CS70 Summer 2018 — Solutions to Homework 8

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Sundry

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up. — Sung Hyun Harvey Woo

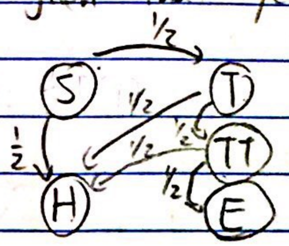
Homework 8

1. Average number of heads = 7

we can solve this problem using Markov Chains (hitting times)

S: start, H: head, T: 1 tail, TT: 2 tails in a row.
E: (TTT) 3 tails in a row

$\beta(i)$ = Expected number of heads to reach state E given that you started at state i



1. $\beta(S) = \frac{1}{2}\beta(T) + \frac{1}{2}\beta(H)$
2. $\beta(T) = \frac{1}{2}\beta(TT) + \frac{1}{2}\beta(H)$
3. $\beta(TT) = \frac{1}{2}\beta(H) + \frac{1}{2}\beta(E)$
4. $\beta(H) = 1 + \frac{1}{2}\beta(H) + \frac{1}{2}\beta(T)$ [only adding 1 here because being at state H, add 1 head to count]
5. $\beta(E) = 0$

(3, 5) $\beta(TT) = \frac{1}{2}\beta(H) + \frac{1}{2}\cdot 0 = \frac{1}{2}\beta(H)$

(2) $\beta(T) = \frac{1}{4}\beta(H) + \frac{1}{2}\beta(H) = \frac{3}{4}\beta(H)$

(4) $\beta(H) = 1 + \frac{1}{2}\beta(H) + \frac{3}{8}\beta(H) = \frac{7}{8}\beta(H) + 1 \rightarrow \frac{1}{8}\beta(H) = 1$

$\beta(H) = 8$

$\beta(T) = \frac{3}{4}\beta(H) = 6$

$\beta(TT) = \frac{1}{2}\beta(H) = 4$

$\beta(S) = \frac{1}{2}\beta(T) + \frac{1}{2}\beta(H) = 3 + 4 = 7$

$\therefore$ we expect an average of 7 heads before reaching 3 consecutive heads

2. (a) $P(X_3=i_3 \mid X_2=i_2, X_1=i_1) = P(X_3=i_3 \mid X_2=i_2)$ [S is state space]

$P(X_3=i_3 \mid X_2=i_2, X_1=i_1) =$

$= \sum_{i\in S} P(X_3=i_3 \mid X_2=i_2, X_1=i_1, X_0=i) \cdot P(X_0=i \mid X_2=i_2, X_1=i_1)$

$= \sum_{i\in S} P(X_3=i_3 \mid X_2=i_2) \cdot P(X_0=i \mid X_2=i_2, X_1=i_1)$ [Markov's property]

$= P(X_3=i_3 \mid X_2=i_2) \cdot \underbrace{\sum_{i\in S} P(X_0=i \mid X_2=i_2, X_1=i_1)}_{\to 1}$ [Total probability theorem]

$= P(X_3=i_3 \mid X_2=i_2) \cdot 1 = P(X_3=i_3 \mid X_2=i_2)$

$\therefore P(X_3=i_3 \mid X_2=i_2, X_1=i_1) = P(X_3=i_3 \mid X_2=i_2)$

(b) $P(X_3=i_3 \mid X_1=i_1, X_0=i_0) = P(X_3=i_3 \mid X_1=i_1)$

$\hookrightarrow = \sum_{i\in S} P(X_3=i_3 \mid X_2=i, X_1=i_1, X_0=i_0)\, P(X_2=i \mid X_1=i_1, X_0=i_0)$

$= \sum_{i\in S} P(X_3=i_3 \mid X_2=i) \cdot P(X_2=i \mid X_1=i_1)$

[using Markov's property]

$= \sum_{i\in S} P(X_3=i_3 \mid X_2=i, X_1=i_1) \cdot P(X_2=i \mid X_1=i_1)$

[Markov's property]

