CS70 Summer 2018 — Solutions to Homework 2

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Sundry

I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up. — Sung Hyun Harvey Woo

1. Hit or Miss?

(a) Incorrect.
The claim asserts that $n^2 \geq n$ for all positive real numbers. The base case starts at 1, which means that the real numbers from 0 to 1 have not been considered in this proof regardless of if it is valid for numbers bigger than 1.

(b) Correct.

(c) Incorrect.
The process of converting $n+1$ to $a+b(0 < a, b \leq n)$ is flawed since it cannot be applied to when $n+1=0$. $0+1 \neq a+b(0 < a, b \leq n)$.

2. Coin Game

Proof by strong induction.

Base Case : $n = 1$.
If n = 1, the game ends immediately since all stacks have 1 coin. The total score would be 0. $\frac{1(1-0)}{2} = 0$, so we know that the claim holds for when $n = 1$.

Induction Hypothesis: $1 \leq n \leq k$.
Assume that for $1 \leq n \leq k$, where k is an arbitrary integer, no matter how we choose to split the stacks, the total score will always be $\frac{n(n-1)}{2}$.

Induction Step: $n = k+1$.
Given $k+1$ coins, we know that no matter how we split the stack into stack a and stack b, $1 \leq a, b \leq k$. According to our induction hypothesis, we know that the total score for both stack a and stack b would be $\frac{a(a-1)}{2}$, and $\frac{b(b-1)}{2}$ respectively. The score achieved from splitting $k+1$ coins is $a * b$. Therefore, using $k+1 = a+b$, we know that the total score for the game when $n = k+1$ is :

$$ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} = \frac{a^2 + b^2 + 2ab - a - b}{2} \tag{1}$$

The score from the claim for when $n = k+1$ is:

$$\frac{(k+1)k}{2} = \frac{(a+b)(a+b-1)}{2} = \frac{a^2 + b^2 + 2ab - a - b}{2} \tag{2}$$

$ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} = \frac{(k+1)k}{2}$, therefore we know that for any n coins, no matter how we split the stacks, the total score will always be $\frac{n(n-1)}{2}$.

3. Calculator Enigma

Proof by well-ordering principle on the number of operators, n.
For simplicity, I will consider a number that can be expressed as $\frac{3k}{10^l}$ as being a part of set A.

We have a base case of $n = 0$. When there are no operators (+, -, *), the only values that are available are the combinations of the numbers 3, 6, 9, in which case we know that it is part of A. since we can split the number into its digits such that the form is $a_n * 10^n + a_{n-1} * 10^{n-1} + ... + n_0 * 10^0$, where a_n to a_0 are multiples of 3 and thus a part of A. It can then be grouped into $3(\frac{a_n}{3} * 10^n + \frac{a_{n-1}}{3} * 10^{n-1} + ... + \frac{a_0}{3} * 10^0)$, showing that a number with 0 operators is a part of A. Given this base case, we know that there is at least one value of n in which the claim holds true.

Assume that we have a set of the values of n, denoted as R, where there exists a number that can be displayed with n operators, but is not a part of A. Since $R \subseteq \mathbb{N}$, we know that there is a least element in set R which we will denote n_0. Given this least element, we know that any numbers displayed with $n_0 - 1$ operators and below will be a part of A.
From here, we have a hypothetical number, denoted G, that is not a part of A, displayed by using n_0 operators, which we can divide into 3 cases according to the operators: addition, subtraction, multiplication. To avoid repetition, I will combine the three cases into one. Each of the cases are can be made using the same logic but using each respective operator.

If we have a G displayed using addition such that $G = G_0 + G_1 + G_2 + ... + G_{n-1}$, where G_0 to G_{n-1} are expressions on their own, we can separate the addition into two parts where one term is $G_0 + G_1 + G_2 + ... + G_{n-2}$ and the other term is simply G_{n-1}. The first term is made up of $n_0 - 1$ operators while the second term is made up of 0 operators. Both 0 and $n_0 - 1$ is not included in the set R and therefore both the first and second terms are a part of A. We add the two terms together, increasing the number of operators by 1, giving us the original G value using n_0 operators. If we add, subtract, or mutliply elements in A, we can see below that it yields a value that is also in A (B and C are arbitrary numbers that are a part of A).

$$B + C = 3(\frac{B}{3} + \frac{C}{3}) \tag{3}$$

and for when $G = G_0 - G_1 - G_2 - ... - G_{n-1}$

$$B - C = 3(\frac{B}{3} - \frac{c}{3}) \tag{4}$$

and for when $G = G_0 * G_1 * G_2 * ... * G_{n-1}$

$$B * C = 3(\frac{B}{3} * \frac{C}{3}) \tag{5}$$

Therefore, for all operators, we can express G, displayed through n_0 operators, in terms of $\frac{3k}{10^l}$, which contradicts our well-ordering principle in the set R. Since there cannot be a least element in R, we can prove the claim that the numbers that can be displayed on this calculator can be expressed as $\frac{3k}{10^l}$.

