Log-normal simulations

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Here we give some details on log-normal simulations. The Appendix gives a power spectrum estimator with μ-leakage correction.

Keywords: cosmology; large-scale structure

CONTENTS

I. INTRODUCTION

The density contrast field $\delta(\boldsymbol{x})$ is often treated as Gaussian. However, this can lead to $\delta < -1$, which is forbidden by construction. A more realistic model, therefore, is a log-normal distribution for δ, which is naturally constrained to $\delta \geq -1$.

In this note we investigate how to derive a log-normal simulation. We mostly follow **?**. This was first done by **?**.

Other codes doing log-normal simulations are FLASK (**?**) and NBodyKit (**?**). FLASK seems to be overly complicated. You provide it $C_\ell(r, r')$ with all the tomographic cross-correlations to make a simulation.

Log-normal is not fully described by all its moments (**?**). That is, there is another distinct distribution that has the same moments.

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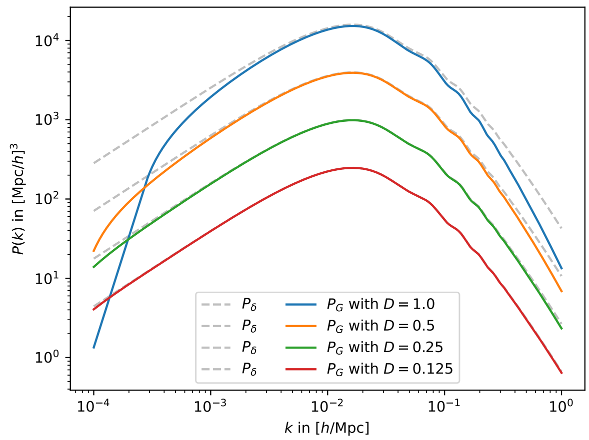

FIG. 1. Power spectrum for δ and G.

? describe the effect of non-Gaussian distribution on the power spectrum estimation. They provide approximations for the distribution of the C_ℓ, and a nice iterative procedure for estimating cosmological parameters from the estimated C_ℓ. **?** apply this to the APM survey.

II. LOG-NORMAL DENSITY FIELD

We define the log-transformed density field

$$G(\boldsymbol{x}) = \ln(1 + \delta(\boldsymbol{x})) - \langle \ln(1 + \delta(\boldsymbol{x})) \rangle_{\boldsymbol{x}} \,. \tag{1}$$

Then,

$$\delta(\boldsymbol{x}) = e^{-\frac{1}{2}\sigma_G^2 + G(\boldsymbol{x})} - 1 \,, \tag{2}$$

where $\sigma_G^2 \equiv \left\langle G(\boldsymbol{x})^2 \right\rangle_{\boldsymbol{x}}$ follows from the requirement that the average density contrast $\langle \delta \rangle_{\boldsymbol{x}} = 0$ must vanish, and we used that $G(\boldsymbol{x})$ is a Gaussian field with vanishing mean so that

$$\left\langle e^G \right\rangle = \sum_n \frac{\langle G^n \rangle}{n!} = \sum_n \frac{\left\langle G^{2n} \right\rangle}{(2n)!} = \sum_n \frac{\# \left\langle G^2 \right\rangle^n}{(2n)!} \tag{3}$$

$$= \sum_n \frac{\left(\frac{1}{2} \left\langle G^2 \right\rangle\right)^n}{n!} = e^{\frac{1}{2}\left\langle G^2 \right\rangle} \,, \tag{4}$$

where Wick's theorem was used. Then, defining

$$\xi(\boldsymbol{r}) = \langle \delta(\boldsymbol{x}) \delta(\boldsymbol{x} + \boldsymbol{r}) \rangle_{\boldsymbol{x}} \,, \tag{5}$$

we get

$$\xi_G(\boldsymbol{r}) = \ln(1+\xi(\boldsymbol{r}))\,, \tag{6}$$

where we assumed a homogeneous universe. To get the normally-distributed power spectrum $P_G(\boldsymbol{q})$, we first Fourier-transform the power spectrum $P(\boldsymbol{k})$, apply Eq. (6), and then transform to $P_G(\boldsymbol{k})$. That is,

$$\begin{aligned} P_G(k) &= \int \mathrm{d}^3\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, \xi_G(\boldsymbol{r}) \\ &= (2\pi)^3 \int_0^\infty \frac{r^2\mathrm{d}r}{2\pi^2}\, j_0(kr)\, \xi_G(r) \end{aligned} \tag{7}$$

$$\begin{aligned} \xi(r) &= \int \frac{\mathrm{d}^3q}{(2\pi)^3}\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\, P(\boldsymbol{q}) \\ &= \int_0^\infty \frac{q^2\mathrm{d}q}{2\pi^2}\, j_0(qr)\, P(q) \end{aligned} \tag{8}$$

This allows us to draw the Gaussian field $\delta_G(\boldsymbol{k})$, which we transform to configuration space, then apply Eq. (2) to obtain the density field.

Fig. 1 shows a linear matter power spectrum with different amplitudes, comparing the power spectrum for the lognormal field δ and the normal field G.

A. Predicting sigma G squared

$$\begin{aligned} \sigma_G^2 &= \left\langle G^2(\boldsymbol{x}) \right\rangle = \xi^G(0) = \ln(1+\xi(0)) \\ &= \ln\left(1 + \int \frac{k^2\mathrm{d}k}{2\pi^2}\, P(k)\right). \end{aligned} \tag{9}$$

This would be the estimate for the fluctuations if the resolution of our simulation was infinite. To get the relevant σ_G^2 at finite resolution, we include the pixel window,

$$\sigma_G^2 = \int \mathrm{d}^3r\, W(\boldsymbol{r}) \int \mathrm{d}^3x\, W(\boldsymbol{x})\, \langle G(\boldsymbol{r})G(\boldsymbol{x})\rangle \tag{10}$$

$$= \int \mathrm{d}^3r\, W(\boldsymbol{r}) \int \mathrm{d}^3x\, W(\boldsymbol{x})\, \xi_G(\boldsymbol{r}-\boldsymbol{x}) \tag{11}$$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, |\widetilde{W}(\boldsymbol{k})|^2\, P_G(k)\,. \tag{12}$$

Numerically, I find that σ_G^2 from one simulation is about 24% higher for the matter field, and 20% for the galaxy field. This may due to the calculation assuming spherical cells, while our cells are cubic.

B. The Velocity field

The velocity field to first order is given by

$$\boldsymbol{v}(\boldsymbol{k}) = aH\boldsymbol{u}\,, \tag{13}$$

where

$$\boldsymbol{u}(\boldsymbol{k}) = if\,\frac{\boldsymbol{k}}{k^2}\,\delta_m(\boldsymbol{k})\,. \tag{14}$$

We assume that galaxies have the same velocity as the matter. RSD puts galaxies at

$$\boldsymbol{s} = \boldsymbol{r} + \frac{\boldsymbol{v}\cdot\hat{\boldsymbol{r}}}{aH}\,\hat{\boldsymbol{r}} = \boldsymbol{r} + (\boldsymbol{u}\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}}\,. \tag{15}$$

Hence, calculating $\boldsymbol{u}$ is needed to get velocities.

C. Drawing galaxies

We draw galaxies from the real-space density contrast. The number of galaxies in each grid cell is drawn from a Poisson distribution, and the galaxy positions are uniformly drawn inside the grid cell. We assign the same velocity to each galaxy within a grid cell. This is similar to NGP assignment.

As a second option, we also allow drawing the position twice: First, uniformly within the grid cell, then uniformly within a cubic cloud around the current position. This is a convolution of two top-hat distributions. Combined with the probability of the neighboring cells, this is the same as drawing from a tri-linear interpolation distribution (since Poisson probabilities add), and it is similar to CiC on the estimator side.

Velocities can be assigned by several options: Using the nearest grid point, or by linearly interpolating neighbors. To simulate a FoG, we add Gaussian noise to the velocities.

III. CONCLUSION

Log-normals for the win!

Appendix A: Log-normal probability distribution

We assume that $\delta(\boldsymbol{r})$ is log-normal distributed. The average must vanish, by definition, and the power spectrum measures its variance. The log-normal is given by the probability distribution

$$p(\delta) = \frac{1}{(1+\delta)\sqrt{2\pi\sigma^2}}\, e^{-\frac{(\ln(1+\delta)-\mu)^2}{2\sigma^2}}\,. \tag{A1}$$

The mean and variance are

$$\langle\delta\rangle = e^{\mu+\frac{1}{2}\sigma^2} - 1\,, \tag{A2}$$

$$\left\langle\delta^2\right\rangle - \langle\delta\rangle^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right). \tag{A3}$$

Requiring $\langle\delta\rangle = 0$, we get

$$\left\langle\delta^2\right\rangle = e^{\sigma^2} - 1\,. \tag{A4}$$

To get the correlation function, we need to get the covariance between δ's at different locations in space. The log-normal distribution for this is given in **?**:

$$p(\vec{\delta}) = \frac{1}{\sqrt{(2\pi)^N \det \Sigma_G}} \left[\prod_s \frac{1}{1+\delta_s}\right] \times e^{-\frac{1}{2}\sum_{ij}[\ln(1+\delta_i)-\mu_i]\left(\Sigma_G^{-1}\right)_{ij}[\ln(1+\delta_j)-\mu_j]} \tag{A5}$$

for the density contrast $\delta_i = \delta(\boldsymbol{r}_i)$. Define

$$G_i = \ln(1+\delta_i) - \mu_i\,, \tag{A6}$$

$$p_G(\vec{G}) = \frac{1}{\sqrt{(2\pi)^N \det \Sigma_G}}\, e^{-\frac{1}{2}\sum_{ij} G_i \left(\Sigma_G^{-1}\right)_{ij} G_j}\,. \tag{A7}$$

