ECE367

2.1) We claim that $q_1, q_2, \dots q_n$ are simply the standard basis vectors; i.e. $q_i = e_i$ where e_i is the i^{th} basis vector (has 0 in all positions, except 1 at the i^{th} position).

We will prove by induction, and use the fact that $a_k = \sum_{j=1}^{k} e_j$.

Base case: $q_1 = a_1 = e_1$

Inductive Hypothesis: Assume $q_i = e_i$ for $i = 1, 2 \dots n$. We will show $q_{n+1} = e_{n+1}$

Inductive Step: By Gram-Schmidt:

$$q^{(m)} = w^{(m)} / \|w^{(m)}\|_2 \quad \text{where} \quad w^{(m)} = v^{(m)} - \sum_{i=1}^{m-1} \langle v^{(m)}, q^{(i)} \rangle q^{(i)}$$

For the $n+1^{th}$ orthonormal vector, we have:

$$w^{(n+1)} = v^{(n+1)} - \sum_{i=1}^{n} \langle v^{(n+1)}, q^{(i)} \rangle q^{(i)}$$

$$= v^{(n+1)} - \sum_{i=1}^{n} \langle v^{(n+1)}, e^{(i)} \rangle e^{(i)}$$

$v^{(n+1)} = \sum_{j=1}^{n+1} e_j$, so $\langle v^{(n+1)}, e^{(i)} \rangle = \langle \sum_{j=1}^{n+1} e_j, e^{(i)} \rangle = 1$ if $n+1 \geq i$, 0 otherwise. So:

$$w^{(n+1)} = v^{(n+1)} - \sum_{i=1}^{n} e^{(i)} = \sum_{j=1}^{n+1} e^{(j)} - \sum_{i=1}^{n} e^{(i)} = \cancel{e^{(1)}} + \cancel{e^{(2)}} + \dots + \cancel{e^{(n)}} + e^{(n+1)} - \cancel{e^{(1)}} - \cancel{e^{(2)}} \dots - \cancel{e^{(n)}}$$

$$= e^{(n+1)}, \quad \text{QED}$$

$a_1, a_2, \dots a_n$ is a basis for $\mathbb{R}^n$ because $q_1, \dots q_n$ is clearly a basis (standard basis)

2.2a) In general, if we have d basis vectors in our subspace S, and want to project x onto S, then we have

$$x_S = \sum_{i=1}^{d} \alpha_i v^{(i)}, \text{ where } \alpha_i \text{ are determined by the matrix equation:}$$

$$\begin{bmatrix} \langle v^{(1)}, v^{(1)} \rangle & \langle v^{(1)}, v^{(2)} \rangle \dots & \langle v^{(1)}, v^{(d)} \rangle \\ \vdots & & \\ \vdots & & \\ \langle v^{(d)}, v^{(1)} \rangle & \langle v^{(d)}, v^{(2)} \rangle \dots & \langle v^{(d)}, v^{(d)} \rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_d \end{bmatrix} = \begin{bmatrix} \langle v^{(1)}, x \rangle \\ \langle v^{(2)}, x \rangle \\ \vdots \\ \vdots \\ \langle v^{(d)}, x \rangle \end{bmatrix}$$

i) $x_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $V_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} = \text{span}\{v^{(1)}\}$

$$\langle v^{(1)}, v^{(1)} \rangle \alpha_1 = \langle v^{(1)}, x \rangle$$

$$\alpha_1 \left(\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rangle\right) = \langle \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \rangle$$

$$\alpha_1 (1+4) = (3-2)$$

$$5\alpha_1 = 1$$

$\alpha_1 = 1/5$, so $x_S = \text{Proj}_{V_1}(x_1) = \frac{1}{5} v^{(1)} = \frac{1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} \Rightarrow \boxed{\text{Proj}_{V_1}(x_1) = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}}$

ii) $x_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$ $V_2 = \text{span}\left\{\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} = \text{span}\{v^{(1)}, v^{(2)}\}$

$$\begin{bmatrix} \langle v^{(1)}, v^{(1)} \rangle & \langle v^{(1)}, v^{(2)} \rangle \\ \langle v^{(2)}, v^{(1)} \rangle & \langle v^{(2)}, v^{(2)} \rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \langle v^{(1)}, x_2 \rangle \\ \langle v^{(2)}, x_2 \rangle \end{bmatrix}$$

$\langle v^{(1)}, v^{(1)} \rangle = \langle \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \rangle = 4$ $\quad \langle v^{(1)}, v^{(2)} \rangle = \langle \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rangle = -2$

$\langle v^{(2)}, v^{(1)} \rangle = \langle v^{(1)}, v^{(2)} \rangle = -2$

$\langle v^{(2)}, v^{(2)} \rangle = \langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rangle = 3$

$<v^{(1)}, x_2> = < \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} > = 10$

$<v^{(2)}, x_2> = < \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} > = -2$

$$\begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$$

Row reduction:

$$\left[\begin{array}{cc|c} 4 & -2 & 10 \\ -2 & 3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 5 \\ -2 & 3 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 5 \\ 0 & 2 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 5 \\ 0 & 1 & 3/2 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 0 & 13/2 \\ 0 & 1 & 3/2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 13/4 \\ 0 & 1 & 3/2 \end{array}\right]$$

$\alpha_1 = 13/4, \ \alpha_2 = 3/2$

So, $x_{2,S}$, the component of x_2 in S, i.e $\text{Proj}_{V_2}(x_2)$ can be written as: $\boxed{\text{Proj}_{V_2}(x_2) = \frac{13}{4}\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$

iii) $x_3 = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}$ $\quad V_3 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span}\{v^{(1)}, v^{(2)}, v^{(3)}\}$

$$\begin{bmatrix} <v^{(1)},v^{(1)}> & <v^{(1)},v^{(2)}> & <v^{(1)},v^{(3)}> \\ <v^{(2)},v^{(1)}> & <v^{(2)},v^{(2)}> & <v^{(2)},v^{(3)}> \\ <v^{(3)},v^{(1)}> & <v^{(3)},v^{(2)}> & <v^{(3)},v^{(3)}> \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} <v^{(1)},x_3> \\ <v^{(2)},x_3> \\ <v^{(3)},x_3> \end{bmatrix}$$

$<v^{(1)},v^{(1)}> = 1+4 = 5$ $\quad <v^{(1)},v^{(2)}> = 3$ $\quad <v^{(1)},v^{(3)}> = 11$ $\quad <v^{(1)},x_3> = -2$

$<v^{(2)},v^{(1)}> = 3$ $\quad <v^{(2)},v^{(2)}> = 11$ $\quad <v^{(2)},v^{(3)}> = 3$ $\quad <v^{(2)},x_3> = 5$

$<v^{(3)},v^{(1)}> = 11$ $\quad <v^{(3)},v^{(2)}> = 3$ $\quad <v^{(3)},v^{(3)}> = 27$ $\quad <v^{(3)},x_3> = -3$

$$\begin{bmatrix} 5 & 3 & 11 \\ 3 & 11 & 3 \\ 11 & 3 & 27 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ -3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 5 & 3 & 11 & -2 \\ 3 & 11 & 3 & 5 \\ 11 & 3 & 27 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 15 & 9 & 33 & -6 \\ 15 & 55 & 15 & 25 \\ 11 & 3 & 27 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 15 & 9 & 33 & -6 \\ 0 & 46 & -18 & 31 \\ 11 & 3 & 27 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3/5 & 11/5 & -2/5 \\ 0 & 1 & -9/23 & 31/46 \\ 11 & 3 & 27 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 3/5 & 11/5 & -2/5 \\ 0 & 1 & -9/23 & 31/46 \\ 0 & -18/5 & 14/5 & 7/5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3/5 & 11/5 & -2/5 \\ 0 & 1 & -9/23 & 31/46 \\ 0 & 0 & 32/23 & 88/23 \end{array}\right]$$

$\alpha_1 + \frac{3}{5}\alpha_2 + \frac{11}{5}\alpha_3 = -\frac{2}{5}$ $\quad \alpha_1 = -\frac{3}{5}\alpha_2 - \frac{11}{5}\alpha_3 - \frac{2}{5} = -\frac{3}{5}\left(\frac{7}{4}\right) - \frac{11}{5}\left(\frac{11}{4}\right) - \frac{2}{5} = -\frac{15}{2}$

$\alpha_2 - \frac{9}{23}\alpha_3 = \frac{31}{46}$ $\quad \alpha_2 = \frac{31}{46} + \frac{9}{23}\alpha_3 = \frac{31}{46} + \frac{9}{23}\left(\frac{11}{4}\right) = \frac{7}{4}$

$\frac{32}{23}\alpha_3 = \frac{88}{23} \Rightarrow \alpha_3 = \frac{88}{32} = \frac{11}{4}$

So, $\boxed{\text{Proj}_{V_3}(x_3) = -\frac{15}{2}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \frac{7}{4}\begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{11}{4}\begin{bmatrix} 0 \\ 1 \\ 5 \\ 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} 7/4 \\ 17/2 \\ 55/4 \\ 7/4 \\ 11/4 \end{bmatrix}$

2.2b) $Proj_{A_i}(x_i) = Proj_{V_i}(x_i - b_i) + b_i$

i) $y_1 = x_1 - b_1 = \begin{bmatrix}4\\-2\end{bmatrix}$ $\quad V_1 = span\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

