1) $K_n$: complete graph. $Aut(K_n) = S_n$.

(2) $C_n = (V, E)$: a cycle of length $n$. $Aut(C_n) = D_n$.