ECE367 Pset 3

Theory

3.1)

a) $A=\begin{bmatrix}3&1\\2&2\end{bmatrix}$

i) $\det(A-\lambda I)=0 \Rightarrow \det\left(\begin{bmatrix}3-\lambda&1\\2&2-\lambda\end{bmatrix}\right)=0$

$(3-\lambda)(2-\lambda)-2=0$

$6-3\lambda-2\lambda+\lambda^2-2=0$

$\lambda^2-5\lambda+4=0$

$(\lambda-4)(\lambda-1)=0$

$\boxed{\lambda_1=4 \text{ and } \lambda_2=1}$

ii) $\begin{bmatrix}3&1\\2&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4x\\4y\end{bmatrix}$ $\qquad$ $\begin{bmatrix}3&1\\2&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$

$3x+y=4x \qquad 2x+2y=4y \qquad 3x+y=x \qquad 2x+2y=y$

$x=y \qquad x=y \qquad y=-2x \qquad y=-2x$

$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\alpha\\\alpha\end{bmatrix}=\alpha\begin{bmatrix}1\\1\end{bmatrix}$ $\qquad$ $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\alpha\\-2\alpha\end{bmatrix}=\alpha\begin{bmatrix}1\\-2\end{bmatrix}$

$\boxed{E_{\lambda_1}=\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}}$ $\qquad$ $\boxed{E_{\lambda_2}=\left\{\begin{bmatrix}1\\-2\end{bmatrix}\right\}}$

iii) $AM(\lambda=1)=GM(\lambda=1)=1$

$AM(\lambda=4)=GM(\lambda=4)=1$

iv) It is indeed diagonalizable. $\boxed{V=\begin{bmatrix}1&1\\1&-2\end{bmatrix},\ \Lambda=\begin{bmatrix}4&0\\0&1\end{bmatrix}} \Rightarrow A=V\Lambda V^{-1}$

b) $A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$

i) $\det(A-\lambda I)=0 \Rightarrow \det\left(\begin{bmatrix}-\lambda&1\\1&-\lambda\end{bmatrix}\right)=0$

$\lambda^2-1=0$

$(\lambda+1)(\lambda-1)=0$

$\boxed{\lambda_1=1,\ \lambda_2=-1}$

ii) $\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$ $\qquad$ $\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-x\\-y\end{bmatrix}$

$y=x \qquad x=y \qquad y=-x \qquad x=-y$

$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\alpha\\\alpha\end{bmatrix}=\alpha\begin{bmatrix}1\\1\end{bmatrix}$ $\qquad$ $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\alpha\\-\alpha\end{bmatrix}=\alpha\begin{bmatrix}1\\-1\end{bmatrix}$

$\boxed{E_{\lambda_1}=\begin{bmatrix}1\\1\end{bmatrix}}$ $\qquad$ $\boxed{E_{\lambda_2}=\begin{bmatrix}1\\-1\end{bmatrix}}$

iii) $AM(\lambda_1)=GM(\lambda_1)=1$

$AM(\lambda_2)=GM(\lambda_2)=1$

iv) It is indeed diagonalizable. $\boxed{V=\begin{bmatrix}1&1\\1&-1\end{bmatrix},\ \Lambda=\begin{bmatrix}1&0\\0&-1\end{bmatrix}}$

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c) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

i) $\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}\right) = 0$

$\lambda^2 + 1 = 0$

$\boxed{\lambda_1 = i,\ \lambda_2 = -i}$

ii) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ix \\ iy \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -ix \\ -iy \end{bmatrix}$

$y = ix \quad -x = iy$ $\qquad$ $y = -ix \quad -x = -iy$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ix \end{bmatrix} = x\begin{bmatrix} 1 \\ i \end{bmatrix}$ $\qquad$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -ix \end{bmatrix} = x\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$\boxed{E_{\lambda_1} = \begin{bmatrix} 1 \\ i \end{bmatrix}}$ $\qquad$ $\boxed{E_{\lambda_2} = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$

iii) $AM(\lambda_1) = GM(\lambda_1) = 1$

$AM(\lambda_2) = GM(\lambda_2) = 1$

iv) $V = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $A = V\Lambda V^{-1}$

d) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

i) $\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = 0$

$(1-\lambda)\left[(1-\lambda)^2\right] \Rightarrow (1-\lambda)^3 = 0$

$\lambda_1 = 1$

ii) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$x = x$ $\qquad$ $x + z = 0$

$x + y + z = y$ $\qquad$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -x \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$z = z$

$\boxed{E_{\lambda_1} = \left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}}$

iii) $AM(\lambda_1) \neq GM(\lambda_1) \Rightarrow$ NOT Diagonalizable

3.2) 1) $A = pq^T + qp^T$

i) $A(p+q) = (pq^T + qp^T)(p+q)$

$= pq^Tp + pq^Tq + qp^Tp + qp^Tq$

$(c = p^Tq = q^Tp)$ $\quad = cp + p(1) + q(1) + cq$

$= (c+1)(p+q)$

ii) $A(p-q) = (pq^T + qp^T)(p-q)$

$= pq^Tp - pq^Tq + qp^Tp - qp^Tq$

$= cp - p + q - cq$

$= (c-1)(p-q)$

2) $R(A) = (pq^T + qp^T)x = (q^Tx)p + (p^Tx)q$

$\dim R(A) = \dim\{(q^Tx)p + (p^Tx)q \mid x \in \mathbb{R}^n\} = 2$

Nullspace: $q^Tx = 0$ and $p^Tx = 0$

$\sum x_i q_i = 0 \quad \sum x_i p_i = 0$

$N(A) = \{x \in \mathbb{R}^n \mid \langle x, q\rangle = 0 \text{ and } \langle x, p\rangle = 0\}$

i.e nullspace is set of vectors orthogonal to p and q.

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3) Eigenvectors across different eigenspaces are orthogonal. For $\lambda = 0$, we can pick an orthonormal basis $\{a_3, a_4, \dots a_n\}$ which allows $\{p+q, p-q, a_3, \dots a_n\}$ to be a basis for $\mathbb{R}^n$. Now, let $a_1 = \frac{p+q}{\|p+q\|}$ and $a_2 = \frac{p-q}{\|p-q\|}$. Then $\{a_1, a_2, \dots a_n\}$ is an orthonormal basis. It follows that

$V = [a_1 \ a_2 \dots a_n]$ is an orthogonal matrix with columns all eigenvectors

So, $V^{-1} = V^T$ and $A = V\Lambda V^T$, where $\Lambda = \text{diag}(c+1, c-1, 0, 0, \dots 0)$

4) The nullspace of A is still given by:

$N(A) = \{x \in \mathbb{R}^n \mid (pq^T + qp^T)x = 0 \Rightarrow (q^Tx)p + (p^Tx)q = 0 \Rightarrow q^Tx = p^Tx = 0\}$, ie x orthogonal to both p and q.

So $\lambda = 0$ and we can select an orthonormal basis $\{a_3, a_4, \dots a_n\}$

$R(A) = \{Ax \mid x \in \mathbb{R}^n\} = \{(pq^T + qp^T)x \mid x \in \mathbb{R}^n\} = \{(q^Tx)p + (p^Tx)q \mid x \in \mathbb{R}^n\} = \text{span}\{p, q\}$.

So rank $A = 2$. We will use this to find eigenvalues, and show they are different from part 3.

$$Ax = \lambda x$$

$x \in \text{span}\{p, q\} \Rightarrow x = c_1 p + c_2 q$

$$Ax = A(c_1p + c_2q) = (pq^T + qp^T)(c_1p + c_2q) = c_1pq^Tp + c_2pq^Tq + c_1qp^Tp + c_2qp^Tq = c_1cp + c_2\|q\|^2p + c_1\|p\|^2q + c_2cq$$
$$= (c_1c + c_2\|q\|^2)p + (c_2\|p\|^2 + c_2c)q$$

$$\lambda = \frac{c_1c + c_2\|q\|^2}{c_1} = \frac{c_2\|p\|^2 + c_2c}{c_2}$$

$$c + \frac{c_2}{c_1}\|q\|^2 = c + \frac{c_1}{c_2}\|p\|^2$$
$$\frac{c_2}{c_1}\|q\|^2 = \frac{c_1}{c_2}\|p\|^2$$
$$\frac{c_1}{c_2} = \pm\frac{\|q\|}{\|p\|}$$

$$\lambda = c + \frac{c_2}{c_1}\|q\|^2 = c \pm \frac{\|p\|}{\|q\|}\|q\|^2 = c \pm \|p\|\|q\|$$

$\lambda_1 = c + \|p\|\|q\|$ $\quad E_{\lambda_1} = \text{span}\{\|q\|p + \|p\|q\}$

$\lambda_2 = c - \|p\|\|q\|$ $\quad E_{\lambda_2} = \text{span}\{\|q\|p - \|p\|q\}$

So we have $V = [a_1, a_2, \dots a_n]$ where $a_1 = \frac{v_1}{\|v_1\|}$ where $v_1 = \|q\|p + \|p\|q$, $a_2 = \frac{v_2}{\|v_2\|}$ where $v_2 = \|q\|p - \|p\|q$ and $a_3, \dots a_n$ unchanged. This is orthogonal matrix so:

$A = V\Lambda V^T$ where $V^{-1} = V^T$ and $\Lambda = \text{diag}(c + \|p\|\|q\|, c - \|p\|\|q\|, 0, 0, \dots 0)$

3.5) Affine approximation:

$f(x) = f(\bar{x}) + \nabla f(\bar{x})^T (x-\bar{x})$

$f(x) = -(x_1 x_2)^{1/2}, \quad x_1, x_2 > 0$

a) $\hat{f}_1(x) = f(\bar{x}) + \nabla f(\bar{x})^T (x-\bar{x})$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial f}{\partial x_1} = -\tfrac{1}{2}(x_1 x_2)^{-1/2}(x_2) = -\tfrac{1}{2} x_1^{-1/2} x_2^{1/2} = -\tfrac{1}{2}\sqrt{\tfrac{x_2}{x_1}}$$

$$\frac{\partial f}{\partial x_2} = -\tfrac{1}{2}(x_1 x_2)^{-1/2}(x_1) = -\tfrac{1}{2}\sqrt{\tfrac{x_1}{x_2}} = -\tfrac{1}{2} x_1^{1/2} x_2^{-1/2}$$

$$\boxed{\hat{f}_1(x) = f(\bar{x}) + \begin{bmatrix} -\tfrac{1}{2}\sqrt{\tfrac{\bar{x}_2}{\bar{x}_1}} & -\tfrac{1}{2}\sqrt{\tfrac{\bar{x}_1}{\bar{x}_2}} \end{bmatrix} \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix}}$$

$$\hat{f}_1(x) = -\sqrt{\bar{x}_1 \bar{x}_2} - \tfrac{1}{2}\sqrt{\tfrac{\bar{x}_2}{\bar{x}_1}}(x_1 - \bar{x}_1) - \tfrac{1}{2}\sqrt{\tfrac{\bar{x}_1}{\bar{x}_2}}(x_2 - \bar{x}_2)$$

$x = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} 9 \\ 1 \end{bmatrix} \qquad x_1 = 6, \; x_2 = 3 \quad \bar{x}_1 = 4, \; \bar{x}_2 = 1$

$$\hat{f}_1\left(\begin{bmatrix} 6 \\ 3 \end{bmatrix}\right) = -\sqrt{(4)(1)} - \tfrac{1}{2}\sqrt{\tfrac{1}{4}}\,(2) - \tfrac{1}{2}\sqrt{\tfrac{4}{1}}\,(2)$$

$$= -2 - \tfrac{1}{2} - 2$$

$$\boxed{\hat{f}_1\left(\begin{bmatrix} 6 \\ 3 \end{bmatrix}\right) = -4.5}$$

b) $\hat{f}_2(x) = f(\bar{x}) + \nabla f(\bar{x})^T (x-\bar{x}) + \tfrac{1}{2}(x-\bar{x})^T \nabla^2 f(\bar{x})(x-\bar{x})$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1 \partial x_1} & \frac{\partial f}{\partial x_1 \partial x_2} \\ \frac{\partial f}{\partial x_2 \partial x_1} & \frac{\partial f}{\partial x_2 \partial x_2} \end{bmatrix}$$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x_1^2} &= \tfrac{1}{4} x_1^{-3/2} x_2^{1/2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} &= -\tfrac{1}{4} x_1^{-1/2} x_2^{-1/2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} &= -\tfrac{1}{4} x_1^{-1/2} x_2^{-1/2} \\ \frac{\partial^2 f}{\partial x_2^2} &= \tfrac{1}{4} x_1^{1/2} x_2^{-3/2} \end{aligned}\right\} \quad \nabla^2 f(x) = \frac{1}{4}\begin{bmatrix} x_1^{-3/2} x_2^{1/2} & -x_1^{-1/2} x_2^{-1/2} \\ -x_1^{-1/2} x_2^{-1/2} & x_1^{1/2} x_2^{-3/2} \end{bmatrix}$$

$$\boxed{\hat{f}_2(x) = f(\bar{x}) + \left(-\tfrac{1}{2}\right)\begin{bmatrix} \bar{x}_2^{1/2}\bar{x}_1^{-1/2} & \bar{x}_1^{1/2}\bar{x}_2^{-1/2} \end{bmatrix}\begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix} + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)\begin{bmatrix} x_1 - \bar{x}_1 & x_2 - \bar{x}_2 \end{bmatrix}\begin{bmatrix} \bar{x}_1^{-3/2}\bar{x}_2^{1/2} & -\bar{x}_1^{-1/2}\bar{x}_2^{-1/2} \\ -\bar{x}_1^{-1/2}\bar{x}_2^{-1/2} & \bar{x}_1^{1/2}\bar{x}_2^{-3/2} \end{bmatrix}\begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix}}$$

$\bar{x}$ $\begin{bmatrix} 9 \\ 1 \end{bmatrix}$

$$\hat{f}_2\left(\begin{bmatrix} 6 \\ 3 \end{bmatrix}\right) = -\sqrt{(4)(1)} + \left(-\tfrac{1}{2}\right)\begin{bmatrix} 1^{1/2}4^{-1/2} & 4^{1/2}1^{-1/2} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \tfrac{1}{8}\begin{bmatrix} 2 & 2 \end{bmatrix}\begin{bmatrix} 4^{-3/2} & -4^{-1/2} \\ 4^{-1/2} & 4^{1/2} \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$\tfrac{1}{4} \cdot 1$

$$= -2 - \tfrac{1}{2}\begin{bmatrix} \tfrac{1}{2} & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \tfrac{1}{8}\begin{bmatrix} 2 & 2 \end{bmatrix}\begin{bmatrix} \tfrac{1}{8} & -\tfrac{1}{2} \\ -\tfrac{1}{2} & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$-1 \cdot 4$

$$= -2 - \tfrac{1}{2}(5) + \tfrac{1}{8}\begin{bmatrix} 2 & 2 \end{bmatrix}\begin{bmatrix} -3/4 \\ 3 \end{bmatrix}$$

$$= -2 - 2.5 + \tfrac{1}{8}\left(-\tfrac{3}{2} + \tfrac{12}{2}\right) = -4.5 + \tfrac{1}{8}\left(\tfrac{9}{2}\right) = \boxed{-4.5 + \tfrac{9}{16}}$$

$$\nabla^2 f(x) = \frac{1}{4}\begin{bmatrix} \frac{\sqrt{x_2}}{\sqrt{x_1^3}} & -\frac{1}{\sqrt{x_1 x_2}} \\ \frac{-1}{\sqrt{x_1 x_2}} & \frac{\sqrt{x_1}}{\sqrt{x_2^3}} \end{bmatrix}$$

$$[a\ b]\begin{bmatrix} \sqrt{\frac{x_2}{x_1^3}} & -\frac{1}{\sqrt{x_1 x_2}} \\ \frac{-1}{\sqrt{x_1 x_2}} & \sqrt{\frac{x_1}{x_2^3}} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$[a\ b]\begin{bmatrix} a\sqrt{\frac{x_2}{x_1^3}} - b\frac{1}{\sqrt{x_1 x_2}} \\ \frac{-a}{\sqrt{x_1 x_2}} + b\sqrt{\frac{x_1}{x_2^3}} \end{bmatrix}$$

$$a^2\sqrt{\frac{x_2}{x_1^3}} - ab\frac{1}{\sqrt{x_1 x_2}} - ab\frac{1}{\sqrt{x_1 x_2}} + b^2\sqrt{\frac{x_1}{x_2^3}}$$

by AM-GM, or $= \frac{(ax_2 - bx_1)^2}{\sqrt{x_1^3 x_2^3}}$

$$a^2\sqrt{\frac{x_2}{x_1^3}} - 2ab\frac{1}{\sqrt{x_1 x_2}} + b^2\sqrt{\frac{x_1}{x_2^3}}$$

$$\frac{a^2\sqrt{\frac{x_2}{x_1^3}} + b^2\sqrt{\frac{x_1}{x_2^3}}}{2} \geq \sqrt{a^2b^2\sqrt{\frac{x_2}{x_1^3}\frac{x_1}{x_2^3}}} = \sqrt{a^2b^2\sqrt{\frac{1}{x_1^2x_2^2}}} = \sqrt{a^2b^2\frac{1}{x_1x_2}} = ab\frac{1}{\sqrt{x_1x_2}}$$

$$\Rightarrow a^2\sqrt{\frac{x_2}{x_1^3}} + b^2\sqrt{\frac{x_1}{x_2^3}} \geq 2ab\frac{1}{\sqrt{x_1 x_2}}$$

$$\Rightarrow a^2\sqrt{\frac{x_2}{x_1^3}} - 2ab\frac{1}{x_1 x_2} + b^2\sqrt{\frac{x_1}{x_2^3}} \geq 0$$

d) $x = \bar{x} + \lambda v$

Want second-order term $= 0$

$$(x-\bar{x})^T \nabla^2 f(\bar{x})(x-\bar{x}) = 0$$

$$(\lambda v)^T \nabla^2 f(\bar{x})(\lambda v) = 0$$

$$v^T \nabla^2 f(\bar{x}) v = 0$$

$$\Rightarrow \frac{(v_1\bar{x}_2 - v_2\bar{x}_1)^2}{\sqrt{x_1^3 x_2^3}} = 0 \text{ from part c}$$

$$\Rightarrow v_1\bar{x}_2 - v_2\bar{x}_1 = 0$$

Pick $\bar{x} = (\bar{x}_1, \bar{x}_2) = (1,2)$ $\quad \boxed{\bar{x} = (1,2)}$

$$2v_1 - v_2 = 0$$

$v_1 = 1,\ v_2 = 2 \Rightarrow$ $\quad \boxed{v = (1,2)}$

3.4) a) $f(x) = \Pi_A x = \left(\Pi_S (x - x^{(0)})\right) + x^{(0)}$

$$\Pi_S(x-x^{(0)}) = \frac{\langle x-x^{(0)}, v\rangle}{\langle v, v\rangle} v \Rightarrow \Pi_A x = \frac{(x-x^{(0)})^T v}{\|v\|^2} v + x^{(0)} = ((x-x^{(0)})^T v) v + x^{(0)}$$

$$= \frac{(x-x^{(0)})^T v}{\|v\|^2} v,$$

$f(x) = 2p(x) - x$

$$= 2((x-x^{(0)})v)v + 2x^{(0)} - x$$

$$= 2((x^T - x^{(0)T})v)v + 2x^{(0)} - x$$

$$= 2(x^T v - x^{(0)T} v) v + 2x^{(0)} - x$$

$$= 2v(x^T v) - 2v(x^{(0)T} v) + 2x^{(0)} - x$$

$$= 2v(v^T x) - 2v(v^T x^{(0)}) + 2x^{(0)} - x$$

$$= 2vv^T x - x + 2x^{(0)} - 2vv^T x^{(0)}$$

$$= (2vv^T - I)x + 2(I - vv^T)x^{(0)}$$

$$\boxed{f(x) = (2P - I)x + 2(I-P)x^{(0)}}$$

b) The shift we have to make ~~to obtain the reflection~~ if instead of reflecting x on $A = \{x^{(0)} + \lambda v, \lambda \in R\}$, we reflect $x - x^{(0)}$ across $S = \{\lambda v, \lambda \in R\}$.

