Problem Set 4

4.2) Quadratic Constraints

1) $A=\begin{bmatrix}2&1\\1&2\end{bmatrix}$

$\det(A-\lambda I)=\det\left(\begin{bmatrix}2-\lambda&1\\1&2-\lambda\end{bmatrix}\right)$

$=(2-\lambda)^2-1$

$=4-4\lambda+\lambda^2-1$

$=\lambda^2-4\lambda+3$

$=(\lambda-3)(\lambda-1)$

$\lambda=3$ and $\lambda=1$

$\lambda=3$:

$\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3x\\3y\end{bmatrix}$

$2x+y=3x \quad x+2y=3y$

$x=y \quad x=y$

$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\x\end{bmatrix}=\alpha\begin{bmatrix}1\\1\end{bmatrix}$

$=\alpha\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix}$

$\lambda=1$:

$\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$

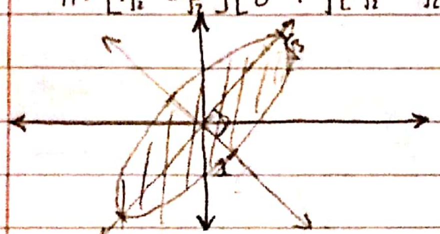
$2x+y=x \Rightarrow x=-y$

$\alpha\begin{bmatrix}1\\-1\end{bmatrix}$

$\alpha=\begin{bmatrix}\frac{1}{\sqrt2}\\-\frac{1}{\sqrt2}\end{bmatrix}$

$A=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\end{bmatrix}\begin{bmatrix}3&0\\0&1\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\end{bmatrix}$

$\left(\frac{\tilde{x}_1}{\sqrt3}\right)^2+\left(\frac{\tilde{x}_2}{\sqrt1}\right)^2\le1$



2) $A=\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$

$A-\lambda I=\begin{bmatrix}1-\lambda&-1\\-1&1-\lambda\end{bmatrix}$

$\det(A-\lambda I)=(1-\lambda)^2-1=1-2\lambda+\lambda^2-1$

$=\lambda^2-2\lambda=\lambda(\lambda-2)$

$\lambda=0$ and $\lambda=2$

$\lambda=0$:

$\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$x-y=0\Rightarrow x=y$

$\alpha\begin{bmatrix}1\\1\end{bmatrix}\Rightarrow\alpha\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix}$

$\lambda=2$:

$\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2x\\2y\end{bmatrix}$

$x-y=2x \rightarrow -x=y$

$\alpha\begin{bmatrix}1\\-1\end{bmatrix}\rightarrow\alpha\begin{bmatrix}\frac{1}{\sqrt2}\\-\frac{1}{\sqrt2}\end{bmatrix}$

$A=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\end{bmatrix}\begin{bmatrix}0&0\\0&2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\end{bmatrix}$

$x^TAx=\begin{bmatrix}x_1&x_2\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}0&0\\0&2\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

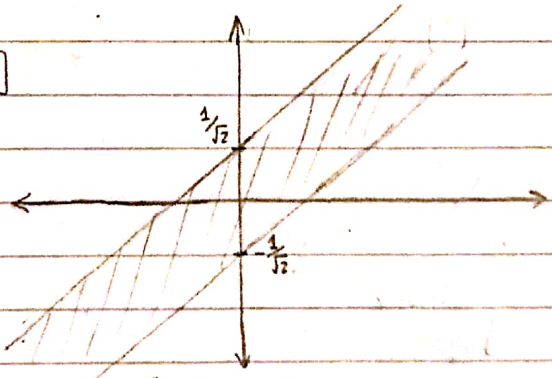
$=\begin{bmatrix}x_1+x_2&x_1-x_2\end{bmatrix}\begin{bmatrix}0&0\\0&2\end{bmatrix}\begin{bmatrix}x_1+x_2\\x_1-x_2\end{bmatrix}$

$=\begin{bmatrix}0&2(x_1-x_2)\end{bmatrix}\begin{bmatrix}x_1+x_2\\x_1-x_2\end{bmatrix}$

