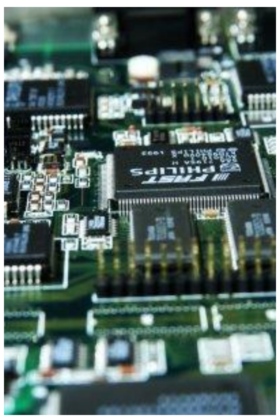
# CSE 401

# Computer Engineering (2)

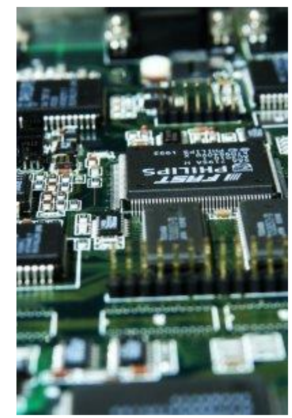
# هندسة الحاسبات (2)



4th year, Comm. Engineering

Winter 2016

**Lecture #10**



Dr. Hazem Ibrahim Shehata

Dept. of Computer & Systems Engineering

Credits to Dr. Ahmed Abdul-Monem Ahmed for the slides

# Adminstrivia

- Midterm:
  - To be rescheduled for next week.
  - Same coverage: lectures #1 ➔ #7.
- Assignment #2:
  - For those who emailed me a softcopy, please hand in the original hardcopy this week!!
- Tutorial:
  - To be held on Wednesday at 12:00pm.

# Chapter 10. Computer Arithmetic (*Cont.*)

# Outline

- Integer Representation
  - Sign-Magnitude, Two's Complement, Biased
- Integer Arithmetic
  - Negation, Addition, Subtraction
  - Multiplication, Division
- Floating-Point Representation
  - IEEE 754
- Floating-Point Arithmetic
  - Addition, Subtraction
  - Multiplication, Division
  - Rounding

# Real Numbers

- Numbers with fractions.
- Could be done in pure binary
  - $1001.1010 = 2^3 + 2^0 + 2^{-1} + 2^{-3} = 9.625$
- Where is the binary point?
- Fixed? **0 0 1 0 1 1 0 1 0 0 . 1 1 1 0 1 0**
  - Very large/small numbers cannot be represented.
    - e.g., 0.0000001, 10000000000
  - Fractional part of the quotient in dividing very large numbers will be lost.
- Moving/floating?
  - How do you show where it is?
  - $976{,}000{,}000{,}000{,}000 = 9.76 \times 10^{14}$
  - $0.0000000000000976 = 9.76 \times 10^{-14}$

**Can do the same with binary numbers. What do we need to store?**

# Floating-Point Representation

$$\pm S \times 2^{E}$$

| Sign | Exponent | Significand (Mantissa) |
|---|---|---|

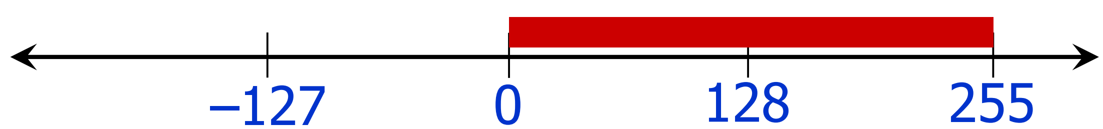
- The base 2 is the same for all numbers ➔ need not be stored.
- Number is stored in a binary word with 3 fields:
  - Sign: +/–
  - Significand S
  - Exponent E
- Normal number: most significant digit of the significand (mantissa) is nonzero ➔ 1 for base 2 (binary).
- What number to store in the significand field? **0 . 0 0 1 0 1 1**
  - Normal form: $1.011 \times 2^{-3}$ ➔ Store only 011 in the significand field!
- There is an implicit 1 to the left of the binary point (normalized).
- Exponent indicates place value (floating-point position).

