Mathematical Methods for Neurosciences.
Paris 6 - Master Maths-Bio
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Outline

Outline

Hodgkin-Huxley Model

$$C\dot{V} + I_{\mathrm{Na}} + I_{\mathrm{K}} + I_{\mathrm{L}} = I,$$

with $I_k := G_k(V - E_k)$ (intensity of the ionic current k for Na, K or L)

$$\begin{aligned} G_{\mathrm{L}} &:= \bar{g}_{\mathrm{L}} \\ G_{\mathrm{K}} &:= \bar{g}_{\mathrm{K}} n^4 \\ G_{\mathrm{Na}} &:= \bar{g}_{\mathrm{Na}} m^3 h. \end{aligned}$$

The proportion of open channels satisfy

$$\begin{aligned} \dot{n} &= \alpha_n(V)(1-n) - \beta_n(V)n \\ \dot{m} &= \alpha_m(V)(1-m) - \beta_m(V)m \\ \dot{h} &= \alpha_h(V)(1-h) - \beta_h(V)h. \end{aligned}$$

Motivation

- What is *really* random?
- Stochastic Models in general
- Sources of noise in neuronal activities
- Monte Carlo Methods
- Efficiency

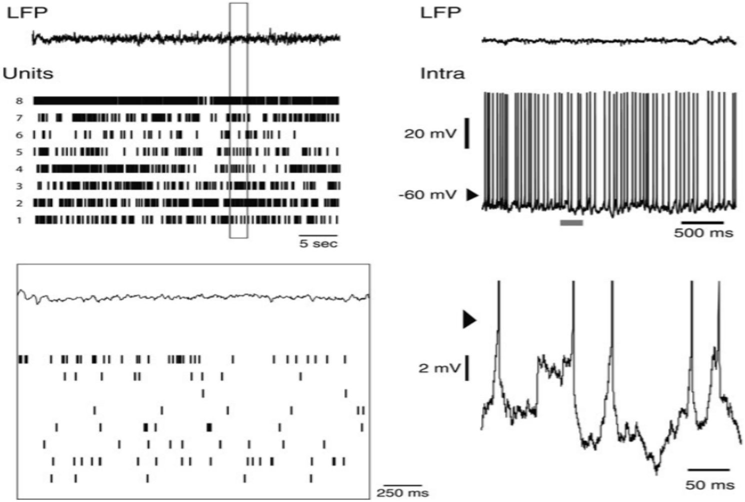

Figure: From “Neuronal Noise”, Alain Destexhe and Michelle Rudolph-Lilith

Noise in neuronal activity

- Thermal noise
- Channel noise
- Electrical noise
- Synaptic noise

Two approaches: strong link between deterministic and stochastic approaches

Toy example

$$A = \int_0^1 f(\theta)\, d\theta$$

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$$\tilde{A} = \iiint_{[0,1]^3} f(\theta_1, \theta_2, \theta_3)\, d\theta_3 d\theta_2 d\theta_1 \qquad \tilde{A} = \mathbb{E}\left[f(U_1, U_2, U_3)\right]$$

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Heat Equation and Brownian Motion

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2} u(t,x) = 0 \\ \qquad\qquad\qquad u(T,x) = \Psi(x) \end{cases}$$

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Heat Equation and Brownian Motion

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$$u(t,x) = \mathbb{E}\left[\Psi(W_T) | W_t = x\right]$$

Outline

Strong Law of Large Numbers

Theorem (Law of Large Numbers)

Consider a random variable X, such that $\mathbb{E}(|X|) < \infty$. We denote the mean $\mu := \mathbb{E}(X)$. We consider a sample of n independent random variables $X_1, \cdots, X_n$ with the same law as X. Then