$x^TAx=2(x_1-x_2)^2\le1$

$(x_1-x_2)^2\le\frac12 \quad -\frac{1}{\sqrt2}\le x_1-x_2\le\frac{1}{\sqrt2}$

$x_1-\frac{1}{\sqrt2}\le x_2\le x_1+\frac{1}{\sqrt2}$



3) $\det(A-\lambda I)=\det\left(\begin{bmatrix}-1-\lambda&0\\0&-1-\lambda\end{bmatrix}\right)=(-1-\lambda)^2=(\lambda+1)^2\rightarrow\lambda=-1 \quad \lambda=-1$

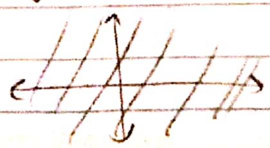
$\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-x\\-y\end{bmatrix}$

$\begin{bmatrix}x\\y\end{bmatrix}=\alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\1\end{bmatrix}$

$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$x^TAx=\begin{bmatrix}x_1&x_2\end{bmatrix}\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-x_1&-x_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=-x_1^2-x_2^2=-(x_1^2+x_2^2)\le1$

$x_1^2+x_2^2\ge-1\Rightarrow$ All plane



4.2)

$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

$A = U\Sigma V^T$

$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \quad A^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$

$A^TA = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$

$\begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} - \lambda I = \begin{bmatrix} 13-\lambda & 12 & 2 \\ 12 & 13-\lambda & -2 \\ 2 & -2 & 8-\lambda \end{bmatrix} = (13-\lambda)\begin{vmatrix} 13-\lambda & -2 \\ -2 & 8-\lambda \end{vmatrix} - 12\begin{vmatrix} 12 & 2 \\ -2 & 8-\lambda \end{vmatrix} + 2\begin{vmatrix} 12 & 2 \\ 13-\lambda & -2 \end{vmatrix}$

$= (13-\lambda)[(13-\lambda)(8-\lambda)-4] - 12[(12)(8-\lambda)+4] + 2[-24-2(13-\lambda)]$

$= (13-\lambda)(100-21\lambda+\lambda^2) - 12(96-12\lambda+4) + 2(-24-26+2\lambda)$

$= (13-\lambda)(100-21\lambda+\lambda^2) - 12(100-12\lambda) + 2(-50+2\lambda)$

$= \cancel{1300} - 273\lambda + 13\lambda^2 - \cancel{100\lambda} + 21\lambda^2 - \lambda^3 - \cancel{1200} + 144\lambda - \cancel{100} + 4\lambda$

$= -\lambda^3 + 34\lambda^2 - 225\lambda$

$= -\lambda(\lambda^2 - 34\lambda + 225)$

$= -\lambda(\lambda-25)(\lambda-9)$

$\begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$13x + 12y + 2z = 0 \rightarrow 25x + 25y = 0$

$12x + 13y - 2z = 0 \rightarrow 48x + 52y - 8z = 0$

$2x - 2y + 8z = 0$

$50x + 50y = 0 \rightarrow x + y = 0$

$2x - 2(-x) + 8z = 0 \rightarrow 2x + 2x + 8z = 0 \rightarrow 8z = -4x \rightarrow z = -\frac{1}{2}x$

$x = 0$, $y = 4x$, $z = 0$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -x \\ -\frac{1}{2}x \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \\ -\frac{1}{2} \end{bmatrix}$ or $\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$

$\lambda = 0, \vec{v} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

$13x + 12y + 2z = 25x \quad 12y + 2z = 12x$

$12x + 13y - 2z = 25y \quad 12x - 2z = 12y \quad 12y + 2z = 12x$

$2x - 2y + 8z = 25z \quad 2x - 2y = 17z \quad 12x - 12y = 102z$

$12y + 2z = 12x \rightarrow 12y = 102z$

$12x + 2z = 102z + 12$

$z = 0$

$12x = 100z + 12$

$y = x \quad x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 25, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$13x + 12y + 2z = 9x$

