# CSE 620: Selective Topics

# Introduction to Formal Verification

Master Studies in CSE

Winter 2017

**Lecture #2**

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# Course Outline

- Computational Boolean Algebra
  - Basics
    - Shannon Expansion
    - Boolean Difference
    - Quantification Operators
      - + Application to Logic Network Repair
  - Validity Checking (Tautology Checking)
  - Satisfiability Checking (SAT solving)
  - Binary Decision Diagrams (BDD's)
- Model Checking
  - Temporal Logics ➔ LTL - CTL
  - SMV: Symbolic Model Verifier
  - Model Checking Algorithms ➔ Explicit CTL

# VLSI CAD:
## Logic to Layout

# Lecture 2.3

## Computational Boolean Algebra: Quantification Operators

**Rob A. Rutenbar**

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# Computational Boolean Algebra, Cont…

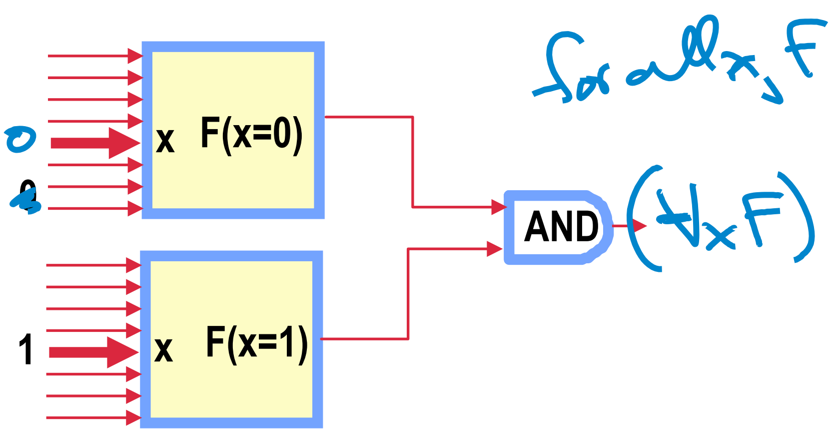
- **What you know**
  - Shannon expansion lets you decompose a Boolean function
  - Combinations of cofactors do interesting things, e.g., the Boolean difference $\frac{\partial f}{\partial x}$ $\oplus$
- **What you don't know**
  - Other combinations of cofactors that do useful things
    - The big ones: **Quantification operators** (this lecture)
    - **Applications:** Being able to do something impressive (next lecture)

# AND: Fx • Fx' is *Universal Quantification*

- **Have F(x1, x2, …, xi, … xn)**
- **AND cofactors: $F_{xi} \bullet F_{xi'}$**
  - Name: **Universal Quantification** of function **F** with respect to (wrt) variable **xi**

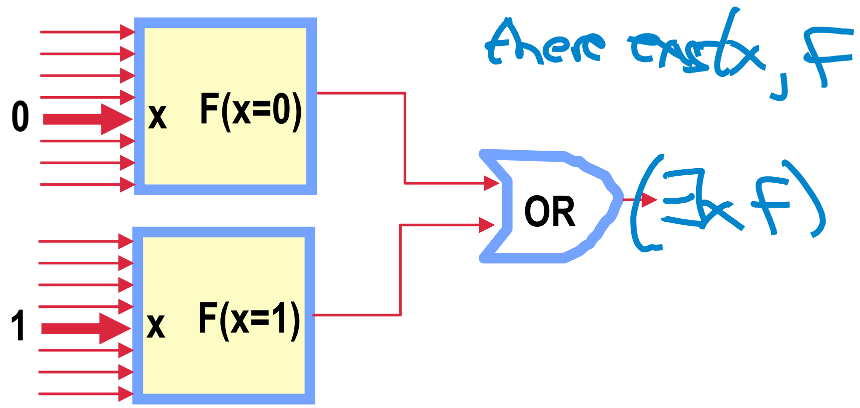
**(∀xi F) [x1,x2, …, xi-1, xi+1, … xn]**

- **"(∀xi F)" is a new function**
  - Yes, the '∀' sign is the "for all" symbol from logic (predicate calculus)
  - And, it does not depend on **xi**…



# OR: Fx + Fx' is *Existential Quantification*

- **Have F(x1, x2, …, xi, … xn)**
- **OR the cofactors: $F_{xi} + F_{xi'}$**
  - Name: **Existential Quantification** of function F wrt variable xi
  - **($\exists$xi F) [x1, x2, …, xi-1, xi+1, … xn]**

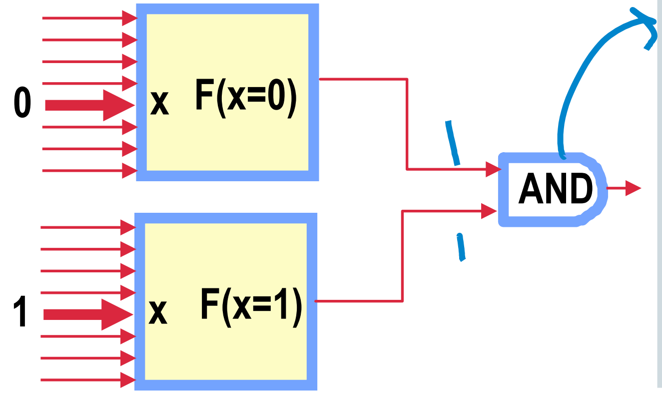


- **"($\exists$xi F)" is a new function**
  - "$\exists$" sign is "there exists" from logic; and function also does not depend on **xi**

**Note**, like anything involving cofactors, both these new functions **do not** depend on **xi**

So: **($\forall$xi F)** and **($\exists$xi F)** both omit **xi**

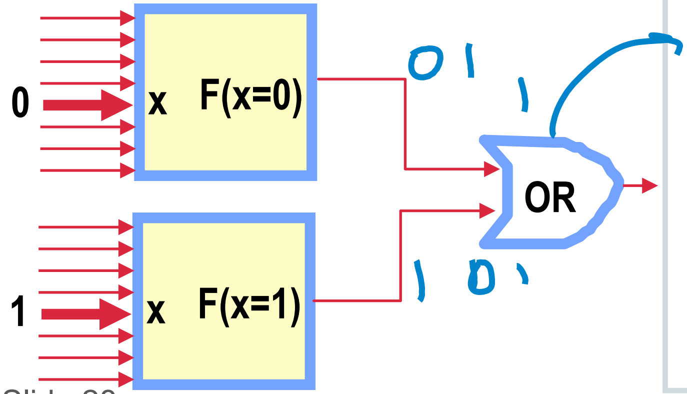
# Quantification Notation Makes Sense…



**(∀x F) ( all original vars but x) ==1** ***when?***

these other inputs make F=1

For All values of x



**(∃x F) ( all original vars but x) ==1** ***when?***

There exists a value of x that makes F=1 for this input pattern of other vars

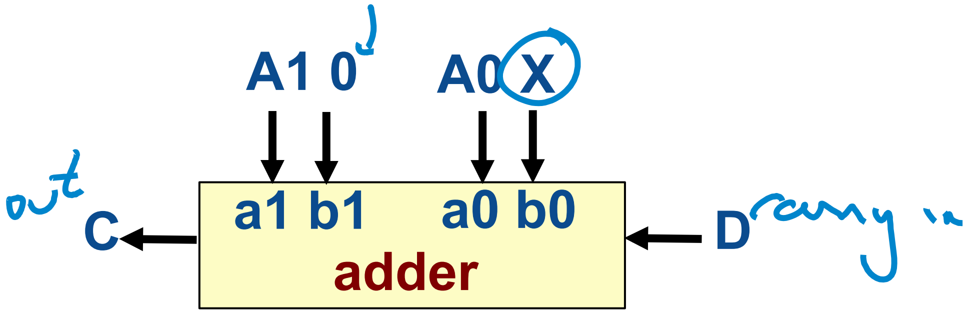
# Extends to More Variables in Obvious Way

- **Additional properties**
  - Like Boolean difference, can do with respect to **more** than 1 var
  - Suppose we have **$F(x,y,z,w)$**
  - Example: $(\underline{\forall x}y\ \ F)[z,w] = (\underline{\forall}x\ \ \underline{(}\forall y\ \ F)) = Fxy \bullet Fx'y \bullet Fxy' \bullet Fx'y'$ AND
  - Example: $(\underline{\exists x}y\ \ F)[z,w] = (\exists x\ \ (\exists y\ \ F)) = Fxy + Fx'y + Fxy' + Fx'y'$ OR

