Econ 8400 Labor Economics: Problem Set 1
Work Hours Choices with Static Labor Supply

Haosi Shen

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Model Setup

Suppose that individuals have the choice to work $h \in \mathcal{H} = \{0, 10, 20, 30, 40\}$ hours per week. Preferences over these discrete alternatives may be described by a parametric utility function:

$$u(c, h) = \gamma \cdot (\theta^{-1} c^{\theta} - \alpha h) + \epsilon_h$$

where the state-specific errors ϵ_h follow a standard Type-I extreme value (Gumbel) distribution. Consumption is given by the budget constraint

$$c = y + wh - T(wh)$$

where y is the non-labor income, $T(\cdot)$ is the tax system, and w is the gross hourly wage rate generated by the following log-linear relationship

$$\ln w = \mu_w + \epsilon_w$$

where the unobserved component of wages $\epsilon_w \sim \mathcal{N}(0, \sigma_w)$.

1 Simulation

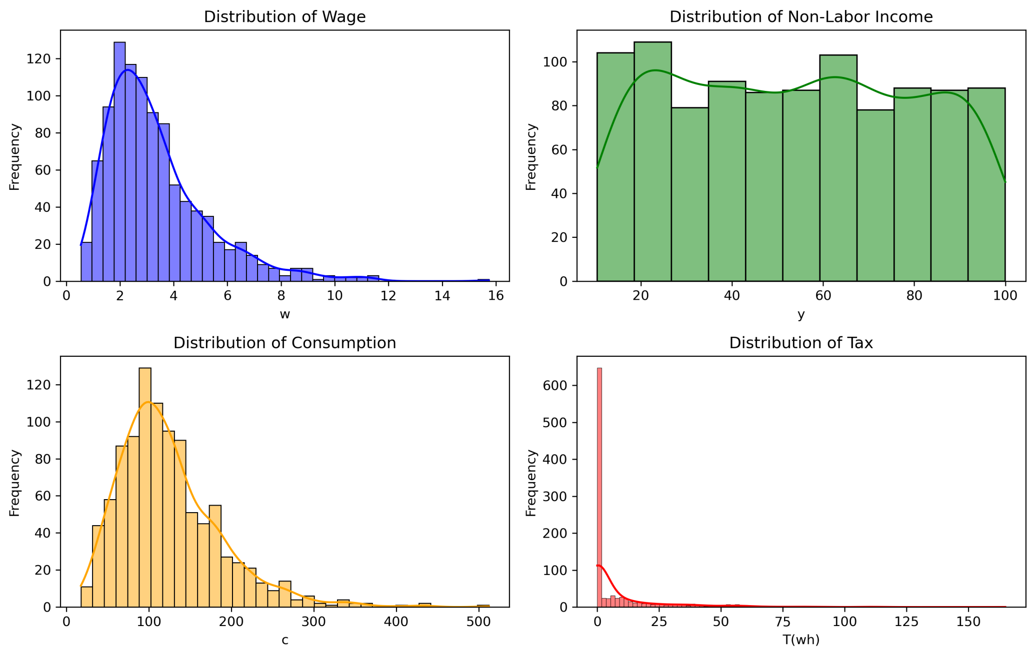
Suppose that the parameter values are $\sigma_w = 0.55$, $\mu_w = 1$, $\theta = 0.3$, $\alpha = 0.1$, $\gamma = 2$, and that $Y \sim \text{Uniform}(10, 100)$.