c) Linear function: $f(\alpha x) = \alpha f(x)$

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

$$f(\alpha x) = (2P-I)\alpha x + 2(I-P)x^{(0)} = \alpha[(2P-I)x] + 2(I-P)x^{(0)}$$

$$f(x_1+x_2) = (2P-I)(x_1+x_2) + 2(I-P)x^{(0)} = (2P-I)x_1 + (2P-I)x_2 + 2(I-P)x^{(0)}$$

If line passes through origin ($x^{(0)} = 0$):

$$f(\alpha x) = \alpha[(2P-I)x] = \alpha f(x)$$

$$f(x_1+x_2) = [(2P-I)x_1] + [(2P-I)x_2] = f(x_1) + f(x_2)$$

If linear fctn:

$$f(\alpha x) = \alpha[(\cancel{2P-I})x] + 2(I-P)x^{(0)} = \alpha[(2P-I)x + 2(I-P)x^{(0)}] = \alpha(\cancel{2P-I})x + 2\alpha(I-P)x^{(0)}$$

$$2(\alpha-1)(I-P)x^{(0)} = 0 \quad \forall \alpha \in R$$

$$\Rightarrow (I-P)x^{(0)} = 0$$

and $f(x_1+x_2) = f(x_1) + f(x_2) \Rightarrow 2(I-P)x^{(0)} = 4(I-P)x^{(0)} \Rightarrow (I-P)x^{(0)} = 0$

$(I-P)x^{(0)} = 0 \Rightarrow x^{(0)} = Px^{(0)} \Rightarrow x^{(0)} = vv^T x^{(0)} = (v^T x^{(0)})v \Rightarrow x^{(0)} = cv, c \in R$, i.e.

$$L = \{x^{(0)} + \lambda v, \lambda \in R\} = \{cv + \lambda v, \lambda, c \in R\} = \{(c+\lambda)v, c, \lambda \in R\}$$

$L = \{kv, k \in R\}$ ← goes thru origin

d) $f(x) = 2p(x) - x = (2P-I)x + 2(I-P)x^{(0)}$

$$f(f(x)) = (2P-I)[(2P-I)x + 2(I-P)x^{(0)}] + 2(I-P)x^{(0)}$$

$$= (4PP - 2P - 2P + I)x + 2(2P - 2PP - I + P)x^{(0)} + 2(I-P)x^{(0)}$$

$$PP = vv^T vv^T = v\|v\|^2 v^T = vv^T = P$$

$$= (\cancel{4P} - \cancel{2P} - \cancel{2P} + I)x + 2(\cancel{2P} - \cancel{2P} - I + P)x^{(0)} + 2(I-P)x^{(0)}$$

$$= x + 2(P - \cancel{I})x^{(0)} + 2(\cancel{I} - P)x^{(0)} = x$$

Geometrically, reflecting, then reflecting back along same line leaves x unchanged

3e) Let $\{u^{(1)}, \ldots, u^{(n-1)}\}$ be orthonormal basis for subspace orthogonal to v.

For each of these vectors, $u^{(i)} \perp v$, we have:

$Av^{(i)} = (2vv^T - I)u^{(i)} = 2vv^Tu^{(i)} - u^{(i)} = 2(v^Tu^{(i)})v - u^{(i)} = 2\langle v, u^{(i)}\rangle v - u^{(i)} = 0 - u^{(i)} = -u^{(i)} \Rightarrow \lambda = -1$

Now, $\|v\| = 1$, and

$Av = (2vv^T - I)v = 2(v^Tv)v - v = 2v - v = v \Rightarrow \lambda = 1$

Now, let's prove A is symmetric. This is easy:

$A = 2vv^T - I$

$A^T = (2vv^T - I)^T = 2(v^T)^Tv^T - I^T = 2vv^T - I = A \Rightarrow A$ symmetric

Now, since A is symmetric, eigenvectors across eigenspaces are orthogonal. So,

$U = [v, u^{(1)}, u^{(2)}, \ldots u^{(n)}]$ is an orthogonal matrix (orthogonal and normalized columns).

So, we can write spectral decomposition as:

$A = U \Lambda U^T$

where $U = [v, u^{(1)} \ldots u^{(n)}]$ and $\Lambda = \mathrm{diag}(1, -1, -1, \ldots -1)$

f) For any vector on line $\vec{x} = \lambda v$, reflecting will keep it unchanged, so $\lambda = 1$

For any vector orthogonal to line, reflecting will flip it, i.e. $\lambda = -1$.

3.5) $A \in S_+^n$ $\quad$ PSD $\Rightarrow \lambda_i \geq 0$

i) $A = U \Lambda U^T$

$= [\lambda_1 u_1 \; \lambda_2 u_2 \; \ldots \; \lambda_n u_n] \begin{bmatrix} - u^{(1)T} - \\ - u^{(2)T} - \\ \vdots \\ - u^{(n)T} - \end{bmatrix}$

$= \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \ldots + \lambda_n u_n u_n^T$

$\langle A, B\rangle = \mathrm{trace}(A^TB)$

$\mathrm{trace}(u_i u_i^T) = \|u_i\|^2$:

$\mathrm{trace}(A) = \lambda_1 \|u_1\|^2 + \lambda_2 \|u_2\|^2 + \ldots + \lambda_n \|u_n\|^2$

$= \lambda_1 + \lambda_2 + \ldots + \lambda_n$ $\quad$ since $\|u_i\| = 1$ (spectral decomposition $\Rightarrow U$ orthogonal matrix)

$\boxed{\mathrm{trace}(A) = \sum_{i=1}^n \lambda_i = \|\vec{\lambda}\|_1}$ since $\lambda_i \geq 0$

$\|A\|_F^2 = \langle A, A\rangle = \mathrm{trace}(A^TA) = \mathrm{trace}((U\Lambda U^T)^T U \Lambda U^T) = \mathrm{trace}(U\Lambda^T U^T U \Lambda U^T) = \mathrm{trace}(U \Lambda^T \Lambda U^T)$

$= \mathrm{trace}(U \Lambda^2 U^T)$

$\Rightarrow \|A\|_F = \sqrt{\mathrm{trace}(U\Lambda^2U^T)} \Rightarrow \boxed{\|A\|_F = \sqrt{\sum_{i=1}^n \lambda_i^2} = \|\vec{\lambda}\|_2}$

2) We need to prove:

$$\|\vec{\lambda}\|_1^2 \le \text{rank}(A)\, \|\vec{\lambda}\|_2^2$$

Cauchy: $|\langle u,v\rangle|^2 \le \langle u,u\rangle\langle v,v\rangle$

For $u,v \in \mathbb{R}^n$:

$$(u^T v)^2 \le \|u\|_2^2 \|v\|_2^2$$

So we pick $\vec{u} = \vec{\lambda}$ and $v = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, i.e. $v_i = 1$ if $u_i > 0$, $v_i = 0$ if $u_i = 0$ (PSD $\to$ each $\lambda_i \ge 0$)

Then: $u^T v = \vec{\lambda}^T v = [\lambda_1 \lambda_2 \dots \lambda_k\ \lambda_{k+1} \dots \lambda_n] \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{n} \lambda_i$ (since $\lambda_{k+1} = \dots = \lambda_n = 0$) $= \|\vec{\lambda}\|_1$

And: $\|u\|^2 = \sum_{i=1}^{n} \lambda_i^2 = \|\vec{\lambda}\|^2$.

$$\|v\|^2 = \sum_{i=1}^{k} 1^2 = k$$

Now, notice that k, the number of non-zero eigenvalues, is same as rank, i.e. rank(A) = # non-zero eigenvalues (can be repeated). So:

$$|u^T v|^2 \le \|u\|_2^2 \|v\|_2^2$$

$$\|\vec{\lambda}\|_1^2 \le k \|\vec{\lambda}\|^2$$

$\|\vec{\lambda}\|_1^2 \le \text{rank}(A) \|\vec{\lambda}\|^2$, and we are done

3.6) a) $x_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$

$\det \begin{bmatrix} +\frac{3}{2} - \lambda & -\frac{1}{2} \\ -\frac{1}{2} & +\frac{3}{2} - \lambda \end{bmatrix} = (\frac{3}{2} - \lambda)^2 - \frac{1}{4} = 0$

$$(\lambda - \tfrac{3}{2})^2 = \tfrac{1}{4}$$

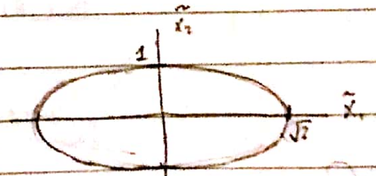
$$\lambda - \tfrac{3}{2} = \pm \tfrac{1}{2} \Rightarrow \lambda = \tfrac{3}{2} \pm \tfrac{1}{2}$$

$$\lambda_1 = 2,\ \lambda_2 = 1$$

$\begin{bmatrix} +\frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \frac{3}{2}x - \frac{1}{2}y = 2x \Rightarrow x = -y$ $\quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

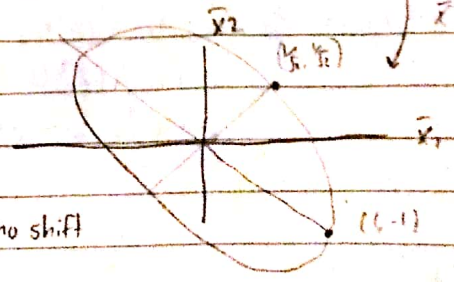
$-\frac{1}{2}x + \frac{3}{2}y = 2y \Rightarrow x = -y$

$\lambda_2 = 1 \Rightarrow x = y \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$



$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$\bar{x} = U\tilde{x}$



$\boxed{\left(\frac{\tilde{x}_1}{\sqrt{\lambda_1}}\right)^2 + \left(\frac{\tilde{x}_2}{\sqrt{\lambda_2}}\right)^2 \le 1} \Rightarrow \left(\frac{\tilde{x}_1}{\sqrt{2}}\right)^2 + \left(\frac{\tilde{x}_2}{\sqrt{1}}\right)^2 \le 1$

$\bar{x} = U\tilde{x}$; if $\tilde{x} = \tilde{x}_1 + \tilde{x}_2$, $\bar{x} = \tilde{x}_1 u_1 + \tilde{x}_2 u_2$

