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ORIGINAL RESEARCH

Healthcare inventory management in the presence of supply disruptions and a reliable secondary supplier

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Abstract
We study the inventory review policy for a healthcare facility to minimize the impact of inevitable drug shortages. Usually, healthcare facilities do not rely on a single source of supply, and alternative mechanisms are present. When the primary supplier is not available, items are produced in-house or supplied through another supplier, albeit with additional cost. Our aim in this study is to determine how optimal inventory parameters are adjusted depending on the availability of the primary supplier. We show that an approximation provides trivial results, yet fails to capture the nuances therein. Our proposed Markov chain model overcomes these issues, and numerical results illustrate the significant economic impact of inventory parameter optimization. Furthermore, we simulate uncertainty scenarios and provide sensitivity analyses concerning fixed ordering cost for the secondary supplier, shortage frequency, shortage duration, and demand rates.

Keywords Healthcare supply chain · Inventory management · Supply disruption · Markov chain

1 Introduction

Supply chain management of a healthcare facility differs from commercial supply chains as healthcare facilities aim to improve patients' well-being and minimize associated risks. Medicine shortages, the lack of stock visibility between hospitals and suppliers, the non-delivery risk of medicines, unexpected peaks in demand, warehouse capacity issues, forecasting errors, and stock holding problems are some of these risks, which directly or indirectly impair the quality of care provided to patients. These risks also limit the adoption of empirical supply chain management studies in healthcare settings (Bialas et al., 2020). In addition, managing these items in the healthcare supply chain has massive costs. Medications

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are the second main source of expenditures in the hospital supply chain, making them crucial for the overall efficiency of hospitals (Gebicki et al., 2014). The total drug expenditures for all hospitals in the US in 2019 were \$507.9 billion, which is expected to increase by 4.0% annually (Tichy et al., 2020). Considering the total number of registered hospitals in the US (6,090 hospitals) and the average supply chain expense to total expense ratio of 15% (Abdulsalam & Schneller, 2019), an estimated total of \$12.5 million per hospital is spent annually on drug supplies.

Drug shortages are a major challenge for hospitals, compromising their ability to provide efficient care for patients. Most healthcare organizations are not immune to drug shortages. As of February 18, 2020, the US Food and Drug Administration (2020) reports 105 ongoing shortages and another 39 that were recently resolved in the United States US Food and Administration (2022). According to the United States Food and Drug Administration (FDA), drug shortages are an ongoing public health concern and pose a serious and growing threat to public health (Headquarters & Trusts, 2018; Urahn et al., 2017; US Food and Drug Administration, 2019). United States Department of Health and Human Services and the Department of Homeland Security have been urged by multiple medical societies to view the drug shortage crisis as a national security initiative (Solnick et al., 2019). Based on a report by Premier, 90% of hospitals experience at least one drug shortage in a 6-month period that affect patient safety, and 99% are forced to purchase a more expensive alternative (Gu et al., 2011). This suggests a need for a thorough analysis of the drug supply chain management and drug shortages.

Drug shortages occur in four general cases: (i) issues in manufacturing processes such as poor quality control (ii) extreme cases and disasters (iii) marketplace factors such as withdrawal of a competitor from the market and (iv) poor supply chain management practices such as inventory-related issues and demand planning (Headquarters & Trusts, 2018; Solnick et al., 2019; Urahn et al., 2017; US Government Accountability Office, 2016). The first three factors are directly controlled and regulated by the government; however, improved supply chain management both in manufacturer and hospital level can reduce the frequency of the shortages or their impact, in case the shortage occurs in supplier level. An estimated 48% of the supply chain costs in pharmaceutical industry can be avoided by improved management (Landry & Philippe, 2004), which translates to a total saving of \$36 billion annually.

The FDA focuses on initiating regulations that resulted in a significant reduction in oncology drug shortages since 2011 (Headquarters & Trusts, 2018). However, more than 72% of shortages are for low-margin generic drugs (Urahn et al., 2017; US Government Accountability Office, 2016). These drugs, which include blood pressure medications and parenteral nutrition, impact almost every area of hospitals and acute care centers and are the most troubling shortages resulting in near \$230 million additional annual costs per hospital because of the higher costs of substitute drugs (Headquarters & Trusts, 2018; Urahn et al., 2017; US Food and Drug Administration, 2019). Thus, the role of hospital inventory management becomes crucial to mitigate the impact of these shortages and reduce the costs and risks associated with them.

Hospital drug inventories are difficult to manage as each hospital manages a combination of drugs with different characteristics (demand, shortage frequency, etc.). Drug shortages and criticality level of drugs influence the performance of the system with respect to total cost and number of stock-outs and directly impacts the customer service level (availability of drug) and overall system efficiency (Dixit et al., 2019). In this paper, we propose a mathematical model for the inventory management of the hospitals considering their most important uncertainty, i.e., shortages, and their most important tool against shortage, i.e., substitute drug or supply alternatives. This new model considers the trade-off between *inventory holding cost* and *usage of the secondary supply source* as the latter affects secondary channel usage frequency when a disruption occurs. Considering the different characteristics of all types of drugs, including oncology and generic drugs, we consider three different models depending on the demand rate, shortage frequency, shortage duration as well as shortage cost associated with each type of drug. Under each condition, an exact stochastic formulation is presented and the optimal solutions, by minimizing the total cost of the hospital, for all realistic test cases are obtained. Finally, a simple and effective heuristic algorithm is proposed and the insights are provided through numerical results.

The remainder of this paper is organized as follows: Sect. 2 reviews relevant studies addressed the problem of inventory management in the presence of supply disruptions. Section 3 formally defines the problem and highlights our assumptions. Section 4 formulates the problem under different scenarios and presents alternative solution approaches. Section 5 validates our approach, presents experiments, simulation results. Finally, Sect. 6 provides concluding remarks and directions for future work on this class of problems.

2 Related literature

In the past decade, there have been several studies in the healthcare inventory management literature. These studies consider different aspects of inventory management in a healthcare system. The common goal of all these studies is to maximize the patient service level while keeping the associated costs as low as possible. This is, indeed, the most important goal of any healthcare problem, including healthcare inventory management, unlike commercial supply chains where usually cost minimization approaches are followed. Researchers could reach the maximum service level in two ways: (i) change the objective of the problem to service maximization, rather than cost minimization (ii) modify the model assumptions to ensure reaching a certain (or maximum) service level in the optimal solution. Sawik (2014) and Bozkir et al. (2022) optimize the inventory of a healthcare system by maximizing the service level subject to their problem-specific assumptions. An alternative way for service level maximization is to translate the associated risks of a stock-out situation into a cost and minimize the total cost of the system (Abu Zwaida et al., 2021; Gebicki et al., 2014). However, Saedi et al. (2016) assumes a minimum required service level while Lin et al. (2000) assumes a fixed service level in their model.

Frequent supply shortages are the second most important part of a healthcare inventory system that is inevitable (Azghandi et al., 2018; Gebicki et al., 2014). However, only a handful of the studies model the inventory management of hospitals in the presence of supply disruptions (shortages). Gebicki et al. (2014) simulate the inventory of a hospital with a central storage location, the main pharmacy along with some dispensing machines. They consider the total cost of the system along with the total number of stock-outs to evaluate the performance of different inventory policies for the described system. Azghandi et al. (2018) study a single-item inventory system with a manufacturer, distributor, wholesaler, and health care center. They simulate the system to obtain the optimal order quantities in a periodic review inventory model.

Saedi et al. (2016) use a continuous-time Markov chain, which initially was proposed by Arreola-Risa and DeCroix (1998) for a commercial setting, to minimize the inventory cost of a hospital with two suppliers. They consider a multi-drug inventory system with a minimum service level constraint where each drug has a more expensive alternative, and both regular and alternative supplies are open to disruptions. This results in a model where the lost demand could mathematically reach infinity. However, according to Premier's report (Premier Inc, 2022) on drug shortages, only 56% of hospital treatments are delayed due to drug shortages. This means that even *backlogging* is often not possible in a hospital inventory management setting, yet alone a lost demand. The demand often needs to be fulfilled immediately by various alternative sources, such as other nearby hospitals or local suppliers, and at any cost. If the demand for a particular drug could not be met with any of the above options, for a patient in a critical condition a less effective substitute drug would be physician's only choice. This was the case in 47% of the healthcare providers which according to Premier's report (Premier Inc, 2022), used less effective drugs in patient care due to drug shortages (Gu et al., 2011). Saedi et al. (2016) also assume no preference over primary or the substitute drug which results in higher-order quantities for the alternative drug. As discussed, although the substitute drug is inevitable in some cases, they might be less effective or pose additional risks to patients (US Food and Drug Administration, 2019). Thus, their order quantities should be limited.

Bozkir et al. (2022) consider a chain of hospitals where proactive shipments are allowed. They make inventory and sharing policy-related decisions while maximizing the service level in the considered healthcare system under uncertain demand and shortage occurrences. A slightly different problem is inventory management during disasters. Balcik et al. (2016) review the inventory management studies for a humanitarian disaster in detail. The key assumption in a humanitarian disaster inventory management system is that the shortages are due to demand explosion rather than supply disruptions. This results in different modeling assumptions between the two sets of problems. For instance, Paul and Venkateswaran (2018) study an inventory system during an epidemic in which as they state, the demand is a bell-shaped demand curve over time. Beamon and Kotleba (2006) focus on a case study of long-term humanitarian emergency relief operations by developing a stochastic inventory control model with two suppliers under a (Q, r) policy. An emergency order is never placed before a normal order and backlogging is allowed. Emergency orders are placed with a secondary supplier when there is a demand explosion. Furthermore, Azizi et al. (2021) work on inventory and transshipment decisions in a refugee camp network under exponentially distributed arrival and cycle times. They define transshipment thresholds for a reactive sharing policy and determine the amounts to be positioned at each location to minimize the expected costs incurred in the system due to sharing and holding inventory considering replenishment cycles of uncertain length.

3 Inventory management with supply disruptions and a reliable supply mechanism

In practice, when a national shortage or a supply disruption occurs for a drug, hospitals usually try and find the drug or a substitute through alternative channels, which might be another hospital or warehouse. This approach calls for additional effort and man-hours every time a secondary channel is sought. However, most of the healthcare facilities we observe in the United States are reactive, rather than proactive, in responding to drug shortages. Our aim in this study is to avoid excessive costs associated with conventional approaches and propose a proactive inventory management scheme in the presence of supply disruption.

We study a single-item continuously reviewed (Q, R) inventory control policy due to its frequent use in healthcare inventory control (see e.g., Priyan & Uthayakumar 2014; Roni et al., 2015, 2016; Uthayakumar & Priyan, 2013) with a primary supply channel that may

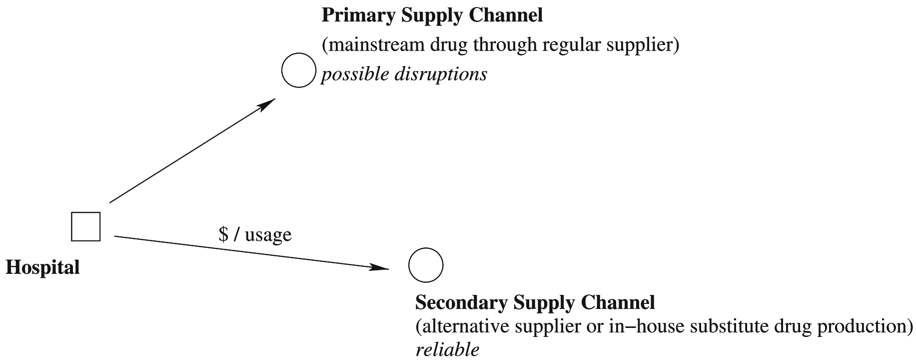


Fig. 1 A reliable secondary supply channel model

not always be available, and a secondary channel that can always be used for purchases yet is costly. In fact, this secondary channel can possibly represent any type of ordering mechanism that is inferior. This inferiority might be due to the alternative supplier or the alternative drug. Alternative drug-related additional costs occur because it might be more expensive or might not be medically preferred. Alternative supplier-related additional costs might be due to the effort spent in finding an alternative supplier or (in-house) production. We consolidate the cost of this inferior supply mechanism, and our approach can be used in any of these scenarios (see Fig. 1). The system simultaneously reduce the average inventory level and expected number of secondary channel searches during a supply disruption. The first term in our objective minimizes the *expected inventory holding cost*. The second term, though, increases the quality of care through *minimization of expected secondary supplier usage cost*. Unlike traditional inventory studies that consider the trade-off between inventory and ordering costs, we consider inventory holding versus secondary supplier usage. With the proposed replenishment policy, we are able to overcome the shortcomings of classical methodologies that do not consider uncertain shortages.

Note that, aside from the fixed cost associated with the secondary (reliable) channel usage, hospitals also aim to minimize holding costs. Healthcare facilities, particularly hospitals, have to bill patients (or insurance companies) based on their purchasing cost for drugs—without added profit. Therefore, the purchase cost of a drug is not a major concern in managing inventory for a hospital. On the other hand, loss of money due to tied-up capital during the time drugs are stored, and associated inventory risks cannot be billed to the patient. Therefore, in contrast to the purchase cost, holding costs should be considered for the inventory management of a hospital.

The real-life scenario in a hospital modeling problem requires consideration of stochastic demand (Saha & Ray, 2019). Thus, we consider demand for the item to arrive in accordance with a Poisson process having rate λ. Considering the stochastic nature of the shortages, supply disruptions on the primary channel assumed to occur according to a Poisson process with rate μ, and disruption periods are assumed to be distributed exponentially with rate α. We would like to clarify that in our model, during a disruption period, the primary focus is on the recovery process, which occurs in an exponentially distributed time. The occurrence of additional disruptions is not a consideration within this specific period of recovery.

Based on Abdulsalam and Schneller (2019) hospitals on average own up to 35,000 stock-keeping units worth of products, but only 6000–8000 stock-keeping units are available at the hospital. The rest of these items are being held by distributors or suppliers which make daily trips to hospital to replenish the required drugs. Thus, once the drug is available, hospital gets the item almost spontaneously which implies that the lead time is effectively zero (Nicholson et al., 2004; Saedi et al., 2016).

Under continuous-review inventory control policy, an order of size Q_1 will be placed through the primary channel when inventory level hits reorder level R_1, and the primary supplier is available. During a disruption period, an order of size Q_2 will be placed through the secondary supply channel when the inventory level hits reorder level R_2. When the primary supplier is unavailable, and the hospital has less than $Q_1 + R_1$ units, upon supplier's transition into the available state, it will immediately deliver units to the hospital so as to hit $Q_1 + R_1$. This assumption ensures that the supplier is *not* imposing the quantized ordering rule (Q_1 units per each order) when recovery from a shortage period occurs. Moreover, the hospital avoids the use of less preferred substitute drug. Note that an R_1 value of zero is feasible and expected as lead time is zero, and that reorder point would minimize the inventory cost alone. On the other hand, the optimal reorder point for the primary supplier might be greater than zero because that would reduce secondary supply channel usage during a disruption period. Despite the fact that multiple items share the same limited warehouse space in practice, we study an uncapacitated model in order to find the ideal inventory levels for each item, to understand the behavior of the model, and to derive meaningful insights. To sum up, our model tries to balance secondary supplier usage frequency and inventory holding cost. Next, we define the mathematical model using a continuous-time Markov chain that captures the aspects mentioned above.

4 Mathematical approaches

We model stochastic processes that consider two supply channels' reorder points and order quantities as well as the disruption status of the primary supply channel. It should be noted that the reorder point for secondary supply, R_2, would always be zero at optimality. This is because the secondary supply channel is always available and zero reorder point ensures the minimum stock level. Another important observation is that, because the lead times are zero, there is never more than one outstanding order. Note that the model represents a single-item inventory system.

4.1 Naïve approach

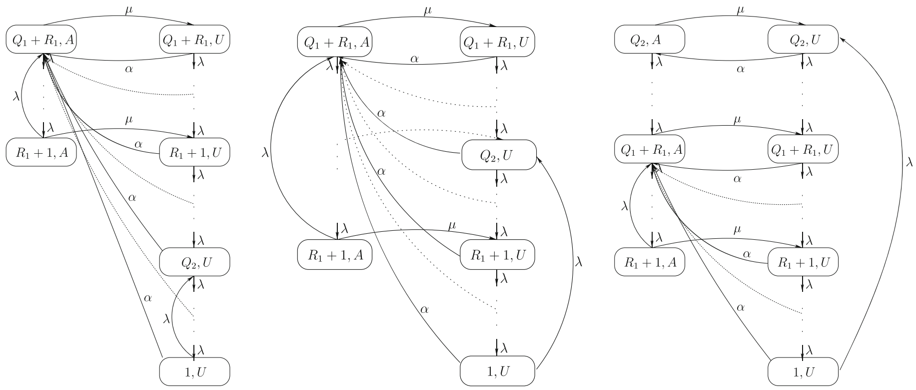
A simple method to evaluate the proposed system is to approximate the cost function considering the two states related to availability of the primary supply channel. The transition rate from available to unavailable state is μ and from unavailable to available is α. It is easy to show that the limiting probabilities of these states would be

$$P_{\text{Available}} = \frac{\alpha}{\alpha + \mu}, \tag{1}$$

$$P_{\text{Unavailable}} = \frac{\mu}{\alpha + \mu}. \tag{2}$$

When the system is in the available (unavailable) state the inventory level is between R_1 (0) and $Q_1 + R_1$ (Q_2). Therefore, the holding cost (HC) for the system can be approximated as

$$HC(Q_1; Q_2; R_1) \approx h\left[(R_1 + Q_1/2)P_{\text{Available}} + (Q_2/2)P_{\text{Unavailable}}\right]. \tag{3}$$



Case 1: $Q_2 < R_1$ **Case 2:** $R_1 \le Q_2 \le Q_1 + R_1$ **Case 3:** $Q_2 > Q_1 + R_1$

Fig. 2 Transition rate diagrams for the inventory model considering alternative intervals of Q_2

To approximate the ordering cost of the system (OC), we consider a scenario where the primary supply channel is unavailable and the system has only one item on hand. In this case, the primary supply channel is unavailable for a fraction of time, specifically $1/Q_2$. As a result, the rate of placing an order from the secondary channel can be calculated as λ/Q_2 multiplied by the limiting probability of being in the unavailable state. Thus, we compute the ordering cost of the system for an order quantity of Q_2 as

$$OC(Q_2) \approx \frac{K\lambda}{Q_2} P_{\text{Unavailable}}, \tag{4}$$

where K is the fixed ordering cost from the secondary supplier.

