ECE557 Lab 3: State Feedback Stabilization of a Cart-Pendulum Robot

Group: PRA04 – Group 6

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Introduction

The purpose of this lab is to investigate the performance of a state feedback controller on a cart pendulum system; first using pole placement and then the Linear Quadratic Regulator (LQR) technique. As LQR allows us to tune the weightings of specific state variables in the cost function, we expect it to allow for more control over the fine tuning of the system response. We will compare the performance of three different controllers (pole placement and LQR with 2 different sets of weights) on both the original non-linear model and a linearization of the system. We will also investigate the effects of saturation on the system's response, particularly in the nonlinear system when the initial condition is far away from equilibrium.

Block Diagram

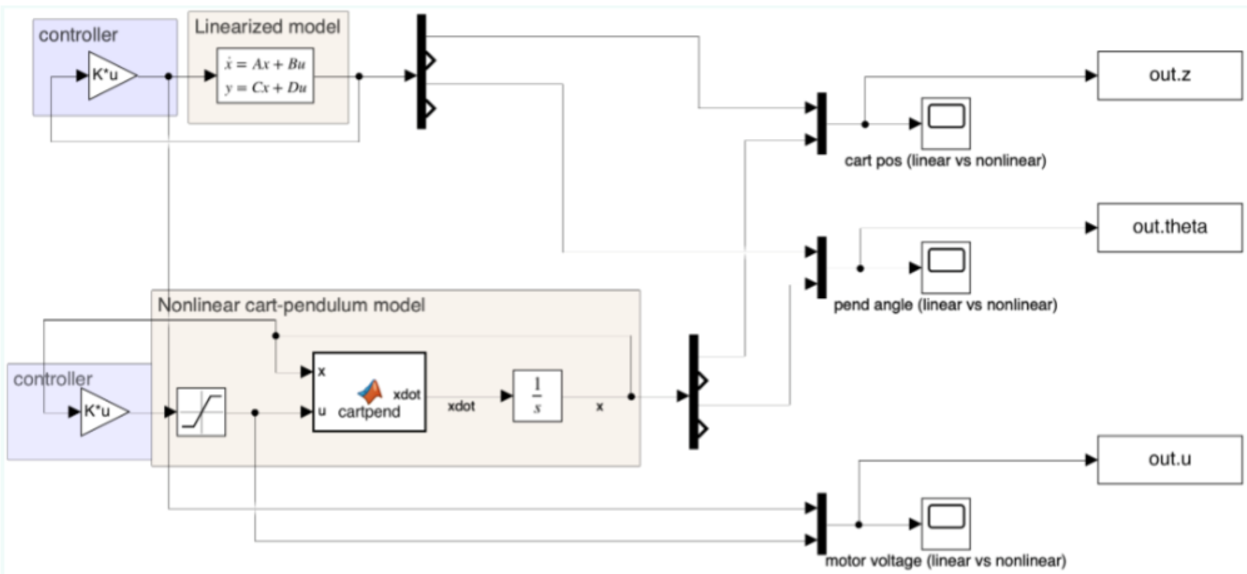

Figure 1. Block Diagram of Closed-Loop Linear & Non-Linear Plant

Figure 1. describes the block diagram. For the linearized model,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -8.82 & 1.89 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -26.70 & 35.41 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1.45 \\ 0 \\ 4.39 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the linearized model has the state equations:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The input to the linearized and nonlinear system is the feedback control u = -K*x and the output is the state which consists of cart position, cart velocity, pendulum angle, and pendulum angle velocity.

Output 1 – Simulation of Controllers on Linearized Plant

Check and compare eigenvalues of A + BK_{LQR1} and A + BK_{LQR2}

eig_lqr1 = [-9.56, -6.04, -4.52, -0.52]

eig_lqr2 = [-9.66, -5.74, -4.72, -0.05]

The two systems have different eigenvalues, especially the final eigenvalue -0.52 vs. -0.052, they are approximately a factor of 10. The other 3 eigenvalues are similar.

The matrices A, B of the linearization are as follows:

A = $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -8.82 & 1.89 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -26.70 & 35.41 & 0 \end{bmatrix}$

B = $\begin{bmatrix} 0 \\ 1.45 \\ 0 \\ 4.39 \end{bmatrix}$

To determine whether the system (A, B) is controllable, we check the rank of the controllability matrix for (A, B) using the MATLAB function `ctrb(A, B)`. It turns out that controllability matrix has a rank of 4, which is the same as the dimension of A. Therefore, the system (A, B) is controllable.

The 3 gains associated with the three controllers are as follows:

$K_{place} = [-14.53 \quad -17.70 \quad 47.03 \quad 8.39]$

$K_{LQR1} = [-3.16 \quad -13.70 \quad 40.71 \quad 7.21]$

$K_{LQR2} = [-0.32 \quad -12.32 \quad 37.59 \quad 6.65]$

The 3 gains are different as they are derived using different methods. K_{place} is derived by setting the poles of the closed-loop feedback system. LQR, in contrast, is a control technique that minimizes a cost function associated with the system’s state and control input. It considers not just the state, but also the control effort, thereby penalizing deviation from the desired state while optimizing the control effort. Generally, pole placement is used when we want certain dynamic response characteristics, while LQR is used when we want more control over individual state variables and optimal control effort.