$\tilde{x} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$; $\bar{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad x_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$ no shift

$\tilde{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\bar{x} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

b) $x_c = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $P = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

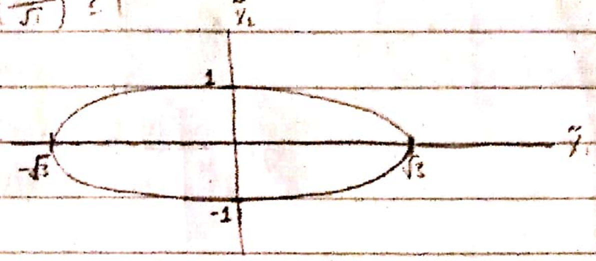
$\det\left(\begin{bmatrix} 3-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}\right) = 0 \Rightarrow (\lambda-3)(\lambda-1) = 0 \quad \lambda = 1, \lambda = 3$

$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad 3x = x \rightarrow x = 0 \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix} \rightarrow y = 0 \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

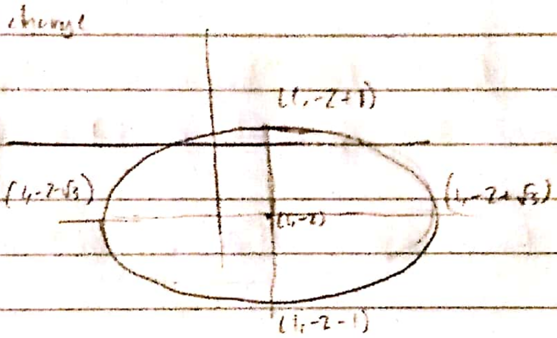
$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad (\lambda_1 = 3, \lambda_2 = 1)$

$\left(\frac{\tilde{x}_1}{\sqrt{3}}\right)^2 + \left(\frac{\tilde{x}_2}{\sqrt{1}}\right)^2 \leq 1$



$\bar{x} = Ux = I\,x \rightarrow$ no change

$x_c = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \rightarrow$ shift



c) $x_c = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $P = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$

$\det\left(\begin{bmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{bmatrix}\right) = (\lambda-9)(\lambda-6) - 4 = 0$

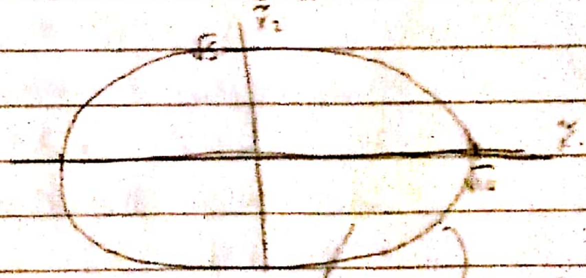
$\lambda^2 - 15\lambda + 50 = 0$

$(\lambda - 10)(\lambda - 5) = 0$

$\lambda_1 = 10 \quad \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10x \\ 10y \end{bmatrix} \rightarrow 9x - 2y = 10x \rightarrow x = -2y \qquad \begin{bmatrix} -2 \\ 1 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$

$\lambda_2 = 5: \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix} \rightarrow 9x - 2y = 5x \rightarrow 2y = 4x \rightarrow y = 2x \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

$U = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$

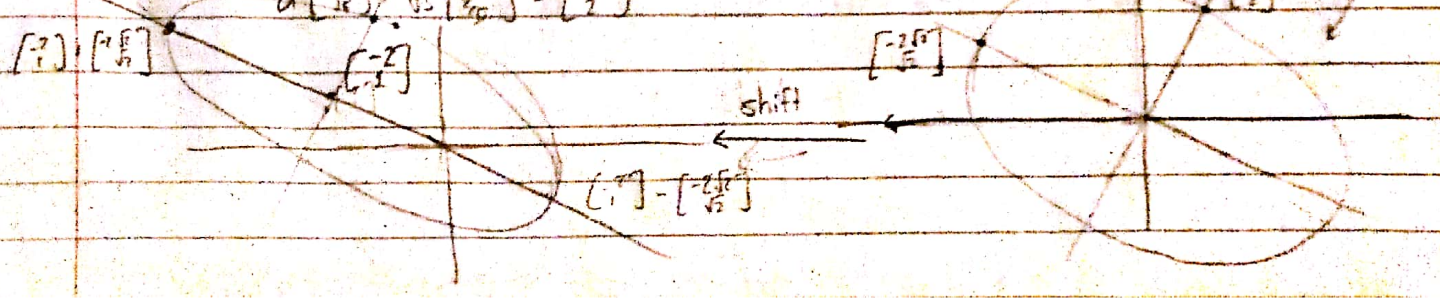


$\left(\frac{\tilde{x}_1}{\sqrt{\lambda_1}}\right)^2 + \left(\frac{\tilde{x}_2}{\sqrt{\lambda_2}}\right)^2 \leq 1 \Rightarrow \left(\frac{\tilde{x}_1}{\sqrt{10}}\right)^2 + \left(\frac{\tilde{x}_2}{\sqrt{5}}\right)^2 \leq 1$

$\bar{x} = U\tilde{x} \qquad U\begin{bmatrix} \sqrt{10} \\ 0 \end{bmatrix} = \sqrt{10}\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -2\sqrt{2} \\ \sqrt{2} \end{bmatrix}$

$U\begin{bmatrix} 0 \\ \sqrt{5} \end{bmatrix} = \sqrt{5}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\bar{x} = U\tilde{x}$



d) If any λ is $= 0$, then A is not invertible. Thus, there exists a mapping to some ~~exponent~~ from $\mathbb{R}$ to $\mathbb{P}$. ~~If~~ If x is eigenvector with $\lambda = 0$, then all points λx map to 0, so we lose this dimension. The volume of a ~~n-1~~ dimensional shape in n-dimensions is 0, so $\det A$ ~~det P~~ $= 0$.

3.7) a) $f_1 = 2x + 3y + 1$

$$\nabla f_1 = \begin{bmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_1}{\partial y} \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\nabla^2 f_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$f_2 = x^2 + y^2 - xy - 5$

$$\frac{\partial f_2}{\partial x} = 2x - y$$

$$\frac{\partial f_2}{\partial y} = 2y - x$$

$$\nabla f_2 = \begin{bmatrix} 2x - y \\ 2y - x \end{bmatrix}$$

$$\nabla^2 f_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$f_3 = (x-5)\cos(y-5) - (y-5)\sin(x-5)$

$$\frac{\partial f_3}{\partial x} = \cos(y-5) - (y-5)\cos(x-5)$$

$$\frac{\partial f_3}{\partial y} = -(x-5)\sin(y-5) - \sin(x-5)$$

$$\nabla f_3 = \begin{bmatrix} \cos(y-5) - (y-5)\cos(x-5) \\ -(x-5)\sin(y-5) - \sin(x-5) \end{bmatrix}$$

$$\nabla^2 f_3 = \begin{bmatrix} \boxed{(y-5)}\sin(x-5) & -\sin(y-5) - \cos(x-5) \\ -\sin(y-5) - \cos(x-5) & -(x-5)\cos(y-5) \end{bmatrix}$$

b) See Matlab code

c) See Matlab code

d) Approximations are perfect for f_1 and f_2 because they are linear and quadratic respectively.
For f_3, approximation is reasonable around approximation point, but not further away from approximation point.

3.8) a) Sum of transition probabilities from one page to all others must be 1. This makes sense. At every step we have to move to new page.

b) See MATLAB code

c) See MATLAB code

Generally speaking, my code matches figure 7.1 in OptM

d) See MATLAB

Max: library / resources / journals...

Makes sense → people want to access library resources; shared value amongst people

Min: things people don't commonly need

Maximum Indices: 303, 731, 732, 733, 735

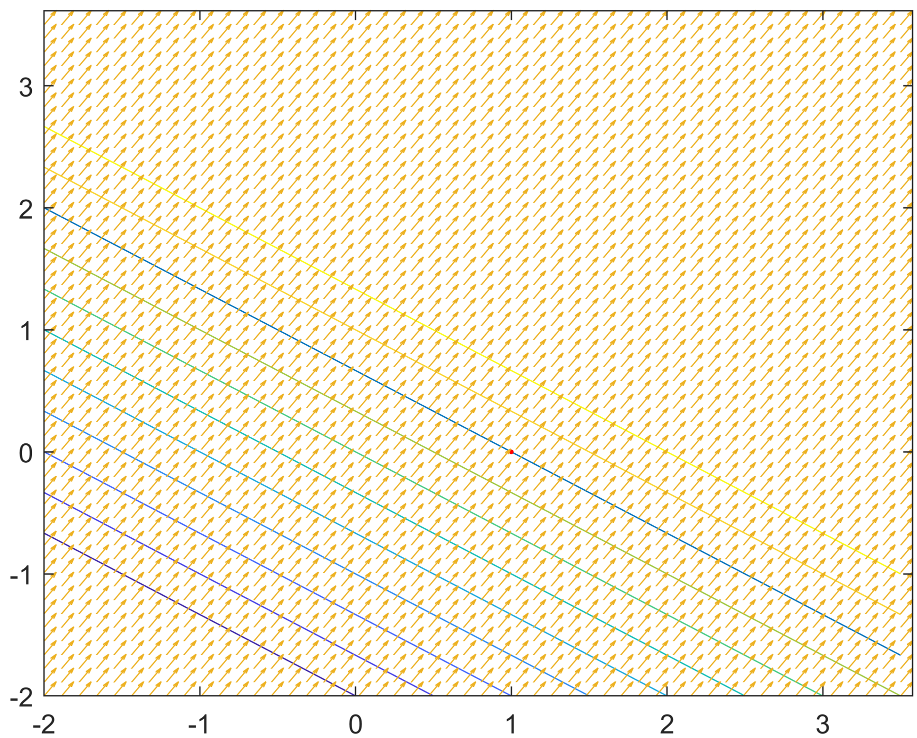
Minimum Indices: 479, 7459, 930, 986, 987

I used indices of last method (Rayleigh Quotient) because it gave least error. I got negative ranks, but maybe that's ok since it's a different method from power iteration

