University of Toronto
Department of Electrical and Computer Engineering

ECE367 MATRIX ALGEBRA AND OPTIMIZATION

Problem Set #4
Autumn 2021

Prof. S. C. Draper **Due:** 8pm (Toronto time) Saturday, 30 October 2021

Homework policy: Problem sets must be turned by the due date and time. Late problem sets will not be accepted. See the information sheet for further details. The course text "Optimization Models" is abbreviated as "OptM" and "Introduction to Applied Linear Algebra" as "IALA".

Problems are categorized as:

- **"Theory" problems:** These are mostly mathematical questions designed to give you deeper insight into the fundamentals of the ideas introduced in this class.
- **"Application" problems:** These questions are designed to expose you to the breadth of application of the ideas developed in class and to introduce you to useful numerical toolboxes. Problems of this sort often ask you to produce plots and discuss your results; said plots and discussions should be included in and form part of your submission – think of your submitted solution like a lab book. Your attached code simply provides back-up evidence.
- **"Optional" problems:** Optional problems provide extra practice or introduce interesting connections or extensions. They need not be turned in. I will assume you have reviewed and understood the solutions to the optional problems when designing the exams.

Hand-in procedure:

- **Initial submission:** Your initial submission of the "Theory" and "Application" questions must be submitted via Quercus upload by the due date. Click on the `Assignments` tab, then look for the `Initial submission` tab and upload under the correct problem set number.
- **Self-assessment:** After the problem set is due we will post solutions. You will have one week from the initial due date to submit a commented version of your assignment in which, using as a reference the posted solutions, you highlight your errors or omissions in red. Annotate the PDF you initially submitted. If you have not submitted a solution you cannot submit the self-assessment. To submit the self-assessment on Quercus, click on the `Assignments` tab, then look for the `Self-assessment` tab and upload under the correct problem set number.
- **Late problem sets are not accepted**
- **Grading:** Per the course handout problem sets are graded for completion only. Points are assigned to (i) Initial submission of theory part, (ii) Submission of application part, (iii) Self-assessment of theory part. The relative points breakdown is detailed in each problem set.

Points allocation

- Theory parts (initial assessment): 1 pt
- Application parts (initial assessment): 1 pt
- Theory parts (self-assessment): 1 pt

Problem categorization and main concepts covered

Theory

- Quadratics: Problem 4.1
- Singular value decomposition: Problems 4.2-4.3

Application

- Latent semantic indexing: Problem 4.4

Optional

- None on this problem set

THEORY

Problem 4.1 (Quadratic constraints)
OptM Problem 4.2.

Problem 4.2 (Practice computing SVDs)
Compute by hand the singular value decomposition of

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

In other words, express A as $A = U\tilde{\Sigma}V^T$ where $U \in \mathbb{R}^{2,2}$ and $V \in \mathbb{R}^{3,3}$ are orthogonal matrices and $\tilde{\Sigma} \in \mathbb{R}^{2,3}$, specify the values of

(a) Specify the singular values by specifying $\tilde{\Sigma}$.

(b) Specify the right singular vectors by specifying V.

(c) Specify the left singular vectors by specifying U.

(d) Reassemble your calculations of parts (a)-(c) to show that $U\tilde{\Sigma}V^T$ in fact does equal the A you started with. Also express A as a sum of rank-one matrices where each matrix is the outer product of a left singular vector and a right singular value scaled by a singular value.

Problem 4.3 (SVD of an orthogonal matrix)
OptM Problem 5.1.

APPLICATION

Problem 4.4 (Latent semantic indexing)
In this problem you build off the problem "Angles between word vectors" from PS01, making a connection to the singular value decomposition. First complete the following two parts of OptM problem 5.5:

(a) OptM problem 5.5 part 3.

(b) OptM problem 5.5 part 4.

In the following parts we connect OptM problem 5.5 to the "Angles between word vectors" (ABWV) problem in PS01, and apply the latent semantic indexing method to the Wikipedia article collection. We start by briefly re-describing the ABWV problem set-up below, but please feel free to go back and read the original problem statement.

In ABWV, we considered a set of documents $\mathcal{D}$ the where the number of documents is $|\mathcal{D}|$. The set $\mathcal{W}$ denotes the union of words in all articles, i.e., the lexicon of the set of documents where the cardinality of $\mathcal{W}$ is $|\mathcal{W}|$. We assume the lexicon is ordered "lexiographically" (e.g., alphabetically) so that there is a one-to-one mapping from each word $w \in \mathcal{W}$ to an element of the index set $t \in [|\mathcal{W}|]$. Let $f_{\text{term}}(t, d)$ denote the number of times the word $w \in \mathcal{W}$ that is indexed as $t \in [|\mathcal{W}|]$ appears in the dth article where $d \in [|\mathcal{D}|]$. For ABWV, you were provided with a pre-processed MATLAB data file `wordVecV.mat`. Please re-use the same data file for this problem. You can load the content in the second file into MATLAB by using command `load 'wordVecV.mat'`. After loading, you will see a variable `V` of dimensions 1651×10. We refer to this matrix as V. The value in the tth row and dth column of this matrix is $f_{\text{term}}(t, d)$. Note that in the given dataset $|\mathcal{D}| = 10$ and $|\mathcal{W}| = 1651$.

Now we connect OptM problem 5.5 to the ABWV problem set-up. You will immediately notice that $m = |\mathcal{D}|$ and $n = |\mathcal{W}|$. You can compute the $n \times m$ "(raw) term-by-document matrix" M by noting that $[M]_{i,j} = \mathbb{1}([V]_{i,j})$, where $\mathbb{1}(x)$ is 1 if $x > 0$ and 0 otherwise. The OptM problem 5.5 also describes how to obtain $\tilde{M}$, a normalized version of M.

(c) Use MATLAB `svd` command to compute the singular value decomposition of $\tilde{M}$. List the 10 largest singular values in sorted order.

(d) In part (b) you assumed a low-rank approximation of $\tilde{M}$ and found an expression for the document similarity. Let the distance between ith and jth documents be $d(i, j)$ as per your expression from part (b). Let the rank of your approximation be k where $0 < k \leq \min(m, n)$. Compute $d(i, j)$ for $i, j \in [m]$ by assuming $k = 9$. Write down the titles of two most similar documents.

(e) Repeat what you did in part (d) with $k = 8, 7, \ldots, 1$. What is the lowest k that does not change your answer for part (d)? If your answer for lowest k is greater than 1 what is the pair of most similar documents for $k - 1$?

<u>**OPTIONAL**</u>

None on this problem set