In sum, the expected total cost (TC) for the system can be approximated as

$$\text{TC}(Q_1; Q_2; R_1) \approx h(R_1 + Q_1/2)\frac{\alpha}{\alpha + \mu} + \left(\frac{K\lambda}{Q_2} + \frac{hQ_2}{2}\right)\frac{\mu}{\alpha + \mu}. \tag{5}$$

It is easy to see that $R_1 = 0$ and Q_1 should take the minimum possible value, which is 1. Taking the derivative of $\text{TC}(Q_1; Q_2; R_1)$ with respect to Q_2 and checking convexity, we see that the optimal value of Q_2 is equal to the economic order quantity (EOQ).

It should be noted that this approach fails to capture the difference between the primary supplier unavailability and the actual drug unavailability. Higher Q_1 and R_1 values help the system survive for a while, although the primary supplier is not available. This challenging calculation is what we address with the exact formulation in the next section.

4.2 Stochastic model and exact formulation

In the light of our assumptions, a more precise continuous-time Markov chain for given Q_1, Q_2, and R_1 can be modeled considering the interval that Q_2 falls. Figure 2 shows how replenishment quantity for the secondary supplier affects the Markov chain. States are denoted by pairs where the first value represents the number of drugs on hand, and the letter A or U indicates whether the primary supply channel is available or unavailable, respectively.

Next, we compute steady-state probabilities for these Markov chains. Using limiting probabilities for given Q_1, Q_2, and R_1, we calculate the expected inventory level $I(Q_1; Q_2; R_1)$ and the expected number of times the secondary supply channel is used $SF(Q_1; Q_2; R_1)$.

It should be noted that the derivation of limiting probabilities depends on the relationship between α and μ. Therefore, we analyze Markov chains under two separate conditions: (a) rare shortage with fast recovery ($\mu < \alpha$) and (b) frequent shortages with slow recovery ($\mu > \alpha$). For each condition that we need to compute, we have three cases to analyze; $Q_2 < R_1$ (Case 1), $R_1 \leq Q_2 \leq Q_1 + R_1$ (Case 2), and $Q_2 > Q_1 + R_1$ (Case 3). Detailed derivations of the limiting probabilities and expected inventory levels for the two conditions and three cases for each condition are presented in the "Appendix".

The objective of this model is to minimize the expected total cost that consists of inventory-related costs and ordering costs incurred by utilizing the secondary supply channel, i.e.,

$$\mathrm{TC}(Q_1; Q_2; R_1) = h \times I(Q_1; Q_2; R_1) + K \times SF(Q_1; Q_2; R_1),$$

where h is the annual holding cost per item, and K is the fixed ordering cost from the secondary supplier. The expected number of times that the secondary supply channel is used per year, $SF(Q_1; Q_2; R_1)$, is the same as finding the annual rate of traversing the arc $(1, U)$ to (Q_2, U) regardless of the case considered. This is equal to the limiting probability of node $(1, U)$ multiplied by the rate over the corresponding arc, i.e., $SF(Q_1; Q_2; R_1) = \lambda P_{1,U}$, and the expected total cost can be rewritten as

$$\mathrm{TC}(Q_1; Q_2; R_1) = h \times I(Q_1; Q_2; R_1) + K\lambda P_{1,U}. \tag{6}$$

Given input data, we first identify if this is a rare shortage with fast recovery or frequent shortages with slow recovery conditions. Next, we compute limiting probabilities and expected inventory levels for the three cases, as explained in the "Appendix". At this point, we do not know for certain, which Markov chain reflects the optimal strategy. We enumerate every single value of Q_1, Q_2 and R_1 from 1 to 12λ (annual demand). The expected total cost is computed using (6) for a combination of Q_1, Q_2 and R_1 depending on which of the three cases it fits. The minimum cost strategy is chosen for each case, and the overall best is chosen among the three as the one with minimum cost. Detailed calculations of limiting probabilities, expected inventory levels, and expected total costs for each of the cases are provided in the "Appendix".

4.3 Heuristic algorithm

The cost function is quite complex, and it is virtually impossible to work with the Jacobian and Hessian matrix to find critical points and prove the convexity of the function. This complexity makes finding the optimal solution computationally challenging. Therefore, we enumerate all possible decision variable values in a reasonable range to find the best solutions for the model. We find the optimal value for Q_1, Q_2, and R_1, using a modified bisection method. Fixing two variables at their current best values, we solve the single-variable optimization problem by iteratively updating the interval of interest. At each iteration, we choose the endpoint with lower cost and midpoint of the current interval. In order not to get stuck at a local optimum and increase the efficiency of search, the algorithm starts with larger step size and gradually decreases the step size at each iteration. The decreased pace of the step size can be regulated by changing the parameter p. Algorithm 1 presents the details of our heuristic approach.

Algorithm 1 Heuristic Algorithm

procedure HEURISTIC
  Get the initial solution, stepsize (a), and parameter p
  Calculate the cost function for initial solution, $TC = TC(Q_1, Q_2, R_1)$
  while $a \geq 1$ **do**
    while $TC \neq TC_{updated}$ **do**
      $TC_{updated} = \min(TC(Q_1 + a, Q_2, R_1), TC(Q_1, Q_2, R_1), TC(Q_1 - a, Q_2, R_1))$
      if $TC_{updated} = TC(Q_1 + a, Q_2, R_1)$
      $Q_1 \leftarrow Q_1 + a$
      elseif $TC_{updated} = TC(Q_1 - a, Q_2, R_1)$
       $Q_1 \leftarrow Q_1 - a$
    while $TC \neq TC_{updated}$ **do**
      $TC_{updated} = \min(TC(Q_1, Q_2 + a, R_1), TC(Q_1, Q_2, R_1), TC(Q_1, Q_2 - a, R_1))$
      if $TC_{updated} = TC(Q_1, Q_2 + a, R_1)$
      $Q_2 \leftarrow Q_2 + a$
      elseif $TC_{updated} = TC(Q_1, Q_2 - a, R_1)$
      $Q_2 \leftarrow Q_2 - a$
    while $TC \neq TC_{updated}$ **do**
      $TC_{updated} = \min(TC(Q_1, Q_2, R_1 + a), TC(Q_1, Q_2, R_1), TC(Q_1, Q_2, R_1 - a))$
      if $TC_{updated} = TC(Q_1, Q_2, R_1 + a)$
      $R_1 \leftarrow R_1 + a$
      elseif $TC_{updated} = TC(Q_1, Q_2, R_1 - a)$
      $R_1 \leftarrow R_1 - a$
    $round(a \leftarrow a \times p)$

5 Numerical analysis

In this section, we present our data generation mechanism and optimal and near-optimal strategies over the data generated. With this analysis, we understand how optimal strategy and optimal stock levels behave with respect to different problem parameters. We also highlight the difference in the total cost and optimal stock levels between exact and approximate approaches. Furthermore, we consider cases where the system does not possess Markovian property, i.e., shortages are not observed as Poisson processes, and shortage durations are not exponentially distributed. In these cases, providing analytical results becomes virtually impossible. Nonetheless, we use simulation to provide a detailed analysis of how the system behaves in practice with respect to different distributions and parameters.

The hospital system we consider in this study has 17 outpatient facilities and two hospitals, and the inpatient (outpatient) holding cost is around \$5.6M (\$2.3M). This constitutes around 10% of the total budget, which clearly demonstrates the need for healthcare-specific inventory studies that aim to increase efficiency, amid real-world challenges such as shortages. We generate test problems using combinations of each of the parameters shown in Table 1, a total of 72 instances. The rates used in this study are estimated based on the real data that is partially available in Saedi et al. (2016) related to the frequency of shortages, speed of recovery, rate of demand, and fixed/holding cost ratios observed in critical drugs.

5.1 Evaluation of methods

We first solve our proposed model using the exact approach presented in Sect. 4.2. Table 2 presents the minimum cost solution for each of the three possible cases (i.e., $Q_2 < R_1$,

Table 1 Annual rates used in numerical study

Recovery rate (α)	Demand rate (λ)	Shortage rate (μ)	Fixed/annual holding cost
Slow—12	Low—144	Rare—1	Low—10
(lasts a months on avg.)	Medium—720	(once a year on avg.)	Medium—100
Fast—36	High—3600	Medium—9	High—1000
(lasts 10 days on avg.)	(once every 40 days on avg.)	Colossal—10,000	
		Frequent—27	
		(once every 2 weeks on avg.)	

$R_1 \leq Q_2 \leq Q_1 + R_1$, $Q_2 > Q_1 + R_1$). The case with the best solution is highlighted for each instance.

It is observed that in the majority of cases, either Case 2 or Case 3 gives the best result. Last column in Table 2 shows the percentage gap between objective function values of Cases 2 and 3. It can be seen that this gap becomes insignificant when the ratio of fixed to holding cost is large. As the ratio of fixed to holding cost increases, the gap decreases and Case 2 tends to perform better.

Remark 1 It is unlikely for Case 1 to be the best strategy. Cases 2 and 3 should be considered separately to decide on the best strategy. As the fixed to holding cost ratio increases, the inventory system tends to hedge against the risk of stock-out by utilizing both $Q_1 + R_1$ and Q_2, whereas with a small ratio of fixed to holding cost, Q_2 is used instead of keeping a large inventory for the whole time horizon.

When the fixed cost gets larger compared to holding cost, one might naturally expect the model to order based on the EOQ during a shortage. However, unexpectedly, there are drastic differences between order quantities and total costs of the exact solution and naïve approach in Sect. 4.1, which essentially orders in EOQ. Table 3 shows these results together with the heuristic solution presented in Sect. 4.3.

We observe that the heuristic solution is close to optimality, with an average gap of 1.1%. In fact, except for only one instance ($\alpha = 12, \lambda = 3,600, \mu = 9, K/h = 10{,}000$), the heuristic performed exceptionally well. We also present a traditional approach where daily demand is to be kept in stock, and orders are placed to *cover* the demand of a period, which is adopted by many hospitals (see Saedi et al., 2016). We modified this traditional approach by covering three days during availability (i.e., $Q_1 = \frac{3}{360}\lambda$). Furthermore, order quantity for the secondary channel during a shortage of mainstream channel is assumed to be $Q_2 = \frac{\mu}{\mu+\alpha} \times \text{EOQ}$, which converges to EOQ when the recovery rate is slow (i.e., $\alpha \to 0$) and to 0 when the recovery rate is faster (i.e., $\alpha \to \infty$). We did not report the optimality gap for naïve or traditional approaches as the heuristic method consistently outperforms these methods.

Remark 2 Traditional and EOQ-driven approaches perform quite poor compared to our proposed exact and heuristic approaches.

Table 2 Results of mathematical model (TC represents annual cost)

K/h	μ	λ	α	Case 1 ($Q_2 < R_1$)				Case 2($R_1 \le Q_2 \le Q_1 + R_1$)				Case 3 ($Q_2 \ge Q_1 + R_1$)				$\frac{TC_2 - TC_3}{TC_3} \times 100$
				TC	Q_1	Q_2	R_1	TC	Q_1	Q_2	R_1	TC	Q_1	Q_2	R_1	
10	1	144	12	18.024	1	10	11	15.447	14	14	0	**11.900**	1	30	0	29.8
			36	9.969	1	5	6	9.251	3	6	4	**8.193**	1	18	3	12.9
		720	12	43.393	1	23	24	33.865	33	33	0	**17.025**	1	89	0	98.9
			36	24.784	1	13	14	20.262	19	19	0	**11.991**	1	60	0	69.0
		3600	12	100.459	1	53	54	75.023	74	74	0	**28.149**	1	233	0	166.5
			36	58.967	1	31	32	44.647	44	44	0	**15.704**	1	185	0	184.3
	9	144	12	29.620	1	22	23	29.400	5	24	20	**28.987**	1	32	21	1.4
			36	16.663	1	11	12	16.471	3	13	11	**16.431**	1	19	12	0.2
		720	12	80.164	1	57	58	77.597	44	69	26	**74.274**	1	90	38	4.5
			36	48.804	1	34	35	48.147	14	39	26	**46.536**	1	62	29	3.5
		3600	12	199.844	1	136	137	177.916	170	170	0	**151.869**	1	233	0	17.2
			36	132.609	1	85	86	121.143	116	116	0	**106.067**	1	186	24	14.2
	27	144	12	34.281	18	16	17	**32.768**	17	32	16	33.978	33	36	1	− 3.6
			36	19.005	1	14	15	18.998	3	16	14	**18.924**	1	20	15	0.4
		720	12	96.2198	53	51	52	**88.198**	1	73	73	95.9115	65	91	22	− 8.0
			36	58.141	1	46	47	57.994	5	48	44	**57.573**	1	62	45	0.7
		3600	12	247.907	142	140	141	**220.761**	126	207	81	232.718	52	234	51	− 5.1
			36	166.671	1	124	125	164.101	60	143	84	**159.738**	1	187	101	2.7

Table 2 continued

K/h	μ	λ	α	Case 1 ($Q_2 < R_1$)				Case 2($R_1 \leq Q_2 \leq Q_1 + R_1$)				Case 3 ($Q_2 \geq Q_1 + R_1$)				$\frac{TC_2 - TC_3}{TC_3} \times 100$
				TC	Q_1	Q_2	R_1	TC	Q_1	Q_2	R_1	TC	Q_1	Q_2	R_1	
100	1	144	12	38.535	1	26	27	38.074	4	27	25	**37.286**	1	56	25	2.1
			36	18.502	1	12	13	**18.150**	5	13	11	18.217	1	28	13	− 0.4
		720	12	110.730	1	70	71	105.028	75	95	21	**93.084**	1	190	36	12.8
			36	56.005	1	37	38	55.279	16	44	29	**52.618**	1	104	32	5.1
		3600	12	286.466	1	167	168	236.521	233	233	0	**137.706**	1	580	0	71.8
			36	156.512	1	95	96	140.135	138	138	0	**107.054**	1	369	8	30.9
	9	144	12	54.603	1	45	46	54.570	3	47	45	**54.506**	1	57	46	0.1
			36	26.515	1	21	22	**26.412**	4	22	20	26.491	1	29	22	− 0.3
		720	12	177.108	1	144	145	176.868	16	151	136	**175.822**	1	192	140	0.6
			36	90.468	1	72	73	90.206	5	73	71	**90.157**	1	105	73	0.1
		3600	12	517.772	1	391	392	511.131	171	447	277	**498.982**	1	581	326	2.4
			36	296.435	1	223	224	295.452	42	241	200	**290.794**	1	371	209	1.6
	27	144	12	59.6899	31	29	30	**56.631**	1	50	50	59.4352	59	62	1	− 4.7
			36	29.002	1	24	25	29.041	3	26	24	**28.993**	1	30	25	0.2
		720	12	199.51	105	103	104	**188.752**	1	169	169	198.857	203	206	1	− 5.1
			36	101.278	1	88	89	101.250	4	89	87	**101.204**	1	106	88	0.0
		3600	12	609.07	328	326	327	**566.863**	1	490	490	610.927	613	616	1	− 7.2
			36	345.764	1	289	290	345.608	16	297	282	**344.522**	1	371	286	0.3

Table 2 continued

K/h	μ	λ	α	Case 1 ($Q_2 < R_1$)				Case 2($R_1 \le Q_2 \le Q_1 + R_1$)				Case 3 ($Q_2 \ge Q_1 + R_1$)				$\frac{TC_2 - TC_3}{TC_3} \times 100$
				TC	Q_1	Q_2	R_1	TC	Q_1	Q_2	R_1	TC	Q_1	Q_2	R_1	
1000	1	144	12	65.589	1	51	53	**65.172**	6	52	50	65.428	1	84	53	− 0.4
			36	28.532	1	21	23	**28.225**	5	23	21	28.493	1	38	23	− 0.9
		720	12	225.456	1	170	172	224.907	6	171	169	**222.654**	1	320	166	1.0
			36	99.603	1	77	79	**99.044**	7	78	76	99.175	1	150	78	− 0.1
		3600	12	701.149	1	477	479	688.636	262	577	317	**648.817**	1	1140	370	6.1
			36	336.293	1	244	246	335.257	44	264	222	**327.881**	1	590	229	2.2
	9	144	12	82.679	1	73	74	82.674	3	75	73	**82.665**	1	85	74	0.0
			36	36.772	1	31	32	**36.692**	4	32	30	36.769	1	39	32	− 0.2
		720	12	305.619	1	269	270	305.579	4	271	269	**305.463**	1	321	270	0.0
			36	136.769	1	118	119	**136.515**	7	118	116	136.735	1	152	119	− 0.2
		3600	12	1062.684	1	897	898	1062.341	46	919	874	**1059.434**	1	1141	886	0.3
			36	512.391	1	431	432	512.065	9	430	428	**511.719**	1	592	430	0.2
	27	144	12	87.6747	45	43	44	87.5658	45	88	44	**87.522**	88	91	1	0.0
			36	39.297	1	35	36	39.338	3	36	34	**39.296**	1	40	36	0.1
		720	12	328.903	169	167	168	**326.195**	167	332	166	327.816	334	337	1	− 0.5
			36	147.791	1	134	135	**147.779**	3	136	134	147.784	1	153	135	0.0
		3600	12	1175.836	611	609	610	**1151.752**	589	1176	588	1171.364	1206	1209	1	− 1.7
			36	565.596	1	506	507	565.553	5	507	505	**565.455**	1	592	507	0.0