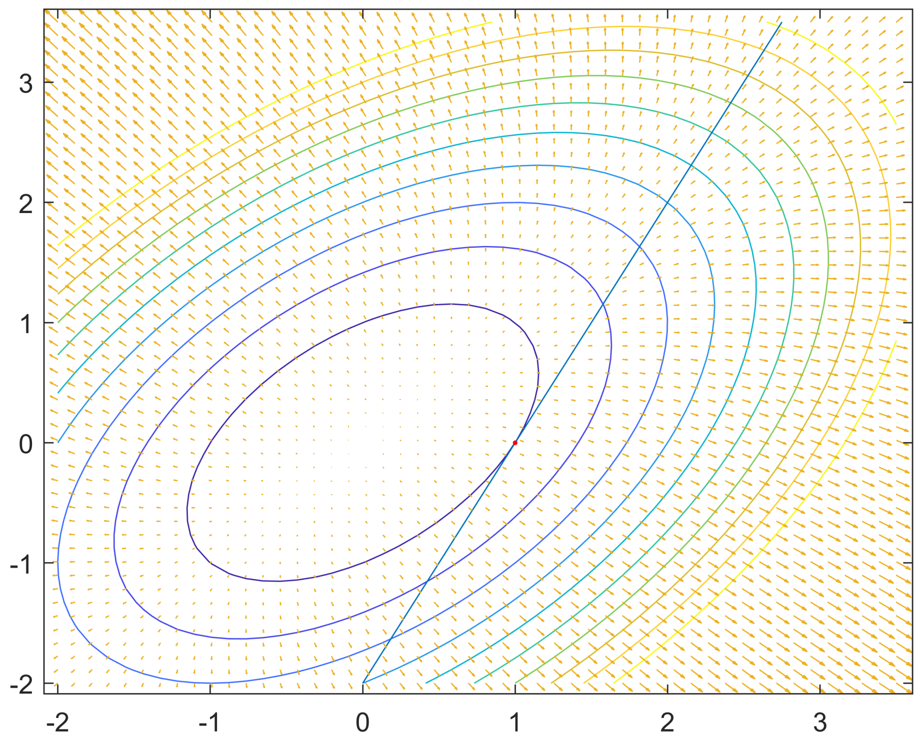
Max Page Ranks: 0.0432 (all)

Min Page Rank: -0.4979, -0.7698, -0.7543, -0.7740, -0.7240

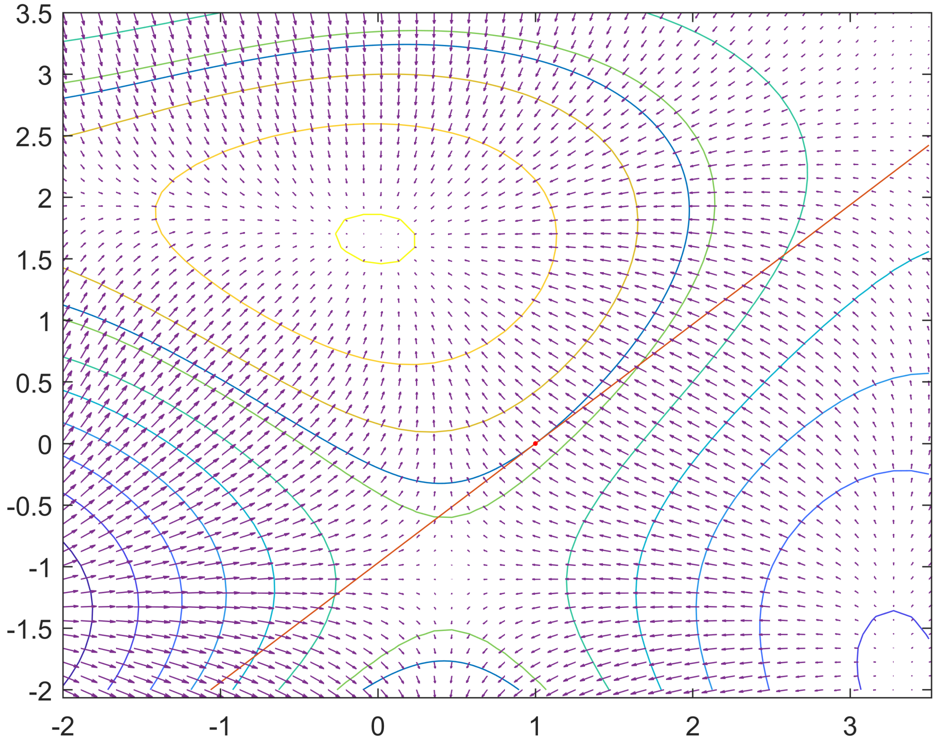
```
x = linspace(-2, 3.5, 50);
y = linspace(-2, 3.5, 50);
[X,Y] = meshgrid(x,y);
z = @(x,y) 2.*x+3.*y+1;
Z = z(X,Y);
[px,py] = gradient(Z);
figure
[c,h]=contour(X,Y,Z,-5:1:5);
hold on
fimplicit(@(x,y) 2.*x + 3.*y - 2, [-2 3.5 -2 3.5])
plot(1,0,'r.')
hold on
quiver(X,Y,px,py)
```



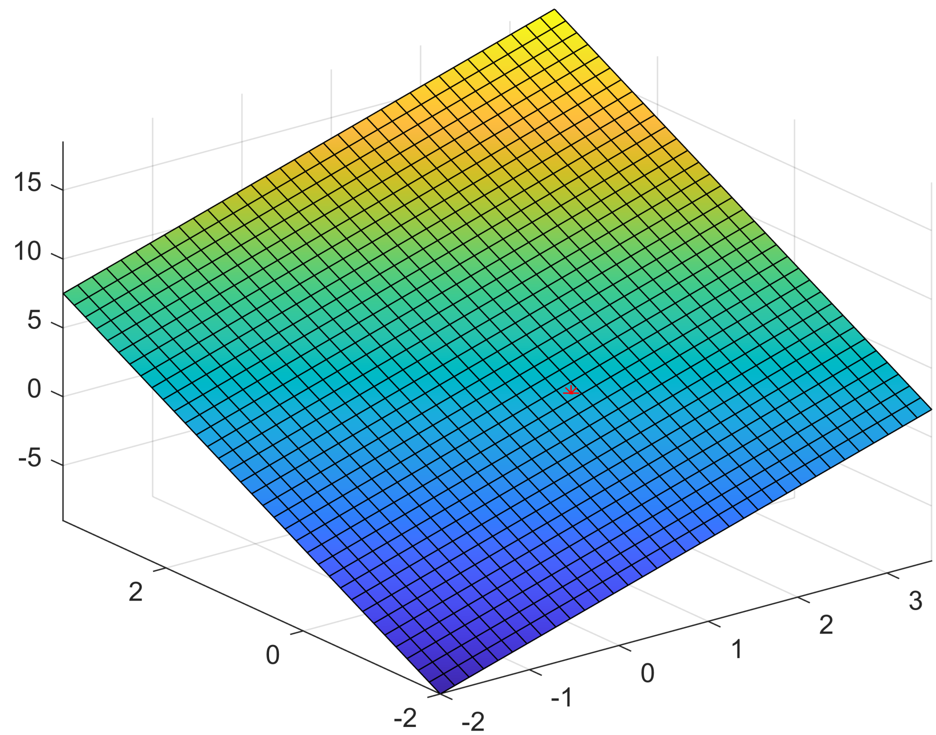
```
x = linspace(-2, 3.5, 50);
y = linspace(-2, 3.5, 50);
[X,Y] = meshgrid(x,y);
z = @(x,y) x.^2+y.^2-x.*y-5;
Z = z(X,Y);
[px,py] = gradient(Z);
figure
[c,h]=contour(X,Y,Z,-5:1:5);
hold on
fimplicit(@(x,y) 2.*x - y - 2, [-2 3.5 -2 3.5])
plot(1,0,'r.')
hold on
quiver(X,Y,px,py)
```



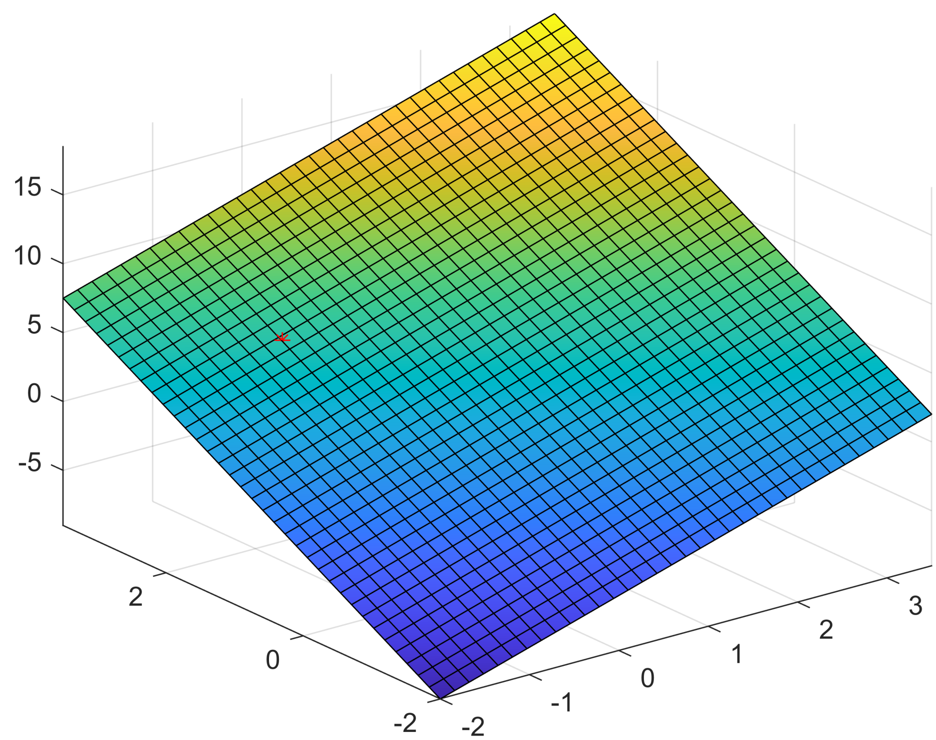

```
x = linspace(-2, 3.5, 50);
y = linspace(-2, 3.5, 50);
[X,Y] = meshgrid(x,y);
z = @(x,y) (x-5).*cos(y-5) - (y-5).*sin(x-5);
Z = z(X,Y);
[px,py] = gradient(Z);
figure
contour(X,Y,Z);
hold on
fimplicit(@(x,y) (x-5).*cos(y-5) - (y-5).*sin(x-5) + 4.*cos(5) + 5.*sin(4), [-2 3.5 -2 3.5])
fimplicit(@(x,y) (x-1)*(cos(5)+5*cos(4)) + y*(sin(4)-4*sin(5)), [-2 3.5 -2 3.5])
plot(1,0,'r.')
hold on
quiver(X,Y,px,py)
```