Then, the mean and variance are

$$\langle \delta_i \rangle = e^{\mu_i + \frac{1}{2}\Sigma_{ii}} - 1\,, \tag{A8}$$

$$\langle (\delta_i - \langle \delta_i \rangle)(\delta_j - \langle \delta_j \rangle) \rangle = e^{\mu_i + \mu_j + \frac{1}{2}(\Sigma_{ii}+\Sigma_{jj})} \left(e^{\Sigma_{ij}} - 1\right)\,. \tag{A9}$$

Requiring the average density contrast to vanish, we get

$$\mu_i = -\frac{1}{2}\Sigma_{ii}\,, \tag{A10}$$

$$\langle \delta_i \delta_j \rangle = e^{\Sigma_{ij}} - 1\,. \tag{A11}$$

1. Bispectrum of Lognormal Simulations

First, the skew is defined as

$$\Gamma = \left\langle \left(\frac{\delta - \langle\delta\rangle}{\sigma_\delta}\right)^3 \right\rangle = \left\langle \left(\frac{\delta}{\sqrt{\langle \delta^2 \rangle}}\right)^3 \right\rangle, \tag{A12}$$

where we assumed that the average $\langle\delta\rangle = 0$ vanishes. Then the skew of the lognormal distribution is

$$\Gamma = \left(e^{\sigma^2} + 2\right)\sqrt{e^{\sigma^2} - 1}\,. \tag{A13}$$

Or,

$$\langle \delta^3 \rangle = \sigma_\delta^3\, \Gamma = \left(e^{\sigma^2} - 1\right)^2 \left(e^{\sigma^2} + 2\right) \tag{A14}$$

$$= e^{3\sigma^2} - 3\, e^{\sigma^2} + 2\,. \tag{A15}$$

Or, when the average does not vanish (**?**),

$$\mu_3 = e^{3\mu + \frac{3}{2}\sigma^2} \left[e^{\sigma^2} - 1\right]^2 \left[e^{\sigma^2} + 2\right]. \tag{A16}$$

The generalization is

$$\mu_3^{ijk} = e^{\mu_i + \mu_j + \mu_k + \frac{1}{2}[\Sigma_{ii} + \Sigma_{jj} + \Sigma_{kk}]} \times \left[e^{\Sigma_{ij} + \Sigma_{jk} + \Sigma_{ki}} - e^{\Sigma_{ij}} - e^{\Sigma_{jk}} - e^{\Sigma_{ki}} + 2\right]. \tag{A17}$$

Or, for a field with vanishing mean,

$$\langle \delta_i \delta_j \delta_k \rangle = e^{\Sigma_{ij} + \Sigma_{jk} + \Sigma_{ki}} - e^{\Sigma_{ij}} - e^{\Sigma_{jk}} - e^{\Sigma_{ki}} + 2 \tag{A18}$$

$$= [\xi_{ij} + 1][\xi_{jk} + 1][\xi_{ki} + 1] - 1 - \xi_{ij} - \xi_{jk} - \xi_{ki} \tag{A19}$$

$$= \xi_{ij}\xi_{jk}\xi_{ki} + \xi_{jk}\xi_{ki} + \xi_{ij}\xi_{ki} + \xi_{ij}\xi_{jk}\,, \tag{A20}$$

which agrees with **?**.

To get the bispectrum, transform to Fourier space. In Eq. (A20) the terms with two 2-point functions can be converted to Fourier space in the following way. For example,

$$B_{12,23} = (2\pi)^3 \delta^D(\boldsymbol{k}_{123}) \int \mathrm{d}^3 s_{12}\, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{s}_{12}}\, \xi(\boldsymbol{s}_{12}) \times \int \mathrm{d}^3 s_{23}\, e^{i\boldsymbol{k}_3\cdot\boldsymbol{s}_{23}}\, \xi(\boldsymbol{s}_{23}) \tag{A21}$$

$$= (2\pi)^3 \delta^D(\boldsymbol{k}_{123})\, P(\boldsymbol{k}_1)\, P(\boldsymbol{k}_3)\,. \tag{A22}$$

The term with three 2-point functions involves a convolution,

$$B_{12,23,31} = \int \mathrm{d}^3 r_1 \int \mathrm{d}^3 r_2 \int \mathrm{d}^3 r_3\, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}_1}\, e^{-i\boldsymbol{k}_2\cdot\boldsymbol{r}_2}\, e^{-i\boldsymbol{k}_3\cdot\boldsymbol{r}_3} \times \xi(\boldsymbol{s}_{12})\, \xi(\boldsymbol{s}_{23})\, \xi(\boldsymbol{s}_{31}) \tag{A23}$$

$$= \int \mathrm{d}^3 r_1 \int \mathrm{d}^3 r_2 \int \mathrm{d}^3 r_3\, e^{-i\boldsymbol{k}_1\cdot\boldsymbol{r}_1}\, e^{-i\boldsymbol{k}_2\cdot\boldsymbol{r}_2}\, e^{-i\boldsymbol{k}_3\cdot\boldsymbol{r}_3} \times \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3}\, e^{i\boldsymbol{q}_1\cdot\boldsymbol{s}_{12}}\, P(\boldsymbol{q}_1) \times \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3}\, e^{i\boldsymbol{q}_2\cdot\boldsymbol{s}_{23}}\, P(\boldsymbol{q}_2) \times \int \frac{\mathrm{d}^3 q_3}{(2\pi)^3}\, e^{i\boldsymbol{q}_3\cdot\boldsymbol{s}_{31}}\, P(\boldsymbol{q}_3) \tag{A24}$$

$$= \int \frac{\mathrm{d}^3 q_1}{(2\pi)^3} \int \frac{\mathrm{d}^3 q_2}{(2\pi)^3} \int \frac{\mathrm{d}^3 q_3}{(2\pi)^3}\, P(\boldsymbol{q}_1)\, P(\boldsymbol{q}_2)\, P(\boldsymbol{q}_3) \times (2\pi)^3 \delta^D(\boldsymbol{k}_1 + \boldsymbol{q}_3 - \boldsymbol{q}_1) \times (2\pi)^3 \delta^D(\boldsymbol{k}_2 + \boldsymbol{q}_1 - \boldsymbol{q}_2) \times (2\pi)^3 \delta^D(\boldsymbol{k}_3 + \boldsymbol{q}_2 - \boldsymbol{q}_3) \tag{A25}$$

$$= (2\pi)^3 \delta^D(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3) \times \int \frac{\mathrm{d}^3 q_3}{(2\pi)^3}\, P(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{q}_3)\, P(\boldsymbol{k}_2 + \boldsymbol{q}_3)\, P(\boldsymbol{q}_3) \tag{A26}$$

The bispectrum multipoles are given by first putting the multipole on the first delta, then binning in k_i shells. We get

$$\begin{aligned} B_\ell^{\rm meas}(k_1,k_2,k_3) &= \frac{1}{(2\pi)^3 N_{\rm tri}^{123}} \int_{k_1} \mathrm{d}^3 k_1' \int_{k_2} \mathrm{d}^3 k_2' \int_{k_3} \mathrm{d}^3 k_3' \, \langle \delta_\ell(\boldsymbol{k}_1')\, \delta(\boldsymbol{k}_2')\, \delta(\boldsymbol{k}_3')\rangle \\ &= \frac{1}{(2\pi)^3 N_{\rm tri}^{123}} \int_{k_1} \mathrm{d}^3 k_1' \int_{k_2} \mathrm{d}^3 k_2' \int_{k_3} \mathrm{d}^3 k_3' \int \mathrm{d}^3 x \, e^{-i\boldsymbol{k}_1'\cdot\boldsymbol{x}}\, \mathcal{L}_\ell(\hat{\boldsymbol{k}}_1'\cdot\hat{\boldsymbol{x}}) \int \frac{\mathrm{d}^3 q_1'}{(2\pi)^3}\, e^{i\boldsymbol{q}_1'\cdot\boldsymbol{x}}\, \langle \delta(\boldsymbol{q}_1')\, \delta(\boldsymbol{k}_2')\, \delta(\boldsymbol{k}_3')\rangle \end{aligned} \tag{A27}$$

Appendix B: Power Spectrum Multipoles with FoG

Here we aim to incorporate the FoG effect in the multipoles for the plane parallel limit. Also, see **?**. Our model for the power spectrum is

$$P(k,\mu) = b^2(1+\beta\mu^2)^2\, e^{-\sigma_u^2 k^2 \mu^2}\, P(k)\,, \tag{B1}$$

where $\beta = f/b$, and to first order we expect

$$\sigma_u^2 = \frac{f}{3}\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \frac{P_m(k)}{k^2}\,. \tag{B2}$$

The multipole power spectrum is, then,

$$P_\ell(k) = \int_{-1}^{1} \frac{\mathrm{d}\mu}{2}\, \mathcal{L}_\ell(\mu)\, P(k,\mu) \tag{B3}$$

$$= b^2\left[I_0^\ell(x) + 2\beta I_2^\ell(x) + \beta^2 I_4^\ell(x)\right] P(k)\,, \tag{B4}$$

where we defined $x = \sigma_u k$ and

$$I_n^\ell(x) = \int_{-1}^{1} \frac{\mathrm{d}\mu}{2}\, \mathcal{L}_\ell(\mu)\, \mu^n\, e^{-x^2\mu^2}\,. \tag{B5}$$

Appendix C: NBodyKit

(This section very incomplete!) NBodyKit does the following in its LogNormalCatalog class. Note, that this account is likely incomplete. First, generate white noise density contrast and displacement fields

$$\delta_G(\boldsymbol{k}) = \sqrt{\frac{P(k)}{2}}\,\left[n_1(\boldsymbol{k}) + i\, n_2(\boldsymbol{k})\right], \tag{C1}$$

$$\boldsymbol{\Psi}(\boldsymbol{k}) = i\,\frac{\boldsymbol{k}}{k^2}\,\delta_G(\boldsymbol{k})\,, \tag{C2}$$

where $n_i(\boldsymbol{k})$ are sampled from a normal distribution with unit variance. Convert to real space,

$$\delta_G(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,\delta_G(\boldsymbol{k})\,, \tag{C3}$$

$$\boldsymbol{\Psi}(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,\boldsymbol{\Psi}(\boldsymbol{k})\,. \tag{C4}$$