$= \sum_{i\in S} P[X_3=i_3, X_2=i \mid X_1=i_1] \to \left[\sum_{i\in S} P_{X_2=i} = 1\right]$ (total probability)

$= P[X_3=i_3 \mid X_1=i_1]$

$\therefore P(X_3=i_3 \mid X_1=i_1, X_0=i_0) = P(X_3=i_3 \mid X_1=i_1)$

(c) $P(X_1=i_1 \mid X_2=i_2, X_3=i_3) = P(X_1=i_1 \mid X_2=i_2)$

$$\hookrightarrow = \frac{P[X_3=i_3, X_2=i_2, X_1=i_1]}{P(X_3=i_3, X_2=i_2)}$$

$$= \frac{P(X_3=i_3 \mid X_2=i_2, X_1=i_1)\, P(X_2=i_2, X_1=i_1)}{P(X_2=i_2, X_3=i_3)}$$

$$= \frac{P(X_3=i_3 \mid X_2=i_2)\, P(X_1=i_1 \mid X_2=i_2)\, P(X_2=i_2)}{P(X_3=i_3 \mid X_2=i_2)\, P(X_2=i_2)}$$ [Markov's Property

$= P(X_1=i_1 \mid X_2=i_2)$

3. Blackjack

(a) State space ($\mathcal{X}$): $\mathcal{X} = \{0, 100, 200, 300, 400\}$

0 : current balance is \$0

100 : current balance is \$100

200 : current balance is \$200

300 : current balance is \$300

400 : current balance is \$400

transition probabilities (matrix)

$$
\begin{array}{c|ccccc|}
 & 0 & 100 & 200 & 300 & 400 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 \\
100 & 1-p & 0 & p & 0 & 0 \\
200 & 0 & 1-p & 0 & p & 0 \\
300 & 0 & 0 & 1-p & 0 & p \\
400 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Starting state is 100 (balance \$100)

(b) recurrent states are 0, 400

When we get to 0, 400, you can only get to 0, 400, respectively with a probability of 1.

meaning you cannot go to another state from 0 or 400.

Therefore, they are recurrent and absorbing.

The transient states are 100, 200, and 300 because they are not recurrent. For example, you can go from 300 to 400, but not from 400 to 300.

(C) Assume $\alpha(i)$ is the probability of going to state 400 starting at state i

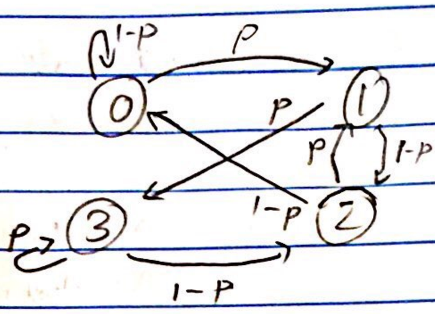
$$\alpha(0)=0, \quad \alpha(100)=p\alpha(200)$$
$$\alpha(200)=(1-p)\alpha(100)+p\alpha(300)$$
$$\alpha(300)=(1-p)\alpha(200)+p\alpha(400)$$
$$\alpha(400)=1.$$

$$\alpha(300)=(1-p)\alpha(200)+p\cdot 1=(1-p)\alpha(200)+p.$$
$$\alpha(200)=(1-p)p\alpha(200)+p((1-p)\alpha(200)+p)$$
$$=(1-p)p\alpha(200)+p(1-p)\alpha(200)+p^2.$$
$$=2p(1-p)\alpha(200)+p^2.$$
$$[1-(2p(1-p))]\alpha(200)=p^2$$
$$\alpha(200)=\frac{p^2}{1-2p+2p^2}, \quad \alpha(100)=p\alpha(200)$$

$$\alpha(100)=p\cdot\frac{p^2}{1-2p+2p^2}=\frac{p^3}{2p^2-2p+1}$$

$\therefore$ The probability of starting at state 100 and ending the game at 400 is $\frac{p^3}{2p^2-2p+1}$.

