Integrator Lab: Solving First Order ODEs in MATLAB and Picard Approximation

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This lab will teach you to numerically solve first order ODEs using a built in MATLAB integrator, `ode45`. `ode45` is a good, general purpose tool for integrating first order equations (and first order systems). It is not always the right algorithm, but it is usually the right algorithm to try first. This lab will also teach you how to manipulate symbolic functions in MATLAB.

You will learn how to use the `ode45` routine, how to interpolate between points, and how MATLAB handles data structures. You will also learn how to use MATLAB for exact symbolic calculations and write your own Picard approximation code.

Opening the m-file lab2.m in the MATLAB editor, step through each part using cell mode to see the results. Compare the output with the PDF, which was generated from this m-file.

There are eight exercises in this lab that are to be handed in at the end of the lab. Write your solutions in the template, including appropriate descriptions in each step. Save the .m file and submit it online using Quercus.

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Set up an inline function representation of an ODE and solve it

MATLAB has many built in routines for solving differential equations of the form

```
y' = f(t,y)
```

We will solve them using `ode45`, a high precision integrator. To do this, we will need to construct an inline function representation of `f`, an initial condition, and specify how far we want MATLAB to integrate the problem. Once we have set these, we pass the information to `ode45` to get the solution.

For a first example, we will solve the initial value problem

```
y' = y, y(0) = 1
```

which has as its answer `y = e^t`.

```
% Set up the right hand side of the ODE as an inline function
f = @(t,y) y;

% The initial conditions
t0 = 0;
y0 = 1;

% The time we will integrate until
t1 = 2;

soln = ode45(f, [t0, t1], y0);
```

Examining the output

When we execute the `ode45`, it returns a data structure, stored in soln. We can see the pieces of the data structure with a display command:

```
disp(soln);
```

```
     solver: 'ode45'
    extdata: [1×1 struct]
          x: [0 0.2000 0.4000 0.6000 0.8000 1 1.2000 1.4000 1.6000
 1.8000 2]
          y: [1×11 double]
      stats: [1×1 struct]
      idata: [1×1 struct]
```

Understanding the components of the solution data structure

The most important elements of the data structure are stored in the `x` and `y` components of the structure; these are vectors. Vectors `x` and `y` contain the points at which the numerical approximation to the initial vlaue problem has been computed. In other words, `y(j)` is the approximate value of the solution at `x(j)`.

NOTE: Even though we may be studying a problem like `u(t)` or `y(t)`, MATLAB will always use `x` for the independent variable and `y` for the dependent variable in the data structure.

Pieces of the data structure can be accessed using a period, as in C/C++ or Java. See the examples below:

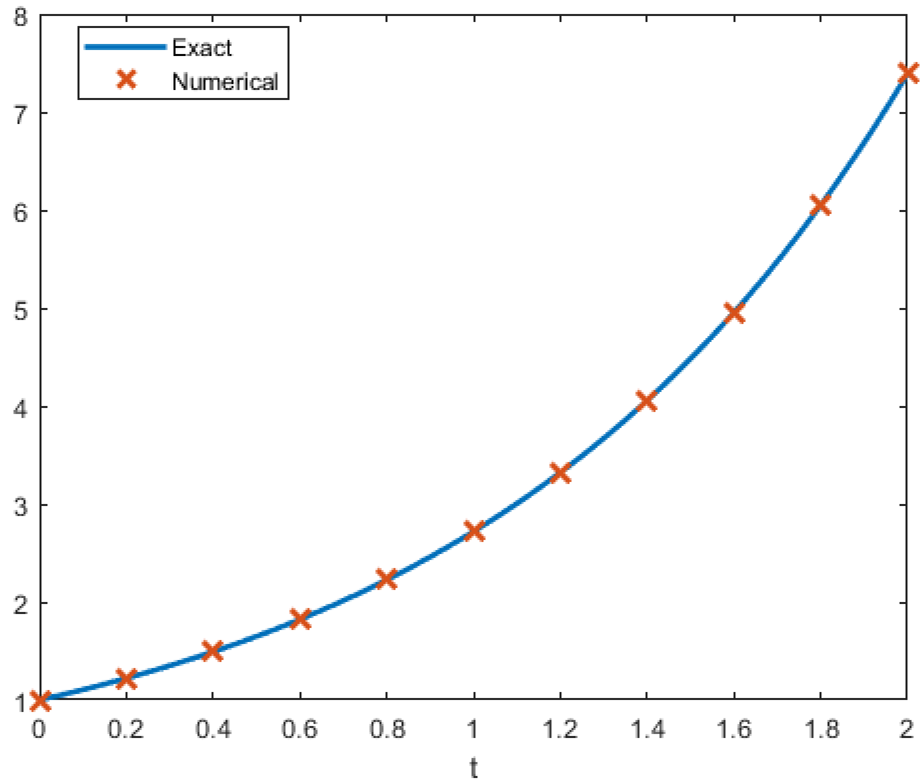
```
% Display the values of |t| at which |y(t)| is approximated
fprintf(' Vector of t values: ');
disp(soln.x);
% Display the the corresponding approximatations of |y(t)|
fprintf(' Vector of y values: ');
disp(soln.y);

% Display the approximation of the solution at the 3rd point:
fprintf(' Third element of the vector of t values: %g\n',soln.x(3));
fprintf(' Third element of the vector of y values: %g\n',soln.y(3));
```

```
 Vector of t values:   Columns 1 through 7

         0    0.2000    0.4000    0.6000    0.8000    1.0000    1.2000

  Columns 8 through 11

    1.4000    1.6000    1.8000    2.0000

 Vector of y values:   Columns 1 through 7

    1.0000    1.2214    1.4918    1.8221    2.2255    2.7183    3.3201

  Columns 8 through 11

    4.0552    4.9530    6.0496    7.3891

 Third element of the vector of t values: 0.4
 Third element of the vector of y values: 1.49182
```

Visualizing and comparing the solution

We can now visualize the solution at the computed data points and compare with the exact solution.

```
% Construct the exact solution
tt = linspace(0,2,50);
yy = exp(tt);

% Plot both on the same figure, plotting the approximation with x's
plot(tt, yy, soln.x, soln.y, 'x', 'MarkerSize',10, 'LineWidth', 2);
% NOTE: the MarkerSize and LineWidth are larger than their defaults of
 6
% and 1, respectively.  This makes the print out more readable.

% Add a label to the axis and a legend
xlabel('t');
legend('Exact', 'Numerical','Location','Best');
```



Exercise 1

Objective: Solve an initial value problem and plot both the numerical approximation and the corresponding exact solution.

