A Bayesian Framework for Joint Morphometry of Surface and Curve meshes in Multi-Object Complexes[1]

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Géométrie et espaces de formes

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[1] Pietro Gori et al. (Jan. 2017). "A Bayesian Framework for Joint Morphometry of Surface and Curve meshes in Multi-Object Complexes". In: *Medical Image Analysis* 35, pp. 458–474. URL: https://hal.inria.fr/hal-01359423.

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Consists of estimating an average shape complex of a population called template complex, and the deformations of the embedding which warp template complex to the shape complexes of every subject.

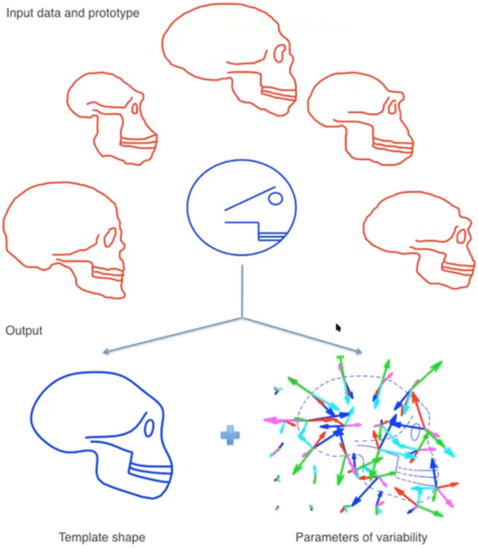

Figure 1: Principle of atlas construction.

Limitations

- Most of the works are about registration and atlas construction of images.
- Even then, data noise and regularity level are usually fixed by the user.
- Curve-specific or Surface-specific methods.
- "currents cancelling effect"
- Typically done using Fréchet mean as in Durrleman et al. 2014
    - Simple underlying statistical model.
    - Trying to optimize its weighting factors σ_j^{-2} can leads to only focusing on one structure, ignoring others.

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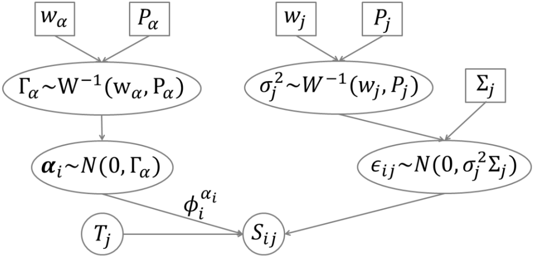


$$S_{ij} = \phi_i^{\alpha_i}(T_j) + \epsilon_{ij}$$

Figure 2: Generative model for shape variability.

i: subject index
j: structure index
S_{ij}: shape
$\boldsymbol{\alpha}_i$: deformation parameter
$\phi_i^{\alpha_i}$: deformation
T_j: template
ϵ_{ij}: noise

This gives MAP estimator

$$\left\{\boldsymbol{T}^*, \Gamma_\alpha^*, \sigma_j^{2*}\right\} = \arg\max_{T,\Gamma_\alpha,\sigma_j^2} \left[\prod_i^N \prod_j^M \int p\left(T_j, \Gamma_\alpha, \sigma_j^2, \alpha_i, S_{ij}\right) d\alpha_i\right]$$

which is approximated for high SNR with

$$\{\boldsymbol{\Theta}^*, \boldsymbol{Z}^*\} = \underset{\boldsymbol{\Theta},\boldsymbol{Z}}{\operatorname{argmin}} - \log\left[p\left(\{\boldsymbol{S}_i\}/\boldsymbol{Z}, \boldsymbol{\Theta}\right)\right] - \log[p(\boldsymbol{Z}/\boldsymbol{\Theta})] - \log[p(\boldsymbol{\Theta})]$$

where $\boldsymbol{\Theta} = \left\{\Gamma_\alpha, \boldsymbol{T}, \left\{\sigma_j^2\right\}\right\}$ are the parameters of interest and $\boldsymbol{Z} = \{\{\boldsymbol{\alpha}_i\}\}$ the unobserved nuisance variables.

Thus the loss to minimize becomes

$$\mathcal{L}(\boldsymbol{\Theta}, \boldsymbol{Z}) = \sum_{j=1}^M \sum_{i=1}^N \frac{1}{2\sigma_j^2}\left(\left\|S_{ij} - \varphi_i\left(T_j\right)\right\|^2 + \frac{P_j w_j}{N}\right) + \sum_{j=1}^M \frac{1}{2}\left(w_j + \Lambda_j N\right)\log\left(\sigma_j^2\right)$$

$$+ \frac{1}{2}\sum_{i=1}^N \left(\boldsymbol{\alpha}_i\right)^T \left(\Gamma_\alpha\right)^{-1} \alpha_i + \frac{1}{2}\left(w_\alpha + N\right)\log\left(\left|\Gamma_\alpha\right|\right) + \frac{w_\alpha}{2}\operatorname{tr}\left(\left(\Gamma_\alpha\right)^{-1} P_\alpha\right)$$

Rest to define $\|S_{ij} - \varphi_i(T_j)\|^2$ i.e. what is $\|\cdot\|$ and φ ?

