CS70 Summer 2018 — Solutions to Homework 1

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I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up. — Sung Hyun Harvey Woo

1. Always True or Always False?

(a) False for all combinations of x and y (Contradiction)

x	y	$x \land (x \implies y) \land (\neg y)$
T	F	F
T	T	F
F	F	F
F	T	F

(b) True for all combinations of x and y (Tautology)

x	y	$x \implies (x \lor y)$
T	F	T
T	T	T
F	F	T
F	T	T

(c) True for all combinations of x and y (Tautology)

x	y	$(x \lor y) \lor (x \lor \neg y)$
T	F	T
T	T	T
F	F	T
F	T	T

(d) True for all combinations of x and y (Tautology)

x	y	$(x \implies y) \lor (x \implies \neg y)$
T	F	T
T	T	T
F	F	T
F	T	T

(e) Neither

x	y	$(x \lor y) \land (\neg(x \land y))$
T	F	T
T	T	F
F	F	F
F	T	T

(f) False for all combinations of x and y (Contradiction)

x	y	$(x \implies y) \land (\neg x \implies y) \land (\neg y)$
T	F	F
T	T	F
F	F	F
F	T	F

2. Propositional Practice

(a) $(\exists x \in \mathbb{R})(x \notin \mathbb{Q})$
True. Real numbers encompass both rational and irrational numbers. Any irrational number would prove this claim true.

(b) $(\forall x \in \mathbb{Z})((x \in \mathbb{N}) \vee (x < 0)) \wedge \neg((x \in \mathbb{N}) \wedge (x < 0))$
True. Natural numbers cover 0 and all positive integers. If an integer is not 0 nor positive, it can only be a negative integer.

(c) $(\forall x \in \mathbb{N})((6|x) \implies ((2|x) \vee (3|x)))$
True. Since 6 is a multiple of, and is divisible by 2 and 3, any natural number divisible by 6 will also be divisible by 2 or 3.

(d) All real numbers are complex numbers.
True. Complex numbers encompass both real and imaginary numbers. Complex numbers are in the form a+bi, where $a, b \in \mathbb{R}$, and b can be set as 0. Since a can be any real number, real numbers can be expressed in complex numbers and are considered complex numbers.

(e) Any integer divisible by 2 or divisible by 3, is divisible by 6.
False. This does not apply to all integers. A counter-example would be 2. 2 is divisible by 2 $(2/2 = 1)$ but not divisible by 6 $(2/6 = 0.\bar{3})$.

(f) If a natural number is bigger than 7, it can be expressed as $a + b$, where a and b are natural numbers.
True. Since any natural number bigger than 7 is, by definition, a natural number, and 0 is a natural number, any natural number bigger than 7 can be expressed as the sum of itself and 0.

3. Prove or Disprove

(a) True. Proceed by direct proof.
$n = 2k + 1$, for some integer k. Then,

$$n^2 + 2n = (2k+1)^2 + 2(2k+1) = 4k^2 + 4k + 1 + 4k + 2 = 4k^2 + 8k + 3 = 2(2k^2 + 4k + 1) + 1 \quad (1)$$

Since integers are closed under addition,

$$n^2 + 2n = 2l + 1 \quad (2)$$

where integer $l = 2k^2 + 4k + 1$. Therefore $n^2 + 2n$ is odd.

(b) True. Proceed by cases.

Case 1 : $x < y, min(x, y) = x$

$$min(x, y) = \frac{(x + y - |x - y|)}{2} = \frac{(x + y - (y - x))}{2} = \frac{x + y - y + x}{2} = \frac{2x}{2} = x \tag{1}$$

Case 2 : $x > y, min(x, y) = y$

$$min(x, y) = \frac{(x + y - |x - y|)}{2} = \frac{(x + y - (x - y))}{2} = \frac{x + y - x + y}{2} = \frac{2y}{2} = y \tag{2}$$

Case 3 : $x = y, min(x, y) = x$ or y

$$min(x, y) = \frac{(x + y - |x - y|)}{2} = \frac{x + x - |x - x|}{2} = \frac{2x}{2} = x \tag{3}$$

$$min(x, y) = \frac{(x + y - |x - y|)}{2} = \frac{y + y - |y - y|}{2} = \frac{2y}{2} = y \tag{4}$$

In all cases, $min(x, y)$ results in $\frac{(x+y-|x-y|)}{2}$.