Details: Solve the IVP

```
y' = y tan t + sin t, y(0) = -1/2
```

from `t = 0` to `t = pi`.

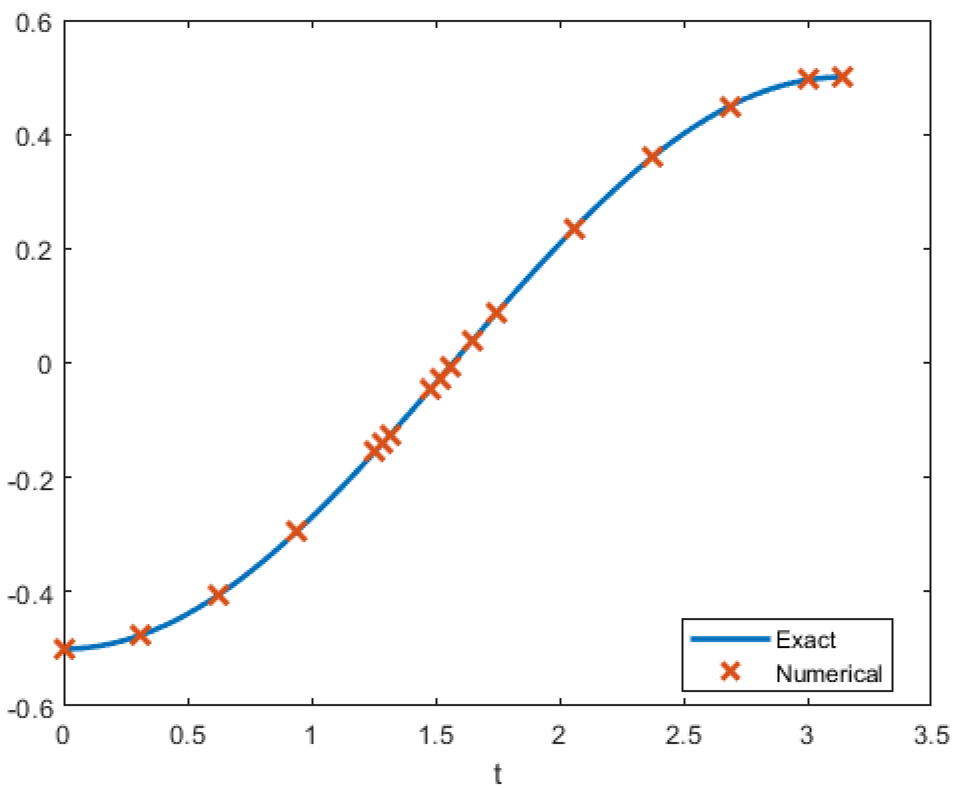
Compute the exact solution (by hand), and plot both on the same figure for comparison, as above.

Your submission should show the construction of the inline function, the use of ode45 to obtain the solution, a construction of the exact solution, and a plot showing both. In the comments, include the exact solution.

Label your axes and include a legend.

```
% use ode45 to compute approximate solution
f = @(t,y) y*tan(t)+sin(t); %inline function
t0 = 0; %lower limit of integration
t1 = pi; %upper limit of integration
y0 = -0.5; %initial condition
solnex1= ode45(f, [t0, t1], y0);
```

```
% exact solution: y=-0.5cost
tt = linspace(0,pi); %x-values
%yy = cos(tt)*-0.5; %exact y values
yy = (sin(tt).^2-1)./(2*cos(tt));

%plot solutions
plot(tt, yy, solnex1.x, solnex1.y, 'x', 'MarkerSize', 10, 'LineWidth',
 2);
xlabel('t');
legend('Exact', 'Numerical', 'Location', 'Best');
```



Computing an approximation at a specific point

As you should be able to see by examining `soln.x`, ode45 returns the solution at a number of points between `t0` and `t1`. But sometimes we want to know the solution at some intermediate point.

To obtain this value, we need to interpolate it in a consistent way. Fortunately, MATLAB provides a convenient function, `deval`, specifically for this.

```
% Compute the solution at t = .25:
deval(soln, .25)

% Compute the solution at t = 1.6753:
fprintf(' Solution at 1.6753: %g\n', deval(soln, 1.6753));
```

```
% Compute the solution at 10 grid points between .45 and 1.65:
tinterp = linspace(.45, 1.65, 10);
deval(soln, tinterp)

% Alternatively:
deval(soln, linspace(.45, 1.65, 10))
```

```
ans =

    1.2840

 Solution at 1.6753: 5.3404

ans =

  Columns 1 through 7

    1.5683    1.7920    2.0476    2.3396    2.6734    3.0547    3.4903

  Columns 8 through 10

    3.9882    4.5570    5.2070


ans =

  Columns 1 through 7

    1.5683    1.7920    2.0476    2.3396    2.6734    3.0547    3.4903

  Columns 8 through 10

    3.9882    4.5570    5.2070
```

Exercise 2

Objective: Interpolate a solution at a number of grid points

Details: For the solution you computed in exercise 1, use deval to compute the interpolated values at 10 grid points between 2 and 3.

```
% We will follow the example above
tinterp = linspace(2, 3, 10); % define interval of interpolation
deval(solnex1, tinterp) %use deval function to compute approximations
 at tinterp
```

```
ans =

  Columns 1 through 7
```

```
    0.2081    0.2572    0.3032    0.3454    0.3833    0.4166    0.4447

  Columns 8 through 10

    0.4673    0.4841    0.4950
```