$\langle v^{(1)}, v^{(1)}\rangle \alpha_1 = \langle v^{(1)}, y_1\rangle$

$5\alpha_1 = 0 \Rightarrow \alpha_1 = 0$

$Proj_{A_1}(x_1) = 0 + b_1 = b_1 = \begin{bmatrix}-1\\1\end{bmatrix}$

ii) $y_2 = x_2 - b_2 = \begin{bmatrix}2\\4\\-4\end{bmatrix}$ $\quad V_2 = span\left\{\begin{bmatrix}0\\0\\-2\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

$$\begin{bmatrix}4 & -2\\-2 & 3\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}8\\2\end{bmatrix}$$

$$\left[\begin{array}{cc|c}4 & -2 & 8\\-2 & 3 & 2\end{array}\right] \to \left[\begin{array}{cc|c}2 & -1 & 4\\-2 & 3 & 2\end{array}\right] \to \left[\begin{array}{cc|c}2 & -1 & 4\\0 & 2 & 6\end{array}\right] \to \left[\begin{array}{cc|c}1 & -\frac{1}{2} & 2\\0 & 2 & 6\end{array}\right] \to \left[\begin{array}{cc|c}1 & -\frac{1}{2} & 2\\0 & 1 & 3\end{array}\right] \to \left[\begin{array}{cc|c}1 & 0 & \frac{7}{2}\\0 & 1 & 3\end{array}\right]$$

$\alpha_2 = 3 \quad \alpha_1 = \frac{7}{2}$

$Proj_{A_2}(x_2) = Proj_{V_2}(y_2) + b_2 = \frac{7}{2}\begin{bmatrix}0\\0\\-2\end{bmatrix} + 3\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\-3\\-1\end{bmatrix} = \begin{bmatrix}3\\0\\-5\end{bmatrix}$

iii) $y_3 = x_3 - b_3 = \begin{bmatrix}4\\0\\-2\\4\\1\end{bmatrix}$ $\quad V_3 = span\left\{\begin{bmatrix}0\\1\\2\\0\\0\end{bmatrix}, \begin{bmatrix}1\\3\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\5\\0\\1\end{bmatrix}\right\}$

$$\begin{bmatrix}5 & 3 & 11\\3 & 11 & 3\\11 & 3 & 27\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}-4\\8\\-9\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}5 & 3 & 11 & -4\\3 & 11 & 3 & 8\\11 & 3 & 27 & -9\end{array}\right] \to \left[\begin{array}{ccc|c}15 & 9 & 33 & -12\\15 & 55 & 15 & 40\\11 & 3 & 27 & -9\end{array}\right] \to \left[\begin{array}{ccc|c}15 & 9 & 33 & -12\\0 & 46 & -18 & 52\\11 & 3 & 27 & -9\end{array}\right] \to \left[\begin{array}{ccc|c}1 & \frac{3}{5} & \frac{11}{5} & -\frac{4}{5}\\0 & 1 & -\frac{9}{23} & \frac{86}{23}\\11 & 3 & 27 & -9\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1 & \frac{3}{5} & \frac{11}{5} & -\frac{4}{5}\\0 & 1 & -\frac{9}{23} & \frac{26}{23}\\0 & -\frac{18}{5} & \frac{14}{5} & -\frac{1}{5}\end{array}\right] \to \left[\begin{array}{ccc|c}1 & \frac{3}{5} & \frac{11}{5} & -\frac{4}{5}\\0 & 1 & -\frac{9}{23} & \frac{26}{23}\\0 & 0 & \frac{32}{23} & \frac{84}{23}\end{array}\right]$$

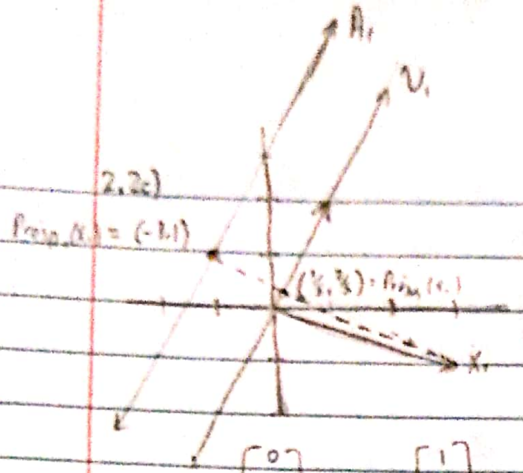
$\alpha_1 + \frac{3}{5}\alpha_2 + \frac{11}{5}\alpha_3 = -\frac{4}{5} \to \alpha_1 = -\frac{4}{5} - \frac{11}{5}\alpha_3 - \frac{3}{5}\alpha_2 = -\frac{33}{4}$

$\alpha_2 - \frac{9}{23}\alpha_3 = \frac{26}{23} \qquad \alpha_2 = \frac{26}{23} + \frac{9}{23}\left(\frac{84}{32}\right) = \frac{71}{32}$

$\alpha_3 = \frac{84}{32}$

$$\boxed{Proj_{A_3}(x_3) = -\frac{33}{4}\begin{bmatrix}0\\1\\2\\0\\0\end{bmatrix} + \frac{71}{32}\begin{bmatrix}1\\3\\0\\1\\0\end{bmatrix} + \frac{84}{32}\begin{bmatrix}0\\1\\5\\0\\1\end{bmatrix} + \begin{bmatrix}-1\\0\\1\\-2\\1\end{bmatrix}} = \begin{bmatrix}71/32\\19/16\\-83/32\\71/32\\84/32\end{bmatrix} + \begin{bmatrix}-1\\0\\1\\-?\end{bmatrix}$$

2.2c)



Point closest to x_1 in V_1 (using standard inner product / 2-norm) is $(\frac{1}{5}, \frac{2}{5})$

$\text{Proj}_{A_1}(x_1) = (-1,1)$

d) $v^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ $v^{(2)} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $v^{(3)} = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 0 \\ 1 \end{bmatrix}$

$$z^{(1)} = \frac{v^{(1)}}{\|v^{(1)}\|} = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{bmatrix}$$

$$w^{(2)} = v^{(2)} - \sum_{k=1}^{1} \langle v^{(2)}, z^{(k)} \rangle z^{(k)} = v^{(2)} - \langle v^{(2)}, z^{(1)} \rangle z^{(1)}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{\sqrt{5}} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 3/5 \\ 6/5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 12/5 \\ -6/5 \\ 1 \\ 0 \end{bmatrix}$$

$$z^{(2)} = \frac{w^{(2)}}{\|w^{(2)}\|} = \begin{bmatrix} 1/\sqrt{\frac{46}{5}} \\ 12/5\sqrt{\frac{46}{5}} \\ -6/5\sqrt{\frac{46}{5}} \\ 1/\sqrt{\frac{46}{5}} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{5}/\sqrt{46} \\ 12/\sqrt{5 \cdot 46} \\ -6/\sqrt{5 \cdot 46} \\ \sqrt{5}/\sqrt{46} \\ 0 \end{bmatrix}$$

$$w^{(3)} = v^{(3)} - \langle v^{(3)}, z^{(1)} \rangle z^{(1)} - \langle v^{(3)}, z^{(2)} \rangle z^{(2)}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} - \frac{11}{\sqrt{5}} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{bmatrix} - \left(\frac{-18}{\sqrt{230}}\right) \begin{bmatrix} \sqrt{5}/\sqrt{46} \\ 12/\sqrt{230} \\ -6/\sqrt{230} \\ \sqrt{5}/\sqrt{46} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 11/5 \\ 22/5 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 9/23 \\ 108/115 \\ -54/115 \\ 9/23 \\ 0 \end{bmatrix} = \begin{bmatrix} 9/23 \\ -6/23 \\ 3/23 \\ 9/23 \\ 1 \end{bmatrix}$$

$$z^{(3)} = \frac{w^{(3)}}{\|w^{(3)}\|} = \begin{bmatrix} 9 \\ -6 \\ 3 \\ 9 \\ 23 \end{bmatrix} / \sqrt{736}$$

$\text{Proj}_{A_1}(x_1)$:

$$\alpha_5 = \alpha_1 z^{(1)} + \alpha_2 z^{(2)} + \alpha_3 z^{(3)}$$

$$\alpha_1 = \langle x_1, z^{(1)} \rangle = \left\langle \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{bmatrix} \right\rangle = -\frac{2}{\sqrt{5}} \Rightarrow \boxed{\alpha_1 = -\frac{2}{\sqrt{5}}}$$

$$\alpha_2 = \langle x_1, z^{(2)} \rangle = \left\langle \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} \sqrt{5}/\sqrt{46} \\ 12/\sqrt{5}\sqrt{46} \\ -6/\sqrt{5}\sqrt{46} \\ \sqrt{5}/\sqrt{46} \\ 0 \end{bmatrix} \right\rangle = \frac{3\sqrt{5}}{\sqrt{46}} + \frac{6}{\sqrt{5}\sqrt{46}} + \frac{2\sqrt{5}}{\sqrt{46}} = \frac{3\sqrt{5}}{\sqrt{46}} + \frac{6\sqrt{5}}{5\sqrt{46}} + \frac{2\sqrt{5}}{\sqrt{46}}$$

$$= \frac{15\sqrt{5} + 6\sqrt{5} + 10\sqrt{5}}{5\sqrt{46}} = \frac{31\sqrt{5}}{5\sqrt{46}} \Rightarrow \boxed{\alpha_2 = \frac{31}{\sqrt{5}\sqrt{46}}}$$

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$$\alpha_3 = \langle x_1, z^{(3)} \rangle = \left\langle \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 9 \\ -6 \\ 3 \\ 9 \\ 23 \end{bmatrix} \right\rangle / \sqrt{736} = \frac{27+3+18+46}{\sqrt{736}} \Rightarrow \boxed{\alpha_3 = \frac{88}{\sqrt{736}}}$$

$$-\frac{2}{\sqrt{5}} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \\ 0 \end{bmatrix} + \frac{31}{\sqrt{5}\sqrt{46}} \begin{bmatrix} \sqrt{5}/\sqrt{46} \\ 12/\sqrt{5}\sqrt{46} \\ -6/\sqrt{5}\sqrt{46} \\ \sqrt{5}/\sqrt{46} \\ 0 \end{bmatrix} + \frac{11}{92} \begin{bmatrix} 9 \\ -6 \\ 3 \\ 9 \\ 23 \end{bmatrix} = \begin{bmatrix} 31/46 + 99/92 \\ -2/5 + 31 \cdot 12/5 \cdot 46 - 66/92 \\ \\ 31/46 + 99/92 \\ 11/4 \end{bmatrix} = \begin{bmatrix} 7/4 \\ 1/2 \\ -5/4 \\ 7/4 \\ 11/4 \end{bmatrix}$$ (same as before)