Next, we ensure that the density contrast is within $-1 \le \delta(\boldsymbol{x}) < \infty$. We start with the Lagrangian bias

$$b_L = b - 1\,, \tag{C5}$$

and we perform a lognormal transform

$$1 + \delta(\boldsymbol{q}) = \frac{1}{A}\, e^{b_L \delta_G(\boldsymbol{x})}\,, \tag{C6}$$

where A is chosen such that

$$\langle 1 + \delta(\boldsymbol{q})\rangle = 1\,. \tag{C7}$$

We can derive it to get

$$\begin{aligned} A = \left\langle e^{b_L\delta_G(\boldsymbol{q})}\right\rangle &= \sum_m \frac{\langle [b_L\delta_G(\boldsymbol{q})]^m\rangle}{m!} \\ &= \sum_n \frac{(2n-1)!!\,\left\langle [b_L\delta_G(\boldsymbol{q})]^2\right\rangle^n}{(2n)!} \\ &= \sum_n \frac{(2n-1)(2n-3)\cdots 1\,\left\langle [b_L\delta_G(\boldsymbol{q})]^2\right\rangle^n}{(2n)(2n-1)(2n-2)\cdots 1} \\ &= \sum_n \frac{\left\langle [b_L\delta_G(\boldsymbol{q})]^2\right\rangle^n}{(2n)(2n-2)\cdots 1} = \sum_n \frac{\left\langle [b_L\delta_G(\boldsymbol{q})]^2\right\rangle^n}{2^n(n)(n-1)\cdots 1} \\ &= \sum_n \frac{\left\langle \frac{1}{2}[b_L\delta_G(\boldsymbol{q})]^2\right\rangle^n}{n!} = \exp\!\left(\frac{1}{2}\, b_L^2\, \left\langle \delta_G^2(\boldsymbol{q})\right\rangle\right). \end{aligned} \tag{C8}$$

Then we draw Poisson points at each Lagrangian position $\boldsymbol{q}$ with mean $[1+\delta(\boldsymbol{q})]\,\bar{n}$. That is, the number of galaxies within a cell centered at $\boldsymbol{q}$ is

$$N(\boldsymbol{q}) \sim \mathrm{Poisson}\left([1+\delta(\boldsymbol{q})]\,\bar{n}\right). \tag{C9}$$

Within each cell the galaxies are shifted by a uniform random shift. That gives the number of particles at a given Lagrangian position $\boldsymbol{q}_i$ with displacement $\boldsymbol{\Psi}_i$. Finally, the particles are moved with the Zel'dovich displacement

$$\boldsymbol{x}_i = \boldsymbol{q}_i + \boldsymbol{\Psi}_i\,, \tag{C10}$$

and the velocity is

$$\boldsymbol{v}_i = f a H \boldsymbol{\Psi}_i\,. \tag{C11}$$

There are some differences. First, the displacement field is added to the Lagrangian position. Second, no Hankel transform of the power spectrum. The Hankel transfrom is there to ensure the power spectrum of the density field equals the input. NBodyKit does not seem to make any such guaranty. For small power spectrum amplitudes that shouldn't matter.

Appendix D: Measuring the Power Spectrum and its Multipoles

In this section we show how to measure the multipoles of the power spectrum. The presentation here largely follows ?.

1. Density Contrast

Here we will focus on the density contrast

$$\delta(\boldsymbol{x}) \equiv \frac{n(\boldsymbol{x})}{\bar{n}} - 1 \tag{D1}$$

If we assign a galaxy to its nearest grid point, then we estimate the density contrast as

$$1 + \delta^{\rm obs}(\boldsymbol{x}) = \int \mathrm{d}^3x' \, W(\boldsymbol{x} - \boldsymbol{x}') \, (1 + \delta(\boldsymbol{x}')) \tag{D2}$$

More on grid assignment in Appendix D 5 Then,

$$\delta^{\rm obs}(\boldsymbol{k}) = \widetilde{W}(\boldsymbol{k}) \, \delta(\boldsymbol{k}) \, . \tag{D3}$$

For discrete galaxies at positions $\boldsymbol{x}_g$ the density is modeled as $(1 + \delta(\boldsymbol{x}'))\,\bar{n} = \sum_g \delta^D(\boldsymbol{x}' - \boldsymbol{x}_g)$. Hence, if we assign galaxies to their nearest grid point $\boldsymbol{x}$, then the observed density contrast at $\boldsymbol{x}$ will be given by Eq. (D2) with a top-hat window function,

$$W(\boldsymbol{x} - \boldsymbol{x}') = \prod_i \frac{1}{H} \, \mathcal{T}\left(\frac{x_i - x_i'}{H}\right) , \tag{D4}$$

where the product is over the three dimensions x, y, and z, and the top-hat is defined as

$$\mathcal{T}(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2} \, , \\ \frac{1}{2} & \text{if } |x| = \frac{1}{2} \, , \\ 0 & \text{otherwise.} \end{cases} \tag{D5}$$

Then,

$$\begin{aligned} \widetilde{W}(\boldsymbol{k}) &= \prod_i \int_{-1/2}^{1/2} \mathrm{d}u_i \, e^{-iHk_iu_i} \\ &= \prod_i j_0\left(\frac{Hk_i}{2}\right) \\ &= \prod_i j_0\left(\frac{L_ik_i}{2n_i}\right) \end{aligned} \tag{D6}$$

Including the alias effect, we have $\widetilde{W}(\boldsymbol{k}) = 1$ for NGP.

2. Discrete Fourier Transform

We definet the Fourier transform as

$$\begin{aligned} \delta(k) &= \int \mathrm{d}x \, e^{-ikx} \, \delta(x) \\ &= \frac{L}{N} \sum_n e^{-iLkn/N} \, \delta(x_n) \\ &= \frac{L}{N} \, \mathrm{FFT}\left[\delta(x)\right](k) \end{aligned} \tag{D7}$$

with $k_F = 2\pi/L$. and the inverse

$$\begin{aligned} \delta(x) &= \int \frac{\mathrm{d}k}{2\pi} \, e^{ikx} \, \delta(k) \\ &= \frac{1}{L} \frac{N}{N} \sum_k e^{ikx} \, \delta(k) \\ &= \frac{N}{L} \, \mathrm{IFFT}\left[\delta(k)\right](x) \end{aligned} \tag{D8}$$

3. Measuring the full Power Spectrum

The power spectrum is

$$\begin{aligned} \hat{P}(k, \mu, \phi) = {} & W^{-1}_{\rm mesh,a}(\boldsymbol{k}) \, W^{-1}_{\rm mesh,b}(\boldsymbol{k}) \\ & \times \left[\Re\left(\delta_a^*(\boldsymbol{k})\delta_b(\boldsymbol{k})\right) - P_{\rm shot,ab} \right] , \end{aligned} \tag{D9}$$

where

$$W_{\rm mesh} = \left[j_0\left(\frac{k_xH}{2}\right) j_0\left(\frac{k_yH}{2}\right) j_0\left(\frac{k_zH}{2}\right) \right]^p , \tag{D10}$$

where $j_0(x) = \sin(x)/x$ is the spherical Bessel function. For NGP, $p = 1$, for CIC, $p = 2$, for TSC, $p = 3$. The shot noise is

$$P_{\rm shot,ab} = \frac{\delta^K_{ab}}{\bar{n}_a} \, . \tag{D11}$$

4. Measuring Multipoles of the Power Spectrum

The multipole power spectrum $P_\ell(k)$ is defined s.t.

$$P(k, \mu) = \sum_{l=0}^{\infty} P_l(k) \mathcal{L}_l(\mu) \, , \tag{D12}$$

where $\mathcal{L}_m(\mu)$ are the Legendre polynomials. The Legendre polynomials obey the orthogonality relation

$$\int_{-1}^{1} \mathrm{d}\mu \, \mathcal{L}_m(\mu) \, \mathcal{L}_n(\mu) = \frac{2}{2m+1} \, \delta^K_{mn} \, . \tag{D13}$$

This allows us to invert Eq. (D12). When measuring the power spectrum from data on a grid, however, we must discretize the integral. Then, as a first estimate for the multipoles, we obtain

$$\hat{P}_m(k) = \sum_{\mu,\phi} \hat{P}(k,\mu,\phi)\,\mathcal{L}_m(\mu) = \sum_{l=0}^{\infty} P_l(k)\,\mathcal{M}_{lm}(k)\,, \tag{D14}$$

where we assume that the power spectrum is independent of ϕ and

$$\mathcal{M}_{lm}(k) = \sum_{\mu,\phi} \mathcal{L}_l(\mu)\,\mathcal{L}_m(\mu) \tag{D15}$$

is the μ-leakage matrix. The k-dependence comes from the sum over μ, which depends on the grid of the data. Inverting Eq. (D14), we get a precise estimate for the multipole moments

$$P_l(k) = \sum_{m=0}^{l_{\max}} \hat{P}_m(k)\,\mathcal{M}^{-1}_{lm}(k)\,. \tag{D16}$$

The sum over m should go from zero to infinity. However, if there are only $N_k = \mathcal{M}_{00}$ modes, then we can only measure up to $l = N_k - 1$. For larger multipoles, $\mathcal{M}$ will be singular and not invertible. Furthermore, we only consider multipoles up to some $l_{\max}$. Thus, for a given mode, the μ-leakage matrix $\mathcal{M}_{lm}$ will only consist of entries $l, m \le \min(N_k - 1, l_{\max})$.