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow[n\to\infty]{a.s.} \mu.$$

Central Limit Theorem

Theorem

Consider a random variable X*, such that* $\mathbb{E}\left(|X|^2\right) < \infty$*. Denote the mean and variance by* $\mu := \mathbb{E}\left(X\right)$, $\sigma^2 = \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right] = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$*. Let us consider a sample of* n *i.i.d random variables* $X_1, \cdots, X_n$ *with the same law as* X*. Then*

$$\frac{\sqrt{n}}{\sigma}\left(\frac{X_1 + \cdots + X_n}{n} - \mu\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1).$$

Confidence intervals

Assume $\mu = \mathbb{E}(X)$. An estimator of μ is

$$\hat{\mu}^n := \frac{1}{n}(X_1 + \cdots + X_n)$$

- Assume n is large enough to be in the asymptotic regime.
- $\mathbb{P}\left[\frac{\sqrt{n}}{\sigma}(\hat{\mu}^n - \mu) \in A\right] \approx \mathbb{P}(G \in A)$ where $G \sim \mathcal{N}(0,1)$
- $\forall \alpha$ there exists y_α such that $\mathbb{P}(|G| \leq y_\alpha) = \alpha$

An example of the size of the confidence interval

For $\alpha = 95\%$, $y_\alpha = 1.96$.

$$\mathbb{P}\left(\mu \in \left[\hat{\mu}^n - \frac{1.96\sigma}{\sqrt{n}}, \hat{\mu}^n + \frac{1.96\sigma}{\sqrt{n}}\right]\right) \geq 95\%$$

A non asymptotic estimate

Theorem (Berry-Esseen)

Let $(X_i)_{i\geq 1}$ *be a sequence of independent and identically distributed random variables with zero mean. Denote by* σ *the common standard deviation. Suppose that* $\mathbb{E}|X|^3 < +\infty$*. Then*

$$\begin{aligned}\varepsilon_N := \sup_{x\in\mathbb{R}} &\left|\mathbb{P}\left(\frac{X_1+\cdots+X_N}{\sigma\sqrt{N}} \leq x\right) - \int_{-\infty}^{x} e^{-u^2/2}\frac{du}{\sqrt{2\pi}}\right| \\ &\leq \frac{C\mathbb{E}\left|X_1\right|^3}{\sigma^3\sqrt{N}}.\end{aligned}$$

In addition, $0.398 \leq C \leq 0.8$*.*

For a proof, see, e.g., Shiryayev (1984).

A more precise result

We now give a result which is slightly more precise than the Berry-Esseen Theorem: the estimate is non uniform in x. See Petrov (1975) for a proof and extensions.

Theorem (Bikelis)

Let $(X_i)_{i\geq 1}$ be a sequence of independent real random variables, which are not necessarily identically distributed. Suppose that $\mathbb{E}X_i = 0$ for all i, and that there exists $0 < \delta \leq 1$ such that $\mathbb{E}|X_i|^{2+\delta} < +\infty$ for all i. Set

$$\sigma_i^2 := \mathbb{E}X_i^2, \quad B_N := \sum_{i=1}^{N} \sigma_i^2, \quad F_N(x) := \mathbb{P}\left[\frac{\sum_{i=1}^{N} X_i}{\sqrt{B_N}} \leq x\right].$$

Denote by Φ the distribution function of a Gaussian law with zero mean and unit variance. There exists a universal constant A in $(\frac{1}{\sqrt{2\pi}}, 1)$ independent of N and of the sequence $(X_i)_{i\geq 1}$, such that, for all x,

$$|F_N(x) - \Phi(x)| \leq \frac{A}{B_N^{1+\delta/2}(1+|x|)^{2+\delta}} \sum_{i=1}^{N} \mathbb{E}|X_i|^{2+\delta}$$

Outline

Uniform Law

Generating a sequence $U_1, \cdots, U_n$ of i.i.d. uniform random variables

Properties

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{a \leq U_i \leq b} \approx b - a \quad \forall a, b \in [0,1]$$

$$\frac{1}{n}\sum_{i=1}^{n} \left(U_{2i+1} - \frac{1}{2}\right)\left(U_{2i+2} - \frac{1}{2}\right) \approx 0$$

etc.