- **Remember!**
  - $(\forall x\ \ F)$, $(\exists x\ \ F)$, and $\partial F / \partial x$ are all **functions**...
  - ..but they are functions of all the vars **except x**
  - We got rid of variable x and made 3 **new** functions

# Quantification Example

- **Consider this circuit, it adds X=0 or X=1 to a 2-bit number A1A0**
  - It's just a 2-bit adder, but instead of **B1B0** for the second operand, it is just **0X**
  - It produces a carry out called **C** and also has a carry in called **D**

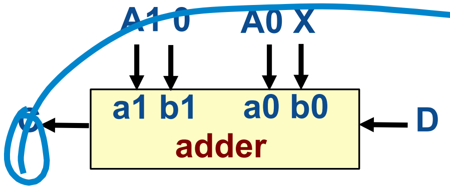


  - What is **(∀A1,A0 C)[X,D]…?**
    - A function of only **X,D**. Makes a **1** for values of **X,D** that make carry **C=1** for ***all values*** of operand input **A1A0**, i.e., makes a carry **C=1** for all values of **A1A0**

independent

  - What is **(∃A1,A0 C)[X,D]…?**
    - A function of just **X,D**. Makes a **1** for values of **X,D** that make carry **C=1**, for ***some value*** of **A1A0**, i.e., ***there exists*** some **A1A0** that, for this **X**,**D** makes **C=1**

# Quantification Example



**Do the math: Need all 4 cofactors:**

**$C = A1A0X + A1(A0+X)D$**

| | | | |
|---|---|---|---|
| $C_{A1A0}$ | $C_{A1'A0}$ | $C_{A1A0'}$ | $C_{A1'A0'}$ |
| X+D | 0 | XD | 0 |

- **Compute ($\forall$ A1,A0 C)[X,D]**
  - $C_{A1A0} \bullet C_{A1'A0} \bullet C_{A1A0'} \bullet C_{A1'A0'}$ ) AND

0

- **Compute ($\exists$ A1,A0 C)[X,D]**
  - $C_{A1A0} + C_{A1'A0} + C_{A1A0'} + C_{A1'A0'}$

$X+D+0+XD+0 = X+D$

- **In words: No** values of **X,D** that make **C=1** independent of **A1,A0**
- **In words: Yes**, if at least one of **X,D =1 → C=1** indep of **A1,A0**

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# Lecture 2.4

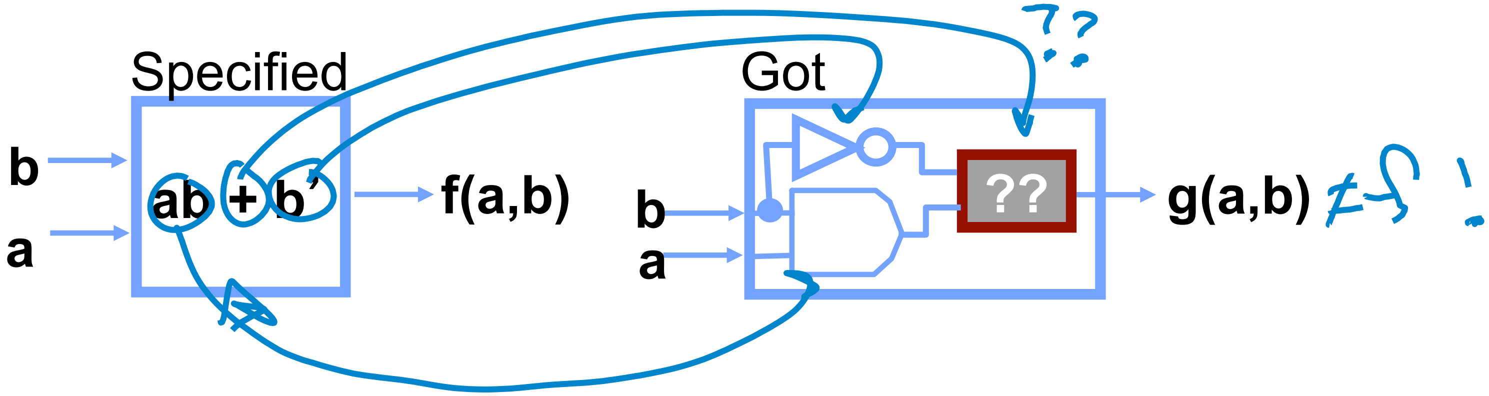
## Computational Boolean Algebra: Application to Logic Network Repair

**Rob A. Rutenbar**
University of Illinois

# Quantification App: Network Repair

- **Suppose ...**
  - I specified a logic block for you to implement... **f(a,b) = ab + b'**
  - …but you implemented it **wrong**: in particular, you got **ONE** gate wrong

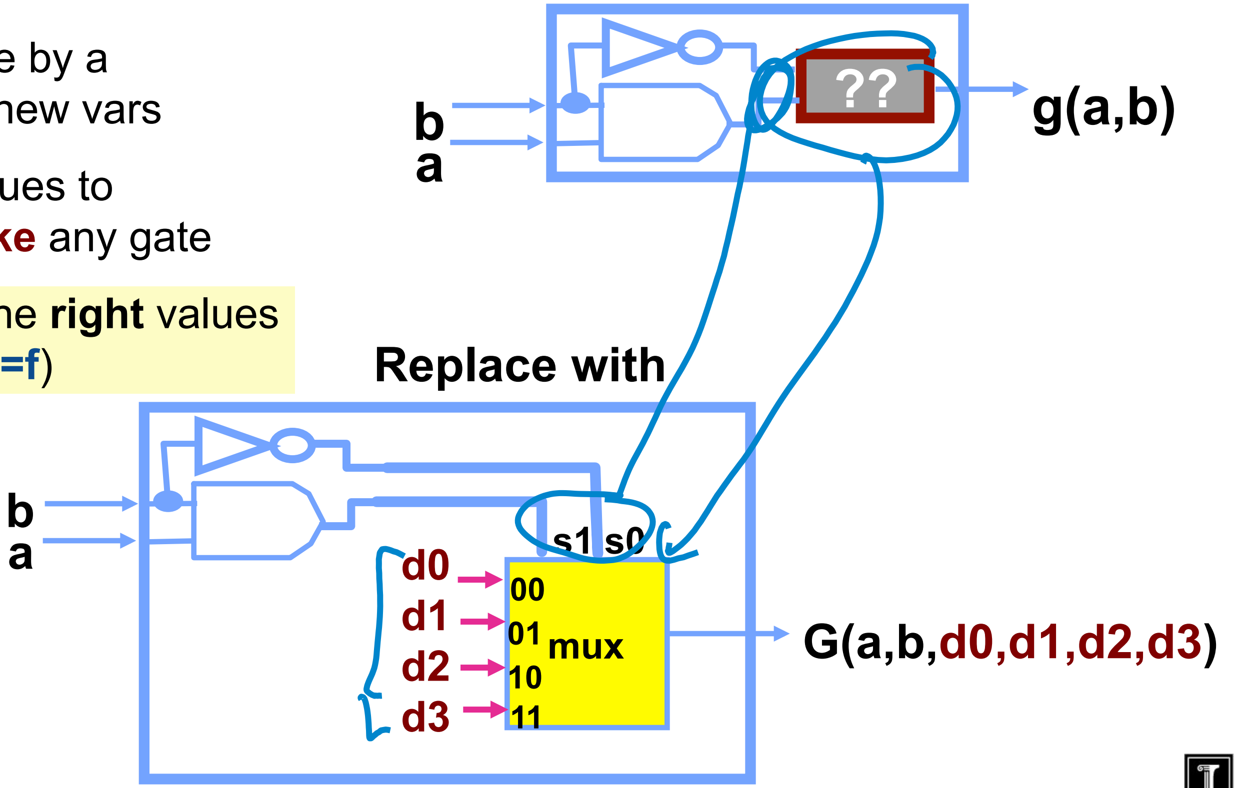
→ $a+\bar{b}$ ignore



- **Goal**
  - Can we deduce how precisely to **change this gate** to restore correct function?
  - Lets go with this very trivial test case to see how mechanics work...