Due to the finite width of the k-bins, we must estimate the mean k at which the multipole moment is measured. We do this by taking the average k within the bin, giving each mode the same weight. That is

$$k = \frac{1}{N_k} \sum_{\mu,\phi} k_i\,, \tag{D17}$$

where the sum goes over all the modes k_i within the k-bin, and $N_k = \mathcal{M}_{00}$ is the number of modes.

5. Grid Assignment

Here we detail the grid assignment schemes NGP, CiC, TSC, PSC. The key question is how to estimate the number density $n(\boldsymbol{r})$ on a grid from discrete points.

Following **??**, each galaxy is modeled as having a shape $S(\boldsymbol{x} - \boldsymbol{x}_i)$. Whichever portion of that shape falls within a given grid cell is then the weight assigned to that grid cell.

The shape function for a galaxy at position $\boldsymbol{x}_i$ considered are

$$S_{\mathrm{NGP}}(\boldsymbol{x} - \boldsymbol{x}_i) = \delta^D\left(\frac{\boldsymbol{x} - \boldsymbol{x}_i}{H}\right)\,, \tag{D18}$$

$$S_{\mathrm{CIC}}(\boldsymbol{x} - \boldsymbol{x}_i) = [\delta^D \star \mathcal{T}^3]\left(\frac{\boldsymbol{x} - \boldsymbol{x}_i}{H}\right)\,, \tag{D19}$$

$$S_{\mathrm{TSC}}(\boldsymbol{x} - \boldsymbol{x}_i) = [\delta^D \star \mathcal{T}^3 \star \mathcal{T}^3]\left(\frac{\boldsymbol{x} - \boldsymbol{x}_i}{H}\right)\,, \tag{D20}$$

$$S_{\mathrm{PSC}}(\boldsymbol{x} - \boldsymbol{x}_i) = [\delta^D \star \mathcal{T}^3 \star \mathcal{T}^3 \star \mathcal{T}^3]\left(\frac{\boldsymbol{x} - \boldsymbol{x}_i}{H}\right)\,, \tag{D21}$$

where $\star$ denotes a convolution. Explicitly,

$$S_{\mathrm{CIC}}(\boldsymbol{x} - \boldsymbol{x}_i) = \int \frac{\mathrm{d}^3 r}{H^3}\,\delta^D\left(\frac{\boldsymbol{x} - \boldsymbol{x}_i - \boldsymbol{r}}{H}\right)\mathcal{T}^3\left(\frac{\boldsymbol{r}}{H}\right) \tag{D22}$$

$$= \mathcal{T}^3\left(\frac{\boldsymbol{x} - \boldsymbol{x}_i}{H}\right)\,. \tag{D23}$$

Or,

$$S_{\mathrm{TSC}}(\boldsymbol{x} - \boldsymbol{x}_i) = \int \frac{\mathrm{d}^3 r}{H^3}\,\mathcal{T}^3(\boldsymbol{x} - \boldsymbol{x}_i - \boldsymbol{r})\mathcal{T}^3\left(\frac{\boldsymbol{r}}{H}\right). \tag{D24}$$

And

$$
\begin{aligned}
S_{\mathrm{PSC}}(\boldsymbol{x}-\boldsymbol{x}_i) &= \int \mathrm{d}^3r'\, S_{\mathrm{TSC}}(\boldsymbol{x}-\boldsymbol{x}_i-\boldsymbol{r}')\mathcal{T}^3(\boldsymbol{r}') && \text{(D25)}\\
&= \int \mathrm{d}^3r' \int \mathrm{d}^3r \mathcal{T}^3(\boldsymbol{x}-\boldsymbol{x}_i-\boldsymbol{r}'-\boldsymbol{r})\mathcal{T}^3(\boldsymbol{r})\mathcal{T}^3(\boldsymbol{r}') && \text{(D26)}\\
&= \int \mathrm{d}^3r''\, \mathcal{T}^3(\boldsymbol{r}'') \int \mathrm{d}^3r'\, \mathcal{T}^3(\boldsymbol{r}') \int \mathrm{d}^3r\, \mathcal{T}^3(\boldsymbol{r})\, \delta^D(\boldsymbol{x}-\boldsymbol{x}_i-\boldsymbol{r}-\boldsymbol{r}'-\boldsymbol{r}'') && \text{(D27)}\\
&= \prod_{\mu\in\{x,y,z\}} \int \mathrm{d}r''_\mu\, \mathcal{T}(r''_\mu) \int \mathrm{d}r'_\mu\, \mathcal{T}(r'_\mu) \int \mathrm{d}r_\mu\, \mathcal{T}(r_\mu)\, \delta^D(x_\mu - x^i_\mu - r_\mu - r'_\mu - r''_\mu) && \text{(D28)}\\
&= \prod_{\mu\in\{x,y,z\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}r''_\mu \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}r'_\mu \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}r_\mu\, \delta^D(x_\mu - x^i_\mu - r_\mu - r'_\mu - r''_\mu) && \text{(D29)}\\
&= \prod_{\mu\in\{x,y,z\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}r'_\mu \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}r_\mu \begin{cases} 1 & \text{for } -\frac{1}{2} < x_\mu - x^i_\mu - r_\mu - r'_\mu < \frac{1}{2}\,, \\ 0 & \text{otherwise.} \end{cases} && \text{(D30)}\\
&= \prod_{\mu\in\{x,y,z\}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}r'_\mu \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}r_\mu \begin{cases} 1 & \text{for } -\frac{1}{2} < x_\mu - x^i_\mu - r_\mu - r'_\mu < \frac{1}{2}\,, \\ 0 & \text{otherwise.} \end{cases} && \text{(D31)}
\end{aligned}
$$

Nah, there's got to be a better way. The Fourier transform of the top-hat is

$$
\widetilde{\mathcal{T}}(k_\mu) = \frac{\sin\left(\frac{Hk_\mu}{2}\right)}{\frac{Hk_\mu}{2}}. \tag{D32}
$$

Since the shape function is a convolution of several top-hats,

$$
S_p(\boldsymbol{x}-\boldsymbol{x}_i) = [\star\mathcal{T}]^{p-1}(\boldsymbol{x}-\boldsymbol{x}_i)\,, \tag{D33}
$$

by the convolution theorem we get in Fourier space

$$
\widetilde{S}_p(\boldsymbol{k}) = \widetilde{\mathcal{T}}^{p-1}(\boldsymbol{k})\,. \tag{D34}
$$

And

$$
\begin{aligned}
S_p(\boldsymbol{x}-\boldsymbol{x}_i) &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}_i)}\, \widetilde{S}_p(\boldsymbol{k}) && \text{(D35)}\\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}_i)}\, \widetilde{\mathcal{T}}^p(\boldsymbol{k}) && \text{(D36)}
\end{aligned}
$$

In the x-direction, substituting $\kappa = Hk_x/2$,

$$
\begin{aligned}
S_p(x-x_i) &= \int \frac{\mathrm{d}k_x}{2\pi}\, e^{ik_x(x-x_i)}\, \widetilde{\mathcal{T}}^p(k_x) && \text{(D37)}\\
&= \frac{2}{H} \int \frac{\mathrm{d}\kappa}{2\pi}\, e^{i\kappa\frac{2}{H}(x-x_i)} \left[\frac{\sin(\kappa)}{\kappa}\right]^p. && \text{(D38)}
\end{aligned}
$$

