CS70 Summer 2018 — Solutions to Homework 6

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I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up. — Sung Hyun Harvey Woo

Homework 6

1. (a) Sample space

Each die has k possible values, and there are n rolls. Each roll is independent from the others, therefore the sample space is:

$$k_1 \cdot k_2 \cdot k_3 \cdot k_4 \cdots k_n = k^n$$

(b) Distribution for X_i, $1 \le i \le k$

Each die has an equal $\frac{1}{k}$ chance of getting any i face

so $P(X_i = j)$ $1 \le j \le n$

$$P_X(X_i = j) = \binom{n}{j}\left(\frac{1}{k}\right)^j \left(\frac{k-1}{k}\right)^{n-j}$$

(c) Joint distribution

$P_X(X_1 = a_1, X_2 = a_2, X_3 = a_3 \ldots X_k = a_i)$ [dubbed $P_X(A)$]

If $a_1 + a_2 + \ldots + a_i > n$ (rolls) or $a_1 + a_2 + \ldots + a_i < n$

$P_X(A) = 0$, there cannot be more or less than n faces total

If $a_1 + a_2 + \ldots + a_i = n$, there are $n!$ total outcomes of n rolls, and each face is the same as another die with the same face, therefore

$$P_X(X_1 = a_1, X_2 = a_2 \ldots X_k = a_i) = \begin{cases} \dfrac{\frac{n!}{x_1! x_2! \ldots x_k!}}{k^n}, & \text{if } a_1 + a_2 + \ldots + a_i = n \\ 0, & \text{if } a_1 + a_2 + \ldots a_i \neq n. \end{cases}$$

(d) No. They are dependent

They are dependent because the result of one changes the probability of another. In other words,

$P_{x_1,x_2}(X_1 = a_1, X_2 = b_2) \neq P(X_1 = a) \cdot P(X_2 = b)$ [Independence]

$P(X_1 = a) = \binom{n}{a}\left(\frac{1}{k}\right)^a\left(1-\frac{1}{k}\right)^{n-a}$

$P(X_2 = b) = \binom{n}{b}\left(\frac{1}{k}\right)^b\left(1-\frac{1}{k}\right)^{n-b}$

$P_{x_1,x_2}(X_1 = a_1, X_2 = b_2) = P(X_1 = a)\, P(X_2 = b \mid X_1 = a)$

$= \binom{n}{a}\left(\frac{1}{k}\right)^a\left(1-\frac{1}{k}\right)^{n-a} \cdot \binom{n-a}{b}\left(\frac{1}{k}\right)^b\left(\frac{1}{k-1}\right)^{n-b}$

$\neq P(X_1 = a) \cdot P(X_2 = b)$

2\. (a) $3^{|V|}$

Each vertex has 3 options, A, B, C, with uniform distribution. There are $|V|$ vertices. Our sample space is all configurations of $|V|$ vertices, each going to partition A, B, or C. The size of our sample space is $3^{|V|}$.

(b) $2/3$

$P(u, v \text{ in different sets}) = P(u \text{ in any set}, v \text{ not in } u\text{'s set})$

[independent events] $= P(u \text{ in any set}) \cdot P(v \text{ not in } v\text{'s set})$

$= 1 \cdot \left(1 - \frac{1}{3}\right) = 1 \cdot \frac{2}{3} = 2/3.$

Each vertex being sorted is independent from another therefore the probability that u, v are in different sets is equal to the probability that v is not in u's set. Therefore the probability is $1 \cdot \left(1-\frac{1}{3}\right) = \frac{2}{3}$.

(c) Let us have an indicator variable X_e for each edge in E. X is a random variable for the expected number of edges that cross between sets.

$$X_e = \begin{cases} 1, & \text{if } e \text{ goes from one set to another} \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{e \in E} X_e = \sum_{e \in E} P(X_e = 1)$$

$$P(X_e = 1) = 1 \cdot \frac{2}{3} = \frac{2}{3} \quad \text{[as seen in part 2.b]}$$

$$\sum_{e \in E} P(X_e = 1) = \sum_{e \in E} \frac{2}{3} = \frac{2|E|}{3}.$$

$$\therefore \frac{2|E|}{3}$$

(d) Given that the number of edges crossing is at least $\frac{2|E|}{3}$, as the weighted average is $\frac{2|E|}{3}$. Therefore, if there exists a subset of the sample space that has less than $\frac{2|E|}{3}$ edges that cross, there must be a subset of the sample space such that there are more than $\frac{2|E|}{3}$ edges that cross between sets. In the case that a subset has all $\frac{2|E|}{3}$ edges, the rest must have an average of $\frac{2|E|}{3}$. Thus, there exists some partition A, B, C of V such that the number of edges that cross between the sets is at least $\frac{2|E|}{3}$.