$$\begin{aligned}\langle V_X, V_Y\rangle_{W^*} &= \sum_{p=1}^{P}\sum_{q=1}^{Q} k_x(x_p, y_q)\, k_\beta\left(\overleftrightarrow{\beta_p}, \overleftrightarrow{\gamma_q}\right) c_p d_q \\ &= \sum_{p=1}^{P}\sum_{q=1}^{Q} \underbrace{\exp\left(\frac{-\|x_p - y_q\|^2}{\lambda_W^2}\right)}_{\text{Gaussian}} \underbrace{\left(\frac{\beta_p^T \gamma_q}{c_p d_q}\right)^2}_{\text{Cauchy-Binet}} c_p d_q\end{aligned}$$

And then $\|V_X - V_Y\|_{W^*}^2 = \langle V_X, V_X\rangle_{W^*} + \langle V_Y, V_Y\rangle_{W^*} - 2\langle V_X, V_Y\rangle_{W^*}$

- For discretization, take the closest point y and closest direction $\overleftrightarrow{\alpha}$ in a joint grid and assign the scalar $|\alpha|$ to that particular couple of grid points.
- Varifold kernel matrix is thus only computed once at the beginning.
- This is done instead of relying on potentially different curves and surfaces discretization points that would need to be recomputed each time those points has moved under the action of diffeomorphisms.

$\{\varphi_t\}_{t\in[0,1]}$ is an invertible flow of diffeomorphisms induced by an RKHS V, supported by control points $\{c_p\}$ i.e.

$$\dot{x}(t) = v_t\left(\varphi_t(x)\right) = \sum_{p=1}^{C_p} K_V\left(x(t), c_p(t)\right)\alpha_p(t) \quad \varphi_0(x) = x(0) = x$$

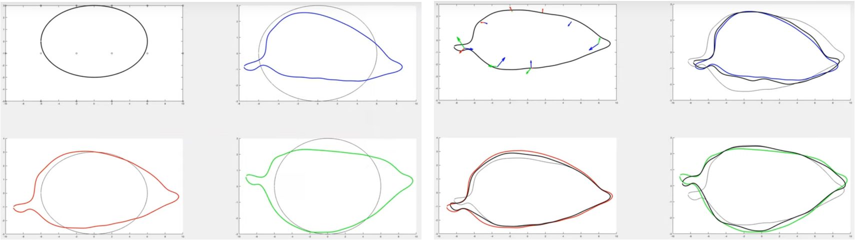
Among all the possible paths connecting φ_0 to φ_1 we use the geodesic one, which means the one that minimizes the total kinetic energy along the path

$$\int_0^1 \|v_t\|_V^2\, dt = \int_0^1 \sum_{k=1}^{C_p}\sum_{p=1}^{C_p} \alpha_k(t)^T K_V\left(c_k(t), c_p(t)\right)\alpha_p(t)dt$$

It has been demonstrated in that the extremal paths follow the Hamiltonian system

$$\begin{cases} \dot{c}_k(t) = \sum_{p=1}^{C_p} K\left(c_k(t), c_p(t)\right)\alpha_p(t) \\ \dot{\alpha}_k(t) = -\sum_{p=1}^{C_p} \alpha_k(t)^T\alpha_p(t)\nabla_1 K\left(c_k(t), c_p(t)\right) \end{cases}$$

We optimize on $\{\alpha_{i0}\}$ the initial momentum, and on template complex T. We also optimize on control points $\{c_p\}$ shared among patients. Their optimization is done by integrating backward in time from $t = 1$ to $t = 0$ a set of linearised ODEs called adjoint equations.



(a) Initialization (b) Termination

Whereas due to the use of conjugate priors, the optimal values for $\{\sigma_j^2\}$ and Γ_α can be computed in closed form:

$$\hat{\Gamma}_\alpha = \frac{\sum_{i=1}^{N} \left[(\boldsymbol{\alpha}_{i0}) (\boldsymbol{\alpha}_{i0})^T \right] + w_\alpha P_\alpha^T}{(w_\alpha + N)} \quad \hat{\sigma}_j^2 = \frac{\sum_{i=1}^{N} \| S_{ij} - \varphi_i (T_j) \|_{W_{\lambda j}^*}^2 + w_j P_j}{(w_j + N \Lambda_j)}$$

Priors introduce previously missing (or a posteriori added) regularization.

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- 20 Gilles de la Tourette patients and 20 control subjects, using three sub-cortical regions and their incident white matter fiber bundles.
- Subset of fiber bundle is chosen according to the varifold metric approximately optimized through a greedy algorithm.
- Their are not homotopic.