4. Build-Up Error?

The inductive step for this proof begins with a graph with n vertices, and builds its way up to a graph with (n+1) vertices. This method is acceptable if the process took a graph with n + 1 vertices, removed a vertex, then added the vertex back, but not when it starts with an n-vertex graph. The process in the incorrect proof allows the author to construct the graph according to a custom set of conditions, which can cause issues when it comes to proving a claim for an arbitrary number or variable such as an arbitrary (n + 1) vertex graph.

A counterexample would be a disconnected graph. Given vertices A, B, C, D with edge A,B and C,D, it is not possible to make a connected graph when a degree one vertex E is added to the graph.

5. Proofs in Graphs

(a) Proof by induction.

Base Case : $n = 2$.
The base case for this proof is 2 cities since the condition $n \geq 2$ exists for n. We will denote a city that is reachable from every other city by traveling through at most 2 roads to be a "special city". If we have two cities, there is only one road. Either it goes from city 1 to city 2, in which case the special city would be city 2, or it goes from city 2 to city 1, in which case the special city would be city 1. Therefore when n is 2, the claim holds.

Inductive Hypothesis: Assume that the claim holds true for when there are k cities where k is an arbitrary integer.

Inductive Step:
Assume that we have $k + 1$ cities. If we take one city, denoted z, away from the map along with all its roads, we have a map with k cities, which we know has a special city as per our induction hypothesis. We will denote this special city of k cities as A. Every city in the k-city map can reach A using at most 2 roads. We make two sets. The first set, A_1 is the set of cities that can reach the special city A with one road, and the second set, A_2, is the set of cities that can only reach the special city with two roads (by using one of the A_1 elements).
We add city z back into the map and we add back the roads that either go from z to a city or go from a city to z. If the road between the original A and z goes directly from z to A, our special city remains A. However, if we have the road go from A to z, we will need further proof that a special city exists.
Given city z, assume A has a road to it. Every element in set A_1 either has a road to z or z has a road to it. If all elements in A_1 has a road to z, we know that z is the new special city since elements in A_2 can get to z using A_1. Conversely, If z has a road to all elements in A_1, then z can go through the elements in A_1 to get to A in two roads, meaning our special city will remain A. However, if there is a mix of roads, such that some elements of A_1 have roads to z, while other elements will have a road from z to it, z can use one of the roads that it has to an element to A_1 to get to A within two roads, meaning that the special city will again remain A. In all possible cases, we see that there exists a special city for when there are $k + 1$ cities, meaning that through induction, the claim holds for all $n \geq 2$.

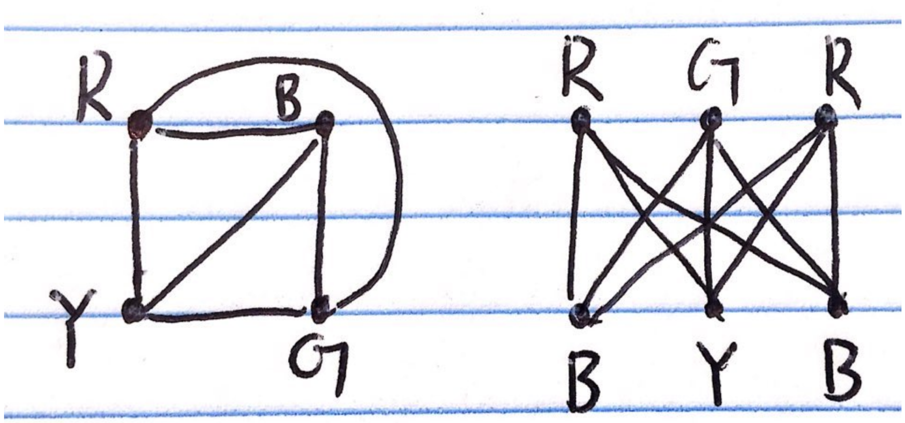
(b) If we are given a connected graph G with n vertices, of which $2m$ vertices have an odd degree, we can create an Eulerian tour of the graph by adding a hypothetical edge between a pair of odd vertices for all $2m$ odd vertices, giving us m hypothetical edges. Now that these m hypothetical edges are in place, the graph is connected and all vertices are of even degree. Therefore there is an Eulerian tour for the new graph, which we will denote as G'.