For the cart-pendulum system, pole placement is the preferred technique when we want certain characteristics like rapid settling time. LQR provides the potential for better overall performance by balancing control effort with deviations from desired state. When we want to consider tradeoffs between different control objectives (cart position tracking, pendulum angle tracking, control effort), LQR gives us more flexibility. In this lab, we have desired states for both cart position and pendulum angle, and want to eventually implement in a real system, so we care about control effort as well. So, we justify tuning the LQR parameters as a way to decouple these different objectives.

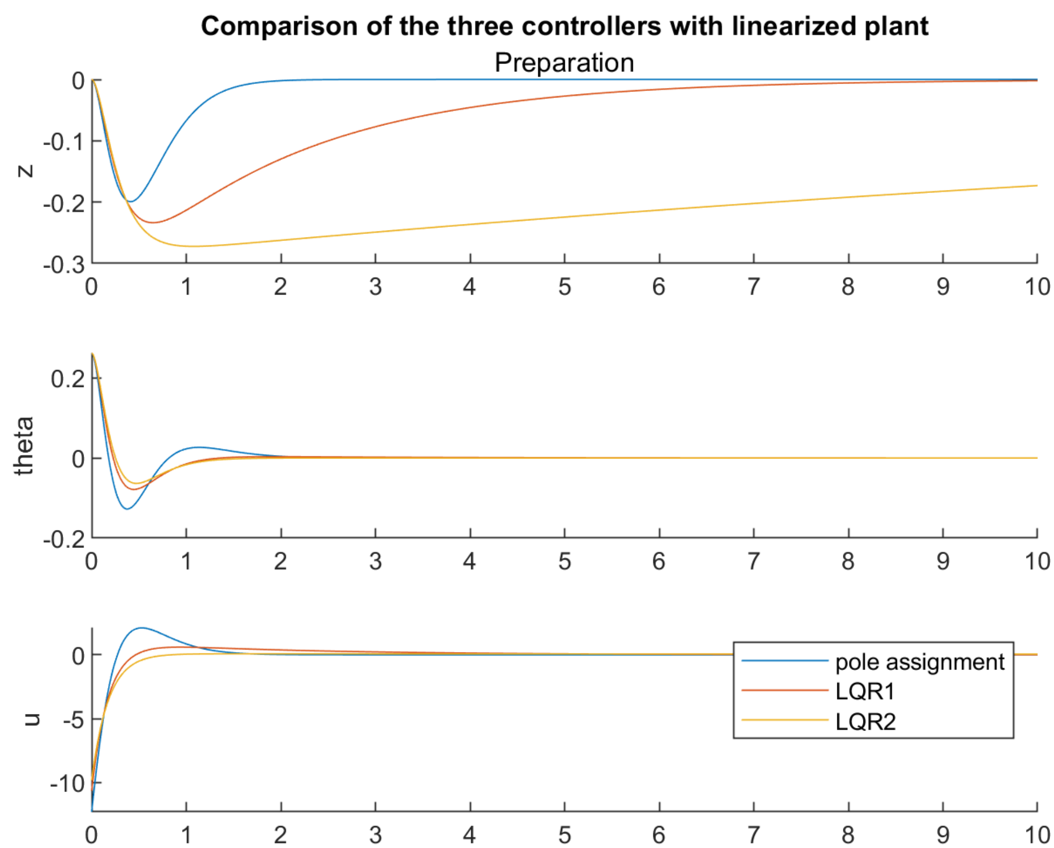

Figure 2. Comparison of Three Controllers with Linearized Plant

Figure 2 provides a comparison of the three controllers. First, we compare the z dimension of the state. The pole assignment controller has the fastest settling time for z when compared to LQR1 and LQR2. We expect this as setting the poles directly in pole placement gives full control over performance of the system (state converges to desired state and can easily be made to go faster by making eigenvalues more negative). In contrast, LQR considers not only the state but also the control effort, so it does not directly optimize for z, and is harder to tune. LQR1 has a faster settling time than LQR2, because it has a relatively higher value for q1, which controls the importance the controller gives to getting cart position right. Next, we can compare the theta dimension of the state. The pole assignment controller has a slight overshoot, which could be explained as the pole assignment does not take control effort (motor voltage) into consideration. LQR1 and LQR2 share similar results in tracking theta, due to their weights being relatively the same in this dimension. Finally, we can compare the input signal. We see that pole assignment, as expected, does not optimize for smaller motor voltages. However, the motor voltages for LQR1 and LQR2 are closer to 0. LQR1 has a slight overshoot compared to LQR2; this makes sense as LQR2 has a higher weight for the input signal, which means LQR2 gives more importance to getting u right.