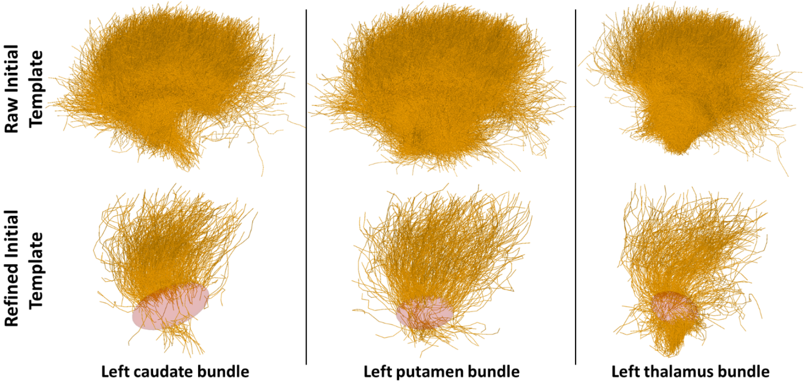

Figure 4: Template initialisation for three brain fiber bundle.

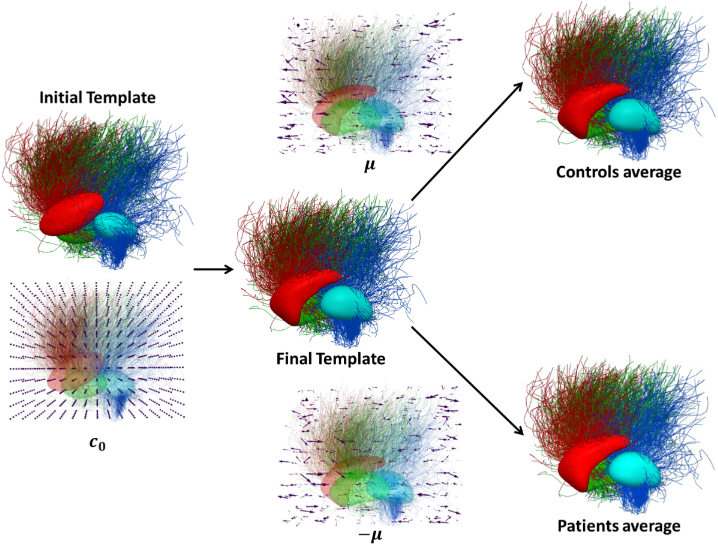

Figure 5: Atlas construction procedure

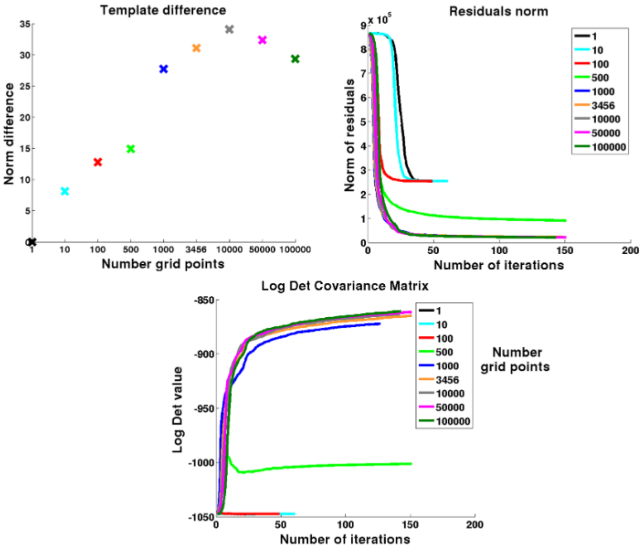

Figure 6: Robustness to level of discretization.

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Conclusion

- Handle both curves and surfaces.
- Only kernel bandwidth need to be selected by the user.
- Prior induce regularization and stabilize weights.
- No cancelling effect.
- Easily tuned for multi-population atlas construction with or without shared variability.

Perspectives

- Automatic bandwidths selection.
- MCMC computation for low SNR.
- Full Bayesian approach quantifying uncertainty.

Durrleman, Stanley et al. (2014). "Morphometry of anatomical shape complexes with dense deformations and sparse parameters". In: *NeuroImage*. DOI: 10.1016/j.neuroimage.2014.06.043. URL: https://hal.inria.fr/hal-01015771.

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Thank you!

Adjoint equations

$$\nabla_T E = \sum_{i=1}^{N} \boldsymbol{\theta}_i(0)$$

$$\nabla_{\boldsymbol{\alpha}_{i0}} E = \xi_i^{\alpha}(0) + \Gamma_{\alpha}^{-1} \boldsymbol{\alpha}_{i0}$$

$$\nabla_{c_0} E = \sum_{i=1}^{N} \xi_i^{c}(0)$$

where the auxiliary variables $\xi_i(t) = \{\xi_i^{\alpha}(t), \xi_i^{c}(t)\}$ and $\theta_i(t)$ satisfy the linearised ODEs:

$$\dot{\boldsymbol{\theta}}_i(t) = -\left(\partial_{T_i} G\left[T_i(t), L_i(t)\right]\right)^T \boldsymbol{\theta}_i(t) \quad \boldsymbol{\theta}_i(1) = \frac{1}{2\sigma^2} \nabla_{T_i(1)}\left[D_i\right]$$

$$\dot{\xi}_i(t) = -\left(\partial_{L_i} G\left[T_i(t), L_i(t)\right]\right)^T \boldsymbol{\theta}_i(t) + d_{L_i} F\left[L_i(t)\right]^T \xi_i(t) \quad \xi_i(1) = 0$$