# Floating-Point Representation
## Biased Exponent

$\pm S \times 2^{E}$

| Sign | Biased Exponent | Significand (Mantissa) |
|---|---|---|

- k-bit unsigned exponent E′ ranges from 0 to $2^k-1$
  - e.g., 8-bit exponent: $0 \le E' \le 255$
- The stored exponent E′ is a biased exponent
  - $E' = E + (2^{k-1}-1)$ bias
  - e.g., for 8-bit exponent, $E' = E + 127$
  - $-127 \le E \le 128$




- Why?
  - Nonnegative floating-point numbers can be treated as unsigned integers for comparison purposes.
  - This is not true for 2′s comp. or sign-magnitude representations.

# Normalization

- FP numbers are usually normalized.
  - i.e., exponent is adjusted so that leading bit (MSB) of mantissa is non-zero, i.e., 1.
  - c.f., Scientific notation where numbers are normalized to give a single digit before the decimal point, e.g. $3.123 \times 10^3$.
- Since the MSB of mantissa is always 1, there is no need to store it!

# Floating-Point Examples

| Sign | Biased Exponent (8 bits) | Significand (Mantissa) (23 bits) |
|---|---|---|

1717698.56

$1.638125 \times 2^{20}$

$1.1010001 \times 2^{10100}$

Positive ➔ sign bit = 0
E′ = E + 127 = 10100 + 1111111 = 10010011
Mantissa = 1010001 0000000000000000

**0 10010011 10100010000000000000000**

-1717698.56

$-1.638125 \times 2^{20}$

$-1.1010001 \times 2^{10100}$

Negative ➔ sign bit = 1
E′ = E + 127 = 10100 + 1111111 = 10010011
Mantissa = 1010001 0000000000000000

**1 10010011 10100010000000000000000**

. . .

$1.638125 \times 2^{-20}$

$1.1010001 \times 2^{-10100}$

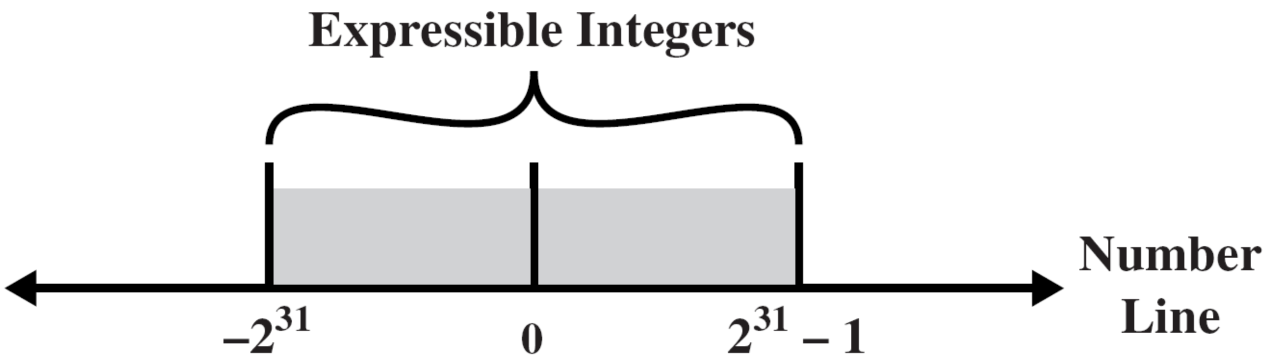
Positive ➔ sign bit = 0
E′ = E + 127 = −10100 + 1111111 = 01101011
Mantissa = 1010001 0000000000000000

