CSE 620: Selective Topics

Introduction to Formal Verification

Master Studies in CSE

Fall 2017

Lecture #5

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Course Outline

- Computational Boolean Algebra
  - Basics
    - Shannon Expansion
    - Boolean Difference
    - Quantification Operators
      - Application to Logic Network Repair
  - Validity Checking (Tautology Checking)
  - Binary Decision Diagrams (BDD's)
  - Satisfiability Checking (SAT solving)
- Model Checking
  - Temporal Logics → LTL - CTL
  - SMV: Symbolic Model Verifier
  - Model Checking Algorithms → Explicit CTL

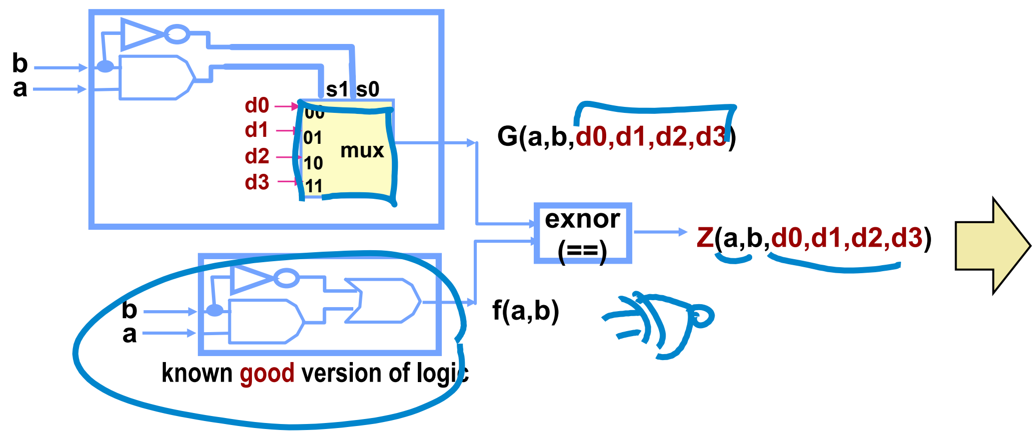

VLSI CAD:
Logic to Layout
Lecture 4.1
Computational Boolean Algebra Representations: Satisfiability (SAT), Part 1
Rob A. Rutenbar
University of Illinois
Chris Knapton/Digital Vision/Getty Images

Some Terminology

- **Satisfiability (called "SAT" for short)**
  - Give me an appropriate representation of function **F(x1, x2, … xn)**
  - Find an assignment of the variables ("**vars**" for short) **(x1, x2, … xn)** so **F( ) = 1**
  - Note – this assignment need not be unique. Could be many satisfying solutions
  - But if there are no satisfying assignments at all – prove it, and return this info

- **Some things you can do with BDDs, can do *easier* with SAT**
  - SAT is aimed at scenarios where you just need **one satisfying assignment**…
  - …or prove that there is **no such satisfying assignment**

Example: Network Repair



$(\forall a,b\ Z)[d0,d1,d2,d3] = 1$?

- **Assuming you can build the quantification function $(\forall a,b\ Z)$**
  - Go find a SAT assignment to get you the **d** values to repair the network
  - Or, if unSAT, know that there is no network repair possible

Standard SAT Form: CNF

- **Conjunctive Normal Form (CNF) = Standard POS form** (SOP)

$$\phi = \underbrace{(a + c)}_{\text{clause}}\,(b + \underbrace{c}_{\text{positive literal}})\,(\neg a + \neg b + \underbrace{\neg c}_{\text{negative literal}})$$

¬ = not

$= \bar{c} = c'$

- **Why CNF is useful**
  - Need only determine that ***one*** clause evaluates to **“0”** to know whole formula = **“0”**
  - Of course, to satisfy the whole formula, you must make ***all*** clauses identically **“1”**.

