# Combinatorics, 2018 Fall, USTC
# Homework14

- **The due is on Thursday, Jan. 03 .**
- **Please sign your name and student number.**
- **Please solve as many problems as you can.**

1. Prove that for all integers $n$ and $p \in [0,1]$,
$$R(k,l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{l} (1-p)^{\binom{l}{2}}.$$
and show $R(4,k) \geq c \cdot \left(\frac{k}{\ln k}\right)^2$ for some constant $c > 0$.

2. Consider the family of all pairs $(A, B)$ of disjoint $k-$elements subsets of $[n]$. A set $Y$ separates the pair $(A, B)$ if $A \subset Y$ and $B \cap Y = \phi$. Prove that there exist $l = 2k4^k \ln n$ sets such that every pair $(A, B)$ is separated by at least one of them.

3. If a bipartite $n \times n$ graph has minimum degree $n - d$, then its edges can be covered by at most $O(d \log n)$ complete bipartite subgraphs.

4. Let $\mathcal{F}$ be an $r$-uniform family of subsets on an $n$-element set. Suppose that, each element belongs to $d$ members of $\mathcal{F}$. Prove that there exists a set $S$ of elements such that $S$ contains no member of $\mathcal{F}$ and has size $|S| \geq (1 - \frac{1}{r}) \cdot n \cdot d^{-\frac{1}{r-1}}$.

5. Recall a set system $\mathcal{F} \subset 2^{[n]}$ is an antichain, if $\forall A, B \in \mathcal{F}$ and $A \neq B$,then $A \nsubseteq B$. Let $\pi \in S_n$ be a random permutation of $[n]$. By considering the random variable $X = |\{i : \{\pi(1), \pi(2), \cdots \pi(i)\} \in \mathcal{F}\}|$, give a new proof that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.