Congruencial generator

- $N_{max} \in \mathbb{N}$
- $n_0 \in \mathbb{N}$
- $n_{k+1} \equiv a n_k + b \pmod{N_{max}}$
- $u_k = \dfrac{n_k}{N_{max}}$

Remarks

1. This sequence $(u_k)_{k\geq 1}$ mimics a sequence of independent random variable uniformly distributed on $(0,1)$ but it is a deterministic sequence.
2. It allows us to compare several methods with the same random events
3. These sequences are periodic
4. We have to take care to the period: as long as possible.
5. A good choice: Mersenne Twister.

Do not forget

If you want to use a software or a given language in order to apply stochastic numerical methods, you have to find its own uniform random generator or to download a good uniform generator.

Rejection Procedure

Principle

- Our aim is to estimate $\mathbb{E}\left[FG\right]$ with $0 \leq G \leq 1$ almost surely.
- The idea : write $G = \tilde{\mathbb{P}}(X)(= \mathbb{P}(X|F,G))$
- $$\begin{aligned} \mathbb{E}\left[FG\right] &= \mathbb{E}\left[F \mathbb{1}_X\right] \\ &= \mathbb{E}\left[F|X\right] \mathbb{E}\left[G\right] \end{aligned}$$

Simulation of a random variable with a rejection procedure

- Let X be a *r.v.* with density f. We do not know how to simulate it.
- Let Y be a *r.v.* with density g. We know how to simulate it.
- Assumption: $\forall x \in \mathbb{R} \quad 0 \leq f(x) \leq Cg(x)$. We set $h(x) := \frac{f(x)}{Cg(x)} \mathbb{1}_{\{g(x)>0\}}$

$$
\begin{aligned}
\mathbb{E}[\varphi(X)] &= \int \varphi(x) f(x) dx = \int \varphi(x) \frac{f(x)}{g(x)} g(x) dx \\
&= \mathbb{E}\left[\varphi(Y) \frac{f(Y)}{g(Y)}\right] = C\mathbb{E}\left[\varphi(Y) \frac{f(Y)}{Cg(Y)}\right] = C\mathbb{E}[\varphi(Y) h(Y)] \\
&= C\mathbb{E}\left[\varphi(Y) \mathbb{1}_{\{U \leq h(Y)\}}\right] \\
&= C\mathbb{E}[\varphi(Y) | U < h(Y)] \, \mathbb{P}[U \leq h(Y)]
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{P}\left[U \le h(Y)\right] &= \mathbb{E}\left[h(Y)\right] \\
&= \int \frac{f(y)}{Cg(y)} g(y) dy = \int \frac{f(y)}{C} dy \\
&= \frac{1}{C}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\varphi(X)\right] &= C\mathbb{E}\left[\varphi(Y) | U < h(Y)\right] \mathbb{P}\left[U \le h(Y)\right] \\
&= \mathbb{E}\left[\varphi(Y) | U < h(Y)\right]
\end{aligned}
$$

Algorithm

1. Generate Y
2. Compute for this realisation $\dfrac{f(Y)}{Cg(Y)}$
3. Generate a random variable U, indep. of Y, with uniform law on $(0,1)$.
4. If $U \leq \dfrac{f(Y)}{Cg(Y)}$, accept the realisation, that is $X = Y$
5. Else (if $U > \dfrac{f(Y)}{Cg(Y)}$), reject the realisation and start again from first step.

Remark

- You have to wait a random time to obtain each realisation
- The probability of acceptance is equal to $\dfrac{1}{C}$.
- Smaller is C, better is the algorithm.