Output 2 – Simulation of Controllers on Non-Linear Plant

Table 1. Limiting Initial Conditions for the Three Controllers in the Case of No Saturation

Controller	**Limiting Initial Condition for Theta**
Pole Assignment	$\frac{7\pi}{24}$
K_{LQR1}	$\frac{8\pi}{24}$
K_{LQR2}	$\frac{9\pi}{24}$

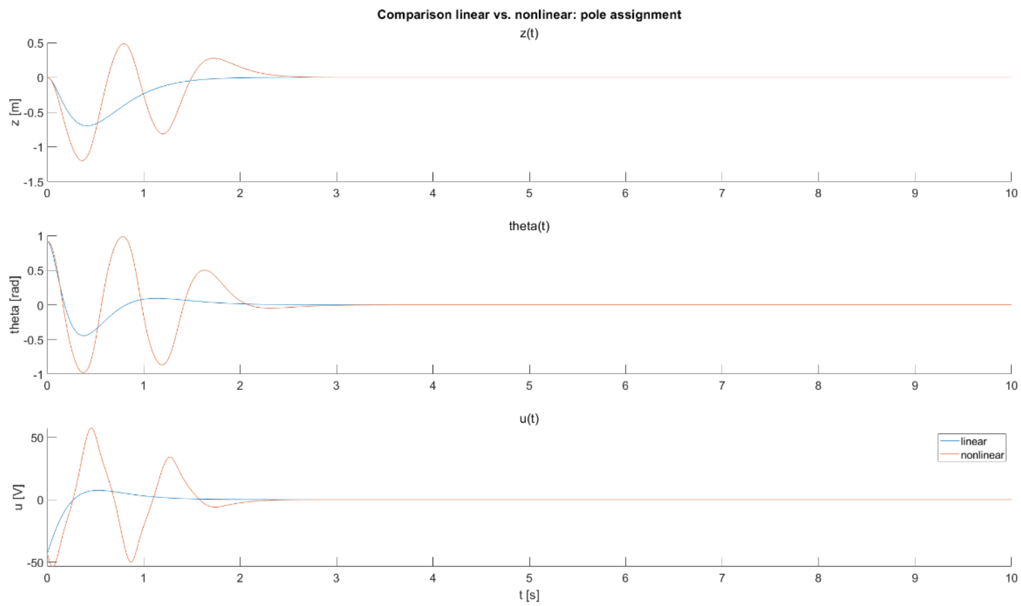

Figure 3. Pole Assignment: z(t), theta(t), and u(t) for Initial Condition $\theta = \frac{7\pi}{24}$ with No Saturation, for both Linear and Non-linear Models

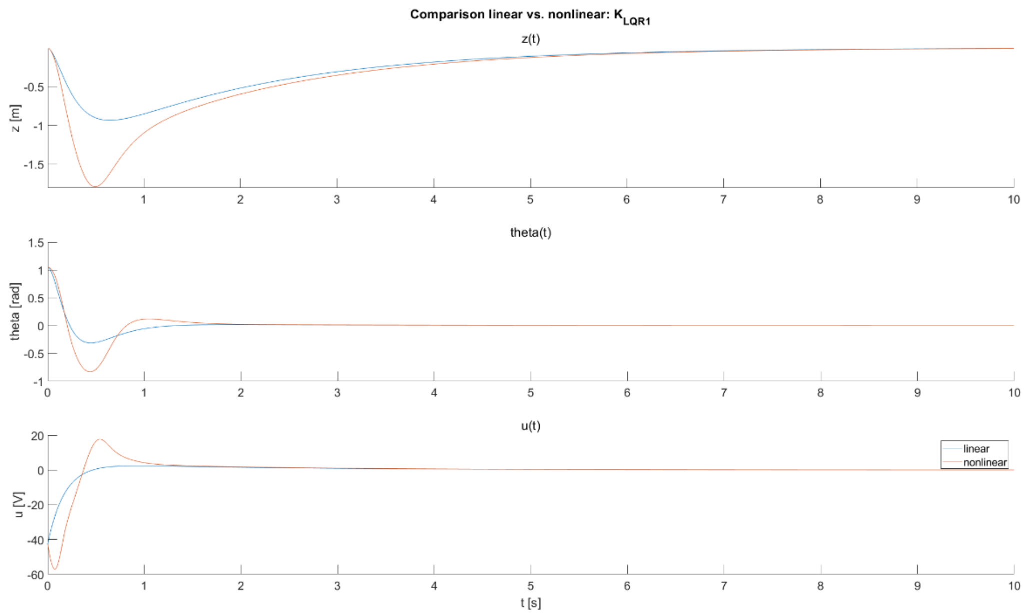

Figure 4. LQR 1: z(t), theta(t), and u(t) for Initial $\theta = \frac{8\pi}{24}$ with No Saturation, for both Linear and Non-linear Models