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$$\mathrm{Proj}_{A_3}(x_3) = \mathrm{Proj}_{\theta_3}(x_3 - b_3) + b_3$$

$$= \mathrm{Proj}_{V_3}\left(\begin{bmatrix}4\\0\\-2\\4\\1\end{bmatrix}\right) + \begin{bmatrix}-1\\0\\1\\-2\\1\end{bmatrix}$$

$$\alpha_1 = \left\langle \begin{bmatrix}4\\0\\-2\\4\\1\end{bmatrix}, \begin{bmatrix}0\\1/\sqrt5\\2/\sqrt5\\0\\0\end{bmatrix}\right\rangle = -4/\sqrt5 \qquad \alpha_2 = \left\langle \begin{bmatrix}4\\0\\-2\\4\\1\end{bmatrix}, \begin{bmatrix}\sqrt5/\sqrt{46}\\12/\sqrt5\sqrt{46}\\-6/\sqrt5\sqrt{46}\\\sqrt5/\sqrt{46}\\0\end{bmatrix}\right\rangle = \frac{4\sqrt5}{\sqrt{46}} + \frac{12}{\sqrt5\sqrt{46}} + \frac{4\sqrt5}{\sqrt{46}} = \frac{20}{\sqrt5\sqrt{46}} + \frac{12}{\sqrt5\sqrt{46}} + \frac{20}{\sqrt5\sqrt{46}}$$

$$\alpha_3 = \left\langle \begin{bmatrix}4\\0\\-2\\4\\1\end{bmatrix}, \begin{bmatrix}9\\-6\\3\\4\\23\end{bmatrix}\right\rangle/\sqrt{736} = (36-6+36+23)/\sqrt{736} = 89/\sqrt{736}$$

$$-\frac{4}{\sqrt5}\begin{bmatrix}0\\1/\sqrt5\\2/\sqrt5\\0\\0\end{bmatrix} + \frac{52}{\sqrt5\sqrt{46}}\begin{bmatrix}\sqrt5/\sqrt{46}\\12/\sqrt5\sqrt{46}\\-6/\sqrt5\sqrt{46}\\\sqrt5/\sqrt{46}\\0\end{bmatrix} + \frac{89}{736}\begin{bmatrix}9\\-6\\3\\9\\23\end{bmatrix} = \begin{bmatrix}52/46+801/736\\-4/5+624/230-534/736\\-8/5-312/230+267/736\\\cancel{52/46+801/736}\end{bmatrix} = \begin{bmatrix}71/32\\19/16\\-83/32\\71/32\\89/32\end{bmatrix}$$

Same as LA

2.3) $f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x)$

$$f(x) = f(x_0) + \nabla f(x_0)^T (x-x_0) + (x-x_0)^T \nabla^2 f(x_0)(x-x_0)/2$$

$$\nabla f(x) = \begin{bmatrix}\partial f/\partial x_1\\ \partial f/\partial x_2\\ \vdots\\ \partial f/\partial x_n\end{bmatrix} = \begin{bmatrix}\sum_{i=1}^{m} \frac{a_1^{(i)}}{b_i - a_i^T x}\\ \vdots\\ \sum_{i=1}^{m}\frac{a_n^{(i)}}{b_i - a_i^T x}\end{bmatrix}$$

$$\boxed{\frac{\partial f}{\partial x_i} = -\sum_{i=1}^{m} \frac{-a_i^{(i)}}{b_i - a_i^T x} = \sum_{i=1}^{m} \frac{a_i^{(i)}}{b_i - a_i^T x}}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(\sum_{i=1}^{m} \frac{a_j^{(i)}}{b_i - a_i^T x}\right) = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\frac{a_j^{(i)}}{b_i - a_i^T x} = \sum_{i=1}^{m} \frac{a_j^{(i)} a_i^{(i)}}{(b_i - a_i^T x)^2}$$

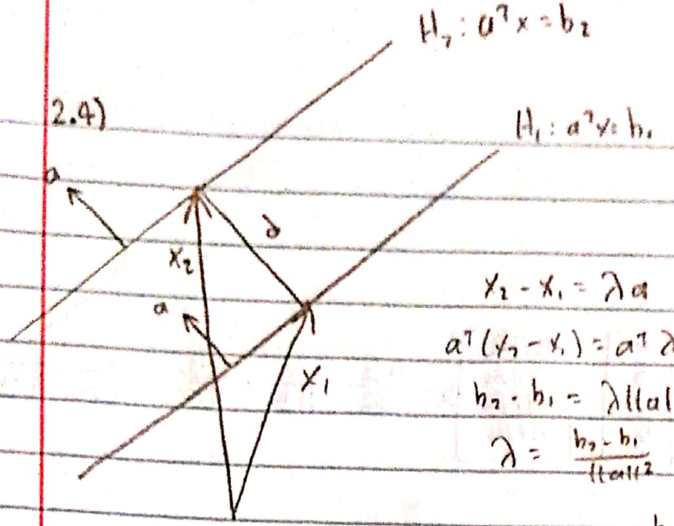
$$\boxed{\nabla^2 f(x) = \begin{bmatrix} \sum_{i=1}^{m} \frac{a_1^{(i)} a_1^{(i)}}{(b_i - a_i^T x)^2} & \sum_{i=1}^{m} \frac{a_1^{(i)} a_2^{(i)}}{(b_i - a_i^T x)^2} & \cdots & \sum_{i=1}^{m} \frac{a_1^{(i)} a_n^{(i)}}{(b_i - a_i^T x)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{m} \frac{a_n^{(i)} a_1^{(i)}}{(b_i - a_i^T x)^2} & \sum_{i=1}^{m} \frac{a_n^{(i)} a_2^{(i)}}{(b_i - a_i^T x)^2} & \cdots & \sum_{i=1}^{m} \frac{a_n^{(i)} a_n^{(i)}}{(b_i - a_i^T x)^2}\end{bmatrix}}$$

$$\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_i - x_{0,i})(x_j - x_{0,j}) = (x-x_0)^T \nabla^2 f(x)(x-x_0)$$

$$\therefore f(x) \approx f(x_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_i - x_{0,i}) + \frac12 \sum_{i,j}^{n,n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_i - x_{0,i})(x_j - x_{0,j})$$

$$\boxed{f(x) \approx f(x_0) + \nabla f(x)^T (x-x_0) + \frac12 (x-x_0)^T \nabla^2 f(x)(x-x_0)}$$

2.4)



$$x_2 - x_1 = \lambda a$$
$$a^T(x_2 - x_1) = a^T \lambda a$$
$$b_2 - b_1 = \lambda \|a\|^2$$
$$\lambda = \frac{b_2 - b_1}{\|a\|^2}$$

$$d = \lambda a = \frac{b_2 - b_1}{\|a\|^2} a$$
$$\|d\| = \frac{|b_2 - b_1|}{\|a\|}$$

So, distance between hyperplanes is $\boxed{\frac{|b_2 - b_1|}{\|a\|}}$

2.5) a)

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 4 & 6 & 9 \\ 1 & 4 & 4 & 4 \\ 1 & 0 & 10 & 4 \end{bmatrix}$$

$$rrf(A) = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{basis for } R(A^T)}$: $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $\dim(R(A^T)) = 2$

$\boxed{\text{basis for } R(A)}$: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $\dim(R(A)) = 2$

basis for $N(A)$: $\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

$\boxed{\text{basis for } N(A)}$: $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\dim(N(A)) = 2$

basis for $N(A^T)$: $\boxed{\text{basis for } N(A^T)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\dim(N(A^T)) = 1$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$rrf(A^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_1 = 0$
$y_2 = 0$
$y_3 =$ anything

$$\begin{bmatrix} 0 \\ 0 \\ y_3 \end{bmatrix} = y_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\text{rank}(A) = \dim(R(A)) = 2$

$$B = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$rref(B) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{basis for } R(B^T)}$: $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \leftarrow \dim(R(B^T)) = 2$

$\dim(R(B)) = 2$

$\boxed{\text{basis for } R(B)}$: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2x_3 - x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$x_1 - 2x_3 + x_4 = 0 \rightarrow x_1 = 2x_3 - x_4$

$x_2 + x_3 = 0$ $\quad\boxed{\text{basis for } N(B)}$: $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad \dim(N(B)) = 2$

$x_2 = -x_3$

$$B^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad rref(B^T) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\text{basis for } N(B^T)} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\quad \dim(N(B^T)) = 1$

$x_1 + x_3 = 0 \quad x_1 = -x_3 \quad \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\quad rank(B) = \dim(R(B)) = 2$

$x_2 = 0$

350
70
420

70

c)

$$C = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 4 & 6 & 9 \\ 1 & 4 & 4 & 4 \\ 1 & 0 & 0 & 4 \end{bmatrix}$$

$$rref(C) = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -2 & 3 & 0 \\ 0 & -2 & 3 & 1 \\ 0 & -2 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & -2 & 3 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & -2 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 12 & 9 \\ 0 & -8 & 12 & 0 \\ 0 & 0 & 12 & 2 \\ 0 & 0 & -12 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 0 & 7 \\ 0 & -8 & 0 & -2 \\ 0 & 0 & 12 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 30 & 0 & 0 & 70 \\ 0 & -280 & 0 & -70 \\ 0 & 0 & 420 & 70 \\ 0 & 0 & 0 & 70 \end{bmatrix} \rightarrow \begin{bmatrix} 30 & 0 & 0 & 0 \\ 0 & -280 & 0 & 0 \\ 0 & 0 & 420 & 0 \\ 0 & 0 & 0 & 70 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -28 & 0 & 0 \\ 0 & 0 & 42 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$\boxed{\text{basis for } R(C^T)}$: $\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -28 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 42 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7 \end{bmatrix}$ $\quad \dim(R(C^T)) = 4$