Errors, Step Sizes, and Tolerances

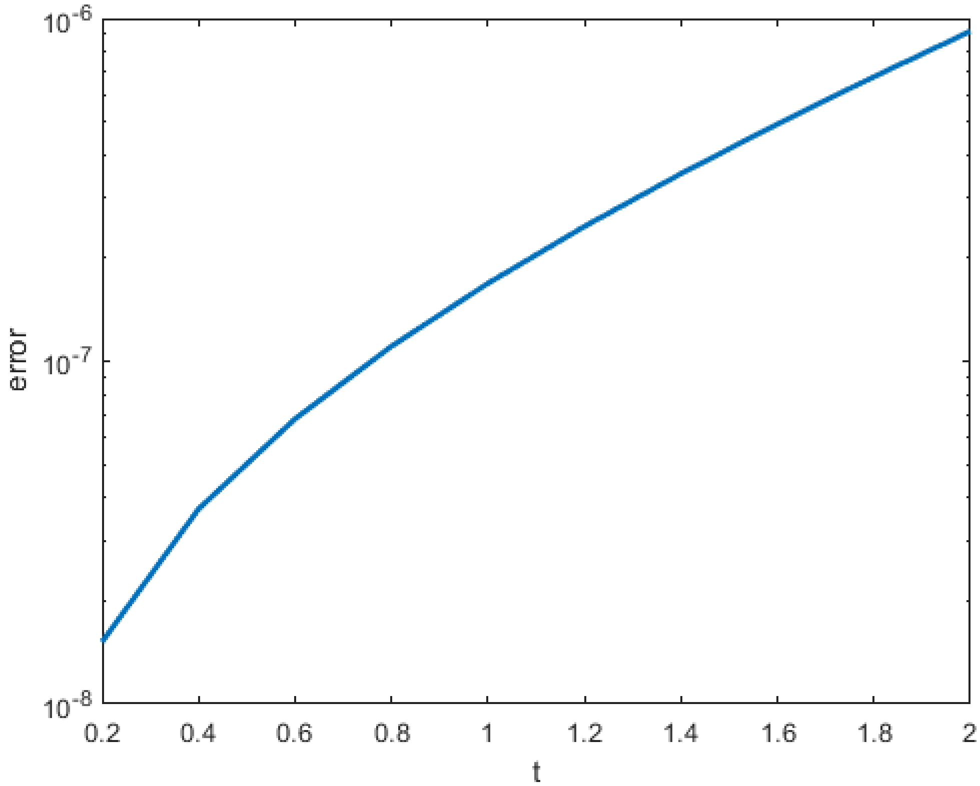
As you may have noticed, in contrast to the IODE software, at no point do we set a step size for our solution. Indeed, the step size is set adaptively to conform to a specified error tolerance.

Roughly speaking, given the solution at `(t_j, y_j)`, `ode45` computes two approximations of the solution at `t_{j+1} = t_j + h`; one is of greater accuracy than the other. If the difference is below a specified tolerance, the step is accepted and we continue. Otherwise the step is rejected and the smaller step size, `h`, is used; it is often halved.

We can compute the global truncation error at each solution point, figure out the maximum error, and visualize this error (on a linear-log scale):

```
% Compute the exact solution
yexact = exp(soln.x);

% Compute the pointwise error; note the use of MATLAB's vectorization
err = abs(yexact - soln.y);

disp(err);

fprintf('maximum error: %g \n', max(err));

semilogy(soln.x, err, 'LineWidth', 2);
xlabel('t');
ylabel('error');
```

```
   1.0e-06 *

  Columns 1 through 7

         0    0.0152    0.0371    0.0679    0.1106    0.1688    0.2475

  Columns 8 through 11

    0.3526    0.4922    0.6764    0.9179

maximum error: 9.17923e-07
```



