PHYSICAL REVIEW E **89**, 060903(R) (2014)

Multiplicative-cascade dynamics in pole balancing

Henry S. Harrison,[1] Damian G. Kelty-Stephen,[2,1] Daniela V. Vaz,[3] and Claire F. Michaels[1]
[1]*Center for the Ecological Study of Perception and Action, Department of Psychology, University of Connecticut, 406 Babbidge Road, Unit 1020, Storrs, Connecticut 06269-1020, USA*
[2]*Department of Psychology, Grinnell College, 1116 8th Avenue, Grinnell, Iowa 50112, USA*
[3]*Department of Physical Therapy, Universidade Federal de Minas Gerais, Avenida Presidente Antônio Carlos, 6627, Belo Horizonte, MG, 31270-901, Brazil*
(Received 17 January 2014; published 17 June 2014)

Pole balancing is a key task for probing the prospective control that organisms must engage in for purposeful action. The temporal structure of pole-balancing behaviors will reflect the on-line operation of control mechanisms needed to maintain an upright posture. In this study, signatures of multifractality are sought and found in time series of the vertical angle of a pole balanced on the fingertip. Comparisons to surrogate time series reveal multiplicative-cascade dynamics and interactivity across scales. In addition, simulations of a pole-balancing model generating on-off intermittency [J. L. Cabrera and J. G. Milton, Phys. Rev. Lett. **89**, 158702 (2002)] were analyzed. Evidence of multifractality is also evident in simulations, though comparing simulated and participant series reveals a significantly greater contribution of cross-scale interactivity for the latter. These findings suggest that multiplicative-cascade dynamics are an extension of on-off intermittency and play a role in prospective coordination.

DOI: 10.1103/PhysRevE.89.060903 PACS number(s): 05.45.Df, 87.19.lr, 05.65.+b

I. INTERACTIVITY IN PROSPECTIVE ACTION

Prospectivity, the ability to coordinate current behavior with future states, is a fundamental characteristic of purposeful action for organisms across all phyla [1,2]. Even the most elementary activities, such as maintaining an upright posture, are plagued by perturbations with many possible effects. Some of these perturbations are negligible; others could possibly destabilize upright posture. An animal that can anticipate threatening perturbations can act to prevent them from overwhelming balance. The problem of control is often equated with overcoming the delay imposed by the transduction of sensory signals and emission of motor commands [3–7]. Anticipation seems to skirt these limitations, and how it does so has been a classic puzzle in control theory [8]. A traditional solution has been to endow the organism with an internal model, generating future values of a control variable [9], though the computational requirements for such predictive internal models remain difficult to resolve in practical applications [10]. Alternative approaches propose that perceptual information usefully constrains behavior without requiring explicit prediction of future states [1,11–13].

Recent explanatory proposals for prospectivity draw on the recognition that complex behavior derives from numerous substructures and subfunctions distributed over multiple spatiotemporal scales [14]. Although the transduction rates between sensors, the central nervous system, and muscles are undoubtedly critical, this is not the only scale relevant for prospective control. For example, external forces propagate along stress-bearing elements of the musculoskeletal system at the speed of sound. These forces trigger context-dependent reactions [15], providing a potential biomechanical explanation for fast corrective movements. Of interest is how neural, biomechanical, and cognitive processes interact in the service of prospective behavior. Interactions among processes at multiple scales generate concurrent short- and long-range influences on behavior. Perceptual-motor fluctuations during motor and cognitive tasks exhibit such interactions across time scales [16–18]. Such multiscale coordination supports organism-information linkages to guide unfolding coordinations [19–22]. More specifically, attunement of current behavior to future possibilities might develop not at a single scale but across many nested, interacting scales. These cross-scale dependencies extend from the inner physiological workings of the anticipating organism to the farthest reaches of the context for action. Interactions among scales weave together concurrent fast and slow processes, allowing anticipation to emerge as a lawful regularity of an organism's embedding in its context [11,23,24].