Suppose also that earnings below \$80 per week are not taxed; any earnings greater than \$80 are taxed at the constant marginal tax rate $\tau = 0.3$. Non-labor income is not taxed. So,

$$T(wh) = \begin{cases} 0 & \text{if } wh < 80 \\ \tau \cdot (wh - 80) & \text{if } wh \geq 80 \end{cases}$$

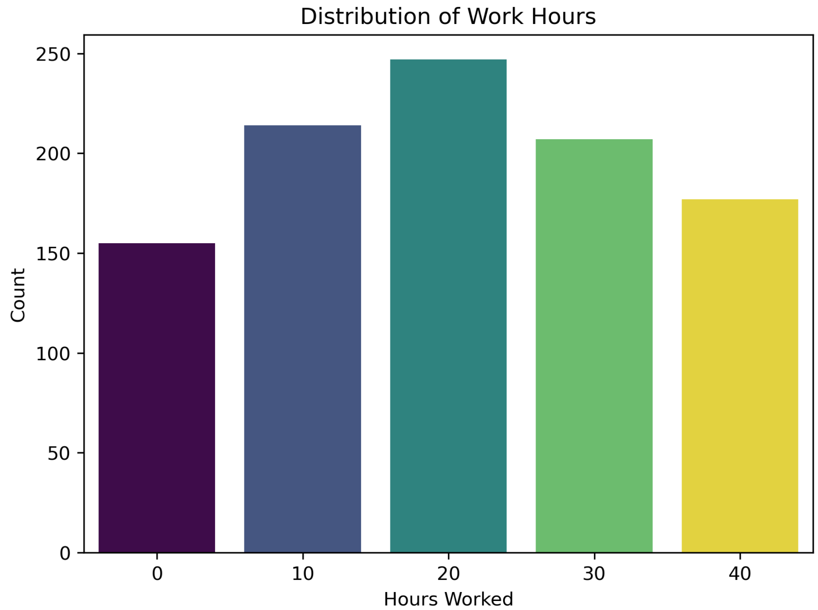
Under these assumptions, simulate a dataset of 1,000 observations.

Figure 1: Distributions of Wage, Non-Labor Income, Consumption, and Tax



To assess how the distribution of non-labor income and (offered) wages varies with hours, I group the dataset by work hour choices. Figure 3 shows the distribution of y and w for each work hours group, and Table 1 presents the

Figure 2: Distributions of Work Hours



Hours Worked	Non-Labor Income (y)			Wage w		
	Mean	Median	Std Dev	Mean	Median	Std Dev
0	70.35	72.27	20.90	1.85	1.70	0.90
10	60.73	62.70	26.01	2.70	2.38	1.39
20	52.91	53.87	24.66	3.53	3.07	1.71
30	46.51	41.11	25.60	3.84	3.41	1.78
40	42.52	35.29	24.81	4.48	3.99	2.38

Table 1: Summary Statistics of y and w by Work Hours

summary statistics.

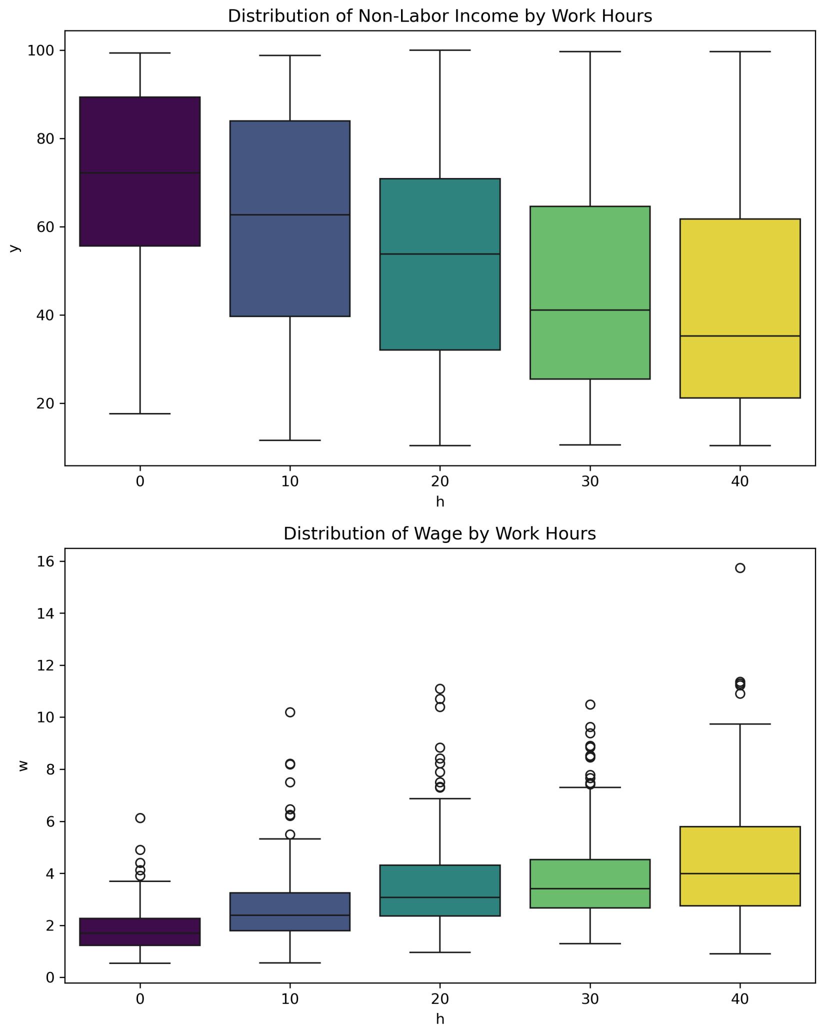
Overall, the simulation results reveal that