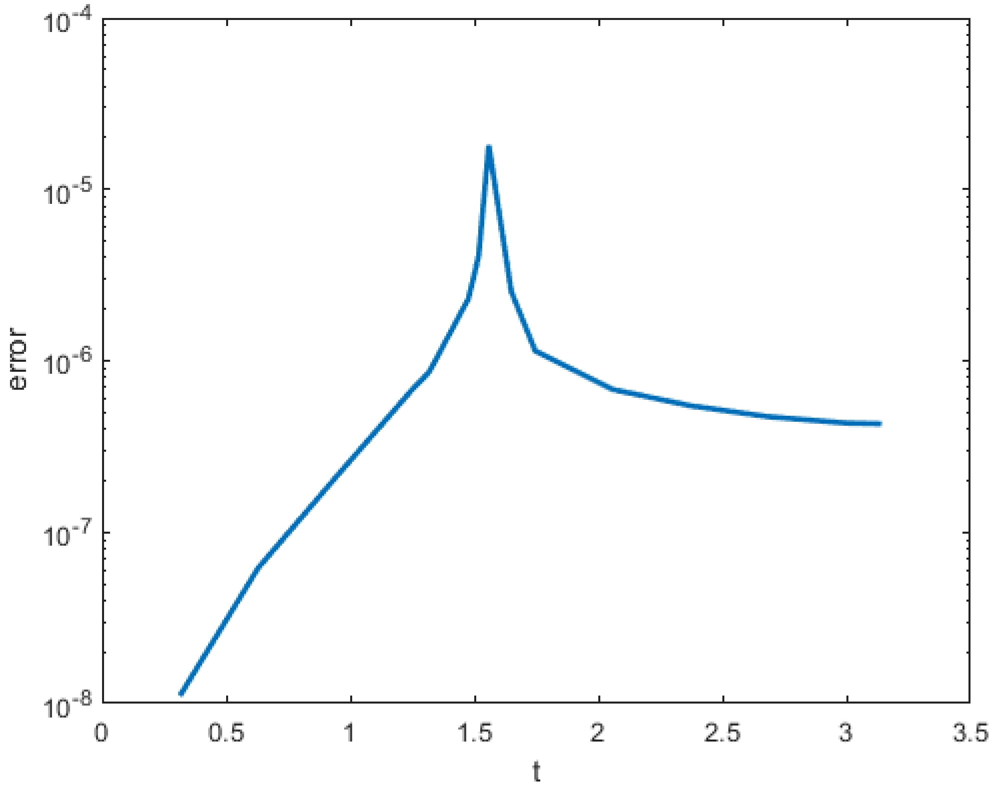
Exercise 3

Objective: Examine the error of a solution generated by `ode45`

Details: For your solution to exercise 1, compute the pointwise error, identify the maximum value of the error, and visualize the error on a linear-log plot (use semilogy to plot the log of the error vs. t). Write in the comments where the error is largest, and give a brief (1-2 sentences) explanation of why it is largest there. Make sure to label your axes.

```
%yexact = cos(solnex1.x)*-0.5;
yexact = (sin(solnex1.x).^2-1)./(2*cos(solnex1.x));
err = abs(yexact-solnex1.y);
disp(err);
fprintf("maximum error: %g \n", max(err));
semilogy(solnex1.x, err, 'LineWidth', 2);
xlabel('t');
ylabel('error');
disp(solnex1.x);
% The error is largest at x=1.5588, which corresponds to
 solnex1.x(10). Here, the error
% is 1.8068*10^-5. This makes sense because at pi/2, the magnitude of
 the derivate
% of the function y = -0.5cosx is maximum. So, any tangent
 approximation
% would predict the value of the function at the next step as if it
 was on the tangent line.
```

```
% However, the point (pi/2, 0) is a a point of inflection, and the
 graph
% changes from concave up to concave down. So, the behaviour changes
 after
% the point (pi/2,0), in a way that is inconsistent and not predicted
 by
% thte tangent approximation(assuming the approximation is indeed a
 tangent approximation)

   1.0e-04 *

  Columns 1 through 7

         0    0.0001    0.0006    0.0021    0.0070    0.0077    0.0087

  Columns 8 through 14

    0.0230    0.0408    0.1807    0.0254    0.0114    0.0068    0.0055

  Columns 15 through 17

    0.0047    0.0043    0.0043

maximum error: 1.8068e-05
  Columns 1 through 7

         0    0.3142    0.6283    0.9425    1.2566    1.2881    1.3195

  Columns 8 through 14

    1.4765    1.5177    1.5588    1.6491    1.7455    2.0597    2.3739

  Columns 15 through 17

    2.6880    3.0022    3.1416
```



Exercise 4

Objective: Solve and visualize a nonlinear ode using ode45

Details: Solve the IVP

```
y' = 1 / y^2 , y(1) = 1
```

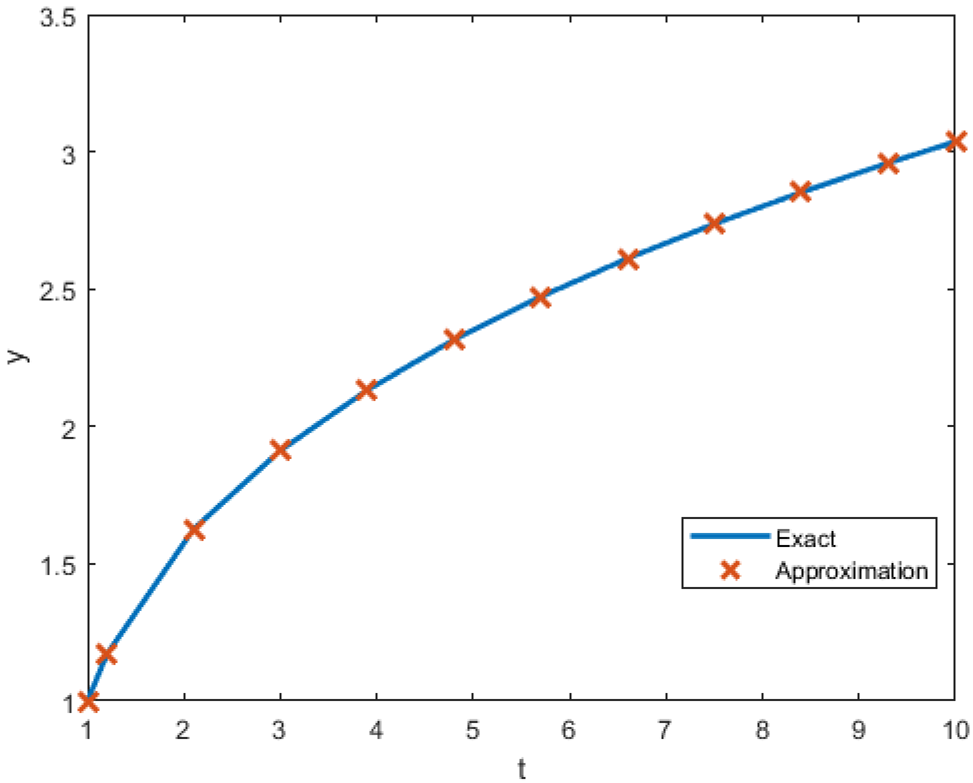
from `t=1` to `t=10` using `ode45`. Find the exact solution and compute the maximum pointwise error. Then plot the approximate solution and the exact solution on the same axes.

Your solution should show the definition of the inline function, the computation of its solution in this interval, the computation of the exact solution at the computed grid points, the computation of the maximum error, and a plot of the exact and approximate solutions.

```
%Your axes should be appropriately labeled and include a legend.

f = @(t,y) 1/(y^2); %definition of inline function
t0 = 1;
t1 = 10;
y1 = 1;
soln = ode45(f, [t0, t1], y1); %computation of approximated solution
 in this interval

% Exact solution is: y=(3x-2)^1/3
yexact = (3*soln.x-2).^(1/3); %computation of exact solution at
 computed grid points
```

```
err = abs(yexact-soln.y);
fprintf("maximum error: %g \n", max(err)); % computation of maximum
 error

% Plot of exact and approximate solutions
plot(soln.x, yexact, soln.x, soln.y, 'x', 'MarkerSize',
 10, 'LineWidth', 2); % Is this the right way, or should i do linspace
xlabel('t'); % label axis
ylabel('y'); % label axis
legend('Exact', 'Approximation', 'Location', 'Best') % legend
```

maximum error: 0.0017118



Exercise 5

Objective: Solve and visualize an ODE that cannot be solved by hand with `ode45`.