Assignment to a CNF Formula

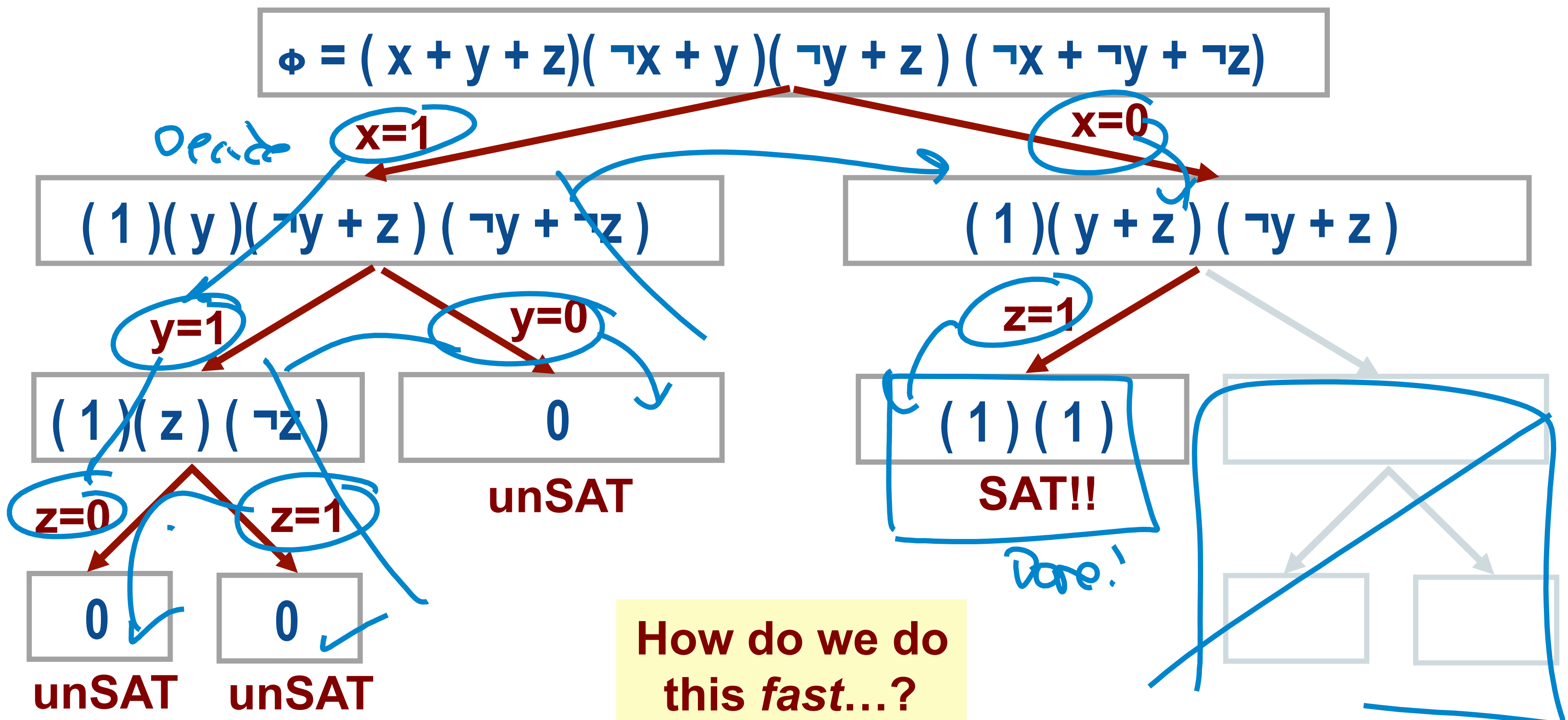
- **An *assignment*…**
  - …gives values to some, not necessarily all, of variables (vars) **xi** in **(x1, x2, … , xn)**.
  - ***Complete*** assignment: assigns value to all vars. ***Partial***: some, not all, have values
- **Assignment means we can evaluate status of the clauses**
  - Suppose $\boxed{\textbf{a=0, b=1} \text{ but } \textbf{c, d}}$ are **unassigned**

$$\Phi = \underbrace{(\cancel{a} + \cancel{\neg b})}_{\text{clause = 0 conflicting}}\ \underbrace{(\cancel{\neg a} + \cancel{b} + \neg c)}_{\text{clause = 1 satisfied}}\ \underbrace{(\cancel{a} + c + d)}_{\text{unresolved}}\ \underbrace{(\cancel{\neg a} + \cancel{\neg b} + \neg c)}_{\text{SAT}}$$

How Do We “Solve” This?

- **Recursively (…surprised?)**
  - Strategy has two big ideas
  - **DECISION:**
    - ***Select*** a variable and ***assign*** its value; ***simplify*** CNF formula as far as you can
    - Hope you can decide if it’s SAT, yes/no, without further work
  - **DEDUCTION:**
    - Look at the newly simplified clauses
    - ***Iteratively simplify***, based on structure of clauses, and value of partial assignment
    - Do this until nothing simplifies. If you can decide SAT yes/no, great.
    - If not, then you have to ***recurse*** some more, back up to DECIDE

How Do We "Solve" This?



How do we do this *fast*…?