4. (a) Markov chain



0 : both compartments empty
1 : top empty, bottom full
2 : top full, bottom empty
3 : both compartments full

Assume $\beta(i)$ = Expected number of transitions until both compartments are full.

$$\beta(0) = 1 + (1-P)\,\beta(0) + p\beta(1)$$
$$\beta(1) = 1 + (1-P)\beta(2) + p\beta(3)$$
$$\beta(2) = 1 + (1-P)\beta(0) + p\beta(1)$$
$$P(3) = 0$$

where $\beta(0)$ is the expected number of transitions (seconds) before both compartments are full

(b) same equations as part 4.a, except the seconds required to carry out certain transitions are different.

$$\beta(0) = (1-P)(\beta(0)+1) + p(\beta(1)+2)$$ $P_{0,1} = 2$
$$\beta(1) = (1-P)(\beta(2)+2) + P(\beta(3)+3)$$ $P_{1,2} = 2$, $P_{2,3} = 3$
$$\beta(2) = (1-P)[\beta(0)+1) + P(\beta(1)+2)$$ $P_{2,1} = 2$.
$$\beta(3) = 0$$

(C) to get $\pi_0 \sim \pi_3$ we set up the system of linear equations

$$\begin{cases} \pi_0 = (1-p)\pi_2 + (1-p)\pi_0 \\ \pi_1 = p\pi_2 + p\pi_0 \\ \pi_2 = (1-p)\pi_1 + (1-p)\pi_3 \\ \pi_3 = p\pi_1 + p\pi_3 \end{cases} \quad [\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_0 = (1-p)\pi_2 + (1-p)\pi_0$$

$$1-(1-p)\pi_0 = 1-p\,\pi_2 \rightarrow \pi_0 = \frac{1-p}{p}\pi_2$$

$$\pi_3 = p\pi_1 + p\pi_3$$

$$(1-p)\pi_3 = p\pi_1 \rightarrow \pi_3 = \frac{p}{1-p}\pi_1$$

$$\pi_1 = p\pi_2 + p\pi_0 = p\pi_2 + p\left(\frac{1-p}{p}\right)\pi_2 = (p+1-p)\pi_2 = \pi_2$$

$$\pi_1 = \pi_2$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

$$\frac{1-p}{p}\pi_1 + \pi_1 + \pi_1 + \frac{p}{1-p}\pi_1 = 1$$

$$\left(\frac{(1-p)^2 + p^2}{p(1-p)} + 2\right)\pi_1 = 1$$

$$\pi_1 = \frac{p(1-p)}{(1-p)^2 + p^2 + 2p(1-p)} = \frac{p(1-p)}{(1-p)(1-p+2p)+p^2} = \frac{p(1-p)}{(1-p)(1+p)+p^2}$$

$$= \frac{p(1-p)}{1^2 - p^2 + p^2} = p(1-p)$$

$$\pi_1 = p(1-p) \qquad \pi_3 = \left(\frac{p}{1-p}\right)\cdot p(1-p) = p^2$$

$$\pi_2 = p(1-p) \qquad \pi_0 = \frac{1-p}{p}\cdot p(1-p) = (1-p)^2$$

$$\pi = [\,(1-p)^2 \quad p(1-p) \quad p(1-p) \quad p^2\,]$$

only states 2,3 can lead to Won Fung swallowing boba therefore we add the two together and multiply by 10 (calories)

$$= p(1-p) + p^2 = p - p^2 + p^2 = p \qquad \therefore \boxed{10p}$$

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(d) average (long run) number of boba inside straw can be found by using the invariant probabilities

$$\pi = [\ (1-p)^2 \quad p(1-p) \quad p(1-p) \quad p^2\ ]$$

$\downarrow$ 0 balls, $\downarrow$ 1 ball, $\downarrow$ 1 ball, $\downarrow$ 2 balls

The avg number of boba inside straw is:

$$[0 \cdot (1-p)^2] + [1 \cdot p(1-p)] + [1 \cdot p(1-p)] + [2 \cdot p^2]$$

$$= 2p(1-p) + 2p^2 = 2p - 2p^2 + 2p^2 = 2p$$

$\therefore \boxed{2p}$