(c) True. Proceed by contraposition.
The contraposition of this claim is $\forall a, b \in \mathbb{R}$, if $a > 7$ and $b > 3$, then $a + b > 10$.
Given $a > 7$ and $b > 3$, we can add the inequalities to get

$$(a > 7) + (b > 3) = a + b > 7 + 3 = a + b > 10 \tag{1}$$

The contraposition of the claim holds, therefore, the original claim holds.

(d) True. Proceed by contraposition.
The contraposition of this claim is $\forall r \in \mathbb{R}$, if $r + 1$ is rational, then r is rational.
Assuming that $r + 1$, it can be expressed as $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$,
The difference of two rational numbers is rational, therefore $(r - 1) - 1 = r$ and r is rational.
The contraposition of the claim holds, therefore, the original claim holds.

(e) False. Proceed by counter example.
Take $n = 10$ as an example.

$$10n^2 > n! = 10 * 10 * 10 > 10 * 9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1 = 1000 > 3628800 \tag{1}$$

$1000 > 3628800$ is false.
The claim does not hold for $n = 10$, therefore the claim does not hold.

4. Preserving Set Operations

(a) Assume a hypothetical $f(x) \in f^{-1}(A \cup B)$.

$$\begin{aligned} f(x) \in f^{-1}(A \cup B) &\implies x \in (A \cup B) \implies (x \in A) \vee (x \in B) && (1)\\ &\implies (f(x) \in f^{-1}(A)) \vee (f(x) \in f^{-1}(B)) && (2)\\ &\implies f(x) \in (f^{-1}(A) \cup f^{-1}(B)) && (3)\end{aligned}$$

Now, assume a hypothetical $f(x) \in (f^{-1}(A) \cup f^{-1}(B))$

$$\begin{aligned} f(x) \in (f^{-1}(A) \cup f^{-1}(B)) &\implies (f(x) \in f^{-1}(A)) \vee (f(x) \in f^{-1}(B)) && (4)\\ &\implies (x \in A) \vee (x \in B) \implies x \in (A \cup B) && (5)\\ &\implies f(x) \in f^{-1}(A \cup B) && (6)\end{aligned}$$

$f^{-1}(A \cup B) \implies f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cup f^{-1}(B) \implies f^{-1}(A \cup B)$, therefore they are equal.

(b) Assume a hypothetical $f(x) \in f^{-1}(A \cap B)$.

$$\begin{aligned} f(x) \in f^{-1}(A \cap B) &\implies x \in (A \cap B) \implies (x \in A) \wedge (x \in B) && (1)\\ &\implies (f(x) \in f^{-1}(A)) \wedge (f(x) \in f^{-1}(B)) && (2)\\ &\implies f(x) \in (f^{-1}(A)) \cap (f^{-1}(B)) && (3)\end{aligned}$$

Now assume a hypothetical $f(x) \in f^{-1}(A) \cap f^{-1}(B)$

$$\begin{aligned} f(x) \in f^{-1}(A) \cap f^{-1}(B) &\implies (f(x) \in f^{-1}(A)) \wedge (f(x) \in f^{-1}(B)) && (4)\\ &\implies (x \in A) \wedge (x \in B) \implies x \in (A \cap B) && (5)\\ &\implies f(x) \in f^{-1}(A \cap B) && (6)\end{aligned}$$

$f^{-1}(A \cap B) \implies f^{-1}(A) \cap f^{-1}(B))$ and $f^{-1}(A) \cap f^{-1}(B) \implies f^{-1}(A \cap B)$, therefore they are equal.

