MATH421 Note

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This Note is a part of MATH421, Spring 2023. The reading material and following contexts are from Prof. Grace Work's class notes and Spivak (2008), one of the most classic pre-Analysis textbooks for undergraduate students.

Contents

1 Basic Properties of Numbers

Theorem 1.1 (Triangle Inequality). For all numbers a, b, we have $|a+b| \leq |a| + |b|$.

Remark. Discuss the four cases of a, b in each sign.

Theorem 1.2. For all $\xi \in \mathbb{R}$, we have $|\xi| = \sqrt{\xi^2}$.

Definition 1.1 (Truth Table). We have the following logical relationships:

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$\neg p$	$\neg q$	$\neg q \Rightarrow \neg p$
T	T	T	T	T	T	F	F	T
T	F	F	T	F	F	F	T	F
F	T	T	F	F	F	T	F	T
F	F	T	T	T	T	T	T	T

Remark. We notice that $p \Leftrightarrow q$ is equivalent to $(p \Rightarrow q) \wedge (q \Rightarrow p)$, meaning that we have to prove both directions for **"iff"** statement. While $p \Rightarrow q$ and $q \Rightarrow p$ are **"converse"** statement, we notice that $p \Rightarrow q$ is equivalent to $\neg q \Rightarrow \neg p$, which is called **"contrapositive"** statement.

2 Numbers in Various Sorts

Definition 2.1 (Set). A collection of objects: S. The following are some specific sets:

- $\mathbb{R}$: set of real numbers
- $\mathbb{Q}$: set of rational numbers $\{\frac{m}{n} | \, m, n \in \mathbb{Z}, n \neq 0, m, n \text{ relatively prime}\}$
- $\mathbb{Z}$: set of integers $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$
- $\mathbb{N}$: set of natural numbers $\{1, 2, 3, \cdots\}$ ($\mathbb{N}$=$\mathbb{Z}_+$)

We also note that $\mathbb{Q}$ has the properties of closure under $(+)$ and $(\cdot)$.

Theorem 2.1. If $x \in \mathbb{R} \setminus \mathbb{Q}$, $y \in \mathbb{Q}$, then $x + y \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. Assume, by contradiction, that $x + y \in \mathbb{Q}$. Then, there exist $m, n \in Z$ s.t. $x + y = \frac{m}{n} \in \mathbb{Q}$. This means $x = \frac{m}{n} + (-y) \in \mathbb{Q}$ by closure under addition $(+)$, contradiction! $\square$

Definition 2.2 (Set operations). Let $\mathcal{A}$ and $\mathcal{B}$ be two sets. We define:

- Intersection: $\mathcal{A} \cap \mathcal{B} = \{x | \, x \in \mathcal{A} \wedge x \in \mathcal{B}\}$
- Union: $\mathcal{A} \cup \mathcal{B} = \{x | \, x \in \mathcal{A} \vee x \in \mathcal{B}\}$
- Complement: $\mathcal{A}^c = \{x | \, x \notin \mathcal{A}\}$
- Set minus: $\mathcal{A} \setminus \mathcal{B} = \{x | \, x \in \mathcal{A} \wedge x \notin \mathcal{B}\}$

Example 2.1 (Spring 2023 Midterm 1 #3). Prove the following statement:
DeMorgan's Law: Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of a set S, then $(\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c$

Proof. We need to show ① "$\Rightarrow$" $(\mathcal{A} \cap \mathcal{B})^c \subseteq \mathcal{A}^c \cup \mathcal{B}^c$ and ② "$\Leftarrow$" $(\mathcal{A}^c \cup \mathcal{B}^c) \subseteq (\mathcal{A} \cap \mathcal{B})^c$.

"$\Rightarrow$" pick arbitrary $x \in (\mathcal{A} \cap \mathcal{B})^c$, then:

$$\begin{aligned} x \in (\mathcal{A} \cap \mathcal{B})^c &\Rightarrow x \notin \mathcal{A} \cap \mathcal{B} \\ &\equiv \{x | x \notin \mathcal{A} \wedge x \notin \mathcal{B}\} \\ &\Rightarrow \{x | x \in \mathcal{A}^c \vee x \in \mathcal{B}^c\} \\ &\equiv x \in \mathcal{A}^c \cup \mathcal{B}^c \end{aligned}$$

Since x is arbitrary, it means every $x \in (\mathcal{A} \cap \mathcal{B})^c$ is an element of $\mathcal{A}^c \cup \mathcal{B}^c$, i.e., $(\mathcal{A} \cap \mathcal{B})^c \subseteq \mathcal{A}^c \cup \mathcal{B}^c$.

"$\Leftarrow$" pick arbitrary $y \in \mathcal{A}^c \cup \mathcal{B}^c$, then:

$$\begin{aligned} y \in \mathcal{A}^c \cup \mathcal{B}^c &\equiv \{y | y \in \mathcal{A}^c \vee y \in \mathcal{B}^c\} \\ &\Rightarrow \{y | y \notin \mathcal{A} \wedge y \notin \mathcal{B}\} \\ &\equiv y \notin \mathcal{A} \cap \mathcal{B} \\ &\Rightarrow y \in (\mathcal{A} \cap \mathcal{B})^c \end{aligned}$$

Since y is arbitrary, it means every $y \in \mathcal{A}^c \cup \mathcal{B}^c$ is an element of $(\mathcal{A} \cap \mathcal{B})^c$, i.e., $\mathcal{A}^c \cup \mathcal{B}^c \subseteq (\mathcal{A} \cap \mathcal{B})^c$.

Since we have shown that $(\mathcal{A} \cap \mathcal{B})^c \subseteq \mathcal{A}^c \cup \mathcal{B}^c$ and $\mathcal{A}^c \cup \mathcal{B}^c \subseteq (\mathcal{A} \cap \mathcal{B})^c$, we conclude that, for $\mathcal{A}, \mathcal{B} \subseteq S$, $(\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c$, Q.E.D. $\square$

3 Functions

Definition 3.1 (Unique mapping)**.** Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}$. A function f is a rule which assigns each element of $\mathcal{A}$ to **exactly one** element of $\mathcal{B}$. Here, $\mathcal{A}$ is the domain of f and $\mathcal{B}$ is the codomain of f.

Definition 3.2 (Image and Preimage)**.** For $f : \mathcal{A} \to \mathcal{B}$, we define:

- Image: if $\mathcal{E} \subseteq \mathcal{A}$, then $f(\mathcal{E}) \equiv \{f(x) | x \in \mathcal{E}\}$
  (i.e., $f(\mathcal{E})$ is image of $\mathcal{E}$ under f. If $\mathcal{E} = \mathcal{A}$, then $f(\mathcal{E}) = f(\mathcal{A}) \subseteq \mathcal{B}$)
- Preimage: if $\mathcal{E} \subseteq \mathcal{B}$, then $f^{-1}(\mathcal{E}) \equiv \{x \in \mathcal{A} | f(x) \in \mathcal{E}\}$
  (i.e., $f^{-1}(\mathcal{E})$ is preimage of $\mathcal{E}$ under f. $f^{-1}(\mathcal{E}) \subseteq \mathcal{A}$)

Example 3.1 (Input $\subseteq$ Output)**.** Suppose $f : \mathcal{A} \to \mathcal{B}$ is a function, where $\mathcal{A}, \mathcal{B}$ are sets. Furthermore, let $R \subseteq \mathcal{A}$ be given, prove that $R \subseteq f^{-1}(f(R))$.

Proof. Strategy: pick $x \in R$ and show $x \in f^{-1}(f(R))$.
Let $x \in R$, since $R \subseteq \mathcal{A}$, then $x \in \mathcal{A}$. Since $f : \mathcal{A} \to \mathcal{B}$, by definition of image (**Definition 3.2**), there exists a unique $y \in \mathcal{B}$ s.t. $f(x) = y$. Note that $y = f(x)$ is in image of R, i.e., $\{y\} = \{f(x)\} \subseteq f(R)$, then $x \in f^{-1}(\{y\}) \subseteq f^{-1}(f(R))$. Since $x \in R$ is arbitrary, every element in R is an element of $f^{-1}(f(R))$, meaning $R \subseteq f^{-1}(f(R))$, Q.E.D. $\square$

Example 3.2 (Output $\subseteq$ Input)**.** Suppose $f : \mathcal{A} \to \mathcal{B}$ is a function, where $\mathcal{A}, \mathcal{B}$ are sets. Furthermore, let $U \subseteq \mathcal{B}$ be given, prove that $f(f^{-1}(U)) \subseteq U$.