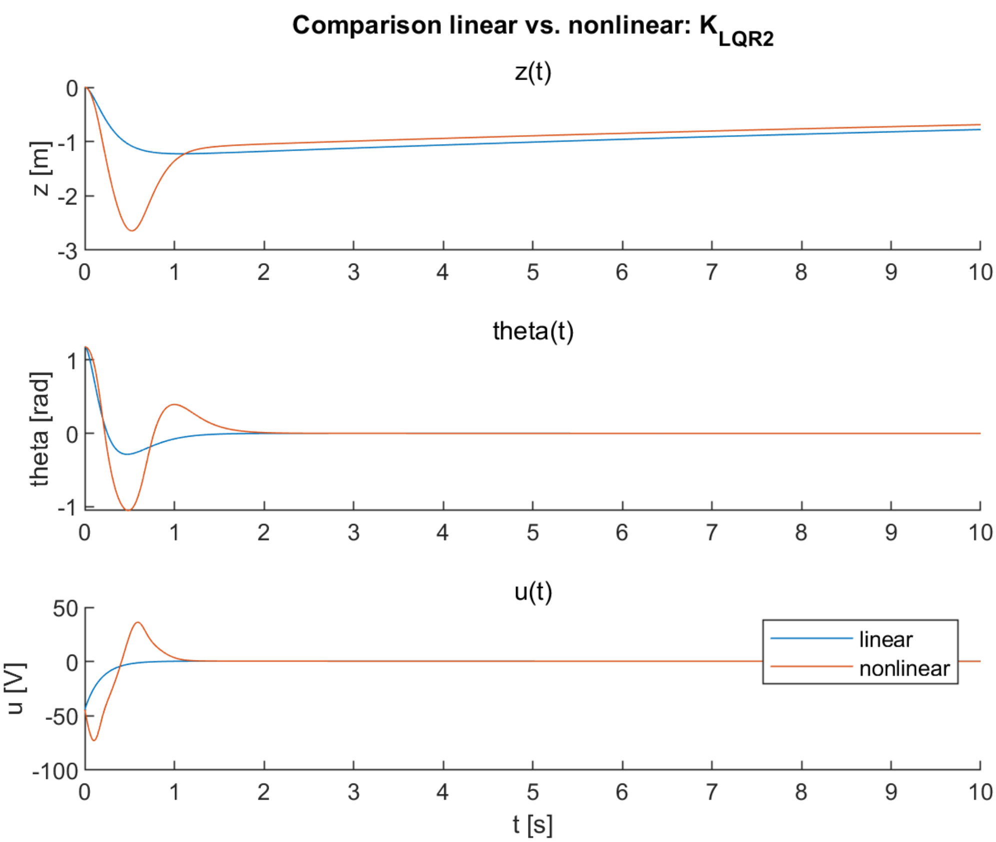

Figure 5. LQR 2: z(t), theta(t), and u(t) for Initial Condition $\theta = \frac{9\pi}{24}$ with No Saturation, for both Linear and Non-linear Models

Note that in Figure 3 -5, the system still converges but once we increase the initial theta to be 8*pi/24, 9*pi/24, and 10*pi/24 correspondingly, MATLAB fails to solve the system. Therefore, we conclude that the limiting initial condition of theta for pole assignment is somewhere between 7*pi/24 and 8*pi/24. The limiting initial condition of theta for LQR 1 is somewhere between 8*pi/24 and 9*pi/24. The limiting initial condition of theta for LQR 2 is somewhere between 9*pi/24 and 10*pi/24.

Similarly in Figure 6-8, in the case for saturation, part a) illustrates the smallest initial theta for the system to be unstable, and part b) illustrates the largest initial theta for the system to be stable. We interpret the limiting initial condition for theta to be the largest initial theta for the system to be stable.

Table 2. Limiting Initial Conditions for the Three Controllers in the Case of Saturation Ulim = 5.875V

Controller	**Limiting Initial Condition for Theta**
Pole Assignment	$\frac{2\pi}{24}$
K_{LQR1}	$\frac{2\pi}{24}$
K_{LQR2}	$\frac{2\pi}{24}$