- **Negative Relationship Between Non-Labor Income and Work Hours:** Individuals with higher non-labor income tend to work fewer hours, while those with lower non-labor income tend to work more. This observation aligns with the *income effect*—individuals with sufficient non-labor income (or wealth) may choose to work less, as they have less need to supplement their consumption through labor earnings.

- **Positive Relationship Between Wages and Work Hours:** Individuals are more likely to choose longer work hours when offered higher wages. This aligns with the intuition of the *substitution effect*, as allocating time to work rather than leisure becomes more valuable when wages (i.e., labor market returns) are higher. Additionally, note that the variability in wages increases for individuals who work more hours. This reflects risk aversion, as individuals may choose to work more to compensate for the higher income risk.

- **Trade-Off Between Non-Labor Income and Wages:** The trade-off between y and w in determining work hours aligns with the idea that individuals optimize their labor supply based on their total income. This behavior reflects a balance between the income effect and the substitution effect in determining the labor elasticity.

Figure 3: Distributions y, w by h



2 Log-likelihood function

Define the utility with respect to each work hour choice:

$$U_h = \gamma \cdot (\theta^{-1} c(h)^{\theta} - \alpha h) + \epsilon_h$$

where ϵ_h follows the Type I Extreme Value (Gumbel) distribution and satisfies independence across alternatives.
Hours are chosen to solve

$$\begin{aligned} h^*(w, y) = \ & \arg\max_h \ U_h \\ \text{s.t.} \quad & c(h) = wh + y - T(wh) \\ & \ln w = \mu_w + \epsilon_w, \quad \epsilon_w \sim \mathcal{N}(0, 1) \end{aligned}$$

Define the full likelihood function as the product of the likelihood contributions for all individuals:

$$L(\theta) = \prod_{i=1}^{N} L_i$$

Following from the Gumbel distribution of preference shocks ϵ_h, the probability of choosing a particular hours level $h_j \in \mathcal{H}$ is given by the multinomial logit model:

$$Pr(h = h_j \mid w, y) = \frac{\exp\left(u(c(h_j), h_j)\right)}{\sum_k \exp\left(u(c(h_k), h_k)\right)}$$

For worker i ($h_i > 0$), her likelihood contribution L_i is the joint probability of her wage realization and hours choice, which is the product of the choice probability and the wage density due to independence between the preference and wage shocks:

$$\begin{aligned} L_i^W &= Pr(h_i \mid w_i, y_i) \cdot f_w(w_i) \\ &= \frac{\exp\left(u(c(h_i), h_i)\right)}{\sum_k \exp\left(u(c(h_k), h_k)\right)} \cdot \frac{1}{w_i \sigma_w \sqrt{2\pi}} \exp\left(\frac{-(\ln w_i - \mu_w)^2}{2\sigma_w^2}\right) \end{aligned}$$

where the wage density $f_w(w_i)$ is derived from the log-normal distribution.