Proof. Strategy: pick $y \in f(f^{-1}(U))$ and show $y \in U$.
Let $y \in f(f^{-1}(U))$, then $y \in \mathcal{B}$. By **Definition 3.2**, there exists a $x \in f^{-1}(U)$ s.t. $f(x) = y$ (preimage). Note that, again, $x \in f^{-1}(U)$ implies that x is in preimage of U, i.e., $f(x) = z \in U$. By **Definition 3.1**, we conclude that $f(x) = y = z \Rightarrow y \in U$. Since $y \in f(f^{-1}(U))$ is arbitrary, every element in $f(f^{-1}(U))$ is an element of U, meaning $f(f^{-1}(U)) \subseteq U$, Q.E.D. $\square$

Definition 3.3 (Surjective, Injective, Bijective)**.** Let $f : \mathcal{A} \to \mathcal{B}$ be a function:

- if $f(\mathcal{A}) = \mathcal{B}$, then f is said to be **surjective** (onto).
- if $f(x) = f(y) \Leftrightarrow x = y$, then f is said to be **injective** (1-1).
- f is surjective and injective $\Leftrightarrow$ f is **bijective**.

Example 3.3 (Spring 2023 Midterm 1 Review)**.** Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathbb{C}$ be functions. Prove the following statement: if $g \circ f$ is bijective, then f is injective and g is surjective.

Proof. We consider:

(i) Prove f injective: let $x_1, x_2 \in A$ and that $f(x_1) = f(x_2)$. Since $g \circ f$ bijective ($\Rightarrow$ injective), we know $g \circ f(x_1) = g \circ f(x_2)$ yields $f(x_1) = f(x_2)$. So we now have $f(x_1) = f(x_2)$ and $x_1 = x_2$, i.e., f injective (1-1).

(ii) Prove g surjective: want to show $\forall y \in \mathbb{C}, \exists x' \in B$ s.t. $y = g(x)$. We can pick such $x' = f(x)$ for $x \in A$, which yields $g(x') = g(f(x)) \equiv g \circ f(x)$ always having a y since $g \circ f$ is assumed bijective ($\Rightarrow$ surjective). Thus, g is surjective.

□

Example 3.4 ([Spring 2023 HW4 #1).] Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be bijective functions. Prove that $(f \circ g)$ is also a bijective function from $\mathbb{R} \to \mathbb{R}$.

Proof. We want to show: ① $f \circ g$ is surjective (onto) and ② $f \circ g$ is injective (1-1). Note that $(f \circ g)(x) = f(g(x))$:

① **surjective** : We want to show that for any $y \in \mathbb{R}$ there exists $x_{f \circ g} \in \mathbb{R}$ such that $y = (f \circ g)(x) = f(x_{f \circ g}) \in \mathbb{R}$. We can thus pick this $x_{f \circ g} = g(x) \in \mathbb{R}$, where $x, g(x) \in \mathbb{R}$ since $g : \mathbb{R} \to \mathbb{R}$, so that:

$$y = (f \circ g)(x) = \overbrace{f(\underbrace{g(x)}_{\in \mathbb{R}})}^{\in \mathbb{R}} = f(x_{f \circ g}) \in \mathbb{R}, \text{ given that } f, g \text{ both } \mathbb{R} \to \mathbb{R}$$

Since $g(x) \in \mathbb{R}$, it also lies in the domain of f ($f : \mathbb{R} \to \mathbb{R}$) and make $f(x_{f \circ g}) \in \mathbb{R}$. Therefore, for any $y \in \mathbb{R}$, we can always find a $x_{f \circ g} \in \mathbb{R}$ that maps to this particular y, i.e., $f \circ g$ is surjective.

② **injective** : let $x_1, x_2 \in \mathbb{R}$ and $(f \circ g)(x_1) = (f \circ g)(x_2) \in \mathbb{R}$, we want to show $x_1 = x_2$. By definition of composition, $(f \circ g)(x_1) = (f \circ g)(x_2)$ is equivalent to $f(g(x_1)) = f(g(x_2))$. Since f is bijective, it immediately implies $g(x_1) = g(x_2)$. Then, again, since g is also bijective, this implies that $x_1 = x_2$. Thus, $f \circ g$ is injective.

We have shown that $f \circ g$ is both surjective and injective, i.e., $f \circ g$ is bijective, Q.E.D. □

4 Graphs

Definition 4.1 (Graph). Let $f : \mathcal{A} \to \mathcal{B}$, then the graph of f is the set: $\mathcal{G}(f) = \{(x, f(x)) | x \in \mathcal{A}\} \subseteq \mathcal{A} \times \mathcal{B}$.

Theorem 4.1. If $\mathcal{G}$ is any subset $\mathbb{R} \times \mathbb{R}$ with property that for each $x \in \mathbb{R}$, the set $\{y \in \mathbb{R} | (x, y) \in \mathcal{G}\}$ (graph) has exactly one element (def of function), then there is exactly one function $f : \mathbb{R} \to \mathbb{R}$ whose graph is $\mathcal{G}$.

Example 4.1 (Spring 2023 HW4 #2). If $f : X \to Y$ is a function, define the graph of f to be the subset of $X \times Y$ defined by $\{(x, f(x)) \mid x \in X\}$. Show that two functions $f : X \to Y$, $g : X \to Y$ are equal (that is, $f(x) = g(x) \forall x \in X$) if and only if they have the same graph.

Proof. We want to show: ① $f(x) = g(x)\ \forall x \in X \Rightarrow$ same graph, and ② same graph $\Rightarrow f(x) = g(x)\ \forall x \in X$.

① :⇒ given $f(x) = g(x)\ \forall x \in X$, then the graph:

$$\mathcal{G}(f) \equiv \{(x, f(x)) \mid x \in X\} = \{(x, g(x)) \mid x \in X\} \equiv \mathcal{G}(g)$$

We have just shown that if $f(x) = g(x)\ \forall x \in X$ yields $\mathcal{G}(f) = \mathcal{G}(g)$ the same graph.

② :⇒ Know $\mathcal{G}(f) = \mathcal{G}(g)$. Suppose that $\mathcal{G}(f) = \mathcal{G}(g) = \mathcal{G}$, by definition of graph we have $\{(x, f(x)) \mid x \in X\} = \{(x, g(x)) \mid x \in X\}$, where $f(x), g(x) \in Y$. By definition, for two sets to be equal, it immediately implies all element in one set is contained in the other set, vice versa. In this case, we should have each ordered pair $(x, f(x)) = (x, g(x))$ for all x in X. This thus yields $f(x) = g(x)\ \forall x \in X$.

Therefore, we have proved that $f(x) = g(x)\ \forall x \in X \Leftrightarrow \mathcal{G}(f) = \mathcal{G}(g)$, Q.E.D. □

5 Limits

Definition 5.1 (Limit: $\lim_{x \to a} f(x) = L$)**.** we have:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. for all } x, \text{ if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \varepsilon$$

Example 5.1. Prove that $\lim_{x \to 1} x^2 + 3x = 3$.

Proof. First rewrite as $|x^2 + 2x - 3| < \varepsilon \Rightarrow |x - 1||x + 3| < \varepsilon$. Set $\delta_1 = 1$, then $0 < |x - 1| < \delta_1 = 1 \Rightarrow -1 < x - 1 < 1 \Rightarrow 3 < x + 3 < 5$. So if $\delta_1 = 1$, then $|x + 3| < 5$ and that $|x - 1||x + 3| < 5|x - 1| < \varepsilon$. We now set $\delta_2 = \frac{1}{5}\varepsilon$.
Thus, let $\varepsilon > 0$, we can pick $\delta = \min\{\delta_1, \delta_2\} = \min\{1, \frac{1}{5}\varepsilon\}$. If $0 < |x - 1| < \delta$, then $|x^2 + 2x - 3| = |x - 1||x + 3| < 5|x - 1| < 5 \cdot \frac{1}{5}\varepsilon = \varepsilon$. Therefore, $\lim_{x \to 1} x^2 + 3x = 3$. □

Definition 5.2 (One-sided Limit)**.** We have the following:
① $L = \lim_{x \to a^+} f(x) = L$: for every $\varepsilon > 0$, there exists a $\delta > 0$ s.t. if $0 < x - a < \delta$, then $|f(x) - L| < \varepsilon$.
② $L = \lim_{x \to a^-} f(x) = L$ we have: for every $\varepsilon > 0$, there exists a $\delta > 0$ s.t. if $0 < a - x < \delta$, then $|f(x) - L| < \varepsilon$

Theorem 5.1 (Uniqueness of limit)**.** A function cannot approach two different limits near a. In other words, if f approaches L near a and f approaches M near a, then $L = M$.