**0 01101011 10100010000000000000000**

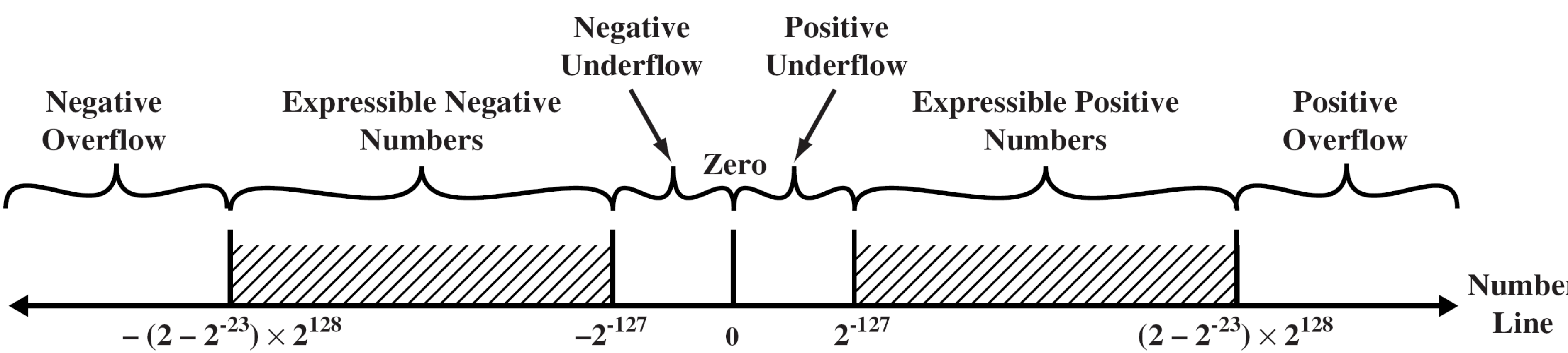
# FP Ranges (32-bit)

- 32-bit FP number, 8-bit exponent, 23-bit mantissa.
- Largest +ve number $(2-2^{-23}) \times 2^{128}$
  - —Largest true exponent: 128
  - —Largest mantissa: $1 + (1 - 2^{-23}) = 2 - 2^{-23}$ 0.111…11
- Smallest +ve number $2^{-127}$
  - —Smallest true exponent: –127
  - —Smallst mantissa: 1
- Smallest –ve number $-(2-2^{23}) \times 2^{128}$
- Largest –ve number $-2^{-127}$
- Accuracy
  - —The effect of changing LSB of mantissa.
  - —23-bit mantissa $2^{-23} \approx 1.2 \times 10^{-7}$
  - —About 6 decimal places.

# Expressible Numbers (32-bit)



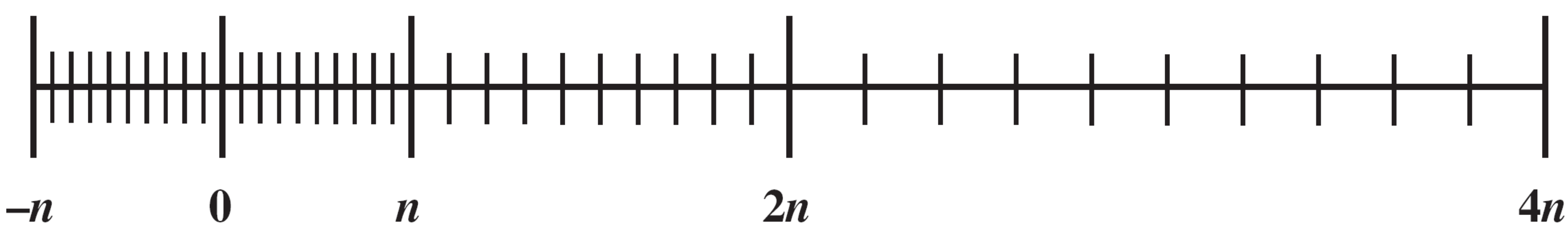

(a) Twos Complement Integers




(b) Floating-Point Numbers

# Density of Floating Point Numbers

- 32-bit FP number ➔ $2^{32}$ different values represented.
- No more individual values are represented with floating-point numbers. Numbers are just spread out.
- Numbers represented in the FP representation are not spaced evenly along the line number. Why?
- Range-precision tradeoff
  - More bits for exponent ➔ wider range & less precision
  - Reason: there is a fixed number of values that can be represented!
  - To increase both range and precision ➔ use more bits!!!