To determine whether coordination phenomena reflect interactions across multiple scales, we draw on the concept of multiplicative cascades [14,25]. Originally developed to model turbulence [26], multiplicative cascades are abstractions of natural cascades describing the evolution of probability distributions as the successive multiplication (i.e., interaction) of random variables across nested measurement scales [27]. Multiplicative cascade processes produce a generalization of fractal patterning in which more than one singularity exponent describes the growth of fluctuations with time scales. In this multifractal case, a continuous spectrum of exponents is needed. Multifractal methods allow us to probe the continuity and coordination among fluctuations of different sizes. In what follows, we take pole balancing as a representative case of anticipatory coordination and seek evidence of multifractal interactivity across time scales.

II. POLE BALANCING

Pole balancing is a classic problem in control theory [28–30]. The task involves balancing an inherently unstable inverted pendulum, for example, on the fingertip. Stabilizing an unstable fixed point is one of only a few modes of behavioral dynamics [12], and so pole balancing serves as a proxy for a wide range of functional activities, such as maintaining an upright posture. One proposed control strategy for overcoming the limitation of sensorimotor delay is on-off intermittency [7]. In this strategy, stochastic forcing of a control parameter across a stability boundary produces corrective movements patterned irregularly over time, with long intervals containing small movements interspersed with shorter intervals containing larger movements [31]. On-off intermittency may allow for corrective movements at all time scales, including those shorter than the delay, and thus may provide the basis for anticipatory behaviors [7].

On-off intermittency arises from a delay-differential equation describing an inverted pendulum stabilized by a stochastic restoring force [7,32]:

$$m\ddot{\theta}(t) + \gamma\dot{\theta}(t) - k\sin\theta(t) + [r_0 + \xi(t)]\theta(t-\tau) = 0, \quad (1)$$

where θ is the angle of the pole from vertical, τ is the time delay, m is the pole's mass, γ is a friction constant, k is a gravity constant, and r_0 is a gain constant. The random function $\xi(t)$, zero-mean white noise, determines the coefficient for the restoring force and is therefore termed parametric or multiplicative noise. While embodying the known utility of multiplicative coupling in rhythmic coordination [33], Eq. (1) generates fractal scaling laws for time intervals between corrective movements [7].

The correcting term in Eq. (1), $[r_0 + \xi(t)]\theta(t-\tau)$, is a promising step toward modeling multiplicative-cascade dynamics, and thereby interactivity. The impact of $\xi(t)$ is state-dependent, depending on an *interaction* between the time-delayed signal $\theta(t-\tau)$ and the noise parameter $r_0 + \xi(t)$. Multiplicative-cascade dynamics comprise many more embedding layers of such interactions, not simply interactions between two signals. If the hypothesis of cross-scale interactions is correct, Eq. (1) should capture some, but not all, of the cross-scale contingencies found in empirical data. In this case, multiplicative-cascade dynamics may extend on-off intermittency and offer a similar, though more general, understanding of prospective coordination.

The speculation that multiplicative-cascade dynamics generalize on-off intermittency demands evidence of interactions among time scales in pole-balancing data as well as simulations of Eq. (1). On-off intermittency might be sufficiently interactive to produce some evidence of multifractality. However, multifractal signatures that are stronger in participant data than in simulated data would suggest that pole balancing enlists more than a single layer of multiplicative noise, namely a cascade of such multiplicative interactions, across many nested time scales.

III. METHODS

Five participants balanced a wooden dowel (38.4 g, 62 cm) on the tip of their index finger while standing upright, and they were unconstrained except for the requirement to remain inside the motion-capture region of approximately 3 m × 3 m. An Optotrak Certus optical motion-capture system sampled the three-dimensional position of two markers on the dowel at 600 Hz. Marker positions allowed calculating the pole's angle from the vertical at each time step.

Data collection began after a short practice period, beginning after the first attempt of at least 1 min, and ended after 14 total minutes of balancing data had been accumulated, discounting trials of less than 1 min. This criterion was achieved in an average of 13.8 trials. Discarding trials under 1 min, a total of 34 trials were analyzed. The mean survival time was 148.8 s. Postprocessing of data involved cropping recorded series to begin with the pole's first deviation from the vertical and to end with its final reversal preceding the fall.

For the simulations, a fourth-order Runge-Kutta algorithm with simulation time $h = 1/600$ s approximated 34 solutions of Eq. (1) with model parameters $\tau = 0.07$ s, $r_0 = 0.227$, $\sigma = 0.13$, and $\gamma = 100$ [7] (see the supplemental material [34] for an analysis with different parameters). Simulation length was matched for each trial of participant data. Comparing the multifractal spectra of simulated data to those of participant data should reveal the extent to which Eq. (1) captures all the multiplicative dynamics in pole balancing.