Table 2 continued

K/h	μ	λ	α	Case 1 ($Q_2 < R_1$)				Case 2($R_1 \le Q_2 \le Q_1 + R_1$)				Case 3 ($Q_2 \ge Q_1 + R_1$)				$\frac{TC_2 - TC_3}{TC_3} \times 100$
				TC	Q_1	Q_2	R_1	TC	Q_1	Q_2	R_1	TC	Q_1	Q_2	R_1	
10,000	1	144	12	94.126	1	81	82	**93.722**	7	80	78	94.112	1	113	82	−0.4
			36	38.830	1	32	33	**38.516**	5	33	31	38.826	1	48	33	−0.8
		720	12	360.610	1	303	304	**360.064**	10	301	299	360.315	1	457	303	−0.1
			36	146.325	1	124	125	**145.766**	7	124	122	146.281	1	197	125	−0.4
		3600	12	1318.660	1	1048	1049	1318.110	10	1046	1044	**1311.820**	1	1803	1033	0.5
			36	558.534	1	460	461	557.857	11	457	455	**557.617**	1	819	459	0.0
	9	144	12	111.344	1	102	103	**111.343**	3	104	102	**111.343**	1	114	103	0.0
			36	47.101	1	41	42	**47.000**	4	43	41	47.101	1	50	42	−0.2
		720	12	443.147	1	406	407	**443.112**	5	407	405	443.130	1	459	407	0.0
			36	183.848	1	165	166	**183.592**	7	165	163	183.844	1	199	166	−0.1
		3600	12	1318.660	1	1048	1049	1318.110	10	1046	1044	**1311.820**	1	1803	1033	0.5
			36	741.535	1	659	660	**741.191**	10	657	655	741.466	1	821	660	0.0
	27	144	12	116.257	59	57	58	116.375	60	117	58	**116.186**	116	119	1	0.2
			36	**49.587**	1	45	46	49.639	3	47	45	**49.587**	1	50	46	0.1
		720	12	465.88	237	235	236	**465.00**	237	472	236	465.256	472	475	1	−0.1
			36	194.891	1	181	182	**194.806**	1	182	182	194.891	1	200	182	0.0
		3600	12	1836.370	940	938	939	**1812.912**	554	1848	1295	1831.570	1871	1874	1	−0.3
			36	795.218	1	736	737	**795.176**	6	736	734	795.204	1	822	737	0.0

Bold values are the minimum total cost achieved among the three cases for each parameter combination

Table 3 Results of all methods

K/h	μ	λ	α	Optimal				Naïve method				Traditional coverage ordering				Heuristic				
				TC	Q_1	Q_2	R_1	TC	Q_1	Q_2 (EOQ)	R_1	TC	Q_1 $(\frac{3\lambda}{360})$	Q_2 $(\frac{\mu}{\mu+\alpha}$ EOQ)	R_1 $(\frac{\lambda}{360})$	TC	Q_1	Q_2	R_1	Opt. gap
10	1	144	12	11.900	1	30	0	13.029	1	53	0	30.168	2	4	1	11.999	1	29	1	0.8
			36	8.193	1	18	3	9.904	1	53	0	23.828	2	1	1	8.414	5	19	1	2.7
		720	12	17.025	1	89	0	18.284	1	120	0	66.476	6	9	2	17.687	1	88	1	3.9
			36	11.991	1	60	0	13.676	1	120	0	58.128	6	3	2	12.442	1	59	1	3.8
		3600	12	28.149	1	233	0	29.237	1	268	0	149.892	30	21	10	28.982	1	232	1	3.0
			36	15.704	1	185	0	17.309	1	268	0	137.605	30	7	10	16.531	1	184	1	5.3
	9	144	12	28.987	1	32	21	59.628	1	53	0	54.760	2	23	1	29.040	4	33	19	0.2
			36	16.431	1	19	12	52.571	1	53	0	46.117	2	11	1	16.522	4	13	11	0.6
		720	12	74.274	1	90	38	91.462	1	120	0	96.532	6	51	2	74.285	3	89	37	0.0
			36	46.536	1	62	29	85.194	1	120	0	85.941	6	24	2	46.550	2	63	29	0.0
		3600	12	151.869	1	233	0	153.348	1	268	0	187.823	30	115	10	151.943	1	232	1	0.0
			36	106.067	1	186	24	114.606	1	268	0	160.425	30	54	10	106.073	3	187	24	0.0
	27	144	12	32.768	17	32	16	93.606	1	53	0	84.098	2	37	1	33.162	15	36	22	1.2
			36	18.924	1	20	15	104.838	1	53	0	86.009	2	23	1	19.270	6	21	12	1.8
		720	12	88.198	1	73	73	146.079	1	120	0	135.527	6	83	2	88.790	30	79	50	0.7
			36	57.573	1	62	45	176.985	1	120	0	140.149	6	51	2	57.579	2	61	44	0.0
		3600	12	220.761	126	207	81	246.450	1	268	0	244.259	30	186	10	221.545	168	216	50	0.4
			36	159.738	1	187	101	242.120	1	268	0	213.833	30	115	10	159.748	4	188	99	0.0

Table 3 continued

K/h	μ	λ	α	Optimal				Naïve method				Traditional coverage ordering				Heuristic				
				TC	Q_1	Q_2	R_1	TC	Q_1	Q_2 (EOQ)	R_1	TC	Q_1 ($\frac{3\lambda}{360}$)	Q_2 ($\frac{\mu}{\mu+\alpha}$ EOQ)	R_1 ($\frac{\lambda}{360}$)	TC	Q_1	Q_2	R_1	Opt. gap
100	1	144	12	37.286	1	56	25	91.569	1	169	0	118.792	2	13	1	37.373	3	55	25	0.2
			36	18.150	5	13	11	70.145	1	169	0	85.748	2	5	1	18.436	5	29	11	1.6
		720	12	93.084	1	190	36	115.917	1	379	0	227.274	6	29	2	93.090	3	189	35	0.0
			36	52.618	1	104	32	101.127	1	379	0	198.817	6	10	2	52.674	3	105	32	0.1
		3600	12	137.706	1	580	0	145.632	1	848	0	462.095	30	65	10	138.175	1	581	1	0.3
			36	107.054	1	369	8	117.977	1	848	0	393.672	30	23	10	107.058	3	370	7	0.0
	9	144	12	54.506	1	57	46	474.459	1	169	0	421.062	2	73	1	54.667	7	56	43	0.3
			36	26.412	4	22	20	444.020	1	169	0	380.944	2	34	1	26.460	5	22	20	0.2
		720	12	175.822	1	192	140	627.164	1	379	0	545.547	6	163	2	175.823	1	193	140	0.0
			36	90.157	1	105	73	702.914	1	379	0	566.231	6	76	2	90.227	6	104	70	0.1
		3600	12	498.982	1	581	326	799.773	1	848	0	772.412	30	364	10	498.985	4	580	324	0.0
			36	290.794	1	371	209	852.089	1	848	0	722.065	30	170	10	290.800	4	372	208	0.0
	27	144	12	56.631	1	50	50	755.631	1	169	0	695.729	2	117	1	57.450	5	49	50	1.4
			36	28.993	1	30	25	918.556	1	169	0	799.126	2	73	1	29.041	3	26	24	0.2
		720	12	188.752	1	169	169	1009.166	1	379	0	890.285	6	263	2	191.880	37	200	164	1.7
			36	101.204	1	106	88	1489.311	1	379	0	1188.912	6	163	2	101.220	3	105	87	0.0
		3600	12	566.863	1	490	490	1289.901	1	848	0	1143.615	30	587	10	567.360	92	507	416	0.1
			36	344.522	1	371	286	1817.641	1	848	0	1327.203	30	364	10	344.532	5	370	284	0.0

Table 3 continued

K/h	μ	λ	α	Optimal				Naïve method				Traditional coverage ordering				Heuristic				
				TC	Q_1	Q_2	R_1	TC	Q_1	Q_2 (EOQ)	R_1	TC	Q_1 $(\frac{3\lambda}{360})$	Q_2 $(\frac{\mu}{\mu+\alpha}$ EOQ)	R_1 $(\frac{\lambda}{360})$	TC	Q_1	Q_2	R_1	Opt. gap
1000	1	144	12	65.172	6	52	50	798.021	1	536	0	776.012	2	41	1	65.558	4	85	50	0.6
			36	28.225	5	23	21	608.730	1	536	0	576.671	2	15	1	28.738	3	37	21	1.8
		720	12	222.654	1	320	166	977.373	1	1200	0	1085.007	6	92	2	222.659	1	321	165	0.0
			36	99.044	7	78	76	908.658	1	1200	0	948.401	6	32	2	99.191	3	149	77	0.1
		3600	12	648.817	1	1140	370	1102.016	1	2683	0	1740.684	30	206	10	648.817	1	1141	371	0.0
			36	327.881	1	590	229	1026.978	1	2683	0	1492.657	30	73	10	327.891	5	591	227	0.0
	9	144	12	82.665	1	85	74	4310.504	1	536	0	4030.994	2	230	1	82.735	6	74	72	0.1
			36	36.692	4	32	30	4176.625	1	536	0	3703.782	2	107	1	36.838	2	40	31	0.4
		720	12	305.463	1	321	270	5381.168	1	1200	0	4810.793	6	514	2	305.466	1	320	269	0.0
			36	136.515	7	118	116	6510.956	1	1200	0	5370.204	6	240	2	136.766	4	153	118	0.2
		3600	12	1059.434	1	1141	886	6112.714	1	2683	0	5175.565	30	1150	10	1059.436	3	1140	885	0.0
			36	511.719	1	592	430	7515.899	1	2683	0	5681.646	30	537	10	511.725	3	593	430	0.0
	27	144	12	87.522	88	91	1	6899.199	1	536	0	6533.637	2	372	1	88.829	29	93	67	1.5
			36	39.296	1	40	36	8761.875	1	536	0	7822.720	2	230	1	40.382	12	31	29	2.8
		720	12	326.195	167	332	166	8678.230	1	1200	0	7953.846	6	831	2	328.349	295	326	33	0.7
			36	147.779	3	136	134	13884.342	1	1200	0	11525.380	6	514	2	147.833	5	154	133	0.0
		3600	12	1151.752	589	1176	588	9868.921	1	2683	0	8620.935	30	1858	10	1168.371	1156	1244	89	1.4
			36	565.455	1	592	507	16078.747	1	2683	0	12227.300	30	1150	10	565.456	2	591	506	0.0

Table 3 continued

K/h	μ	λ	α	Optimal				Naïve method				Traditional coverage ordering				Heuristic				
				TC	Q_1	Q_2	R_1	TC	Q_1	Q_2 (EOQ)	R_1	TC	Q_1 ($\frac{3\lambda}{360}$)	Q_2 ($\frac{\mu}{\mu+\alpha}$ EOQ)	R_1 ($\frac{\lambda}{360}$)	TC	Q_1	Q_2	R_1	Opt. gap
10,000	1	144	12	93.722	7	80	78	7580.259	1	1697	0	7289.452	2	131	1	93.936	11	77	75	0.2
			36	38.516	5	33	31	5720.856	1	1697	0	5433.564	2	46	1	38.570	6	33	31	0.1
		720	12	360.064	10	301	299	9115.933	1	3794	0	8473.909	6	292	2	360.336	3	456	301	0.1
			36	145.766	7	124	122	8680.001	1	3794	0	7480.756	6	103	2	146.290	1	197	126	0.4
		3600	12	1311.820	1	1803	1033	9785.350	1	8485	0	9620.138	30	653	10	1311.821	1	1804	1033	0.0
			36	557.617	1	819	459	9727.953	1	8485	0	8463.254	30	229	10	557.617	1	818	459	0.0
	9	144	12	111.343	3	104	102	41650.940	1	1697	0	39724.670	2	727	1	112.425	18	94	92	1.0
			36	47.000	4	43	41	40917.682	1	1697	0	36876.688	2	339	1	47.828	8	51	38	1.8
		720	12	443.112	5	407	405	50553.536	1	3794	0	46879.111	6	1626	2	443.237	12	401	399	0.0
			36	183.592	7	165	163	63382.352	1	3794	0	53321.618	6	759	2	183.724	5	168	165	0.1
		3600	12	1311.820	1	1803	1033	54419.998	1	8485	0	48425.541	30	3637	10	1722.487	9	1806	1545	31.3
			36	741.191	10	657	655	71609.346	1	8485	0	55822.771	30	1697	10	741.203	14	655	653	0.0
	27	144	12	116.186	116	119	1	66781.439	1	1697	0	63873.728	2	1175	1	116.273	88	119	28	0.1
			36	49.587	1	50	46	86265.197	1	1697	0	77698.994	2	727	1	49.844	5	47	43	0.5
		720	12	465.000	237	472	236	81597.174	1	3794	0	76235.084	6	2627	2	465.405	448	471	21	0.1
			36	194.806	1	182	182	135504.266	1	3794	0	114051.963	6	1626	2	194.943	7	180	178	0.1
		3600	12	1812.912	554	1848	1295	87890.835	1	8485	0	79534.583	30	5874	10	1834.646	1754	1824	71	1.2
			36	795.176	6	736	734	153351.793	1	8485	0	119779.739	30	3637	10	795.270	14	729	727	0.0

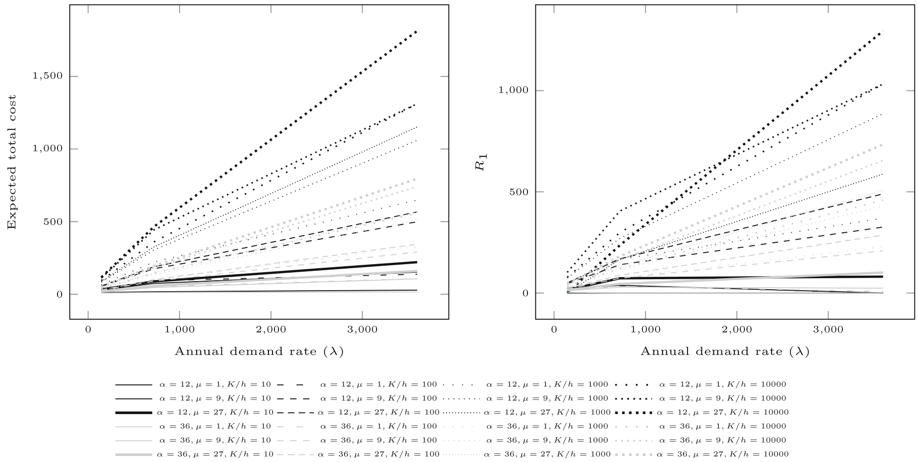


Fig. 3 Expected total cost and R_1, respectively, with respect to annual demand rate (λ)

5.2 Sensitivity analysis

In this section, we try to capture the behavior of optimal stock levels. Clearly, order quantities behave somewhat uniquely, unlike EOQ. Our aim is to identify how order quantities and reorder point behave with respect to different parameters and provide managerial insights.

Remark 3 As demand rate increases, expected total cost increases except for the case with rare shortages and fast recovery (see Fig. 3). With demand rate, Q_2 decreases (except for the case with frequent shortages and fast recovery), R_1 increases (except when shortages are not frequent and fixed cost is not large), as shown in Table 3.

According to changes in shortage occurrence, recovery rate, demand, and costs, our findings from Table 3 can be summarized as follows:

Remark 4 As recovery rate increases, expected total cost decreases, Q_2 decreases, R_1 usually decreases.

Remark 5 In all fast recovery cases, Q_1 is below daily demand, practically orders are placed every day. In slow recovery cases, Q_1 only becomes an important factor when shortages are frequent.

Although decision variables increase with demand as expected, it is difficult to claim that it possesses a special structure. There are multiple factors that affect the quantities, and there are instances where all quantities do not increase simultaneously with increased fixed to holding cost ratio. Next, we investigate the behavior of all decision variables in detail with respect to this ratio. Figures 4 and 5 presents how Q_1/λ and Q_2/λ changes with respect to fixed to holding cost ratio, respectively.

Based on these figures, we do not see a trend at Q_1. This may be because Q_1 does not have a dominant effect on the expected total cost, hence the haphazardness. On the other hand, we can derive meaningful results for the order quantity from the alternative channel.

Remark 6 In spite of exceptions, secondary channel order quantity in days of demand (i.e., Q_2/λ) increase proportional to the logarithm of K/h in a majority of test instances.