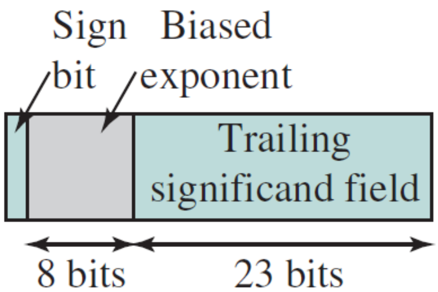


# IEEE 754

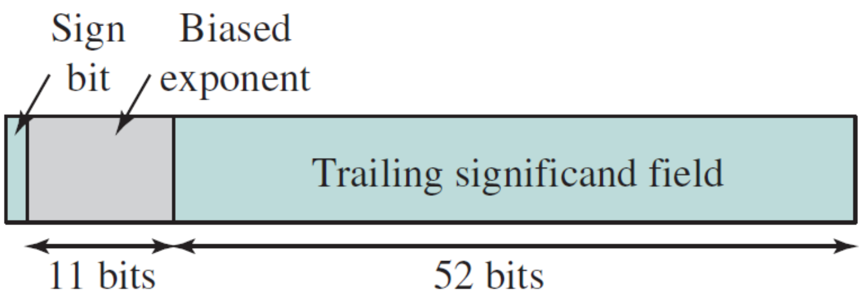
- Standard for floating-point representation.
- Adopted 1985 and revised 2008.
- IEEE 754-2008 defines many FP formats for different purposes:

| Format | Format Type | | |
|---|---|---|---|
| | Arithmetic Format | Basic Format | Interchange Format |
| binary16 | | | X |
| binary32 | X | X | X |
| binary64 | X | X | X |
| binary128 | X | X | X |
| binary{$k$}<br>($k = n \times 32$ for $n > 4$) | X | | X |
| decimal64 | X | X | X |
| decimal128 | X | X | X |
| decimal{$k$}<br>($k = n \times 32$ for $n > 4$) | X | | X |
| extended precision | X | | |
| extendable precision | X | | |

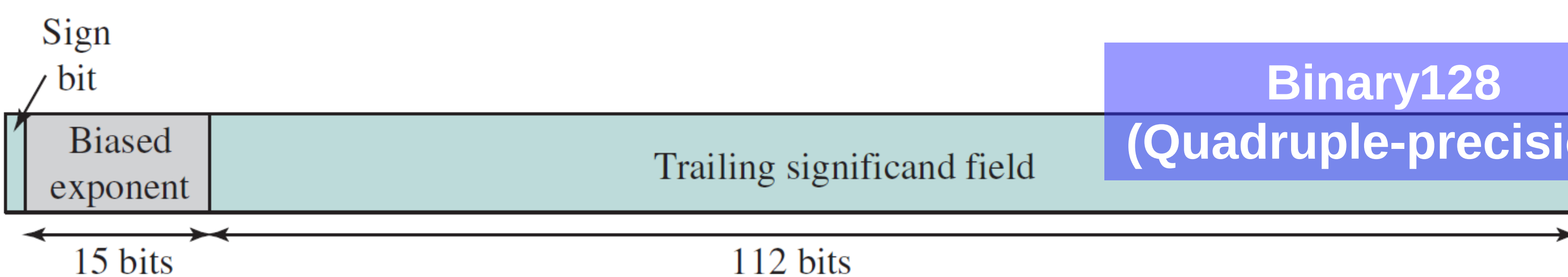
# IEEE 754 - Binary32/64/128 Formats




**Binary32 (Single-precision)**




**Binary64 (Double-precision)**




**Binary128 (Quadruple-precision)**

# IEEE 754 - Binary32/64/128 Interpretations

| | Sign | Biased Exponent | Fraction | Value |
|---|---|---|---|---|
| positive zero | 0 | 0 | 0 | 0 |
| negative zero | 1 | 0 | 0 | $-0$ |
| plus infinity | 0 | all 1s | 0 | $\infty$ |
| minus infinity | 1 | all 1s | 0 | $-\infty$ |
| quiet NaN | 0 or 1 | all 1s | $\neq 0$; first bit $= 1$ | qNaN |
| signaling NaN | 0 or 1 | all 1s | $\neq 0$; first bit $= 0$ | sNaN |