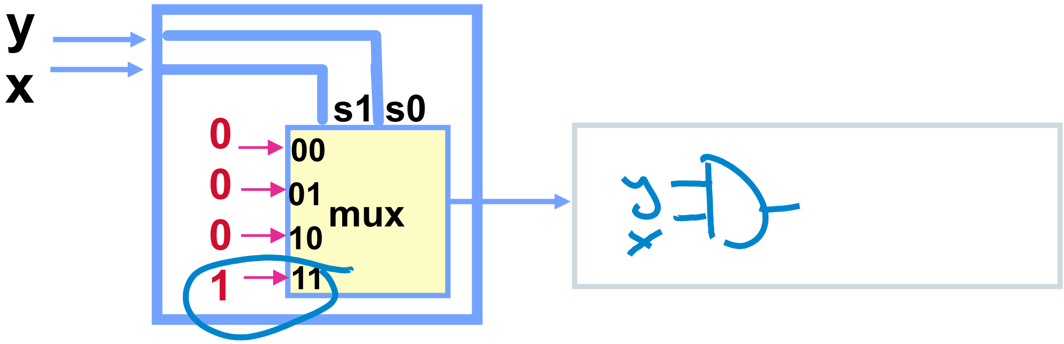
# Network Repair

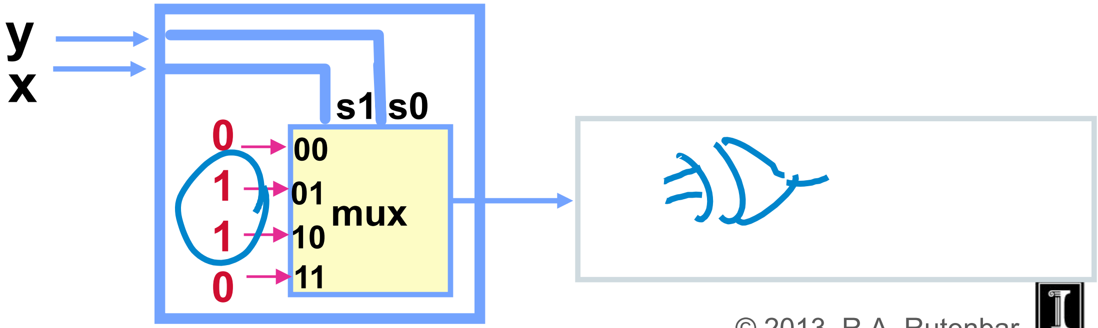
- **Clever trick**
  - Replace our suspect gate by a **4:1** mux with **4** arbitrary new vars
  - By cleverly assigning values to **d0 d1 d2 d3**, we can **fake** any gate
  - **Question is:** what are the **right** values of **d's** so **g** is repaired (**==f**)



# Aside: Faking a Gate with a MUX

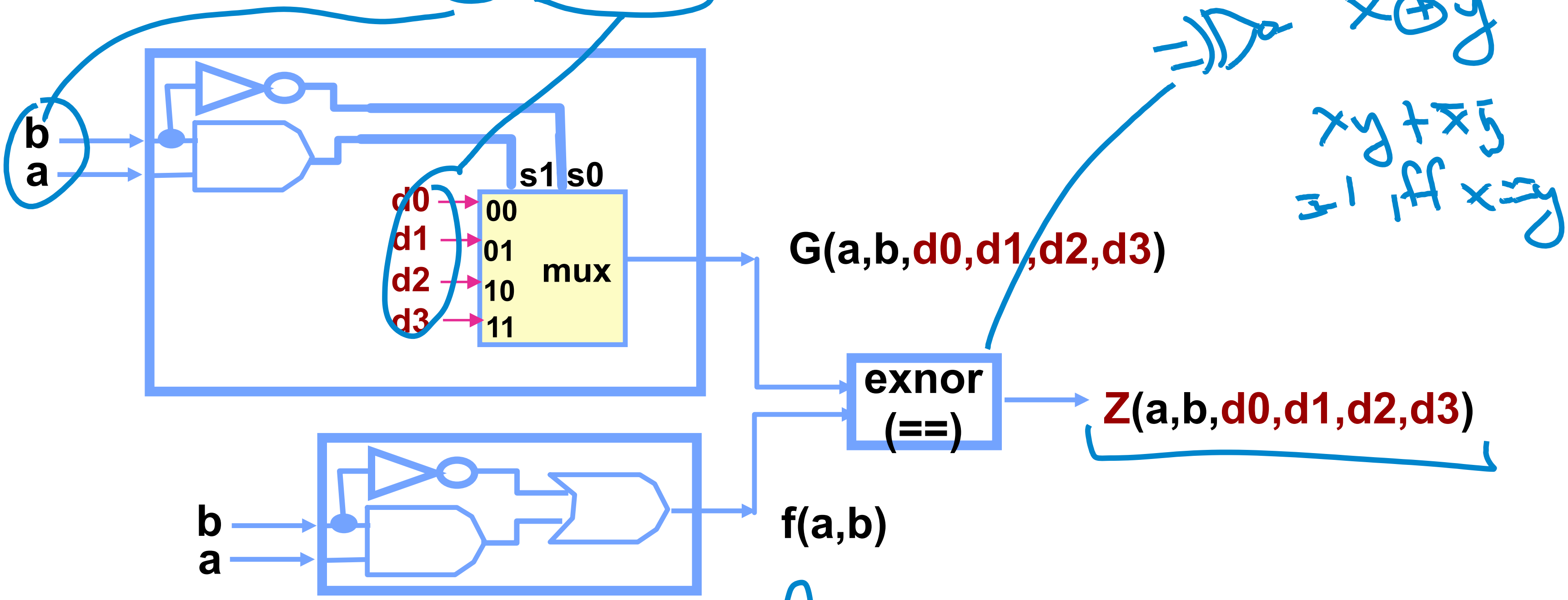
- **Remember…**
  - You can do **any** function of 2 vars with one 4 input multiplexor (MUX)





# Network Repair: Using Quantification

- **Next trick**
  - Make new function Z(a,b,d0,d1,d2,d3) that =1 just when G == f



$x \overline{\oplus} y$

$xy + \bar{x}\bar{y}$

$=1$ iff $x == y$

known good version of logic

# Using Quantification

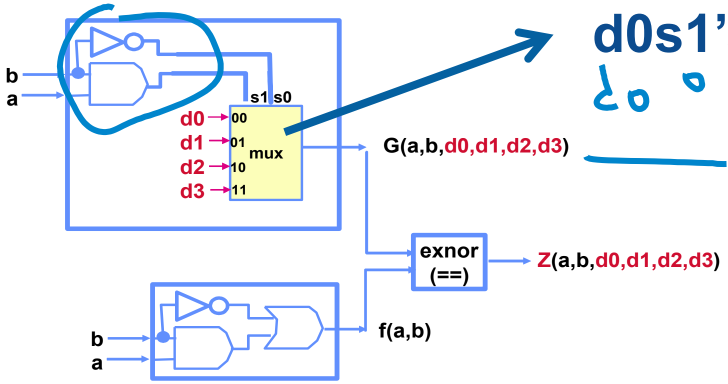
- **What now?**
  - Think hard about exactly what we want:

values of $d_0 d_1 d_2 d_3$ that make $Z==1$
for all possible values of inputs a,b.

$(\forall_{ab}, Z)[d_0 d_1 d_2 d_3]==1$ ?

- **But this is something we have seen!**
  - **Universal quantification** of function **Z** wrt variables **a,b**!
  - Any pattern of **(d0 d1 d2 d3)** that makes **(∀ab Z)(d0,d1,d2,d3)==1** will do it!
  - (Aside: do you know where **a, b** went??)