6. Assignment results

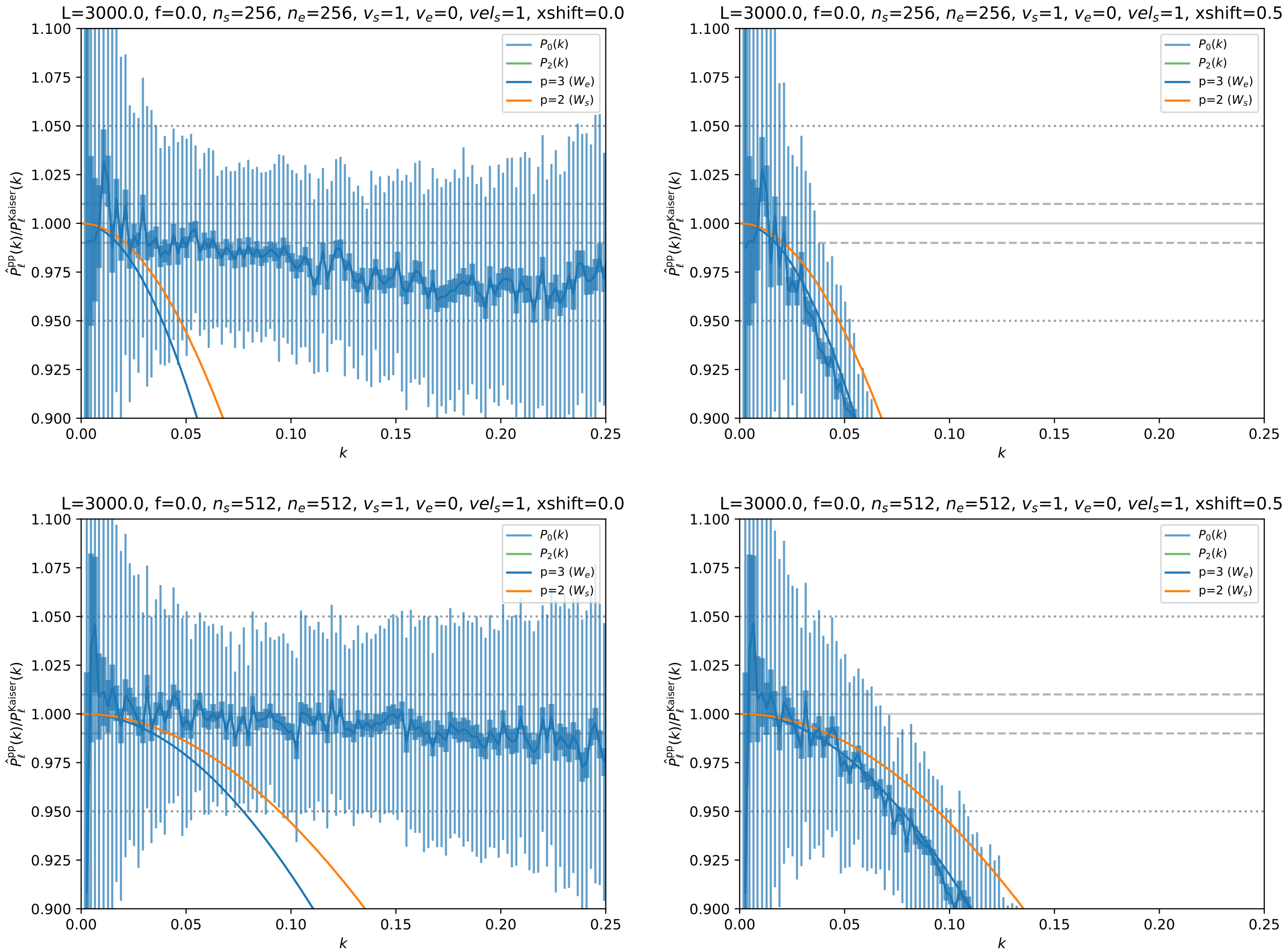

FIG. 2. Real space, sim:NGP, est:NGP. Top: Nmesh=256. Bottom: Nmesh=512

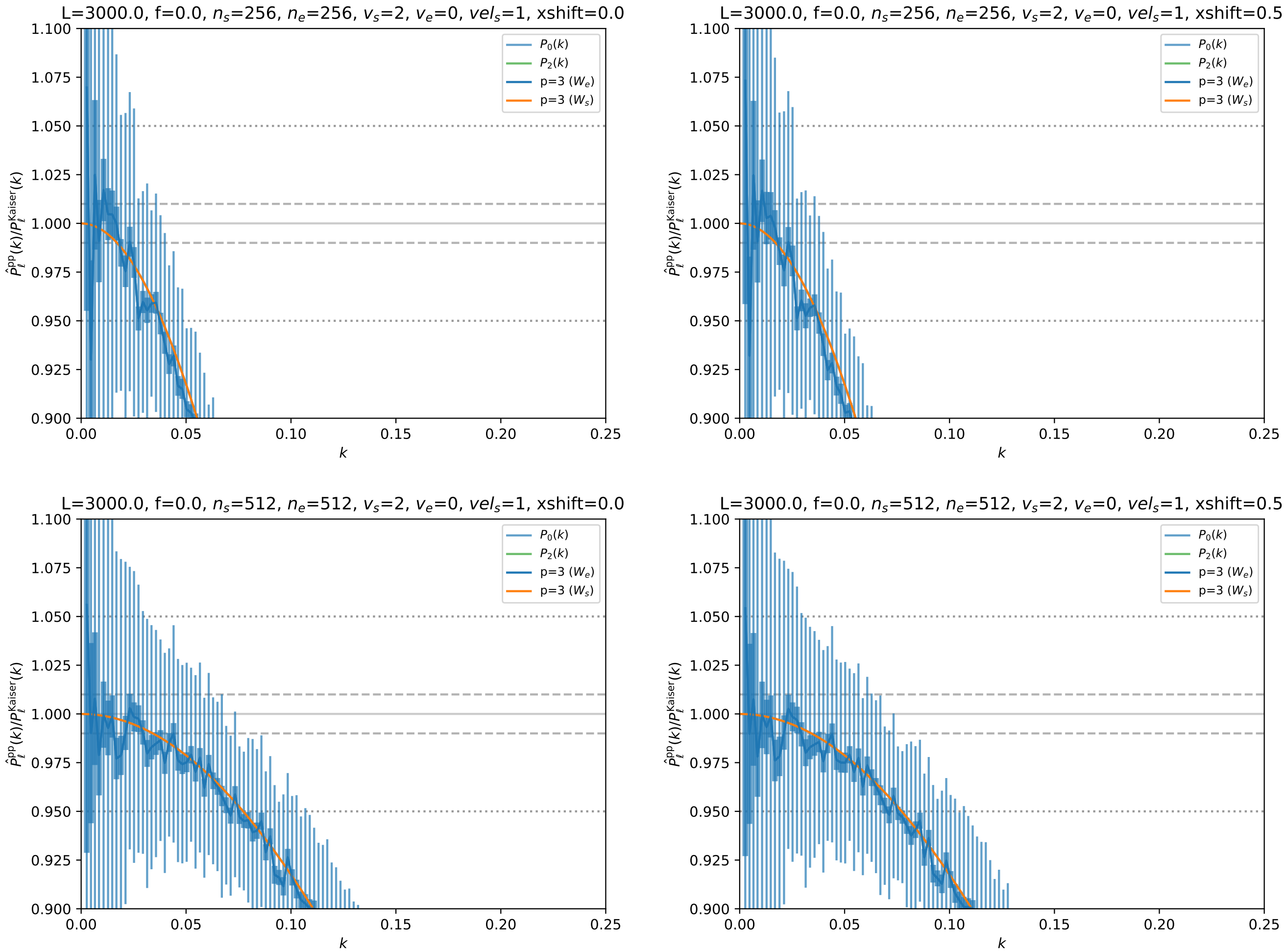

FIG. 3. Real space, sim:CIC, est:NGP. Top: Nmesh=256. Bottom: Nmesh=512.

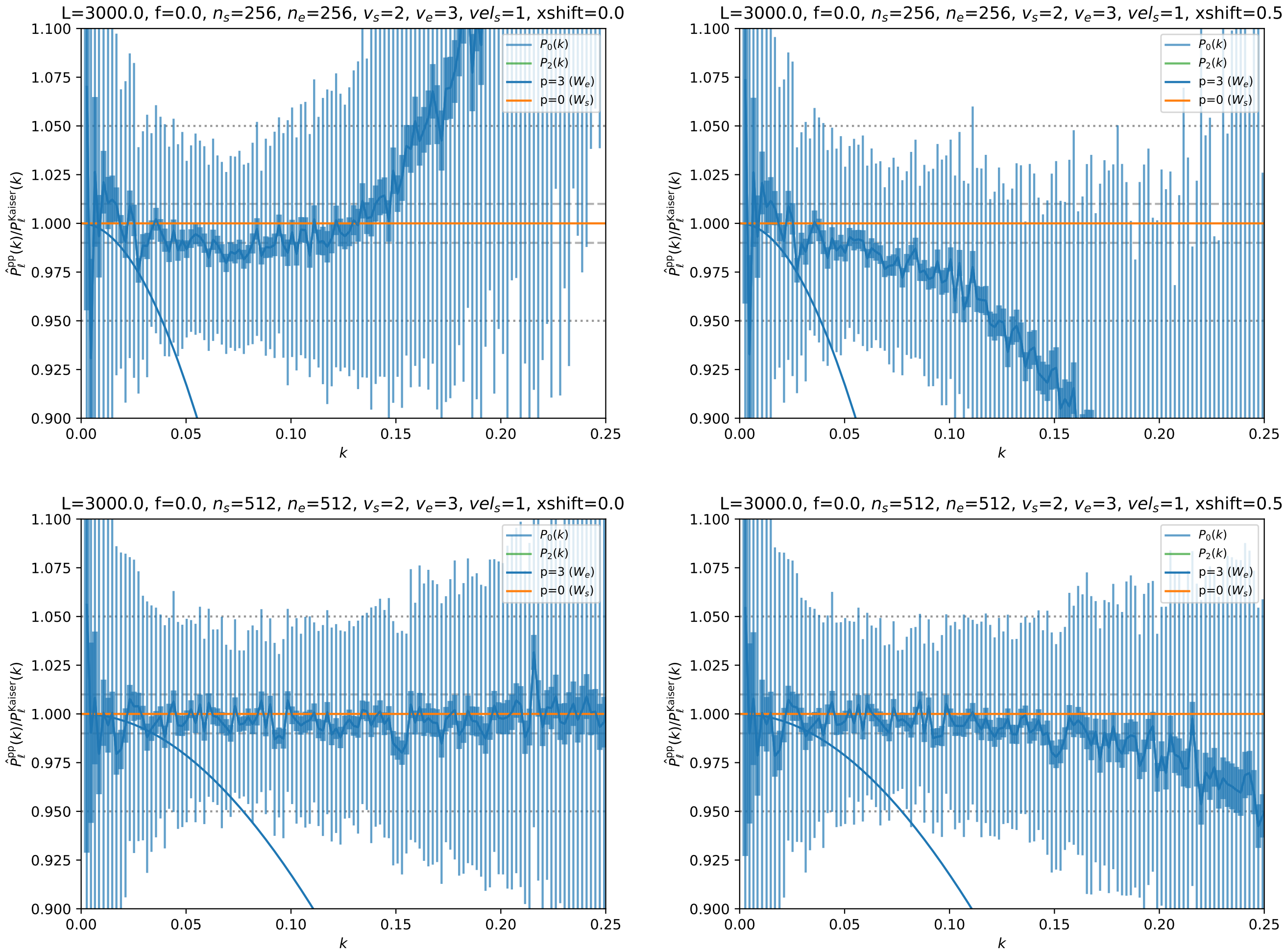

FIG. 4. Real space, sim:CIC, est:NGP. Top: Nmesh=256. Bottom: Nmesh=512, compensated.