VLSI CAD:
Logic to Layout

Lecture 4.2

Computational Boolean Algebra Representations: Boolean Constraint Propagation (BCP) for SAT

Rob A. Rutenbar
University of Illinois

BCP: Boolean Constraint Propagation

- **To do "deduction", use BCP**
  - Given a set of ***fixed*** variable assignments, what else can you "***deduce***" about necessary assignments by "***propagating constraints***"
- **Most famous BCP strategy is "Unit Clause Rule"**
  - A clause is said to be "**unit**" if it has exactly one unassigned literal
  - Unit clause has exactly **one** way to be satisfied, ie, pick polarity that makes clause=**"1"**
  - This choice is called an **"implication"**

$$\phi = (\cancel{a}^{1} + c)(\cancel{b}^{1} + c)(\cancel{\neg a}^{0} + \cancel{\neg b}^{0} + \neg c)$$

SAT SAT

Assume:
a=1, b=1

UNIT: 1 unassigned literal $= \bar{c}$
c must be 0 → SAT!

BCP Iterative – Go Until No More Implications

$\phi = (\omega_1)(\omega_2)(\omega_3)(\omega_4)(\omega_5)(\omega_6)(\omega_7)(\omega_8)(\omega_9)$

$\omega_1 = (\neg x1 + x2)$

$\omega_2 = (\neg x1 + x3 + \overset{0}{\cancel{x9}})$

$\omega_3 = (\neg x2 + \neg x3 + x4)$

$\omega_4 = (\neg x4 + x5 + \overset{0}{\cancel{x10}})$

$\omega_5 = (\neg x4 + x6 + \overset{0}{\cancel{x11}})$

$\omega_6 = (\neg x5 + \neg x6)$

$\omega_7 = (x1 + x7 + \overset{0}{\cancel{\neg x12}})$

$\omega_8 = (x1 + x8)$

$\omega_9 = (\neg x7 + \neg x8 + \overset{0}{\cancel{\neg x13}})$

No SAT
No BCP
Now what?

- **Example from**
  - J.P. Marques-Silva and K. Sakallah, “GRASP: A Search Algorithm for Propositional Satisfiability”, *IEEE Trans. Computers*, Vol 8, No 5, May’99
- **Partial assignment is:**
  - **x9=0 x10=0 x11=0 x12=1 x13=1**
- **To start…**
  - What are ***obvious*** simplifications when we assign these variables?

BCP Iterative – Go Until No More Implications

$\phi = (\omega_1)(\omega_2)(\omega_3)(\omega_4)(\omega_5)(\omega_6)(\omega_7)(\omega_8)(\omega_9)$

Unit

$\omega_1 = (\cancel{\neg x1}^{0} + x2)$

$\omega_2 = (\cancel{\neg x1}^{0} + x3 + \cancel{x9}^{0})$

$\omega_3 = (\neg x2 + \neg x3 + x4)$

$\omega_4 = (\neg x4 + x5 + \cancel{x10}^{0})$

$\omega_5 = (\neg x4 + x6 + \cancel{x11}^{0})$

$\omega_6 = (\neg x5 + \neg x6)$

SAT

$\omega_7 = (\cancel{x1}^{1} + x7 + \cancel{\neg x12}^{0})$

$\omega_8 = (\cancel{x1}^{1} + x8)$

$\omega_9 = (\neg x7 + \neg x8 + \cancel{\neg x13}^{0})$

- **Partial assignment is:**
  - x9=0 x10=0 x11=0 x12=1 x13=1
- **Next: Assign a variable to value**
  - Assign x1=1

Implications!

x1=1 → x2=1 && x3=1

BCP Iterative – Go Until No More Implications

$\phi = (\omega_1)(\omega_2)(\omega_3)(\omega_4)(\omega_5)(\omega_6)(\omega_7)(\omega_8)(\omega_9)$

$\omega_1 = (\cancel{\neg x1}^{0} + \cancel{x2}^{1})$ — SAT

$\omega_2 = (\cancel{\neg x1}^{0} + \cancel{x3}^{1} + \cancel{x9}^{0})$ — SAT

$\omega_3 = (\cancel{\neg x2}^{0} + \cancel{\neg x3}^{0} + x4)$ — UNIT

$\omega_4 = (\neg x4 + x5 + \cancel{x10}^{0})$

$\omega_5 = (\neg x4 + x6 + \cancel{x11}^{0})$

$\omega_6 = (\neg x5 + \neg x6)$

$\omega_7 = (\cancel{x1}^{1} + x7 + \cancel{\neg x12}^{0})$ — SAT

$\omega_8 = (\cancel{x1}^{1} + x8)$ — SAT

$\omega_9 = (\neg x7 + \neg x8 + \cancel{\neg x13}^{0})$

- **Partial assignment is:**
  - **x9=0 x10=0 x11=0 x12=1 x13=1**
- **Next: Assign a var to value**
  - Assign **x1=1**
  - **Assign (implied): x2=1, x3=1**