Remark. It is equivalent to say: $\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

Theorem 5.2. If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$, then: ① $\lim_{x \to a}(f + g)(x) = L + M$
② $\lim_{x \to a}(f \cdot g)(x) = L \cdot M$ ③ $\lim_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{M} (M \neq 0)$.

Remark. Basically for computational efficiency.

Theorem 5.3 (Horizontal Asymptotes). We say that the limit of f as x approaches $\infty(-\infty)$ is L.

$$\lim_{x\to\pm\infty} f(x) = L \text{ if } \forall\varepsilon > 0, \exists M \in \mathbb{R} \text{ s.t. if } X \gtrless M, \text{ then } |f(x) - L| < \varepsilon.$$

Theorem 5.4 (Vertical Asymptotes). We say that the limit of f as x approaches a is $\pm\infty$.

$$\lim_{x\to a} f(x) = \pm\infty \text{ if } \forall M \in \mathbb{R}, \exists\delta > 0 \text{ s.t. if } 0 < |x - a| < \delta, \text{ then } f(x) \gtrless M.$$

Example 5.2. Prove $\lim_{x\to 0^+} \frac{1}{x} = \infty$

Proof. Note that this is an one-sided lim with vertical asymptote. Want $f(x) = \frac{1}{x} > M \Rightarrow x < \frac{1}{M} \Rightarrow 0 < x - 0 < \frac{1}{M}$. Can this pick $\delta = \frac{1}{M}$ to complete the statement. □

Example 5.3 (Squeeze Theorem). Suppose that $f(x) \leq g(x) \leq h(x)$ and that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. Prove that $\lim_{x\to a} g(x)$ exist and that $\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x)$.

Scratch Work: We know there exist $\delta_f, \delta_h > 0$ such that if $0 < |x - a| < \delta_{f,g}$, $0 < |f(x) - L| = |h(x) - L| < \varepsilon$. If $\lim_{x\to a} g(x)$ exists and equals to $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$, we want to find there also exists a $\delta_g > 0$ such that if $0 < |x - a| < \delta_g$, $0 < |g(x) - L| < \varepsilon$.

Proof. Let $\varepsilon > 0$, we can pick $\delta_g = \min\{\delta_f, \delta_h\} > 0$. Since $f(x) \leq g(x) \leq h(x)$, we have:

$$\begin{aligned} f(x) - L \leq \quad & g(x) - L \quad \leq h(x) - L \\ -\varepsilon < f(x) - L \leq \quad & g(x) - L \quad \leq h(x) - L < \varepsilon \\ -\varepsilon < \quad & g(x) - L \quad < \varepsilon \\ \Rightarrow \quad & |g(x) - L| \quad < \varepsilon \end{aligned}$$

Therefore, we have proved that $\lim_{x\to a} g(x)$ exist and that $\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = \lim_{x\to a} h(x)$, Q.E.D. □

6 Continuous Function

Definition 6.1 (Continuity). The function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$. Using $\delta - \varepsilon$ definition, we have:

$$\forall\varepsilon > 0, \ \exists\delta > 0 \text{ s.t. if } |x - a| < \delta, \text{ then } |f(x) - f(a)| < \varepsilon.$$

Theorem 6.1. If f, g is continuous at a, then: ① $f + g$ is conti at a, ② $f \cdot g$ is conti at a, and ③ $\frac{1}{g}$ is conti at a ($g(a) \neq 0$)

Remark. This is basically for computational efficiency. For instance, constant function, sin / cos functions, polynomials ($y = x^n$), rational functions ($\frac{P(x)}{Q(x)}$, $Q(x) \neq 0$), and any linear combination of the above functions (e.g., $y = \cos(x) + x^3$) are conti on domains.

Theorem 6.2 (Continuity on Open Interval)**.** A function f is continuous on (a,b) if f is continuous at every $x \in (a,b)$.

Remark. What if on a closed interval $[a,b]$? We then need ① continuous on (a,b), and ② checking end points: $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

Theorem 6.3. $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$.

Remark. Set $x = a+h$ to prove "$\Rightarrow$" and $h = x-a$ to prove "$\Leftarrow$".
For "$\Rightarrow$", set $x = a+h$, we have: if $|(a+h)-a| = |h| = |h-0| < \delta$, then $|f(a+h)-L| < \varepsilon$.

Example 6.1 (Triangle Inequalities)**.** Prove that if f is conti at a, then for any $\varepsilon > 0$ there exists a $\delta > 0$ so that whenever $|x-a| < \delta$ and $|y-a| < \delta$, we have $|f(x)-f(y)| < \varepsilon$.

Scratch Work: We know f is continuous at a. That is, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all x, y, if $|x-a| < \delta$, $|y-a| < \delta$, $|f(x)-f(a)| < \frac{\varepsilon}{2}$ and $|f(y)-f(a)| < \frac{\varepsilon}{2}$. We then have:

$$\begin{aligned}
|f(x)-f(y)| &= |f(x)\underline{-f(a)+f(a)}-f(y)| \\
&\leq |f(x)-f(a)| + |f(a)-f(y)| \text{ (by triangle inequality)} \\
&= |f(x)-f(a)| + |f(y)-f(a)| \text{ (changing signs in abs)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Example 6.2 (Spring 2023 HW6 #2)**.** Use the $\delta - \varepsilon$ definition of continuity to prove that the function $f : [0,\infty) \to [0,\infty)$ defined by $f(x) = \sqrt{x}$ is continuous on its domain.

Scratch Work: Pick a $c \in [0,\infty)$. We want to show for any $\varepsilon > 0$ there exist $\delta > 0$ such that, for $x \in [0,\infty)$, if $|x-c| < \delta$, $|\sqrt{x}-\sqrt{c}| < \varepsilon$. Consider ① arbitrary $c \in (0,\infty)$ and ② $c = 0$. For ① $c \in (0,\infty)$ case, we have:

$$\begin{aligned}
&|\sqrt{x}-\sqrt{c}| < \varepsilon \\
(\text{times } \frac{|\sqrt{x}+\sqrt{c}|}{|\sqrt{x}+\sqrt{c}|}) \quad \Rightarrow \quad &\frac{|x-c|}{|\sqrt{x}+\sqrt{c}|} = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} < \frac{|x-c|}{\sqrt{c}} < \varepsilon \Rightarrow |x-c| < \sqrt{c}\varepsilon
\end{aligned}$$

In this case, we can thus pick $\delta = \sqrt{c}\varepsilon$.
For ② $c = 0$ case, we have: $|\sqrt{x}-0| = |\sqrt{x}| < \epsilon \Rightarrow |x| < \varepsilon^2$. We can therefore pick $\delta = \varepsilon^2$ in this case.

7 Three Hard Theorem

Theorem 7.1 (Intermediate Value Theorem: IVT). If f is continuous on $[a,b]$ and $f(a) < 0 < f(b)$, then there exists some $\xi \in [a,b]$ s.t. $f(\xi) = 0$.

Remark. Must cross through 0 at some point in this close interval.

Theorem 7.2 (Bounded Function). Let f be continuous on $[a,b]$. Then f is bounded above and below, i.e., $\exists M, N \in \mathbb{R}$ s.t. $f(x) \leq M$ (bounded above) and $f(x) \geq N$ (bounded below) for all $x \in [a,b]$.

Remark. We need continuity here to avoid vertical asymptotes. Why in a closed interval? Consider this counterexample of $f(x) = \frac{1}{x}$ on $(0,1)$ which is not bounded above as $x \to 0$.

Corollary. Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$. f is bounded if there exists an $M_1 \geq 0$ such that $|f(x)| \leq M_1$ for all $x \in D$.

Remark. Choose this $M_1 = \max\{|M|, |N|\}$ from Theorem 7.2. This Corollary implies that: Conti on closed $\Rightarrow$ Bounded.

Theorem 7.3 (Extreme Value Theorem). Let f be continuous on $[a,b]$. Then, f attains a global minimum and maximum on $[a,b]$ (on this interval), i.e. there exists a $c, d \in [a,b]$ such that: $f(c) \leq f(x) \leq f(d)$ for all $x \in [a,b]$.

Remark. Can a function have a global maximum and minimum on this interval but NOT bounded? F- Can use that global max and min as M, N to prove bounded. This idea contributes to next theorem.

Theorem 7.4. Given a function $f : D \to \mathbb{R}$ for $D \subseteq \mathbb{R}$. If f has a global minimum and maximum, then f is bounded.

