ECE368

Lab 3: Hidden Markov Model

1a) Write down the formulas of the forward-backward algorithm to compute the marginal distribution $p(z_i|(\hat{x}_0,\hat{y}_0),(\hat{x}_1,\hat{y}_1),\ldots(\hat{x}_{N-1},\hat{y}_{N-1}))$ for $i=0,1,2,\ldots N-1$. Your answer should contain the initializations of the forward and backward messages, the recursion relations of the messages, and the computation of the marginal distribution based on these messages.

$$p(z_i|(\hat{x}_0,\hat{y}_0),\ldots,(\hat{x}_{N-1},\hat{y}_{N-1}))=\gamma(z_i)=\frac{\alpha(z_i)\beta(z_i)}{\sum_{z_i}\alpha(z_i)\beta(z_i)}$$

Forward Message:

$\alpha(z_0)=p(z_0)p((\hat{x}_0,\hat{y}_0)|z_0)$ Note: When $(\hat{x}_i,\hat{y}_i)$ is not observed, $p((\hat{x}_i,\hat{y}_i)|z_i)=1$

$\alpha(z_i)=p((\hat{x}_i,\hat{y}_i)|z_i)\sum_{z_{i-1}}p(z_i|z_{i-1})\alpha(z_{i-1})$, $i=1,2,\ldots N-1$

Backwards Message:

$\beta(z_{N-1})=1$

$\beta(z_i)=\sum_{z_{i+1}}\beta(z_{i+1})p((\hat{x}_{i+1},\hat{y}_{i+1})|z_{i+1})p(z_{i+1}|z_i)$, $i=N-2,N-3,\ldots 0$

b)

$$p(z_{99}|(\hat{x}_0,\hat{y}_0),\ldots,(\hat{x}_{99},\hat{y}_{99}))=\begin{cases}0.8103 & z_{99}=(11,0,\text{stay})\\ 0.1796 & z_{99}=(11,0,\text{right})\\ 0.0101 & z_{99}=(10,1,\text{down})\end{cases}$$

Probabilities are rounded to 4 decimal places.

2)

$$p(z_{30}|(\hat{x}_0,\hat{y}_0),(\hat{x}_1,\hat{y}_1)\ldots(\hat{x}_{99},\hat{y}_{99}))=\begin{cases}0.9130 & z_{30}=(6,7,\text{right})\\ 0.0435 & z_{30}=(5,7,\text{right})\\ 0.0435 & z_{30}=(5,7,\text{stay})\end{cases}$$

Probabilities are rounded to 4 decimal places

3a) Forward Path:

$w_0(z_0)=\ln(p((\hat{x}_0,\hat{y}_0)|z_0)p(z_0))$

$w_i(z_i)=\ln(p((\hat{x}_i,\hat{y}_i)|z_i))+\max_{z_{i-1}}\{\ln(p(z_i|z_{i-1}))+w_{i-1}(z_{i-1})\}$

$z_i^*=\arg\max_{z_i} w_i(z_i)$

Backwards Path:

Use traceback procedure. $z_{i-1}^*=\phi_i(z_i)=\arg\max w_i(z_i^*)$

Note: When $(\hat{x}_i,\hat{y}_i)$ is not observed:

$p((\hat{x}_i,\hat{y}_i)|z_i)=1$

b) Last 10 hidden states in the MAP estimate:

$z_{90} = (11, 5, \text{down})$

$z_{91} = (11, 6, \text{down})$

$z_{92} = (11, 7, \text{down})$

$z_{93} = (11, 7, \text{stay})$

$z_{94} = (11, 7, \text{stay})$

$z_{95} = (10, 7, \text{left})$

$z_{96} = (9, 7, \text{left})$

$z_{97} = (8, 7, \text{left})$

$z_{98} = (7, 7, \text{left})$

$z_{99} = (6, 7, \text{left})$

4) The error probability of $\{\tilde{z}_i\}$ is 0.03

The error probability of $\{\check{z}_i\}$ is 0.02

5) Sequence $\{\check{z}_i\}$ is not a valid sequence

$z_{64} = (3, 7, \text{stay})$

$z_{65} = (2, 7, \text{stay})$

Given rover is at position (3,7) at timestep 64, and the recorded previous action at timestep 65 is "stay", then the rover should remain at (3,7) at timestep 65. However, we see it changes from (3,7) to (2,7); a contradiction, and we are done.