Implications!
x2=1,x3=1 → x4=1

BCP Iterative – Go Until No More Implications

$\phi = (\omega_1)(\omega_2)(\omega_3)(\omega_4)(\omega_5)(\omega_6)(\omega_7)(\omega_8)(\omega_9)$

$\omega_1 = (\cancel{\neg x1}^{0} + \cancel{x2}^{1})$

$\omega_2 = (\cancel{\neg x1}^{0} + \cancel{x3}^{1} + \cancel{x9}^{0})$

$\omega_3 = (\cancel{\neg x2}^{0} + \cancel{\neg x3}^{0} + \cancel{x4}^{1})$

SAT

$\omega_4 = (\cancel{\neg x4}^{0} + x5 + \cancel{x10}^{0})$

$\omega_5 = (\cancel{\neg x4}^{0} + x6 + \cancel{x11}^{0})$

UNIT

$\omega_6 = (\neg x5 + \neg x6)$

$\omega_7 = (\cancel{x1}^{1} + x7 + \cancel{\neg x12}^{0})$

$\omega_8 = (\cancel{x1}^{1} + x8)$

SAT

$\omega_9 = (\neg x7 + \neg x8 + \cancel{\neg x13}^{0})$

- **Partial assignment is:**
  - **x9=0 x10=0 x11=0 x12=1 x13=1**
- **Next: Assign a var to value**
  - Assign **x1=1**
  - **Assign (implied): x2=1, x3=1**
  - **Assign (implied): x4=1**

Implications!
x4=1 → x5=1 && x6=1

BCP Iterative – Go Until No More Implications

$\phi = (\omega_1)(\omega_2)(\omega_3)(\omega_4)(\omega_5)(\omega_6)(\omega_7)(\omega_8)(\omega_9)$

$\omega_1 = (\neg x1 + x2)$
$\omega_2 = (\neg x1 + x3 + x9)$
$\omega_3 = (\neg x2 + \neg x3 + x4)$
SAT

$\omega_4 = (\neg x4 + x5 + x10)$
$\omega_5 = (\neg x4 + x6 + x11)$
UNIT

$\omega_6 = (\neg x5 + \neg x6)$
CONFLICT!

$\omega_7 = (x1 + x7 + \neg x12)$
$\omega_8 = (x1 + x8)$
SAT

$\omega_9 = (\neg x7 + \neg x8 + \neg x13)$

- **Partial assignment is:**
  - **x9=0 x10=0 x11=0 x12=1 x13=1**
- **Next: Assign a var to value**
  - Assign **x1=1**
  - **Assign (implied): x2=1, x3=1**
  - **Assign (implied): x4=1**
  - **Assign (implied): x5=1 && x6=1**

Conflict →unSAT
x5=1 && x6=1 → clause ω6==0!

BCP Iterative – Go Until No More Implications

$\phi = (\omega_1)(\omega_2)(\omega_3)(\omega_4)(\omega_5)(\omega_6)(\omega_7)(\omega_8)(\omega_9)$

$\omega_1 = (\overset{0}{\cancel{\neg x1}} + \overset{1}{\cancel{x2}})$	**SAT**
$\omega_2 = (\overset{0}{\cancel{\neg x1}} + \overset{1}{\cancel{x3}} + \overset{0}{\cancel{x9}})$	
$\omega_3 = (\overset{0}{\cancel{\neg x2}} + \overset{0}{\cancel{\neg x3}} + \overset{1}{\cancel{x4}})$	
$\omega_4 = (\overset{0}{\cancel{\neg x4}} + \overset{1}{\cancel{x5}} + \overset{0}{\cancel{x10}})$	**SAT**
$\omega_5 = (\overset{0}{\cancel{\neg x4}} + \overset{1}{\cancel{x6}} + \overset{0}{\cancel{x11}})$	
$\omega_6 = (\overset{0}{\cancel{\neg x5}} + \overset{0}{\cancel{\neg x6}})$	**CONFLICT!**
$\omega_7 = (\overset{1}{\cancel{x1}} + x7 + \overset{0}{\cancel{\neg x12}})$	**SAT**
$\omega_8 = (\overset{1}{\cancel{x1}} + x8)$	
$\omega_9 = (\neg x7 + \neg x8 + \overset{0}{\cancel{\neg x13}})$	**UNRESOLVED**