From here we begin to create m Eulerian walks for G' similar to splicing together smaller disconnected Eulerian tours to create a Eulerian tour. We erase each of the m hypothetical edges that we created from G', disconnecting the Eulerian tour into different sub-sections. We use the two odd degree vertices that the edge was connected to to create a walk from one odd vertex to the other odd vertex. This walk can be any walk that would create a smaller cycle from the starting odd degree vertex to itself (without repeating edges) if the hypothetical edge between the two odd degree vertices existed. We repeat this process, eliminating one hypothetical edge between two odd degree vertices every time a walk is created until m walks are created and all 2m odd degree vertices are used up. Each walk eliminates 2 of the 2m odd degree vertices, and creates a walk. When m walks are created, we can be sure that because these walks were part of the Eulerian tour of G', the walks combined cover all the edges of G without any repeating edges.

6. Always, Sometimes, or Never

(a) Either

A planar example would be a K_4 complete graph, and a non-planar example would be a $K_{3,3}$ complete graph as shown below.

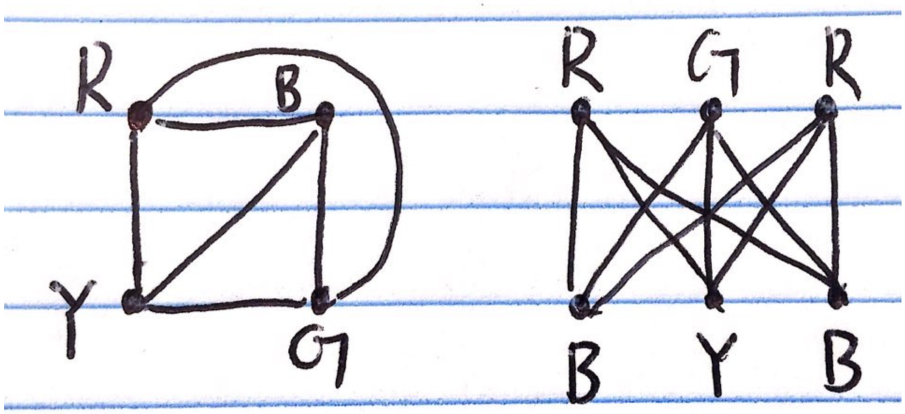


Both graphs can be colored with 4 colors, but one is planar and one is non-planar

(b) Non-planar.

Theorem 5.4 in note 5 proved that every planar graph can be colored with five colors. The contrapositive of that would be that if a graph cannot be colored with five colors, it must be non-planar. Any graph G in which 7 colors are required would be non-planar. An example would be a K_7 complete graph, which requires 7 colors, but contains a K_5 complete graph meaning that it is non-planar.

(c) Either.
It was proven that all planar graphs satisfy this equation, however satisfying this equation does not necessarily mean that the graph is planar. An example of a planar graph that satisfies this equation is the K_4 complete graph whereas an example of a non-planar graph that satisfies this equation would be a $K_{3,3}$ graph, as shown below.



More specifically, $e \leq 3v - 6$ is satisfied for both graphs.
For K_4, $4 \leq 3(4) - 6$, which is equal to $4 \leq 6$, satisfying the inequality.
For $K_{3,3}$, $9 \leq 3(6) - 6$, which is equal to $9 \leq 12$, satisfying the inequality.

(d) Planar.
G is connected, and each vertex has a degree of at most 2. Assume a special case, G' where all vertices of graph has a degree of 2. If we prove that G' is planar, we can prove that the claim is true(Planar), because we can get all graphs G, by removing a number of edges from G'. We know that removing an edge does not affect planarity, therefore, if we prove G', where the degree of all vertices is 2, is planar, we can prove the claim to be planar.
Given G', we know that the sum of the degrees of all vertices is equal to $2 *$(Number of edges).

$$2|V| = 2|E|, |V| = |E| \tag{6}$$

where V is the set of all vertices, and E is the set of all edges.
We have the same number of vertices and edges, and all vertices have a degree of 2, so we know that the only possible graphs that satisfies these conditions are cycles, in which case we know that this graph is planar regardless of the number of vertices.
Therefore, we can claim that if G is connected and each vertex has a degree of at most 2, G will be planar.

(e) Planar.
If each vertex in graph G has a degree of at most 2, we have two cases. Either graph G is connected or it is disconnected. Since we proved that if G is connected and each vertex has a degree of at most 2, the graph will be planar, all we have to do is consider the case for when graph G is disconnected.

If G is disconnected, we can use the same method as problem 6.d and use the fact that the planarity of a graph is not affected when removing an edge. Assume that we have a graph G', which is disconnected and each vertex has a degree of 2. We can get all possible graphs for G by removing edges from G', and since planarity is not affected by the removal of edges, if we can prove G' is planar, we can prove the claim to be planar.