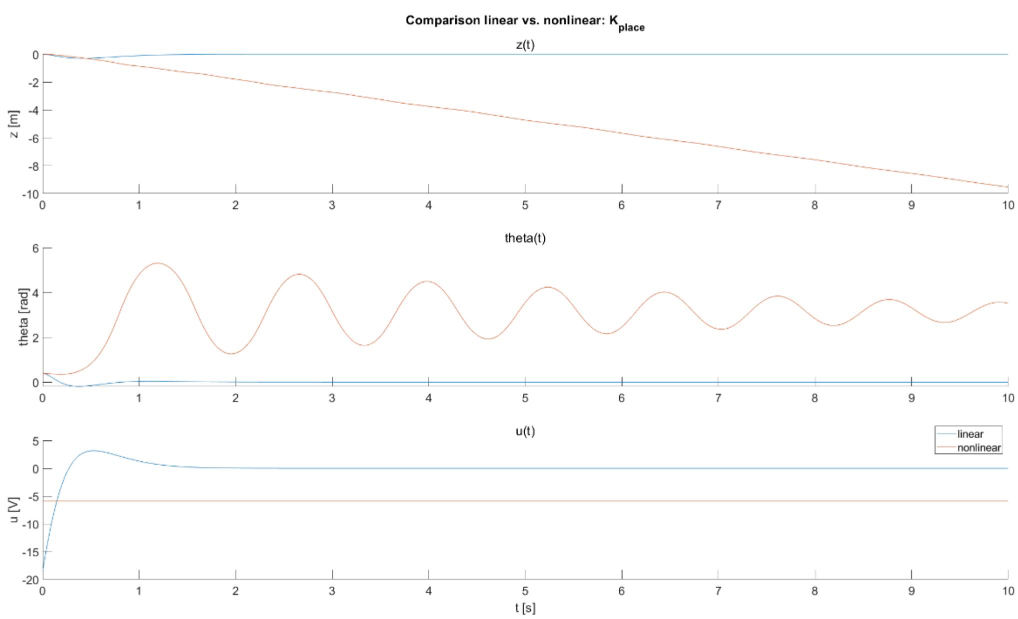

Figure 6a). Pole Assignment: z(t), theta(t), and u(t) for Initial Condition $\theta = \frac{3\pi}{24}$ with saturation Ulim = 5.875V, for both Linear and Non-linear Models

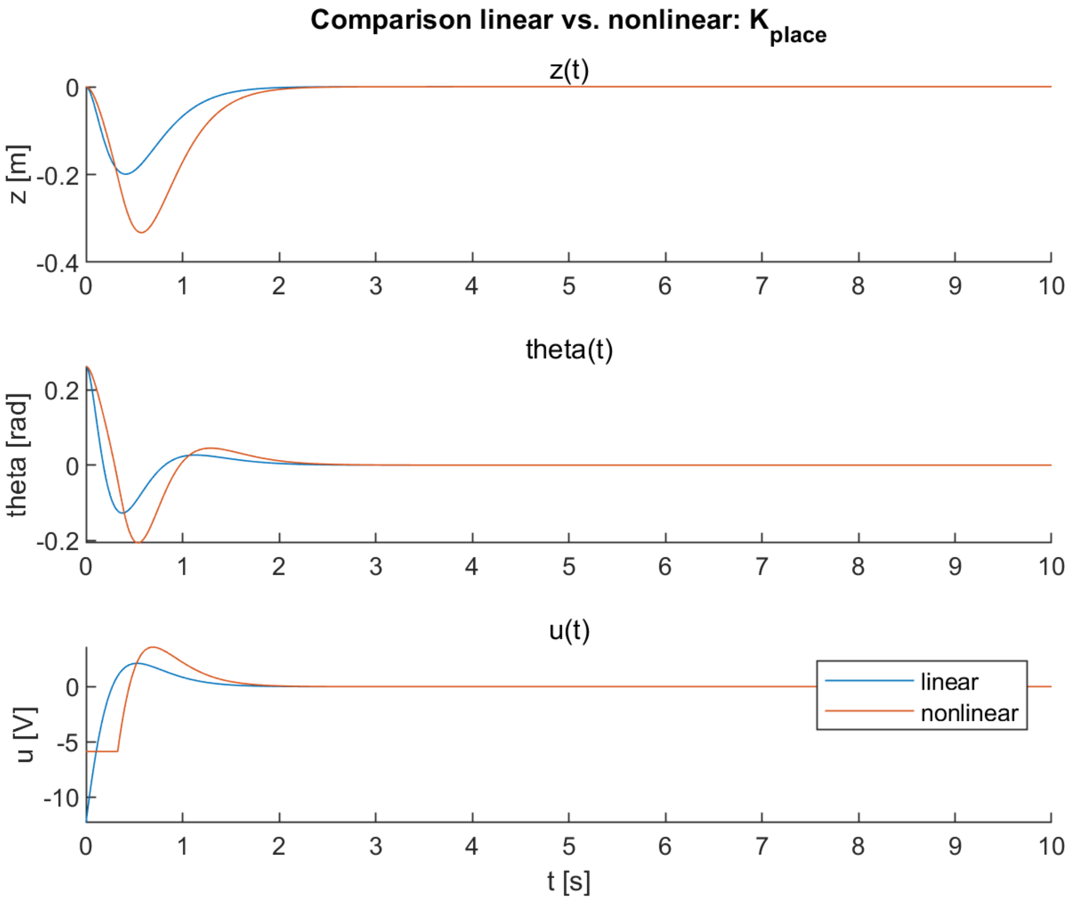

Figure 7b). Pole Assignment: z(t), theta(t), and u(t) for Initial Condition $\theta = \frac{2\pi}{24}$ *with saturation Ulim = 5.875V, for both Linear and Non-linear Models*