Details: Solve the IVP

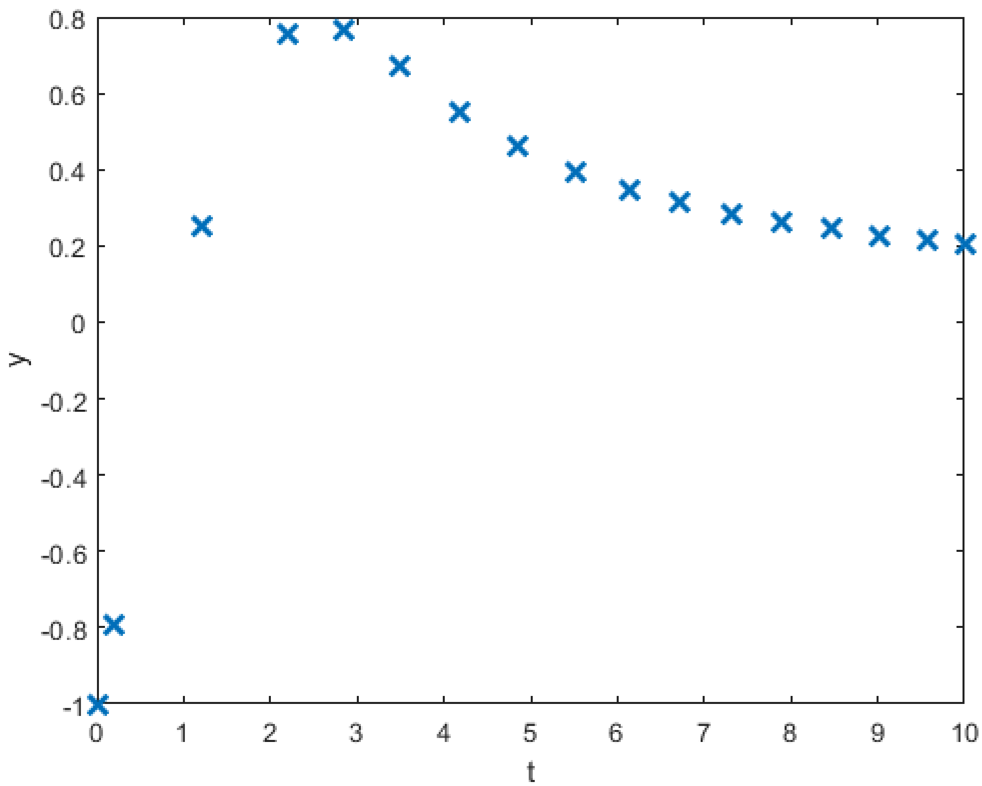
```
y' = 1 - t y / 2, y(0) = -1
```

from `t=0` to `t=10`.

Your solution should show you defining the inline function, computing the solution in this interval, and plotting it.

Your axes should be appropriately labeled

```
f = @(t,y) 1 - t*y/2; % define inline function
t0 = 0;
t1 = 10;
y0 = -1;
soln = ode45(f, [t0, t1], y0); %compute solution in this interval

% Plot solution
plot (soln.x, soln.y, 'x', 'MarkerSize', 10, 'LineWidth', 2)
xlabel('t'); %label axis
ylabel('y'); % albel axis
```



Exercise 6 - When things go wrong

Objective: Solve an ode and explain the warning message

Details: Solve the IVP:

```
y' = y^3 - t^2, y(0) = 1
```

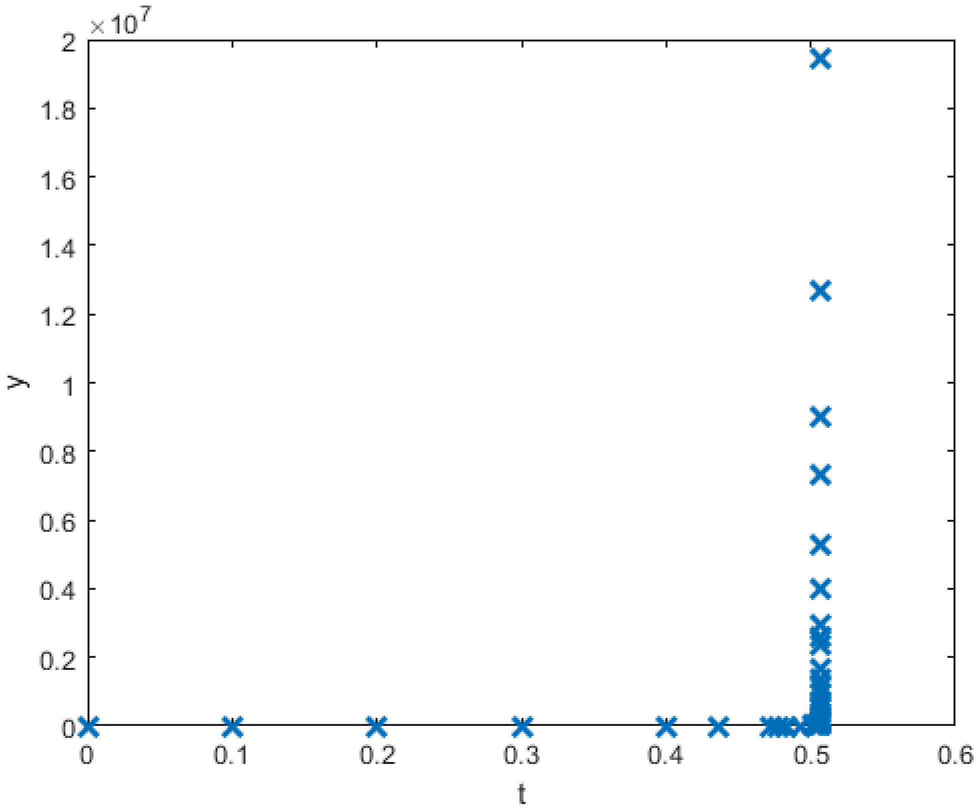
from `t=0` to `t=1`.

Your solution should show you defining the inline function, and computing the solution in this interval.

If you try to plot the solution, you should find that the solution does not make it all the way to t = 1.

In the comments explain why MATLAB generates the warning message that you may see, or fails to integrate all the way to t=1. HINT: Try plotting the direction field for this with IODE.

```
f = @(t,y) y^3 - t^2; %define inline function
t0 = 0;
t1 = 1;
y0 = 1;
soln = ode45(f, [t0, t1], y0); %compute solution in this interval
plot (soln.x, soln.y, 'x', 'MarkerSize', 10, 'LineWidth', 2) % try
 plotting
xlabel('t');
ylabel('y');


% MATLAB fails to integrate to t=1 because y->infinity as t->k, where
 k is
% some number that satisfies 0<k<1. This result was obtained by
 examining the
% direction field on iode, and observing the graph of y(t)
 corresponding to
% our initial values. t=1 is not in the interval of validity for this
% solution. The discontinuity ensures that the function is not
 integrable
% over the entire interval

% Another way to explain this: From the direction field, we see that
 if we
% start at (t, y) = (0, 1), then y'->infinity as t->k, 0<k<1. So, if
 we
% were to integrate y' from 0 to 1 to obtain y, it makes sense for y
 to go to infinity
% before it reaches 1(in general, y'->inf does not guarantee y->inf
 but here
% we know it does, and so y'->inf gives some intuitive sense to why y-
>inf).

% To drill our point further, lets consider this from the perspective
 of
% the solver, ode45. The solver works such given y at some t_j, it
 predicts
% y at t_j+1. But since there is a discontinuity somewhere in (0,1),
 it
% doesnt make sense to use y at t_j<k to predict y at some t_j+1>k,
 where k
% in (0,1) is our point of discontnuity.

%Evidence of htis: points are closer and closer together as we
 approach the
%asymptote. So, ode45 realizes we are going ot infinity and keeps
%decreasing the delta x that gives its approximation
```