3. males and females have the same average number of sisters
Both binomial distributions

For boys, we have $P(B)=\frac{1}{2}$, $E(B)=\frac{n}{2}$.
and $X=B(N-B)=NB-B^2$.

$E[X]=nE(B)-E[B^2]$ $\quad var(x)=np(1-p)$

$var(X)=E[B^2]-E[B]^2$

$E[B^2]=var(x)+E[B]^2$, $E[B]^2=\frac{n}{2}\cdot\frac{n}{2}=\frac{n^2}{4}$

$=np(1-p)+\frac{n^2}{4}$

$=\frac{n}{2}-\frac{n}{4}+\frac{n^2}{4}$

$=\frac{n^2}{4}+\frac{n}{4}$.

$E[X]=n\cdot\frac{n}{2}-(\frac{n^2}{4}+\frac{n}{4})=\frac{n^2}{2}-\frac{n^2}{4}-\frac{n}{4}=\frac{n^2}{4}-\frac{n}{4}$

For girls, using the same process,
$P(G)=\frac{1}{2}$ $E(G)=\frac{n}{2}$ and $Y=G(G-1)$, since
you do not count yourself as a sister.

$E[Y]=G(G-1)=G^2-G$

$E[Y]=E(G^2)-E[G]$ $\quad var(Y)=np(1-p)$

$E(G^2)=var(x)+E[G]^2$

$=np(1-p)+\frac{n^2}{4}$

$=\frac{n}{2}-\frac{n}{4}+\frac{n^2}{4}$

$=\frac{n^2}{4}+\frac{n}{4}$

$E[Y]=E(G^2)-E(G)=\frac{n^2}{4}+\frac{n}{4}-\frac{n}{2}=\frac{n^2}{4}-\frac{n}{4}$

Both boys an girls have an expected $(\frac{n^2}{4}-\frac{n}{4})$ sisters on average. Thus, males and females have the same number of sisters on average.

4. (a) $P_z(X+Y=j) = e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^j}{j!}$

$P_x(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$

$P_y(Y=l) = e^{-\mu}\frac{\mu^l}{l!}$

$P_z(X+Y=j) = \sum_{i=0}^{j} P(X+Y=j, X=i)$

$= \sum_{i=0}^{j} P(X=i)P(X+Y=j)$ [independent]

$= \sum_{i=0}^{j} P(X=i)\cdot P(X+Y=j)$

$= \sum_{i=0}^{j} e^{-\lambda}\frac{\lambda^i}{i!}\cdot e^{-\mu}\frac{\mu^{j-i}}{(j-i)!}$

$= e^{-(\lambda+\mu)}\sum_{i=0}^{j}\frac{\lambda^i}{i!}\frac{\mu^{j-i}}{(j-i)!}$

$= e^{-(\lambda+\mu)}\frac{1}{j!}\sum_{i=0}^{j}\binom{j}{i}\lambda^i\mu^{j-i}$

$= e^{-(\lambda+\mu)}\frac{1}{j!}\sum_{i=0}^{j}\binom{j}{i}\lambda^i\mu^{j-i}$ [binomial theorem]

$= e^{-(\lambda+\mu)}\frac{1}{j!}(\lambda+\mu)^j$

$$P_z(X+Y=j) = e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^j}{j!}$$

Using X, Y independence and the binomial theorem we can see that $X+Y \sim Pois(\lambda+\mu)$.

(b) $(X|X+Y) \sim Bin(l, \frac{\lambda}{\lambda+\mu})$

$P(X|X+Y) = \frac{P(X=k)P(Y=l-k)}{P(X+Y=l)} = \frac{e^{-\lambda}\frac{\lambda^k}{k!}\cdot e^{-\mu}\frac{\mu^{l-k}}{(l-k)!}}{P(X+Y=l)} = \frac{e^{-(\lambda+\mu)}\frac{\lambda^k\mu^{l-k}}{k!(l-k)!}}{P(X+Y=l)}$

$P(X+Y=l) = \sum_{k=0}^{l} e^{-\lambda}\frac{\lambda^k}{k!}\cdot e^{-\mu}\frac{\mu^{l-k}}{(l-k)!}$

$= e^{-(\lambda+\mu)}\sum_{k=0}^{l}\frac{\lambda^k\mu^{l-k}}{k!(l-k)!}$

using the binomial theorem:

$\sum_{i=0}^{n}\binom{n}{i}\lambda^i\mu^{n-i} = (\lambda+\mu)^n \rightarrow \sum_{i=0}^{n}\frac{\lambda^i\mu^{n-i}}{i!(n-i)!} = \frac{(\lambda+\mu)^n}{n!}$