$12x + 13y - 2z = 9y$

$2x - 2y + 8z = 9z \rightarrow 2x - 2y = z$

$25x + 25y = 9x + 9y \rightarrow 16(x+y) = 0 \rightarrow x + y = 0$

$2z = 12x + 4y \rightarrow z = 6x + 2y$

$2x - 2(-x) = 4x$

$2x - 2y = 6x + 2y \rightarrow 4x = -4y \rightarrow x = -y$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -x \\ 4x \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

$\lambda = 9, \vec{v} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix}$

Singular values = sqrt of non-zero eigenvalues : $\sigma_1 = \sqrt{25} = 5$

$\sigma_2 = \sqrt{9} = 3$

a) $\Rightarrow \Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

$u_1 = \frac{1}{\sigma_1} A \cdot v_1 = \frac{1}{5}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} \frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} + \frac{3}{\sqrt{2}} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} \frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

b) V = matrix where columns = eigenvectors $\Rightarrow V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$

$u_2 = \frac{1}{3} A \cdot v_2 = \frac{1}{3}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{6} \\ -\frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} \frac{3\sqrt{2}}{6} - \frac{2\sqrt{2}}{6} + \frac{8\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{6} - \frac{3\sqrt{2}}{6} - \frac{8\sqrt{2}}{6} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} \frac{3\sqrt{2}}{2} \\ -\frac{3\sqrt{2}}{2} \end{bmatrix}$

$= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$

Sum of rank-1 matrices

$A = 5\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad 3\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \quad 0$

c) $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

$\boxed{A = 5\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} + 3\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \end{bmatrix}}$

d) $U\tilde{\Sigma}V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}^T$

$= \begin{bmatrix} \frac{5}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ \frac{5}{\sqrt{2}} & -\frac{3}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

$= \begin{bmatrix} \frac{5}{2}+\frac{3}{6}+0 & \frac{5}{2}-\frac{3}{6}+0 & 2 \\ \frac{5}{2}-\frac{3}{6}+0 & \frac{5}{2}+\frac{3}{6}+0 & -2 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = A$ ✓

5.1)

1) $A = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

A orthogonal: $A^TA = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\frac{1}{9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

2)

$A^TA = I$

$\lambda = 1 \quad v = e_1, e_2, e_3$

$V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\tilde{\Sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$u_i = \frac{1}{\sigma}A \cdot v_i \quad \sigma = 1$

$u_1 = A \cdot v_1 = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ $\quad U = A$

$u_2 = A \cdot v_2 = \frac{1}{3}A \cdot e_2 = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$u_3 = A v_3 = \frac{1}{3}A e_3 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

$U\tilde{\Sigma}V^T = A\tilde{\Sigma}I = A$

$\boxed{U = A,\ \tilde{\Sigma} = I,\ V = I}$

Scanned with CamScanner

$M = \begin{bmatrix} \quad \end{bmatrix}$

Application

4.4a)

$\tilde{M} = U \Sigma V^T$ ← overall importance of σ

$= \sum \sigma_i u_i v_i^T$

relevance of word i to each of the other words — relevance of word r to each of the documents

Guess that σ_i is overall importance of σ among documents

u_i is the relevance of each word relative to each other word; i.e. a similarity index.

v_i is relevance of word k, $0 \le k \le r$ to each of the documents (importance measure)

b) For a general document D_j in the text collection, we have

$$d_j = Me_j = \sum_{i=1}^{r} \sigma_i (v_i^T e_j) u_i$$

Thus, the coordinates of d_j in the orthonormal basis $(u_1, \dots u_n)$ are $(\sigma_i (v_i^T e_j))_{1 \le i \le n}$, with the convention $\sigma_{r+1} = \dots = \sigma_n = 0$. The projection onto the subspace spanned by the first k vectors of the basis is the k-vector $\hat{d}_j$ with the same k first coordinates as d:

$$\boxed{\hat{d}_j = \sum_{i=1}^{k} \sigma_i (v_i^T e_j) u_i} \qquad \boxed{M_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T}$$

For a query document q, we can write:

$$q = \sum_{i=1}^{n} (q^T u_i) u_i$$

$$\Rightarrow \boxed{\hat{q} = \sum_{i=1}^{k} (q^T u_i) u_i}$$

The latent semantic indexing approach is based on evaluating the angles between the projected query $\hat{q}$ and the projected documents $\hat{d}_j$; $j = 1, \dots m$

c) See code for svd command / computation

10 largest σ-values in sorted order:

1.5366, 1.0192, 0.9587, 0.9539, 0.9413, 0.9289, 0.8977, 0.8919, 0.8687, 0.8161

d) See code for computation.