Given G', we know that the sum of the degrees of all vertices is equal to (2 * Number of edges).

$$2|V| = 2|E|, |V| = |E| \tag{7}$$

Even if G' is disconnected, each vertex of each disconnected portion has to have a degree of 2, which means that for this smaller disconnected graph, the number of vertices is equal to the number of edges. Therefore, even if G' is disconnected, it will consist of segments of cycles, which we know is planar. Since all elements of the disconnected graph G' is planar, graph G' is planar. Consequently, if each vertex in G has a degree of at most 2, G will be planar.

7. Bipartite Graphs

(a) Since G is bipartite, every edge has to be between one element in L and one element in R. Therefore, each edge will increment both $\sum_{v \in L} deg(v)$ and $\sum_{v \in R} deg(v)$ by 1. Since both sums start at 0, the two sums will be equal in a bipartite graph G.

(b) The sum of the degrees of the vertices in L is equal to the sum of the degrees of the vertices in R, as proved in 7.a. We can denote this value to be A for simplicity since both sums are of equal value. Since s and t are the average degree of vertices in L and R respectively,

$$s = \frac{A}{|L|} \tag{8}$$

$$t = \frac{A}{|R|} \tag{9}$$

$$\frac{s}{t} = \frac{\frac{A}{|L|}}{\frac{A}{|R|}} = \frac{|R|}{|L|} \tag{10}$$

Therefore, we know that this claim is true.

(c) Proof by Strong induction.

Base Case : G_1 is bipartite.
Inductive Hypothesis : Assume that G_n is bipartite for $1 \leq n \leq k$.
Inductive Step:
We have a G_{k+1} which is a double of G_k. Assume that we have G_{k1} and G_{k2}, which are the two bipartite graphs that make up G_{k+1}. Since we know that G_{k1} and G_{k2} are bipartite from the inductive hypothesis, we can safely assume that G_{k1} has sets A and B, which divides the vertices into two groups such that edges can only exist between one member of A and one member of B. Similarly, G_{k2} has sets A$'$ and B$'$.
In G_{k+1}, each vertex in A is connected to its mirroring counter part in A' via an edge and each vertex in B is connected to its mirroring counter part in B' via an edge. Here, we can see that there exists no edges between A and B$'$, and A$'$ and B. Since no edges are newly formed in G_{k+1} other than between mirroring vertices, all edges in G_{k1} are formed between sets A and B, and all edges in G_{k2} are formed between sets A$'$ and B$'$, we can combine sets A and B$'$ to

form one set of vertices for G_{k+1} and combine A′ and B to form the other set of vertices, and because all edges of this graph are between these two new sets, it proves that G_{k+1} is bipartite, and thus $\forall n \geq 1, G_n$ is bipartite.

8. Modular Arithmetic Solutions

(a)

$$2x \equiv 5(\text{mod } 15) \tag{11}$$
$$8(2x) \equiv 8(5)(\text{mod } 15) \tag{12}$$
$$x \equiv 40(\text{mod } 15) \tag{13}$$
$$x \equiv 10(\text{mod } 15) \tag{14}$$

Since 2 and 15 are relatively prime, we can find the multiplicative inverse, which in this case is 8. We multiply both sides by 8, leaving us with an x coefficient of 1 on one side and 40 (mod 15) on the other side. Since 40 (mod 15) is the same as 10 (mod 15), we know that all solutions is defined by:

$$x \equiv 10(\text{mod } 15) \tag{15}$$

There are no other solutions to this problem since the greatest common divisor of 2 and 15 is 1, meaning that the multiplicative inverse of a is unique to the solution, therefore the solution is unique and there are no other solutions.

(b)

$$2x \equiv 5(\text{mod } 16) \tag{16}$$
$$2x = 16k + 5(\text{ for integer k}) \tag{17}$$
$$x = \frac{16k+5}{2} \tag{18}$$
$$x = 8k + \frac{5}{2} \tag{19}$$

We can express x as $8k + \frac{5}{2}$ for an integer k. However, since x must be an integer in a modular setting, there are no solutions that would satisfy the equation since it adds 2.5 to each integer $8k$. Therefore, there are no solutions.

(c)

$$5x \equiv 10(\text{mod } 25) \tag{20}$$
$$5x = 25k + 10(\text{ for integer k}) \tag{21}$$
$$x = 5k + 2 \tag{22}$$

We can express all solutions of this equation to $5k + 2$ or $x \equiv 2$ (mod 5). There are no other solutions can exist because in a modular setting the only solutions that can satisfy the problems is 2 more than multiples of 5.