IV. ASSESSING MULTIFRACTALITY AND INTERACTIVITY

A. Direct estimation of the singularity spectrum

Singularity spectra of pole-angle time series of both human data and simulations were estimated using Chhabra and Jensen's [35,36] direct method, in which a time series $u(t)$ is sampled at progressively larger scales. The proportion $P_i(L)$ falling within the ith bin of scale L is

$$P_i(L) = \frac{\sum_{k=(i-1)L+1}^{iL} u(k)}{\sum u(t)}. \quad (2)$$

On average, as the scale increases, these sums represent larger proportions of $u(t)$, such that

$$P(L) \propto L^{\alpha}. \quad (3)$$

For monofractal dynamics, $P(L)$ grows homogeneously across many time scales according to a single, potentially noninteger singularity strength α [37]. $P(L)$ exhibits multifractal dynamics when it grows heterogeneously with time scale L, according to a range of singularity strengths,

$$P_i(L) \propto L^{\alpha_i}, \quad (4)$$

where each ith bin sized less than L may show a different relationship of $P(L)$ with L. The width of this singularity spectrum, $\alpha_{\max} - \alpha_{\min}$, indicates the heterogeneity of relationships between $P(L)$ and L [38,39].

The Chhabra and Jensen method [35,36] involves estimating $P(L)$ for N_L nonoverlapping L-sized bins of a time series and transforming them with a parameter q, which emphasizes higher or lower proportions for $q > 1$ and $q < 1$, respectively. For each bin i of size L, the proportions $P_i(L)$ are transformed into mass $\mu_i(q,L)$:

$$\mu_i(q,L) = \frac{[P_i(L)]^q}{\sum_{i=1}^{N_L} [P_i(L)]^q}, \quad (5)$$

reflecting different q-weighted distributions of fluctuations at scale L.

For each q, each estimated value of $\alpha(q)$ is part of the singularity spectrum only when the Shannon entropy of $\mu(q,L)$ scales with L according to the Hausdorff dimension

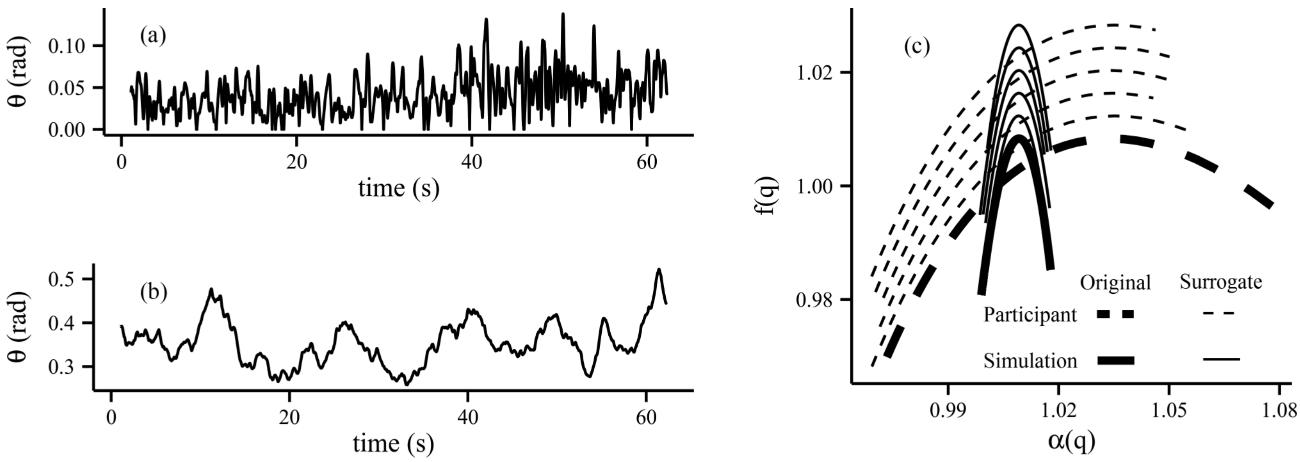

FIG. 1. (a) Time series of θ from a typical trial and (b) a length-matched solution of Eq. (1). (c) The singularity spectra ($\alpha(q)$, $f(q)$) of these time series and five representative IAAFT surrogates of each. Dashed lines represent spectra derived from participant data; unbroken lines represent spectra derived from simulations. The spectrum of each original series is drawn with a thick line, and those of surrogates with a thin line. Surrogate spectra are vertically offset for clarity.