Outline

Low discrepancy Sequences

Using sequences of points *more regular* than random points may sometimes improve Monte Carlo methods. We look for deterministic sequences $(x_i, i \geq 1)$ such that

$$\int_{[0,1]^d} f(x)dx \approx \frac{1}{n}\left(f(x_1) + \cdots + f(x_n)\right)$$

for all function f in a large enough set.

Definition

These methods with deterministic sequences are called quasi Monte Carlo methods.

One can find sequences such that the speed of convergence of the previous approximation is of order $K\dfrac{\log(n)^d}{n}$

Low discrepancy Sequences (2)

Definition (Uniformly distributed sequences)

For all $y, z \in [0,1]^d$, we say that $y \leq z$ if $\forall i = 1, .., d, y^i \leq z^i$.
A sequence $(x_1, i \geq 1)$ is said to be uniformly distributed on $[0,1]^d$ if one of the following equivalent properties is fulfilled:

1. For all $y = (y_1, \cdots, y_d) \in [0,1]^d$, $\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{x_k \in [0,y]} = \mathrm{Volume}([0,y])$
2. Let $D_n^*(x) = \sup_{y \in [0,1]^d} \left| \frac{1}{n} \sum_{k=1}^{n} \mathbb{1}_{x_k \in [0,y]} - \mathrm{Volume}([0,y]) \right|$ be the discrepancy of the sequence, then
$$\lim_{n \to \infty} D_n^*(x) = 0$$
3. For every (bounded) continuous function f on $[0,1]^d$
$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) = \int_{[0,1]^d} f(x) dx$$

Low discrepancy Sequences(3)

Remark

If $(U_n)_{n\geq 1}$ is a sequence of independent random variables with uniform law on $[0,1]$, the random sequence

$$(U_n(\omega), n \geq 1)$$

- is almost surely uniformly distributed.
- The discrepancy fulfills an iterated logarithm law

$$\limsup_n \sqrt{\frac{2n}{\log(\log n)}} D_n^*(U) = 1 \quad \text{a.s.}$$

Lower bound for the discrepancy: Roth Theorem

The discrepancy of any infinite sequence satisfies the property

$$D_n^* > C_d \frac{(\log n)^{\frac{d-1}{2}}}{n} \text{ for } d \geq 3$$

for an infinite number of values of n, where C_d is a constant which depends on d only.

Low discrepancy Sequences (4)

Koksma-Hlawka inequality

Let g be a finite variation function in the sense of Hardy and Krause and denote by $V(g)$ its variation. Then, for $n \geq 1$,

$$\left| \frac{1}{N} \sum_{k=1}^{N} g(x_k) - \int_{[0,1]^d} g(u) du \right| \leq V(g) D_N^*(x)$$

Finite variation function in the sense of Hardy and Krause

If the function g is d times continuously differentiable, the variation $V(g)$ is given by

$$\sum_{k=1}^{d} \sum_{1 \leq i_1 < .. < i_k \leq d} \int_{\begin{cases} x \in [0,1]^d \\ x_j = 1 \text{ for } j \neq i_1, .., i_k \end{cases}} \left| \frac{\partial^k g(x)}{\partial x_{i_1} \cdots \partial x_{i_k}} \right| dx_{i_1} \cdots dx_{i_k}$$

Popular Quasi Monte Carlo Sequences

1. Faure sequences
2. Halton sequences
3. Sobol sequences
4. van der Corput sequences

An upper bound

For such sequences, we obtain an upper bound of the discrepancy:

$$D_n^* \leq C \frac{(\log n)^d}{n}$$

Remark

- For small d: deterministic methods
- For moderated d: Quasi Monte Carlo methods
- For large d: Monte Carlo methods