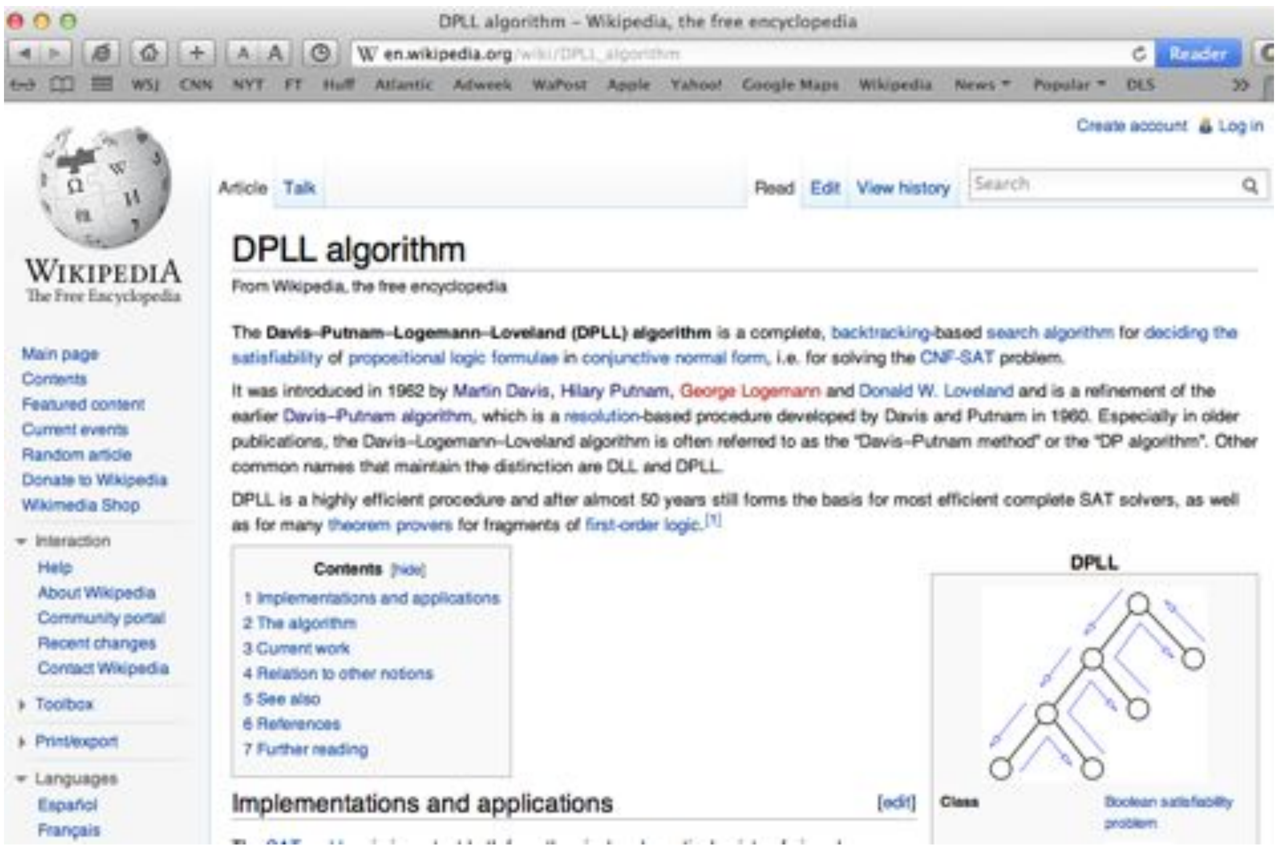
- **3 cases when BCP finishes**
  - **SAT:** Find a SAT assignment, all clauses resolve to "**1**". Return it.
  - **UNRESOLVED:** One or more clauses unresolved. Pick another unassigned var, and recurse more.
  - **UNSAT:** Like this. Found **conflict,** one or more clauses eval to "**0**"
- **Now what?**
  - You need to **undo** one of our variable assignments, try again…

This Has a Famous Name: DPLL

- **Davis-Putnam-Logemann-Loveland Algorithm**
  - Davis, Putnam published the basic recursive framework in 1960 (!)
  - Davis, Logemann, Loveland: found smarter BCP, eg, unit-clause rule, in 1962
  - Often called "Davis-Putnam" or "DP" in honor of the first paper in 1960, or (inaccurately) DPLL (all four of them never did publish this stuff together)

- **Big ideas**
  - A complete, systematic search of variable assignments
  - Useful CNF form for efficiency
  - BCP makes search stop earlier, "resolving" more assignments w/o recursing more

DPLL: Famous Stuff…



SAT: Huge Progress Last ~20 Years

- **But: DPLL is only the start…**
- **SAT has been subject of intense work and great progress**
  - Efficient data structures for clauses (so can search them fast)
  - Efficient variable selection heuristics (so search smart, find lots of implications)
  - Efficient BCP mechanisms (because SAT spends MOST of its time here)
  - Learning mechanisms (find patterns of vars that NEVER lead to SAT, avoid them)

- **Results: Good SAT codes that can do huge problems, fast**
  - Huge means? **50,000** vars; **25,000,000** literals; **50,000,000** clauses (!!)