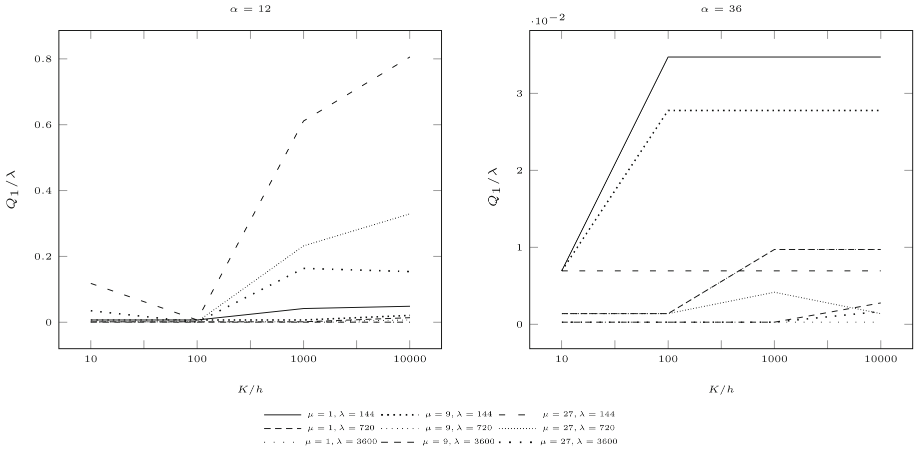


Fig. 4 Change in primary channel order quantity per annual demand (coverage in years, i.e., Q_1/λ) with respect to fixed to holding cost ratio, i.e., K/h

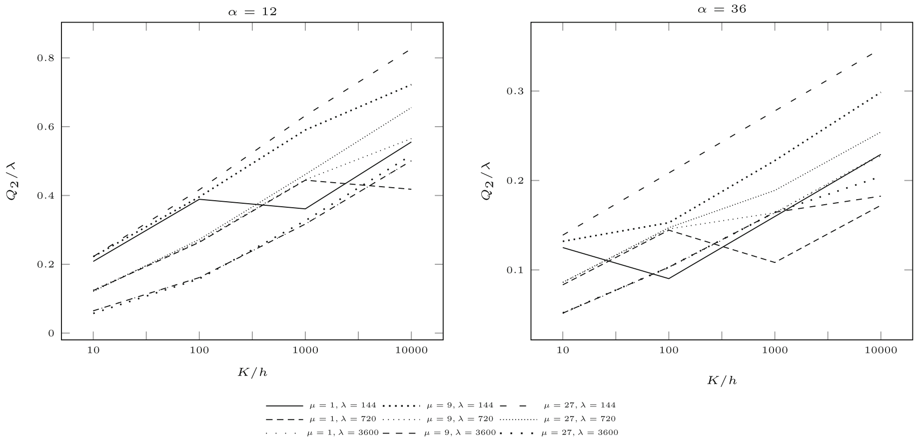


Fig. 5 Change in secondary channel order quantity per annual demand, i.e., Q_2/λ with respect to fixed to holding cost ratio, i.e., K/h

We present a deeper analysis on Q_2. First, we highlight the limits of Q_2, which is expected and can be observed easily from our analysis.

Remark 7 $Q_2 \rightarrow$ EOQ when recovery rate is slow (i.e., $\alpha \rightarrow 0$) and to $Q_2 \rightarrow 0$ when recovery rate is faster (i.e., $\alpha \rightarrow \infty$), hence the proposed secondary channel order quantity for the traditional approach.

It may be more interesting to focus on Q_2/EOQ. Therefore we present the progression of Q_2/EOQ with respect to problem inputs defining shortage patters, i.e., μ/α, in Fig. 6.

From Fig. 6, we have a couple of critical observations. First, smaller K/h or larger λ values consistently lead to larger Q_2/EOQ values.

Remark 8 Q_2 approaches EOQ when λ is increased or K/h is decreased.

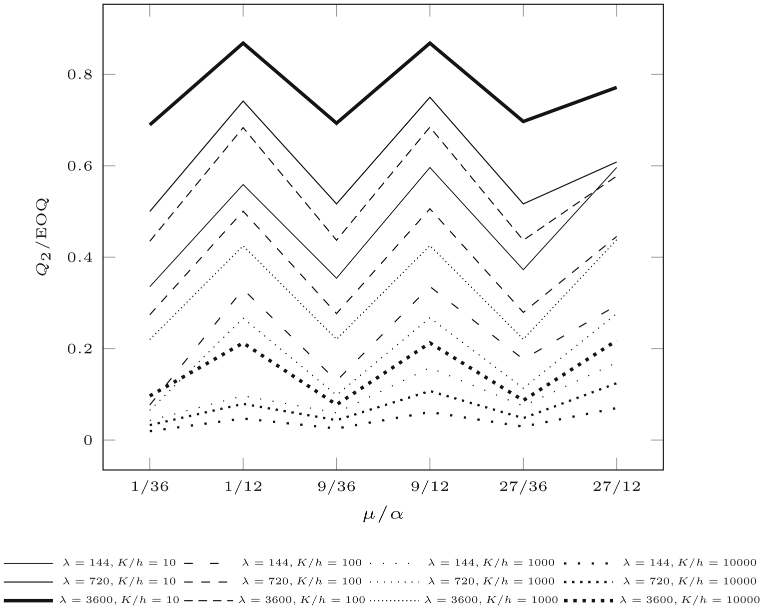


Fig. 6 Q_2/EOQ versus μ/α

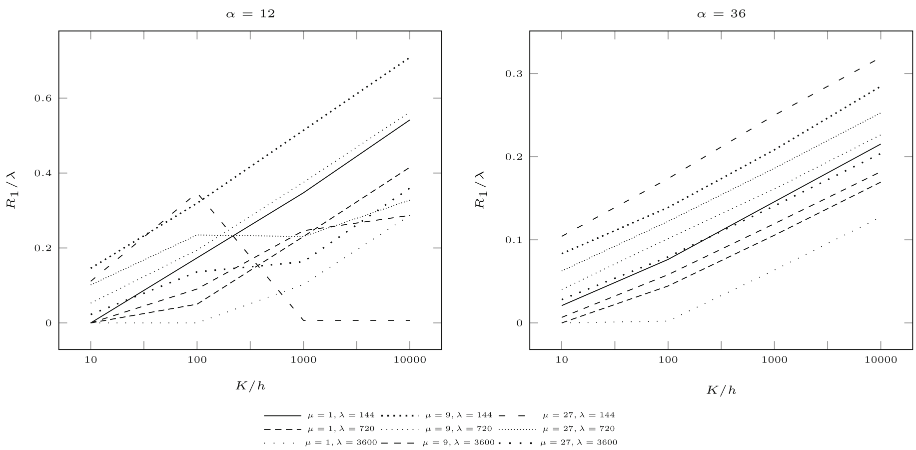


Fig. 7 Change in reorder point per annual demand, i.e., R_1/λ with respect to fixed to holding cost ratio, i.e., K/h

We also observe from Fig. 6 that keeping α fixed, Q_2/EOQ line is almost flat when μ varies. This is the case for any parameter combination.

Remark 9 Q_2 is more sensitive to α compared to μ.

Last variable we investigate is R_1. Figure 7 presents how reorder point is effected with respect to fixed to holding cost ratio. Based on this, we derive the following remark.

Remark 10 In spite of exceptions, reorder point in days of demand (i.e., R_1/λ) increase proportional to the logarithm of K/h in a majority of test instances.

6 Conclusions

In this study, a stochastic model for hospitals' inventory management with a continuous review policy under supply disruption is presented. Supplies can be received from two suppliers, where the primary supply channel is open to disruptions, and the secondary supply channel is always available. Inventory level and shortage status for the primary supply channel denote the states of a continuous-time Markov chain. The two objectives in this model are minimization of expected inventory cost and secondary supply cost, which is measured through the expected secondary supply channel usage frequency.

To the best of our knowledge, there exists no model considering supply disruption for one of the two suppliers and includes the second supplier as a backup to prevent possible supply shortages. We show that traditional or heuristic approaches fail to capture how such an inventory system should be operated under supply shortages. This implies the need for more complex solution methodologies as proposed in this paper.

There are a number of directions for further research. This framework can be extended for two or more hospitals, where transshipment between hospitals is also allowed. In addition, warehouse capacity constraints for suppliers can be introduced to the model. Moreover, analyses performed in this paper can be made for multiple items or with the existence of lead time under certain conditions. Further consequences of using the secondary channel can be studied such as an uncertain (unstructured) lead time or loss of manpower in a hospital during the search for an alternative supplier, a higher (per item) cost of supply, or a higher cost of care due to consequences of using a substitute medicine.

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Declarations

Conflict of interest All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

A Limiting probabilities and expected inventory levels for rare shortages with fast recovery, $\mu < \alpha$

A.1 Case 1: $Q_2 < R_1$

In order to obtain the limiting probabilities of the states, we divide the Markov chain states into four parts. The first part includes all the available states (all the states on the left side of the transition diagram). The second part consists of all the states from $(Q_1 + R_1, U)$ to $(R_1 + 1, U)$; the third includes states (R_1, U) to $(Q_2 + 1, U)$, and the remaining states are in the fourth part. Next, we obtain the limiting probabilities of these parts separately.

All the states of the first part can be written as a function of the limiting probability of the $(Q_1 + R_1, A)$ state:

$$P_{Q_1+R_1-j,A} = \left(\frac{\lambda}{\lambda+\mu}\right)^j P_{Q_1+R_1,A}, \qquad j = 0, \ldots, Q_1 - 1. \tag{7}$$

The sum of the probabilities of these states is

$$\sum_{j=0}^{Q_1-1} P_{Q_1+R_1-j,A} = \left(\frac{\lambda+\mu}{\mu}\right)\left[1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right] P_{Q_1+R_1,A}. \tag{8}$$

Now, let us try to obtain the limiting probabilities of the second part, that is, the states between $(Q_1 + R_1, U)$ and $(R_1 + 1, U)$. Similarly, we can write the limiting probabilities of all the states in this part as a function of the limiting probabilities of $(Q_1 + R_1, U)$ and $(Q_1 + R_1, A)$ states. We have the following equation for the $(Q_1 + R_1, U)$ state:

$$P_{Q_1+R_1,U} = \left(\frac{\mu}{\lambda+\alpha}\right) P_{Q_1+R_1,A}. \tag{9}$$

For the second state of this part, we have the following:

$$P_{Q_1+R_1-1,U} = \left(\frac{\mu}{\lambda+\alpha}\right) P_{Q_1+R_1-1,A} + \left(\frac{\lambda}{\lambda+\alpha}\right) P_{Q_1+R_1,U}.$$

Using Eqs. (7) and (9), we substitute $P_{Q_1+R_1-1,A}$ and $P_{Q_1+R_1,U}$ respectively:

$$P_{Q_1+R_1-1,U} = \left(\left(\frac{\mu}{\lambda+\alpha}\right)\left(\frac{\lambda}{\lambda+\mu}\right) + \left(\frac{\lambda}{\lambda+\alpha}\right)\left(\frac{\mu}{\lambda+\alpha}\right)\right) P_{Q_1+R_1,A}.$$

The generalized form is as follows:

$$P_{Q_1+R_1-j,U} = \frac{\mu\lambda^j}{\lambda+\alpha}\left[\sum_{k=0}^{j} \frac{1}{(\lambda+\alpha)^k(\lambda+\mu)^{j-k}}\right] P_{Q_1+R_1,A}. \tag{10}$$

Let us try to find the sum of series in the last equation. We have

$$\begin{aligned} P_{Q_1+R_1-j,U} &= \frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^j \sum_{k=0}^{j}\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^k P_{Q_1+R_1,A} \\ &= \frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^j \frac{1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{j+1}}{\frac{\alpha-\mu}{\lambda+\alpha}} P_{Q_1+R_1,A} \\ &= \frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^j \left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{j+1}\right) P_{Q_1+R_1,A}, \qquad j = 0, \ldots, Q_1 - 1. \end{aligned} \tag{11}$$

We can obtain the following equation for $(R_1 + 1, U)$ state:

$$\begin{aligned} P_{R_1+1,U} &= \frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right) P_{Q_1+R_1,A} \\ &= \frac{\mu(\lambda+\mu)}{\lambda(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right) P_{Q_1+R_1,A}. \end{aligned} \tag{12}$$

The sum of the probabilities of the states in this part is

$$\sum_{j=0}^{Q_1-1} P_{Q_1+R_1-j,U} = \frac{\mu}{\alpha-\mu}\sum_{j=0}^{Q_1-1}\left(\frac{\lambda}{\lambda+\mu}\right)^j\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{j+1}\right) P_{Q_1+R_1,A}$$

$$= \frac{\mu}{\alpha-\mu}\left[\frac{1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}}{1-\frac{\lambda}{\lambda+\mu}} - \frac{\lambda+\mu}{\lambda+\alpha}\frac{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}}{1-\frac{\lambda}{\lambda+\alpha}}\right] P_{Q_1+R_1,A}$$

$$= \left[\frac{\lambda+\mu}{\alpha-\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right) - \frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right] P_{Q_1+R_1,A} \tag{13}$$

For the third part, we obtain the limiting probabilities in a similar way with what we have done in the first part. All the limiting probabilities of these states can be written as a function of the limiting probability of (R_1+1, U) state:

$$P_{R_1-j,U} = \left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1} P_{R_1+1,U}, \qquad j = 0, \ldots, R_1 - Q_2 - 1. \tag{14}$$

The sum of the probabilities of third part is

$$\sum_{j=0}^{R_1-Q_2-1} P_{R_1-j,U} = \frac{\lambda}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2}\right] P_{R_1+1,U}. \tag{15}$$

Now, let us try to obtain the limiting probabilities of the last part. The structure is similar to previous part, and the only thing we have to consider is to write the probabilities of these states in terms of the probability of (Q_2, U) state. We, then, would have the following:

$$P_{Q_2-j,U} = \left(\frac{\lambda}{\lambda+\alpha}\right)^{j} P_{Q_2,U}, \qquad j = 0, \ldots, Q_2 - 1. \tag{16}$$

We can write the following equation for the limiting probability of (Q_2, U) state:

$$(\lambda+\alpha)\, P_{Q_2,U} = \lambda\left[P_{1,U} + P_{Q_2+1,U}\right]. \tag{17}$$

Using Eqs. (14) and (16), we substitute for $P_{Q_2+1,U}$ and $P_{1,U}$, respectively, in the last equation:

$$(\lambda+\alpha)\, P_{Q_2,U} = \lambda\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1} P_{Q_2,U} + \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2} P_{R_1+1,U}\right]$$

$$P_{Q_2,U} = \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2} P_{Q_2,U} + \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1} P_{R_1+1,U}$$

$$P_{Q_2,U} = \frac{\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}} P_{R_1+1,U}. \tag{18}$$

Then, we have the following equation to obtain the limiting probabilities of the last part:

$$P_{Q_2-j,U} = \frac{\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1+j}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}} P_{R_1+1,U}, \qquad j = 0, \ldots, Q_2 - 1. \tag{19}$$

The sum of the probabilities of these states is

$$\sum_{j=0}^{Q_2-1} P_{Q_2-j,U} = \frac{\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}} . \frac{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)} P_{R_1+1,U}$$
$$= \frac{\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1}}{\frac{\alpha}{\lambda+\alpha}} P_{R_1+1,U}$$
$$= \frac{\lambda}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2} P_{R_1+1,U} \tag{20}$$

Considering the fact that the limiting probabilities of all of the states must add up to 1, we obtain the limiting probability of (Q_1+R_1, A) state for case 1 as follows:

$$\left[\frac{\lambda+\mu}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)+\frac{\lambda+\mu}{\alpha-\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\right.$$
$$\left.-\frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right] P_{Q_1+R_1,A}$$
$$+\frac{\lambda}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2}\right] P_{R_1+1,U}+\frac{\lambda}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2} P_{R_1+1,U}=1$$
$$1=\left[\frac{\alpha(\lambda+\mu)}{\mu(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)-\frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right]$$
$$P_{Q_1+R_1,A}+\frac{\lambda}{\alpha} P_{R_1+1,U}$$
$$1=\left[\frac{\alpha(\lambda+\mu)}{\mu(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)-\frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right.$$
$$\left.+\frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}-\frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right] P_{Q_1+R_1,A}$$
$$1=\frac{(\lambda+\mu)(\alpha+\mu)}{\alpha\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right) P_{Q_1+R_1,A}$$
$$P_{Q_1+R_1,A}=\frac{\alpha\mu}{(\lambda+\mu)(\alpha+\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1}. \tag{21}$$

Plugging (21) in (12) we obtain $P_{1,U}$ to be used in the expected total cost function in (6) as follows:

$$P_{1,U} = \frac{\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}} \frac{\mu(\lambda+\mu)}{\lambda(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)$$
$$\times \frac{\alpha\mu}{(\lambda+\mu)(\alpha+\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1}. \tag{22}$$

Expected inventory level of this case can be computed as follows:

$$I(Q_1;Q_2;R_1) = \sum_{j=0}^{Q_1-1}(Q_1+R_1-j)\,P_{Q_1+R_1-j,A} + \sum_{j=0}^{Q_1-1}(Q_1+R_1-j)\,P_{Q_1+R_1-j,U} + \sum_{j=0}^{R_1-Q_2-1}(R_1-j)\,P_{R_1-j,U} + \sum_{j=0}^{Q_2-1}(Q_2-j)P_{Q_2-j,U}, \tag{23}$$

which can be rewritten as

$$\begin{aligned} I(Q_1;Q_2;R_1) = (Q_1+R_1)&\left(\sum_{j=0}^{Q_1-1} P_{Q_1+R_1-j,A} + \sum_{j=0}^{Q_1-1} P_{Q_1+R_1-j,U}\right) \\ &- \sum_{j=0}^{Q_1-1} jP_{Q_1+R_1-j,A} - \sum_{j=0}^{Q_1-1} jP_{Q_1+R_1-j,U} + R_1\sum_{j=0}^{R_1-Q_2-1} P_{R_1-j,U} \\ &- \sum_{j=0}^{R_1-Q_2-1} jP_{R_1-j,U} + Q_2\sum_{j=0}^{Q_2-1} P_{Q_2-j,U} - \sum_{j=0}^{Q_2-1} jP_{Q_2-j,U}. \end{aligned} \tag{24}$$

Knowing that the sum of the limiting probabilities of the all the states is 1, we rewrite the last equation as

$$\begin{aligned} I(Q_1;Q_2;R_1) = (Q_1+R_1) - Q_1&\sum_{j=0}^{R_1-Q_2-1} P_{R_1-j,U} - (Q_1+R_1-Q_2)\sum_{j=0}^{Q_2-1} P_{Q_2-j,U} \\ &- \sum_{j=0}^{Q_1-1} jP_{Q_1+R_1-j,A} - \sum_{j=0}^{Q_1-1} jP_{Q_1+R_1-j,U} \\ &- \sum_{j=0}^{R_1-Q_2-1} jP_{R_1-j,U} - \sum_{j=0}^{Q_2-1} jP_{Q_2-j,U}. \end{aligned} \tag{25}$$