For a non-worker i ($h_i = 0$), however, her wage offer is unobserved, so her likelihood contribution L_i involves integrating over the wage distribution while using the multinomial logit choice probability:

$$\begin{aligned}
L_i^{NW} &= \int_{-\infty}^{\infty} Pr(h_i = 0 \mid w_i, y_i) \cdot f_w(w_i) dw_i \\
&= \int_{-\infty}^{\infty} \frac{\exp\left(u(c(0), 0)\right)}{\sum_k \exp\left(u(c(h_k), h_k)\right)} \cdot \frac{1}{w_i \sigma_w \sqrt{2\pi}} \exp\left(\frac{-(\ln w_i - \mu_w)^2}{2\sigma_w^2}\right) dw_i \\
&= \int_{-\infty}^{\infty} \frac{\exp\left(u(y_i, 0)\right)}{\sum_k \exp\left(u(c(h_k), h_k)\right)} \cdot \frac{1}{w_i \sigma_w \sqrt{2\pi}} \exp\left(\frac{-(\ln w_i - \mu_w)^2}{2\sigma_w^2}\right) dw_i
\end{aligned}$$

Combining the above, the log-likelihood function is

$$\begin{aligned}
\ln L(\theta) &= \sum_{i=1}^{N} \ln L_i \\
&= \sum_{i=1}^{N} \left[\ln \Big(\int_{-\infty}^{\infty} Pr(h_i \mid w_i, y_i) \cdot f_w(w_i) dw_i \Big) \right] \\
&= \sum_{i=1}^{N} \left[\ln \Big(\int_{-\infty}^{\infty} \frac{\exp\left(u(c(h_i), h_i)\right)}{\sum_k \exp\left(u(c(h_k), h_k)\right)} \cdot \frac{1}{w_i \sigma_w \sqrt{2\pi}} \exp\left(\frac{-(\ln w_i - \mu_w)^2}{2\sigma_w^2}\right) dw_i \Big) \right]
\end{aligned}$$

which accounts for both observed wages for workers and unobserved wages for non-workers. For workers ($h_i > 0$), the log-likelihood contribution is

$$\ln L_i^W = \ln \Big(\frac{\exp\left(u(c(h_i), h_i)\right)}{\sum_k \exp\left(u(c(h_k), h_k)\right)} \Big) + \ln \Big(\frac{1}{w_i \sigma_w \sqrt{2\pi}} \Big) - \frac{(\ln w_i - \mu_w)^2}{2\sigma_w^2}$$

3 MLE

For numerical estimation of the model parameters using MLE, I employ a quasi-Newton method (BFGS), which approximates the Hessian of the likelihood function. This method is generally robust and efficient for smooth, unconstrained optimization problems.

Table 2 displays results of the estimation. Overall, the MLE estimates are reasonably accurate, with percentage errors below 10% for all parameters.

Parameter	**Actual Value**	**MLE Estimate**	**Error (%)**
μ_w	1.0	1.0660	6.60
σ_w	0.55	0.5266	4.25
θ	0.3	0.3167	5.57
α	0.1	0.1096	9.60
γ	2.0	1.8884	5.58

Table 2: Comparison of Actual Parameters and MLE Estimates

4 Tax Reform

Using the true parameter values (i.e., those given in part 1), simulate the impact on the distribution of work hours of changing the tax system so that all earnings are taxed at the constant marginal tax rate $\tilde{\tau} = 0.2$.

$$T(wh) = \tilde{\tau} \cdot wh = 0.2 \cdot wh$$

Figures 4 and 5 display the distribution of the labor supply (work hours), wage, non-labor income, consumption, and tax after moving from a progressive to a uniform tax system. Figure 6 shows the distribution of y and w for each work hours group, and Table 3 presents the summary statistics. Overall, the tax reform has ambiguous effects on labor supply as it reduces redistribution:

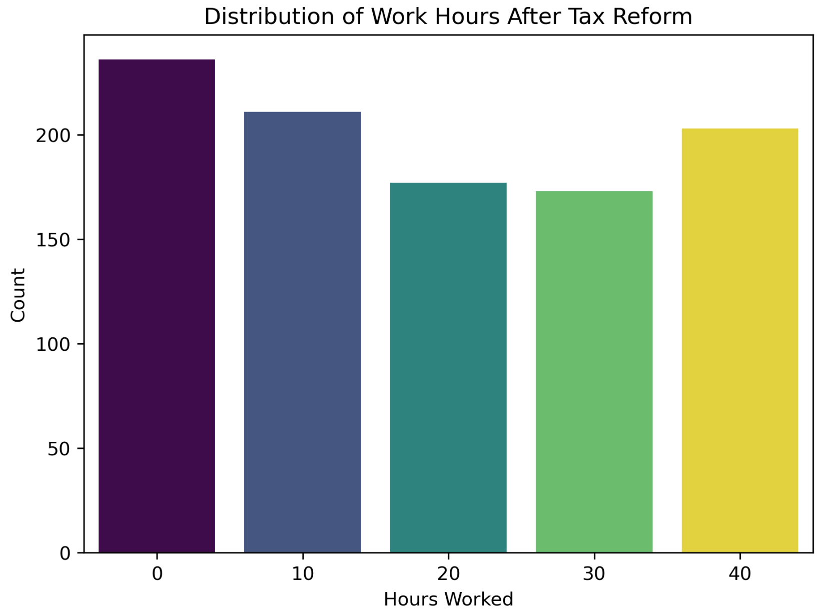
- **Income effect**: The distribution of work hours shows an increase in non-participation in the labor market (i.e., zero work hours). This suggests that the income effect dominates for a portion of the population—likely **those with high non-labor income and low wages**. After switching to a uniform tax system, the low earners see a decrease in after-tax income from the same work hours, which disincentivizes them to work more to compensate for lower wages, especially if their purchasing power is already sufficiently covered by non-labor income.

- **Subsitution effect**: Furthermore, **higher-wage individuals increase work hours**, indicating dominance of the substitution effect. With the shift to a uniform tax rate, the marginal tax rate for high earners decreased, which increases the after-tax return to working longer.

How does the amount of tax revenue raised by the government change?

Table 4 summarizes the statistics, and Figure 7 compares the distributions of tax revenues under the progressive and uniform tax systems, respectively. The results suggest that the uniform tax system expands the tax base by taxing more individuals, including those with lower earnings. Additionally, the dispersion of taxes collected from individuals has slightly narrowed after