# Network Repair via Quantification: Try It…



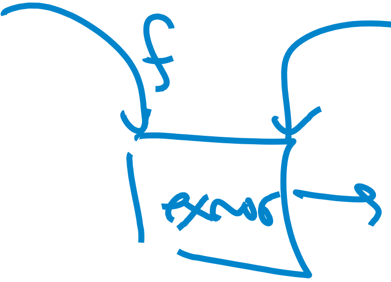
**ab+b'**

**d0s1's0' + d1s1's0 + d2s1s0' + d3s1s0** mux

$d_0 \; 0 \; 0$

$G = d_0 \overline{ab}\,\bar{\bar{b}} + d_1 \overline{ab}\,\bar{b} + d_2\, ab\, \bar{\bar{b}}$
$\cancel{+ d_3\, ab\, \bar{b}}$

$G = d_0 \bar{a} b + d_1 \bar{b} + d_2\, ab$



$Z = G \overline{\oplus} f = Gf + \bar{G}\bar{f}$

$\forall (\forall ab\ Z) = Z_{ab} \cdot Z_{\bar{a}b} \cdot Z_{a\bar{b}} \cdot Z_{\bar{a}\bar{b}}$

# Network Repair via Quantification: Continued

$$Z = \overbrace{[d0a'b + d1b' + d2ab]}^{G} \;\overline{\oplus}\; \overbrace{[ab + b']}^{f} = G \text{ exnor } f$$

**Reminder:**
Q exnor 0 = **Q'**
Q exnor 1 = **Q**

Use nice property: *cofactor of exnor is exnor of cofactors!*

$Z_{a'b'} = G_{a'b'} \;\overline{\oplus}\; f_{a'b'}$ → set a=0,b=0 → $d_1 \overline{\oplus} 1 = d_1$

$Z_{a'b} = G_{a'b} \;\overline{\oplus}\; f_{a'b}$ → set a=0,b=1 → $d_0 \overline{\oplus} 0 = \overline{d_0}$

$Z_{ab'} = G_{ab'} \;\overline{\oplus}\; f_{ab'}$ → set a=1,b=0 → $d_1 \overline{\oplus} 1 = d_1$

$Z_{ab} = G_{ab} \;\overline{\oplus}\; f_{ab}$ → set a=1,b=1 → $d_2 \overline{\oplus} 1 = d_2$

$[\forall ab\, Z](d_0, d_1, d_2) = \underline{\underline{AND}} = \overline{d_0}\, d_1 d_2$ !

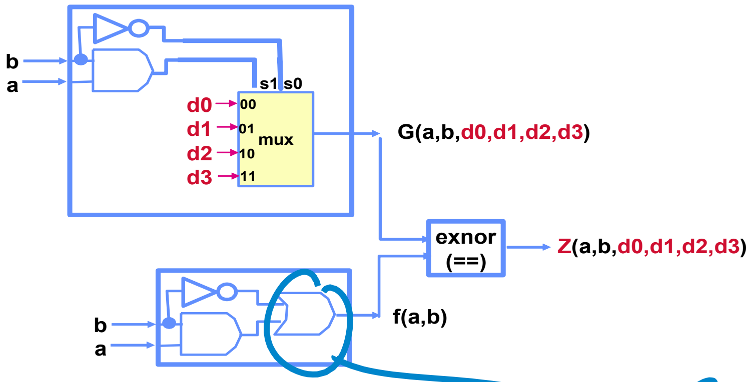
# Repair via Quantification: Continued

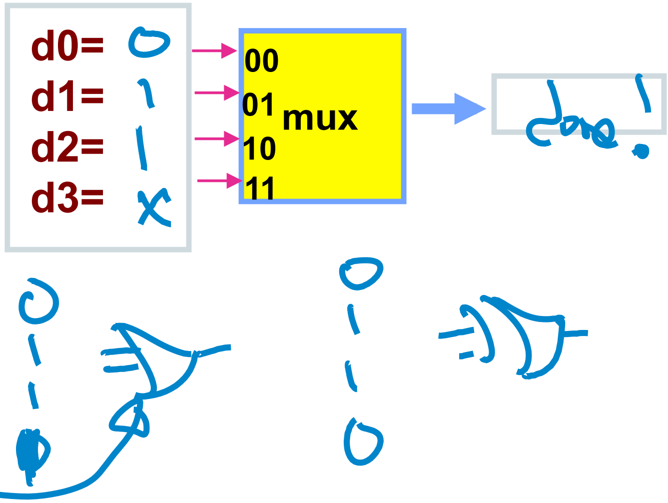
- **So, we got this: (∀ab Z)[d0,d1,d2,d3] = d0’ •d1•d2**
- **And know this: if can solve (∀ab Z)[d0,d1,d2,d3]==1 → repaired!**
- **Hey – this one is not very hard…**

d0 = 0
d1 = 1
d2 = 1
d3 = X don't care!

# Network Repair

- **Does it work? What do these d's represent?**





- **This example is tiny...**
  - But in a real example, you have a **big** network-- 100 inputs, 50,000 gates
  - When it doesn't work, it's a major hassle to go thru in detail
  - This is a mechanical procedure to answer: Can we change 1 gate to **repair**?

# *Computational* Boolean Alg Strategies

- **What haven't we seen yet? Computational strategies**
  - Example: find inputs to make **$(\forall ab\ Z)(d0,d1,d2,d3)$==1** for gate debug
  - This computation is called **Boolean Satisfiability (also called SAT)**

- **Ability to do Boolean SAT efficiently is a big goal for us**
  - **We will see how to do this in later lectures…**

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# Lecture 2.5

## Computational Boolean Algebra: URP Tautology

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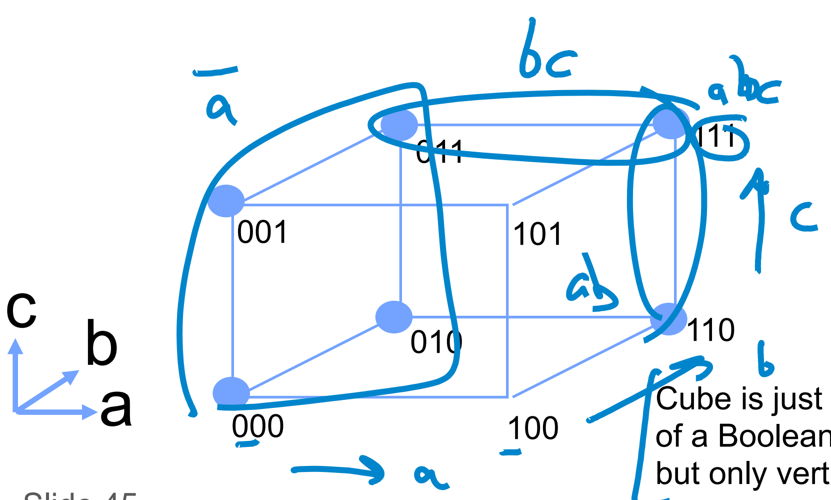
# Important Ex Computation: Tautology

- **Let's build a real computational strategy for a real problem**
- **Tautology:**
  - I will give you a **representation – a *data structure* --** for a Boolean function **f( )**
  - You build an **algorithm** to tell – yes or no – if this function **f( ) == 1** for every input
- **You might be thinking: Hey, how *hard* can that be…??**
  - Very, *very* hard.
  - What happens if I give you a function with **50 variables**…?
  - Turns out this is a great example: illustrates all the stuff we need to know...

# Start with: Representation

- **We use a simple, early representation scheme for functions**
  - Represent a function as a set of OR'ed product terms (i.e., a sum of products)
  - **Simple visual:** use a 3-var **Boolean cube**, with solid circles where **f() = 1**
  - So: each product term (circle in a Kmap) called a "**cube**" == $2^k$ corners circled



Cube is just a "graph" of a Boolean function, but only vertices have values for the function

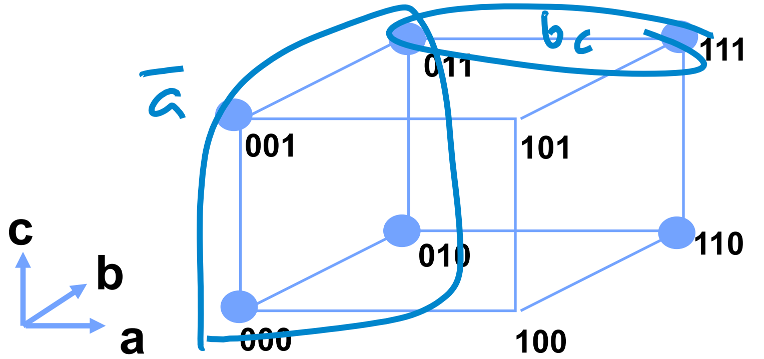
$\bar{a} + bc + ab$

Δ not nec minimal

# Positional Cube Notation (PCN)

- **So, we say ‘cube’ and mean ‘product term’**
  - So, how to represent each cube? **PCN:** one slot per variable, 2 bits per slot
  - Write each cube by just noting which variables are true, complemented, or absent
    - In slot for var **x**: put **01** if product term has **...x...** in it
    - In slot for var **x**: put **10** if product term has **...x’...** in it
    - In slot for var **x**: put **11** if product terms has **no x or x’** in it

~~[00]~~ !