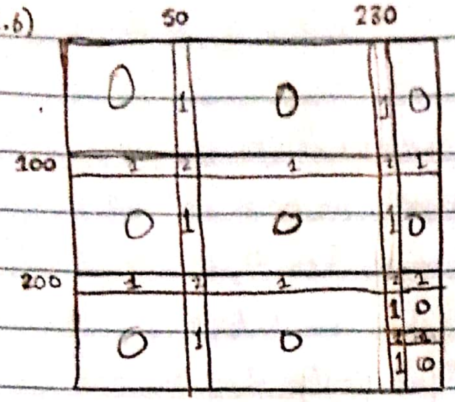
$\boxed{\text{basis for } R(C)}$: $\begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 4 \end{bmatrix}$ $\quad \dim(R(C)) = 4$

$\dim(R(C^T)) + \dim(N(C)) = 4 \rightarrow \dim(N(C)) = 0 \quad N(C) = \{\}$ (no vectors in basis) $\quad \dim(N(C)) = 0$

$\dim(R(C)) + \dim(N(C^T)) = 4 \rightarrow \dim(N(C^T)) = 0 \quad N(C^T) = \{\}$ (no vectors in basis) $\quad \dim(N(C^T)) = 0$

$rank(C) = \dim(R(C)) = 4$

2.6)



It is not clear how the colouring on this thing works. Here's my guess:

$$\begin{matrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 & 0 \end{matrix}$$

$\boxed{\text{rank} = 3}$ (3 linearly independent rows)

$\text{rank}(A) + \dim(N(A)) = 3 + \dim N(A) = 256 \Rightarrow \dim N(A) = 256 - 3 = 253$

$X_{50} + X_{230} = 0$

$X_1 + \dots + X_{49} + X_{51} + \dots + X_{229} + X_{231} + \dots + X_{256} = 0$ — 3 equations, so $256 - 3 = 253$ free variables

$X_{50} + 2X_{230} + X_{231} + \dots + X_{256} = 0$

$\hookrightarrow \boxed{X_{231} + \dots + X_{256} = 0}$

2.7) $x(t+1) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$

1) $x(0) = 0$

$y(T) = Cx(T) =$

$X(T) = Ax(T-1) + Bu(T-1) = A[Ax(T-2) + Bu(T-2)] + Bu(T-1) = A^2x(T-2) + ABu(T-2) + Bu(T-1)$

$= A^2(Ax(T-3) + Bu(T-3)) + ABu(T-2) + Bu(T-1)$

$= A^3x(T-3) + A^2Bu(T-3) + ABu(T-2) + Bu(T-1)$

$= A^Tx(0) + A^{T-1}Bu(0) + A^{T-2}Bu(1) + \dots + ABu(T-2) + Bu(T-1)$

$= A^{T-1}Bu(0) + A^{T-2}Bu(1) + \dots + ABu(T-2) + Bu(T-1)$

$$= \begin{bmatrix} A^{T-1}B & A^{T-2}B & \cdots & AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(T-2) \\ u(T-1) \end{bmatrix}$$

$H = [A^{T-1}B \; A^{T-2}B \cdots AB \; B]$ (could prove by induction)

2) Range of H is set of all outputs possible at time T, since we can choose $u(0), u(1), \dots u(T-1)$ to be anything we want, $HU =$ is set of all $y(T)$ after all possible input sequences of length T.

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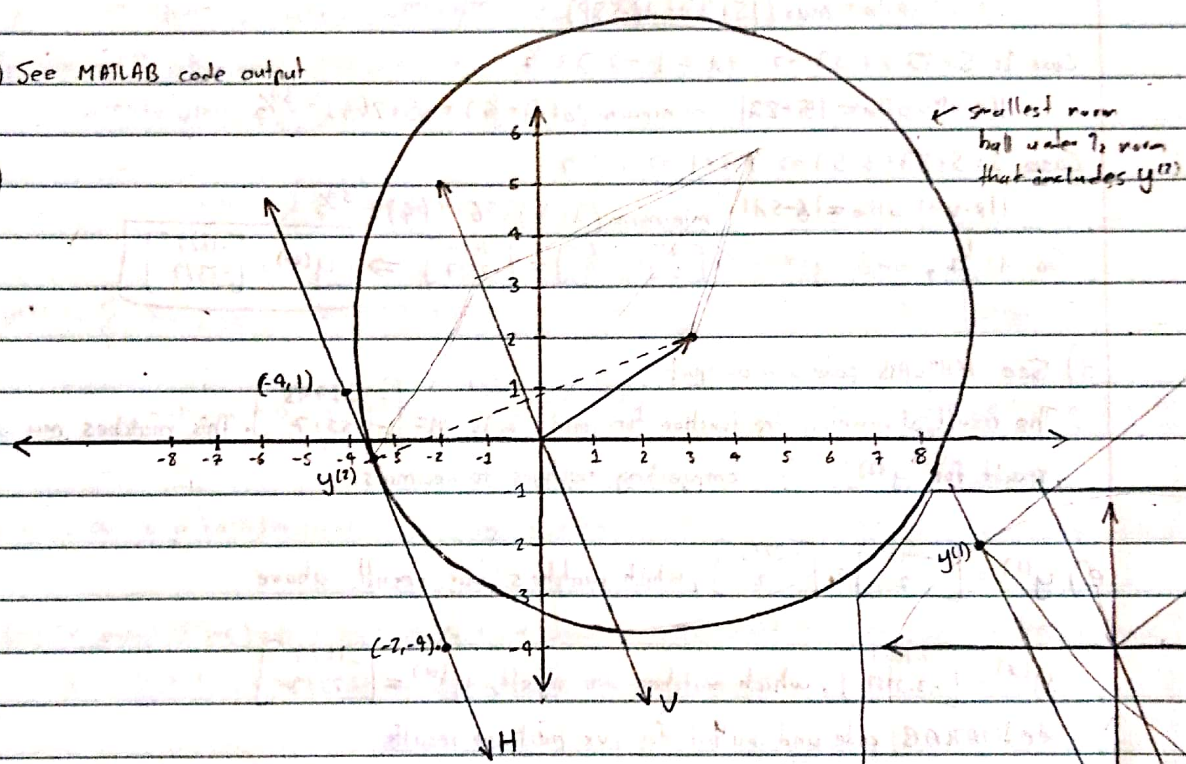
2.8) a) See MATLAB code output

b) I would numerically test for the orthogonality of the basis vectors by computing the inner product between them. Two vectors are orthogonal if and only if their inner product is 0. Since the rows of B are vectors of finite length, and we are using ℓ_2 norm to determine how good any approximation is (implying we are using standard inner product), we simply need to compute a dot product

See MATLAB code output for plots

c) See MATLAB code output

2.9) a)



b) $\min_y \|x-y\|_2 \quad , y \in A$

$y^{(2)} = \text{Proj}_V (x - v^{(0)}) + v^{(0)}$

$x = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad v^{(0)} = \begin{bmatrix} -2 \\ -4 \end{bmatrix} \qquad V = \text{span}\left\{\begin{bmatrix} -2 \\ 5 \end{bmatrix}\right\} = \text{span}\{v_1\}$

$\langle v_1, v_1 \rangle \alpha = \langle x - v^{(0)}, v_1 \rangle$

$(4+25)\alpha = \langle \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \end{bmatrix} \rangle = -10 + 30 = 20$

$\alpha = \frac{20}{29}$

$y^{(2)} = \frac{20}{29}\begin{bmatrix} -2 \\ 5 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

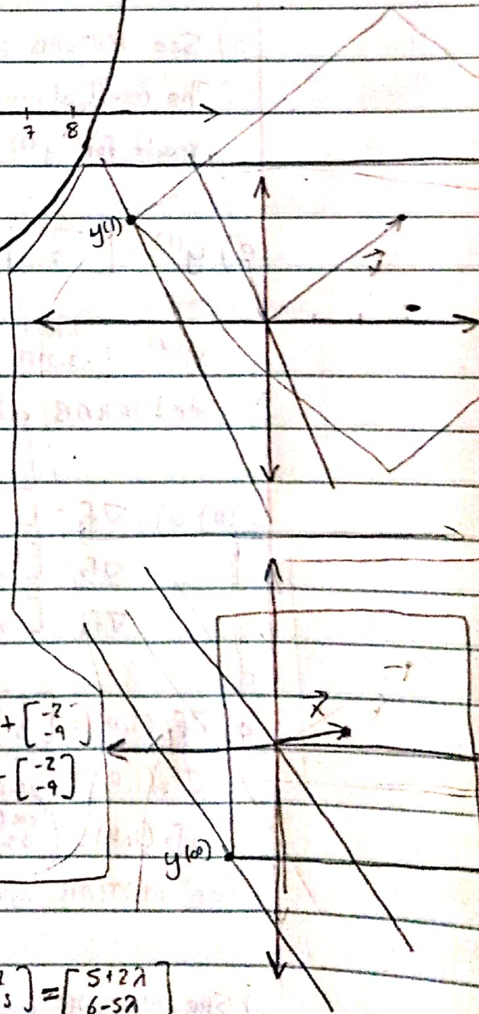
$= \begin{bmatrix} -40/29 \\ 100/29 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

$\boxed{y^{(2)} = \begin{bmatrix} -98/29 \\ -16/29 \end{bmatrix}}$

c) $p=1$:

$\boxed{y^{(p)} = \underset{a \in V}{\text{argmin}} \|x - v^{(0)} - a\|_p + v^{(0)}}$

