Transient critical-level effect for internal gravity waves in a stably stratified shear flow with thermal forcing

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We have examined the transient critical-level effect for internal gravity waves produced by thermal forcing in a stably stratified shear flow. For this, we have solved analytically the equations governing small-amplitude perturbations in a two-dimensional, hydrostatic, nonrotating, inviscid, and Boussinesq flow system. In the case of the point pulse forcing, there is only one transient critical-level line at any time and every internal gravity wave passing through the critical level is attenuated by a factor of $e^{-\pi\mu}$ [$\mu=(\mathrm{Ri}-1/4)^{1/2}$, Ri: Richardson number ($=N^2/\alpha^2$, where N is the buoyancy frequency and α the vertical shear of the basic-state horizontal velocity)]. In the case of the line-type pulse forcing (bell-shaped in the horizontal), there are an infinite number of transient critical-level lines at any time. The attenuation factor for internal gravity waves is a function of space and time. This is because internal gravity waves passing through any point at any time consist of internal gravity waves which have already experienced the transient critical-level effect and those which have not experienced the effect yet.
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The critical level is a level at which the horizontal phase velocity of the wave is equal to the basic-state horizontal velocity and has important dynamical implications to geophysical flows.[1,2] Booker and Bretherton[3] demonstrated that in a linear, steady-state, Boussinesq flow, internal gravity waves passing through a critical level are attenuated by a factor of $e^{-\pi\mu}$. They also analyzed the asymptotic behavior of transient perturbation over a sinusoidal corrugation for large time and obtained the same attenuation factor of $e^{-\pi\mu}$. In our recent work, Baik *et al.*[4] examined the transient, linear dynamics of a stably stratified shear flow with thermal forcing (bell-shaped in the horizontal) in the presence of a critical level. We showed that in a linear, transient, Boussinesq flow, internal gravity waves are attenuated by an exponential factor that is a function of space and time. This raises an important question: ''What causes a spatially and temporally dependent attenuation factor?'' In this study, we try to answer this issue theoretically to further enhance our understanding of transient critical-level behavior for internal gravity waves.

The equations governing small-amplitude perturbations in a two-dimensional, hydrostatic, nonrotating, inviscid, and Boussinesq flow can be written as

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+w\frac{dU}{dz}=-\frac{\partial \phi}{\partial x}, \tag{1}$$

$$\frac{\partial \phi}{\partial z}=b, \tag{2}$$

$$\frac{\partial b}{\partial t}+U\frac{\partial b}{\partial x}+N^2 w=\frac{g}{c_p T_0}q, \tag{3}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0. \tag{4}$$

Here, u is the perturbation horizontal velocity, w the perturbation vertical velocity, ϕ the perturbation kinematic pressure, b the perturbation buoyancy, U the basic-state horizontal velocity, N the buoyancy frequency (assumed to be constant), g the gravitational acceleration, c_p the specific heat of air at constant pressure, T_0 the basic-state temperature, and q the thermal forcing. The basic-state horizontal velocity is given by $U(z)=-\alpha z$ (α; positive constant).

We first consider the line-type pulse thermal forcing given by

$$q(x,z,t)=q_0\frac{a^2}{x^2+a^2}\,\delta(z-\xi)\,\delta(t), \tag{5}$$

where q_0 is the amplitude of the thermal forcing, a the half-width of the bell-shaped function, δ the Dirac delta function, and ξ the height at which the forcing is applied. The term line-type means that the structured horizontal forcing is applied to a height. The thermal forcing is assumed to be located below $z=0(\xi<0)$. In this study, an unbounded domain in the vertical is considered for mathematical simplicity, but essential critical-level behavior as described below remains the same regardless of the domain restriction.

The equations (1)–(4) can be combined to give a single equation for the perturbation vertical velocity, which yields, after taking the Fourier transform in $x(x \to k)$ and the Laplace transform in $t(t \to s)$,

$$\frac{\partial^2 \hat{w}}{\partial z^2} + \frac{N^2 (ik)^2}{(s+ikU)^2} \hat{w} = \frac{g q_0 a (ik)^2 e^{-ak}}{c_p T_0 (s+ikU)^2} \delta(z-\xi), \tag{6}$$

where $\hat{w}$ denotes the perturbation vertical velocity in transformed space. The general solution of (6) is (hereafter, the solution for $z \leqslant \xi$ is denoted by subscript 1 and the solution for $z > \xi$ by subscript 2)

$$\hat{w}_1(k,z,s) = A_1(k,s)(s+ikU)^{1/2+i\mu} + B_1(k,s)(s+ikU)^{1/2-i\mu}, \tag{7}$$

$$\hat{w}_2(k,z,s) = A_2(k,s)(s+ikU)^{1/2+i\mu} + B_2(k,s)(s+ikU)^{1/2-i\mu}. \tag{8}$$