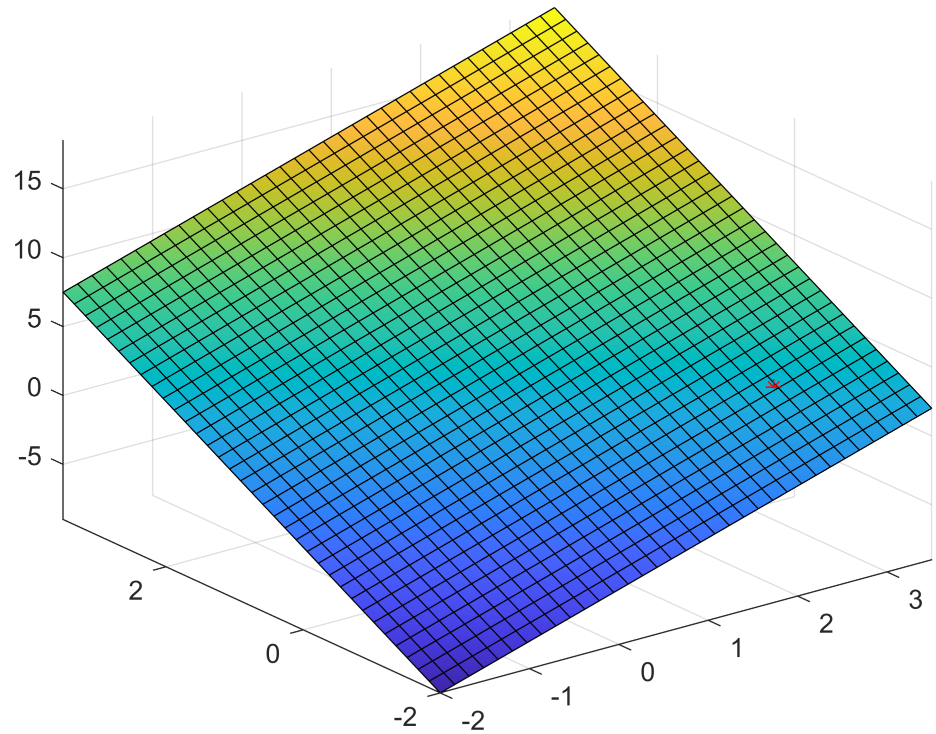

```
syms f1(x,y)
f1(x,y) = 2*x + 3*y + 1;
figure;
drawquadraticapprox(f1, 1, 0)
```



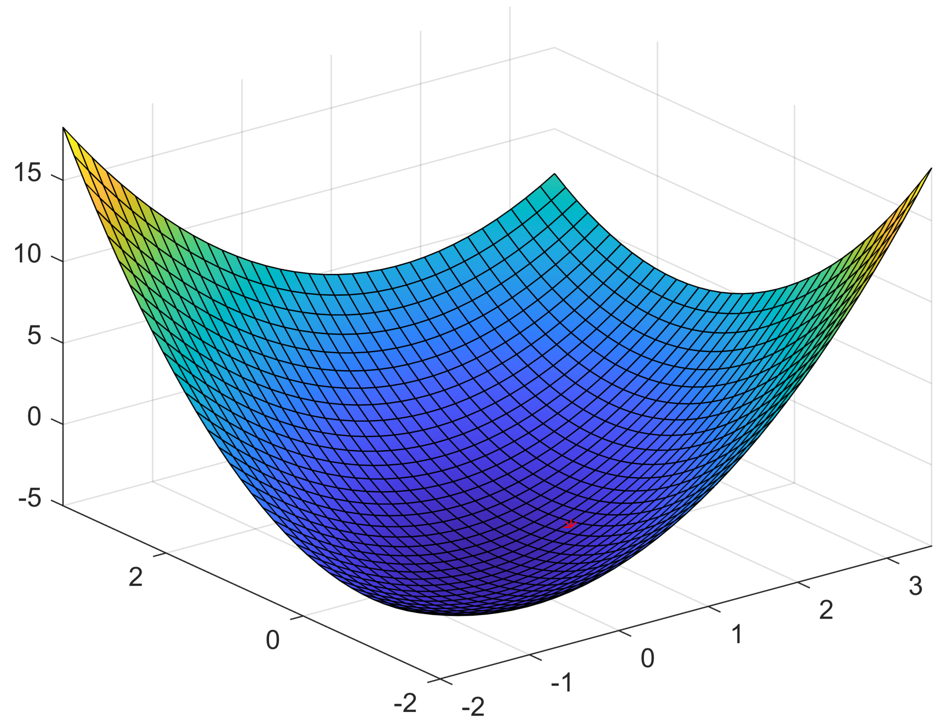

```
figure;
drawquadraticapprox(f1, -0.7, 2)
```



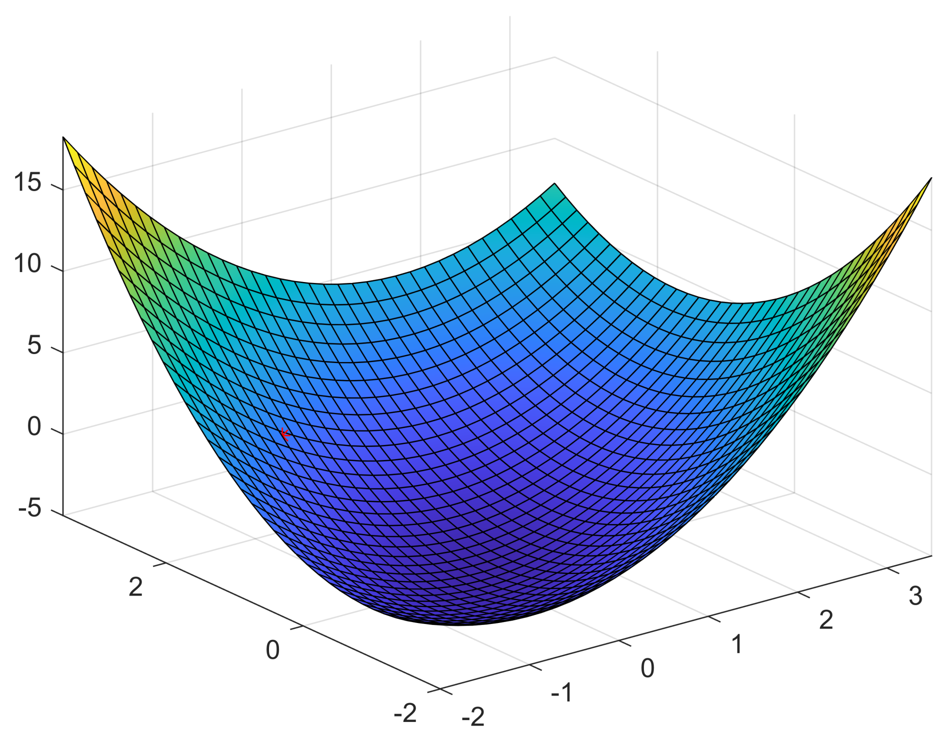
```
figure;
drawquadraticapprox(f1, 2.5, -1)
```



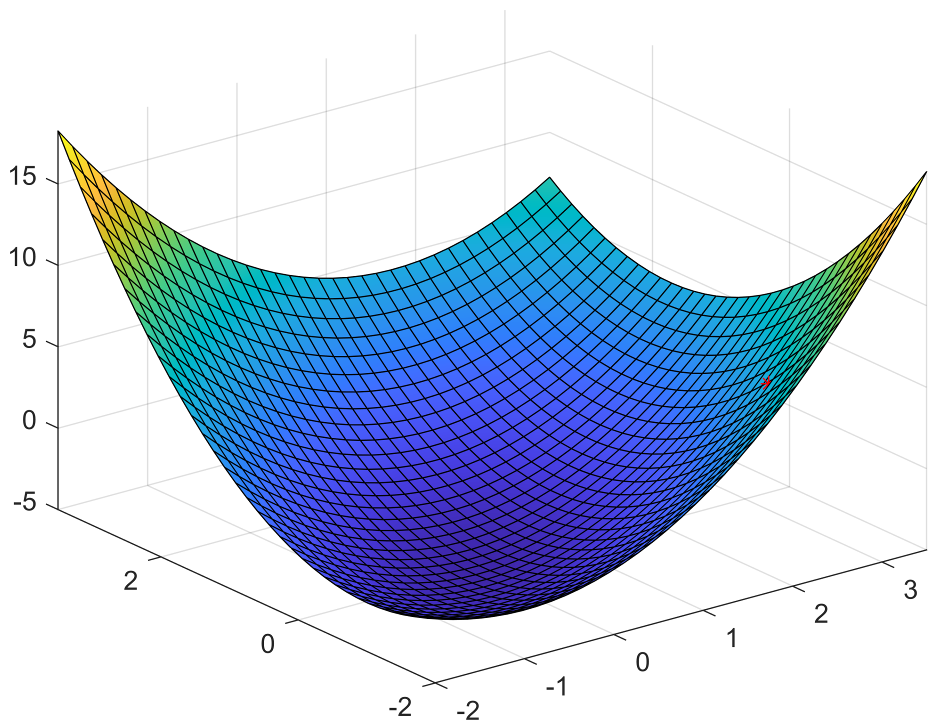

```
syms f2(x,y)
f2(x,y) = x^2 + y^2 -x*y - 5;
figure;
drawquadraticapprox(f2, 1, 0)
```