$x - v^{(0)} - a = x - v^{(0)} - \lambda v = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ -4 \end{bmatrix} - \lambda\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} + \lambda\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 5+2\lambda \\ 6-5\lambda \end{bmatrix}$



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$p=1$:

$$\|x-v^{(0)}-a\lambda\|_1 = |5+2\lambda| + |6-5\lambda|$$

$-\frac{5}{2} \le \lambda \le \frac{6}{5}$: $\|x-v^{(0)}-a\lambda\|_1 = 5+2\lambda+6-5\lambda = 11-3\lambda \Rightarrow$ minimum (at $\lambda = \frac{6}{5}$): $11-3(\frac{6}{5}) = \frac{37}{5}$

$\lambda \ge \frac{6}{5}$: $\|x-v^{(0)}-a\lambda\|_1 = 5+2\lambda+5\lambda-6 = 7\lambda-1 \Rightarrow$ minimum (at $\lambda = \frac{6}{5}$): $7(\frac{6}{5})-1 = \frac{37}{5}$

$\lambda \le -\frac{5}{2}$: $\|x-v^{(0)}-a\lambda\|_1 = -2\lambda-5+6-5\lambda = -7\lambda+1 \Rightarrow$ minimum (at $\lambda = -\frac{5}{2}$): $-7(-\frac{5}{2})+1 = \frac{37}{2}$

So $\lambda = \frac{6}{5}$, and $y^{(1)} = \frac{6}{5}\begin{bmatrix}-2\\5\end{bmatrix} + \begin{bmatrix}-2\\-4\end{bmatrix} = \begin{bmatrix}-12/5\\6\end{bmatrix} + \begin{bmatrix}-2\\-4\end{bmatrix} = \begin{bmatrix}-22/5\\2\end{bmatrix} \Rightarrow \boxed{y^{(1)} = \begin{bmatrix}-22/5\\2\end{bmatrix}}$

$p=\infty$:

$$\|x-v^{(0)}-a\lambda\|_\infty = \max(|5+2\lambda|, |6-5\lambda|)$$

Case 1: $5+2\lambda \ge 6-5\lambda \Rightarrow 7\lambda \ge 1 \Rightarrow \lambda \ge \frac{1}{7}$

$\|x-v^{(0)}-a\lambda\|_\infty = |5+2\lambda|$ minimum (at $\lambda = \frac{1}{7}$) $= 5+2(\frac{1}{7}) = \frac{37}{7}$

Case 2: $5+2\lambda \le 6-5\lambda \Rightarrow 7\lambda \le 1 \Rightarrow \lambda \le \frac{1}{7}$

$\|x-v^{(0)}-a\lambda\|_\infty = |6-5\lambda|$ minimum ($\lambda = \frac{1}{7}$) $= 6-5(\frac{1}{7}) = \frac{37}{7}$

So $\lambda = \frac{1}{7}$, and $y^{(\infty)} = \frac{1}{7}\begin{bmatrix}-2\\5\end{bmatrix} + \begin{bmatrix}-2\\-4\end{bmatrix} = \begin{bmatrix}-16/7\\-23/7\end{bmatrix} \Rightarrow \boxed{y^{(\infty)} = \begin{bmatrix}-16/7\\-23/7\end{bmatrix}}$

d) See MATLAB code and output

The result of running the function for norm $=2$ is $y = \begin{bmatrix}-3.3793\\-0.5517\end{bmatrix}$. This matches my analytic result for $y^{(2)}$ after converting fractions to decimals.

e) $y^{(1)} = \begin{bmatrix}-4.4\\2\end{bmatrix} = \begin{bmatrix}-22/5\\2\end{bmatrix}$, which matches our result above

$y^{(\infty)} = \begin{bmatrix}-2.2857\\-3.2857\end{bmatrix}$, which matches our result, $y^{(\infty)} = \begin{bmatrix}-16/7\\-23/7\end{bmatrix}$

See MATLAB code and output for cvx package results

2.10) a) $\nabla f_1 = \begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\end{bmatrix}^T = [2 \quad 3]^T = \begin{bmatrix}2\\3\end{bmatrix}$

$\nabla f_2 = \begin{bmatrix}\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{bmatrix}^T = [2x-y,\ 2y-x]^T = \begin{bmatrix}2x-y\\2y-x\end{bmatrix}$

$\nabla f_3 = \begin{bmatrix}\frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y}\end{bmatrix}^T = [\cos(y-5)-(y-5)\cos(x-5),\ (5-x)\sin(y-5)-\sin(x-5)]^T = \begin{bmatrix}\cos(y-5)-(y-5)\cos(x\\(5-x)\sin(y-5)-\sin(x-\end{bmatrix}$

b)

b) $\nabla f_1(1,0) = \begin{bmatrix}2\\3\end{bmatrix}$ $\qquad [2 \quad 3]\begin{bmatrix}x_1\\x_1\end{bmatrix} =$

$\nabla f_2(1,0) = \begin{bmatrix}2(1)-0\\2(0)-1\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$

$\nabla f_3(1,0) = \begin{bmatrix}\cos(5)+5\cos(4)\\4\sin(-5)-\sin(-4)\end{bmatrix} = \begin{bmatrix}\cos(5)+5\cos(4)\\\sin(4)-4\sin(5)\end{bmatrix}$

See MATLAB code and output for contour plots.

c) See MATLAB code and output for tangent plane approximation to all functions at the point (1,0).

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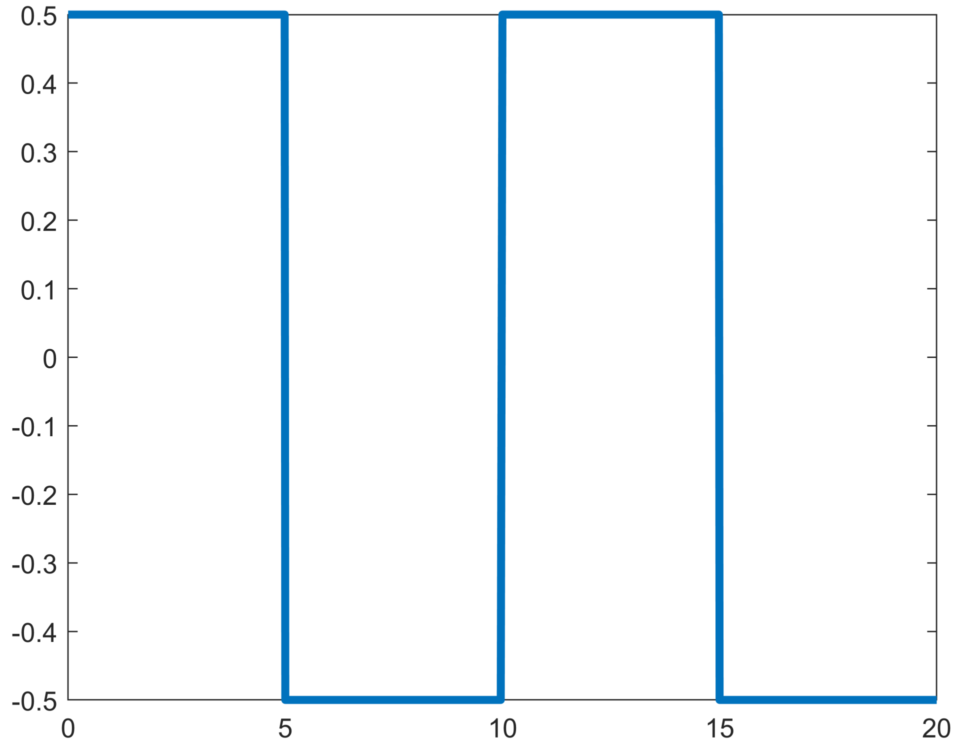
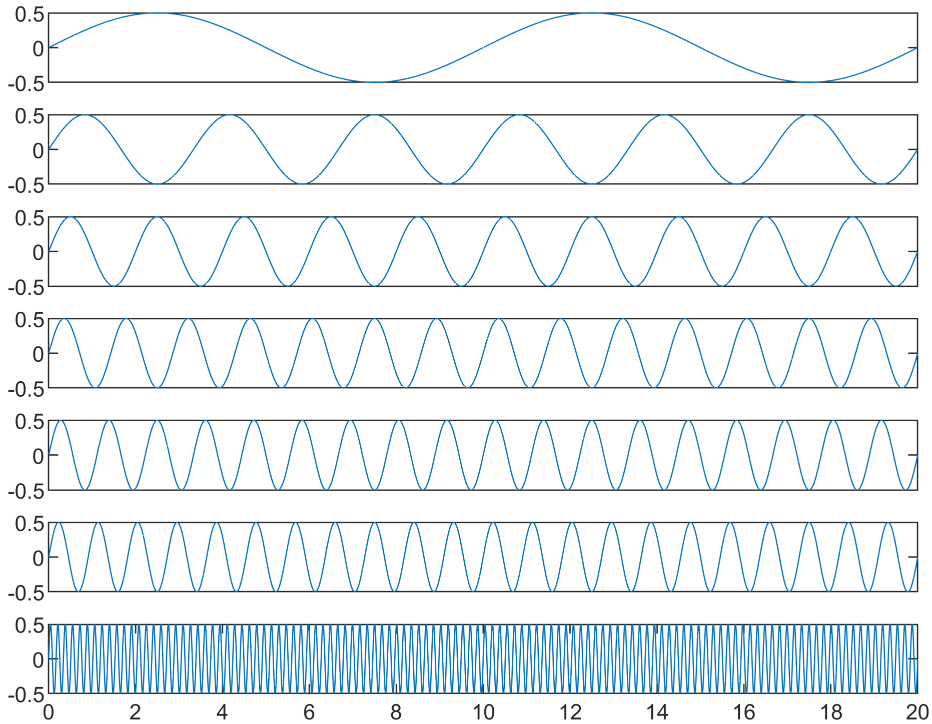
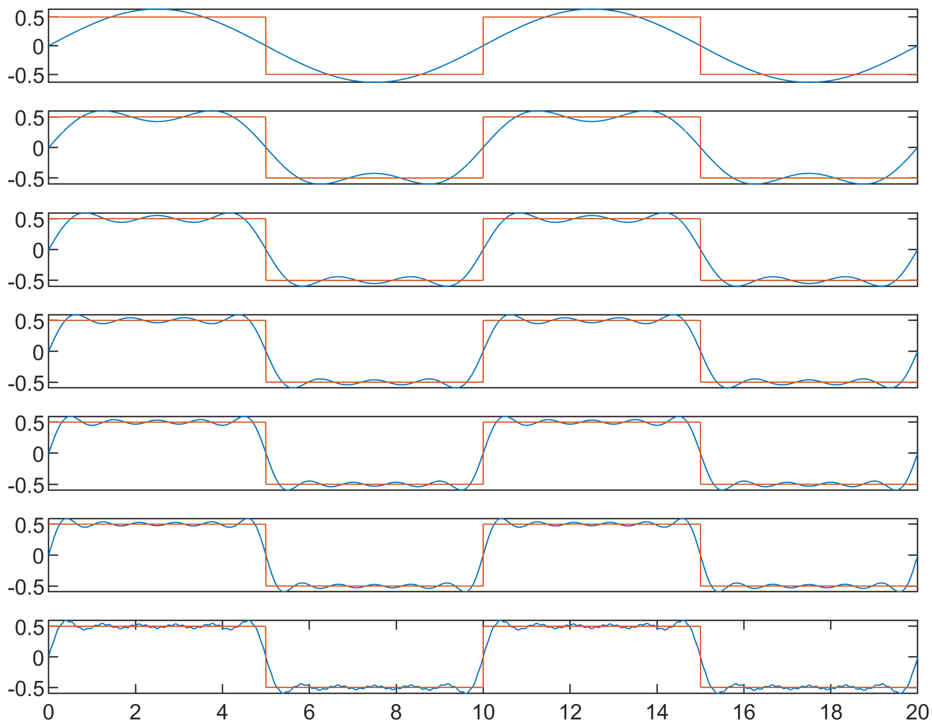
```
%% 2.8a
[ time_pos , sq_wave , B_unnorm ] = generate_data;
figure;
plot(time_pos, sq_wave, 'LineWidth',3)
```




```
%% 2.8b
figure;
row = [1:6,30];
for i = 1:7
  subplot(7,1,i);
  plot(time_pos, B_unnorm(row(i), :));
  set(gca,'xtick',[])
end
set(gca,'xtickMode', 'auto')
```




```
%% 2.8c
B_norm = zeros(size(B_unnorm));
for i = 1:30
    B_norm(i,:) = B_unnorm(i,:)/norm(B_unnorm(i,:));
end

alphas = zeros(30,1);
for i = 1:30
    alphas(i,1) = sum(sq_wave.*B_norm(i,:));
end

[alphasorted,alphasortidx]=sort(alphas, 'descend');
B_norm = B_norm(alphasortidx, :);
B_normT = B_norm';
figure;
for i = 1:6
  subplot(7,1,i);
  approximation = B_normT(:,1:i)*alphasorted(1:i);
  plot(time_pos, approximation, time_pos, sq_wave)
  set(gca,'xtick',[])
end
subplot(7,1,7);
approximation = B_normT(:,[1:6,30])*alphasorted([1:6,30]);
plot(time_pos, approximation, time_pos, sq_wave)
```