**32**

| | Sign | Biased Exponent | Fraction | Value |
|---|---|---|---|---|
| positive normal nonzero | 0 | $0 < e < 255$ | f | $2^{e-127}(1.f)$ |
| negative normal nonzero | 1 | $0 < e < 255$ | f | $-2^{e-127}(1.f)$ |
| positive subnormal | 0 | 0 | $f \neq 0$ | $2^{-126}(0.f)$ |
| negative subnormal | 1 | 0 | $f \neq 0$ | $-2^{-126}(0.f)$ |

**64**

| | Sign | Biased Exponent | Fraction | Value |
|---|---|---|---|---|
| positive normal nonzero | 0 | $0 < e < 2047$ | f | $2^{e-1023}(1.f)$ |
| negative normal nonzero | 1 | $0 < e < 2047$ | f | $-2^{e-1023}(1.f)$ |
| positive subnormal | 0 | 0 | $f \neq 0$ | $2^{-1022}(0.f)$ |
| negative subnormal | 1 | 0 | $f \neq 0$ | $-2^{-1022}(0.f)$ |

**128**

| | Sign | Biased Exponent | Fraction | Value |
|---|---|---|---|---|
| positive normal nonzero | 0 | $0 < e < 32767$ | f | $2^{e-16383}(1.f)$ |
| negative normal nonzero | 1 | $0 < e < 32767$ | f | $-2^{e-16383}(1.f)$ |
| positive subnormal | 0 | 0 | $f \neq 0$ | $2^{-16382}(0.f)$ |
| negative subnormal | 1 | 0 | $f \neq 0$ | $-2^{-16382}(0.f)$ |

# IEEE 754 - Binary32/64/128 Parameters

| Parameter | Format | | |
|---|---|---|---|
| | Binary32 | Binary64 | Binary128 |
| Storage width (bits) | 32 | 64 | 128 |
| Exponent width (bits) | 8 | 11 | 15 |
| Exponent bias | 127 | 1023 | 16383 |
| Maximum exponent | 127 | 1023 | 16383 |
| Minimum exponent | $-126$ | $-1022$ | $-16382$ |
| Approx normal number range (base 10) | $10^{-38}, 10^{+38}$ | $10^{-308}, 10^{+308}$ | $10^{-4932}, 10^{+4932}$ |
| Trailing significand width (bits)* | 23 | 52 | 112 |
| Number of exponents | 254 | 2046 | 32766 |
| Number of fractions | $2^{23}$ | $2^{52}$ | $2^{112}$ |
| Number of values | $1.98 \times 2^{31}$ | $1.99 \times 2^{63}$ | $1.99 \times 2^{128}$ |
| Smallest positive normal number | $2^{-126}$ | $2^{-1022}$ | $2^{-16362}$ |
| Largest positive normal number | $2^{128} - 2^{104}$ | $2^{1024} - 2^{971}$ | $2^{16384} - 2^{16271}$ |
| Smallest subnormal magnitude | $2^{-149}$ | $2^{-1074}$ | $2^{-16494}$ |

*Note*: *not including implied bit and not including sign bit

# IEEE 754 - NaNs

- NaN:
  - Symbolic entity encoded in FP format
  - Types: Signaling (sNaN) or Quiet (qNaN)
  - Both types have the same format:

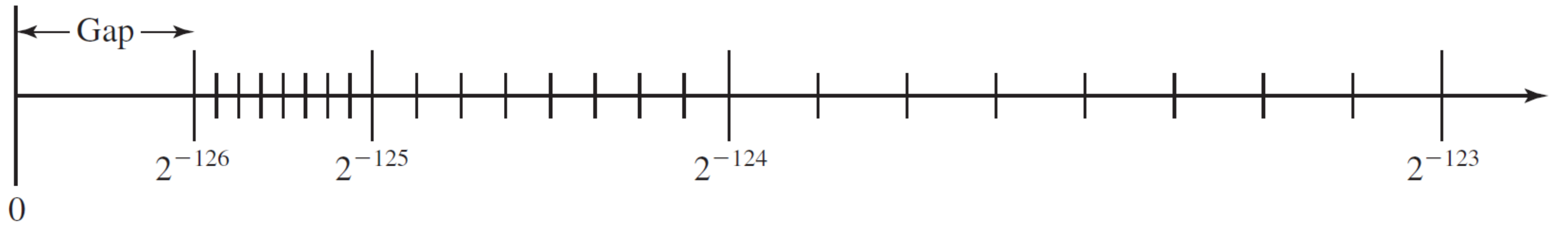
| | | |
|---|---|---|
| **S= 0 or 1** | **E= 1111…11** | **F ≠ 0000..00** |

  - F distinguishes between the two types:
    - F=**0**xxxx..xx ➔ sNaN, F=**1**xxxx..xx ➔ qNaN
- Signaling NaN:
  - Signals an invalid operation exception whenever it appears as an operand. Ex.: uninitialized variables
- Quite NaN:
  - Propagates without signaling exceptions.

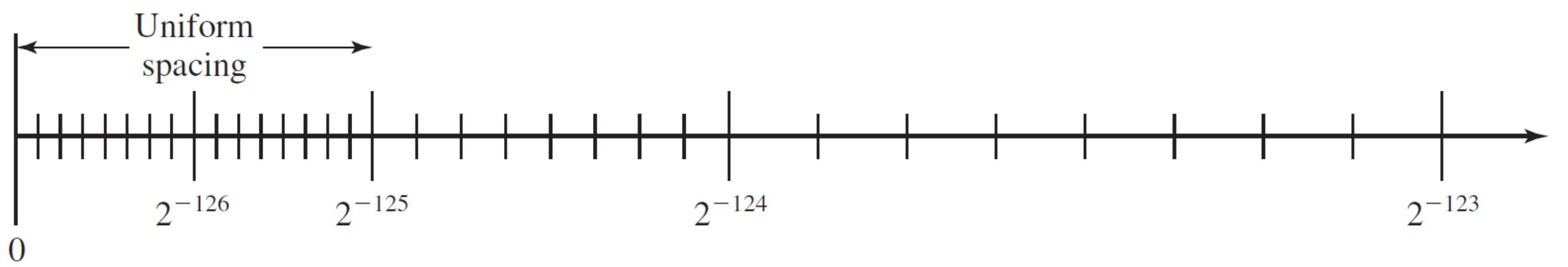
# IEEE 754 - Quiet NaN

| Operation | Quiet NaN Produced By |
|---|---|
| Any | Any operation on a signaling NaN |
| Add or subtract | Magnitude subtraction of infinities:<br>$(+\infty) + (-\infty)$<br>$(-\infty) + (+\infty)$<br>$(+\infty) - (+\infty)$<br>$(-\infty) - (-\infty)$ |
| Multiply | $0 \times \infty$ |
| Division | $\frac{0}{0}$ or $\frac{\infty}{\infty}$ |
| Remainder | $x$ REM 0 or $\infty$ REM $y$ |
| Square root | $\sqrt{x}$, where $x < 0$ |