Low discrepancy Sequences

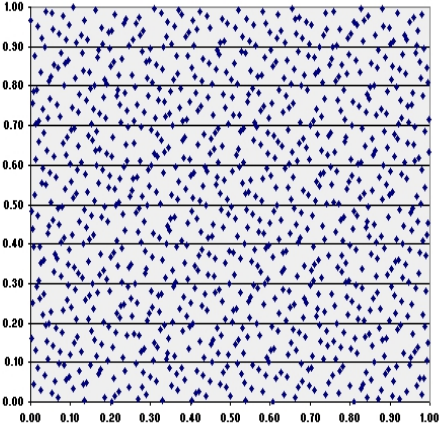

Figure: Halton Points

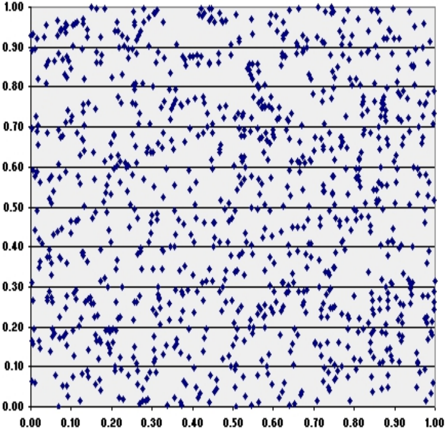

Figure: (Pseudo) uniform points

Outline

Remind on Poisson laws

Definition (Poisson law)

The random variable Y has Poisson law of parameter λ if and only if

- $Y \in \mathbb{N}$ almost surely.
- $\mathbb{P}(Y = k) = \exp(-\lambda)\dfrac{\lambda^k}{k!}$.

Property

Let $Y \sim \mathcal{P}(\lambda)$ and $Z \sim \mathcal{P}(\beta)$ be two independent Poisson random variables.

$$\Lambda := Y + Z \sim \mathcal{P}(\lambda + \beta)$$

Remind on Poisson laws

Proof.

$$
\begin{aligned}
\mathbb{P}(\Lambda = k) &= \sum_{i=0}^{k} \mathbb{P}(Y = i, Z = k - i) \\
&= \sum_{i=0}^{k} \mathbb{P}(Y = i)\mathbb{P}(Z = k - i) \\
&= \sum_{i=0}^{k} \exp(-\lambda)\frac{\lambda^i}{i!} \exp(-\beta)\frac{\beta^{k-i}}{(k-i)!} \\
&= \frac{\exp(-(\lambda+\beta))}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda^i \beta^{k-i} \\
&= \exp(-(\lambda+\beta))\frac{(\lambda+\beta)^k}{k!}.
\end{aligned}
$$

$\square$

Rarefaction Poisson

Notation

- let Y be a Poisson random variable with parameter λ
- let $(\xi_i, i \geq 1)$ a sequence of *i.i.d.* random variables, independent of Y, which take values in a countable set I

$$\mathbb{P}(\xi_1 = i) = p_i.$$

- For any $i \in I$, we introduce

$$Y^{(i)} = \sum_{j=1}^{Y} \mathbb{1}_{\{\xi_j = i\}}$$

Conclusion

The random variables $Y^{(1)}, \cdots, Y^{(i)}, \cdots$ are independent with Poisson laws of parameters λp_i.

Counting Processes

Definition

A counting process $(N(t), t \geq 0)$ is a stochastic process

- $N(0) = 0$ almost surely
- N is almost surely non-decreasing
- $t \mapsto N(t)$ is almost surely cadlag
- N is piecewise constant and has jump of size 1.

Remark

A counting process is used to model the number of times that a particular phenomenon has been observed by time t (typical example in neuroscience is the number of spikes emitted by a neuron).

Poisson Process

Definition

A counting process is a Poisson process if it satisfies the following conditions:

1. Numbers of observations in disjoint time intervals are independent random variables, i.e.,if $t_0 < t_1 < \cdots < t_m$, then $N(t_k) - N(t_{k-1})$, $k = 1, \cdots, m$ are independent random variables.
2. The distribution of $N(t+a) - N(t)$ does not depend on t.