The four unknown coefficients A_1, B_1, A_2, and B_2 in (7) and (8) can be determined by imposing upper and lower radiation conditions,[3] and two interface conditions at $z=\xi$ that can be obtained by integrating (6) twice from $z=\xi^-$ to ξ^+,

$$\frac{\partial \hat{w}_2}{\partial z} - \frac{\partial \hat{w}_1}{\partial z} = \frac{g q_0 a (ik)^2 e^{-ak}}{c_p T_0 (s+ikU)^2}, \tag{9}$$

$$\hat{w}_2 - \hat{w}_1 = 0. \tag{10}$$

Then, the solution in transformed space can be given by

$$\hat{w}_1(k,z,s) = C_1 k e^{-ak} (s+ikU_\xi)^{-3/2-i\mu} (s+ikU)^{1/2+i\mu}, \tag{11}$$

$$\hat{w}_2(k,z,s) = C_1 k e^{-ak} (s+ikU_\xi)^{-3/2+i\mu} (s+ikU)^{1/2-i\mu}, \tag{12}$$

where $C_1 = g q_0 a/(2 c_p T_0 \alpha \mu)$ and U_ξ is the basic-state velocity at $z=\xi$. After taking the inverse Laplace transform in $s(s \to t)$ and the inverse Fourier transform in $k(k \to x)$ of (11) and (12), one can obtain the solution for the perturbation vertical velocity in physical space:

$$w_1(x,z,t) = C_1 R_\xi^{-5/2} R^{1/2} e^{-\mu(\theta-\theta_\xi)} [-3/2 \cos(\beta+S) + \mu \sin(\beta+S)] + C_1 R_\xi^{-3/2} R^{-1/2} e^{-\mu(\theta-\theta_\xi)} \times [1/2 \cos(\gamma+S) - \mu \sin(\gamma+S)], \tag{13}$$

$$w_2(x,z,t) = C_1 R_\xi^{-5/2} R^{1/2} e^{\mu(\theta-\theta_\xi)} [-3/2 \cos(\beta-S) - \mu \sin(\beta-S)] + C_1 R_\xi^{-3/2} R^{-1/2} e^{\mu(\theta-\theta_\xi)} \times [1/2 \cos(\gamma-S) + \mu \sin(\gamma-S)], \tag{14}$$

where

$$R_\xi = (a^2 + D_\xi^2)^{1/2}, \quad R = (a^2 + D^2)^{1/2},$$

$$D_\xi = x - U_\xi t, \quad D = x - Ut,$$

$$\theta_\xi = \tan^{-1}(-D_\xi/a), \quad \theta = \tan^{-1}(-D/a),$$

$$\beta = 1/2\theta - 5/2\theta_\xi, \quad \gamma = -1/2\theta - 3/2\theta_\xi,$$