Figure 4: Distributions of Work Hours After Tax Reform

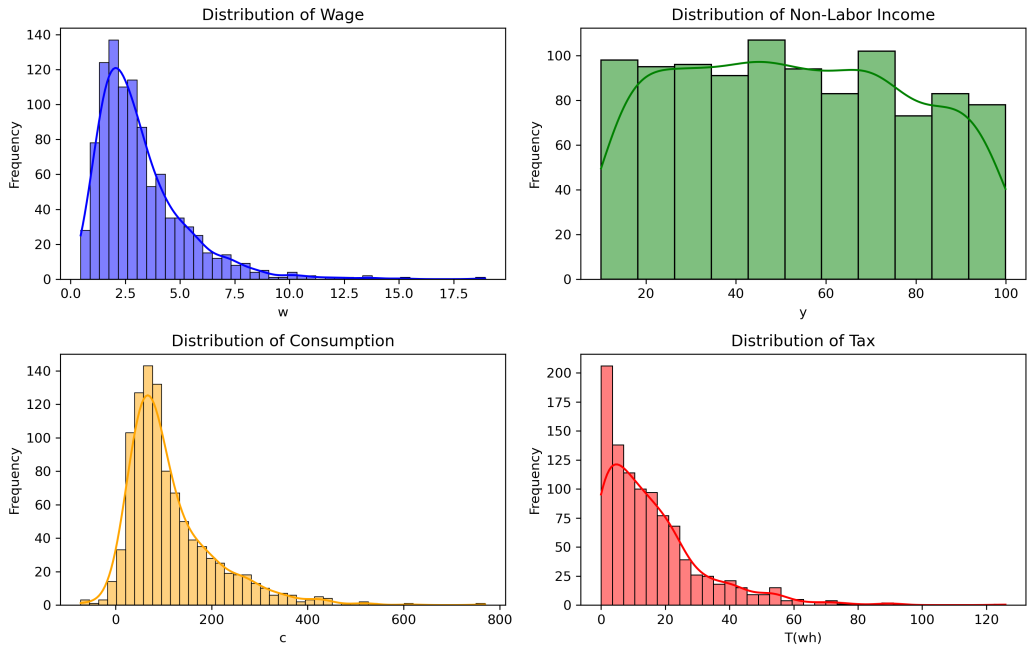


Hours Worked	Non-Labor Income (y)			Wage w		
	Mean	Median	Std Dev	Mean	Median	Std Dev
0	65.77	68.82	22.30	1.91	1.79	0.83
10	56.05	55.03	24.19	2.40	2.21	0.97
20	49.14	45.43	26.19	2.90	2.70	1.36
30	47.83	41.98	26.60	4.11	3.70	2.23
40	43.64	41.49	23.05	4.99	4.58	2.52

Table 3: Labor and Non-Labor Income After Tax Reform

the reform, indicating that tax burdens are more evenly distributed. In summary, the uniform tax system raises more government revenue while reducing tax burden extremes, but at the cost of making lower-income individuals contribute more, which could have negatively impacted total welfare.

Figure 5: Distributions of Wage, Non-Labor Income, Consumption, and Tax After Tax Reform



Statistic	Progressive Tax	Uniform Tax
Mean	8.516	15.748
Std Dev	17.736	15.658
Min	0.000	0.000
25%	0.000	4.362
50% (Median)	0.000	12.001
75%	8.991	21.994
Max	164.88	125.92
Variance	314.556	245.166
Skewness	3.157	1.818
Kurtosis	13.270	5.087
Sum	8515.560	15748.399

Table 4: Government Revenue under Different Tax Systems

Can you determine the value of the constant marginal tax rate which provides the same amount of revenue as the progressive schedule described in part 1?

Since tax revenue is directly proportional to the constant tax rate under uniform taxation, one can find the constant marginal tax rate $\hat{\tau}$ that provides the equivalent amount of government revenue using:

$$\hat{\tau} = \tilde{\tau} \cdot \frac{\mu_{T,\text{ progressive}}}{\mu_{T,\text{ uniform}}} = 0.2 \cdot \frac{8.51556}{15.74840} \approx 10.81451\%$$

where $\tilde{\tau}$ denotes the original constant marginal tax rate, and $\mu_{T,\text{ x}}$ represents the mean tax revenue from each individual under tax system x.

The distribution of tax revenue is summarized in Table 5 and Figure 8. There is a slight discrepancy in the realized aggregate tax revenue due to the presence of extreme-value idiosyncratic shocks in utilities, despite the means being correctly calibrated. Fixing the random seed of the RNG does not eliminate this when there is a shift in the distribution of choices. Increasing the sample size or running multiple simulations may help narrow the gap.

Statistic	Uniform Tax ($\hat{\tau}$)
Count	1000
Mean	8.568
Std Dev	9.530
Min	0.000
25%	1.650
50% (Median)	5.857
75%	12.209
Max	81.906
Variance	90.815
Skewness	2.077
Kurtosis	6.938
Sum	8568.396