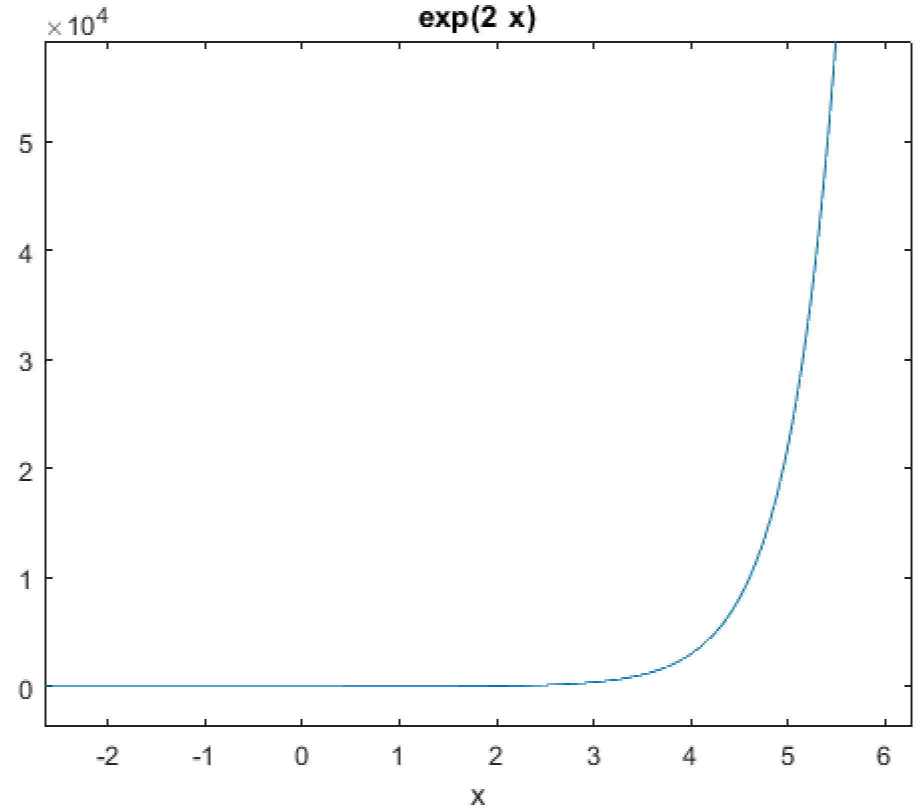
Using symbolic variables to define functions

We can define symbolic variables to let MATLAB know that these variables will be used for exact computations

```
% Start by defining the variables as symbolic
syms t s x y

% Define a function by simply writing its expression

f = cos(t)
g = sin(t)
h = exp(2*x)

% We can manipulate these functions

simplify(f^2+g^2)
diff(h)

% We can plot a function defined symbolically using the command |
ezplot|.
% Learn about the command |ezplot|:

help ezplot

% Plot the function |f(t)| and |h(x)|
```

```
ezplot(f)
ezplot(h)


f =

cos(t)


g =

sin(t)


h =

exp(2*x)


ans =

1


ans =

2*exp(2*x)

 EZPLOT   (NOT RECOMMENDED) Easy to use function plotter

  ==========================================================
  EZPLOT is not recommended. Use FPLOT or FIMPLICIT instead.
  ==========================================================

    EZPLOT(FUN) plots the function FUN(X) over the default domain
    -2*PI < X < 2*PI, where FUN(X) is an explicitly defined function
 of X.

    EZPLOT(FUN2) plots the implicitly defined function FUN2(X,Y) = 0
 over
    the default domain -2*PI < X < 2*PI and -2*PI < Y < 2*PI.

    EZPLOT(FUN,[A,B]) plots FUN(X) over A < X < B.
    EZPLOT(FUN2,[A,B]) plots FUN2(X,Y) = 0 over A < X < B and A < Y <
 B.

    EZPLOT(FUN2,[XMIN,XMAX,YMIN,YMAX]) plots FUN2(X,Y) = 0 over
    XMIN < X < XMAX and YMIN < Y < YMAX.

    EZPLOT(FUNX,FUNY) plots the parametrically defined planar curve
 FUNX(T)
    and FUNY(T) over the default domain 0 < T < 2*PI.
```

```
    EZPLOT(FUNX,FUNY,[TMIN,TMAX]) plots FUNX(T) and FUNY(T) over
    TMIN < T < TMAX.

    EZPLOT(FUN,[A,B],FIG), EZPLOT(FUN2,[XMIN,XMAX,YMIN,YMAX],FIG), or
    EZPLOT(FUNX,FUNY,[TMIN,TMAX],FIG) plots the function over the
    specified domain in the figure window FIG.

    EZPLOT(AX,...) plots into AX instead of GCA or FIG.

    H = EZPLOT(...) returns handles to the plotted objects in H.

    Examples:
    The easiest way to express a function is via a string:
       ezplot('x^2 - 2*x + 1')

    One programming technique is to vectorize the string expression
using
    the array operators .* (TIMES), ./ (RDIVIDE), .\ (LDIVIDE), .^
(POWER).
    This makes the algorithm more efficient since it can perform
multiple
    function evaluations at once.
       ezplot('x.*y + x.^2 - y.^2 - 1')

    You may also use a function handle to an existing function.
Function
    handles are more powerful and efficient than string expressions.
       ezplot(@humps)
       ezplot(@cos,@sin)

    EZPLOT plots the variables in string expressions alphabetically.
       subplot(1,2,1), ezplot('1./z - log(z) + log(-1+z) + t - 1')
    To avoid this ambiguity, specify the order with an anonymous
function:
       subplot(1,2,2), ezplot(@(z,t)1./z - log(z) + log(-1+z) + t - 1)

    If your function has additional parameters, for example k in
myfun:
       %-----------------------%
       function z = myfun(x,y,k)
       z = x.^k - y.^k - 1;
       %-----------------------%
    then you may use an anonymous function to specify that parameter:
       ezplot(@(x,y)myfun(x,y,2))

    See also EZCONTOUR, EZCONTOURF, EZMESH, EZMESHC, EZPLOT3, EZPOLAR,
             EZSURF, EZSURFC, PLOT, VECTORIZE, FUNCTION_HANDLE.

    Documentation for ezplot
       doc ezplot

    Other functions named ezplot

       sym/ezplot
```



If we try to evaluate the function `f(0)`, we get an error message.