$\bar{a} = [\overset{a}{10}\ \ \overset{b}{11}\ \ \overset{c}{11}]$

$bc = [11\ \ 01\ \ 01]$

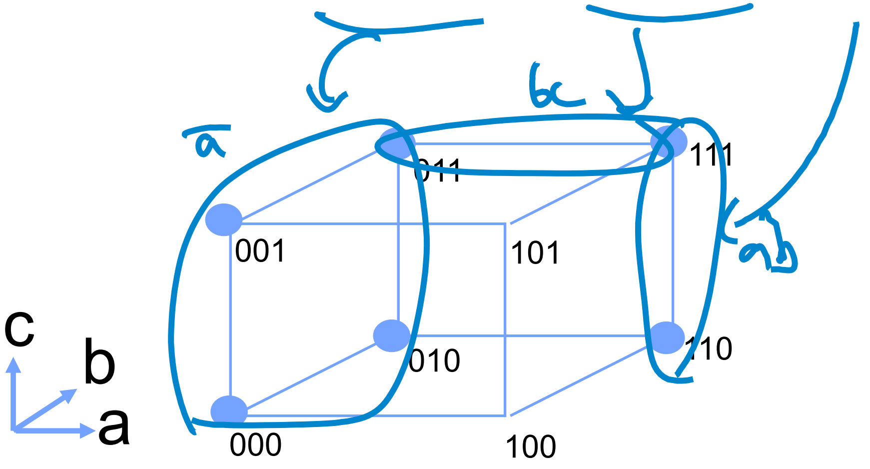
# PCN Cube List = Our Representation

- **So, we represent a function as a cover of cubes (circle its 1's)**
  - This is a **list of cubes** (this is the 'sum') in **positional cube notation** (of products)

OR

products

- Ex: **f(a,b,c)=a' +bc +ab => [01 11 11], [11 01 01], [01 01 11]**

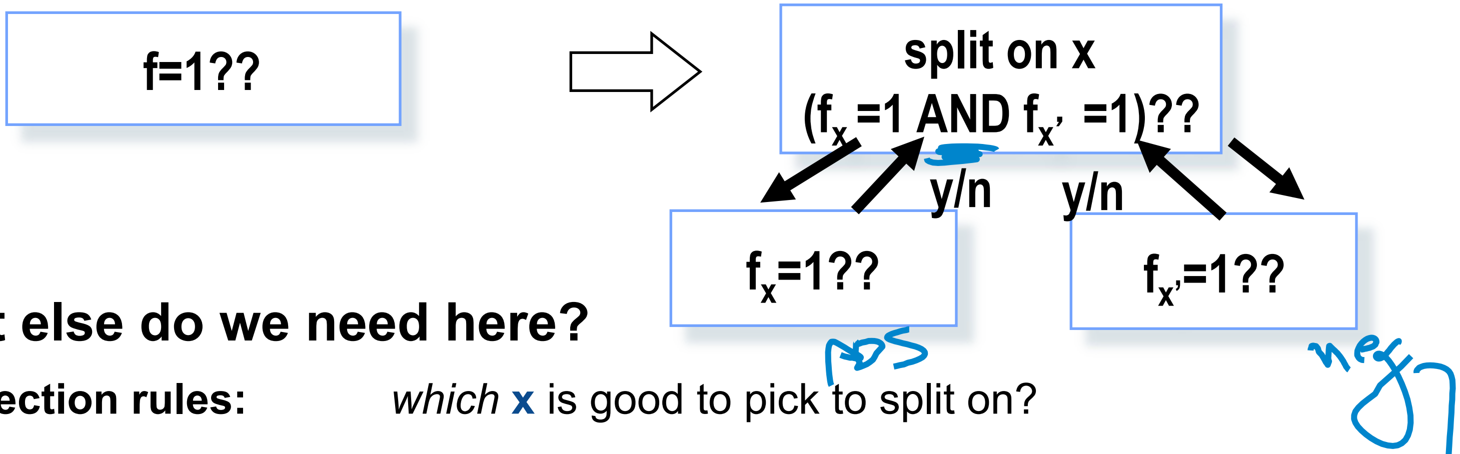


# Tautology Checking

- **How do we approach tautology as a computation?**
  - **Input** = cube-list representing products in an SOP cover of **f**
  - **Output** = yes/no, **f ==1** always or not
- **Cofactors to the rescue**
  - Great result: **f is a tautology *if and only if* fx and fx' are *both* tautologies**
  - This makes sense: f(x=0), f(x=1) = 1
    - If function $\boxed{f() = 1}$ → then cofactors both obviously **= 1** ✓
    - If **both** cofactors **= 1** → $\boxed{x\cdot F(x=1) + x'\cdot F(x=0)} = \underline{x\cdot 1 + x'\cdot 1} = \underline{x+x'} = 1$ ✓

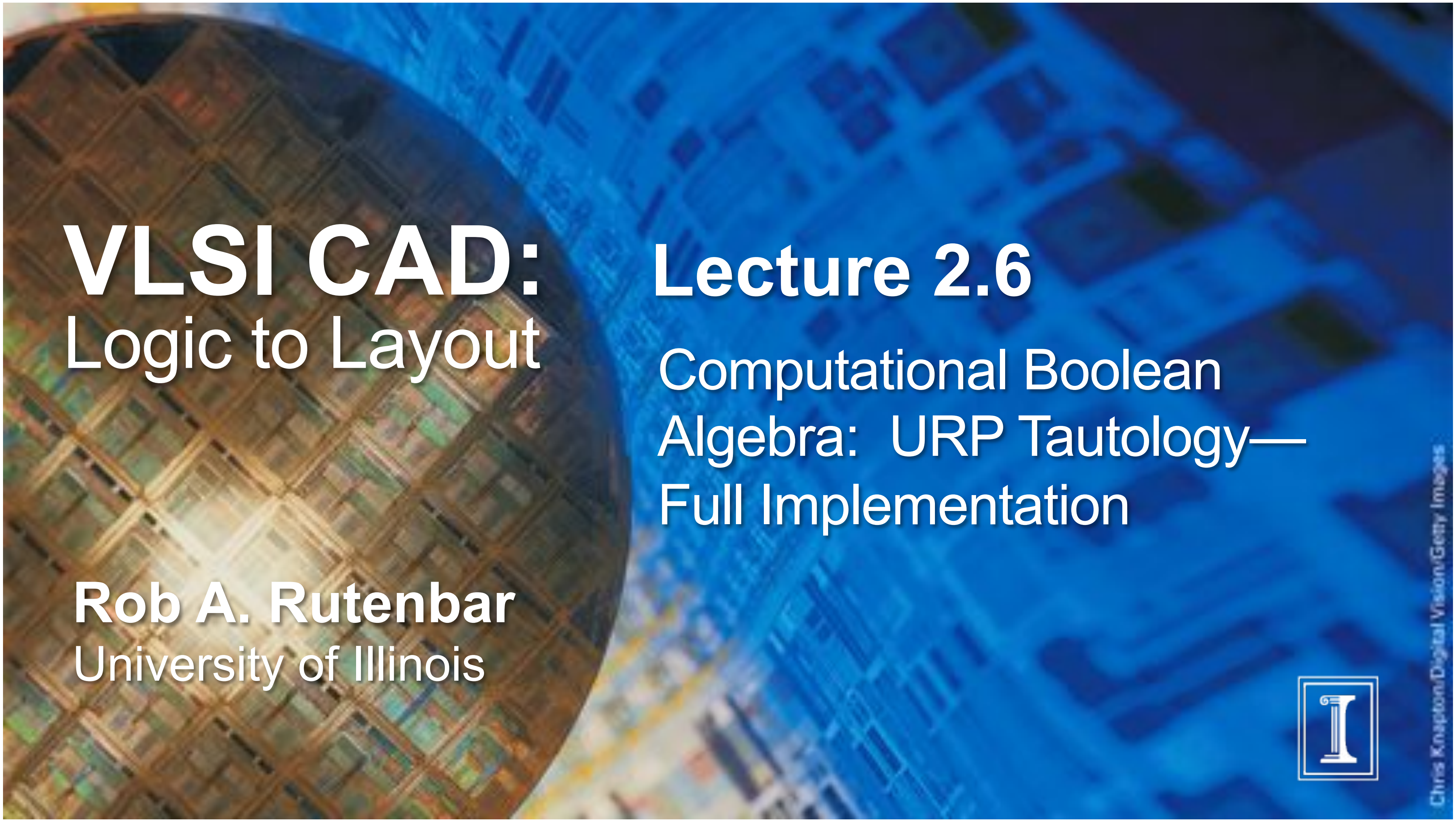
# Recursive Tautology Checking

- **Suggests a recursive computation strategy:**
  - If you cannot tell immediately that **f==1** ...go try to see if each **cofactor == 1** !