Therefore,

$P(X+Y=l) = e^{-(\lambda+\mu)}\sum_{k=0}^{l}\frac{\lambda^k\mu^{l-k}}{k!(l-k)!} = e^{-(\lambda+\mu)}\frac{(\lambda+\mu)^l}{l!}$

$P(X|X+Y) = \frac{e^{-(\lambda+\mu)}\frac{\lambda^k\mu^{l-k}}{k!(l-k)!}}{P(X+Y=l)} = \frac{e^{-(\lambda+\mu)}\frac{\lambda^k\mu^{l-k}}{k!(l-k)!}}{\frac{(\lambda+\mu)^l}{l!}e^{-(\lambda+\mu)}} = \frac{l!}{k!(l-k)!}\frac{\lambda^k\mu^{l-k}}{(\lambda+\mu)^l}$

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$= \binom{l}{k}\frac{\lambda^k\mu^{(l-k)}}{(\lambda+\mu)^k(\lambda+\mu)^{l-k}} = \binom{l}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{l-k}$. $\therefore (X|X+Y) \sim Bin(l, \frac{\lambda}{\lambda+\mu})$

5. (a) $E[X]$

mean

$$Y = \frac{x_1 + x_2 \ldots + x_n}{n} \qquad E[Y] = E\left[\frac{x_1}{n} + \frac{x_2}{n} + \ldots \frac{x_n}{n}\right]$$

$$E[Y] \triangleq E[X]$$

(b) $\frac{\sigma^2}{n}$

$$E(Y^2) = E\left(\left(\frac{x_1 + x_2 + \ldots x_n}{n}\right)^2\right) = E\left(\frac{(x_1 + x_2 + \ldots + x_n)^2}{n^2}\right) = \frac{1}{n^2} E\left((x_1 + x_2 + \ldots + x_n)^2\right)$$

$$= \frac{1}{n^2}\left(\sum_i^n E[X_i^2] + \sum_{i \neq j} E(X_i X_j)\right)$$

$$= \frac{1}{n^2}\left(n E[X^2]\right)$$

$$= \frac{1}{n} E[X^2] \qquad / \quad var(X) = E[X^2] - E(X)^2.$$

$$= \frac{var(Y) + E(Y)^2}{n} \qquad E[X^2] = var(x) + E(x)^2$$

$$= \frac{var(x) + E(x)^2}{n} = \frac{\sigma^2}{n} \qquad (E(x) \text{ is estimated as } 0)$$

(c) $(n-1)\sigma^2$

$$E(Z) = E\left[\sum_{i=1}^n (x_i - Y)^2\right]$$

$$= \sum_{i=1}^n E[x_i^2] - E(nY^2)$$

$$= \sum_{i=1}^n (\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= \sum_{i=1}^n (\sigma^2 + \mu^2) - \sigma^2 - n\mu^2.$$

$$= (n-1)\sigma^2$$

(d) $\frac{E[Z]}{n-1}$

from part 5.c we have $E[Z] = (n-1)\sigma^2$
If we divide this by $(n-1)$ we get
σ^2, giving us a good estimate for σ^2

(e) $\frac{E[Z]}{n}$

I expected it to be simply $\frac{E[Z]}{n}$ since
It was the estimate for $n\sigma^2$, however this
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is not the case because Y does not consider
all samples of X.

6. (a) $var(X) \geq 0$

$var(X) = \sum_{x_i} (x_i - E[X])^2 \, p(X=x)$

since $p(X=x)$ has to be a value between 0 and 1 (probability) and each $(x_i - E[x])^2$ has to be positive because it is a square value, variance (x) is a positive multiplied by a positive number so $var(X) \geq 0$

(b)

$$Var(X_1 + \ldots + X_n) = E[(X_1 + \ldots + X_n)^2] - E[X_1 + \ldots + X_n]^2$$
$$= \sum_{i=1}^{n} E[X_i^2] + 2\sum_{i<j} E[X_i X_j] - [E(X_1) + E(X_2) + \ldots E[X_n]]^2$$

$$[E(X_1) + E(X_2) + \ldots E(X_n)]^2 = \sum_{i=1}^{n} E(X_i)^2 + 2\sum_{i<j} E[X_i][X_j]$$

$$Var(X_1 + \ldots + X_n) = \sum_{i=1}^{n} E[X_i^2] - E(X_i)^2 + 2\sum_{i<j} E(X_i X_j) + 2\sum_{i<j} E[X_i][X_j]$$
$$= \sum_{i=1}^{n} var(X_i) + 2\,cov(X_i X_j)$$