$f(q)$,where

$$f(q) = -\lim_{N\to\infty} \frac{1}{\ln N} \sum_{i=1}^{N} \mu_i(q,L) \ln \mu_i(q,L)$$
$$= \lim_{L\to 0} \frac{1}{\ln L} \sum_{i=1}^{N} \mu_i(q,L) \ln \mu_i(q,L). \tag{6}$$

Finally, $\alpha(q)$ is estimated as

$$\alpha(q) = -\lim_{N\to\infty} \frac{1}{\ln N} \sum_{i=1}^{N} \mu_i(q,L) \ln P_i(L)$$
$$= \lim_{L\to 0} \frac{1}{\ln L} \sum_{i=1}^{N} \mu_i(q,L) \ln P_i(L). \tag{7}$$

For values of q yielding strongly linear relationships in Eqs. (6) and (7), the generally single-humped parametric curve $(\alpha(q), f(q))$ is known as the singularity spectrum [40]. The present study included in each spectrum only those values of q for which the relationships in Eqs. (6) and (7) exhibited a correlation coefficient of $r > 0.9999$ [14,41].

B. Surrogate testing

Nonzero singularity-spectrum width can arise from interactions across time scales, but it can also result from linear autocorrelation or heavy-tailed distributions [25,42]. The contribution of multiplicative interactions to a time series' singularity spectrum can be tested by comparing the original spectrum to those of surrogate time series in which cross-scale contingencies are eliminated [14]. The iterated amplitude adjusted Fourier transformation (IAAFT) [43] generates surrogates that preserve a time series' probability density and autocorrelation function while randomizing the phase ordering of the spectral amplitudes. That is, IAAFT surrogates sort the original values in a time-symmetric fashion around the overall autoregressive structure.

If the original series has a singularity-spectrum width outside the 95% confidence interval of mean width for a sample of IAAFT spectra, then the original series exhibits nonlinear interactions across time scales. This comparison can be captured in a t-statistic: because any difference in width—narrower or wider—is associated with interactions across scales [25], $|t|$ indexes the contribution of interactions across scales to the time series' multifractality.

V. RESULTS

A. Multifractal-spectrum widths

Multifractal spectra were estimated for participant and simulated data. Fifty IAAFT surrogates were generated for each time series. Data from a sample trial and a length-matched simulation are shown in Figs. 1(a) and 1(b), respectively. Figure 1(c) depicts the resulting multifractal spectra and those of five representative surrogates of each series. Evidence for multifractality in the form of singularity-spectrum widths $\alpha_{\max} - \alpha_{\min} > 0$ was found for both participant and simulated time series. Spectra were significantly wider for participant data ($M = 9.09 \times 10^{-2}$, $SD = 3.51 \times 10^{-2}$) than for simulations ($M = 1.49 \times 10^{-2}$, $SD = 4.71 \times 10^{-3}$), $t(33) = 14.19$, $p < 0.001$, indicating greater heterogeneity of scaling exponents in participant data.

B. Surrogate tests

One-sample, two-tailed t-tests contrasted each time series' singularity-spectrum width with those of its surrogates. Table I summarizes the results. A total of 32 of 34 spectra estimated from participant data differed significantly in width from their surrogates ($p < 0.001$), compared to 23 of 34 spectra from

TABLE I. Results of t-tests comparing original spectrum widths to the spectrum widths of IAAFT surrogates. Significance was evaluated at the $p < 0.001$ level.