Theorem 7.5 (Generalized IVT). Suppose f is continuous on $[a,b]$ and $\alpha \in \mathbb{R}$. If ① $f(a) < \alpha < f(b)$ or ② $f(b) < \alpha < f(a)$, then there exists some $\xi \in (a,b)$ s.t. $f(\xi) = \alpha$.

Proof. Let $g(x) = f(x) - \alpha$. Since f is conti on $[a,b]$, g is also conti on $[a,b]$ (by linear combination). We have: $f(a) < \alpha < f(b) \Rightarrow f(a) - \alpha < 0 < f(b) - \alpha \Rightarrow g(a) < 0 < g(b)$. Since g is conti on closed interval, by **Theorem 7.1 IVT**, there exists a $\xi \in [a,b]$ s.t. $g(\xi) = 0$. This means $g(\xi) = f(\xi) - \alpha = 0$, which yields $f(\xi) = \alpha$ (skip another case). □

Example 7.1. Prove the following: Every positive number has a square root, i.e., if $\alpha > 0$, then there is some x s.t. $x^2 = \alpha$

Proof. We can use **Theorem 7.5 Generalized IVT**. Let $f(x) = x^2$, then ① f is continuous (trivial). Now consider $f(a) < \alpha < f(b)$: let $a = 0$, we have $f(0) < \alpha < f(b)$ since $\alpha > 0$. By Generalized IVT, there exists a $x \in [0,b]$ s.t. $f(x) = \alpha \Rightarrow x^2 = \alpha$, Q.E.D. □

Example 7.2 (Spring 2023 HW7 #1)**.** Suppose f is continuous on $\mathbb{R}$. Prove that if $\lim_{x\to-\infty} f(x) = L$ and $\lim_{x\to\infty} f(x) = M$, then f is bounded.

Scratch Work: Want to find a sufficiently large closed interval on $\mathbb{R}$ and identify the global min/max on that interval (two value) by EVT. Then we may choose the maximum of the absolute value of the two values along with $|L|$ and $|M|$.

Proof. Since f is continuous on $\mathbb{R}(f : \mathbb{R} \to \mathbb{R})$, by Corollary, f is said to be bounded if there exists an $B \geq 0$ such that $|f(x)| \leq B$ for all $x \in \mathbb{R}$. Since we know $\lim_{x\to-\infty} f(x) = L$ and $\lim_{x\to\infty} f(x) = M$, we can pick some $a_0, a_1 \in \mathbb{R}$, where $|a_0|, |a_1|$ are sufficiently large. Since f is continuous on $\mathbb{R}$, f should also continuous on $[a_0, a_1]$, i.e., continuous on a closed interval. By Extreme Value Theorem, there exist $c, d \in [a_0, a_1]$ such that $f(c) \leq f(x) \leq f(d)$, which $f(c)$ is minimum and $f(d)$ is maximum on $[a_0, a_1]$. We thus can choose $B = \max\{|L|, |M|, |f(c)|, |f(d)|\}$ so that we ensure $|f(x)| \leq B$ for all value of $x \in \mathbb{R}$, i.e., f is bounded, Q.E.D. $\square$

Example 7.3 (Spring 2023 HW7 #2)**.** Use the Extreme Value Theorem to prove that $f(x) = x^2 - x + 1 + \cos(x)$ has a minimum value on $\mathbb{R}$.

Scratch Work: Can first complete the square: $f(x) = x^2 - x + 1 + \cos(x) = (x - \frac{1}{2})^2 + \frac{3}{4} + \cos(x)$. Since f is linear combination of continuous function, f is thus continuous on $\mathbb{R}$. If specifying a closed interval $[c, d]$ on $\mathbb{R}$, we want to show that there exists a $c \in [a, b]$ s.t. $f(c) \leq f(x)$ for all $x \in \mathbb{R}$ (i.e., have min) and show that it is the global minimum.

Proof. We use $f(x) = (x - \frac{1}{2})^2 + \frac{3}{4} + \cos(x)$ and the fact that, for all value in $\mathbb{R}$, $\cos(x)$ always satisfies $-1 \leq \cos(x) \leq 1$. We then have the following expression: $(x - \frac{1}{2})^2 - \frac{1}{4} \leq f(x)$ and can further denote $g(x) = (x - \frac{1}{2})^2 - \frac{1}{4}$. Since $g(x)$ is continuous and an even function, by algebra and Theorem 10 in textbook, we know it attains a min at $x = \frac{1}{2}$. We now choose the closed interval $[0, \pi]$ that ensures the $\cos(x)$ inequalities hold and $\frac{1}{2}$ is in the interval. The global min argument can be justified by that: if we set any $\varepsilon > 0$, we notice $g(\frac{1}{2} + \varepsilon) = g(\frac{1}{2} - \varepsilon) = \varepsilon^2 > 0 = g(\frac{1}{2})$. Since $f(x)$ is continuous on $[0, \pi]$ and bounded below by $g(x)$, who attains its global min on the same interval, by Extreme Value Theorem, there should also exist a $c \in [0, \pi]$ such that $f(c) \leq f(x)$ for all value of $x \in \mathbb{R}$ and that $f(c)$ is its global min. In other words, $f(x)$ has global minimum value of $f(c)$ for some $c \in [0, \pi]$, Q.E.D. $\square$

8 Least Upper Bounds

Definition 8.1 (Bounded Above/Below & Upper/Lower Bound)**.** A set $\mathcal{A}$ of real number ($\mathbb{R}$) is *bounded above* if there is a x s.t. $x \geq a \; \forall a \in \mathcal{A}$. Here, x is the **upper bound** for $\mathcal{A}$. Similarly, a set $\mathcal{A}$ of real number ($\mathbb{R}$) is *bounded below* if there is a x s.t. $x \leq a \; \forall a \in \mathcal{A}$. Then, x is the **lower bound** for $\mathcal{A}$.

Remark. $\mathcal{A}$ is bounded above $\Leftrightarrow$ $\mathcal{A}$ has an upper bound $\Leftrightarrow$ x is an upper bound for $\mathcal{A}$

Definition 8.2 (Least Upper Bound/Greatest Lower Bound)**.** A number x_u is a *least upper bound* for $\mathcal{A}$ if ① x_u is an upper bound for $\mathcal{A}$, AND ② if x is also an upper bound for $\mathcal{A}$, then $x_u \leq x$. We denote such least upper bound for $\mathcal{A}$: $x_u \equiv \sup \mathcal{A}$. Similarly, the greatest lower bound for $\mathcal{A}$: $x_l \equiv \inf \mathcal{A}$.

Remark. There are several useful equivalent definitions of least upper bound.
We say $\beta = \sup S$ if:

(i) $\forall M < \beta$, there is $s \in S$ s.t. $s > M$ (i.e., $M \neq \sup S$).

(ii) $\forall \varepsilon > 0$, there is $s \in S$ s.t. $\beta - \varepsilon < s \leq \beta$ (i.e., $\beta - s < \varepsilon$).

Example 8.1. Does $\mathcal{A} = \{x \in \mathbb{Q} : x^2 < 2\}$ has rational sup / inf?
$\Rightarrow -\sqrt{2} < x < \sqrt{2}$, but $\sup \mathcal{A} = \sqrt{2} \notin \mathbb{Q}$; $\inf \mathcal{A} = -\sqrt{2} \notin \mathbb{Q}$. So, does not have rational sup, inf (but has real number sup, inf)

Theorem 8.1 (The Least Upper Bound Property)**.** If ① $\mathcal{A} \subseteq \mathbb{R}$, ② $\mathcal{A} \neq \emptyset$, and ③ $\mathcal{A}$ is bounded above, then $\mathcal{A}$ has a least upper bound, i.e., $\sup \mathcal{A}$.

Example 8.2. Prove that $\mathbb{N}$ is NOT bounded above.
$\Rightarrow$ Suppose, by contradiction, $\mathbb{N}$ is bounded above. Then, by Theorem 8.1 it follows that $\exists a \in \mathbb{N}$ s.t. $a \geq n \; \forall n \in \mathbb{N}$ (i.e., $\mathbb{N}$ has a $\sup \mathbb{N} = a$) since ① $\mathbb{N} \subseteq \mathbb{R}$, ② $\mathbb{N} \neq \emptyset$ ($1 \in \mathbb{N}$ at least), and ③ bounded above (we assumed so). yet, $n+1 \in \mathbb{N}$ as well (well-ordering principle), we have $a \geq n+1 \Rightarrow a-1 \geq n$, meaning $a-1$ is also $\sup \mathbb{N}$, contradiction!

Lemma. Suppose f is continuous at a. If $f(a) \gtrless 0$,then there exists $\delta > 0$ s.t. $\forall x \in \mathbb{R}$ if $|x-a| < \delta$, then $f(x) \gtrless 0$.

