# Combinatorics 2018 Fall

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***2018.12.24***

**Key words:** Linearity of Expectation, Deletion Method

**Thm1:** $G = (V, E)$, $|V| = n$, $|E| = m$, then $G$ contains a bipartite subgraph with at least $\frac{m}{2}$ edges.

***proof :***.
Consider a random partition $L \cup R = V$. $\forall v \in V$, put it into $L$ or $R$ with equal probability, independently. Let $X$ be the number of crossing edges from $L$ to $R$. Let $X_{uv}$ be the indicator random variable of the event that the edge $uv$ is crossing, then $X = \sum\limits_{u \sim v} X_{uv}$
$\mathrm{E}[X_{uv}]$ =Pr[$uv$ is crossing]=Pr[$u \in L, v \in R$ or $u \in R, v \in L$]=$\frac{1}{2}$
$\mathrm{E}[X]$=$\sum\limits_{u \sim v}\mathrm{E}[X_{uv}]$=$\frac{m}{2}$
$\Longrightarrow \exists$ a bipartite subgraph with $\geq \frac{m}{2}$ edges.

$\square$

**Recall:** $\alpha(G)$ : independent number

$\alpha(G) \leq \frac{-nd_n}{d_1 - d_n}$

- $\chi(G)\alpha(G) \geq |G|$
- $\alpha(G) \leq n\frac{-\lambda_n}{\lambda_1 - \lambda_n}$

**Thm2:** For any graph $G$, $\alpha(G) \geq \sum\limits_{v \in V} \dfrac{1}{1 + deg(v)}$.

***proof :***.
Let $V(G) = [n]$. For $i \in [n]$, let $N_i = \{j \in [n] : j \sim i\}$, let $S_n$ be the symmetric group over $[n]$.
For given $\pi \in S_n$ we say a vertex $i \in [n]$ is $\pi-$dominating, if $\pi(i) < \pi(j)$ for all $j \in N_i$. Let $M_\pi$ ={all $\pi-$dominating vertices.}

**Claim:** $\forall \pi \in S_n,\ M_\pi$ is a independent set.
**proof of claim:** Suppose not, then $\exists i, j \in M_\pi$ with $i \sim j$. $i \in N(j) \Rightarrow \pi(j) < \pi(i)$ and $j \in N(i) \Rightarrow \pi(i) < \pi(j)$, contradiction!

Pick $\pi \in S_n$ uniformly at random, compute $E[|M_\pi|]$.
For any fixed $i \in [n]$, let $I_{\{i \text{ is } \pi\text{-dominating}\}}$ be a indicator random variable, then $|M_\pi| = \sum\limits_{i\in[n]} I_{\{i \text{ is } \pi\text{-dominating}\}}$

$$\begin{aligned} E[|M_\pi|] &= \sum_{i\in[n]} E[I_{\{i \text{ is } \pi\text{-dominating}\}}] \\ &= \sum_{i\in[n]} Pr[i \text{ is } \pi - \text{dominating}] \\ &= \sum_{i\in[n]} \frac{\binom{n}{deg(i)+1} \cdot deg(i)! \cdot (n - deg(i) - 1)!}{n!} \\ &= \sum_{i\in[n]} \frac{1}{deg(i)+1}. \end{aligned}$$

$\implies \exists$ independent set of size $\geq \sum\limits_{i\in[n]} \frac{1}{deg(i)+1}$

$\square$

**Cor1:** $\forall G$ with $m$ edges and $n$ vertices, then $\alpha(G) \geq \dfrac{n^2}{2m+n}$
$m = \dfrac{nd}{2}$ where $d$ is average degree, then $\alpha(G) \geq \dfrac{n}{1+d}$.

***proof :.***
$2m = \sum\limits_{v\in V} deg(v)$, then $\sum\limits_{v\in V}(deg(v)+1) = 2m+n$.
By Cauchy-Schwarz Inequality, $\alpha(G) \geq \sum\limits_{v\in V} \frac{1}{1+deg(v)} \geq \frac{n^2}{\sum\limits_{v\in V}(1+deg(v))} = \frac{n^2}{2m+n}$

$\square$

### The deletion method

• **Ideas:** A random structure doesn't always have the directed property, and may have some very few "blemishes". After deleting all blemishes, we will obtain the wanted structure.

**Thm1:** Let $G$ be a graph on $n$ vertices and with average degree $d$, then $\alpha(G) \geq \dfrac{n}{2d}$

***proof :.***
Let $S \subset V(G)$ be a random set, for $\forall v \in V, Pr(v \in S) = p$ and value of $p$ will be determined later. Let $X = |S|$ and $Y = \#\text{edges in } S$. Then $E[X] = np, E[Y] = |E(G)| \cdot p^2 = \dfrac{nd}{2}p^2$. Then $E[X - Y] = np - \dfrac{nd}{2}p^2 = n(p - \dfrac{d}{2}p^2)$.
By choosing $p = \dfrac{1}{d}$, we have $E[X - Y] = \dfrac{n}{2d}$ which is maximum.
So there is a particular set $S$ such that $|S| - Y \geq E[X - Y] = \dfrac{n}{2d}$.
Now deleting one vertex form each edge of $S$, leaving a set $S'$. This set $S'$ is independent and has at least $\dfrac{n}{2d}$ vertices.

$\square$

**Recall:** $R(k, k) > \dfrac{1}{e\sqrt{2}}k2^{k/2}$.

**Thm2:** $\forall n,\ R(k, k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$

***proof :.***
Consider a random 2−edge-coloring of $K_n$, where each edge is colored by red or blue with probability $\dfrac{1}{2}$, independently. For $A \in \binom{[n]}{k}$, Let $X_A$ be the indicator random variable of the event that $A$ is monochromatic. Let $X$ be the number of monochromatic $k$ cliques.

$$E[X] = \sum_{A \in \binom{[n]}{k}} E[X_A] = \binom{n}{k}2^{1-\binom{k}{2}}.$$

Then there exists a 2−edge-coloring of $K_n$, s.t. there are $\leq E[X] = \binom{n}{k}2^{1-\binom{k}{2}}$ monochromatic $k$ cliques. Fix such a coloring, remove one vertex from each monochromatic $k$−subset. This will delete $\leq \binom{n}{k}2^{1-\binom{k}{2}}$ vertices, which has No monochromatic $K_k$. So $R(k, k) >$

$$n - \binom{n}{k} 2^{1-\binom{k}{2}}.$$

$\square$

**Cor:** $R(k,k) > \frac{1}{e}(1+O(1))k2^{k/2}$.

**Def:** A clique covering of a graph $G$ is a set $H_1, \cdots, H_t$ of its clique subgraphs such that each edge of $G$ belongs to at least one of those cliques. Denote $cc(G) = \min \sharp$ cliques in clique covering of $G$.

**Thm3** Let $G$ be a graph on $n$ vertices such that every vertex has at least one neighbor and at most $d$ non-neighbors. Then $cc(G) = O(d^2 \ln n)$.

***proof :***.
To be continued.

$\square$