(c)

$$\sum_{i=1}^{n} a_i^2 \cdot Var(X_i) + 2 \sum_{1 \leq i < j \leq n} cov(X_i, X_j) \cdot a_i \cdot a_j$$

using the properties of multiplying constants to variance

$$a_i^2 \cdot Var(X_i) = var(a_i X_i)$$

and also using the properties of constants and covariance we have:

$$2\sum_{1 \leq i < j \leq n} a_i a_j \, Cov(X_i, X_j) = 2\sum_{1 \leq i < j \leq n} Cov(a_i X_i, a_j X_j)$$

$\hookrightarrow \sum_{i=1}^{n} var(X_i) + 2\sum_{1 \leq i < j \leq n} cov(a_i X_i, a_j X_i)$

is simply equal to $var[\sum_{i=1}^{n} a_i X_i]$

Since variance has to be a positive number or 0 7

$$\sum_{i=1}^{n} a_i^2 \cdot var(X_i) + 2\sum_{1 \leq i < j \leq n} cov(X_i, X_j) a_i a_j \geq 0$$

Scanned with CamScanner

7. (a) 50

s can range from $0 \sim 100$

$$E[S] = \frac{\sum_{i=0}^{100} i}{101} = \frac{101 \cdot 50}{101} = 50$$

$\therefore E[S] = 50$

(b) $\frac{s+101}{2}$

$$E[V \mid S=s] = \sum_v v \cdot P_x(v, S=s)$$

$$P_x(v, S=s) = \frac{1}{(101-s)}$$

$$\sum_v v \cdot P_x(v, S=s) = \sum_v v \cdot \frac{1}{(101-s)}$$

$$= \frac{1}{(101-s)} \sum_v v$$

$$= \frac{1}{(101-s)} \sum_{s=v}^{100} v$$

$$= \frac{1}{(101-s)} \frac{(101-s)(101+s)}{2} = \frac{101+s}{2}$$

(c) 75

$$E[V] = \sum_{s=0}^{100} P(s)\, E[V \mid S=s]$$

$$= \frac{1}{101} \sum_{s=0}^{100} E[V \mid S=s] \quad \rightarrow \text{from part 7.b}$$

$$= \frac{1}{101} \sum_{s=0}^{100} \frac{101+s}{2} = \frac{1}{101}\left[\frac{101}{2} + \frac{102}{2} + \ldots + \frac{201}{2}\right]$$

$$= 75$$

8. (a) $\frac{n-k}{n}$

since Bob helping each student does not eliminate them from the possible students from the R.S. button, it is $\frac{n-k}{n}$.

(b) $(1-\frac{1}{n})^r \le e^{-\frac{r}{n}}$

$$P[X_i^r] = \left(\frac{n-1}{n}\right)^r = \left(1-\frac{1}{n}\right)^r \qquad 1-x \le e^{-x}$$

$$\hookrightarrow \left(1-\frac{1}{n}\right)^r \le e^{-\frac{r}{n}}$$

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(c) $\bigcup_{i=1}^{n} X_i^r$

T_r is the same as X_i^r but for each student. Bob could have pressed the button r times but the number of students he has not helped can range from $1 \sim n$. Therefore,

$$T_r = \bigcup_{i=1}^{n} X_i^r$$

(d) $ne^{-r/n}$

$P[T_r] = n(1-\frac{1}{n})^r$ and since $1-x \leq e^{-x}$,

$P[T_r] = n(1-\frac{1}{n})^r \leq ne^{-\frac{r}{n}}$

(e) $n^{-\alpha}$

$r = \alpha n \ln n$

$e^{-\alpha \frac{n \ln n}{n}} = e^{-\alpha \ln n} = (e^{\ln n})^{-\alpha} = n^{-\alpha}$

(f) $n^{-\alpha+1}$

$n(1-\frac{1}{n})^{\alpha n \ln n} \leq n e^{-\alpha \ln n}$ using $1-x \leq e^{-x}$

$ne^{-\alpha \ln n} = n(e^{\ln n})^{-\alpha} = n n^{-\alpha} = n^{-\alpha+1}$

(g) $r \leq 3n \ln(n)$

$ne^{-r/n}$ has to be at most $\frac{1}{n^2}$.

$$ne^{-r/n} \leq \frac{1}{n^2}$$

$$e^{-r/n} \leq \frac{1}{n^3}$$

$$\ln(e^{-r/n}) \leq \ln(\frac{1}{n^3})$$

$$-\frac{r}{n} \leq \ln(\frac{1}{n^3})$$

$$r \leq -n \ln(\frac{1}{n^3})$$ - power rule for logs

$$r \leq 3n \ln(n)$$