0.5
0
-0.5
0
2
4
6
8
10
12
14
16
18
20

```
%% 2.9d
[y2, r2] = proj_cvx([3;2], [-2;-4], [-2;5], 2)
```

```
Calling SDPT3 4.0: 3 variables, 1 equality constraints
------------------------------------------------------------

 num. of constraints =  1
 dim. of socp   var  =  3,   num. of socp blk  =  1
*******************************************************************
   SDPT3: Infeasible path-following algorithms
*******************************************************************
 version  predcorr  gam  expon  scale_data
    NT      1      0.000   1        0
it pstep dstep pinfeas dinfeas  gap      prim-obj      dual-obj    cputime
-------------------------------------------------------------------
 0|0.000|0.000|9.5e-01|1.3e+00|3.4e+01| 9.146713e+00  0.000000e+00| 0:0:03| chol  1  1
 1|1.000|0.836|2.1e-07|2.6e-01|5.3e+00| 9.676653e+00  9.494304e+00| 0:0:06| chol  1  1
 2|1.000|1.000|3.7e-07|5.0e-03|3.2e-01| 7.137307e+00  6.885179e+00| 0:0:06| chol  1  1
 3|0.990|0.990|6.5e-09|5.5e-04|3.4e-03| 6.873512e+00  6.877649e+00| 0:0:06| chol  1  1
 4|0.989|0.989|2.6e-09|5.5e-05|3.7e-05| 6.870758e+00  6.871483e+00| 0:0:06| chol  1  1
 5|0.989|0.989|7.5e-11|6.1e-07|4.1e-07| 6.870728e+00  6.870736e+00| 0:0:06| chol  1  1
 6|0.991|0.999|9.1e-13|6.5e-10|4.7e-09| 6.870728e+00  6.870728e+00| 0:0:06|
  stop: max(relative gap, infeasibilities) < 1.49e-08
-------------------------------------------------------------------
 number of iterations   =  6
 primal objective value =  6.87072752e+00
 dual   objective value =  6.87072752e+00
 gap := trace(XZ)       = 4.72e-09
 relative gap           = 3.20e-10
 actual relative gap    = -2.84e-10
 rel. primal infeas (scaled problem)   = 9.06e-13
 rel. dual     "        "       "      = 6.49e-10
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 9.7e+00, 3.7e-01, 1.4e+00
 norm(A), norm(b), norm(C) = 3.7e+00, 2.0e+01, 2.0e+00
 Total CPU time (secs)  = 6.35
 CPU time per iteration = 1.06
 termination code       =  0
 DIMACS: 9.1e-13  0.0e+00  6.5e-10  0.0e+00  -2.8e-10  3.2e-10
-------------------------------------------------------------------

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +6.87073

y2 = 2×1
   -3.3793
   -0.5517
r2 = 6.8707
```

```
%% 2.9e
[y1, r1] = proj_cvx([3;2], [-2;-4], [-2;5], 1)
```

```
Calling SDPT3 4.0: 4 variables, 1 equality constraints
------------------------------------------------------------

 num. of constraints =  1
 dim. of socp   var  =  4,   num. of socp blk  =  2
*******************************************************************
```

```
   SDPT3: Infeasible path-following algorithms
*******************************************************************
 version  predcorr  gam  expon  scale_data
    NT      1      0.000   1        0
it pstep dstep pinfeas dinfeas  gap      prim-obj      dual-obj    cputime
-------------------------------------------------------------------
 0|0.000|0.000|9.5e-01|1.6e+00|5.5e+01| 1.493652e+01  0.000000e+00| 0:0:00| chol  1  1
 1|1.000|0.790|2.1e-07|3.8e-01|1.3e+01| 1.461784e+01  1.099215e+01| 0:0:00| chol  1  1
 2|1.000|1.000|3.9e-07|5.9e-03|1.1e+00| 8.474405e+00  7.437095e+00| 0:0:00| chol  1  1
 3|0.989|0.989|1.5e-08|6.4e-04|1.2e-02| 7.411687e+00  7.407734e+00| 0:0:00| chol  1  1
 4|0.989|0.989|3.0e-09|6.5e-05|1.3e-04| 7.400128e+00  7.400817e+00| 0:0:00| chol  1  1
 5|0.989|0.989|8.3e-11|7.1e-07|1.5e-06| 7.400001e+00  7.400009e+00| 0:0:00| chol  1  1
 6|0.993|1.000|9.4e-13|1.7e-11|1.9e-08| 7.400000e+00  7.400000e+00| 0:0:00|
  stop: max(relative gap, infeasibilities) < 1.49e-08
-------------------------------------------------------------------
 number of iterations   =  6
 primal objective value =  7.40000002e+00
 dual   objective value =  7.40000000e+00
 gap := trace(XZ)       = 1.86e-08
 relative gap           = 1.17e-09
 actual relative gap    = 1.16e-09
 rel. primal infeas (scaled problem)   = 9.43e-13
 rel. dual     "        "       "      = 1.66e-11
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 1.0e+01, 4.0e-01, 1.8e+00
 norm(A), norm(b), norm(C) = 3.7e+00, 2.0e+01, 2.4e+00
 Total CPU time (secs)  = 0.26
 CPU time per iteration = 0.04
 termination code       =  0
 DIMACS: 9.4e-13  0.0e+00  2.0e-11  0.0e+00  1.2e-09  1.2e-09
-------------------------------------------------------------------

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +7.4

y1 = 2×1
   -4.4000
    2.0000
r1 = 7.4000
```