- **What else do we need here?**

| | |
|---|---|
| **Selection rules:** | *which* **x** is good to pick to split on? |
| **Termination rules:** | how do we know when to quit splitting, so we can answer **==1** or **!=1** for function at this node of tree? |
| **Mechanics:** | how hard is it to actually *represent* the cofactors? |



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# Lecture 2.6

## Computational Boolean Algebra: URP Tautology—Full Implementation

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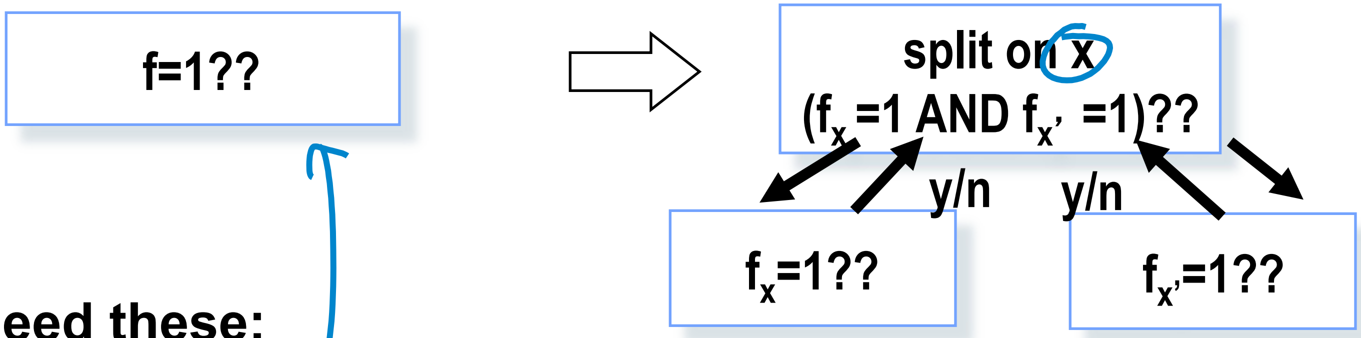
# Recursive Tautology Checking

- **So, we think a recursive computation strategy will do it:**
  - If you cannot tell immediately that **f==1** ...go try to see if each **cofactor == 1** !



$f=1??$

split on x

$(f_x=1 \text{ AND } f_{x'}=1)??$

y/n

y/n

$f_x=1??$

$f_{x'}=1??$

- **We need these:**
  - **Selection rules:** *which* **x** is good to pick to split on?
  - **Termination rules:** how do we know when to quit splitting, so we can answer **==1** or **!=1** for function at this node of tree?
  - **Mechanics:** how hard is it to actually *represent* the cofactors?

# Recursive Cofactoring

recipe

- **Do mechanics first (easy!). For each cube in your list:**
  - If you want cofactor wrt var **x=1**, look at **x** slot in each cube:
    - [... **10** ...] => just remove this cube from list, since it’s a term with an **x’**
    - [... **01** ...] => just make this slot **11** == don’t care, strike the **x** from product term
    - [... **11** ...] => just leave this alone, this term doesn’t have any **x** in it
  - If you want cofactor wrt var **x=0**, look at **x** slot in each cube:
    - [...**01** ...] => just remove this cube from list, since it’s a term with an **x**
    - [...**10** ...] => just make this slot **11** == don’t care, strike the **x’** from product term
    - [...**11** ...] => just leave this alone, this term doesn’t have any **x** in it

| | f = abd+bc’ | $f_a$ a->1 | $f_c$ c->1 |
|---|---|---|---|
| abd | [01 01 11 01] | bd [11 01 11 01] | abd [01 01 11 01] |
| $b\bar{c}$ | [11 01 10 11] | $b\bar{c}$ [11 01 10 11] | delete |

# Unate Functions

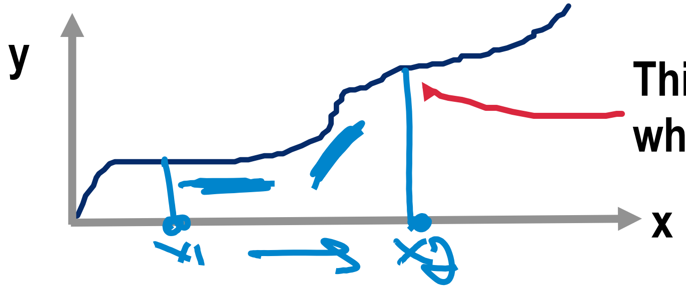
- **Selection / termination, another trick: Unate functions**
  - Special class of Boolean functions
  - **f** is ***unate*** if a SOP representation only has each literal in **exactly one polarity**, either all true, or all complemented

- **Ex:** **ab + ac'd + c'de'** is … unate: $a, b, \bar{c}, d, \bar{e}$
- **Ex:** **xy + x'y + xyz' + z'** is ... NOT unate: $x$, ~~y~~ $\bar{x}$
  is UNATE in var $y$

- **Terminology**
  - **f** is **positive unate** in var **x** -- if changing **x 0→1** keeps **f constant** or makes **f: 0→1**
  - **f** is **negative unate** in var **x** -- if changing **x 0→1** keeps **f constant** or makes **f: 1→0**
  - Function that is not unate is called **binate**

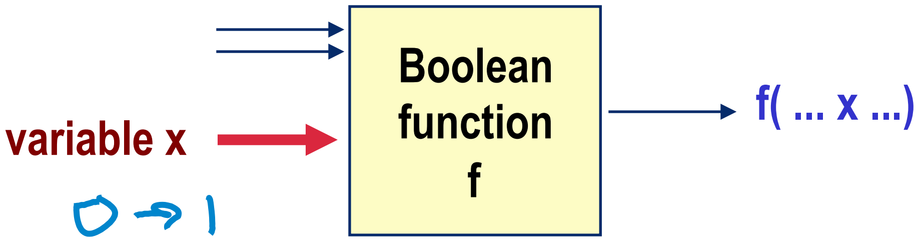
# Unate Functions

- **Analogous to monotone continuous functions**



This function is monotone non-decreasing, whenever $x2 \geq x1$, then $f(x2) \geq f(x1)$

- **Example: for a Boolean function f positive unate in x**



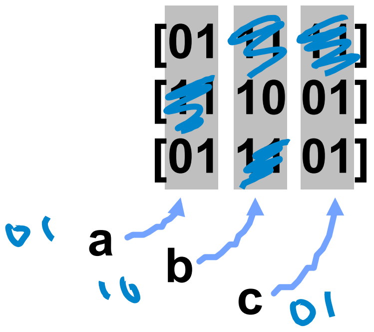
0 → 0
1 → 1 } flat

0 → 1 } increases

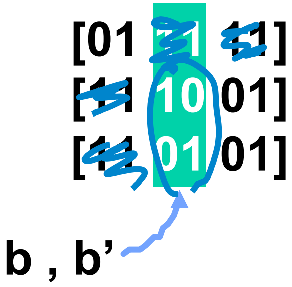
# Can Exploit Unate Func's For Computation

- **Suppose you have a cube-list for f**
  - A cube-list is **unate** if each var in each cube only appears in **one** polarity, not both
  - Ex: **f(a,b,c)=a +bc +ac** → **[01 11 11], [11 01 01], [01 11 01]** is **unate**
  - Ex: **f(a,b,c)=a +b'c +bc** → **[01 11 11], [11 10 01], [11 01 01]** is **not**
  - Easier to see if draw vertically

**a+b'c+ac UNATE**



**a+b'c+bc NOT**



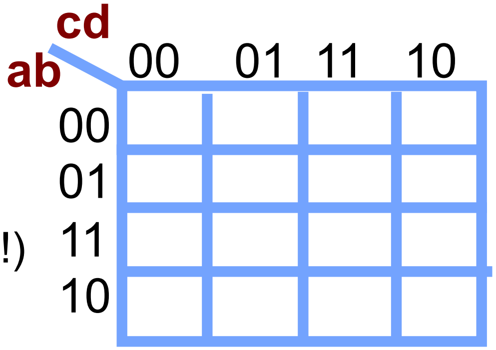
# Using Unate Functions in Tautology Checking

- **Beautiful result**
  - It is very **easy** to check a unate cube-list for tautology
  - **Unate cube-list for f is tautology iff it contains *all don't care* cube = [11 ... 11]**
  - Reminder: what exactly is [11 11 11 ... 11] as a product term?