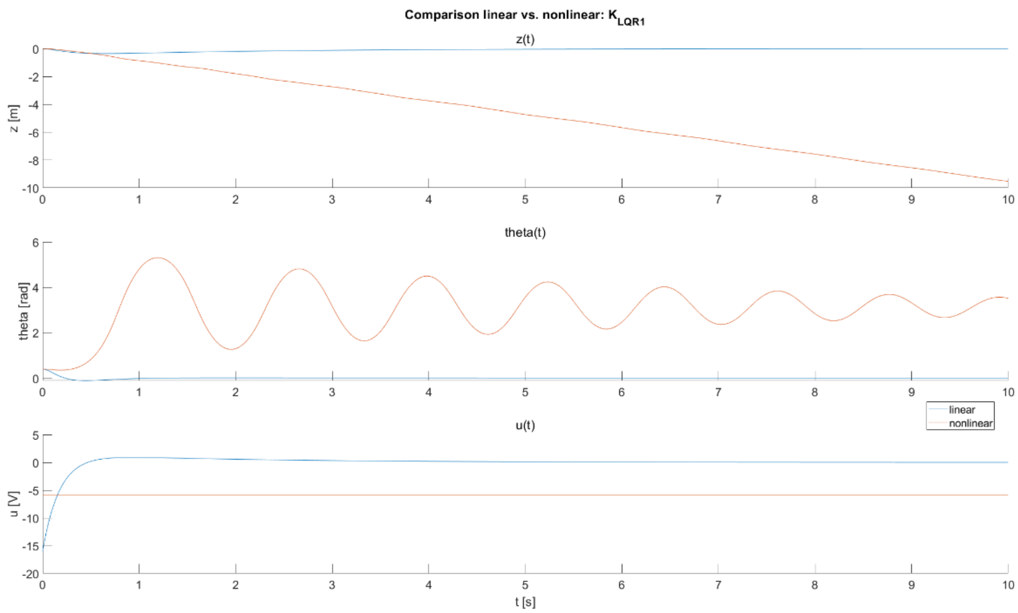

Figure 7a). LQR 1: z(t), theta(t), and u(t) for Initial Condition $\theta = \frac{3\pi}{24}$ *with Saturation Ulim = 5.875V, for both Linear and Non-linear Models*

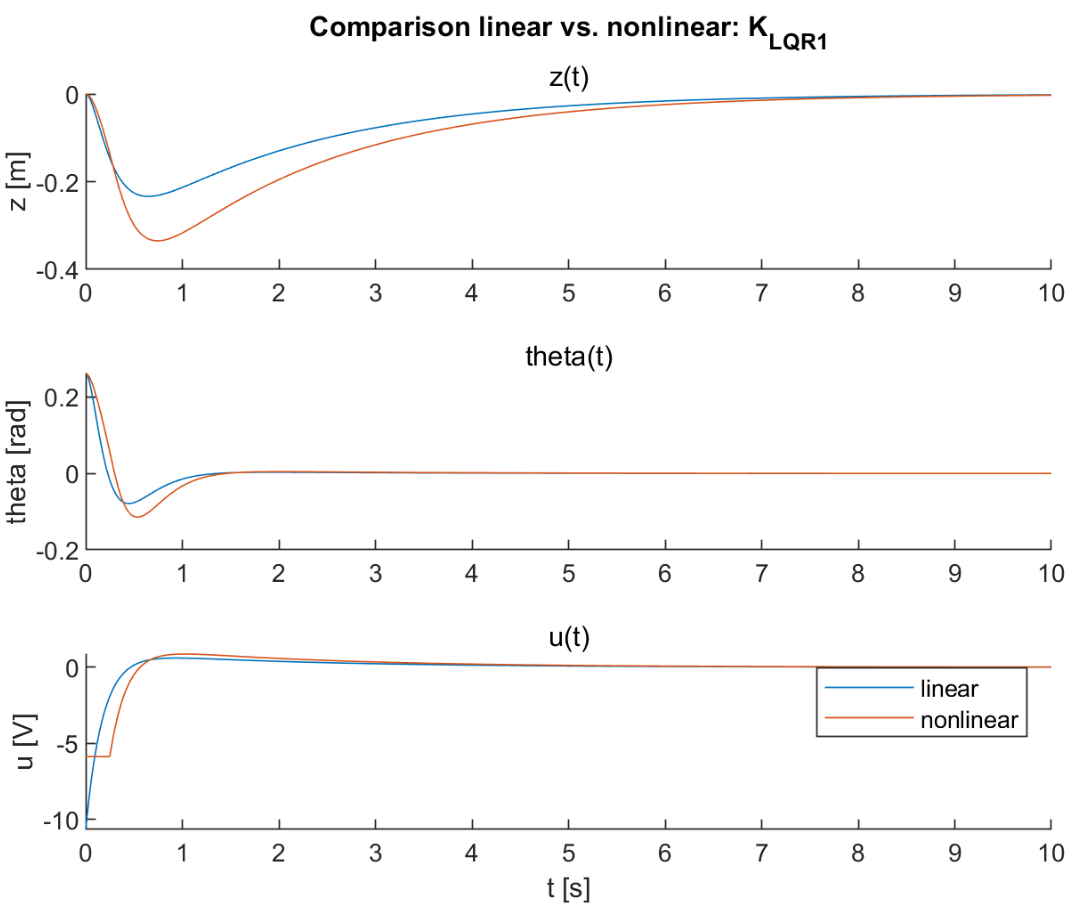

Figure 7b). LQR 1: z(t), theta(t), and u(t) for Initial Condition $\theta = \frac{2\pi}{24}$ *with Saturation Ulim = 5.875V, for both Linear and Non-linear Models*

