

1 Question 1

Suppose $n \geq 3$ and let be G a n -cycle graph. After removing one edge with uniform probability, one gets a $(n-1)$ -chain. Then one can remove a second edge with uniform probability and get

- with probability $\frac{2}{n-1}$, a $(n-2)$ -chain, then there is 1 connected components.
- with probability $\frac{n-3}{n-1}$, 2 chains, then there is 2 connected components.

2 Question 2

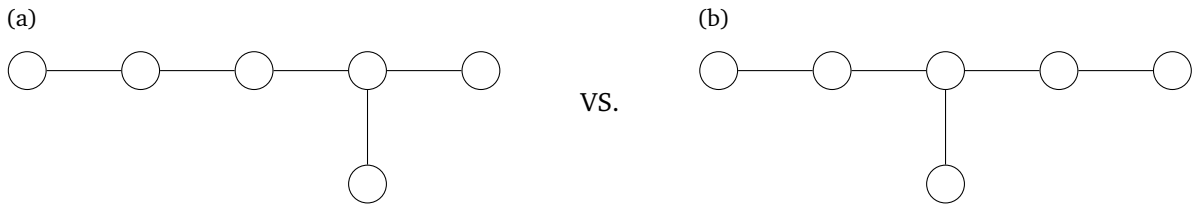


Figure 1: The two graphs above have same degree distribution but are not isomorphic.

3 Question 3

Let be G an undirected graph without self-loops consisting of $n \geq 3$ nodes and $\frac{n(n-1)}{2} - 1$ edges. This means G is a clique with one less edge. Therefore G has $\binom{3}{n} - (n-2)$ triangles, hence $3 \times (\binom{3}{n} - (n-2))$ closed triplets, and $n-2$ open triplets. Finally, the transitivity coefficient of G is

$$\frac{3 \times (\binom{3}{n} - (n-2))}{3 \times (\binom{3}{n} - (n-2)) + n-2} = \frac{n(n-1)-6}{n(n-1)-4}$$

4 Question 4

According to **Proposition 4** from von Luxburg [1], the smallest eigenvalue of L_{rw} is 0 and its multiplicity k equals the number of connected components A_1, \dots, A_k in the graph. Moreover, the eigenspace of 0 is spanned by the indicator vectors $\mathbb{1}_{A_i}$ of those components.

5 Question 5

The output of spectral clustering can be stochastic if the initialisation of k -means is stochastic, which is often the case in practice. Indeed, k -means does not necessarily converge to the same values.

6 Question 6

a) $m = 8$; $l = (4, 3)^\top$; $d = (9, 7)^\top$; hence $Q = \frac{4+3}{8} - (\frac{9}{2 \times 8})^2 - (\frac{7}{2 \times 8})^2 = 752 \times 2^{-11} \simeq 0.367$

b) $m = 8$; $l = (1, 2, 1)^\top$; $d = (4, 8, 4)^\top$; hence $Q = \frac{1+2+1}{8} - 2 \times (\frac{4}{2 \times 8})^2 - (\frac{8}{2 \times 8})^2 = \frac{1}{8} = 0.125$

7 Question 7

$$P_4 \quad \begin{array}{c} \circ \\ \circ \\ \circ \\ \circ \end{array} \text{---} \text{---} \text{---} \text{---} \quad \longrightarrow \quad \Phi(P_4) = (3, 2, 1)^\top$$

$$K_4 \quad \begin{array}{cc} \circ & \circ \\ \circ & \circ \end{array} \quad \longrightarrow \quad \Phi(K_4) = (6, 0, 0)^\top$$

Then,

$$\begin{cases} K_{sp}(P_4, P_4) = \|\Phi(P_4)\|^2 = 14 \\ K_{sp}(P_4, K_4) = \Phi(P_4)^\top \Phi(K_4) = 18 \\ K_{sp}(K_4, K_4) = \|\Phi(K_4)\|^2 = 36 \end{cases}$$

References

- [1] Ulrike von Luxburg. A tutorial on spectral clustering, 2007.