SAT Solvers

- **Many good solvers available online, open source**

- **Examples**
  - **MiniSAT**, from Niklas Eén, Niklas Sörensson in Sweden.
    - **We are using this one for our MOOC**
  - CHAFF, from Sharad Malik and students, Princeton University
  - GRASP, from Joao Marques-Silva and Karem Sakallah, University of Michigan
  - …and many others too. Go google around for them…

Lots of Information on SAT…

VLSI CAD:
Logic to Layout

Lecture 4.3

Computational Boolean Algebra Representations: Using SAT for Logic

Rob A. Rutenbar
University of Illinois

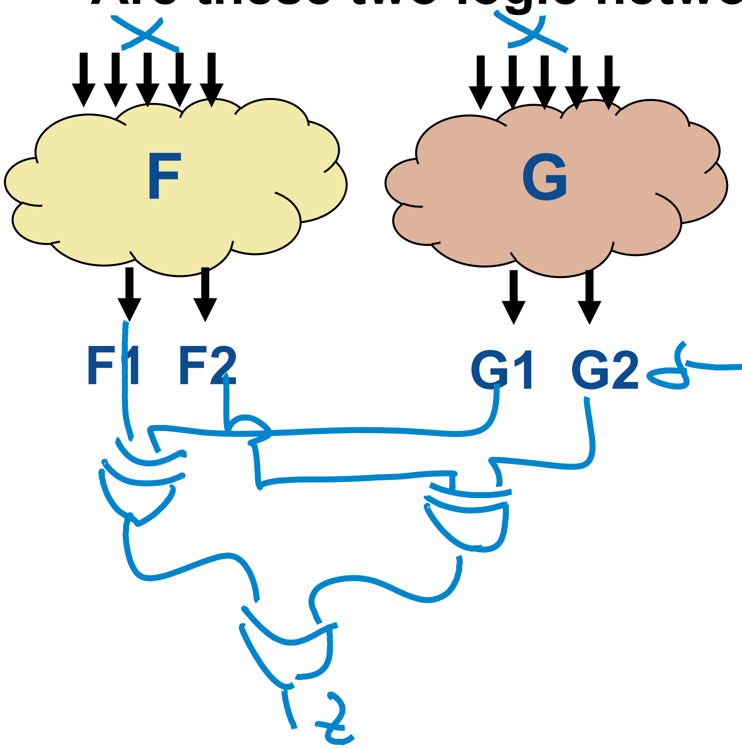
BDDs vs SAT Functionality

- **BDDs** $\neq$
  - Often work well for many problems
  - But – no guarantee it will always work
  - Can build BDD to *represent* function **ϕ**
  - But—sometimes **cannot build BDD** with reasonable computer resources (run out of memory **SPACE)**
  - Yes -- builds a **full** representation of **ϕ**
    - Can do a big set of Boolean *manipulations* on data structure
  - Can build **(∃ xyz F)** and **(∀ xyz F)**

- **SAT**
  - Often works well for many problems
  - But – no guarantee it will always work
  - Can *solve for SAT* (y/n) on function **ϕ**
  - But—sometimes **cannot find SAT** with reasonable computer resources (run out of **TIME** doing search)
  - No – does **not represent** all of **ϕ**
    - Can *solve* for SAT, but does not support big set of operators
  - There are versions of **Quantified SAT** that solve SAT on **(∃ xyz F), (∀ xyz F)**

Typical Practical SAT Problem

- **Are these two logic networks the *same* Boolean function(s)?** $F = G$?