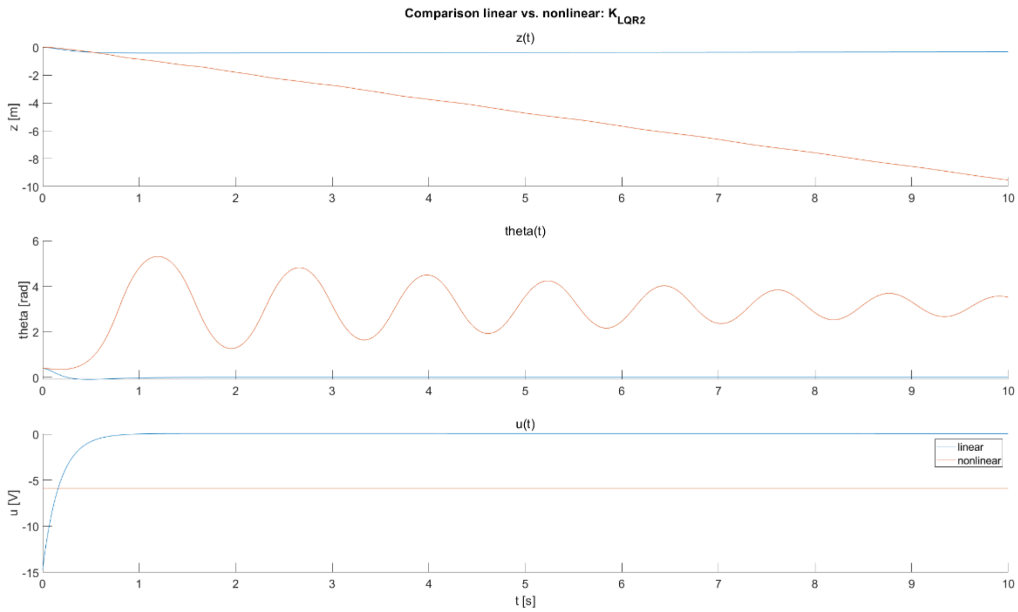

Figure 8a). LQR 2: z(t), theta(t), and u(t) for Initial Condition $\theta = \frac{3\pi}{24}$ *with Saturation Ulim = 5.875V, for both Linear and Non-linear Models*