	Original narrower	Not sig.	Original wider
Participant	0	2	32
Simulation	17	11	6

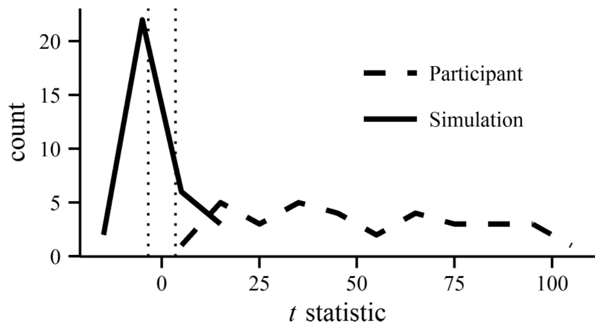

FIG. 2. Frequency plot of t-statistics comparing singularity-spectrum width $\alpha_{\max} - \alpha_{\min}$ of each participant and simulated time series to 50 IAAFT surrogates. $t > 0$ indicates that the original spectrum was wider than that of its surrogates. Dotted vertical lines indicate significance cutoffs at the $p < 0.001$ level with 49 degrees of freedom. One participant time series, with $t \gg 0$, is not shown in order to keep the scale reasonable.

simulated data. Figure 2 shows a histogram of t-statistics, indicating consistent differences in participant time series but not simulations. Absolute differences in spectrum widths between original and surrogate series, as indexed by $|t|$, were significantly greater for participant data ($M = 47.99$, $SD = 27.39$) than for simulations ($M = 6.36$, $SD = 4.15$), $t(30) = 8.61$, $p < 0.001$. Three participant time series whose surrogates were not multifractal ($\alpha_{\max} - \alpha_{\min}$ not significantly different from zero, $p < 0.001$) were excluded to ensure a conservative analysis. Consistently positive t-statistics for participant data indicate that original time series exhibited wider singularity spectra than did their surrogates, for all 32 trials with significant differences. The spectrum width comparisons for simulated time series did not exhibit the same consistency: 6 spectra were significantly wider and 17 were significantly narrower than their surrogates.

VI. MULTIPLICATIVE-CASCADE DYNAMICS SUPPORT PROSPECTIVITY

These results suggest that corrective movements in human pole balancing arise from multiplicative-cascade processes. Pole-balancing series exhibit fluctuations distributed heterogeneously over time, following multiple fractional relationships with time scale. Comparisons to IAAFT surrogates confirm that the multifractal spectra are not simply consequences of linear processes but rather arise from multiplicative interactions between time scales. This corroborates the findings of interactivity in a range of human anticipatory behaviors [16–18,44].

Additionally, simulations of the pole-balancing model Eq. (1) exhibited multifractal spectra. These were narrower than their counterparts estimated from participant data, indicating a lesser degree of multifractality. Although the majority of simulated spectra differed from their surrogates when considered individually, these differences lacked systematicity across simulations. This relatively weak and variable effect of multiplicative-cascade dynamics is consistent with interpreting the multiplicative noise in Eq. (1) as describing a single multiplicative interaction between the time scale of the sensorimotor delay and the arbitrarily fast time scale approximated by white noise. The model, therefore, represents an initial move toward modeling multiplicative-cascade processes, which are constituted by many such interactions reaching across a wide range of scales. Consistent with the above analysis, surrogate comparisons revealed a greater contribution of cross-scale effects for participant data than for simulated time series. This suggests that Eq. (1) captures some, but not all, of the contingencies between processes at multiple time scales involved in pole balancing.

This conclusion suggests a route for extending Eq. (1) by expanding the interaction term to include heterogeneous multiplicativity across a variety of time scales. Similar models include the binomial multiplicative cascade model, which elaborates a simple monofractal cascade with two multiplicative operators [39] (Sec. II.c.4), and a model of multiscale competition and cooperation featuring heterogeneity across scales [45]. Another avenue for exploration is the use of fractional calculus, as suggested by a recent model featuring a fractional derivative [46]. If heterogeneity of the derivative's fractional order could be introduced, this model may reproduce the multifractal spectra found in human pole-balancing time series.

It should be noted that participants in the present study were standing and allowed to move, whereas the original demonstration of on-off intermittency had participants sitting [7]. As fluctuations of the pole are more heterogeneous when the balancer is standing [47], it is possible that some of the present results can be attributed to this difference. To ameliorate this concern, we tested whether the present data reproduce a statistical signature of on-off intermittency. On-off intermittency is characterized by a power law with a scaling exponent of $-3/2$ in the distribution of periods between fluctuations [7,31]. In the present data, the mean scaling exponent across all five participants was -1.57; this extends the finding of on-off intermittency to standing pole-balancers. However, the extent to which multiplicative-cascade dynamics are present under more constrained balancing conditions remains to be seen. A reasonable expectation is that reducing the available degrees of freedom would decrease the potential interactions between disparate processes and thereby result in weaker multifractal signatures. This hypothesis is consistent with previous work comparing sitting and standing balancers [47].