Remark. Continuous function in a *neighborhood* of a positive func value are still positive.

Proof. Say $f(a) > 0$. Since f is continuous at $a \Leftrightarrow \forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon \Rightarrow -\varepsilon < f(x) - f(a) < \varepsilon \Rightarrow f(a) - \varepsilon < f(x) < f(a) + \varepsilon$. We want $0 < f(a) - \varepsilon$, and we can simply just set $\varepsilon = \frac{f(a)}{2}$ so that there is always some δ satisfying the claim, Q.E.D. □

Example 8.3. Let $\mathcal{A}$ be a finite nonempty set of real numbers. Prove that $\sup \mathcal{A} \in \mathcal{A}$.

Proof. Since A is finite set, we can let $n \in \mathbb{N}$ be the number of elements in $\mathcal{A}$. Since $\mathcal{A} \subseteq \mathbb{R}$, $\mathcal{A} \neq \emptyset$, if $\mathcal{A}$ is bounded above, then A has a $\sup \mathcal{A}$. We can proceed the proof by Principle of Induction:

Base case: P(1) $n = 1$: only one element in A, say x_1, then we can simply choose $M = \max\{|x_1|\} = |x_1|$ such that $M \geq x_1$. Thus $\mathcal{A}$ is bounded ($\Rightarrow$ also bounded above). By the Least Upper Bound Property, $\mathcal{A}$ has a $\sup \mathcal{A}$. Since $\max\{x_1\} = x_1 \geq x_1$ is always true, $\sup \mathcal{A} = \max\{x_1\} \in \mathcal{A}$.

Inductive case: P(k) now $\mathcal{A}$ has $\{x_1, \ldots, x_k\}$ elements. Assumed $n = k$ case is also true, meaning $\mathcal{A}$ is bounded and has $\sup \mathcal{A} = \max\{x_1, \ldots, x_k\} \in \mathcal{A}$. Then, when $n = k+1$, we can show that $\{x_1, \ldots, x_{k+1}\} = \{x_1, \ldots, x_k\} \bigcup \{x_{k+1}\}$.

We now can choose $M = \max\{|\max\{x_1, \ldots, x_k\}|, |x_{k+1}|\}$ such that $M \geq x_i, \forall i = 1, \ldots, k+1$. $\mathcal{A}$ is thus bounded and has a $\sup \mathcal{A}$. Then, if ① $x_{k+1} > \max\{x_1, \ldots, x_k\}$, we have $\sup \mathcal{A} = x_{k+1} \in \mathcal{A}$. If ② $x_{k+1} = \max\{x_1, \ldots, x_k\}$, we have $\sup \mathcal{A} = x_{k+1} = \max\{x_1, \ldots, x_k\} \in \mathcal{A}$. Finally, if ③ $x_{k+1} < \max\{x_1, \ldots, x_k\}$, we have $\sup \mathcal{A} = \max\{x_1, \ldots, x_k\} \in \mathcal{A}$. So, the statement also holds true for $n = k+1$.

Therefore, by Principle of Induction, we show that if $\mathcal{A}$ be a finite nonempty set of real numbers, $\sup \mathcal{A} \in \mathcal{A}$, Q.E.D. $\square$

Example 8.4 (Follow-up Question)**.** Suppose $\mathcal{A}$ is a nonempty set of real numbers which is bounded above and suppose that $\beta = \sup \mathcal{A}$. Suppose further that $\beta \notin \mathcal{A}$ and that $\varepsilon > 0$. Then $\{x \in \mathcal{A} : x > \beta - \varepsilon\}$ is an infinite set.

Proof. Suppose $\{x \in \mathcal{A} : \beta - \varepsilon < x\}$ is a finite set, by contradiction. We then notice that $\{x \in \mathcal{A} : x > \beta - \varepsilon\} = \{x \in \mathcal{A} : \beta - \varepsilon < x\}$ is just the set of all $x \in \mathcal{A}$, i.e., the set $\mathcal{A}$, and $\beta - \varepsilon < x$ is the equivalent definition of $\sup \mathcal{A}$. By **Example 8.3**, we know that $\sup \mathcal{A} \in \mathcal{A}$ if $\mathcal{A}$ is finite, non-empty subset of $\mathbb{R}$, but this instantly leads to a contradiction since we have $\beta = \sup \mathcal{A} \notin \mathcal{A}$. Therefore, by Proof of Contradiction, we have shown that $\mathcal{A}$ cannot not be non-empty finite subset of $\mathbb{R}$, i.e., $\{x \in \mathcal{A} : x > \beta - \varepsilon\}$ is an infinite set. $\square$

9 Derivatives

Definition 9.1. f is *differentiable* at a if ① $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$, or ② $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists. We can expand this to function: f is differentiable if ① $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$, or ② $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists.

Remark. Linear Approximation: $f(x) \approx f'(a)(x-a) + f(a)$.

Theorem 9.1 (Diff ⇒ Conti). **If f is differentiable at a, then f is continuous at a.**

Proof. Since f is differentiable at a, $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists. Want $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h) = f(a)$. We can rewrite $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \left(h \left[\frac{f(a+h)-f(a)}{h} \right] + f(a) \right) = \left(\lim_{h \to 0} h \right) \left(\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \right) + \lim_{h \to 0} f(a) = 0 \cdot f'(a) + f(a) = f(a)$, Q.E.D. □

Remark. Some brain teasers (T/F):

- If f is not diff at a, then f is not conti? F- consider $f = |x|$ is conti
- If f is conti at a, then f is diff at a? F- again, consider $f = |x|$ is not diff at 0
- If f is not conti at a, then f is not diff at a? T- this is contra-positive of Theorem 9.1.

10 Differentiation

Theorem 10.1 (Constant function). $f'(x) = 0$ if $f(x) = c$.

Theorem 10.2 (Linear function). $f'(x) = 1$ if $f(x) = x$.

Theorem 10.3 (Addition Rule). $(f+g)'(a) = f'(a) + g'(a)$.

Theorem 10.4 (Product Rule). $(f \cdot g)'(a) = f'(a)g(a) + g'(a)f(a)$.

Proof. $(f \cdot g)'(a) \equiv \lim_{h \to 0} \frac{(f \cdot g)(a+h)-(f \cdot g)(a)}{h} = \lim_{h \to 0} \frac{f(a+h)g(a+h)-f(a)g(a)}{h} = \lim_{h \to 0} \frac{f(a+h)(g(a+h)-g(a))+g(a)f(a+h)-f(a)g(a)}{h} = f(a)g'(a) + f'(a)g(a)$ □

Theorem 10.5. $(cf)'(a) = cf'(a)$.

Theorem 10.6 (Power Rule). $(x^n)' = nx^{n-1}$.

Theorem 10.7. If g is differentiable at $x = a$, and $g(a) \neq 0$, then $\frac{1}{g}$ is differentiable at $x = a$ and $\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{(g(a))^2}$.

Proof. Use limit definition: $\left(\frac{1}{g}\right)'(a) = \left(\frac{1}{g(a)}\right)' = \lim_{h \to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} = \lim_{h \to 0} \frac{g(a)-g(a+h)}{h \cdot g(a+h)}$ (skip). □

Theorem 10.8 (Quotient Rule)**.** If f and g are differentiable at $x = a$ and $g(a) \neq 0$, then $\frac{f}{g}$ is differentiable at $x = a$ and $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - g'(a)f(a)}{(g(a))^2}$

Proof. Idea: use **Theorem 10.4 (Product Rule)** and **Theorem 10.7**
$\left(\frac{f}{g}\right)'(a) = \left(f \cdot \frac{1}{g}\right)'(a) = f'(a)\left(\frac{1}{g}\right)(a) + \left(\frac{1}{g}\right)'(a)f(a) = \frac{f'(a)}{g(a)} + \frac{-f(a)g'(a)}{(g(a))^2} = \frac{f'(a)g(a) - g'(a)f(a)}{(g(a))^2}$ □

Theorem 10.9 (Chain Rule)**.** If g is differentiable at $x = a$ and f is differentiable at $x = g(a)$, then $(f \circ g)$ is differentiable at $x = a$ and $(f \circ g)'(a) = f'(g(a))g'(a)$.