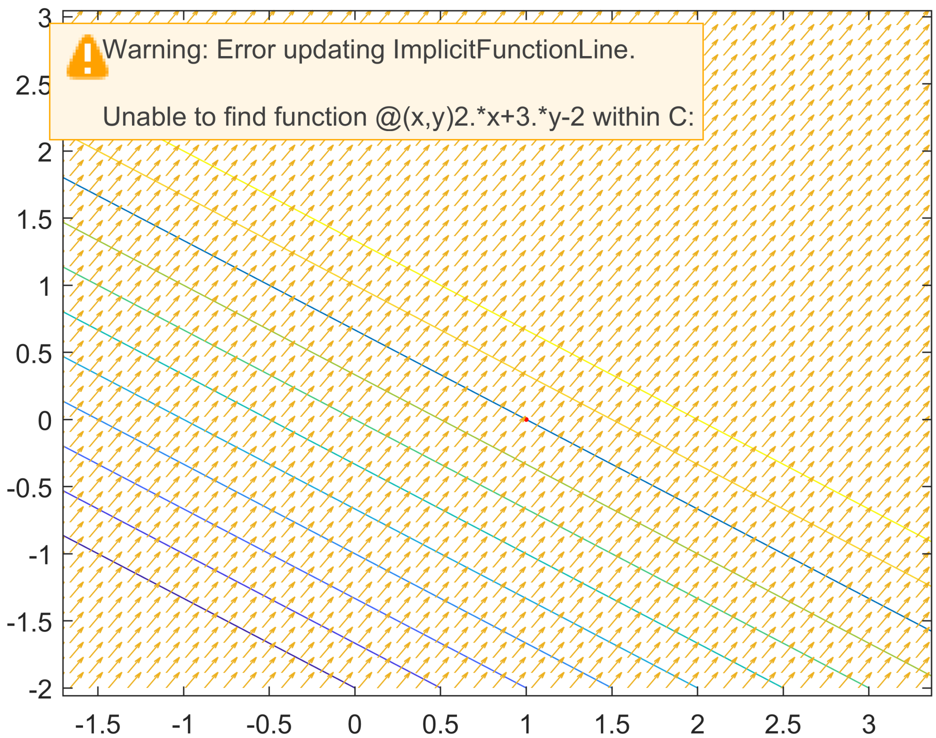
```
[yinf, rinf] = proj_cvx([3;2], [-2;-4], [-2;5], inf)
```

```
Calling SDPT3 4.0: 6 variables, 2 equality constraints
------------------------------------------------------------

 num. of constraints =  2
 dim. of socp   var  =  4,   num. of socp blk  =  2
 dim. of linear var  =  2
*******************************************************************
   SDPT3: Infeasible path-following algorithms
*******************************************************************
 version  predcorr  gam  expon  scale_data
    NT      1      0.000   1        0
it pstep dstep pinfeas dinfeas  gap      prim-obj      dual-obj    cputime
-------------------------------------------------------------------
 0|0.000|0.000|9.5e-01|5.9e+00|6.1e+02| 3.504542e+01  0.000000e+00| 0:0:00| chol  1  1
 1|1.000|0.884|1.1e-07|7.5e-01|1.3e+02| 4.747492e+01  6.702049e+00| 0:0:00| chol  1  1
 2|1.000|1.000|3.2e-07|8.3e-03|1.7e+01| 2.144067e+01  4.385268e+00| 0:0:00| chol  1  1
 3|0.962|0.765|4.6e-08|2.6e-03|9.2e-01| 6.013615e+00  5.130942e+00| 0:0:00| chol  1  1
 4|0.971|0.979|3.0e-08|1.4e-04|3.2e-02| 5.310529e+00  5.280517e+00| 0:0:00| chol  1  1
 5|0.988|0.988|6.2e-10|9.8e-06|3.7e-04| 5.286002e+00  5.285759e+00| 0:0:00| chol  1  1
```

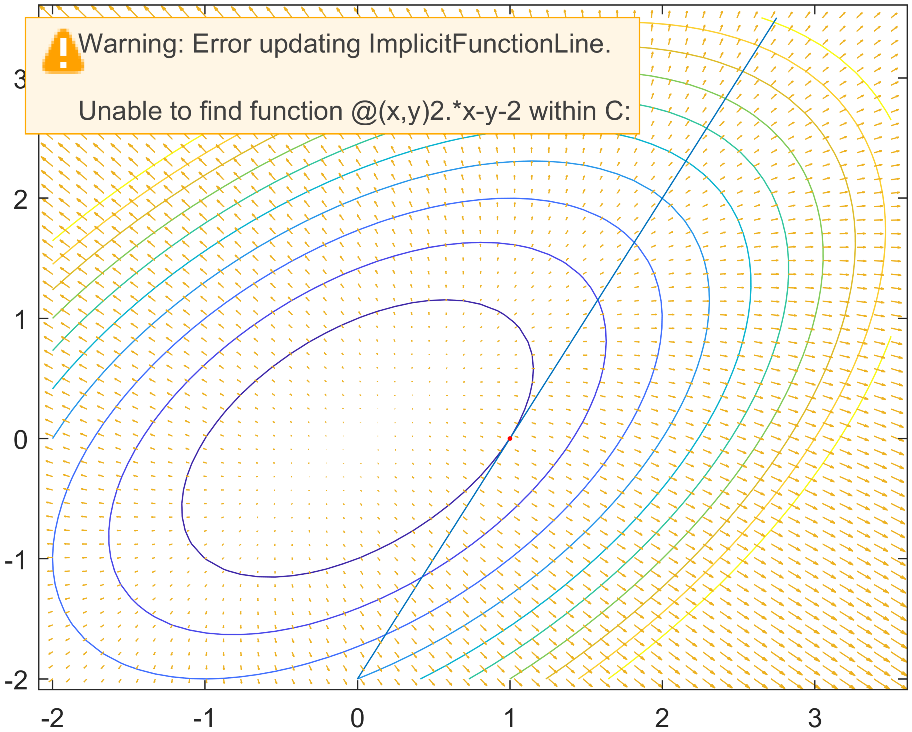
```
 6|0.989|0.989|6.8e-12|1.1e-07|4.0e-06| 5.285717e+00  5.285715e+00| 0:0:00| chol  1  1
 7|0.993|0.995|4.9e-14|5.9e-10|5.6e-08| 5.285714e+00  5.285714e+00| 0:0:00|
  stop: max(relative gap, infeasibilities) < 1.49e-08
-------------------------------------------------------------------
 number of iterations   =  7
 primal objective value =  5.28571433e+00
 dual   objective value =  5.28571428e+00
 gap := trace(XZ)       = 5.57e-08
 relative gap           = 4.81e-09
 actual relative gap    = 4.16e-09
 rel. primal infeas (scaled problem)   = 4.95e-14
 rel. dual     "        "       "      = 5.90e-10
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 1.1e+01, 4.0e-01, 1.3e+00
 norm(A), norm(b), norm(C) = 4.4e+00, 2.0e+01, 2.4e+00
 Total CPU time (secs)  = 0.47
 CPU time per iteration = 0.07
 termination code       =  0
 DIMACS: 4.9e-14  0.0e+00  7.1e-10  0.0e+00  4.2e-09  4.8e-09
-------------------------------------------------------------------

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +5.28571

yinf = 2×1
   -2.2857
   -3.2857
rinf = 5.2857
```

```
function [y, r] = proj_cvx(x, v0, v, nrm)% % x , v0 and v must be column vectors
    objtv = @(y) norm(x-y , nrm); % % objective is L_2 norm
    cvx_begin
        variable y(2) % % 2 - d variable we are optimizing over
        variable t(1) % % real valued parameter that defines
        minimize (objtv(y)) % % defining the objective
        subject to
        v0 + t*v == y; % % the projection y must be in set
    cvx_end
    r = objtv(y); %% minimum  value of the  objectiveend
end
```

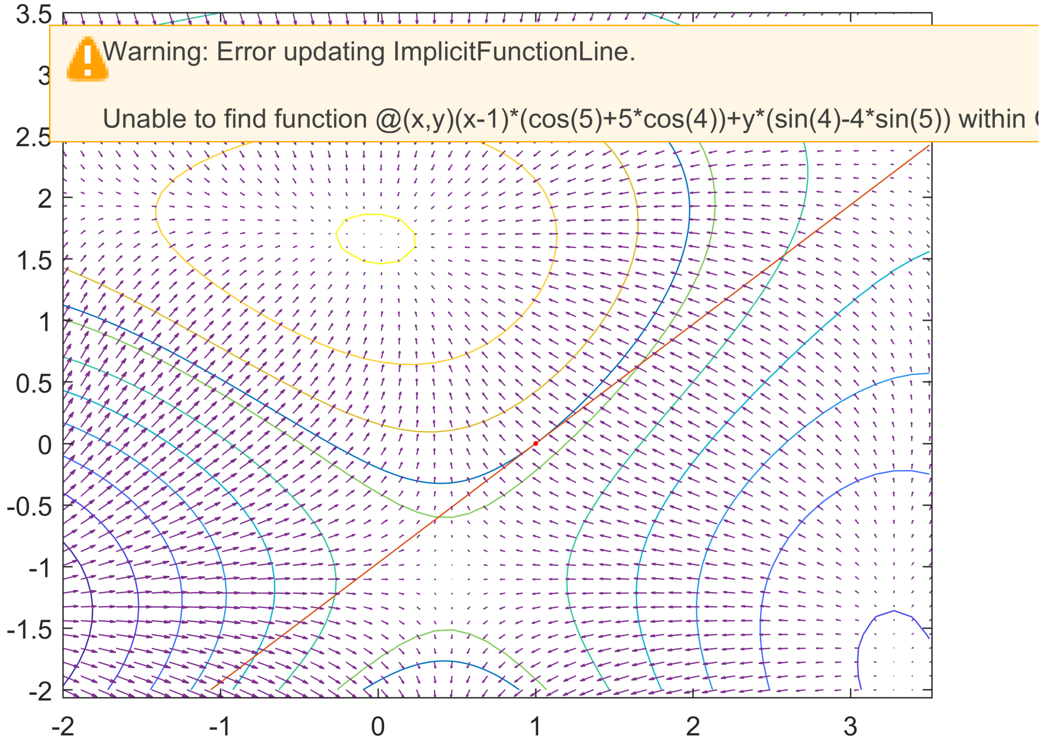
```
x = linspace(-2, 3.5, 50);
y = linspace(-2, 3.5, 50);
[X,Y] = meshgrid(x,y);
z = @(x,y) 2.*x+3.*y+1;
Z = z(X,Y);
[px,py] = gradient(Z);
figure
[c,h]=contour(X,Y,Z,-5:1:5);
hold on
fimplicit(@(x,y) 2.*x + 3.*y - 2, [-2 3.5 -2 3.5])
plot(1,0,'r.')
hold on
quiver(X,Y,px,py)
```