Theorem

If N is a Poisson process, then there is a constant $\lambda > 0$ such that, for $s < t$, $N(t) - N(s)$ is Poisson distributed with parameter $\lambda(t-s)$, i.e

$$\mathbb{P}(N(t) - N(s) = k) = \frac{(\lambda(t-s))^k}{k!} \exp(-\lambda(t-s)).$$

Proof

Step 1
For any $n \geq 1$, we write $p_n = \mathbb{P}\left(N((k+1)/n) - N(k/n) \geq 1\right)$.

$$\begin{aligned} \mathbb{P}(N(1) = 0) &= (1 - p_n)^n \\ \lambda &= -\log(\mathbb{P}(N(1) = 0)) \\ &= -n \log(1 - p_n) \\ &= \lim_n np_n. \qquad \Rightarrow \qquad \mathbb{P}(N(1) = 0) = \exp(-\lambda) \end{aligned}$$

Step 2 Let denote $q_n = \mathbb{P}\left(N(1/n) \geq 2\right)$. Denote Γ_n the number of intervals $[k/n, (k+1)/n]$ containing at least 2 arrivals.

- For $\Gamma_n(\omega) \to_n 0$ for almost all ω (the time arrival are different).
- $\Gamma_n \leq N(1)$
- We have $\mathbb{E}(N(1)) < \infty$(admitted)
- So, we conclude $\mathbb{E}(\Gamma_n) \to_n 0$ (Fubini), that is nq_n tends to 0.

Step 3
We deduce that $\lim_n n\mathbb{P}(N(1/n) = 1) = \lim_n n\mathbb{P}(N(1/n) \geq 1) = \lambda$.

$$
\begin{aligned}
\mathbb{P}(N(1) = 1) &= \binom{n}{1} p_n (1 - p_n)^{n-1} \\
&\approx n\frac{\lambda}{n}(1 - \frac{\lambda}{n})^{n-1} \\
&\approx \lambda \exp(-\lambda)
\end{aligned}
$$

The end of the proof is similar to the next one.

Construction of Poisson Processes

A non-decreasing random walk

Consider the random walk S_n:

$$S_0 = 0, \qquad S_{k+1} = S_k + X_{k+1},$$

where $X_1, \cdots, X_k, \cdots$ are *i.i.d.* random variables

$$\mathbb{P}(X_k = 1) = p = 1 - \mathbb{P}(X_k = 0).$$

Let $\lambda > 0$ and $t_1 < t_2 < \cdots < t_\ell$ and consider sequences, $p(n)$, $n_1(n), \cdots, n_\ell(n)$ such that

$$\lim_n np(n) = \lambda \quad \text{and} \quad \forall i \in 1, \cdots, \ell, \quad \lim_n \frac{n_i(n)}{n} = t_i,$$

Result

$$(S_{n_i(n)} - S_{n_{i-1}(n)}, 1 \leq i \leq \ell) \underset{n\to\infty}{\overset{\mathcal{L}}{\Longrightarrow}} (Y_1, Y_2, \cdots, Y_\ell)$$

where Y_i are independent r.v. with Poisson laws of parameters $\lambda(t_i - t_{i-1})$.

Proof.

$$
\begin{aligned}
\mathbb{P}(S_{n_1(n)} = k) &= \binom{n_1(n)}{k} p(n)^k (1-p(n))^{n_1(n)-k} \\
&\approx \frac{(t_1 n)!}{k!(t_1 n - k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{t_1 n - k}
\end{aligned}
$$