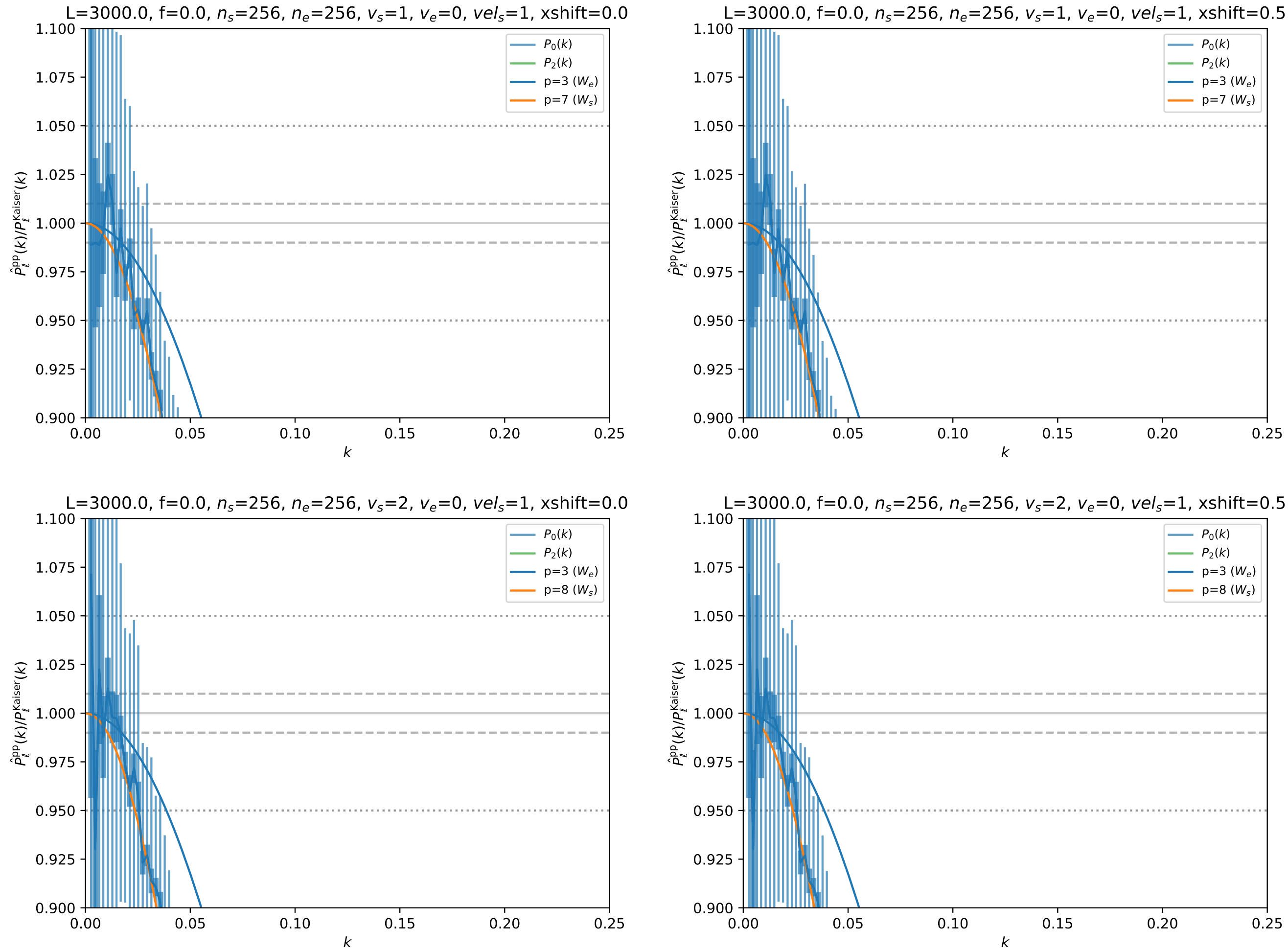

FIG. 5. Real space, Top: sim:NGP est:CIC. Bottom: sim:CIC est:CIC. Nmesh=256.

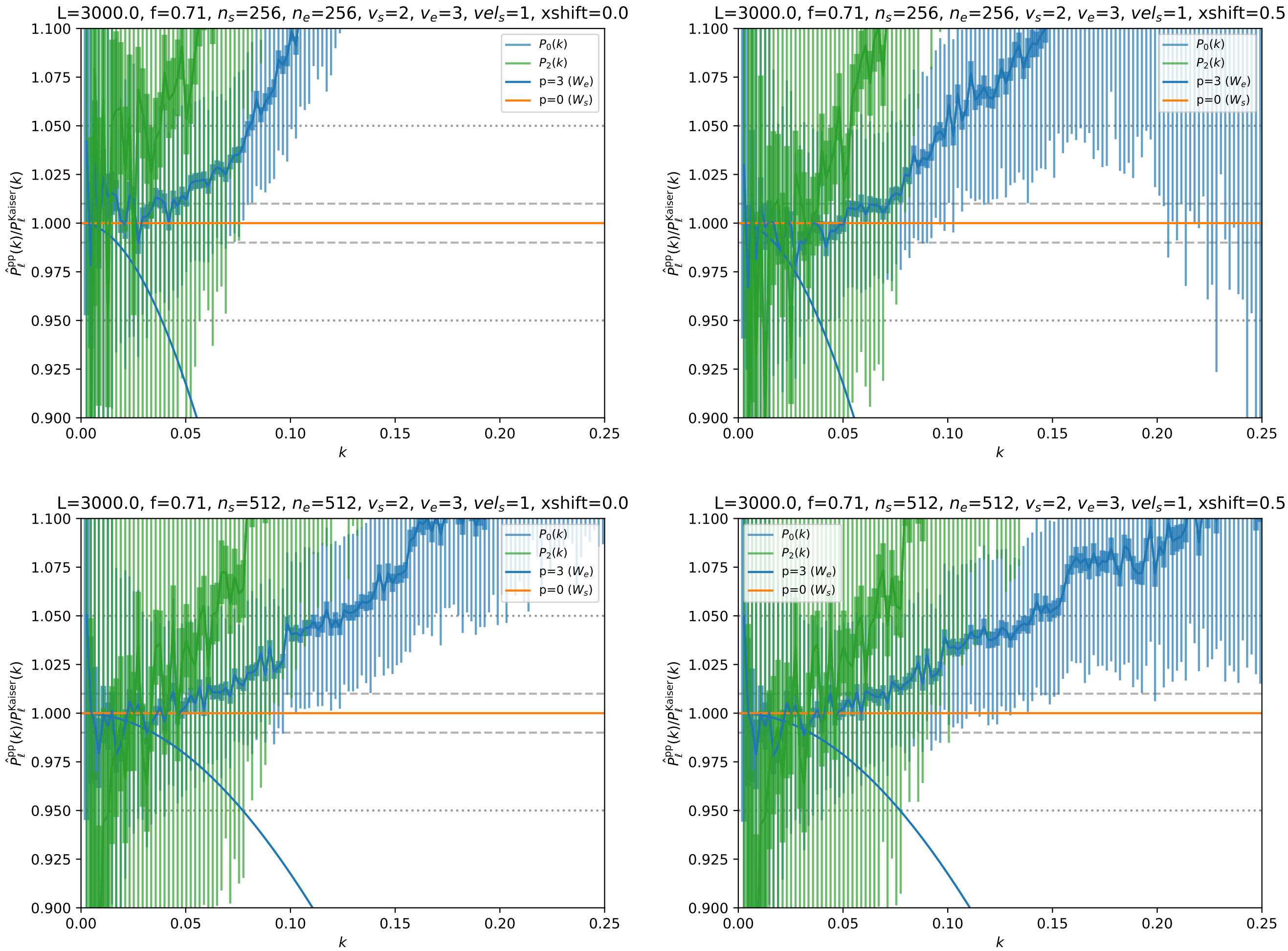

FIG. 6. Redshift space, sim:CIC, est:NGP, velocity 1.

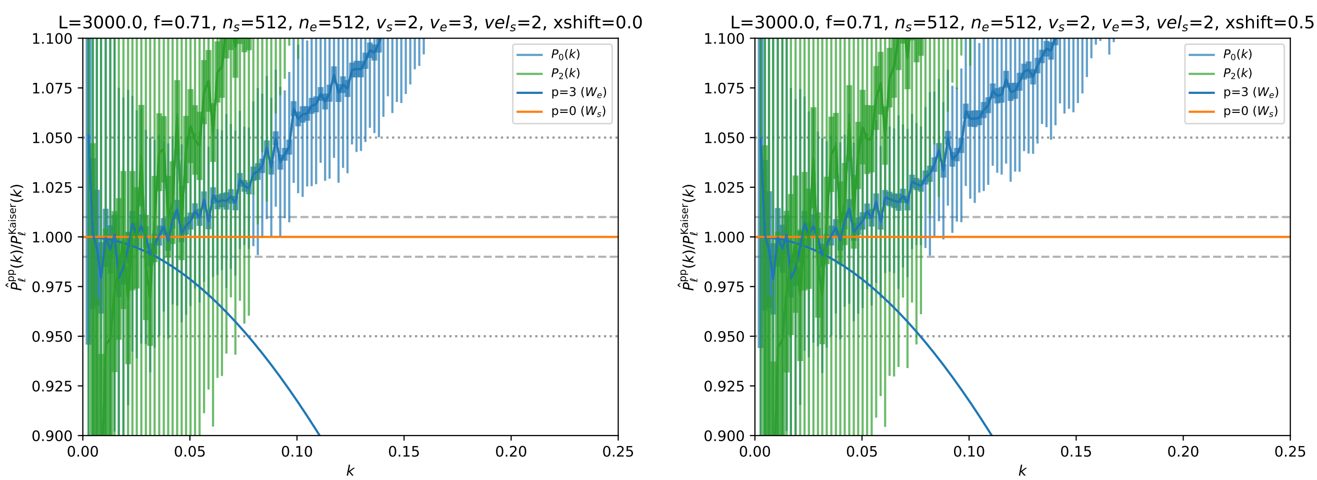

FIG. 7. Redshift space, sim:CIC, est:NGP, velocity 2.