# IEEE 754 - Effect of Subnormal Numbers




(a) 32-Bit format without subnormal numbers

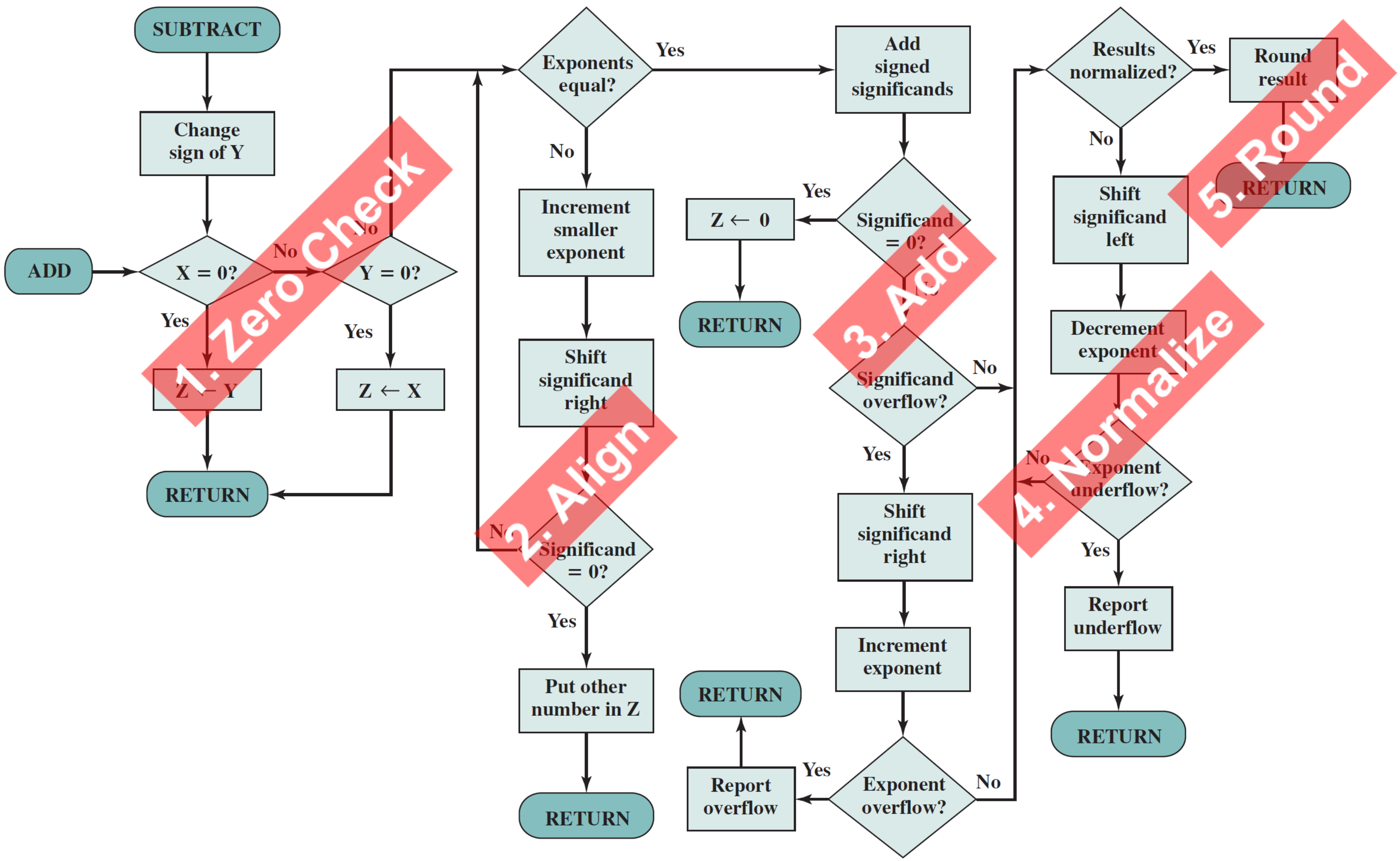



(b) 32-Bit format with subnormal numbers

# FP Arithmetic +/-

- Algorithm:
  1. Check for zeros.
  2. Align significands (adjusting exponents).
  3. Add or subtract significands.
  4. Normalize result.
  5. Round result.

# FP Addition & Subtraction Flowchart



# Reading Material

- Stallings, Chapter 10:
  - Pages 341-352
  - Pages 356-358