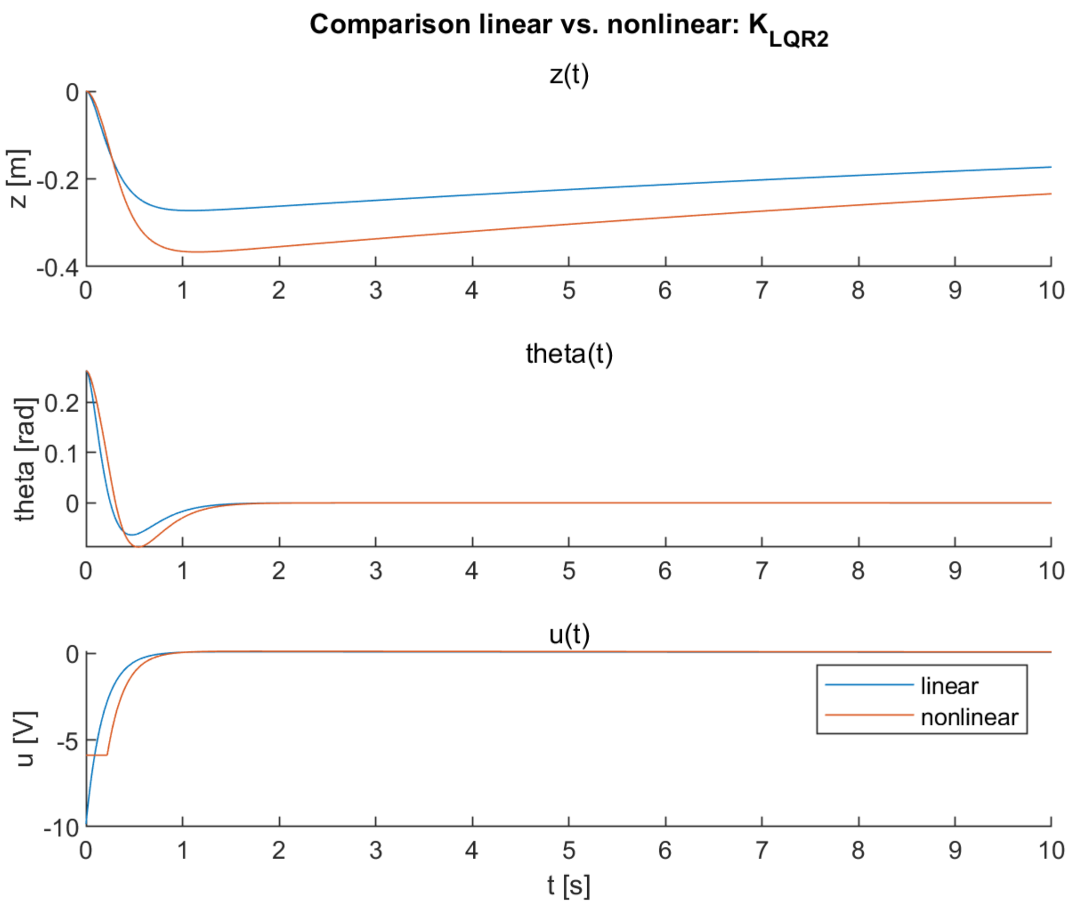

Figure 8b). LQR 2: z(t), theta(t), and u(t) for Initial Condition $\theta = \frac{2\pi}{24}$ *with Saturation Ulim = 5.875V, for both Linear and Non-linear Models*

The non-linear model appears to show more aggressive and oscillatory behavior before stabilizing compared to the linear model. This is because the nonlinear system inherently has nonlinear dynamics that are not captured by the linearized system. Specifically, nonlinearities like gravity, friction, and trigonometric terms in the equations of motion are not accounted for. In fact, the true dynamics of the system (z'' and theta'') both include sine and cosine terms, which are integrated over time to get the state at any time. Since the integral of a trigonometric function is still a trigonometric function, the output of the integrator block in the block diagram (Figure 1), which represents the state, is sinusoidal in nature. This is clearly shown in the plots for the nonlinear system; however, the sinusoidal nature does not appear in the linearized system, as it only approximates the sinusoids to one derivative.

Due to these discrepancies, the value of x dot is naturally different away from the equilibrium point than what is estimated by linearizing. From Figure 3, we can observe that x dot for the nonlinear system causes more aggressive changes in x. To obtain these aggressive changes, we intuitively require a similarly aggressive control input, which we do indeed see in Figure 3. Once the aggressive changes push the state closer towards the desired state, the effects of the nonlinear dynamics of the system become more and more negligible, as the system is moving closer to equilibrium, and thus more closely matches the linearized system. This is clearly shown in Figure 3, as the magnitude of z, theta and u oscillations become smaller and smaller, and eventually, the nonlinear and linearized system variables exhibit the same response.

The discrepancy is expected to grow larger as the initial pendulum angle moves further away from the equilibrium because the linear model is linearized with respect to the equilibrium point. By definition of being a linearization, it is most accurate near the equilibrium point, and becomes a worse approximation of the nonlinear system as we move further away from the equilibrium.

The effect of saturation can be seen in Fig 6b, 7b, 8b, where the stable non-linear model input signal gets clipped at -5.875V. The non-stable case is shown in Figure 6a, 7a, 8a. They become present in these waveforms when the required input signal exceeds what the physical actuators can provide. Specifically, we notice that z starts at 0 and keeps decreasing in a linear fashion. This is because we are consistently providing the maximum amount of voltage we can. The pendulum angle exceeds pi radians, and then oscillates to around pi. This is because the effect of gravity on the pendulum is too strong to make up for with lateral movement of the cart at voltage of only 5.875V. So, the pendulum spins all the way downward and settles at a vertically downwards position. The voltage (u) is always at 5.875V. This is because it will push the cart laterally in an effort to make the angle smaller, but the motion provided by this voltage is not fast enough. To summarize, the drastic effect is that the introduction of saturation restricts us to a smaller set of initial conditions from which we can recover from. In comparison to the no saturation case, the initial condition for theta can go up to 7pi/24 and still be stable. However, with saturation, the system response becomes unstable after 2pi/24. For initial angles greater than this, we only get worse over time with respect to balancing the pendulum because the torque provided by gravity becomes stronger while the voltage is limited to 5.875V.

The difference seen between the three controllers with saturation is similar to the observed behavior with no saturation. LQR1 is seen in Figure 7b to stabilize z with priority when compared to LQR2 (goes to 0 faster, which agrees with q1 being higher). The theta dimension is similar between LQR1 and LQR2, as q2 for is relatively similar. In the u dimension, LQR2 cares more than LQR1 (R higher), which is evident in the overshoot of u in Figure 7b. When comparing LQR with pole placement, we see that z and theta move more aggressively towards 0. This is because pole placement tries to move rapidly towards a zero state, and doesn’t consider the integral of the signal. Also, pole placement doesn’t consider u in the optimization, which agrees with Figure 6b), as we see an even larger overshoot than LQR1. (The reason for the larger overshoot in pole placement is due to the lack of optimization in control signal.) Ultimately, we still observe that we can tune the parameters despite placing a saturation limit on the input.

An additional observation is that without saturation limit, the tolerance on the initial angle is larger for LQR2 than LQR1 than the pole assignment method.