The symbolic variables are not meant to be used to evaluate functions, but to manipulate functions, compute derivatives, etc. To evaluate a function using symbolic variables is a little cumbersome:

```
% We need to substitute the variable by a value:

subs(f,t,pi)
```

```
ans =

-1
```

This expression means: In the expression `f`, substitute the variable `t` by the number `pi`.

```
% If we use a value where the cosine does not have a "nice"
 expression, we
% need to approximate the result:

subs(f,t,2)

% We need to use the command |eval|

eval(subs(f,t,2))
```

```
ans =

cos(2)


ans =

   -0.4161
```

Exercise 7

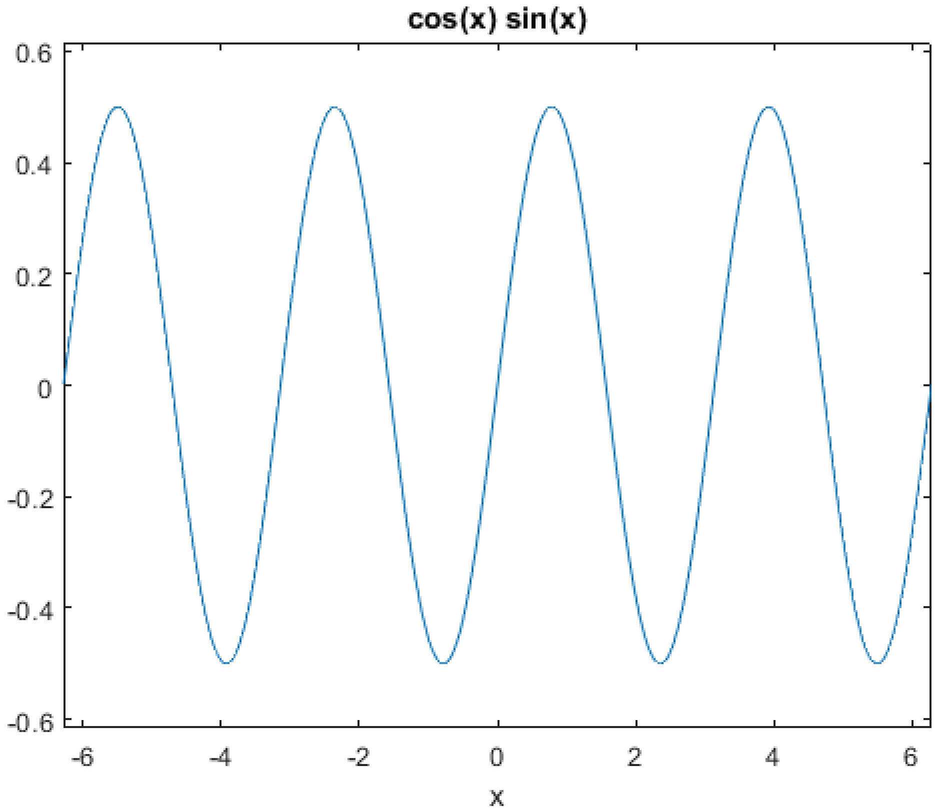
Objective: Define a function using symbolic variables and manipulate it.

Details: Define the function `f(x) = sin(x)cos(x)`

Use MATLAB commands to obtain a simpler form of this function, compute value of this function for `x=pi/4` and `x=1`, and plot its graph.

```
syms x % define variable as symbolic
f = sin(x)*cos(x); % define function by writing its expression
simplify(f) % simplify the function
subs(f,x,pi/4) % compute value for x=pi/4
eval(subs(f, x, pi/4))
subs(f,x,1) % compute value for x=1
eval(subs(f, x, 1))
ezplot(f); % plot the graph
```

```
ans =

sin(2*x)/2


ans =

1/2


ans =

    0.5000


ans =

cos(1)*sin(1)


ans =

    0.4546
```



Obtaining Picard approximations

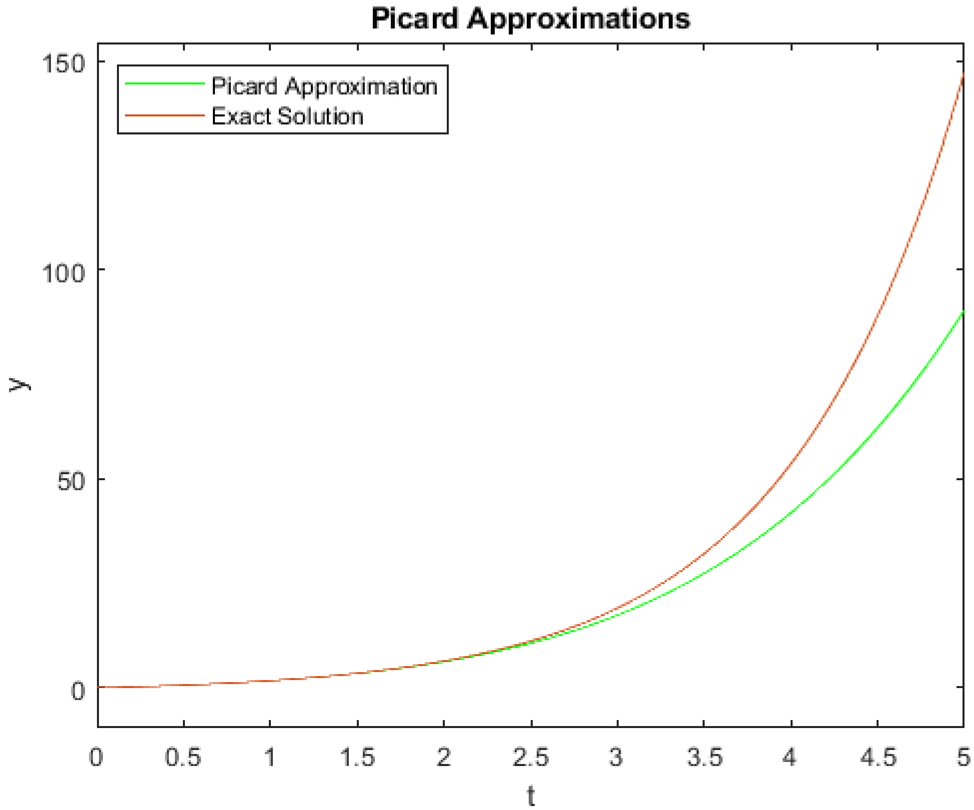
Consider an initial value problem

```
y' = 1 + y y(0) = 0
```