We replace the sum of probabilities from Eqs. (15) and (20), and $P_{Q_1+R_1-j,A}$, $P_{Q_1+R_1-j,U}$, $P_{R_1-j,U}$, and $P_{Q_2-j,U}$ from Eqs. (8), (13), (15), and (20):

$$\begin{aligned} I(Q_1;Q_2;R_1) &= (Q_1+R_1) \\ &- \left[Q_1\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2}\right) + (Q_1+R_1-Q_2)\frac{\lambda}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2}\right]P_{R_1+1,U} \\ &- \left[\sum_{j=0}^{Q_1-1} j\left(\frac{\lambda}{\lambda+\mu}\right)^j + \sum_{j=0}^{Q_1-1} j\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^j\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{j+1}\right)\right]P_{Q_1+R_1,A} \\ &- \left[\sum_{j=0}^{R_1-Q_2-1} j\left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1} + \sum_{j=0}^{Q_2-1} j\frac{\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1+j}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right]P_{R_1+1,U}. \end{aligned} \tag{26}$$

Knowing that

$$\sum_{i=0}^{n} ip^i = p\sum_{i=0}^{n} ip^{i-1} = p\sum_{i=0}^{n}\frac{\partial}{\partial p}p^i = p\frac{\partial}{\partial p}\sum_{i=0}^{n} p^i$$

$$= \frac{np^{n+2} - (n+1)p^{n+1} + p}{(1-p)^2}, \tag{27}$$

we can rewrite the last equation as

$$\begin{aligned} I(Q_1; Q_2; R_1) = (Q_1 + R_1) - \frac{\lambda}{\alpha}\left[Q_1 + (R_1 - Q_2)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2}\right] P_{R_1+1,U} \\ - \left[\left(\frac{\alpha}{\alpha-\mu}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1} + \frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right)\right. \\ \left. - \left(\frac{\mu(\lambda+\mu)}{(\alpha-\mu)(\lambda+\alpha)}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right)\right] P_{Q_1+R_1,A} \\ - \left[\left(\frac{\lambda}{\lambda+\alpha}\right)\left(\frac{(R_1-Q_2-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1} - (R_1-Q_2)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right)\right. \\ \left. + \left(\frac{\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right)\left(\frac{(Q_2-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2+1} - Q_2\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right)\right] P_{R_1+1,U}. \end{aligned} \tag{28}$$

We can plug in limiting probabilities and obtain the final equation to be used in (6):

$$\begin{aligned} I(Q_1; Q_2; R_1) = (Q_1 + R_1) - \frac{\lambda}{\alpha}\left[Q_1 + (R_1 - Q_2)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2}\right] \\ \frac{\mu(\lambda+\mu)}{\lambda(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)\frac{\alpha\mu}{(\lambda+\mu)(\alpha+\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1} \\ - \left[\left(\frac{\alpha}{\alpha-\mu}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1} + \frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right)\right. \\ \left. - \left(\frac{\mu(\lambda+\mu)}{(\alpha-\mu)(\lambda+\alpha)}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right)\right] \\ \times \frac{\alpha\mu}{(\lambda+\mu)(\alpha+\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1} \\ - \left[\left(\frac{\lambda}{\lambda+\alpha}\right)\left(\frac{(R_1-Q_2-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1} - (R_1-Q_2)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right)\right. \\ \left. + \left(\frac{\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right)\left(\frac{(Q_2-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2+1} - Q_2\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right)\right] \\ \times \frac{\mu(\lambda+\mu)}{\lambda(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)\frac{\alpha\mu}{(\lambda+\mu)(\alpha+\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1}. \end{aligned} \tag{29}$$

A.2 Case 2: $R_1 \le Q_2 \le Q_1 + R_1$

Similar to Case 1, we divide the Markov chain states into four parts. The first part includes all the available states, similar to Case 1. However, the second part consists of all the states from $(Q_1 + R_1, U)$ to $(Q_2 + 1, U)$; the third includes states (Q_2, U) to $(R_1 + 1, U)$, and the remaining states are in the fourth part, states between (R_1, U) and $(1, U)$. Next, we would obtain the limiting probabilities of these parts separately.

We can obtain the limiting probabilities of the first part from those of case 1:

$$P_{Q_1+R_1-j,A} = \left(\frac{\lambda}{\lambda+\mu}\right)^j P_{Q_1+R_1,A}, \qquad j = 0, \ldots, Q_1 - 1. \tag{30}$$

The sum of the probabilities of these states is

$$\sum_{j=0}^{Q_1-1} P_{Q_1+R_1-j,A} = \frac{\lambda+\mu}{\mu}\left[1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right] P_{Q_1+R_1,A}. \tag{31}$$

The equations for the second part are similar to part two of case 1, but the range of the states, j, has to be updated. We have the following equation for the $(Q_1 + R_1, U)$ state:

$$P_{Q_1+R_1-j,U} = \left(\frac{\mu}{\alpha-\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right)^j \left(1 - \left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{j+1}\right) P_{Q_1+R_1,A},$$
$$j = 0, \ldots, Q_1 + R_1 - Q_2 - 1. \tag{32}$$

The sum of the states in this part, for case 2, is

$$\sum_{j=0}^{Q_1+R_1-Q_2-1} P_{Q_1+R_1-j,U}$$
$$= \frac{\lambda+\mu}{\alpha-\mu}\left[1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2} - \frac{\mu}{\alpha}\left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right)\right] P_{Q_1+R_1,A}. \tag{33}$$

For the third part, we have the following equation for (Q_2, U) state:

$$P_{Q_2,U} = \frac{\lambda}{\lambda+\alpha}\left(P_{Q_2+1,U} + P_{1,U}\right) + \frac{\mu}{\lambda+\alpha} P_{Q_2,A}. \tag{34}$$

For the remaining states of this part, we obtain the limiting probabilities as a function of $P_{Q_2,U}$ and $P_{Q_1+R_1,A}$:

$$P_{Q_2-j,U} = \left(\frac{\lambda}{\lambda+\alpha}\right)^j P_{Q_2,U} + \frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}$$
$$\sum_{k=0}^{j-1}\left(\frac{\lambda}{\lambda+\mu}\right)^{j-k}\left(\frac{\lambda}{\lambda+\alpha}\right)^k P_{Q_1+R_1,A}$$
$$P_{Q_2-j,U} = \left(\frac{\lambda}{\lambda+\alpha}\right)^j P_{Q_2,U} + \frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2+j}$$
$$\left(1 - \left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^j\right) P_{Q_1+R_1,A},$$

$$j = 1, \ldots, Q_2 - R_1 - 1. \tag{35}$$

Note that the range is starting from *one, not zero*. $P_{R_1+1,U}$ is

$$\begin{aligned} P_{R_1+1,U} &= \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1} P_{Q_2,U} \\ &\quad + \frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right) P_{Q_1+R_1,A}. \end{aligned} \tag{36}$$

The sum of the probabilities of third part is

$$\begin{aligned} \sum_{j=0}^{Q_2-R_1-1} P_{Q_2-j,U} &= \sum_{j=0}^{Q_2-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{j} P_{Q_2,U} \\ &\quad + \sum_{j=1}^{Q_2-R_1-1}\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2+j}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{j}\right) P_{Q_1+R_1,A} \\ &= \frac{\lambda+\alpha}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right] P_{Q_2,U} \\ &\quad + \frac{\lambda}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left[1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right] P_{Q_1+R_1,A} \\ &\quad - \frac{\lambda\mu}{\alpha(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right] P_{Q_1+R_1,A}. \end{aligned} \tag{37}$$

Now, let us try to obtain the limiting probabilities of the last part. As mentioned above, this part would be considered only when the reorder point is greater than zero. The structure is similar to part one, and the only thing we have to consider is to write the probabilities of these states in terms of the probability of $(R_1 + 1, U)$ state. We, then, have the following:

$$P_{R_1-j,U} = \left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1} P_{R_1+1,U}, \qquad j = 0, \ldots, R_1 - 1. \tag{38}$$

The sum of the probabilities of these states is

$$\sum_{j=0}^{R_1-1} P_{R_1-j,U} = \frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right) P_{R_1+1,U}. \tag{39}$$

In order to obtain the closed form equation for the limiting probability of (Q_2, U) state, we have the following equations:

$$\begin{aligned} P_{1,U} &= \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1} P_{R_1+1,U} \\ &= \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1} P_{Q_2,U} + \frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1} \\ &\quad \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right) P_{Q_1+R_1,A}, \end{aligned} \tag{40}$$

$$P_{Q_2+1,U} = \frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right)P_{Q_1+R_1,A}, \tag{41}$$

$$P_{Q_2,A} = \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2} P_{Q_1+R_1,A}. \tag{42}$$

Then from Eq. (34),

$$P_{Q_2,U} = \frac{\mu}{\alpha-\mu}\left[\frac{\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right.$$
$$\left.+\frac{\frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right)+\frac{\alpha-\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right]P_{Q_1+R_1,A}. \tag{43}$$

Considering the fact that the sum of the limiting probabilities of all of the states must add up to 1, we obtain the limiting probability of $(Q_1 + R_1, A)$ state for case 2 as follows:

$$1 = \left[\frac{\lambda+\mu}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)+\frac{\lambda+\mu}{\alpha-\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right.\right.$$
$$\left.\left.-\frac{\mu}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right)\right)\right]P_{Q_1+R_1,A}$$
$$+\frac{\lambda+\alpha}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right]P_{Q_2,U}$$
$$+\frac{\lambda}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left[1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right]P_{Q_1+R_1,A}$$
$$-\frac{\lambda\mu}{\alpha(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right]P_{Q_1+R_1,A}$$
$$+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)P_{R_1+1,U}. \tag{44}$$

Then,

$$P_{Q_1+R_1,A} = \left[\frac{\alpha+\mu}{\alpha\mu}-\frac{(\alpha+\mu)(\lambda+\mu)}{\mu\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right]^{-1}$$
$$= \frac{\alpha\mu}{(\alpha+\mu)(\lambda+\mu)}\left(\frac{1}{\lambda+\mu}-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1}. \tag{45}$$

Expected inventory level can be computed as follows:

$$I(Q_1; Q_2; R_1) = \sum_{j=0}^{Q_1-1}(Q_1+R_1-j)\,P_{Q_1+R_1-j,A}$$
$$+\sum_{j=0}^{Q_1+R_1-Q_2-1}(Q_1+R_1-j)\,P_{Q_1+R_1-j,U}$$

$$+\sum_{j=0}^{Q_2-R_1-1}(Q_2-j)\,P_{Q_2-j,U}+\sum_{j=0}^{R_1-1}(R_1-j)P_{R_1-j,U}, \tag{46}$$

which can be rewritten as

$$\begin{aligned}
I(Q_1;Q_2;R_1) &= (Q_1+R_1)\left(\sum_{j=0}^{Q_1-1}P_{Q_1+R_1-j,A}+\sum_{j=0}^{Q_1+R_1-Q_2-1}P_{Q_1+R_1-j,U}\right)\\
&\quad-\sum_{j=0}^{Q_1-1}jP_{Q_1+R_1-j,A}\\
&\quad-\sum_{j=0}^{Q_1+R_1-Q_2-1}jP_{Q_1+R_1-j,U}+Q_2\sum_{j=0}^{Q_2-R_1-1}P_{Q_2-j,U}\\
&\quad-\sum_{j=0}^{Q_2-R_1-1}jP_{Q_2-j,U}+R_1\sum_{j=0}^{R_1-1}P_{R_1-j,U}-\sum_{j=0}^{R_1-1}jP_{R_1-j,U} \qquad (47)\\
&=(Q_1+R_1)-Q_1\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)P_{R_1+1,U}\\
&\quad-(Q_1+R_1-Q_2)\Bigg[\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right)P_{Q_2,U}\\
&\quad+\frac{\lambda}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)P_{Q_1+R_1,A}\\
&\quad-\frac{\lambda\mu}{\alpha(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)P_{Q_1+R_1,A}\Bigg]\\
&\quad-\sum_{j=0}^{Q_1-1}jP_{Q_1+R_1-j,A}-\sum_{j=0}^{Q_1+R_1-Q_2-1}jP_{Q_1+R_1-j,U}\\
&\quad-\sum_{j=0}^{Q_2-R_1-1}jP_{Q_2-j,U}-\sum_{j=0}^{R_1-1}jP_{R_1-j,U}. \qquad (48)
\end{aligned}$$

We replace the probabilities from Eqs. (15) and (20), and $P_{Q_1+R_1-j,A}$, $P_{Q_1+R_1-j,U}$, $P_{R_1-j,U}$, and $P_{Q_2-j,U}$ from Eqs. (31), (33), (37), and (39):

$$\begin{aligned}
I(Q_1;Q_2;R_1) &= -Q_1\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)P_{R_1+1,U}\\
&\quad-(Q_1+R_1-Q_2)\Bigg[\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right)P_{Q_2,U}\\
&\quad+\frac{\lambda}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)P_{Q_1+R_1,A}\\
&\quad-\frac{\lambda\mu}{\alpha(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)P_{Q_1+R_1,A}\Bigg]
\end{aligned}$$

$$+ (Q_1 + R_1) - \sum_{j=0}^{Q_1-1} j \left(\frac{\lambda}{\lambda+\mu}\right)^j P_{Q_1+R_1,A}$$
$$- \sum_{j=0}^{Q_1+R_1-Q_2-1} j \frac{\mu}{\alpha-\mu} \left(\frac{\lambda}{\lambda+\mu}\right)^j \left(1 - \left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{j+1}\right) P_{Q_1+R_1,A}$$
$$- \sum_{j=0}^{Q_2-R_1-1} j \left(\frac{\lambda}{\lambda+\alpha}\right)^j P_{Q_2,U} - \sum_{j=0}^{Q_2-R_1-1} j \frac{\mu}{\alpha-\mu} \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2+j}$$
$$\left(1 - \left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^j\right) P_{Q_1+R_1,A} - \sum_{j=0}^{R_1-1} j \left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1} P_{R_1+1,U} \tag{49}$$

$$\begin{aligned}
&I(Q_1; Q_2; R_1)\\
&= -Q_1 \frac{\lambda}{\alpha}\left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right) P_{R_1+1,U}\\
&- (Q_1 + R_1 - Q_2)\Bigg[\frac{\lambda+\alpha}{\alpha}\left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right) P_{Q_2,U}\\
&+ \frac{\lambda}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left(1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right) P_{Q_1+R_1,A}\\
&- \frac{\lambda\mu}{\alpha(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right) P_{Q_1+R_1,A}\Bigg]\\
&+ (Q_1 + R_1) - \Bigg[\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1} + \frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\\
&- \left(\frac{\mu(\lambda+\mu)}{(\alpha-\mu)(\lambda+\alpha)}\right)\frac{(Q_1+R_1-Q_2-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2+1} - (Q_1+R_1-Q_2)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\\
&+ \left(\frac{\mu}{\alpha-\mu}\right)\frac{(Q_1+R_1-Q_2-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2+1} - (Q_1+R_1-Q_2)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2} + \frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\Bigg] P_{Q_1+R_1,A}\\
&- \frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1+1} - (Q_2-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2} P_{Q_2,U}\\
&- \frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1+1} - (Q_2-R_1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1} + \frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2} P_{Q_1+R_1,A}\\
&+ \frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1+1} - (Q_2-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2} P_{Q_1+R_1,A}\\
&- \left(\frac{\lambda}{\lambda+\alpha}\right)\frac{(R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1} - R_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2} P_{R_1+1,U}.
\end{aligned} \tag{50}$$

For this case, $I(Q_1; Q_2; R_1)$ and $P_{1,U}$ that is used when calculating $SF(Q_1; Q_2; R_1)$ can be written in terms of α, λ, μ, Q_1, Q_2, and R_1 as follows:

$$I(Q_1; Q_2; R_1) = (Q_1 + R_1) + \frac{\alpha\mu}{(\alpha+\mu)(\lambda+\mu)}\left(\frac{1}{\lambda+\mu} - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1}$$
$$\left[-Q_1\frac{\lambda}{\alpha}\left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)\right[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\frac{\mu}{\alpha-\mu}$$

$$
\times\left[\frac{\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right.
$$

$$
\left.+\frac{\frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right)+\frac{\alpha-\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right]+\frac{\mu}{\alpha-\mu}
$$

$$
\left.\times\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)\right]
$$

$$
-(Q_1+R_1-Q_2)\left[\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right)\frac{\mu}{\alpha-\mu}\right.
$$

$$
\left[\frac{\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right.
$$

$$
\left.+\frac{\frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right)+\frac{\alpha-\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right]
$$

$$
+\frac{\lambda}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)
$$

$$
\left.-\frac{\lambda\mu}{\alpha(\alpha-\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)\right]
$$

$$
-\left[\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+1}-Q_1\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right.
$$

$$
-\left(\frac{\mu(\lambda+\mu)}{(\alpha-\mu)(\lambda+\alpha)}\right)\frac{(Q_1+R_1-Q_2-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2+1}-(Q_1+R_1-Q_2)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}
$$

$$
\left.+\left(\frac{\mu}{\alpha-\mu}\right)\frac{(Q_1+R_1-Q_2-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2+1}-(Q_1+R_1-Q_2)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right]
$$

$$
-\frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1+1}-(Q_2-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}
$$

$$
\frac{\mu}{\alpha-\mu}\left[\frac{\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right.
$$

$$
\left.+\frac{\frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right)+\frac{\alpha-\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right]
$$

$$
-\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1+1}-(Q_2-R_1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}
$$

$$
+\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1+1}-(Q_2-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}
$$

$$
-\left(\frac{\lambda}{\lambda+\alpha}\right)\frac{(R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1}-R_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right.
$$

$$\frac{\mu}{\alpha-\mu}\left[\frac{\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right.$$

$$\left.+\frac{\frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right)+\frac{\alpha-\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right]$$

$$+\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)\Bigg]\Bigg],$$

$$P_{1,U}=\frac{\alpha\mu}{(\alpha+\mu)(\lambda+\mu)}\left(\frac{1}{\lambda+\mu}-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}\right.$$

$$\frac{\mu}{\alpha-\mu}\left[\frac{\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right.$$

$$\left.+\frac{\frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right)+\frac{\alpha-\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right]$$

$$\left.\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)\right].$$

A.3 Case 3: $Q_2 > Q_1 + R_1$

In order to obtain the limiting probabilities of the states we divide the Markov chain states into five parts. The first part includes all the available states from state (Q_2, A) to (Q_1+R_1+1, A). The second part includes all the states from (Q_2, U) to (Q_1+R_1+1, U); the third consists of all the available states from (Q_1+R_1, A) to (R_1+1, A), fourth part includes states (Q_1+R_1, U) to (R_1+1, U), and fifth part includes all the remaining states. Next, we obtain the limiting probabilities of these parts separately.