Remind

$$
\text{(Stirling)} \quad n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad \lim_n \left(1 + \frac{u}{n}\right)^n = \exp(u).
$$

$$
\begin{aligned}
\mathbb{P}(S_{n_1(n)} = k) &\approx \frac{\sqrt{2\pi t_1 n}}{\sqrt{2\pi(t_1 n - k)}} \left(\frac{t_1 n}{e}\right)^{t_1 n} \left(\frac{e}{t_1 n - k}\right)^{t_1 n - k} \frac{1}{n^k} \frac{\lambda^k}{k!} \exp(-\lambda t_1) \\
&\approx \exp(-\lambda t_1) \frac{(\lambda t_1)^k}{k!} e^{-k} \left(1 + \frac{k}{t_1 n - k}\right)^{t_1 n} \approx \exp(-\lambda t_1) \frac{(\lambda t_1)^k}{k!}.
\end{aligned}
$$

A dual approach

- Let N be a Poisson process.
- We define S_k the time of the kth observation, that is $N(S_k-) = k-1$ and $N(S_k) = k$.
- We have

$$\{N(t) \geq k\} = \{S_k \leq t\}$$

- The c.d.f of S_k is

$$\mathbb{P}(S_k \leq t) = 1 - \sum_{i=0}^{k-1} \exp(-\lambda t) \frac{(\lambda t)^i}{i!}$$

- The p.d.f of S_k is

$$\frac{d}{dt}\mathbb{P}(S_k \leq t) = \frac{1}{(k-1)!} \lambda (\lambda t)^{k-1} \exp(-\lambda t).$$

Distribution of the inter-events interval

- Let N be a Poisson process.
- We define S_k the time of the kth observation, that is $N(S_k-) = k-1$ and $N(S_k) = k$.
- We denote $T_k = S_k - S_{k-1}$.

Proposition

The random times $(T_k, k \geq 1)$ are i.i.d. with exponential law of parameter λ.

Proof.

- We have already proved the result for $k = 1$.
- Let us prove for (T_1, T_2)

$$\begin{aligned}
\mathbb{P}(T_1 > t, T_2 > s) &= \mathbb{P}(N(t) < 1, N(T_1 + s) < 2) \\
&= \mathbb{P}(N(t) = 0, N(T_1 + s) - N(T_1) = 0) \\
&= \mathbb{P}(N(t) = 0)\mathbb{P}(N(T_1 + s) - N(T_1) = 0) \\
&= \exp(-\lambda t)\exp(-\lambda s) \\
&= \mathbb{P}(T_1 > t)\mathbb{P}(T_2 > s).
\end{aligned}$$

Poisson process with time dependent intensity

Definition

- Let $(\lambda(t), t \geq 0)$ be a deterministic function.
- The process $(N(t), t \geq 0)$ is a Poisson process with intensity λ if
  1. $N(0) = 0$ a.s.
  2. N is a.s. a non decreasing cadlag process
  3. N is a.s. piecewise constant with jumps of size 1
  4. For any Borel set A, we consider the number of jumps of N in A, i.e.

$$N^{(A)} = \sum_{s \in A} \mathbb{1}_{\{N(s) - N(s-) = 1\}}$$

     then

$$N^{(A)} \sim \mathcal{P}\left(\int_A \lambda(s) ds\right).$$

  5. If $A_1, \cdots, A_\ell$ are Borel set such that $A_i \cap A_j = \emptyset$ if $i \neq j$, then $N^{(A_1)}, \cdots, N^{(A_\ell)}$ are independent.

Dual approach

- Let N be a Poisson process on $\mathbb{R}^+$ with non-negative intensity $(\lambda(t), t \geq 0)$.
- Denote by $(S_k, k \geq 1)$ the (random) jump times of N:

$$N(t) = \begin{cases} 0 & \text{if } 0 \leq t < S_1 \\ 1 & \text{if } S_1 \leq t < S_2 \\ 2 & \text{if } S_2 \leq t < S_3 \\ \dots & \\ k & \text{if } S_k \leq t < S_{k+1} \end{cases}$$

- Denote $(T_k, k \geq 1)$ the inter-arrival times:

$$T_k = S_k - S_{k-1}.$$

- $$\mathbb{P}(T_k \geq t | S_{k-1}) = \exp\left(- \int_{S_{k-1}}^{S_{k-1}+t} \lambda(\theta) d\theta \right)$$

Simulation of $T_1, \cdots, T_k, \cdots$

Inversion of the cumulative distribution function

$$F_{T_1}(t) := \mathbb{P}(T_1 \leq t) = 1 - \exp\left(-\int_0^t \lambda(\theta)d\theta\right)$$

The time T_1 is given by

$$\int_0^{T_1} \lambda(\theta)d\theta \sim -\log(1-U) \sim -\log(U)$$

$\Longrightarrow$ Compute the antiderivative of λ !