(c) Assume a hypothetical $f(x) \in f^{-1}(A \backslash B)$

$$\begin{aligned} f(x) \in f^{-1}(A \backslash B) &\implies x \in (A \backslash B) && (1)\\ &\implies (x \in A) \wedge (x \notin B) \implies (f(x) \in f^{-1}(A)) \wedge (f(x) \notin f^{-1}(B)) && (2)\\ &\implies f(x) \in (f^{-1}(A)) \backslash (f^{-1}(B)) && (3)\end{aligned}$$

Now, assume a hypothetical $f(x) \in (f^{-1}(A) \backslash f^{-1}(B))$

$$\begin{aligned} f(x) \in (f^{-1}(A) \backslash f^{-1}(B)) &\implies x \in (A \backslash B) \implies (x \in A) \wedge x \notin B) && (4)\\ &\implies (f(x) \in f^{-1}(A) \wedge (f(x) \notin f^{-1}(B)) && (5)\\ &\implies f(x) \in f^{-1}(A \backslash B) && (6)\end{aligned}$$

$f^{-1}(A \backslash B) \implies f^{-1}(A) \backslash f^{-1}(B)$ and $f^{-1}(A) \backslash f^{-1}(B) \implies f^{-1}(A \backslash B)$, therefore they are equal.

(d) Assume a hypothetical $x \in f(A \cup B)$

$$
\begin{aligned}
x \in (A \cup B) &\implies f(x) \in f^{-1}(A \cup B) \implies f(x) \in (f^{-1}(A)) \vee (f(x) \in f^{-1}(B)) && (1)\\
&\implies (x \in A) \vee (x \in B) \implies (x \in f(A)) \vee (x \in f(B)) && (2)\\
&\implies x \in f(A) \cup f(B) && (3)
\end{aligned}
$$

Now, assume a hypothetical $x \in (f(A) \cup f(B))$

$$
\begin{aligned}
x \in (f(A) \cup f(B)) &\implies (x \in A) \vee (x \in B) && (4)\\
&\implies x \in (A \cup B) && (5)
\end{aligned}
$$

$f(A \cup B) \implies f(A) \cup f(B)$ and $f(A) \cup f(B) \implies f(A \cup B)$, therefore they are equal.

(e) Assume a hypothetical $f(x) \in f(A \cap B)$

$$
\begin{aligned}
&f(x) \in f(A \cap B) \implies x \in (A \cap B) \implies (x \in A) \wedge (x \in B) && (1)\\
&x \in A \implies f(x) \in f(A) && (2)\\
&x \in B \implies f(x) \in f(B) && (3)\\
&f(x) \in f(A) \cap f(x) \in f(B) \implies f(x) \in f(A) \cap f(B) && (4)
\end{aligned}
$$

A counter example of equality would prove that $f(A) \cap f(B) \implies f(A \cap B)$ does not hold. Assume that $f(x) = 3,\ A = \{1, 2, 3\}, B = \{4, 5, 6\}$. $A \cap B = \varnothing$, therefore $f(A \cap B) = \varnothing$. $f(A) \cap f(B) = \{3, 3, 3\} \cap \{3, 3, 3\} = \{3\}$. Therefore, $f(A \cap B) \supseteq f(A) \cap f(B)$ is false, disproving the equality.

(f) Assume a hypothetical $f(x) \in f(A) \backslash f(B)$

$$
\begin{aligned}
f(x) \in f(A) \backslash f(B) &\implies f(x) \in f(A) - f(B) \implies x \in (A - B) && (1)\\
&\implies (x \in A) \wedge (x \notin B) && (2)\\
&\implies f(x) \in f(A \backslash B) && (3)
\end{aligned}
$$

A counter example of equality would prove that $f(A \backslash B) \subseteq f(A) \backslash f(B)$ does not hold.
Assume that $f(x) = 3,\ A = \{1, 2, 3\}, B = \{3, 4, 5\}$. $A \backslash B = \{1, 2\}$, $f(A) = f(B) = \{3, 3, 3\}$ and $A \backslash B = \{1, 2\}$.
$f(A) \backslash f(B) = \varnothing$ and $f(A \backslash B) = \{3, 3\}$. $\varnothing \neq \{3\}$, disproving the equality.