Other manipulations of task difficulty, such as adjusting the dimensions of the pole, may also be informative. It has been proposed that an increasing number of feedback loops contribute to corrective movements for increasingly larger fluctuations of the pole [48]. The methods employed here would provide an avenue to test this hypothesis; if it is correct, increased heterogeneity of interactions, and therefore a wider multifractal spectrum, would be expected with increases in difficulty.

It remains to be seen whether the present findings can be reconciled with other modeling approaches to pole balancing. These include recognition of the possibility of pivot points other than the fingertip [49], and the modeling of translational rather than rotational movements [46,50]. Another approach, inspired in part by Eq. (1), models intermittency as explicitly regulated by a switching function [4,51,52], perhaps related to sensory thresholds [53]. However, as these models have not reproduced the original finding of on-off intermittency [7], the theoretical connection to multiplicative-cascade dynamics is not as apparent as with Eq. (1). Furthermore, whereas these approaches rely on predictive controllers, multiplicative-cascade dynamics may enable anticipatory behavior without explicit prediction of future states [11,17].

As anticipation has been found to arise from multiplicative-cascade dynamics in other domains [17], such dynamics are likely also responsible for prospectivity in pole balancing. Equation (1) captures a portion of these dynamics, but multiplicative noise and on-off intermittency may not be the entire story. The present findings suggest two conclusions: (i) multiplicative-cascade dynamics generalize and extend on-off intermittency, and (ii) multiplicative-cascade dynamics support prospective coordination.

ACKNOWLEDGMENTS

We thank Till D. Frank for assistance with the laminar-phase analysis and Juan L. Cabrera for helpful discussion regarding the parameters of Eq. (1). D.V.V. was supported by Grant No. 200676/2009-1 of the Brazilian Council of Scientific and Technological Development (CNPq).