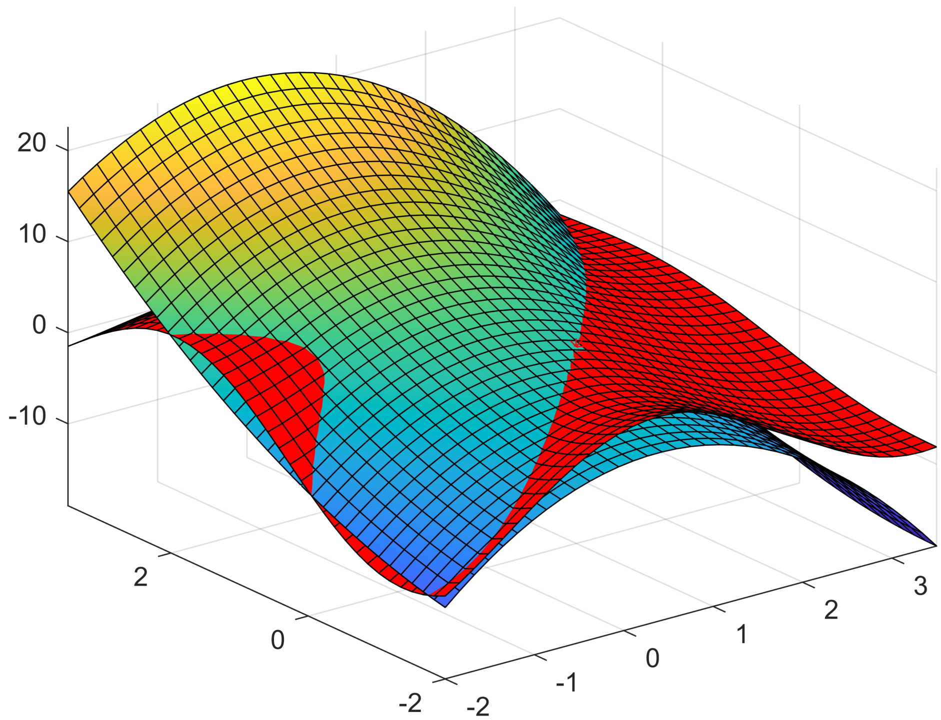

```
figure;
drawquadraticapprox(f2, -0.7, 2)
```



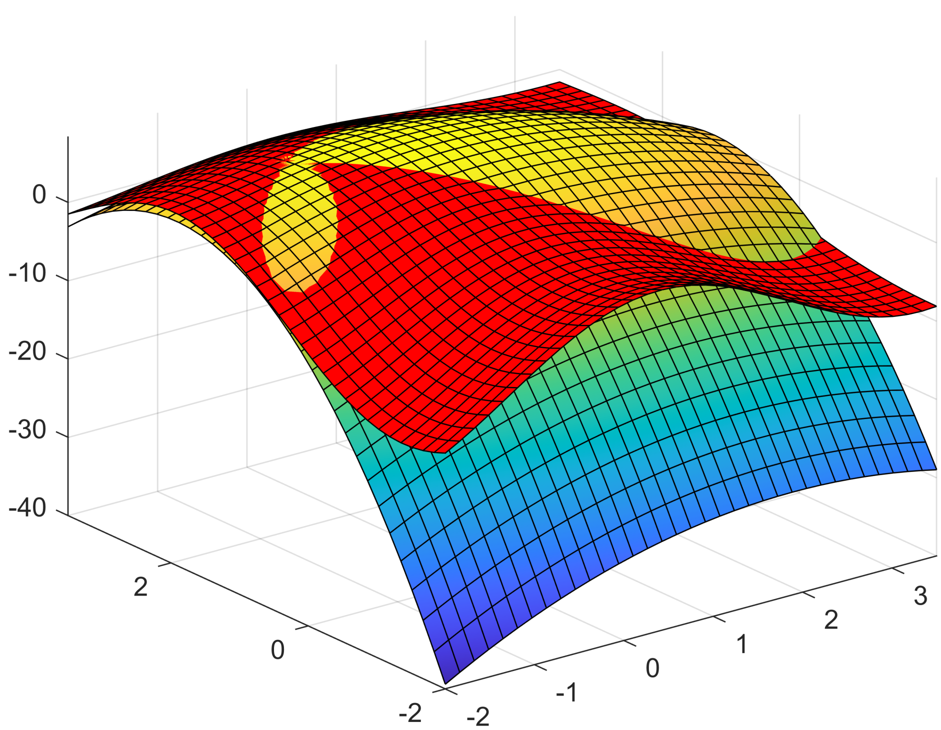
```
figure;
drawquadraticapprox(f2, 2.5, -1)
```



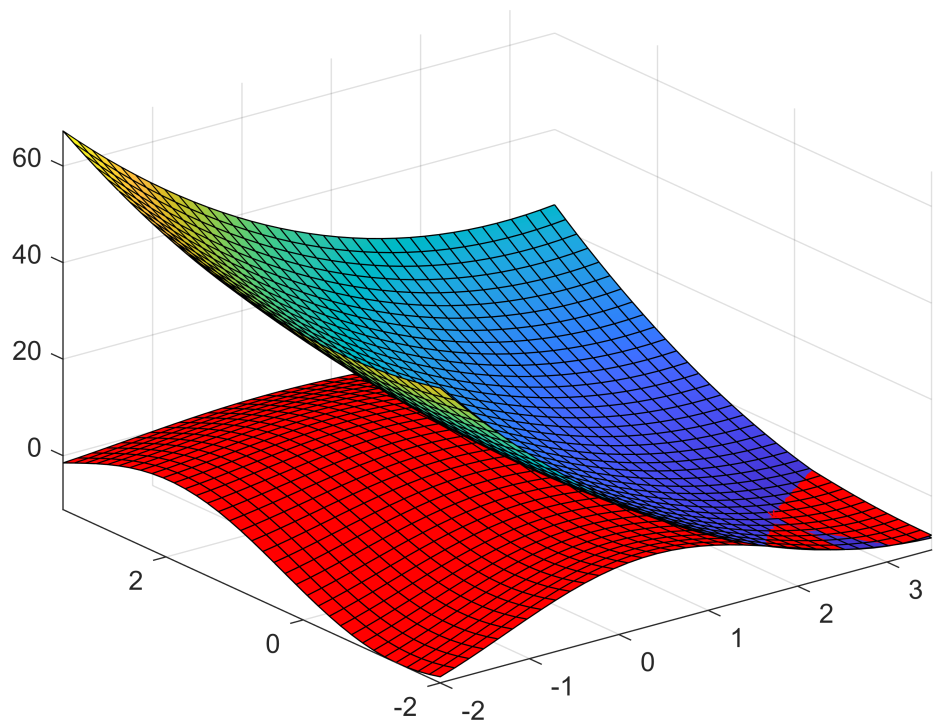

```
syms f3(x,y)
f3(x,y) = (x - 5).*cos(y - 5) - (y - 5).* sin(x - 5);
figure;
drawquadraticapprox(f3, 1, 0)
```




```
figure;
drawquadraticapprox(f3, -0.7, 2)
```



```
figure;
drawquadraticapprox(f3, 2.5, -1)
```




```
function drawquadraticapprox(f, x0, y0)
    grad = gradient(f);
    hess = hessian(f);

    syms approx(x, y)
    approx(x,y) = f(x0,y0) + grad(x0, y0)' * [x-x0; y-y0] + 0.5 * [x-x0; y-y0]' * hess(x0, y0)

    fsurf(f,[-2 3.5 -2 3.5], 'r')
    hold on
    fsurf(approx,[-2 3.5 -2 3.5])
    plot3(x0,y0,f(x0,y0),'r*')
end
```

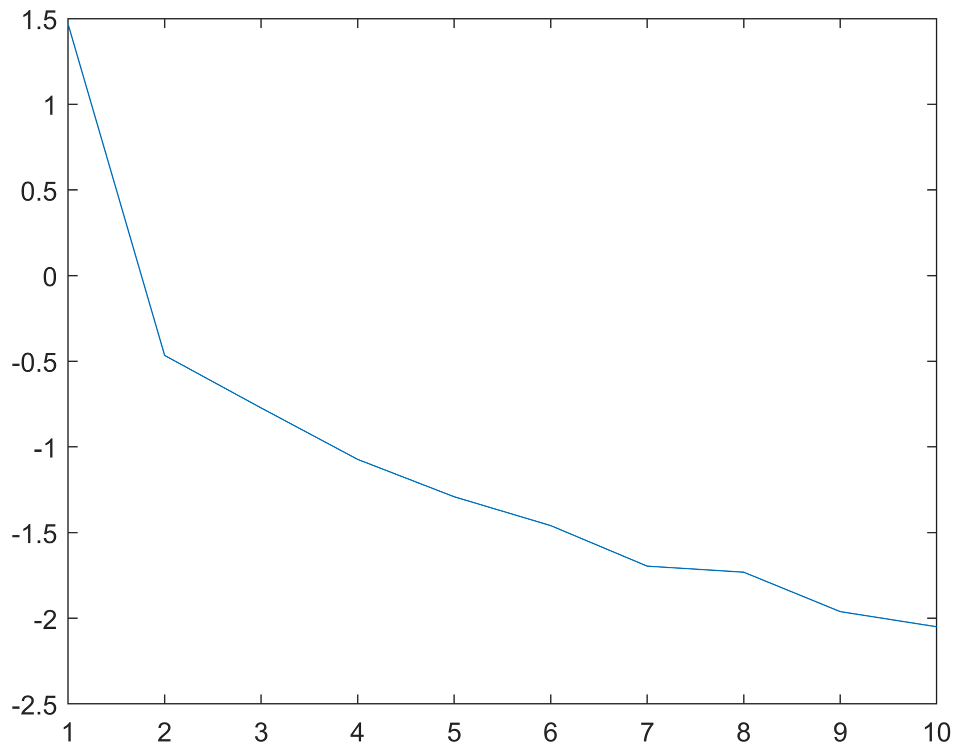
```
load pagerank_adj.mat
A = zeros(size(J));
for i = 1:length(J)
    A(:,i) = J(:,i)/sum(J(:,i));
end

x1(:,1) = ones(length(A), 1);
k = 1;
x1(:,1) = x1(:,1)/norm(x1(:,1));
for i = 1:10
   y(:,k) = A*x1(:,k);
   x1(:,k+1) = y(:,k)/norm(y(:,k));
   lambda(k) = x1(:,k+1)' * A * x1(:,k+1);
   e1(k) = norm(A*x1(:,k)-x1(:,k));
   k = k + 1;
end
[maxis,maxidx] = maxk(x1(:,end),5)
```

```
maxis = 5×1
    0.4394
    0.3296
    0.3073
    0.2996
    0.2751
maxidx = 5×1
     2
    35
    36
    58
    49
```

```
[minis,minidx] = mink(x1(:,end),5)
```

```
minis = 5×1
     0
     0
     0
     0
     0
minidx = 5×1
     1
     3
     4
     5
    10
```

```
figure;
plot(log(e1))
```




```
x2(:,1) = ones(length(A), 1);
k = 1;
sigma = 0.99;
x2(:,1) = x2(:,1)/norm(x2(:,1));
for i=1:10
   y(:,k) = (A-sigma*eye(length(A))) \ x2(:,k);
   x2(:,k+1) = y(:,k)/norm(y(:,k));
   lambda(k) = x2(:,k+1)' * A * x2(:,k+1);
   e2(k) = norm(A*x2(:,k)-x2(:,k));
   k = k + 1;
end
[maxis,maxidx] = maxk(x2(:,end),5)
```