Proof. Idea: define a new function $\phi(h) = \begin{cases} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)}, \text{ if } g(a+h) - g(a) \neq 0 \\ f'(g(a)), \text{ if } g(a+h) - g(a) = 0 \end{cases}$
(skip) □

11 Significance of the Derivative

Definition 11.1 (Maximum/Minimum Point)**.** A point $x \in \mathcal{A}$ is a **maximum point** for f on $\mathcal{A}$ if $f(x) \geq f(y)$ $\forall y \in \mathcal{A}$. The $f(x)$ itself is called the maximum value of f on $\mathcal{A}$. On the other hand, a point $x \in \mathcal{A}$ is a **minimum point** for f on $\mathcal{A}$ if $f(x) \leq f(y)$ $\forall y \in \mathcal{A}$. The $f(x)$ itself is called the minimum value of f on $\mathcal{A}$.

Remark. The max/min value is *unique*, but there may be multiple max/min points that attain such a value.

Definition 11.2. A point $x \in \mathcal{A}$ is a **local max/min point** for f on $\mathcal{A}$ if there is some $\delta > 0$ s.t. x is max/min point for f on $\mathcal{A} \bigcap (x - \delta, x + \delta)$.

Remark. Note that:

- We already know if f is conti on $[a, b]$, then f has a max/min point and value (EVT).
- Max/min points do not always exist.
- if x is a max/min point, then f is also a local max/min point.

Theorem 11.1 (Max/min point & derivative)**.** Let f defined on (a, b). If x is a max/min point for f on (a, b), and f is differentiable at x, then $f'(x) = 0$.

Theorem 11.2. If x is a local max/min point for f on (a, b) and f is differentiable at x, then $f'(x) = 0$.

Remark. Some brain teasers:

(i) **Why must the interval be open?** For end points- we don't know the definition of $f'(x)$ on end point if it happens to be local max.

(ii) **Is the converse of the statement true** (i.e., $f'(x) = 0 \Rightarrow$ local max/min)? NO- consider saddle points where $f'(x) = 0$ but not local min/max (e.g., x^3 at $x = 0$).

(iii) **Must every max/min point of function f has $f'(x) = 0$?** This is equivalent to ask what happens if f is not differentiable at some x. Obviously, if $f(x) = |x|$, $f'(x)$ DNE at $x = 0$ but x is a local min.

Definition 11.3 (Critical point)**.** A critical point of a function f is a number such that $f'(x) = 0$. E.g., if x is a local max/min of f and $f'(x)$ exists, then x is a critical point of f.

Motivation. If f is differentiable at all $x \in (a, b)$ and that $f'(x) = 0$ for all $x \in (a, b)$, what should be true about the function?

Theorem 11.3 (Rolle's Theorem)**.** If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there is a number $x \in (a, b)$ s.t. $f'(x) = 0$

Proof. ① If such max/min $x \in (a, b)$, then by **Theorem 11.2** we have shown that $f'(x) = 0$, done. ② If such max/min on a or b (end points). Say a is the max point and b is the min point, then we have $f(a) = f(b) \Leftrightarrow$ max value = min value. This implies that f is a constant function $\Rightarrow f'(x) = 0$ by **Theorem 10.1**, done. □

Theorem 11.4 (Mean Value Theorem: MVT)**.** If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number $x \in (a, b)$ s.t. $f'(x) = \frac{f(b)-f(a)}{b-a}$.

Remark. $f'(x)$: instantaneous change; $\frac{f(b)-f(a)}{b-a}$: average rate of change

Proof. Idea: define $h(x)$ such that $h'(x) = 0$ we get $f'(x) = \frac{f(b)-f(a)}{b-a}$
$\Rightarrow$ let $h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} = 0$, then $h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x - a)$.
We notice that since f is conti on $[a, b]$ and differentiable on (a, b), then h is also conti on $[a, b]$ and and differentiable on (a, b). Also, $h(a) = f(a) - \frac{f(b)-f(a)}{b-a}(a - a) = f(a)$ and $h(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b - a) = f(b) - (f(b) - f(a)) = f(a)$. By **Theorem 11.3 (Rolle's Theorem)**, we conclude that there exist a number $x \in [a, b]$ s.t. $h'(x) = 0 \Rightarrow f'(x) - \frac{f(b)-f(a)}{b-a} = 0 \Rightarrow f'(x) = \frac{f(b)-f(a)}{b-a}$, Q.E.D. □

Corollary 1. If f is defined on (a, b) and $f'(x) = 0\ \forall x \in (a, b)$, then f is constant function on (a, b).

Remark. ① pick $c, d \in (a, b)$ and apply MVT, or ② assume $\exists m, n \in (a, b)$, $m < n$ s.t. $f(m) \neq f(n)$, apply MVT, and then show contradiction to $f'(x) = 0$.

Corollary 2. If f and g are defined on (a, b) and $f'(x) = g'(x)\ \forall x \in (a, b)$, then there is some number c s.t. $f(x) = g(x) + c\ \forall x \in (a, b)$.

Remark. ① Assume $\forall c, f(x) \neq g(x) + c \Rightarrow f'(x) \neq g'(x) + 0$. Contradiction. Or, ② let $h(x) = f(x) - g(x)$, then $h'(x) = f'(x) - g'(x) = 0 \Rightarrow h(x)$ is constant function by **Corollary 1.** So $h(x) = c = f(x) - g(x)$.

Definition 11.4 (Increasing/Decreasing)**.** Let f be a function and let $\mathcal{A} \subset \text{dom}(f)$.

(i) f is increasing on $\mathcal{A}$ if $f(a) < f(b)$ whenever $a, b \in \mathcal{A}$ and $a < b$.

(ii) f is decreasing on $\mathcal{A}$ if $f(a) > f(b)$ whenever $a, b \in \mathcal{A}$ and $a < b$.

Remark. Non-decreasing on $\mathcal{A}$ if $f(a) \leq f(b)$; Non-increasing on $\mathcal{A}$ if $f(a) \geq f(b)$.

Corollary 3. Suppose f is defined on (a, b).

(i) if $f'(x) > 0 \; \forall x \in (a, b)$, then f is increasing on (a, b).

(ii) if $f'(x) < 0 \; \forall x \in (a, b)$, then f is decreasing on (a, b).

Remark. The converse of statement is NOT true (if f is increasing on (a, b), then $f'(x) > 0 \; \forall x \in (a, b)$). E.g., x^3 is an increasing function, but $f'(x) = 0$ at $x = 0$.

Theorem 11.5 (L'Hopital Rule)**.** Suppose that $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$. If $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ (i.e., exists).

Remark. A brain teaser (T/F): Suppose $f, g : \mathbb{R} \to \mathbb{R}$ differentiable and $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ exists, does $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ exist? F- if $\lim\limits_{x \to a} g'(x) = 0$ then DNE. E.g., take $f = x^2$ and $g = \cos(x)$ as $x \to 0$.

12 Uniform Continuity (Chpt 8 Appendix)

Motivation. want a δ only depending on ε. Consider the following illustrative examples:

(i) find δ s.t. $f(x) = 2x - 1$ conti $\Rightarrow |f(x) - (2a - 1)| = 2|x - a| < \varepsilon \Rightarrow$ can pick a $\delta = \frac{\varepsilon}{2}$

(ii) find δ s.t. $g(x) = x^2$ conti $\Rightarrow$ can pick $\delta = \min\left\{1, \frac{\varepsilon}{2|a|+1}\right\}$ but δ depends on a and ε.

Definition 12.1 (Uniformly Continuous)**.** A function f is be *uniformly continuous* on an interval $\mathcal{A}$ **if** for every $\varepsilon > 0$, $\exists$ some δ s.t. $\forall x, y \in \mathcal{A}$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Remark. Different from the original definition of continuous, we now allow a to dynamically change to arbitrary y in the interval $\Rightarrow \delta$ does not depend on a (i.e., only ε matters).

Example 12.1. Let $S = \{x \in \mathbb{R} : 0 < x < 4\}$. Show $g(x) = x^2$ uniformly conti.
$\Rightarrow |f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| \leq |x - y|(|x| + |y|) < |x - y|(4 + 4) < \varepsilon$
(by triangle inequality) We can thus pick $\delta = \frac{1}{8}\varepsilon$.

Example 12.2. Let $f(x) = \frac{1}{x^2}$, $\mathcal{A} = (0, 1]$. Show that f is NOT uniformly continuous.
$\Rightarrow$ Set $\varepsilon = 1$, choose $\delta > 0$. Want to find $x, y \in \mathcal{A}$ s.t. if $|x - y| < \delta$, but $|f(x) - f(y)| > \varepsilon$.
We can pick $x < \delta$ and $y = \frac{x}{2}$, then $|x - y| = |x - \frac{x}{2}| = \frac{x}{2} < \frac{\delta}{2} < \delta$, but then $|\frac{1}{x^2} - \frac{1}{y^2}| = \frac{3}{x^2} \geq 3 > 1 > \varepsilon$, done.