$z = 1$ SAT iff $F \neq G$

if z SAT: find x such that

$F(x) \neq G(x)$

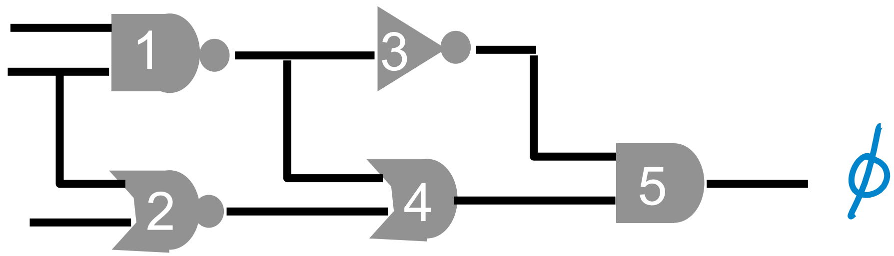
Do SAT solve on this new network

→ If **SAT**: networks **not same**, and this pattern makes them give different outputs

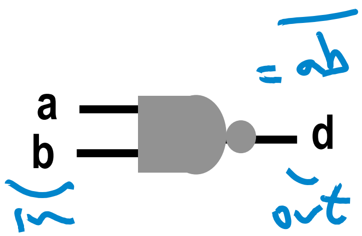
→ If **unSAT**: yes, **same**!

Final Topic: Gates → CNF

- **How do I *start* with a gate-level description and get CNF?**
  - Isn't this hard? Don't I need Boolean algebra or BDDs? No – it's **really easy**



- **Trick: build up CNF one gate at a time**



Gate consistency function (or gate satisfiability function)

$$\Phi_d = [d == (\overline{ab})] = d \,\overline{\oplus}\, \overline{(ab)}$$

$$(a+d)(b+d)(\bar{a}+\bar{b}+\bar{d}) = \Phi_d$$

ASIDE: EXOR vs EXNOR

- **EXOR: Exclusive OR**
  - Write is as: **$Y = A \oplus B$**
  - Output is **1** just if **$A \neq B$**, ie, if **A** is ***different*** than **B**

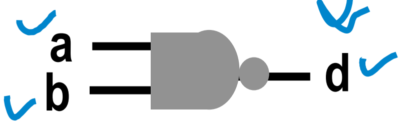
$Y = \bar{A}B + A\bar{B}$

- **EXNOR: Exclusive NOR**
  - Write is as: **$Y = A \overline{\oplus} B$** **(I like *this*)**
  - Or like this: **$Y = A \odot B$**
  - Output is **1** just if **$A == B$**, ie, if **A** is ***same as, equal to*** **B**

$Y = AB + \bar{A}B$

Final Topic: Gates → CNF

- **Gate consistency function == "1" just for combinations of inputs *and the output* that are "consistent" with what gate *actually does***



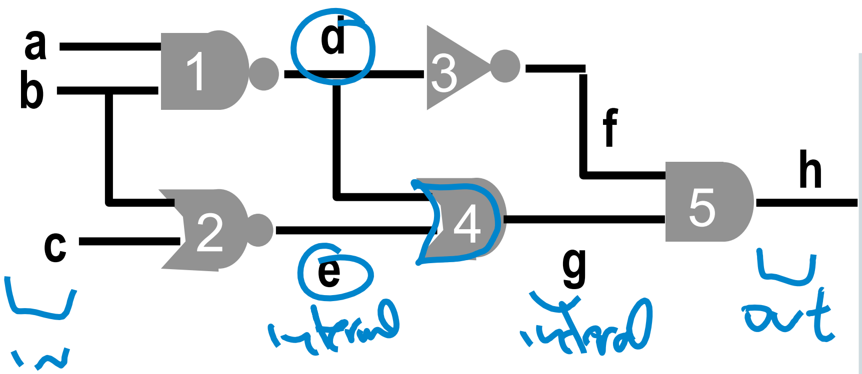
$$\phi_d = (a + d)(b + d)(\neg a + \neg b + \neg d)$$

a=0 b=0 d = 1 →  consistent $\phi_d = 1$

a=1 b=1 d = 1 →  inconsistent $\phi_d = 0$

Final Topic: Gates → CNF

- **For a network: label *each wire*, build *all* gate consistency funcs**



$\Phi_d = \ldots$ do it!

$\Phi_e = \ldots$

$\Phi_f = \ldots$

$\Phi_g = (\bar{d}+g)(\bar{e}+g)(d+e+\bar{g})$

$\Phi_h = \ldots$

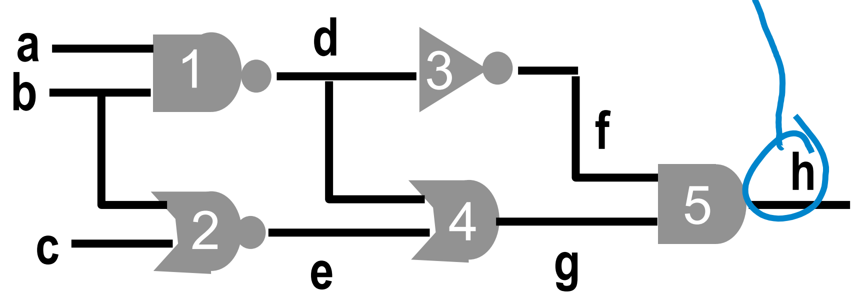
$\phi_g = g \overline{\oplus} [d+e]$

Final Topic: Gates → CNF

- **SAT CNF for network is simple:**
  - Any pattern of **abch** that satisfies this, also makes the gate network output **h=1**