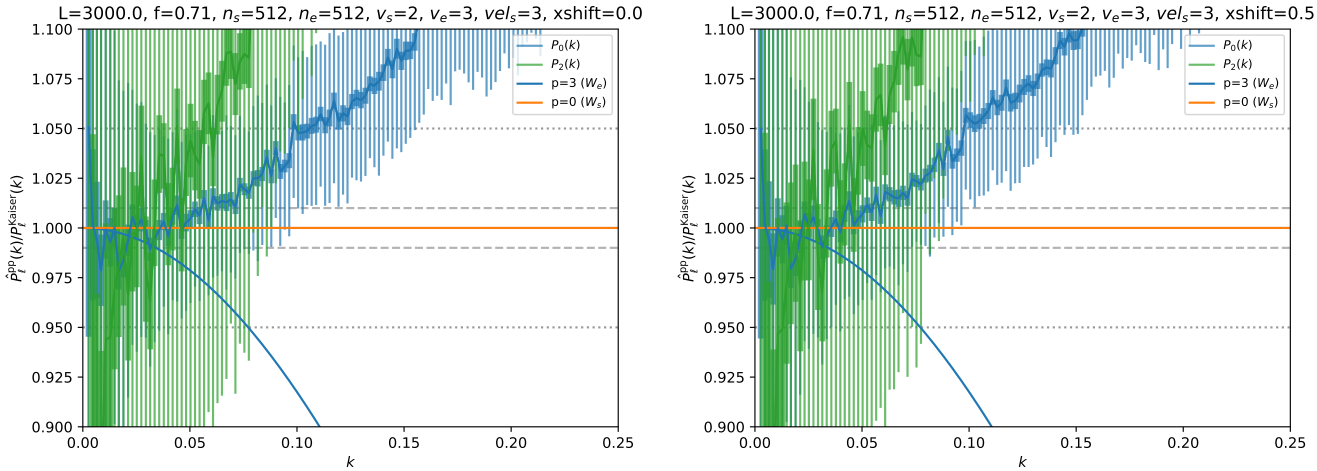

FIG. 8. Redshift space, sim:CIC, est:NGP, velocity 3.

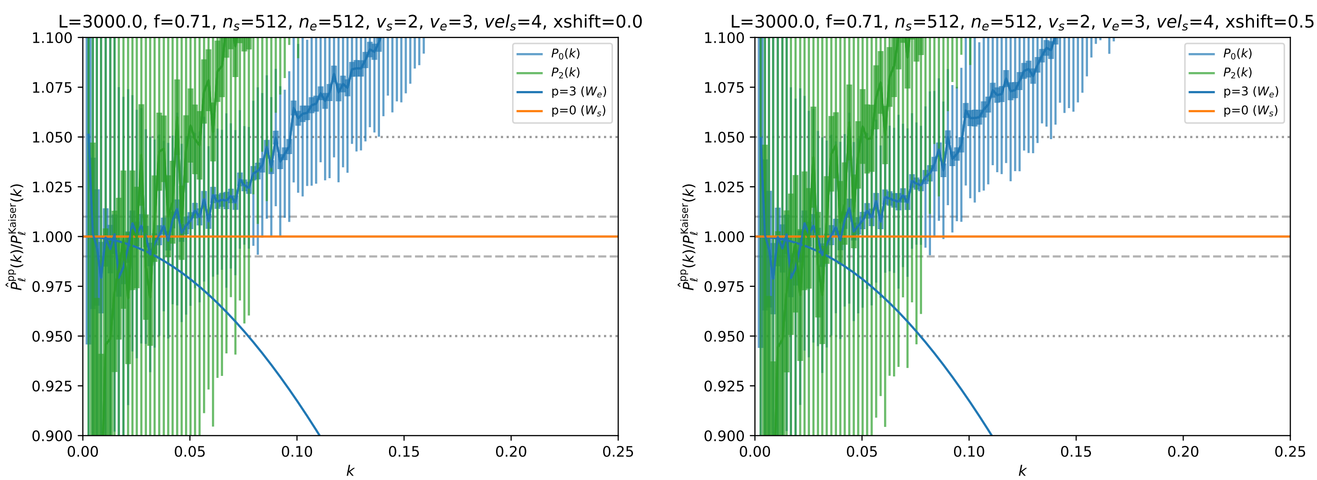

FIG. 9. Redshift space, sim:CIC, est:NGP, velocity 4.

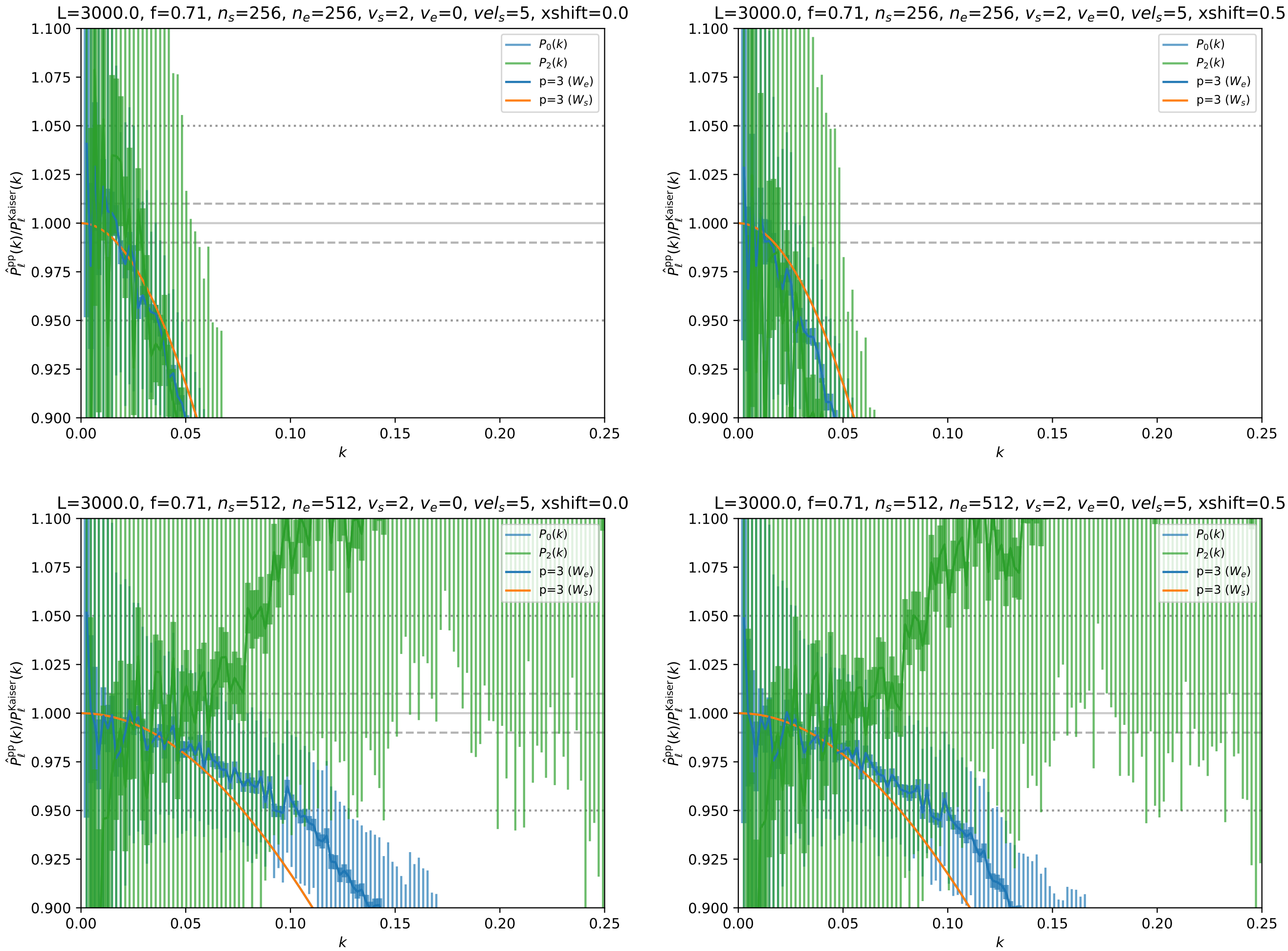

FIG. 10. Redshift space, sim:CIC, est:NGP, velocity 5.