[1] M. T. Turvey, AVANT: The Journal of the Philosophical-Interdisciplinary Vanguard **3**, 128 (2012).
[2] T. Saigusa, A. Tero, T. Nakagaki, and Y. Kuramoto, Phys. Rev. Lett. **100**, 018101 (2008).
[3] G. Stépán and T. Insperger, Ann. Rev. Control **30**, 159 (2006).
[4] T. Insperger, Annual Reviews in Control. **14**, 159 (2006).
[5] J. G. Milton, J. L. Cabrera, and T. Ohira, Europhys. Lett. **83**, 48001 (2008).
[6] J. G. Milton, J. L. Cabrera, T. Ohira, S. Tajima, Y. Tonosaki, C. W. Eurich, and S. A. Campbell, Chaos **19**, 026110 (2009).
[7] J. L. Cabrera and J. G. Milton, Phys. Rev. Lett. **89**, 158702 (2002).
[8] R. Rosen, *Anticipatory Systems: Philosophical, Mathematical, and Methodological Foundations* (Pergamon, Oxford, UK, 1985).
[9] G. Pezzulo, Minds Machines **18**, 179 (2008).
[10] S. Lenhart and J. T. Workman, *Optimal Control Applied to Biological Models* (Chapman & Hall/CRC, Boca Raton, FL, 2007).
[11] N. Stepp and M. T. Turvey, Cogn. Syst. Res. **11**, 148 (2010).
[12] W. H. Warren, Psych. Rev. **113**, 358 (2006).
[13] D. N. Lee, Perception **5**, 437 (1976).
[14] E. Ihlen and B. Vereijken, J. Exp. Psych.: General **139**, 436 (2010).
[15] M. T. Turvey and S. T. Fonseca, J. Motor Behav. **46**, 143 (2014).
[16] D. G. Stephen and D. Mirman, Cognition **115**, 154 (2010).
[17] D. G. Stephen and J. A. Dixon, Chaos Solitons Fractals **44**, 160 (2011).
[18] D. G. Stephen, J. R. Anastas, and J. A. Dixon, Front Physiol **3**, 102 (2012).
[19] D. G. Stephen and J. R. Anastas, Attention Perception Psychophys. **73**, 666 (2010).
[20] D. G. Stephen, R. Arzamarski, and C. F. Michaels, J. Exp. Psych.: Human Perception Perform. **36**, 1161 (2010).
[21] J. A. Dixon, D. G. Stephen, R. Boncoddo, and J. Anastas, in *Psychology of Learning and Motivation*, edited by B. H. Ross (Academic, Waltham, MA, 2010), Vol. 52, pp. 343–384.
[22] D. G. Stephen and J. A. Dixon, J. Prob. Solv. **2**, 6 (2009).
[23] D. G. Stephen, N. Stepp, J. A. Dixon, and M. T. Turvey, Physica A **387**, 5271 (2008).
[24] D. M. Dubois, in *Anticipatory Behavior in Adaptive Learning Systems*, edited by M. V. Butz, O. Sigaud, and P. Gérard (Springer, Berlin, 2003), Vol. 2684, pp. 110–132.
[25] D. G. Kelty-Stephen, K. Palatinus, E. Saltzman, and J. A. Dixon, Ecol. Psych. **25**, 1 (2013).
[26] B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).
[27] T. Bohr, M. H. Jensen, G. Paladin, and A. Vulpiani, *Dynamical Systems Approach to Turbulence*, Cambridge Nonlinear Science Series, Vol. 8 (Cambridge University Press, Cambridge, UK, 1998).
[28] S. Geva and J. Sitte, Control Syst., IEEE **13**, 40 (1993).
[29] P. Foo, J. A. Kelso, and G. C. De Guzman, J. Exp. Psych.: Human Perception Perform. **26**, 1281 (2000).
[30] P. J. Treffner and J. A. S. Kelso, Ecol. Psych. **11**, 103 (1999).
[31] N. Platt, E. A. Spiegel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993).
[32] J. L. Cabrera and J. G. Milton, Nonlin. Stud. **11**, 305 (2004).
[33] V. K. Jirsa, P. Fink, P. Foo, and J. A. S. Kelso, J. Biol. Phys. **26**, 85 (2000).
[34] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.89.060903 for a preliminary exploration of parameter space.
[35] A. B. Chhabra and R. V. Jensen, Phys. Rev. Lett. **62**, 1327 (1989).
[36] A. B. Chhabra, C. Meneveau, R. V. Jensen, and K. R. Sreenivasan, Phys. Rev. A **40**, 5284 (1989).
[37] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
[38] B. B. Mandelbrot, *Fractals and Scaling in Finance* (Springer, New York, 1997).
[39] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).
[40] M. Zamir, J. Theor. Biol. **225**, 407 (2003).
[41] P. C. Ivanov, L. A. Nunes Amaral, A. L. Goldberger, S. Havlin, M. G. Rosenblum, H. E. Stanley, and Z. R. Struzik, Chaos **11**, 641 (2001).
[42] D. Veneziano, G. E. Moglen, and R. L. Bras, Phys. Rev. E **52**, 1387 (1995).
[43] T. Schreiber and A. Schmitz, Phys. Rev. Lett. **77**, 635 (1996).
[44] D. G. Kelty-Stephen and D. Mirman, Cognition **126**, 128 (2013).
[45] S. Solomon and M. Levy, Int. J. Mod. Phys. C **07**, 745 (1996).
[46] D. M. Jacobs, D. V. Vaz, and C. F. Michaels, J. Exp. Psych.: Human Perception Perform. **38**, 1215 (2012).
[47] T. Cluff and R. Balasubramaniam, PLoS ONE **4**, e5998 (2009).
[48] J. L. Cabrera and J. G. Milton, Eur. Phys. J. ST **205**, 231 (2012).
[49] K.-Y. Lee, N. O'Dwyer, M. Halaki, and R. Smith, Exp. Brain Res. **221**, 309 (2012).
[50] G. Stépán, Philos. Trans. R. Soc. A **367**, 1195 (2009).
[51] P. Gawthrop, I. Loram, M. Lakie, and H. Gollee, Biol. Cybern. **104**, 31 (2011).
[52] Y. Asai, Y. Tasaka, K. Nomura, T. Nomura, M. Casadio, and P. Morasso, PLoS ONE **4**, e6169 (2009).
[53] T. Insperger, J. G. Milton, and G. Stépán, J. R. Soc. Interf. **10**, 20120763 (2012).