

ISYE 6740 Homework 2

Q1-1

lets consider a graph with $A_1 \dots A_m$ components.
we can consider the Adjacency matrix and degree
matrix as below:

$$\begin{bmatrix} D_1 & & \\ & D_2 & \\ & & \ddots \\ & & & D_m \end{bmatrix} \quad \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \\ & & & A_m \end{bmatrix}$$

because each component is connected with minimum
one edge then we can consider each A_1 to A_m and
 D_1 to D_m as block of semimetric matrix. \square

hence $L = \begin{bmatrix} L_1 & & \\ & L_2 & \\ & & \ddots \\ & & & L_m \end{bmatrix}$. ①

Now lets consider an eigenvalue decomposition of L_n for
eigenvalue zero and eigen vector V ($V \in \mathbb{R}^n$)

$$\begin{aligned} \lambda = 0 &= V^T L_n V = V^T D_n V - V^T A_n V \\ &= \sum_{i=1}^n d_i v_i^2 - \sum_{ij} a_{ij} v_i v_j \\ &= \frac{1}{2} \left(\sum_{i=1}^n d_i v_i^2 - 2 \sum_{ij} a_{ij} v_i v_j + \sum_{j=1}^n d_j v_j^2 \right) \\ &= \frac{1}{2} \sum_{i,j=1}^n a_{ij} (v_i - v_j)^2 \quad ② \end{aligned}$$

$$\Rightarrow \frac{1}{2} \sum_{i,j=1}^n a_{ij} (v_i - v_j)^2 = 0 \Rightarrow \sum_{i,j=1}^n a_{ij} (v_i - v_j)^2 = 0$$

↑
positive

the equation above can only equal to zero if
 $v_i = v_j$ for each $i \neq j$.

this means for component v_i , the eigenvector
is scaled on identity vector.

if $v_i = v_j = v_n \Rightarrow$ eigenvector = $v_n I_n$ (I_n the indicator
vector)

we can generalize this concept to all components

$$V^T L V = \begin{bmatrix} v_1 I_{A_1} \\ v_2 I_{A_2} \\ \vdots \\ v_m I_{A_m} \end{bmatrix}$$

(v_1, \dots, v_m) are values

this shows that the eigenvectors to eigenvalue
zero are linear combination of I_{A_1}, \dots, I_{A_m} .

Q1-1

I used both my Kmeans and the Kmeans from scikit-learn package. In both cases the false classification rate was close to 5.2%. For details please check the Jupyter file attached to this report.

Q2-1

we have done the PCA in several steps.
we also cleaned up the data and we got to
13 countries and 20 types of foods. using same
model we used in the class (slides)

$$m=13 \quad d=20$$

$$\downarrow \quad \downarrow \\ x^1, \dots, x^{13} \in \mathbb{R}^{d=20} \Rightarrow \text{matrix } X \in \mathbb{R}^{13 \times 20}$$

next step: calculate the mean and covariance
of the matrix

$$\mu = \frac{1}{m} \sum_{i=1}^m x^i \quad \text{and} \quad C = \frac{1}{m} \sum_{i=1}^m (x^i - \mu)(x^i - \mu)^T$$

next step: find the eigenvectors w_1, w_2, \dots
of the covariance matrix and pick
the largest eigenvalue, then the next
one, then the next, ... Depend on the
dimension we want to reduce to.

next step: compute the principle components

$$z_i = \begin{pmatrix} w_1^T (x^i - \mu) / \sqrt{\lambda_1} \\ w_2^T (x^i - \mu) / \sqrt{\lambda_2} \\ \vdots \end{pmatrix}$$

Q2-2

lets assume we have m data points $x^1, \dots, x^m \in \mathbb{R}^d$
and the mean is $\mu = \frac{1}{m} \sum_{i=1}^m x^i$.

We will find direction w in the space \mathbb{R}^d in which
 $\|w\| \leq 1$ and it captures the maximum
variance. Therefore:

$$\underset{\|w\|=1}{\text{Max}} \underbrace{\frac{1}{m} \sum_{i=1}^m (w^T x^i - w^T \mu)^2}_{\text{maximize it.}}$$

$$\frac{1}{m} \sum_{i=1}^m (w^T x^i - w^T \mu)^2 = \frac{1}{m} \sum_{i=1}^m (w^T (x^i - \mu))^2$$

$$= \frac{1}{m} \sum_{i=1}^m w^T (x^i - \mu) (x^i - \mu)^T w$$

$$= w^T \left(\frac{1}{m} \sum_{i=1}^m (x^i - \mu)(x^i - \mu)^T \right) w = w^T C w$$