Lemma. Let $a < b < c$ and f be continuous on $[a, b]$. Suppose f is uniformly continuous on $[a, b]$ and $[b, c]$, then f is uniformly continuous on $[a, c]$.

Proof. let $\varepsilon > 0$,

(i) $\exists \delta_1 > 0$ s.t. if $x, y \in [a, b]$ and $|x - y| < \delta_1$, then $|f(x) - f(y)| < \varepsilon$.

(ii) $\exists \delta_2 > 0$ s.t. if $x, y \in [b, c]$ and $|x - y| < \delta_2$, then $|f(x) - f(y)| < \varepsilon$.

(iii) $\exists \delta_3 > 0$ s.t. if $|x - b| < \delta_3$, then $|f(x) - f(b)| < \frac{\varepsilon}{2}$. (using the fact that f conti at b) $\Rightarrow$ if $|x - b| < \delta_3, |y - b| < \delta_3$, then: $|f(x) - f(y)| = |f(x) - f(b) + f(b) - f(y)| \leq |f(x) - f(b)| + |f(y) - f(b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus, we only need to pick $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, Q.E.D. □

Definition 12.2 (ε-good)**.** Suppose f is conti on $[a, b]$. Let $\varepsilon > 0$ be given. f is *ε-good* on $[a, x]$ if there is a $\delta > 0$ s.t. for all $y, z \in [a, x]$, if $|y - z| < \delta$, then $|f(y) - f(z)| < \varepsilon$.

Remark. Different from uniformly continuity, we are given specific ε here.

Theorem 12.1 (Continuous $\Rightarrow$ Uniformly Continuous)**. If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.**

Proof. Want to use **Lemma** and **Definition 12.2** to show that f is ε-good on $[a, b]$ for all $\varepsilon > 0$. We can consider a specific $\varepsilon > 0$. Let $\mathcal{A} = \{x : a \leq x \leq b$ and f is ε-good on $[a, x]\}$. Since $\mathcal{A} \neq \emptyset$ ($a \in \mathcal{A}$ at least), and $\mathcal{A}$ is bounded above (by b), $\mathcal{A}$ has a least upper bound α by Property of Least Upper Bound (**Theorem 8.1**). Then we only need to show such $\alpha = b$ no matter what ε is.

- Suppose $\alpha < b$: since f is conti at α, there is some $\delta_0 > 0$ s.t. if $|y - \alpha| < \delta_0$, then $|f(y) - f(\alpha)| < \frac{\varepsilon}{2} \Rightarrow$ if $|y - \alpha| < \delta_0 (\Leftrightarrow \alpha - \delta_0 < y < \alpha + \delta_0)$ and $|z - \alpha| < \delta_0|$, then $|f(y) - f(z)| < \varepsilon$ (similar trick to **Lemma** case(iii)). This implies f is ε-good on $[\alpha - \delta_0, \alpha + \delta_0]$. Note that $\alpha = \sup \mathcal{A} \Rightarrow f$ is ε-good on $[a, \alpha - \delta_0]$. By **Lemma**, we know f is then ε-good on $[a, \alpha + \delta_0]$, so $\alpha + \delta_0$ is in $\mathcal{A}$, **contradiction** to the fact that $\alpha = \sup \mathcal{A}$.

- Suppose $\alpha = b$: since f is conti at b, (by one-sided limit from left) there exists δ_0 such that if $0 < b - y < \delta_0 \Rightarrow b - \delta_0 < y < b$, then $|f(y) - f(b)| < \frac{\varepsilon}{2}$. So f is ε-good on $[b - \delta_0, b]$. Note that $b = \alpha = \sup \mathcal{A} \Rightarrow f$ is ε-good on $[a, \alpha - \delta_0] = [a, b - \delta_0]$. By **Lemma**, we confirm f is ε-good on $[a, b]$ and holds true for all $\varepsilon > 0$, Q.E.D.

□

Example 12.3. Let $\mathcal{A} = [a, \infty)$ and $f : \mathcal{A} \to \mathbb{R}$ be differentiable on $\mathcal{A}$. If $\lim\limits_{x\to\infty} f'(x) = \infty$, then f is NOT uniformly continuous on $\mathcal{A}$.

Scratch Work: Consider showing the contra-positive form ($\neg q \Rightarrow \neg p$): "if f is uniformly continuous on $\mathcal{A}$, then $\lim\limits_{x\to\infty} f'(x) \neq \infty$."

Proof. By definition of uniform continuity, we know $\forall \varepsilon > 0$, there exists some $\delta > 0$ s.t. $\forall x, y \in \mathcal{A}$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Suppose $x > y$, since f is differentiable on $[y, x]$ (and thus continuous on $[y, x]$ by **Theorem 9.1**), we can then apply **Theorem 11.4 MVT** so that there exists a number $\phi \in (y, x)$ s.t.

$$f'(\phi) = \frac{f(x) - f(y)}{x - y} \Rightarrow f(x) - f(y) = f'(\phi)(x - y) \Rightarrow |f(x) - f(y)| = |f'(\phi)| \cdot |x - y|$$

Backing to the uniform discontinuity, we obtain: $|f(x) - f(y)| = |f'(\phi)| \cdot |x - y| < |f'(\phi)|\delta = \varepsilon$. Since the existence of δ is secured by the definition, it suggests that we can now pick such $\delta = \frac{\varepsilon}{|f'(\phi)|} > 0$. This requires that:

$$\delta = \frac{\varepsilon}{|f'(\phi)|} > 0 \Rightarrow \lim_{\phi\to\infty} \delta = \lim_{\phi\to\infty} \frac{\varepsilon}{|f'(\phi)|} = \varepsilon \cdot \frac{1}{\lim\limits_{\phi\to\infty} |f'(\phi)|} = \varepsilon \cdot \frac{1}{|\lim\limits_{\phi\to\infty} f'(\phi)|} > 0.$$

Since $\varepsilon > 0, 1 > 0$, we immediately know that $\lim\limits_{\phi\to\infty} f'(\phi) \neq \infty$ to ensure the existence of such $\delta > 0$ (otherwise, $\lim\limits_{\phi\to\infty} \delta = 0 \not> 0$), which completes our contra-positive statement. Therefore, for $\mathcal{A} = [a, \infty)$ and $f : \mathcal{A} \to \mathbb{R}$ be differentiable on $\mathcal{A}$, if f is uniformly continuous on $\mathcal{A}$, then $\lim\limits_{x\to\infty} f'(x) \neq \infty$. Equivalently, if $\lim\limits_{x\to\infty} f'(x) = \infty$, then f is NOT uniformly continuous on $\mathcal{A}$, Q.E.D. $\square$

13 Integral

Definition 13.1 (partition)**.** Let $a < b$. A **partition** of the interval $[a, b]$ is a finite collection of points in $[a, b]$, one of which is a, and one of which is b.

Remark. Simply let $a = t_0 < t_1 < \cdots < t_n = b$.

Definition 13.2 (Lower & Upper Sum)**.** Suppose f is bounded on $[a, b]$ and $P = \{t_0, \cdots, t_n\}$ is a partition of $[a, b]$. Let $m_i = \inf\{f(x) : t_{i-1} \le x \le t_i\}$, $M_i = \sup\{f(x) : t_{i-1} \le x \le t_i\}$. Then, The **lower sum** f for P: $L(f, P) \equiv \sum\limits_{i=1}^{n} m_i(t_i - t_{i-1})$.
The **upper sum** f for P: $U(f, P) \equiv \sum\limits_{i=1}^{n} M_i(t_i - t_{i-1})$.

Remark. Total area of rectangles below (i.e., inf $\Rightarrow$ lower sum L) and above (i.e., sup $\Rightarrow$ upper sum U). Obviously, $L(f, P) \le U(f, P)$.

Lemma. If $P \subseteq Q$ (i.e., Q contains $P \Leftrightarrow$ all points of P are in Q), then $L(f, P) \le L(f, Q)$ and $U(f, P) \ge U(f, Q)$.