All the states of the first part can be written as an equation of the limiting probability of (Q_2, A) state. By doing so, we write the following equation for the limiting probabilities of the first part:

$$P_{Q_2-j,A}=\left(\frac{\lambda}{\lambda+\mu}\right)^{j}P_{Q_2,A}+\frac{\alpha}{\lambda+\mu}P_{Q_2-j,U},\qquad j=1,\ldots,Q_2-Q_1-R_1-1, \tag{51}$$

$$P_{Q_2-j,U}=\left(\frac{\lambda}{\lambda+\alpha}\right)^{j}P_{Q_2,U}+\frac{\mu}{\lambda+\alpha}P_{Q_2-j,A},\qquad j=1,\ldots,Q_2-Q_1-R_1-1, \tag{52}$$

$$P_{Q_2,A}=\frac{\alpha}{\lambda+\mu}P_{Q_2,U}. \tag{53}$$

From Eqs. (51), (52), and (53) we obtain the following for limiting probabilities of the first part:

$$P_{Q_2-j,A}=\frac{\alpha(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left[\left(\frac{\lambda}{\lambda+\mu}\right)^{j}+\left(\frac{\lambda}{\lambda+\alpha}\right)^{j}\right]P_{Q_2,U},$$
$$j=1,\ldots,Q_2-Q_1-R_1-1. \tag{54}$$

The sum of the probabilities of these states is

$$\sum_{j=0}^{Q_2-Q_1-R_1-1} P_{Q_2-j,A}$$
$$= \frac{\alpha}{\lambda+\mu} P_{Q_2,U} + \sum_{j=1}^{Q_2-Q_1-R_1-1} \frac{\alpha(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)} \left[\left(\frac{\lambda}{\lambda+\mu}\right)^j + \left(\frac{\lambda}{\lambda+\alpha}\right)^j \right] P_{Q_2,U}$$
$$= \frac{\lambda+\alpha}{\lambda+\alpha+\mu} \left[\frac{\alpha}{\mu} \left(1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right) + \left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right) \right.$$
$$\left. + \frac{\alpha(\lambda+\alpha+\mu)}{(\lambda+\mu)(\lambda+\alpha)} \right] P_{Q_2,U}. \tag{55}$$

For the second part, we follow steps similar to part one:

$$P_{Q_2-j,U} = \left[\frac{(\mu+\lambda)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)} \left(\frac{\lambda}{\lambda+\alpha}\right)^j + \frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)} \left(\frac{\lambda}{\lambda+\mu}\right)^j \right] P_{Q_2,U},$$
$$j = 1, \ldots, Q_2 - Q_1 - R_1 - 1. \tag{56}$$

For the sum of these states, we have

$$\sum_{j=0}^{Q_2-Q_1-R_1-1} P_{Q_2-j,U}$$
$$= P_{Q_2,U} + \sum_{j=1}^{Q_2-Q_1-R_1-1} \left[\frac{(\mu+\lambda)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)} \left(\frac{\lambda}{\lambda+\alpha}\right)^j + \frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)} \left(\frac{\lambda}{\lambda+\mu}\right)^j \right] P_{Q_2,U}$$
$$= \frac{\lambda+\mu}{\lambda+\alpha+\mu} \left[\frac{\lambda+\alpha}{\alpha} \left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right) + \frac{\alpha}{\lambda+\mu} \left(1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right) \right.$$
$$\left. + \frac{\lambda+\alpha+\mu}{\lambda+\mu} \right] P_{Q_2,U}. \tag{57}$$

Now, let us try to obtain the limiting probabilities of the third part:

$$P_{Q_1+R_1-j,A} = \left(\frac{\lambda}{\lambda+\mu}\right)^j P_{Q_1+R_1,A}, \qquad j = 0, \ldots, Q_1 - 1. \tag{58}$$

The sum of these states is

$$\sum_{j=0}^{Q_1-1} P_{Q_1+R_1-j,A} = \frac{1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}}{1 - \frac{\lambda}{\lambda+\mu}} P_{Q_1+R_1,A}$$
$$= \frac{\lambda+\mu}{\mu} \left(1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right) P_{Q_1+R_1,A}. \tag{59}$$

For the fifth part, we have

$$P_{R_1-j,U} = \left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1} P_{R_1+1,U}, \qquad j = 0, \ldots, R_1 - 1. \tag{60}$$

The sum of the probabilities of the states in the fifth part is

$$\sum_{j=0}^{R_1-1} P_{R_1-j,U} = \frac{\lambda}{\lambda+\alpha}\frac{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}}{1-\frac{\lambda}{\lambda+\alpha}} P_{R_1+1,U} = \frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right) P_{R_1+1,U}. \tag{61}$$

For the fourth part, we have the following:

$$P_{Q_1+R_1,U} = \frac{\mu}{\lambda+\alpha} P_{Q_1+R_1,A} + \frac{\lambda}{\lambda+\alpha} P_{Q_1+R_1+1,U}. \tag{62}$$

Then,

$$\begin{aligned} P_{Q_1+R_1-1,U} &= \frac{\mu}{\lambda+\alpha} P_{Q_1+R_1-1,A} + \frac{\lambda}{\lambda+\alpha} P_{Q_1+R_1,U} \\ &= \left(\frac{\mu}{\lambda+\alpha}\right)\left(\frac{\lambda}{\lambda+\mu}\right) P_{Q_1+R_1,A} \\ &\quad + \left(\frac{\mu}{\lambda+\alpha}\right)\left(\frac{\lambda}{\lambda+\alpha}\right) P_{Q_1+R_1,A} + \left(\frac{\lambda}{\lambda+\alpha}\right)^2 P_{Q_1+R_1+1,U}, \end{aligned} \tag{63}$$

$$\begin{aligned} P_{Q_1+R_1-2,U} &= \frac{\mu}{\lambda+\alpha} P_{Q_1+R_1-2,A} + \frac{\lambda}{\lambda+\alpha} P_{Q_1+R_1-1,U} \\ &= \frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^2 P_{Q_1+R_1,A} \\ &\quad + \frac{\mu}{\lambda+\alpha}\frac{\lambda}{\lambda+\mu}\frac{\lambda}{\lambda+\alpha} P_{Q_1+R_1,A} + \frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^2 P_{Q_1+R_1,A} \\ &\quad + \left(\frac{\lambda}{\lambda+\alpha}\right)^3 P_{Q_1+R_1+1,U}. \end{aligned} \tag{64}$$

The generalized form is as follows:

$$\begin{aligned} P_{Q_1+R_1-j,U} &= \frac{\mu\lambda^j}{\lambda+\alpha}\left[\sum_{k=0}^{j}\frac{1}{(\lambda+\alpha)^k(\lambda+\mu)^{j-k}}\right] P_{Q_1+R_1,A} \\ &\quad + \left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1} P_{Q_1+R_1+1,U} \\ &= \frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^j\left[\sum_{k=0}^{j}\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^k\right] P_{Q_1+R_1,A} \\ &\quad + \left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1} P_{Q_1+R_1+1,U} \\ P_{Q_1+R_1-j,U} &= \frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^j\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{j+1}\right) P_{Q_1+R_1,A} \\ &\quad + \left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1} P_{Q_1+R_1+1,U}, \end{aligned}$$

$$j = 0, \ldots, Q_1 - 1. \tag{65}$$

For $(R_1 + 1, U)$ state, we have

$$P_{R_1+1,U} = \frac{\mu}{\alpha - \mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1 - \left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right) P_{Q_1+R_1,A} + \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} P_{Q_1+R_1+1,U}. \tag{66}$$

Note that we can obtain $P_{Q_1+R_1+1,U}$ from Eq. (56) as follows:

$$P_{Q_1+R_1+1,U} = \left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1} + \frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right] P_{Q_2,U}. \tag{67}$$

Let us try to find the sum of series in last equation. We have

$$\sum_{j=0}^{Q_1-1} P_{Q_1+R_1-j,U} = \left[\frac{\lambda+\mu}{\alpha-\mu}\left(1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right) - \frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right] P_{Q_1+R_1,A} + \frac{\lambda}{\alpha}\left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right) P_{Q_1+R_1+1,U}. \tag{68}$$

For (Q_2, U) state, we have the following equation:

$$P_{Q_2,U} = \frac{\lambda}{\lambda+\alpha} P_{1,U} + \frac{\mu}{\lambda+\alpha} P_{Q_2,A}. \tag{69}$$

We replace $P_{Q_2,A}$ from Eq. (53):

$$P_{Q_2,U} = \frac{\lambda+\mu}{\lambda+\mu+\alpha} P_{1,U}. \tag{70}$$

From the equation for the fifth part, we obtain $P_{1,U}$:

$$P_{1,U} = \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1} P_{R_1+1,U}. \tag{71}$$

Then,

$$P_{Q_2,U} = \frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1} P_{R_1+1,U}. \tag{72}$$

For $P_{R_1+1,U}$, we have,

$$P_{R_1+1,U} = \frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1 - \left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right) P_{Q_1+R_1,A} + \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} \times \left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right.$$

$$+\frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\Bigg]$$
$$\times\frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}P_{R_1+1,U}$$
$$P_{R_1+1,U}=\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)P_{Q_1+R_1,A}+$$
$$\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]P_{R_1+1,U}$$
$$P_{R_1+1,U}=\frac{\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}P_{Q_1+R_1,A} \tag{73}$$

Then for $P_{Q_2,U}$,

$$P_{Q_2,U}=\frac{\frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}P_{Q_1+R_1,A} \tag{74}$$

$$\begin{aligned}1=&\frac{\lambda+\alpha}{\lambda+\alpha+\mu}\left[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)\right.\\&\left.+\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\alpha(\lambda+\alpha+\mu)}{(\lambda+\mu)(\lambda+\alpha)}\right]P_{Q_2,U}\\&+\frac{\lambda+\mu}{\lambda+\alpha+\mu}\left[\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)\right.\\&\left.+\frac{\alpha}{\lambda+\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\lambda+\alpha+\mu}{\lambda+\mu}\right]P_{Q_2,U}\\&+\frac{\lambda+\mu}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)P_{Q_1+R_1,A}+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)P_{R_1+1,U}\\&+\left[\frac{\lambda+\mu}{\alpha-\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)-\frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right]\\&P_{Q_1+R_1,A}+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)P_{Q_1+R_1+1,U}\end{aligned} \tag{75}$$

$$\begin{aligned}1=&\left[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)\right.\\&\left.+\frac{\alpha+\lambda+\mu}{\lambda+\mu}\right]P_{Q_2,U}+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)P_{R_1+1,U}+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)P_{Q_1+R_1+1,U}\end{aligned}$$

$$+\left[\frac{\alpha(\lambda+\mu)}{\mu(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)-\frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right]P_{Q_1+R_1,A}. \tag{76}$$

Then,

$$\begin{aligned}
&\left[\left[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\alpha+\lambda+\mu}{\lambda+\mu}\right]\right.\\
&\times\frac{\frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)\frac{\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right.\\
&\left.+\frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right]\\
&\times\frac{\frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&\left.+\frac{\alpha(\lambda+\mu)}{\mu(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)-\frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right]^{-1}=P_{Q_1+R_1,A}.
\end{aligned} \tag{77}$$

Expected inventory level can be computed as follows:

$$\begin{aligned}
I(Q_1;Q_2;R_1)=&\sum_{j=0}^{Q_2-Q_1-R_1-1}(Q_2-j)(P_{Q_2-j,A}+P_{Q_2-j,U})+\sum_{j=0}^{R_1-1}(R_1-j)P_{R_1-j,U}\\
&+\sum_{j=0}^{Q_1-1}(Q_1+R_1-j)(P_{Q_1+R_1-j,A}+P_{Q_1+R_1-j,U}).
\end{aligned} \tag{78}$$

Then,

$$\begin{aligned}
I(Q_1;Q_2;R_1)=Q_2&\sum_{j=0}^{Q_2-Q_1-R_1-1}(P_{Q_2-j,A}+P_{Q_2-j,U})\\
&+(Q_1+R_1)\sum_{j=0}^{Q_1-1}(P_{Q_1+R_1-j,A}+P_{Q_1+R_1-j,U})\\
&+R_1\sum_{j=0}^{R_1-1}P_{R_1-j,U}-\sum_{j=0}^{Q_2-Q_1-R_1-1}j(P_{Q_2-j,A}+P_{Q_2-j,U})
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=0}^{R_1-1} j P_{R_1-j,U} \\
&- \sum_{j=0}^{Q_1-1} j (P_{Q_1+R_1-j,A} + P_{Q_1+R_1-j,U}). \qquad (79)
\end{aligned}
$$

We replace the probabilities $P_{Q_2-j,A}$, $P_{Q_1+R_1-j,A}$, $P_{Q_2-j,U}$, $P_{Q_1+R_1-j,U}$, and $P_{R_1-j,U}$ from Eqs. (55), (59), (57), (61), and (68):

$$
\begin{aligned}
I(Q_1;Q_2;R_1) = Q_2 &\left[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right) + \frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)\right. \\
&\left. + \frac{\alpha+\lambda+\mu}{\lambda+\mu}\right] P_{Q_2,U} + (Q_1+R_1)\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right) P_{Q_1+R_1+1,U} \\
&+ (Q_1+R_1)\left[\frac{\alpha(\lambda+\mu)}{\mu(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right) - \frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right] P_{Q_1+R_1,A} \\
&+ R_1\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right) P_{R_1+1,U} - \frac{\alpha}{\lambda} \times P_{Q_2,U} \times \\
&\left[\frac{(Q_2-Q_1-R_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1+1} - (Q_2-Q_1-R_1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1} + \frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right] \\
&- \frac{(Q_2-Q_1-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1} - (Q_2-Q_1-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1} + 1}{(1-\frac{\lambda}{\lambda+\alpha})^2} P_{Q_2,U} \\
&- \left(\frac{\lambda}{\lambda+\alpha}\right)\left(\frac{(R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1} - R_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right) P_{R_1+1,U} \\
&- \left(\frac{\alpha}{\alpha-\mu}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1} + \frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right) P_{Q_1+R_1,A} \\
&+ \left(\frac{\mu(\mu+\lambda)}{(\lambda+\alpha)(\alpha-\mu)}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right) P_{Q_1+R_1,A} \\
&- \left(\frac{\lambda}{\lambda+\alpha}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right) P_{Q_1+R_1+1,U}. \qquad (80)
\end{aligned}
$$

Note that the equations for this case would not bear any changes for $R_1 = 0$.