covariance

the optimization

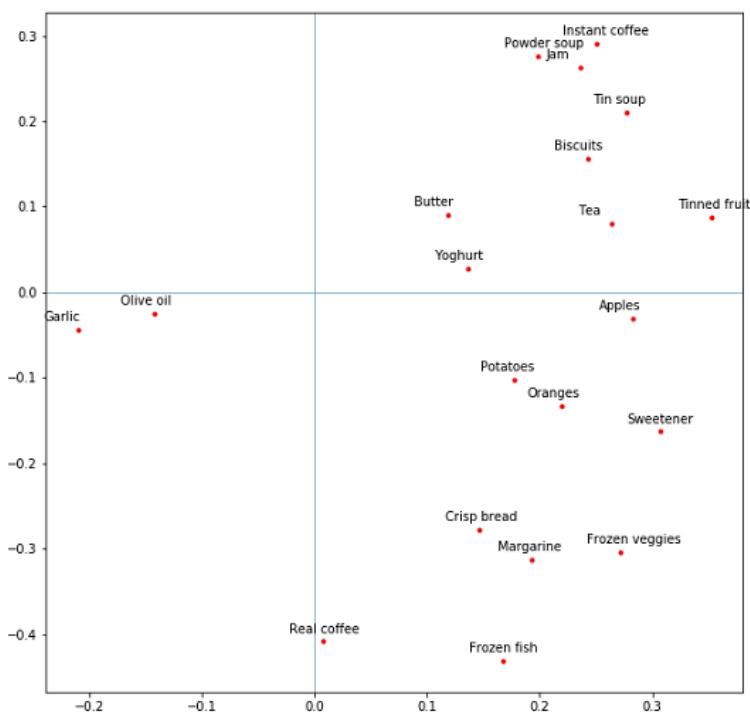
$$\Rightarrow \underset{\|w\|=1}{\text{Max}} w^T C w$$

if λ is the eigenvalue for matrix covariance
 C then :

$$Cw = \lambda w$$

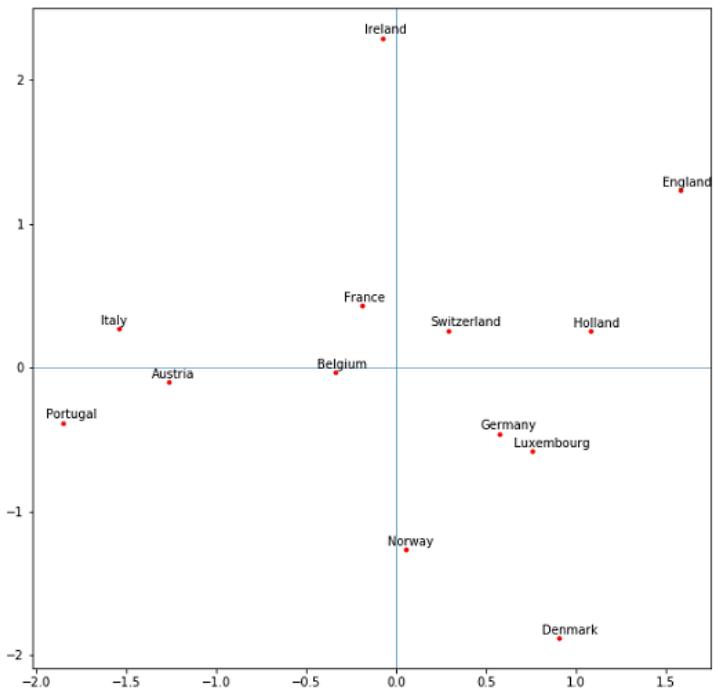
Therefore to get top k principle components
all we need to do is to find the eigenvalues
and its related eigenvectors. Pick the largest
eigenvalue $\overset{(1)}{\lambda_1}$ and related eigenvector as first
component, the second λ_2 and its eigenvector
as second principle, ...

Q2-3



This figure shows the correlation between various foods in the reduced data components. Tea and biscuits are positively correlated therefore, countries (such as England) uses biscuits and tea, the more biscuits are consumed, the more tea is also consumed. It also shows that places that consumes foods such as olive oil which is the opposite to for example apple, they consume less apple. Similarly, if the food are opposite each other from Y axis.

Q2-4



This figure shows the reduced data to two components. Countries close to each other, has similar daily diet. For example, Norway Denmark or Switzerland and Holland.

Countries far from each other has different diets. For example, Italy compared to Denmark. Italy consumes more garlic and olive oil while Denmark consumes more frozen fish and frozen veggie.