Two most similar: Barack Obama and George W. Bush

e) Lowest k that doesn't change answer: k = 3

Documents 1 & 6 are closest for k = 2 (B.J. Cole and John Holland)

Again, see code for computation.

```
load wordVecV.mat
[W, D] = size(V);
M = V;
for i=1:W
    for j = 1:D
        if V(i,j) > 0
            M(i,j) = 1;
        else
            M(i,j) = 0;
        end
    end
end
M
```

```
M = 1651×10
     0     0     0     0     1     0     0     0     0     0
     0     0     0     1     0     1     0     0     0     0
     1     0     0     0     0     0     0     0     0     0
     0     1     0     0     1     0     0     0     0     0
     0     1     0     0     0     0     0     1     0     1
     0     0     0     0     1     0     0     0     0     0
     0     0     0     0     1     0     0     0     0     0
     0     0     0     0     0     0     0     0     1     0
     1     0     0     0     0     0     0     0     0     0
     0     1     0     0     0     0     0     0     0     0
     ⋮
```

```
Mnorm = M;
for i=1:D
    Mnorm(:,i) = M(:,i)/norm(M(:,i));
end
Mnorm
```

```
Mnorm = 1651×10
         0         0         0         0    0.0801         0         0         0 ···
         0         0         0    0.0558         0    0.0567         0         0
    0.0644         0         0         0         0         0         0         0
         0    0.0679         0         0    0.0801         0         0         0
         0    0.0679         0         0         0         0         0    0.0880
         0         0         0         0    0.0801         0         0         0
         0         0         0         0    0.0801         0         0         0
         0         0         0         0         0         0         0         0
    0.0644         0         0         0         0         0         0         0
         0    0.0679         0         0         0         0         0         0
     ⋮
```

```
s = svd(Mnorm)
```

```
s = 10×1
    1.5366
    1.0192
    0.9587
    0.9539
    0.9413
    0.9289
    0.8977
    0.8919
    0.8687
```

```
    0.8161
```

```
k=9;
[U, S, V] = svd(Mnorm);
min_angle(U, V, S, k, D)
```

```
Document 9 and Document 10 have minimum angle of 36.06 degrees between them
```

```
for k = 9:-1:1
    min_angle(U, V, S, k, D)
end
```

```
Document 9 and Document 10 have minimum angle of 36.06 degrees between them
Document 9 and Document 10 have minimum angle of 34.17 degrees between them
Document 9 and Document 10 have minimum angle of 27.84 degrees between them
Document 9 and Document 10 have minimum angle of 27.10 degrees between them
Document 9 and Document 10 have minimum angle of 6.35 degrees between them
Document 9 and Document 10 have minimum angle of 5.15 degrees between them
Document 9 and Document 10 have minimum angle of 5.13 degrees between them
Document 1 and Document 6 have minimum angle of 2.48 degrees between them
Document 1 and Document 2 have minimum angle of 0.00 degrees between them
```

```
function min_angle(U, V, S, k, D)
Mlr = U(:, 1:k) * S(1:k, 1:k) * (V(:, 1:k))';
minangnorm = acos(dot(Mlr(:,1), Mlr(:,2))/(norm(Mlr(:,1))*norm(Mlr(:,2))))*180/pi;
minangnorm_v1 = 1;
minangnorm_v2 = 2;
for i = 1:D-1
   for j = i+1:D
      pair_ang = acos(dot(Mlr(:,i), Mlr(:,j))/(norm(Mlr(:,i))*norm(Mlr(:,j))))*180/pi;
      if pair_ang < minangnorm
          minangnorm = pair_ang;
          minangnorm_v1 = i;
          minangnorm_v2 = j;
      end
   end
end
fprintf('Document %d and Document %d have minimum angle of %.02f degrees between them\n', minar
end
```