Proof. Let $P = \{t_0, \cdots, t_n\}$ and $Q = \{t_0, \cdots, t_{k-1}, u, t_k, \cdots t_n\}$ additional one point than P. Suppose $m' = \inf\{f(x) : t_{k-1} \le x \le u\}$ and $m'' = \inf\{f(x) : u \le x \le t_k\}$ (two segments). Then $L(f, P) = \sum\limits_{i=1}^{n} m_i(t_i - t_{i-1})$ (unchanged), and
$L(f, Q) \equiv \sum\limits_{i=1}^{k-1} m_i(t_i - t_{i-1}) + m'(u - t_{k-1}) + m''(t_k - u) + \sum\limits_{i=k+1}^{n} m_i(t_i - t_{i-1})$. Say $m_k = \inf\{f(x) : t_{k-1} \le x \le t_k\}$, obviously $m_k \le m'$ and $m_k \le m''$. Plug this into to above equation and complete the proof of $L(f, P) \le L(f, Q)$. Prove $U(f, P) \ge U(f, Q)$ in a similar fashion. □

Theorem 13.1. Let P_1 and P_2 be partitions of $[a, b]$ and let f be a function which is bounded on $[a, b]$. Then,

$$L(f, P_1) \le U(f, P_2)$$

Proof. Let P contains both P_1, P_2, by **Lemma.**, we have: $L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2)$. □

Definition 13.3. A function f which is bounded on $[a, b]$ is **integrable** on $[a, b]$ if $\sup\{L(f, P)\} = \inf\{U(f, P)\}$ where P is a partition of $[a, b]$. The common number is called the integral of f on $[a, b]$ and is denoted by $\int_a^b f$.

Remark. $L(f, P) \le \int_a^b f =$ Area of $R(f, a, b) \le U(f, P)$, for $f(x) \ge 0$. $\int_a^b$ is a *unique* number here.

Theorem 13.2. If f is bounded on $[a,b]$, then f is integrable on $[a,b]$ on $[a,b]$ **if and only if** for every $\varepsilon > 0$ there is a partition P of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$

Proof. We need to show both directions are true.
"$\Leftarrow$": Use $\inf\{U(f,P')\} \leq U(f,P)$ and $\sup\{L(f,P')\} \geq L(f,P)$, then $\inf\{U(f,P')\} - \sup\{L(f,P')\} \leq U(f,P) - L(f,P) < \varepsilon$. Since it is true for all $\varepsilon > 0$, it follows that $\inf\{U(f,P')\} = \sup\{L(f,P')\}$. By **Definition 13.3**, done.
"$\Rightarrow$": By **Definition 13.3**, if f is integrable, then $\sup\{L(f,P)\} = \inf\{U(f,P)\} \Leftrightarrow U(f,P'') - L(f,P') < \varepsilon$ for some partitions P', P''. Let P contains P', P'', then by **Lemma.**, $L(f,P') \leq L(f,P)$ and $U(f,P'') \geq U(f,P) \Rightarrow U(f,P) - L(f,P) \leq U(f,P'') - L(f,P') < \varepsilon$, done. $\square$

Remark. We have the following **Properties**:

(i) $\int_a^b f = c \cdot (b-a)$ **if** $f(x) = c \ \forall x$

(ii) $\int_a^b f = \frac{b^2}{2} - \frac{a^2}{2}$ **if** $f(x) = x \ \forall x$

(iii) $\int_a^b f = \frac{b^3}{3} - \frac{a^3}{3}$ **if** $f(x) = x^2 \ \forall x$

$\Rightarrow \int_a^b (x+y)dx = \int_a^b x dx + \int_a^b y dx = \frac{b^2}{2} - \frac{a^2}{2} + y(b-a)$ (i.e., treat y as const c 1st property)

Theorem 13.3 (Conti $\Rightarrow$ Integrable). **If f is conti on $[a,b]$, then f is integrable on $[a,b]$.**

Proof. Since f is conti on $[a,b] \Rightarrow f$ is bounded on $[a,b]$. We can use Theorem 13.2 to show that $\forall \varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$. Then, by **Theorem 12.1**, f is also uniformly continuous on $[a,b]$ since f is conti on $[a,b]$. So there exists some $\delta > 0$ s.t. if $|x-y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$ for all $x, y \in [t_{i-1}, t_i]$. Recall that $m_i = \inf\{f(x) : t_{i-1} \leq x \leq t_i\}$ and $M_i = \sup\{f(x) : t_{i-1} \leq x \leq t_i\}$, we thus have $M_i - m_i \leq \frac{\varepsilon}{2(b-a)} < \frac{\varepsilon}{b-a}$. Since it is true for all i, we then obtain $U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (t_i - t_{i-1}) = \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon$ $\square$

Summary. Diff $\Rightarrow$ Conti $\Rightarrow$ { Unif Conti; Integrable }

Theorem 13.4. Let $a < c < b$. If f is integrable on $[a,b]$, then it follows that f is integrable on $[a,c]$ and $[c,b]$. Conversely if f is integrable on $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$. Notation: $\int_a^b f = \int_a^c f + \int_c^b f$.

Theorem 13.5. If f and g are integrable on $[a,b]$, then $f+g$ is integrable on $[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Theorem 13.6. If f is integrable on $[a,b]$, then for any number c, the function cf is integrable on $[a,b]$ and $\int_a^b cf = c \cdot \int_a^b f$.

Theorem 13.7. Suppose f is integrable on $[a,b]$ and that $m \leq f(x) \leq M$ for all $x \in [a,b]$. Then $m(b-a) \leq \int_a^b f \leq M(b-a)$. (Note: this follows $m(b-a) \leq \sup\{L(f,P)\} = \int_a^b f = \inf\{U(f,P)\}$)

Theorem 13.8. If f is integrable on $[a,b]$ and $F(x) \equiv \int_a^x f$ on $[a,b]$, then F is continuous on $[a,b]$.

14 Fundamental Theorem of Calculus

Theorem 14.1 (The First Fundamental Theorem of Calculus)**.** Let f be integrable on $[a,b]$ and define F on $[a,b]$ as $F \equiv \int_a^x f$. If f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$. (Note: Separate $[a,b]$ into ① $c \in (a,b)$ and ② $c = a$ or b for the proof)

Corollary. If f is conti on $[a,b]$ and $f = g'$ for some function g, then $\int_a^b f = g(b) - g(a)$.

Remark. Why do we need this Corollary? Say $g(x) = \frac{x^3}{3}$ and $f(x) = x^2$. Since $g'(x) = f(x)$, then $\int_a^b f(x) = g(b) - g(a) = \frac{b^3}{3} - \frac{a^3}{3}$ without computing lower/upper sum for f.

Theorem 14.2 (Second Fundamental Theorem of Calculus)**.** If f is integrable on $[a,b]$ and $f = g'$ for some function g, then $\int_a^b f = g(b) - g(a)$.

Example 14.1. Find $f'(x)$ suppose that $f(x) = \int_a^{x^3} \frac{1}{1+\sin^2(t)} dt$.
$\Rightarrow$ we can let $c(x) = x^3$ and $G(x) = \int_a^x \frac{1}{1+\sin^2(t)} dt$, and then we notice that $f(x) = G(c(x))$. Thus, by Chain Rule (**Theorem 10.9**), we obtain:

$$f'(x) = G'(c(x)) \cdot c'(x) = \underbrace{\frac{1}{1+\sin^2(x^3)}}_{\text{by FTC}} \cdot 3x^2$$

.

Example 14.2. Find $f'(x)$ suppose that $f(x) = \int_a^{\left(\int_a^{x^3} \frac{1}{1+\sin^2(t)dt}\right)} \frac{1}{1+\sin^2(t)} dt$.
$\Rightarrow$ we can let $c(x) = x^3$ and $G(x) = \int_a^x \frac{1}{1+\sin^2(t)} dt$, and then we notice that $f(x) = G(G(c(x)))$. Thus, by Chain Rule (**Theorem 10.9**), we obtain:

$$f'(x) = G'(G(c(x))) \cdot G'(c(x)) \cdot c'(x) = \frac{1}{1+\sin^2\left(\int_a^{x^3} \frac{1}{1+\sin^2(t)} dt\right)} \cdot \frac{1}{1+\sin^2(x^3)} \cdot 3x^2$$

Example 14.3. Some brain teasers:

(i) Every integrable function f is the derivative of another function g. F- think $f(x) = |x|$: integrable but not differentiable.

(ii) $\int_a^b x^3 dx$ represents the area of the region bounded by $x = a, x = b$, and the horizontal axis. F- think $[-1,1]$ (in fact any $a < 0, b > 0$).

(iii) Let f be integrable on $[a,b]$, let $c \in (a,b)$, and let $F(x) = \int_a^b f$, where $a \le x \le b$.

(a) If f is differentiable at c, then F is differentiable at c. T- Theorem 14.1

(b) If f is differentiable at c, then F' is continuous at c. T- Theorem 14.1

(c) If f' is continuous at c, then F' is continuous at c. T- $f'(c)$ exists, i.e., f differentiable at c, then follows (b).

References

Spivak, M. (2008). Calculus^{4th} (4th ed.).