φ = (output var) $\bigwedge_{(k \text{ is gate output wire})} \Phi_k$

$\varphi = h$
$(a + d)(b + d)(\neg a + \neg b + \neg d)$ 1
$(\neg b + \neg e)(\neg c + \neg e)(b + c + e)$ 2
$(\neg d + \neg f)(d + f)$ 3
$(\neg d + g)(\neg e + g)(d + e + \neg g)$ 4
$(f + \neg h)(g + \neg h)(\neg f + \neg g + h)$ 5



  - Only need Boolean algebra/simplification for each ***individual*** gate-level function
  - At network level, just **AND** them all together to get **CNF**

Rules for ALL Kinds of Basic Gates

- **Gate consistency rules from:**
  - Fadi Aloul, Igor L. Markov, Karem Sakallah, "MINCE: A Static Global Variable-Ordering Heuristic for SAT Search and BDD Manipulation," *J. of Universal Computer Sci.*, vol. 10, no. 12 (2004), 1562-1596.

z= (x)

(yes this is just a wire)

$$\left[\overline{x}+z\right]\left[x+\overline{z}\right]$$

z=NOR(x1, x2, … xn)

$$\left[\underbrace{\prod_{i=1}^{n}\left(\overline{xi}+\overline{z}\right)}_{\text{product}}\right]\left[\underbrace{\left(\sum_{i=1}^{n}xi\right)}_{\text{sum}}+z\right]$$

z=OR(x1, x2, … xn)

$$\left[\prod_{i=1}^{n}\left(\overline{xi}+z\right)\right]\left[\left(\sum_{i=1}^{n}xi\right)+\overline{z}\right]$$

z=NOT(x)

$$\left[x+z\right]\left[\overline{x}+\overline{z}\right]$$

z=NAND(x1, x2, … xn)

$$\left[\prod_{i=1}^{n}\left(xi+z\right)\right]\left[\left(\sum_{i=1}^{n}\overline{xi}\right)+\overline{z}\right]$$

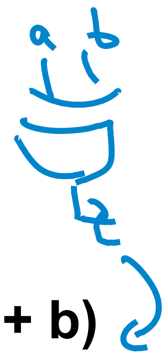
Z=AND(x1, x2, … xn)

$$\left[\prod_{i=1}^{n}\left(xi+\overline{z}\right)\right]\left[\left(\sum_{i=1}^{n}\overline{xi}\right)+z\right]$$

Rules for ALL Kinds of Basic Gates

- **EXOR/EXNOR gates are rather *unpleasant* for SAT**
  - And the basic "either-or" structure makes for some tough SAT search, often
  - And, they have rather large gate consistency functions too
  - Even small 2-input gates create a lot of terms, like this:

z=EXOR(a, b)



$$\phi_z = z \,\overline{\oplus}\, (a \oplus b)$$

$$= (\neg z + \neg a + \neg b)\cdot(\neg z + a + b)$$
$$\cdot(z + \neg a + b)\cdot(z + a + \neg b)$$

z=EXNOR(a, b)

Do it – good practice!

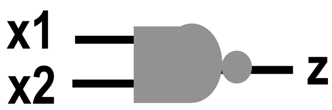
Using the Rules…

z=NAND(x1, x2, … xn)

$$\left[\prod_{i=1}^{n=2}(xi+z)\right]\left[\left(\sum_{i=1}^{n=2}\overline{xi}\right)+\overline{z}\right]$$

product

n==2: z=NAND(x1, x2)



$$\phi_z = (x1 + z)(x2 + z)(\neg x1 + \neg x2 + \neg z)$$

For these formulae, $\prod$ means "AND" $\sum$ means "OR"

Summary

- **SAT has largely displaced BDDs for "just solve it" apps**
  - Reason is scalability: can do *very* large problems faster, more reliably
  - Still, SAT, like BDDs, not guaranteed to find a solution in reasonable time or space
- **40 years old, but still "the" big idea: DPLL**
  - Many recent engineering advances make it *stupendously* fast
- **Acknowledgements for help with earlier versions of this SAT lec**
  - **Karem Sakallah**
    U of Michigan

Joao Marques-Silva
University College, Dublin