Table 5: Uniform Taxation with Revenue Equivalence

Figure 6: Distributions y, w After Tax Reform

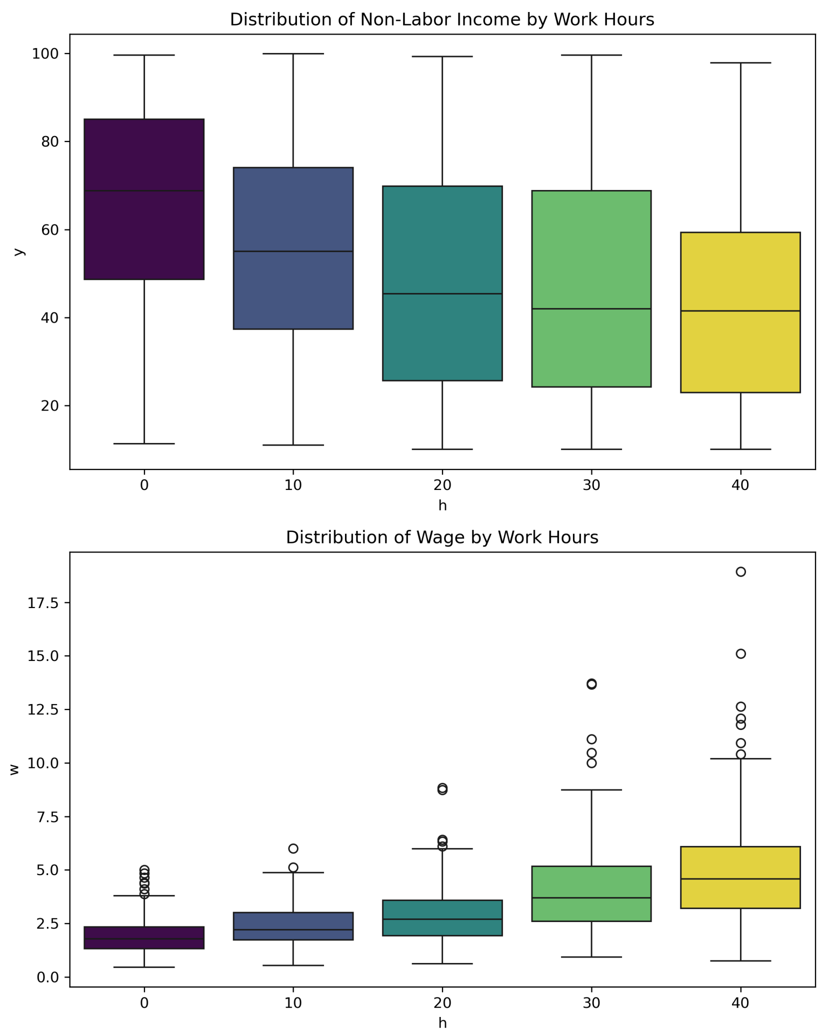


Figure 7: Tax Revenue Distribution Before and After Reform

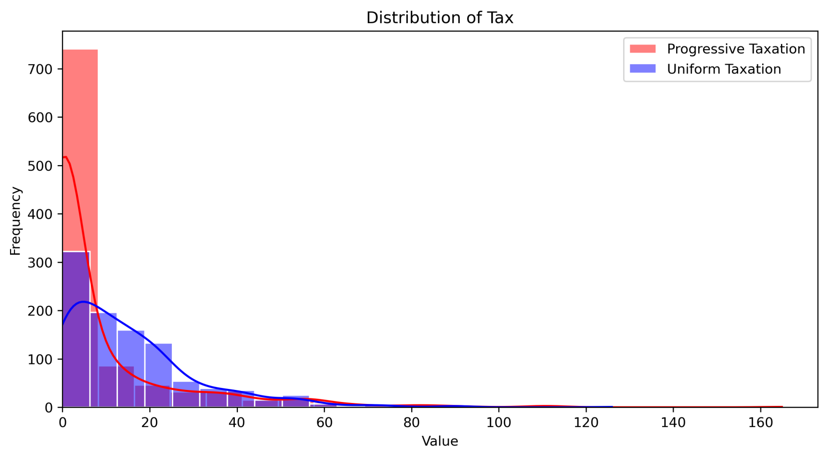


Figure 8: Distribution of Tax with Revenue Equivalence