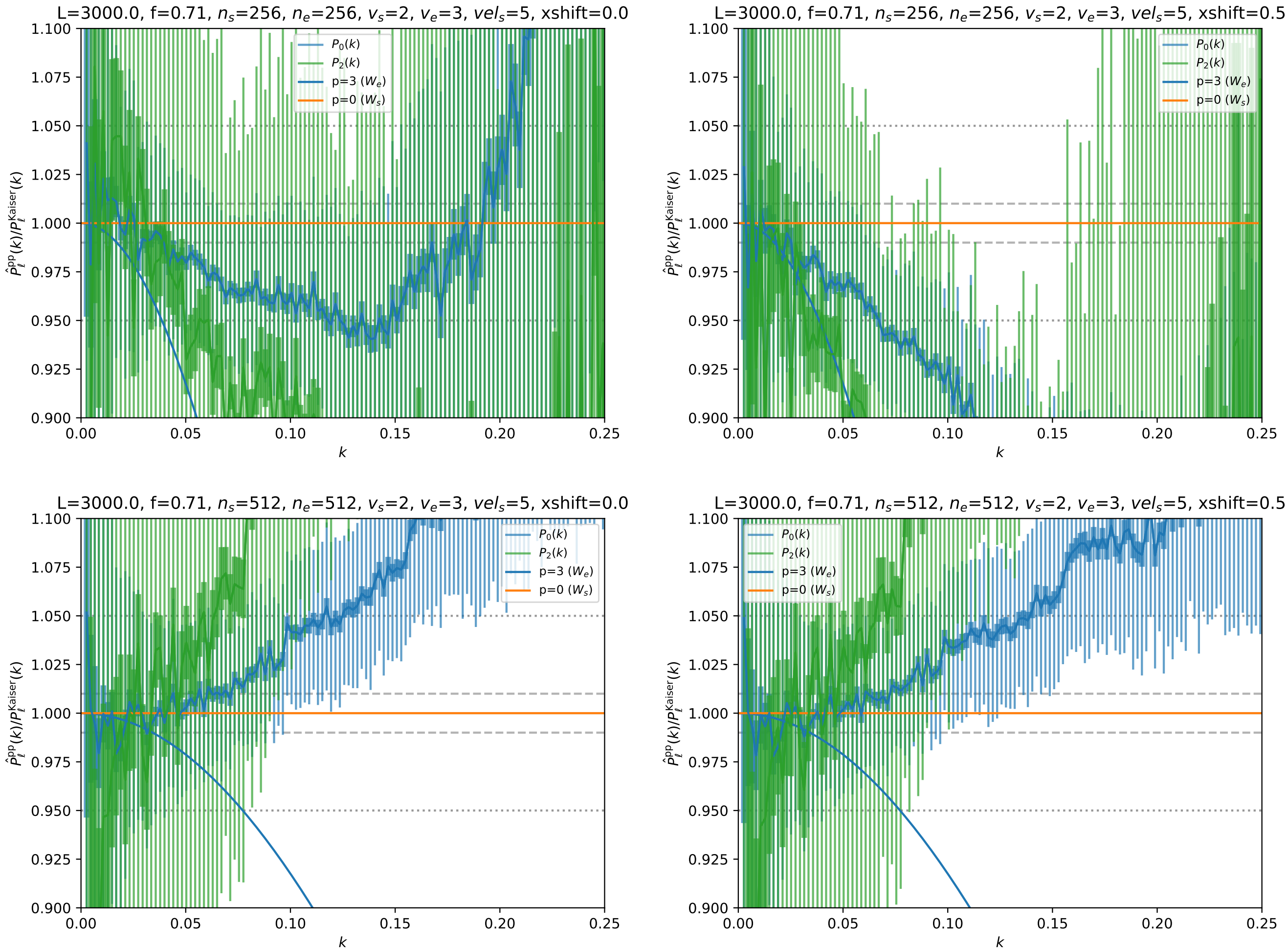

FIG. 11. Redshift space, sim:CIC, est:NGP, velocity 5, corrected.

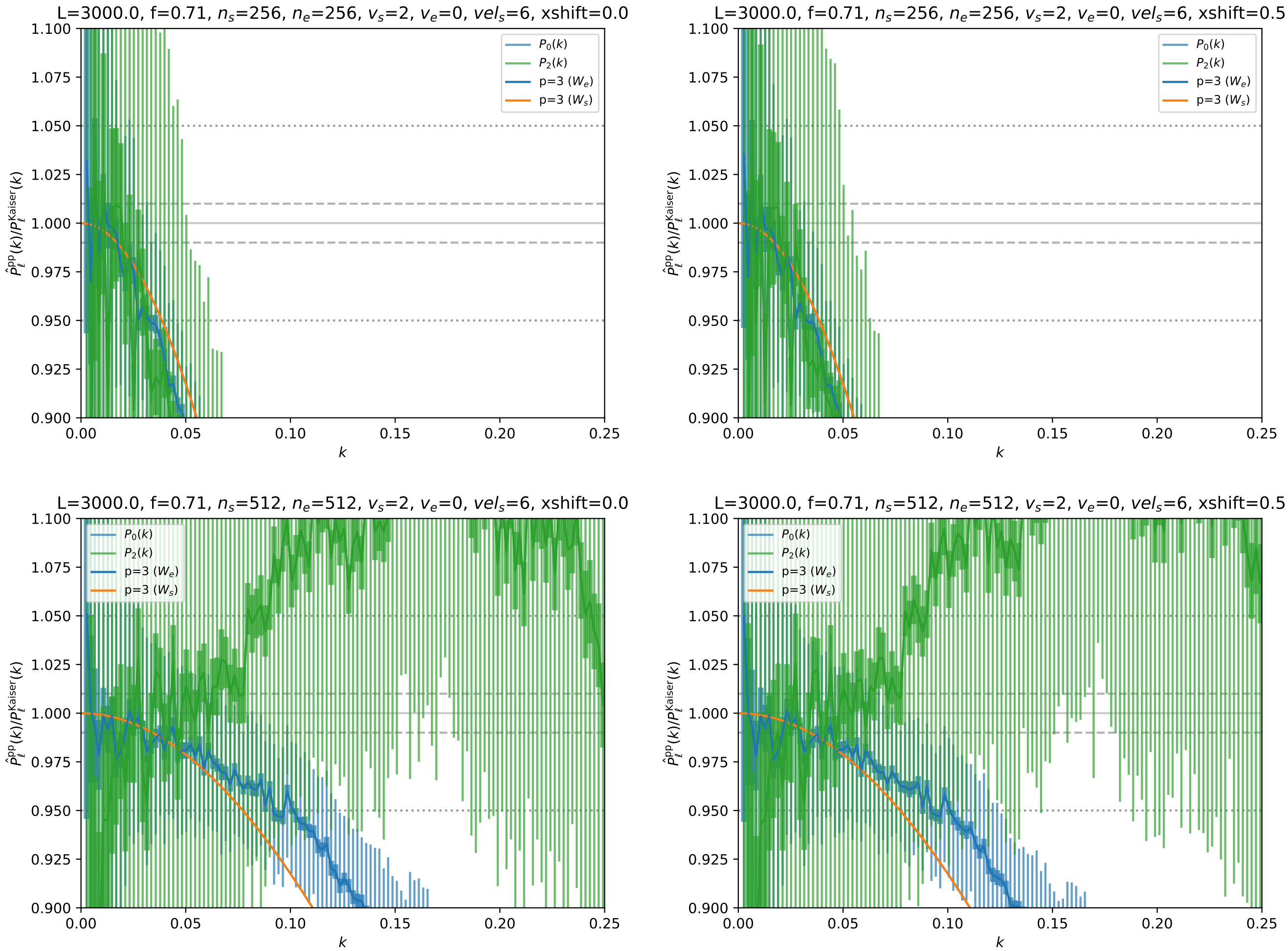

FIG. 12. Redshift space, sim:CIC, est:NGP, velocity 6.

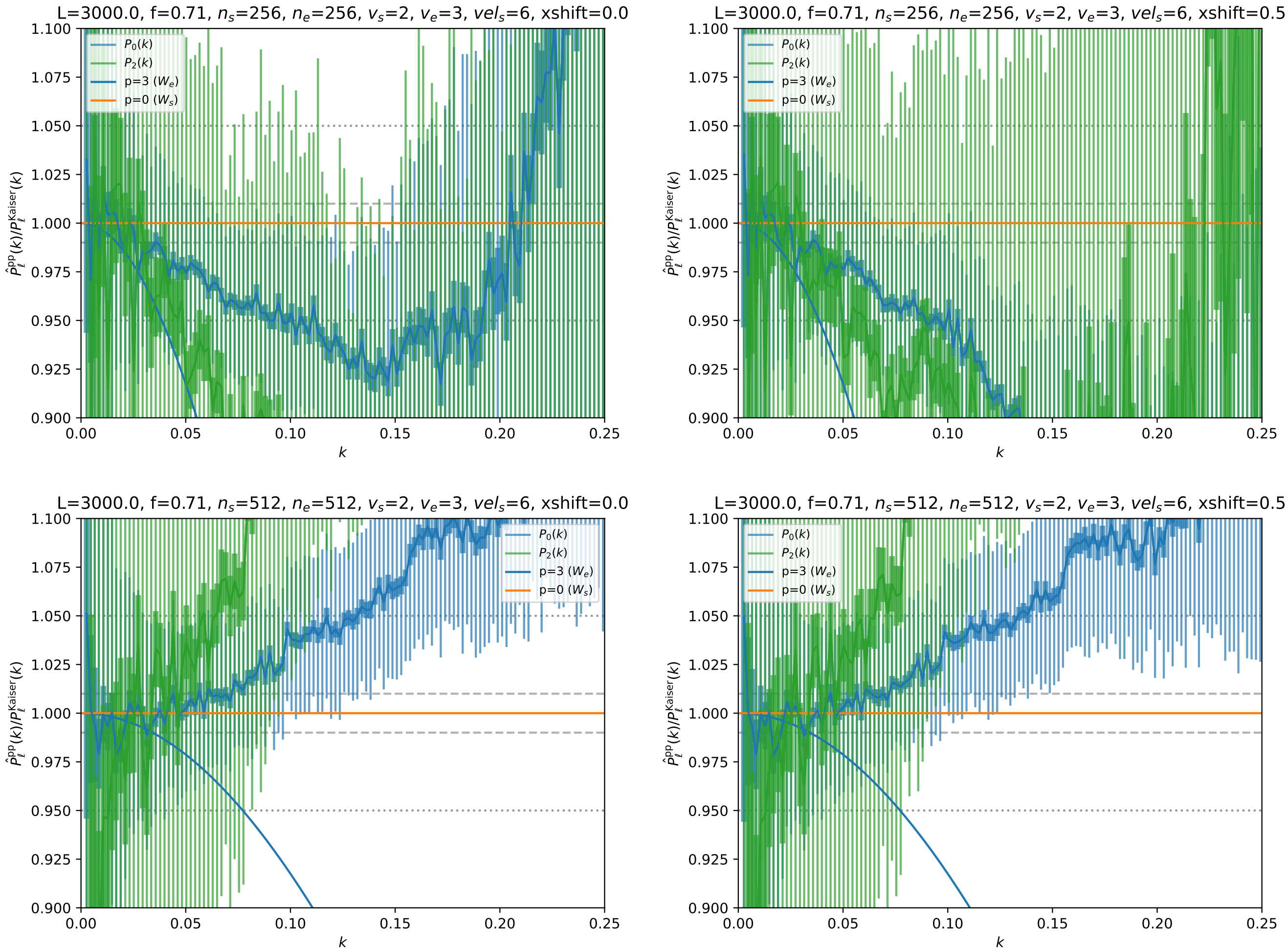

FIG. 13. Redshift space, sim:CIC, est:NGP, velocity 6, corrected.