First we need to define the variables we will be using

```
syms t s y;

% We then need to define the function f

f = 1+y; % we define it without the @(t,y) because it is a symbolic
 function

% We set up our initial approximation phi_0 = 0:

phi=[sym(0)];   % we will keep a list with all the approximations

% Set up a loop to get successive approximations using Picard
 iterations

N=5;

for i = 1:N
```

```
    func=subs(f,y,phi(i));     % prepare function to integrate: y ->
 previous phi
    func=subs(func,t,s);       % variable of integration is s, so we
 need to change
                               % t -> s

    newphi = int(func, s, 0 ,t);    % integrate to find next
 approximation

    phi=cat(2,phi,[newphi]);        % update the list of approximations
 by adding new phi
end

% Show the last approximation

phi(N+1)

% Plot the approximation just found

picard=ezplot(phi(N+1),[0,5]);
set(picard,'Color','green');          % set the color of the graph to
 green

% In this case, the exact solution is
%
% |y=e^t-1|
%
% Compare the approximation and the exact solutions

hold on;
exact=ezplot(exp(t)-1,[0,5]);

xlabel('t');
ylabel('y');
title('Picard Approximations');
legend('Picard Approximation', 'Exact
 Solution','Location','NorthWest');


ans =

(t*(t^4 + 5*t^3 + 20*t^2 + 60*t + 120))/120
```



Exercise 8

```
Objective: Solve your own Picard Approximation and compare it to the
exact solution.

Details: Consider the IVP
   | y' = 1+y^2|
   | y(0) = 1 |
```

Find the Picard approximation phi_5. For better efficiency, do not keep all the previous approximations.

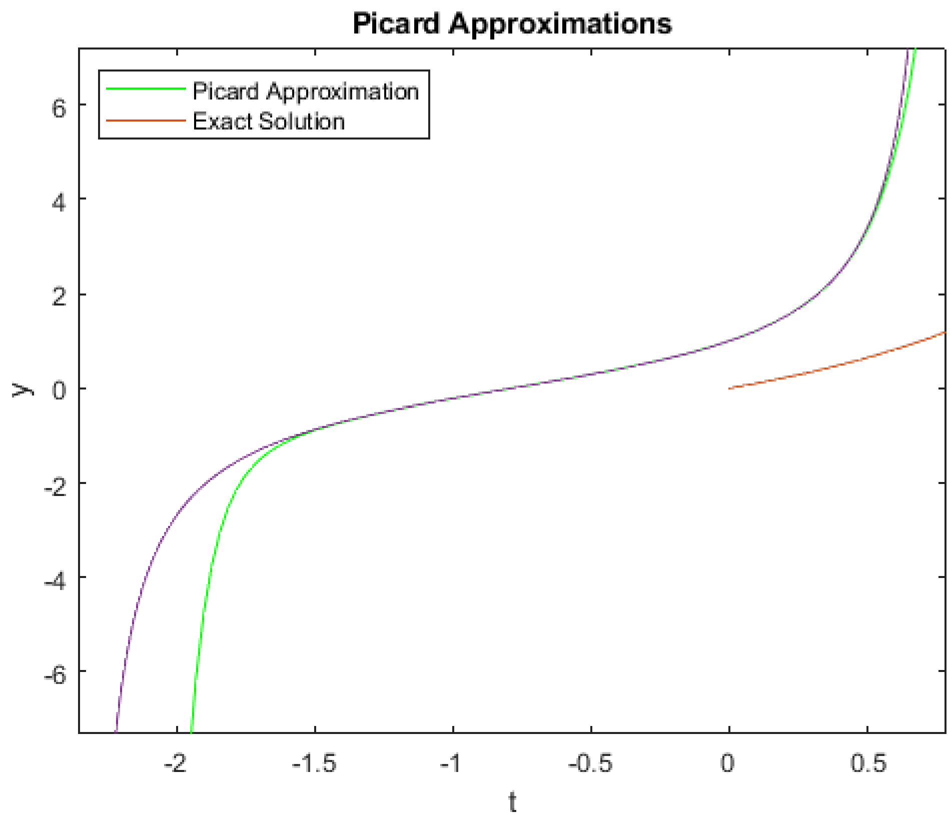
Compute the exact solution (by hand), and plot both on the same figure for comparison, as above.

Label your axes and include a legend.

HINT. The initial condition has 1 instead of 0, so the Picard method needs to be adapted.

```
syms t s y; %define variables we will be using
f = 1+y^2; %symbolic function for derivative
phi = sym(1); % current approximation
N = 5; % number of iterations
for i = 1:N
    func = subs(f, y, phi); % prepare function to integrate: y ->
 previous phi
    func = subs(func, t, s); % change t to s, the desired variable of
 integration
```

```
    phi = 1+int(func, s, 0, t); % integrate and update phi
end

% Plot using ezplot
picard=ezplot(phi);
set(picard,'Color','green');

%exact solution: y=tan(t+pi/4), -3pi/4<=t<=pi/4
hold on;
exact=ezplot(tan(t+pi/4), [-3*pi/4, pi/4]);

%plot details
xlabel('t');
ylabel('y');
title('Picard Approximations');
legend('Picard Approximation', 'Exact
 Solution','Location','NorthWest');
```



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