Algorithm

- Simulate a uniform random variable U_1 on $[0,1]$
- Find T_1 such that $\int_0^{T_1} \lambda(\theta)d\theta = -\log(U_1)$
- Simulate a uniform random variable U_2 on $[0,1]$, independent of U_1
- Find T_2 such that $\int_{T_1}^{T_1+T_2} \lambda(\theta)d\theta = -\log(U_2)$
- etc.

Definition of Point Poisson Processes (PPP)

- Let $D \subset \mathbb{R}^p$ be the domain of the PPP N.
- Let λ be a nonnegative function defined on D, such that $\int_D \lambda < \infty$.
- A PPP N on D with intensity λ is a random set of points

$$N(\omega) = \{X_1(\omega), X_2(\omega), \cdots, X_{n(\omega)}(\omega)\} \qquad X_k \in D,$$

1. $\forall A \subset D$, define N_A the number of points of N belonging in A, i.e. $N_A = \mathrm{Card}(N \cap A)$.

$$N_A \stackrel{\mathcal{L}}{=} \mathcal{P}\left(\int_A \lambda\right)$$

2. $\forall A, \tilde{A} \subset D$: $A \cap \tilde{A} = \emptyset \implies N_A$ and $N_{\tilde{A}}$ are independent.

Properties

- The number of points $n(\omega)$ has a Poisson law of parameter $\int_D \lambda$.
- The number of points is finite if and only in $\int_D \lambda < \infty$.
- If $\tilde{D} \subset D$, the restriction to $\tilde{D}$ of a PPP on D with intensity λ is a PPP on $\tilde{D}$ with intensity λ.
- Assume $D = \mathbb{R}^+ \times [0, K]$ and denote the coordinate of $X_i(\omega) = (t_i, z_i)$, with $t_1 \leq t_2 \leq \cdots \leq t_n$. If the intensity is constant, $\lambda(t, z) \equiv \lambda$ then

$$t_1 \stackrel{\mathcal{L}}{=} \mathcal{E}(K\lambda) \qquad z_1 \stackrel{\mathcal{L}}{=} \mathcal{U}([0, K])$$

$$t_k - t_{k-1} \stackrel{\mathcal{L}}{=} \mathcal{E}(K\lambda) \qquad z_k \stackrel{\mathcal{L}}{=} \mathcal{U}([0, K])$$

- More generally, for any domain D, if the intensity is constant, conditionally on n, the points $X_1, \cdots, X_n$ are independent and uniformly distributed on D.

A very simple algorithm of simulation.

Simulation of a Poisson Process with time dependent intensity $\lambda(t)$

- We assume that the intensity is bounded.

$$\sup_{t\geq 0} \lambda(t) = K < \infty.$$

- Consider a PPP N on $\mathbb{R}^+ \times [0, K]$ and define the hypograph $D^{(\lambda)}$ of λ

$$D^{(\lambda)} := \{(t, z) \in \mathbb{R}^+ \times \mathbb{R}^+, z \leq \lambda(t)\}$$

- Define the restriction of N to $D^{(\lambda)}$ and

$$\bar{N}(t) = \mathrm{Card}(N \cap D^{(\lambda)} \cap ([0, t] \times [0, K]))$$

- $\bar{N}(t)$ is a Poisson Process with time dependent intensity λ.