**[01 01 01] = abc    [01 01 11] = ab    [01 11 11] = a    [11 11 11] =** 1

a b c    a b    a

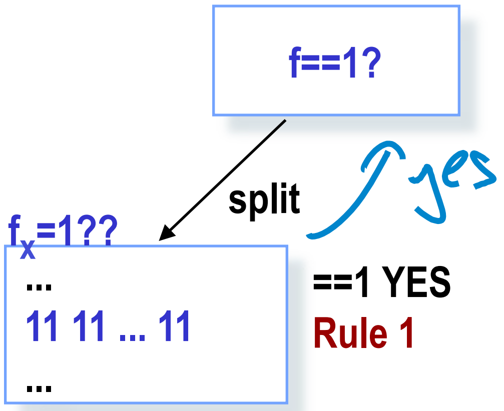
- **This result actually makes sense...**
  - Cannot make a **"1"** with only product terms where all literals are in just **one polarity**. (Try it!)



# So, Unateness Gives Us *Termination Rules*

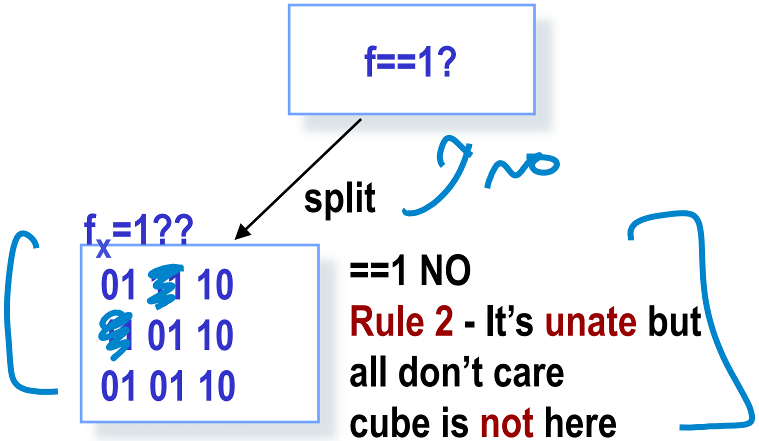
- **We can look for tautology directly, if we have a unate cube-list**
  - If match rule, know immediately if **==1**, or not
    - **Rule 1:** **==1** if cube-list has all don't care cube **[11 11 ... 11]**
      **Why:** function at this leaf is **(stuff + 1 + stuff) == 1**
    - **Rule 2:** **!=1** if cube-list unate and all don't care cube **missing**
      **Why:** unate ==1 if and only if has [11 11 ... 11] cube

easy, trivial



**==1 YES**

**Rule 1**

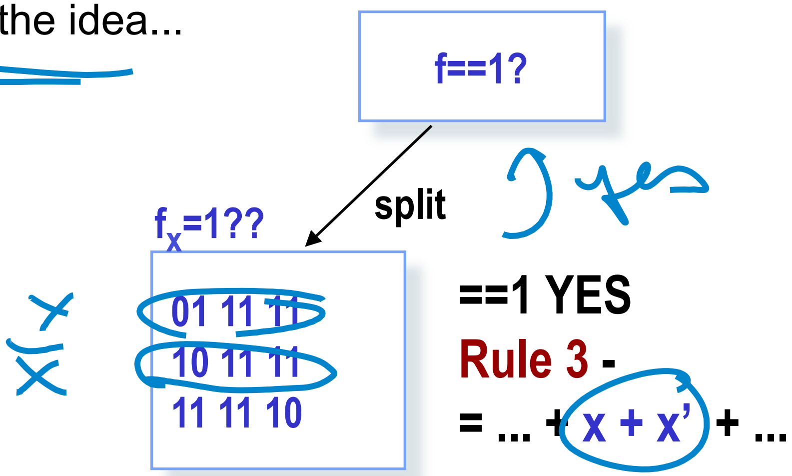


**==1 NO**

**Rule 2 - It's unate but all don't care cube is not here**

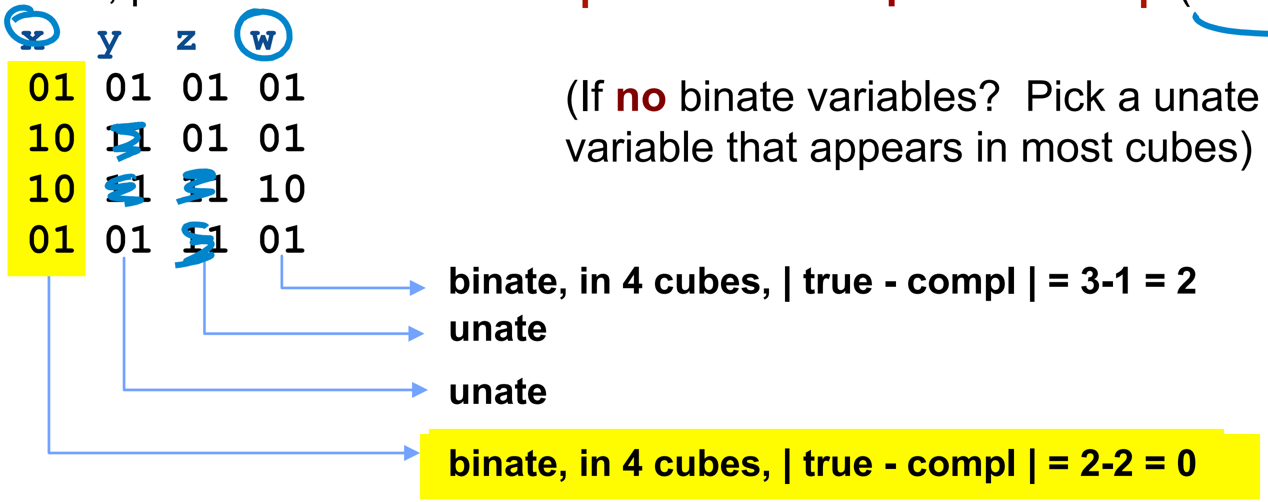
# Recursive Tautology Checking

- **Lots more possible rules...**
  - **Rule 3:** **==1** if cube list has single var cube that appears in both polarities
    **Why:** function at this leaf is **(stuff + x + x' + stuff) == 1**
- You get the idea...



# Recursive Tautology Checking

- **But can't use easy termination rules *unless* unate cubelist**
- **Selection rule…? Pick splitting var to *make* unate cofactors!**
  - **Strategy:** pick "most not-unate" (binate) var as split var ) heuristic!!
  - ① Pick binate var with **most** product terms dependent on it (Why? *Simplify more cubes*)
  - If a tie, pick var with **minimum | true var - complement var |** (L-R *subtree balance*)



| x | y | z | w |
|---|---|---|---|
| 01 | 01 | 01 | 01 |
| 10 | 11 | 01 | 01 |
| 10 | 11 | 11 | 10 |
| 01 | 01 | 11 | 01 |

w: **binate, in 4 cubes, | true - compl | = 3-1 = 2**

z: **unate**

y: **unate**

x: **binate, in 4 cubes, | true - compl | = 2-2 = 0**

(If **no** binate variables? Pick a unate variable that appears in most cubes)

# Recursive Tautology Checking: *Done*!

- **Algorithm**

```
tautology(f represented as cubelist) {
    /* check if we can terminate recursion */
    if (f is unate)  {
        apply unate tautology termination rules directly
        if (==1) return (1)
        else return (0)
    }
    else if (any other termination rules, like rule 3, work?) {
        return the appropriate value if ==1 or ==0
    }
    else {   /* can't tell from this -- find splitting variable */
        x = most-not-unate variable in f
        return (  tautology( $f_x$ ) && tautology( $f_{x'}$ )   )
    }
}
```