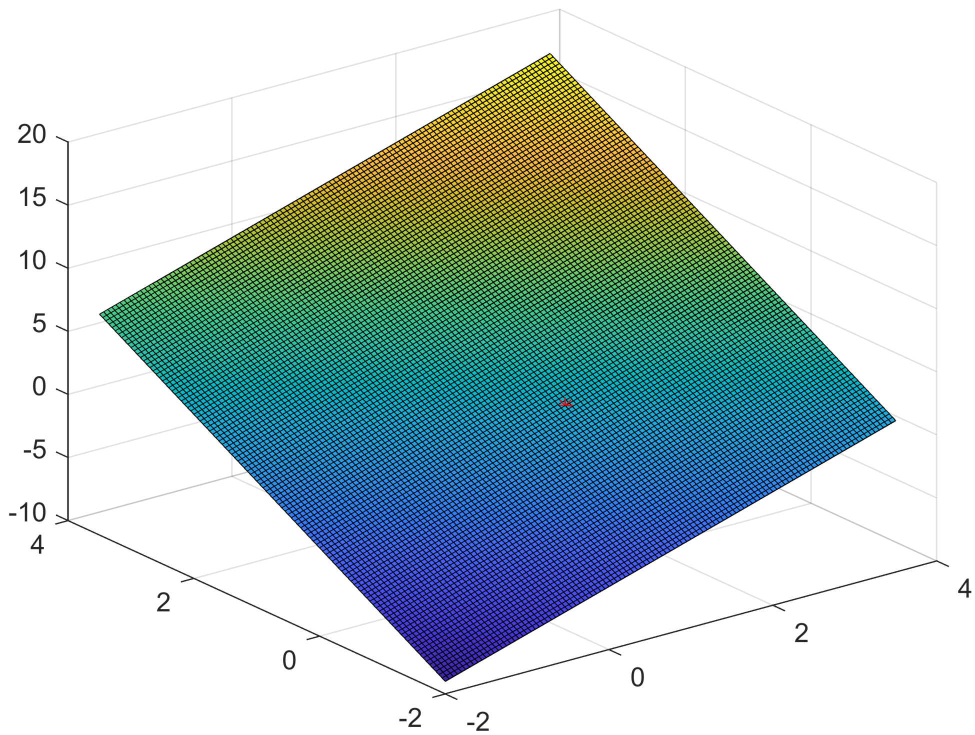

```
x = linspace(-2, 3.5, 50);
y = linspace(-2, 3.5, 50);
[X,Y] = meshgrid(x,y);
z = @(x,y) x.^2+y.^2-x.*y-5;
Z = z(X,Y);
[px,py] = gradient(Z);
figure
[c,h]=contour(X,Y,Z,-5:1:5);
hold on
fimplicit(@(x,y) 2.*x - y - 2, [-2 3.5 -2 3.5])
plot(1,0,'r.')
hold on
quiver(X,Y,px,py)
```



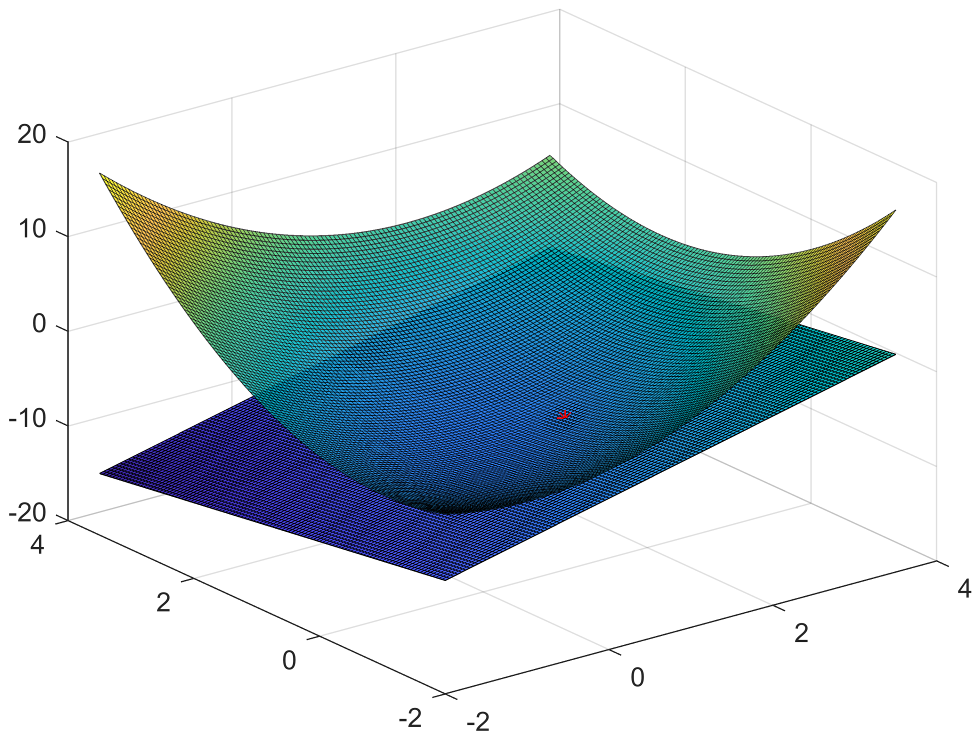

```
x = linspace(-2, 3.5, 50);
y = linspace(-2, 3.5, 50);
[X,Y] = meshgrid(x,y);
z = @(x,y) (x-5).*cos(y-5) - (y-5).*sin(x-5);
Z = z(X,Y);
[px,py] = gradient(Z);
figure
contour(X,Y,Z);
hold on
fimplicit(@(x,y) (x-5).*cos(y-5) - (y-5).*sin(x-5) + 4.*cos(5) + 5.*sin(4), [-2 3.5 -2 3.5])
fimplicit(@(x,y) (x-1)*(cos(5)+5*cos(4)) + y*(sin(4)-4*sin(5)), [-2 3.5 -2 3.5])
plot(1,0,'r.')
hold on
quiver(X,Y,px,py)
```



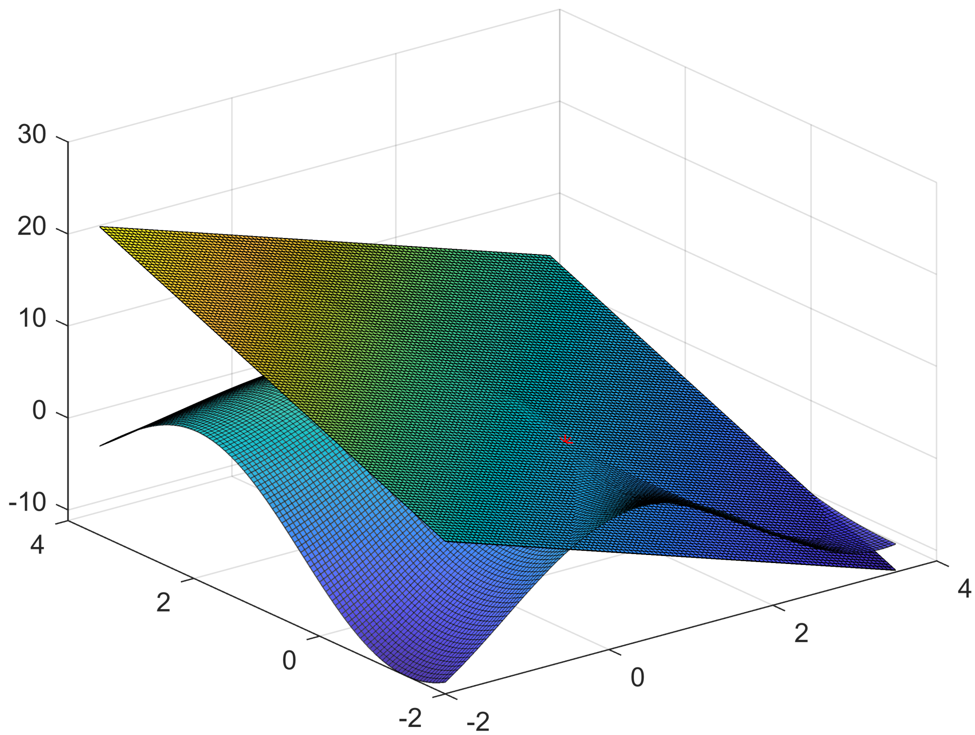

```
% hold on
% quiver(X,Y,px,py)
% hold on
% yTangentLine = 5 * (x - 1);
% plot(x, yTangentLine, 'b-');

f1 = @(x,y) 2.*x + 3.*y + 1;
figure;
drawtangplane(f1)
```




```
f2 = @(x,y) x.^2 + y.^2 -x.*y - 5;
figure;
drawtangplane(f2)
```




```
f3 = @(x,y) (x - 5).*cos(y - 5) - (y - 5).* sin(x - 5);
figure;
drawtangplane(f3)
```




```
function drawtangplane(f)
    [xx,yy] = meshgrid(-2:0.05:3.5);
    [fx,fy] = gradient(f(xx,yy),0.05);
    x0 = 1;
    y0 = 0;
    t = (xx == x0) & (yy == y0);
    indt = find(t);
    fx0 = fx(indt);
    fy0 = fy(indt);
    z = @(x,y) f(x0,y0) + fx0*(x-x0) + fy0*(y-y0);
    surf(xx,yy,f(xx,yy),'EdgeAlpha',0.7,'FaceAlpha',0.9)
    hold on
    surf(xx,yy,z(xx,yy))
    plot3(x0,y0,f(x0,y0),'r*')
end
```