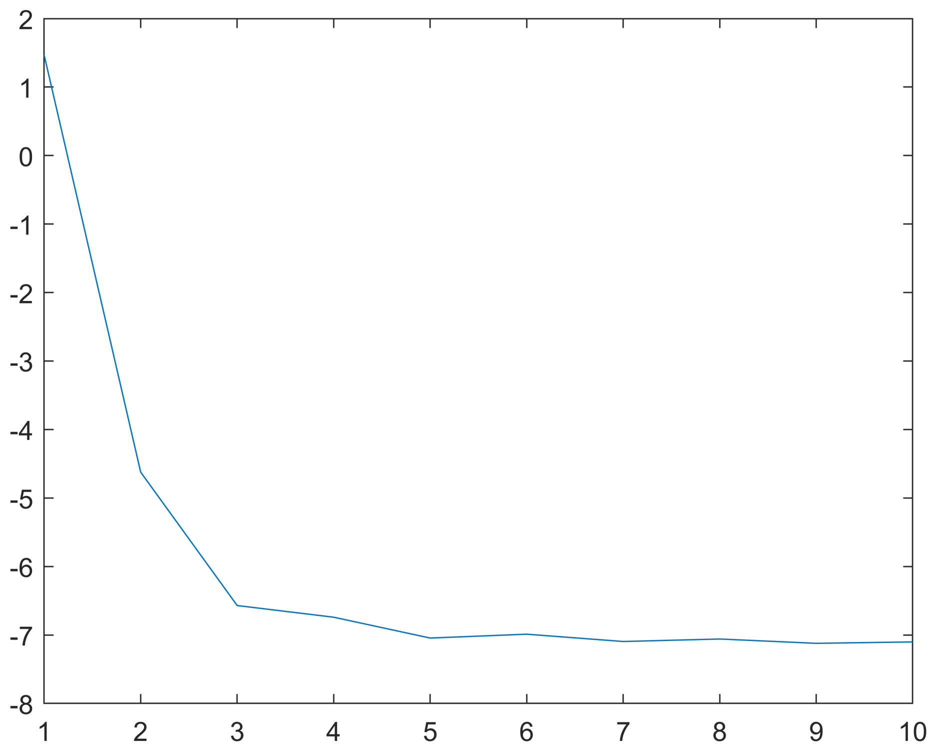
```
maxis = 5×1
    0.3711
    0.3184
    0.2975
    0.2904
    0.2637
maxidx = 5×1
     2
    35
    36
    58
    49
```

```
[minis,minidx] = mink(x2(:,end),5)
```

```
minis = 5×1
```

```
   -0.3196
   -0.1602
   -0.1602
   -0.1200
   -0.0917
minidx = 5×1
   424
   987
   986
   985
   930
```

```
figure;
plot(log(e2))
```




```
x3(:,1) = ones(length(A), 1);
k = 1;
x3(:,1) = x3(:,1)/norm(x3(:,1));
for i=1:10
   if i==1 || i==2
       sigma = 0.99;
   else
       sigma = (x3(:,k)' * A * x3(:,k))/(x3(:,k)' * x3(:,k));
   end
   y(:,k) = (A-sigma*eye(length(A))) \ x3(:,k);
   x3(:,k+1) = y(:,k)/norm(y(:,k));
   e3(k) = norm(A*x3(:,k)-x3(:,k));
   k = k + 1;
end
```

```
Warning: Matrix is close to singular or badly scaled. Results may be inaccurate. RCOND =  6.476301e-17.
Warning: Matrix is close to singular or badly scaled. Results may be inaccurate. RCOND =  6.021051e-18.
```

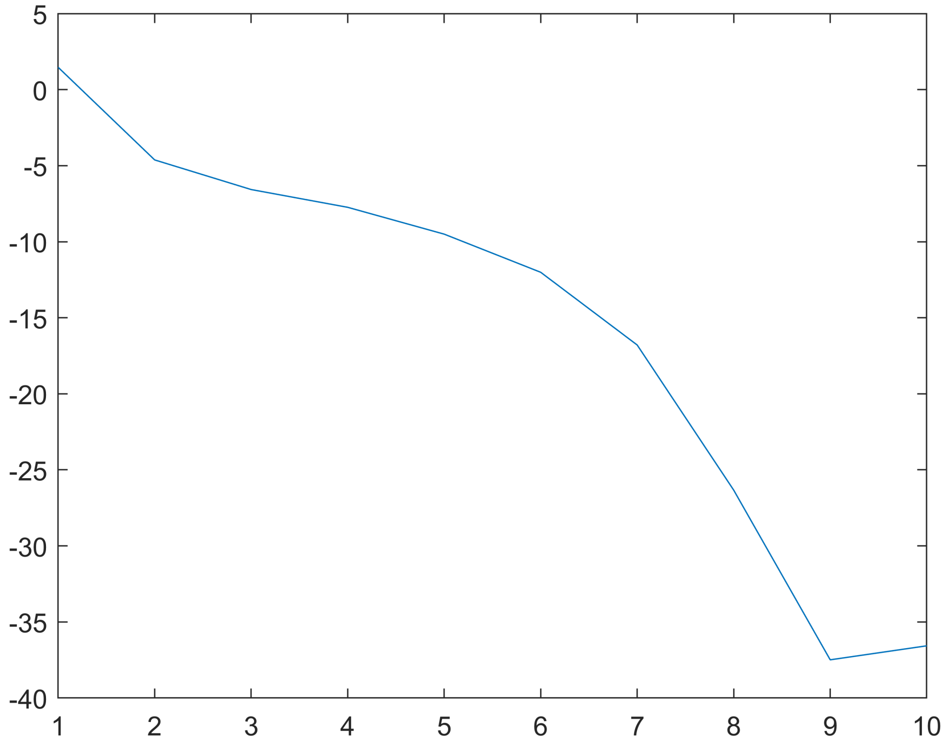
```
[maxis,maxidx] = maxk(x3(:,end),5)
```

```
maxis = 5×1
    0.0432
    0.0432
    0.0432
    0.0432
    0.0432
maxidx = 5×1
   303
   731
   732
   733
   735
```

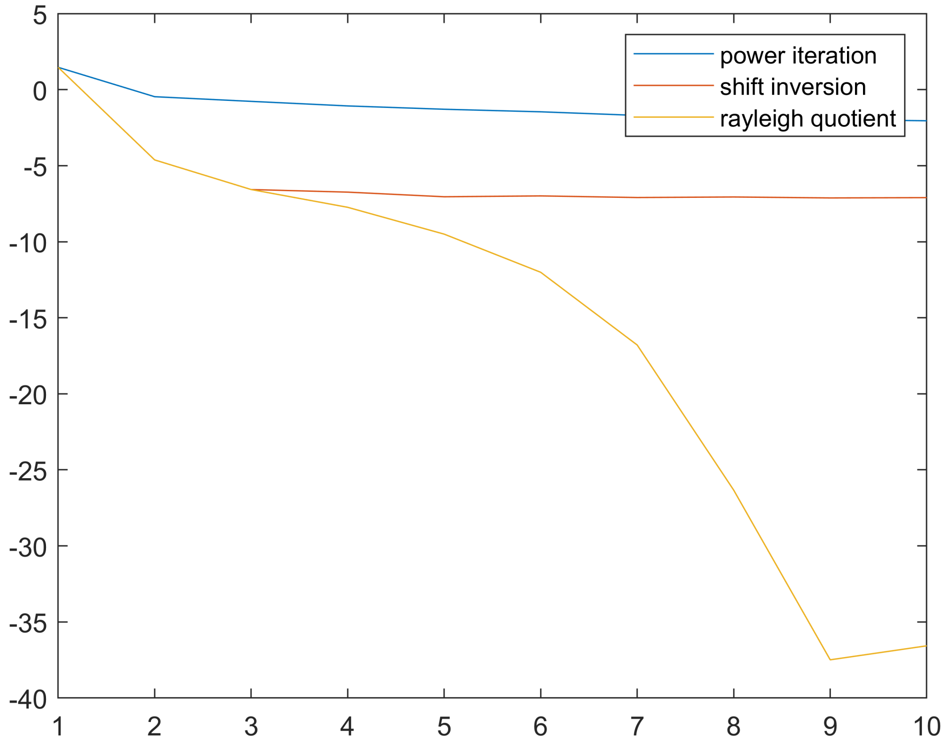
```
[minis,minidx] = mink(x3(:,end),5)
```

```
minis = 5×1
   -0.4479
   -0.2648
   -0.2593
   -0.2240
   -0.2240
minidx = 5×1
         424
        2459
         930
         986
         987
```

```
figure;
plot(log(e3))
```




```
figure;
plot(log(e1))
hold on
plot(log(e2))
hold on;
plot(log(e3))
legend('power iteration', 'shift inversion', 'rayleigh quotient')
```



5
0
-5
-10
-15
-20
-25
-30
-35
-40
1
2
3
4
5
6
7
8
9
10
power iteration
shift inversion
rayleigh quotient