$$S = \mu \ln(R/R_\xi).$$

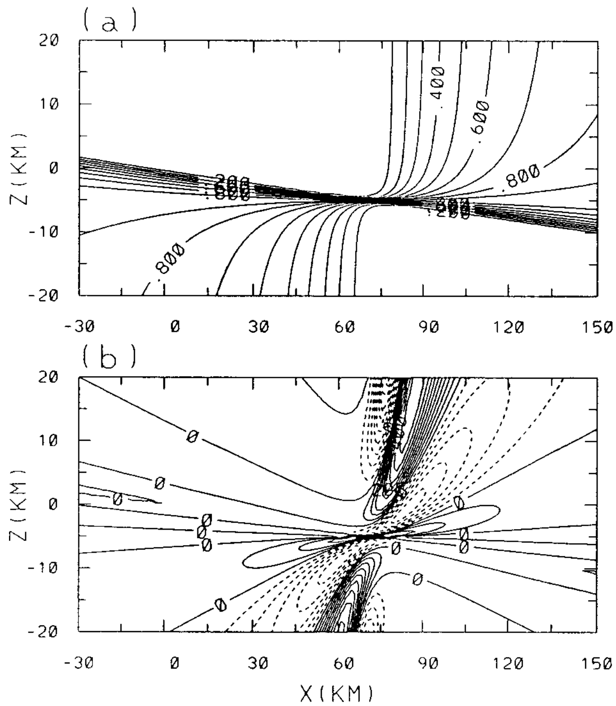

FIG. 1. The fields of (a) the attenuation factor and (b) the perturbation vertical velocity at $t=1$ h calculated in the case of the line-type pulse heating. The parameters specified are $N=0.01\ \mathrm{s^{-1}}$, $T_0=273$ K, $a=10$ km, $\alpha = 4\times10^{-3}\ \mathrm{s^{-1}}$(20 m s^{-1}/5 km), $\xi=-5$ km, and $q_0=3\ \mathrm{J\ kg^{-1}\ s^{-1}}$. The contour interval in (a) is 0.05. The values in (b) are scaled by 10^9 and the actual contour interval is $5\times10^{-8}\ \mathrm{m\ s^{-1}}$.

The above solution in the case of the line-type pulse forcing indicates that the attenuation factor ($e^{-\mu(\theta-\theta_\xi)}$ and $e^{\mu(\theta-\theta_\xi)}$) for internal gravity waves is a function of space and time. Consider a special case of $t \to \infty$ (steady-state limit) for $|x| < \infty$. When $z < z_c$ (prescribed critical level, $z=0$ in this study), $D_\xi \to -\infty$ and $D \to -\infty$, and accordingly $\theta_\xi \to \pi/2$ and $\theta \to \pi/2$. Therefore, $e^{-\mu(\theta-\theta_\xi)} \to 1$ and $e^{\mu(\theta-\theta_\xi)} \to 1$. When $z > z_c$, $D_\xi \to -\infty$ and $D \to \infty$, and accordingly $\theta_\xi \to \pi/2$ and $\theta \to -\pi/2$. Therefore, $e^{\mu(\theta-\theta_\xi)} \to e^{-\pi\mu}$. Hence, our transient solution in a steady-state limit gives the attenuation factor of $e^{-\pi\mu}$. This is consistent with previous steady-state studies.[2,5]

Figure 1 shows the fields of the attenuation factor ($e^{-\mu(\theta-\theta_\xi)}$ and $e^{\mu(\theta-\theta_\xi)}$) and the perturbation vertical velocity at $t=1$ h calculated using (13) and (14). The parameters specified are $N=0.01\ \mathrm{s^{-1}}$, $T_0=273$ K, $a=10$ km, $\alpha=4\times10^{-3}\ \mathrm{s^{-1}}$, $\xi=-5$ km, and $q_0=3\ \mathrm{J\ kg^{-1}\ s^{-1}}$. The center of the dominant moving mode travels downstream with a speed of the basic-state velocity at the level where the thermal forcing is located ($\xi=-5$ km).[4] The transient critical level for internal gravity waves originated at $x=0$ (note that the thermal forcing at $t=0$ is maximum at $x=0$) is given by $z_{ct} = -x/(\alpha t)$. Therefore, the tilt of the transient critical level decreases with time. For $t \to \infty$, z_{ct} approaches the prescribed critical level, that is, $z=0$. At $t=1$ h, the line of the transient critical level is the line connecting the point of $(x,z)=(0,0)$ and the center of the moving mode $(x,z) = (72\ \mathrm{km}, -5\ \mathrm{km})$. When the transient critical level is located above (below) the forcing level to the left (right) region

TABLE I. The parameter values for e^{λ}, β_{δ}, and γ_{δ} in (16) for each divided region for (a) $z \leqslant \xi$ and (b) $z > \xi$.

	Region	e^{λ}	Parameter β_{δ}	γ_{δ}
		(a)		
I.	$D_{\xi}>0, D>0$	1	π	π
II.	$D_{\xi}>0, D<0$	$e^{-\pi\mu}$	$\frac{3}{2}\pi$	$\frac{\pi}{2}$
III.	$D_{\xi}<0, D>0$		Not valid region	
IV.	$D_{\xi}<0, D<0$	1	$-\pi$	$-\pi$
		(b)		
V.	$D_{\xi}>0, D>0$	1	π	π
VI.	$D_{\xi}>0, D<0$		Not valid region	
VII.	$D_{\xi}<0, D>0$	$e^{-\pi\mu}$	$-\frac{3}{2}\pi$	$-\frac{\pi}{2}$
VIII.	$D_{\xi}<0, D<0$	1	$-\pi$	$-\pi$

of the center of the moving mode, the wave attenuation is clearly observed in Fig. 1(b). Figure 1(a) indicates that even when the transient critical level for internal gravity waves originated at $x=0$ is located below (above) the forcing level, upgoing (downgoing) internal gravity waves are also attenuated with an attenuation degree that increases with height.