For this case, $I(Q_1;Q_2;R_1)$ and $P_{1,U}$ that is used when calculating $SF(Q_1;Q_2;R_1)$ can be written in terms of α, λ, μ, Q_1, Q_2, and R_1 as follows:

$$
\begin{aligned}
P_{1,U} = &\left[\frac{\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\left[\left[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)\right.\right.\right. \\
&+\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)+\left.\frac{\alpha+\lambda+\mu}{\lambda+\mu}\right]\frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right) \\
&+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}+\frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right] \\
&\left.\left.\times\frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right]+\frac{\alpha(\lambda+\mu)}{\mu(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)-\frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right]^{-1}
\end{aligned}
$$

$$\times \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1} \frac{\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}$$

$$\begin{aligned}
I(Q_1;Q_2;R_1) = &\left[\frac{\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\left[\left[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)\right.\right.\right. \\
&+\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)+\left.\frac{\alpha+\lambda+\mu}{\lambda+\mu}\right]\frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right) \\
&+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}+\frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right] \\
&\left.\left.\times\frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right]+\frac{\alpha(\lambda+\mu)}{\mu(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)-\frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right]^{-1} \\
&\times\left[\frac{\frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\right. \\
&\times\left[Q_2\left[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\alpha+\lambda+\mu}{\lambda+\mu}\right]\right. \\
&+(Q_1+R_1)\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right. \\
&\left.+\frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right]
\end{aligned}$$

$$\begin{aligned}
&-\frac{\alpha}{\lambda}\times\left[\frac{(Q_2-Q_1-R_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1+1}-(Q_2-Q_1-R_1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right] \\
&-\frac{(Q_2-Q_1-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1}-(Q_2-Q_1-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}+1}{(1-\frac{\lambda}{\lambda+\alpha})^2} \\
&-\left(\frac{\lambda}{\lambda+\alpha}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+1}-Q_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right) \\
&\left.\left.\left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}+\frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right]\right]\right. \\
&+(Q_1+R_1)\left[\frac{\alpha(\lambda+\mu)}{\mu(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)-\frac{\mu(\lambda+\mu)}{\alpha(\alpha-\mu)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right] \\
&+R_1\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)\frac{\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]} \\
&-\left(\frac{\lambda}{\lambda+\alpha}\right)\left(\frac{(R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1}-R_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right)\frac{\frac{\mu}{\alpha-\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\mu}{\lambda+\alpha}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{(\lambda+\mu+\alpha)^2}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-2}+\frac{\mu\alpha}{\lambda^2}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}
\end{aligned}$$

$$-\left(\frac{\alpha}{\alpha-\mu}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+1}-Q_1\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right)$$
$$+\left(\frac{\mu(\mu+\lambda)}{(\lambda+\alpha)(\alpha-\mu)}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+1}-Q_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right)\Bigg]$$

B Limiting probabilities and expected inventory levels for frequent shortages with slow recovery, $\mu > \alpha$

B.1 Case 1: $Q_2 < R_1$

All the equations of Case 1 for frequent shortage case are the same as the Case 1 of rare shortage except the second part. To obtain the limiting probabilities of the second part, we write the limiting probabilities of all the states in this part as a function of the limiting probabilities of $(Q_1 + R_1, U)$ and $(Q_1 + R_1, A)$ states. We have the following equation for the $(Q_1 + R_1, U)$ state:

$$P_{Q_1+R_1,U} = \left(\frac{\mu}{\lambda+\alpha}\right) P_{Q_1+R_1,A}. \tag{81}$$

For the second state of this part, we have the following:

$$P_{Q_1+R_1-1,U} = \left(\frac{\mu}{\lambda+\alpha}\right) P_{Q_1+R_1-1,A} + \left(\frac{\lambda}{\lambda+\alpha}\right) P_{Q_1+R_1,U}.$$

Using Eqs. (7) and (81), we substitute $P_{Q_1+R_1-1,A}$ and $P_{Q_1+R_1,U}$ respectively:

$$P_{Q_1+R_1-1,U} = \left(\frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right) + \frac{\lambda}{\lambda+\alpha}\left(\frac{\mu}{\lambda+\alpha}\right)\right) P_{Q_1+R_1,A}.$$

The generalized form of the probabilities of this case is as follows:

$$P_{Q_1+R_1-j,U} = \frac{\mu\lambda^j}{\lambda+\alpha}\left[\sum_{k=0}^{j}\frac{1}{(\lambda+\alpha)^{j-k}(\lambda+\mu)^k}\right] P_{Q_1+R_1,A}. \tag{82}$$

Let us try to find the sum of series in last equation. We have,

$$\begin{aligned}
P_{Q_1+R_1-j,U} &= \frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^j \sum_{k=0}^{j}\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^k P_{Q_1+R_1,A} \\
&= \frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^j \frac{1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{j+1}}{\frac{\mu-\alpha}{\lambda+\mu}} P_{Q_1+R_1,A} \\
&= \frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^j \left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{j+1}\right) P_{Q_1+R_1,A}, \\
& j = 0, \ldots, Q_1 - 1.
\end{aligned} \tag{83}$$

We can obtain the following equation for (R_1+1, U) state when $\alpha < \mu$ and $Q_2 < R_1$:

$$
\begin{aligned}
P_{R_1+1,U} &= \frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)P_{Q_1+R_1,A} \\
&= \frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right]P_{Q_1+R_1,A}.
\end{aligned} \tag{84}
$$

The sum of the states in this part is

$$
\begin{aligned}
\sum_{j=0}^{Q_1-1} P_{Q_1+R_1-j,U} &= \sum_{j=0}^{Q_1-1}\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{j}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{j+1}\right)P_{Q_1+R_1,A} \\
&= \frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left[\frac{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}}{1-\frac{\lambda}{\lambda+\alpha}}-\frac{\lambda+\alpha}{\lambda+\mu}\frac{1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}}{1-\frac{\lambda}{\lambda+\mu}}\right]P_{Q_1+R_1,A} \\
&= \left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)-\frac{\lambda+\mu}{\mu-\alpha}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\right]P_{Q_1+R_1,A}.
\end{aligned} \tag{85}
$$

Considering the fact the sum of the limiting probabilities of all of the states must add up to 1, we obtain the limiting probability of Q_1+R_1, A state as follows:

$$
\begin{aligned}
&\left[\frac{\lambda+\mu}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)+\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\right. \\
&\left.-\frac{\lambda+\mu}{\mu-\alpha}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\right]P_{Q_1+R_1,A} \\
&+\frac{\lambda}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2}\right]P_{R_1+1,U}+\frac{\lambda}{\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2}P_{R_1+1,U}=1
\end{aligned} \tag{86}
$$

$$
\begin{aligned}
1 &= \left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)-\frac{\alpha(\lambda+\mu)}{\mu(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\right] \\
&\quad P_{Q_1+R_1,A}+\frac{\lambda}{\alpha}P_{R_1+1,U}
\end{aligned} \tag{87}
$$

$$
\begin{aligned}
1 &= \left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)-\frac{\alpha(\lambda+\mu)}{\mu(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\right. \\
&\quad\left.+\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right]\right]P_{Q_1+R_1,A}
\end{aligned} \tag{88}
$$

$$
1 = \frac{(\lambda+\mu)(\alpha+\mu)}{\alpha\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)P_{Q_1+R_1,A} \tag{89}
$$

$$
P_{Q_1+R_1,A} = \frac{\alpha\mu}{(\lambda+\mu)(\alpha+\mu)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1} \tag{90}
$$

As it can be seen, the equations are the same in case 1 for both conditions. As a result, the inventory level of this case is be equal to the inventory level of case 1 for the rare shortage condition.

For this case, $I(Q_1; Q_2; R_1)$ and $P_{1,U}$ that is used when calculating $SF(Q_1; Q_2; R_1)$ can be written in terms of α, λ, μ, Q_1, Q_2, and R_1 as follows:

$$
\begin{aligned}
P_{1,U} = & \frac{\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1+j}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}} \frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)} \left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right] \\
& \frac{\alpha\mu}{(\lambda+\mu)(\alpha+\mu)} \left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1} \\
I(Q_1; Q_2; R_1) = & (Q_1+R_1) - \frac{\lambda}{\alpha}\left[Q_1 + (R_1-Q_2)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2}\right] \\
& \frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)} \left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right] \frac{\alpha\mu}{(\lambda+\mu)(\alpha+\mu)} \left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1} \\
& - \left[\left(\frac{\alpha}{\alpha-\mu}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1} + \frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right)\right. \\
& \left. - \left(\frac{\mu(\lambda+\mu)}{(\alpha-\mu)(\lambda+\alpha)}\right)\left(\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right)\right] \\
& \frac{\alpha\mu}{(\lambda+\mu)(\alpha+\mu)} \left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1} \\
& - \left[\left(\frac{\lambda}{\lambda+\alpha}\right)\left(\frac{(R_1-Q_2-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1} - (R_1-Q_2)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right)\right. \\
& \left. + \left(\frac{\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1-Q_2+1}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right)\left(\frac{(Q_2-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2+1} - Q_2\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right)\right] \\
& \frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)} \left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right] \frac{\alpha\mu}{(\lambda+\mu)(\alpha+\mu)} \left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)^{-1}
\end{aligned}
$$

B.2 Case 2: $R_1 \leq Q_2 \leq Q_1 + R_1$

All the equations of Case 2 for the frequent shortage case are the same as Case 2 for the rare shortage condition except the second and third parts. The equations for the second part are similar to part two of case 1, but the range of the states, j, has to be updated. We have the following equation for the $(Q_1 + R_1, U)$ state:

$$
\begin{aligned}
P_{Q_1+R_1-j,U} = & \frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)} \left(\frac{\lambda}{\lambda+\alpha}\right)^j \left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{j+1}\right) P_{Q_1+R_1,A}, \\
& j = 0, \ldots, Q_1 + R_1 - Q_2 - 1.
\end{aligned} \tag{91}
$$

The sum of the states in this part, for case 2, is

$$\begin{aligned}\sum_{j=0}^{Q_1+R_1-Q_2-1} & P_{Q_1+R_1-j,U} \\ &= \left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right)\right. \\ &\quad \left.-\frac{\lambda+\mu}{\mu-\alpha}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right)\right] P_{Q_1+R_1,A}. \end{aligned} \tag{92}$$

For the third part, we obtain the limiting probabilities as a function of $P_{Q_2,U}$ and $P_{Q_1+R_1,A}$:

$$\begin{aligned} P_{Q_2-j,U} = & \left(\frac{\lambda}{\lambda+\alpha}\right)^{j} P_{Q_2,U} + \frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2+j} \\ & \left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{j}\right) P_{Q_1+R_1,A}, \\ & j=1,\ldots,Q_2-R_1-1. \end{aligned} \tag{93}$$

Note that the range is starting from **one, not zero**. The sum of the probabilities of the third part is

$$\begin{aligned} \sum_{j=0}^{Q_2-R_1-1} P_{Q_2-j,U} = & \sum_{j=1}^{Q_2-R_1-1} \frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2+j} \\ & \left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{j}\right) P_{Q_1+R_1,A} \\ & + \sum_{j=0}^{Q_2-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{j} P_{Q_2,U} \\ = & \frac{\lambda+\alpha}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right] P_{Q_2,U} \\ & + \frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right] P_{Q_1+R_1,A} \\ & - \frac{(\lambda+\mu)^2}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left[1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right] P_{Q_1+R_1,A}. \end{aligned} \tag{94}$$

For $P_{R_1+1,U}$, we have,

$$\begin{aligned} P_{R_1+1,U} = & \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1} P_{Q_2,U} + \frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} \\ & \left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right) P_{Q_1+R_1,A}. \end{aligned} \tag{95}$$

In order to obtain the closed form equation for the limiting probability of (Q_2, U) state, we use the equations from (38) and (95):

$$\begin{aligned} P_{1,U} &= \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1} P_{R_1+1,U} \\ &= \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1} P_{Q_2,U} + \frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1} \\ &\quad \left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right) P_{Q_1+R_1,A}. \end{aligned} \tag{96}$$

From (34), we also know that

$$P_{Q_2,U} = \frac{\lambda}{\lambda+\alpha}\left(P_{Q_2+1,U} + P_{1,U}\right) + \frac{\mu}{\lambda+\alpha} P_{Q_2,A}. \tag{97}$$

From (91), we have the following for $P_{Q_2+1,U}$:

$$P_{Q_2+1,U} = \frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2-1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right) P_{Q_1+R_1,A}. \tag{98}$$

From the second case of rare shortage condition, we know that

$$P_{Q_2,A} = \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2} P_{Q_1+R_1,A}. \tag{99}$$

Then,

$$\begin{aligned} P_{Q_2,U} = \Bigg[& \frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}} + \frac{\frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}} \\ & + \frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\Bigg] P_{Q_1+R_1,A}. \end{aligned} \tag{100}$$

Now, for $P_{Q_1+R_1,A}$, we have,

$$\begin{aligned} 1 &= \frac{\lambda+\mu}{\mu}\left[1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right] P_{Q_1+R_1,A} \\ &\quad + \frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right) P_{R_1+1,U} \\ &\quad + \left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right)\right. \\ &\quad \left. -\frac{\lambda+\mu}{\mu-\alpha}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right)\right] P_{Q_1+R_1,A} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda+\alpha}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right]P_{Q_2,U}\\
&+\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right]P_{Q_1+R_1,A}\\
&-\frac{(\lambda+\mu)^2}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left[1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right]P_{Q_1+R_1,A}
\end{aligned}
$$

$$
\begin{aligned}
P_{Q_1+R_1,A}=&\left[\frac{\lambda+\mu}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)+\frac{\lambda+\mu}{\alpha}-\frac{\lambda}{\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right.\\
&+\frac{\lambda(\lambda+\mu)}{\alpha(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\\
&-\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}+\frac{(\lambda+\mu)^2}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda+\mu}{\lambda}-\frac{\mu}{\alpha}\right)\\
&\left.\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1}\right]^{-1}\\
P_{Q_1+R_1,A}=&\left[\frac{\lambda+\mu}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)+\frac{\lambda+\mu}{\alpha}-\frac{\lambda}{\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right.\\
&+\frac{\lambda+\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\\
&\times\left(\frac{\lambda}{\alpha}+\frac{\lambda+\mu}{\mu-\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)-\frac{\mu(\lambda+\mu)}{\mu-\alpha}\\
&\left.\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(\frac{1}{\lambda}+\frac{1}{\alpha}\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)\right]^{-1}
\end{aligned}
\tag{101}
$$

Expected inventory level can be computed as follows:

$$
\begin{aligned}
I(Q_1;Q_2;R_1)=&\sum_{j=0}^{Q_1-1}(Q_1+R_1-j)\,P_{Q_1+R_1-j,A}\\
&+\sum_{j=0}^{Q_1+R_1-Q_2-1}(Q_1+R_1-j)\,P_{Q_1+R_1-j,U}\\
&+\sum_{j=0}^{Q_2-R_1-1}(Q_2-j)\,P_{Q_2-j,U}+\sum_{j=0}^{R_1-1}(R_1-j)P_{R_1-j,U}.
\end{aligned}
\tag{102}
$$

Then,

$$
\begin{aligned}
I(Q_1;Q_2;R_1)=&(Q_1+R_1)\left(\sum_{j=0}^{Q_1-1}P_{Q_1+R_1-j,A}+\sum_{j=0}^{Q_1+R_1-Q_2-1}P_{Q_1+R_1-j,U}\right)\\
&-\sum_{j=0}^{Q_1-1}jP_{Q_1+R_1-j,A}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=0}^{Q_1+R_1-Q_2-1} j P_{Q_1+R_1-j,U} + Q_2 \sum_{j=0}^{Q_2-R_1-1} P_{Q_2-j,U} - \sum_{j=0}^{Q_2-R_1-1} j P_{Q_2-j,U} \\
&+ R_1 \sum_{j=0}^{R_1-1} P_{R_1-j,U} - \sum_{j=0}^{R_1-1} j P_{R_1-j,U} \\
&= (Q_1+R_1) - Q_1 \left(\frac{\lambda}{\alpha}\right) \left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right) P_{R_1+1,U} \\
&- (Q_1+R_1-Q_2) \left(\frac{\lambda+\alpha}{\alpha}\right) \left[1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right] P_{Q_2,U} \\
&- (Q_1+R_1-Q_2) \left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)} \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2} \left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)\right] P_{Q_1+R_1,A} \\
&+ (Q_1+R_1-Q_2) \left[\frac{(\lambda+\mu)^2}{(\mu-\alpha)(\lambda+\alpha)} \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2} \left(1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)\right] P_{Q_1+R_1,A} \\
&- \sum_{j=0}^{Q_1-1} j P_{Q_1+R_1-j,A} - \sum_{j=0}^{Q_1+R_1-Q_2-1} j P_{Q_1+R_1-j,U} \\
&- \sum_{j=0}^{Q_2-R_1-1} j P_{Q_2-j,U} - \sum_{j=0}^{R_1-1} j P_{R_1-j,U}.
\end{aligned} \tag{103}
$$

We replace the sum of probabilities from Eqs. (94) and (39), and $P_{Q_1+R_1-j,A}$, $P_{Q_1+R_1-j,U}$, $P_{R_1-j,U}$, and $P_{Q_2-j,U}$ from Eqs. (30), (91), (93), and (38):

$$
\begin{aligned}
I(Q_1; Q_2; R_1) &= (Q_1+R_1) - Q_1 \frac{\lambda}{\alpha} \left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right) P_{R_1+1,U} \\
&- (Q_1+R_1-Q_2) \frac{\lambda+\alpha}{\alpha} \left[1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right] P_{Q_2,U} \\
&- (Q_1+R_1-Q_2) \left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)} \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2} \left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)\right] P_{Q_1+R_1,A} \\
&+ (Q_1+R_1-Q_2) \left[\frac{(\lambda+\mu)^2}{(\mu-\alpha)(\lambda+\alpha)} \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\right. \\
&\left. \left(1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)\right] P_{Q_1+R_1,A} \\
&- \left[\sum_{j=0}^{Q_1-1} j \left(\frac{\lambda}{\lambda+\mu}\right)^j + \sum_{j=0}^{Q_1+R_1-Q_2-1} j \left(\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\right)\right. \\
&\left. \left(\frac{\lambda}{\lambda+\alpha}\right)^j \left(1 - \left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{j+1}\right)\right] P_{Q_1+R_1,A} \\
&- \sum_{j=0}^{Q_2-R_1-1} j \left(\frac{\lambda}{\lambda+\alpha}\right)^j P_{Q_2,U} + j \left(\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\right) \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2+j} \\
&\left(1 - \left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^j\right) P_{Q_1+R_1,A} + \sum_{j=0}^{R_1-1} j \left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1} P_{R_1+1,U}.
\end{aligned} \tag{104}
$$

We can rewrite the last equation as

$$
\begin{aligned}
I(Q_1; Q_2; R_1) &= (Q_1+R_1) - Q_1 \frac{\lambda}{\alpha} \left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right) P_{R_1+1,U} \\
&- (Q_1+R_1-Q_2) \frac{\lambda+\alpha}{\alpha} \left[1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right] P_{Q_2,U}
\end{aligned}
$$

$$
\begin{aligned}
&-(Q_1+R_1-Q_2)\left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)\right]P_{Q_1+R_1,A}\\
&+(Q_1+R_1-Q_2)\left[\frac{(\lambda+\mu)^2}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)\right]P_{Q_1+R_1,A}\\
&-\left[\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+1}-Q_1\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right.\\
&+\left(\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\right)\frac{(Q_1+R_1-Q_2-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2+1}-(Q_1+R_1-Q_2)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\\
&\left.-\left(\frac{\mu}{\mu-\alpha}\right)\frac{(Q_1+R_1-Q_2-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2+1}-(Q_1+R_1-Q_2)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right]P_{Q_1+R_1,A}\\
&-\frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1+1}-(Q_2-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}P_{Q_2,U}\\
&-\left(\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\right)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left[\frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1+1}-(Q_2-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}\right.\\
&\left.-\frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1+1}-(Q_2-R_1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right]P_{Q_1+R_1,A}\\
&-\left(\frac{\lambda}{\lambda+\alpha}\right)\frac{(R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1}-R_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}P_{R_1+1,U}.
\end{aligned}\tag{105}
$$