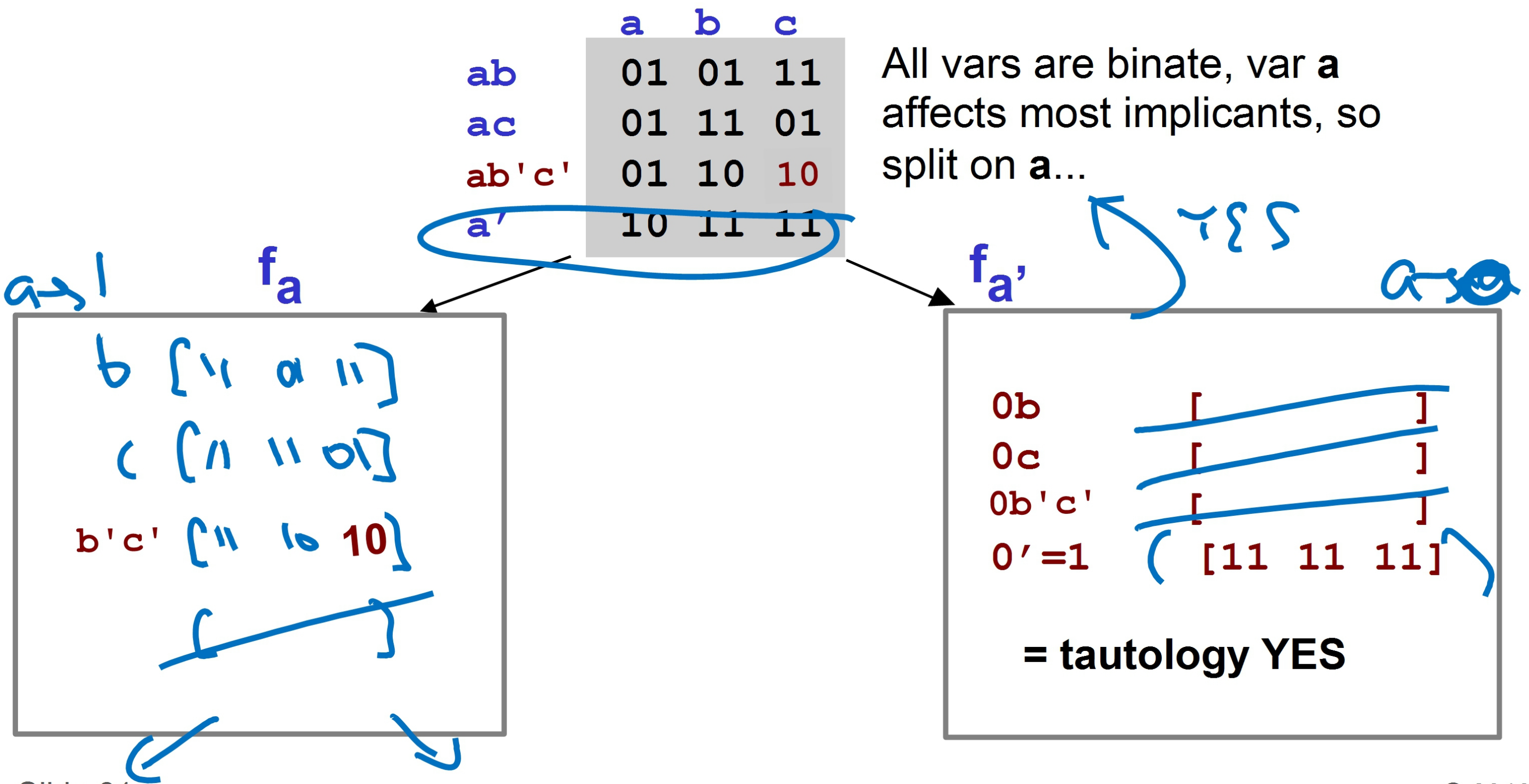
terminate

select

recurse

# Recursive Tautology Checking: Example

- **Tautology example:  f = ab + ac + ab'c' + a'**



| | a | b | c |
|---|---|---|---|
| ab | 01 | 01 | 11 |
| ac | 01 | 11 | 01 |
| ab'c' | 01 | 10 | 10 |
| a' | 10 | 11 | 11 |

All vars are binate, var **a** affects most implicants, so split on **a**...

YES

**$f_a$**

a→1

b [11 01 11]

c [11 11 01]

b'c' [11 10 **10**]

~~[ ]~~

**$f_{a'}$**

a→0

~~0b [ ]~~

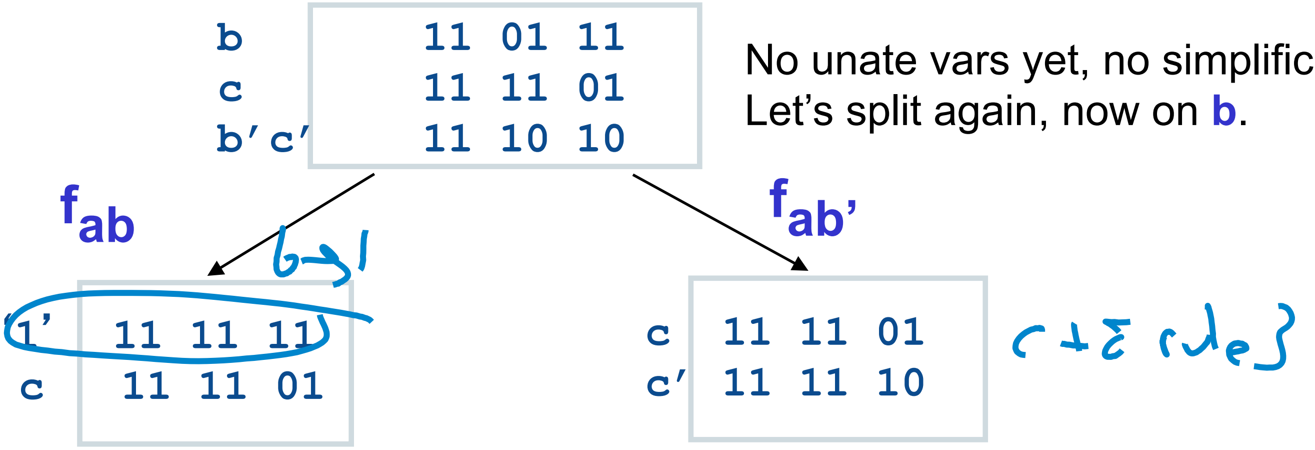
~~0c [ ]~~

~~0b'c' [ ]~~

0'=1 [11 11 11]

**= tautology YES**

# Recursive Tautology Checking: Example



```
b      11 01 11
c      11 11 01
b'c'   11 10 10
```

No unate vars yet, no simplifications.
Let's split again, now on **b**.

$f_{ab}$ b→1

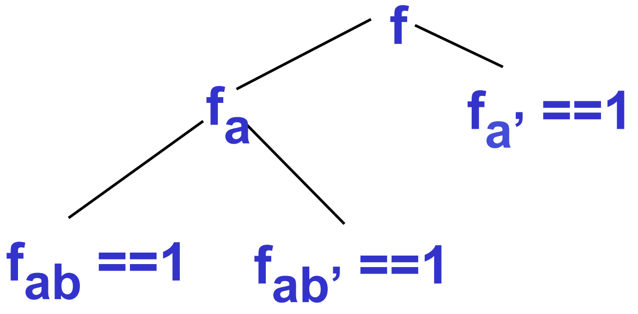
```
'1'    11 11 11
c      11 11 01
```

$f_{ab'}$

```
c      11 11 01
c'     11 11 10
```

$c + \bar{c}$ rule}

- **So we are done:**
  - Our tree has tautologies at all leaves!
  - Note -- if any leaf **!=1**, then **f != 1** too, this is how tautology fails



# Computational Boolean Algebra

- **Computational philosophy revisited**
  - Strategy is so general and useful it has a name: **Unate Recursive Paradigm**
  - Abbreviated usually as **"URP"**

- **Summary**
  - **Cofactors** and **functions of cofactors** are important and useful
    - Boolean difference, quantifications; real applications like network repair
  - **Representations (data structures)** for Boolean functions are critical
    - Truth tables, Kmaps, equations ***cannot be manipulated*** by software
    - Saw one real representation: Cube-list, positional cube notation