Next, we consider the case of the point pulse forcing in which thermal forcing is given by

$$q(x,z,t) = Q_0 \delta(x) \delta(z-\xi) \delta(t). \tag{15}$$

The solution in the case of the above point pulse forcing can be obtained using the solution in the case of the line-type pulse forcing (13) and (14). A property on the delta function tells us that $a/(x^2+a^2) \rightarrow \pi\delta(x)$ as $a \rightarrow 0$. Therefore, from (5) and (15), $Q_0 = \pi q_0 a$. Taking the limit of (13) and (14) as $a \rightarrow 0$ and $q_0 \rightarrow \infty$ while Q_0 remains fixed yields the solution for the perturbation vertical velocity in a unified form of

$$\begin{aligned} w(x,z,t) = {} & C_2 |D_{\xi}|^{-5/2} |D|^{1/2} e^{\lambda} [-3/2 \cos(\beta_{\delta}+S_{\delta}) \\ & + \mu \sin(\beta_{\delta}+S_{\delta})] + C_2 |D_{\xi}|^{-3/2} |D|^{-1/2} \\ & \times e^{\lambda} [1/2 \cos(\gamma_{\delta}+S_{\delta}) - \mu \sin(\gamma_{\delta}+S_{\delta})], \end{aligned} \tag{16}$$

where $C_2 = C_1/\pi$ and $S_{\delta} = \mu \ln(|D|/|D_{\xi}|)$. The parameter values for e^{λ}, β_{δ}, and γ_{δ} for each divided region are given in Table I. This table indicates that in the case of the point pulse forcing, the attenuation factor for internal gravity waves across a critical level is not a function of space and time, but is $e^{-\pi\mu}$. Figure 2 shows a schematic diagram of each region in Table I.

Now, we are in a position of explaining the spatially and temporally dependent attenuation factor in the bell-shaped forcing in the horizontal. In the case of the point forcing $\delta(x)$, there is only one transient critical-level line at any time that passes through the point (0,0) and the center of the moving mode. Therefore, every internal gravity wave across the critical level is attenuated by a factor of $e^{-\pi\mu}$. The response to the bell-shaped forcing ($-\infty < x < \infty$) can be regarded as a sum of an infinite number of responses to point forcings $\delta(x-\zeta)$, but with a different forcing amplitude at each $x = \zeta$, where $-\infty < \zeta < \infty$. Accordingly, there are an infinite number of transient critical-level lines at any time. At $t = t_1$, consider any point (x_1, z_1) and suppose the point is located above the forcing level $z=\xi$. Above and below the point (x_1, z_1), there are an infinite number of transient critical-level lines that satisfy $z_{ct} = (x_0 - x)/\alpha t$, where x_0 is the horizontal position of wave origination. Thus, at the point (x_1, z_1), waves originated in the region of $x < x_1 + \alpha t_1 z_1$ experience the critical-level attenuation before arriving at the point (x_1, z_1), while waves originated in the region of $x > x_1 + \alpha t_1 z_1$ do not. Therefore, internal gravity waves passing through the point consist of internal gravity waves that have already experienced the transient critical-level effect and those that have not experienced the effect yet. The combined degree of these two types of internal gravity waves certainly depends on the location and time. A similar argument can be applied to any point located below the forcing level. Because of the combination of these two types of internal gravity waves, the attenuation degree for internal gravity waves in the case of the line-type pulse forcing appears to be a function of space and time.

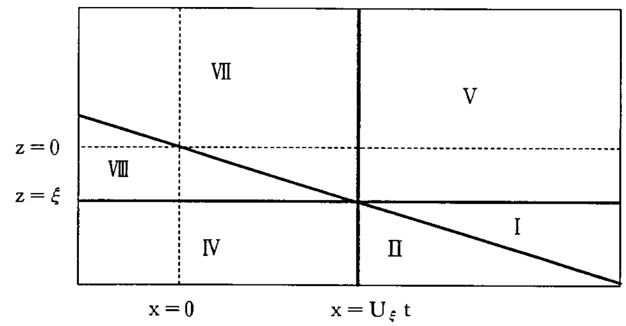

FIG. 2. A schematic diagram of each divided region in Table I. The numeric number in each region corresponds to that in the first column of Table I. The solid lines denote boundaries between the divided regions.

ACKNOWLEDGMENT

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