For this case, $I(Q_1; Q_2; R_1)$ and $P_{1,U}$ that is used when calculating $SF(Q_1; Q_2; R_1)$ can be written in terms of α, λ, μ, Q_1, Q_2, and R_1 as follows:

$$
\begin{aligned}
P_{1,U}=&\left[\frac{\lambda+\mu}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)+\frac{\lambda+\mu}{\alpha}-\frac{\lambda}{\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right.\\
&+\frac{\lambda(\lambda+\mu)}{\alpha(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}-\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}+\frac{(\lambda+\mu)^2}{(\mu-\alpha)(\lambda+\alpha)}\\
&\left.\times\left(\frac{\lambda+\mu}{\lambda}-\frac{\mu}{\alpha}\right)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1}\right]^{-1}\\
&\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}\left[\frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right.\right.\\
&+\frac{\frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\\
&\left.+\frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\right]\\
&\left.+\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
I(Q_1;Q_2;R_1) &= (Q_1+R_1) + \Bigg[\frac{\lambda+\mu}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right) \\
&+\frac{\lambda+\mu}{\alpha}-\frac{\lambda}{\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}+\frac{\lambda(\lambda+\mu)}{\alpha(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2} \\
&-\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}+\frac{(\lambda+\mu)^2}{(\mu-\alpha)(\lambda+\alpha)} \\
&\times\left(\frac{\lambda+\mu}{\lambda}-\frac{\mu}{\alpha}\right)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1}\Bigg]^{-1} \\
&\Bigg[-Q_1\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)\Bigg[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1} \\
&\Bigg[\frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}} \\
&+\frac{\frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}+\frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\Bigg] \\
&+\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)\Bigg] \\
&-(Q_1+R_1-Q_2)\frac{\lambda+\alpha}{\alpha}\left[1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}\right] \\
&\Bigg[\frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}+\frac{\frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}} \\
&+\frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\Bigg] \\
&-(Q_1+R_1-Q_2)\Bigg[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\right)\Bigg] \\
&+(Q_1+R_1-Q_2)\Bigg[\frac{(\lambda+\mu)^2}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)\Bigg] \\
&-\Bigg[\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+1}-Q_1\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2} \\
&+\left(\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\right)\frac{(Q_1+R_1-Q_2-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2+1}-(Q_1+R_1-Q_2)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2} \\
&-\left(\frac{\mu}{\mu-\alpha}\right)\frac{(Q_1+R_1-Q_2-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2+1}-(Q_1+R_1-Q_2)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\Bigg] \\
&-\frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1+1}-(Q_2-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2} \\
&\Bigg[\frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}+\frac{\frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}
\end{aligned}
$$

$$+\frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\Bigg]$$

$$-\left(\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\right)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\Bigg[\frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1+1}-(Q_2-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}$$

$$-\frac{(Q_2-R_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1+1}-(Q_2-R_1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-R_1}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\Bigg]$$

$$-\left(\frac{\lambda}{\lambda+\alpha}\right)\frac{(R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1}-R_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}$$

$$\times\Bigg[\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-R_1-1}\Bigg[\frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}$$

$$+\frac{\frac{\mu}{\lambda+\alpha}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}$$

$$+\frac{\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1-Q_2}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1+R_1-Q_2}\right)}{1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2}}\Bigg]$$

$$+\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_2-R_1-1}\right)\Bigg]\Bigg],$$

B.3 Case 3: $Q_2 > Q_1 + R_1$

All parts of this model, except part 4, are the same as the Case 3 of rare shortage condition. So, we just try to find the equations for fourth part and replace the other parts from case 3 of rare-shortage model. For part 4, we have

$$P_{Q_1+R_1-j,U}=\frac{\mu\lambda^j}{\lambda+\alpha}\left[\sum_{k=0}^{j}\frac{1}{(\lambda+\alpha)^{j-k}(\lambda+\mu)^k}\right]P_{Q_1+R_1,A}+\left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1}P_{Q_1+R_1+1,U} \tag{106}$$

$$P_{Q_1+R_1-j,U}=\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^j\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{j+1}\right)P_{Q_1+R_1,A}+\left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1}P_{Q_1+R_1+1,U},\quad j=0,\ldots,Q_1-1. \tag{107}$$

Let us try to find the sum of series in last equation. We have

$$\sum_{j=0}^{Q_1-1}P_{Q_1+R_1-j,U}=\Bigg[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)$$

$$- \frac{\lambda+\mu}{\mu-\alpha}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\Bigg] P_{Q_1+R_1,A} \tag{108}$$

$$+ \frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right) P_{Q_1+R_1+1,U}. \tag{109}$$

From Eq. (107), we have the following equation for $P_{R_1+1,U}$:

$$P_{R_1+1,U} = \frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)$$
$$P_{Q_1+R_1,A} + \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} P_{Q_1+R_1+1,U}. \tag{110}$$

We know the probability of $(Q_1 + R_1 + 1, U)$ and (Q_2, U) from the probability equations for case 3 of rare shortage model, (56) and (72). Then for $P_{R_1+1,U}$:

$$\begin{aligned}
P_{R_1+1,U} =& \frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right) P_{Q_1+R_1,A} + \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\\
&\times\left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right.\\
&\left.+\frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right]\\
&\times\frac{\lambda+\mu}{\lambda+\mu+\alpha}\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1} P_{R_1+1,U}\\
P_{R_1+1,U} =& \frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1-1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right) P_{Q_1+R_1,A}\\
&+\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}\right.\\
&\left.+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right] P_{R_1+1,U}\\
P_{R_1+1,U} =& \frac{\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]} P_{Q_1+R_1,A}.
\end{aligned} \tag{111}$$

We also know that

$$P_{Q_2,U} = \frac{\frac{\mu(\lambda+\mu)^2}{\lambda(\mu-\alpha)(\lambda+\mu+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]} P_{Q_1+R_1,A}. \tag{112}$$

For $P_{Q_1+R_1,A}$, we have

$$1 = \frac{\lambda+\alpha}{\lambda+\alpha+\mu}\left[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)\right.$$

$$
\begin{aligned}
&+\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\alpha(\lambda+\alpha+\mu)}{(\lambda+\mu)(\lambda+\alpha)}\Bigg]P_{Q_2,U}\\
&+\frac{\lambda+\mu}{\lambda+\alpha+\mu}\Bigg[\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)\\
&+\frac{\alpha}{\lambda+\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\lambda+\alpha+\mu}{\lambda+\mu}\Bigg]P_{Q_2,U}\\
&+\frac{\lambda+\mu}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)P_{Q_1+R_1,A}+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)P_{R_1+1,U}\\
&+\left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)-\frac{\lambda+\mu}{\mu-\alpha}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\right]\\
&P_{Q_1+R_1,A}+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)P_{Q_1+R_1+1,U}.
\end{aligned}\tag{113}
$$

Then,

$$
\begin{aligned}
1=&\left[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)\right.\\
&\left.+\frac{\alpha+\lambda+\mu}{\lambda+\mu}\right]P_{Q_2,U}+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)P_{R_1+1,U}\\
&+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)P_{Q_1+R_1+1,U}\\
&+\left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)-\frac{\alpha(\lambda+\mu)}{\mu(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\right]\\
&P_{Q_1+R_1,A},
\end{aligned}\tag{114}
$$

$$
\begin{aligned}
P_{Q_1+R_1,A}=&\left[\left[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)\right.\right.\\
&\left.+\frac{\alpha+\lambda+\mu}{\lambda+\mu}\right]\times\frac{\frac{\mu(\lambda+\mu)^2}{\lambda(\mu-\alpha)(\lambda+\mu+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)\frac{\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}+\frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right]\\
&\times\frac{\frac{\mu(\lambda+\mu)^2}{\lambda(\mu-\alpha)(\lambda+\mu+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&\left.+\left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)-\frac{\alpha(\lambda+\mu)}{\mu(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\right]\right]^{-1}.
\end{aligned}\tag{115}
$$

Expected inventory level can be computed as follows:

$$
\begin{aligned}
I(Q_1; Q_2; R_1) = & \sum_{j=0}^{Q_2-Q_1-R_1-1} (Q_2 - j, A) P_{Q_2-j,A} + \sum_{j=0}^{Q_2-Q_1-R_1-1} (Q_2 - j)\, P_{Q_2-j,U} \\
& + \sum_{j=0}^{R_1-1} (R_1 - j) P_{R_1-j,U} \\
& + \sum_{j=0}^{Q_1-1} (Q_1 + R_1 - j)\, P_{Q_1+R_1-j,A} \\
& + \sum_{j=0}^{Q_1-1} (Q_1 + R_1 - j)\, P_{Q_1+R_1-j,U}. \qquad (116)
\end{aligned}
$$

Then,

$$
\begin{aligned}
I(Q_1; Q_2; R_1) = Q_2 & \sum_{j=0}^{Q_2-Q_1-R_1-1} (P_{Q_2-j,A} + P_{Q_2-j,U}) \\
& + (Q_1 + R_1) \sum_{j=0}^{Q_1-1} (P_{Q_1+R_1-j,A} + P_{Q_1+R_1-j,U}) \\
& + R_1 \sum_{j=0}^{R_1-1} P_{R_1-j,U} - \sum_{j=0}^{Q_2-Q_1-R_1-1} j(P_{Q_2-j,A} + P_{Q_2-j,U}) \\
& - \sum_{j=0}^{R_1-1} j P_{R_1-j,U} \\
& - \sum_{j=0}^{Q_1-1} j(P_{Q_1+R_1-j,A} + P_{Q_1+R_1-j,U}). \qquad (117)
\end{aligned}
$$

We replace the probabilities $P_{Q_2-j,A}$, $P_{Q_1+R_1-j,A}$, $P_{Q_2-j,U}$, $P_{Q_1+R_1-j,U}$, and $P_{R_1-j,U}$ from Eqs. (55), (59), (57), (108), and (61):

$$
\begin{aligned}
I(Q_1; Q_2; R_1) = Q_2 & \Bigg[\frac{\alpha}{\mu} \left(1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right) \\
& + \frac{\lambda+\alpha}{\alpha} \left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right) \\
& + \frac{\alpha+\lambda+\mu}{\lambda+\mu} \Bigg] P_{Q_2,U} + (Q_1 + R_1) \frac{\lambda}{\alpha} \left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right) P_{Q_1+R_1+1,U} \\
& + (Q_1 + R_1) \left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)} \left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right) - \frac{\alpha(\lambda+\mu)}{\mu(\mu-\alpha)} \left(1 - \left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right) \right] P_{Q_1+R_1,A} \\
& + R_1 \frac{\lambda}{\alpha} \left(1 - \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right) P_{R_1+1,U} \\
& - \sum_{j=1}^{Q_2-Q_1-R_1-1} j \frac{\alpha(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)} \left[\left(\frac{\lambda}{\lambda+\mu}\right)^{j} + \left(\frac{\lambda}{\lambda+\alpha}\right)^{j} \right] P_{Q_2,U}
\end{aligned}
$$

$$- \sum_{j=1}^{Q_2-Q_1-R_1-1} j\left[\frac{(\mu+\lambda)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^j + \frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^j\right]P_{Q_2,U}$$

$$- \sum_{j=0}^{Q_1-1} j\left(\frac{\lambda}{\lambda+\mu}\right)^j P_{Q_1+R_1,A} - \sum_{j=0}^{R_1-1} j\left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1} P_{R_1+1,U}$$

$$- \sum_{j=0}^{Q_1-1} j\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^j\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{j+1}\right)$$

$$P_{Q_1+R_1,A} + j\left(\frac{\lambda}{\lambda+\alpha}\right)^{j+1} P_{Q_1+R_1+1,U} \tag{118}$$

$$\begin{aligned}
I(Q_1;Q_2;R_1) = Q_2\Bigg[&\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right) \\
&+ \frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right) \\
&+ \frac{\alpha+\lambda+\mu}{\lambda+\mu}\Bigg]P_{Q_2,U} + (Q_1+R_1)\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)P_{Q_1+R_1+1,U} \\
&+ (Q_1+R_1)\left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right) - \frac{\alpha(\lambda+\mu)}{\mu(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\right]P_{Q_1+R_1,A} \\
&+ R_1\left(\frac{\lambda}{\alpha}\right)\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)P_{R_1+1,U} - \left(\frac{\alpha}{\lambda}\right)P_{Q_2,U}\times \\
&\left[\frac{(Q_2-Q_1-R_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1+1} - (Q_2-Q_1-R_1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1} + \frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right] \\
&- \frac{(Q_2-Q_1-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1} - (Q_2-Q_1-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1} + 1}{(1-\frac{\lambda}{\lambda+\alpha})^2}P_{Q_2,U} \\
&- \left(\frac{\lambda}{\lambda+\alpha}\right)\frac{(R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1} - R_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}P_{R_1+1,U} \\
&+ \left(\frac{\alpha}{\mu-\alpha}\right)\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1} + \frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}P_{Q_1+R_1,A} \\
&- \left(\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\right)\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}P_{Q_1+R_1,A} \\
&- \left(\frac{\lambda}{\lambda+\alpha}\right)\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+1} - Q_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1} + \frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}P_{Q_1+R_1+1,U}.
\end{aligned} \tag{119}$$

Note that this equation is the same with case 3 of rare shortage condition.

For this case, $I(Q_1;Q_2;R_1)$ and $P_{1,U}$ that is used when calculating $SF(Q_1;Q_2;R_1)$ can be written in terms of α, λ, μ, Q_1, Q_2, and R_1 as follows:

$$P_{1,U} = \left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1} \frac{\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1} + \frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}P_{Q_1+R_1,A}$$

$$I(Q_1;Q_2;R_1) = \Bigg[\Bigg[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right) + \frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right) + \frac{\alpha+\lambda+\mu}{\lambda+\mu}\Bigg]$$

$$
\begin{aligned}
&\times \frac{\frac{\mu(\lambda+\mu)^2}{\lambda(\mu-\alpha)(\lambda+\mu+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)\\
&\times \frac{\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&+\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\\
&\times\left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}+\frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right]\\
&\times \frac{\frac{\mu(\lambda+\mu)^2}{\lambda(\mu-\alpha)(\lambda+\mu+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&+\left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)-\frac{\alpha(\lambda+\mu)}{\mu(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\right]\Bigg]^{-1}\\
&\left[Q_2\left[\frac{\alpha}{\mu}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\lambda+\alpha}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}\right)+\frac{\alpha+\lambda+\mu}{\lambda+\mu}\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{\frac{\mu(\lambda+\mu)^2}{\lambda(\mu-\alpha)(\lambda+\mu+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&+(Q_1+R_1)\frac{\lambda}{\alpha}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)\left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}+\frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right]\\
&\times \frac{\frac{\mu(\lambda+\mu)^2}{\lambda(\mu-\alpha)(\lambda+\mu+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&+(Q_1+R_1)\left[\frac{\mu(\lambda+\mu)}{\alpha(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\right)-\frac{\alpha(\lambda+\mu)}{\mu(\mu-\alpha)}\left(1-\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}\right)\right]\\
&+R_1\left(\frac{\lambda}{\alpha}\right)\left(1-\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}\right)\frac{\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&-\left(\frac{\alpha}{\lambda}\right)\frac{\frac{\mu(\lambda+\mu)^2}{\lambda(\mu-\alpha)(\lambda+\mu+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\\
&\times\left[\frac{(Q_2-Q_1-R_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1+1}-(Q_2-Q_1-R_1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}\right]\\
&-\frac{(Q_2-Q_1-R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1}-(Q_2-Q_1-R_1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}+1}{(1-\frac{\lambda}{\lambda+\alpha})^2}
\end{aligned}
$$

$$
\times \frac{\frac{\mu(\lambda+\mu)^2}{\lambda(\mu-\alpha)(\lambda+\mu+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}
$$

$$
-\left(\frac{\lambda}{\lambda+\alpha}\right)\frac{(R_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1+1}-R_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{R_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}
$$

$$
\times \frac{\frac{\mu(\lambda+\mu)}{\lambda(\mu-\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}
$$

$$
+\left(\frac{\alpha}{\mu-\alpha}\right)\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1+1}-Q_1\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_1}+\frac{\lambda}{\lambda+\mu}}{(1-\frac{\lambda}{\lambda+\mu})^2}
$$

$$
-\left(\frac{\mu(\lambda+\mu)}{(\mu-\alpha)(\lambda+\alpha)}\right)\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+1}-Q_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}
$$

$$
-\left(\frac{\lambda}{\lambda+\alpha}\right)\frac{(Q_1-1)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+1}-Q_1\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1}+\frac{\lambda}{\lambda+\alpha}}{(1-\frac{\lambda}{\lambda+\alpha})^2}
$$

$$
\times\left[\frac{(\lambda+\mu)(\lambda+\alpha)}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-Q_1-R_1-1}+\frac{\mu\alpha}{\lambda(\lambda+\alpha+\mu)}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\right]
$$

$$
\left.\times \frac{\frac{\mu(\lambda+\mu)^2}{\lambda(\mu-\alpha)(\lambda+\mu+\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\left(1-\left(\frac{\lambda+\alpha}{\lambda+\mu}\right)^{Q_1}\right)}{1-\frac{(\lambda+\mu)^2}{\lambda(\lambda+\mu+\alpha)^2}\left[(\lambda+\alpha)\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_2-1}+\frac{\mu\alpha}{\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\right)^{Q_2-Q_1-R_1-1}\left(\frac{\lambda}{\lambda+\alpha}\right)^{Q_1+R_1}\right]}\right]
$$

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