

lecture 2

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Center manifold

Center manifold: the case where $\Re \lambda = 0$

Consider u_{eq} equilibrium for $\dot{u} = \mathbf{F}(u; \alpha)$ at parameter value α_0 , $\mathbf{F}(u_{eq}; \alpha_0) = 0$. Write $u = u_{eq} + x$ and $\alpha = \alpha_0 + \mu$. It solves an equation like:

$$\dot{x} = \mathbf{L}x + \mathbf{R}(x; \mu), \quad \mathbf{L} \in \mathcal{L}(\mathbb{R}^n), \quad \mathbf{R} \in C^k(\mathcal{V}_x \times \mathcal{V}_\mu, \mathbb{R}^n), \quad k \geq 2 \quad (1)$$

$$\mathbf{R}(0; 0) = 0, \quad d\mathbf{R}(0; 0) = 0$$

Theorem 1/2

Write $\mathbb{R}^n = \mathcal{X}_c \oplus \mathcal{X}_h$ where $\mathcal{X}_h = \mathcal{X}_s \oplus \mathcal{X}_u$ and $\dim \mathcal{X}_c > 0$. Then, there is a neighborhood $\mathcal{O} = \mathcal{O}_x \times \mathcal{O}_\mu$ of $(0, 0)$ in $\mathbb{R}^n \times \mathbb{R}^m$, a mapping $\Psi \in C^k(\mathcal{X}_c \times \mathbb{R}^m; \mathcal{X}_h)$ with

$$\Psi(0; 0) = 0, \quad d_1\Psi(0; 0) = 0$$

and a manifold $\mathcal{M}(\mu) = \{u_c + \Psi(u_c, \mu), u_c \in \mathcal{X}_c\}$ for $\mu \in \mathcal{V}_\mu$ such that:

- 1 $\mathcal{M}(\mu)$ is **locally invariant**, i.e., $x(0) \in \mathcal{M}(\mu) \cap \mathcal{O}_x$ and $x(t) \in \mathcal{O}_x$ for all $t \in [0, T]$ implies $x(t) \in \mathcal{M}(\mu)$ for all $t \in [0, T]$.

Center manifold

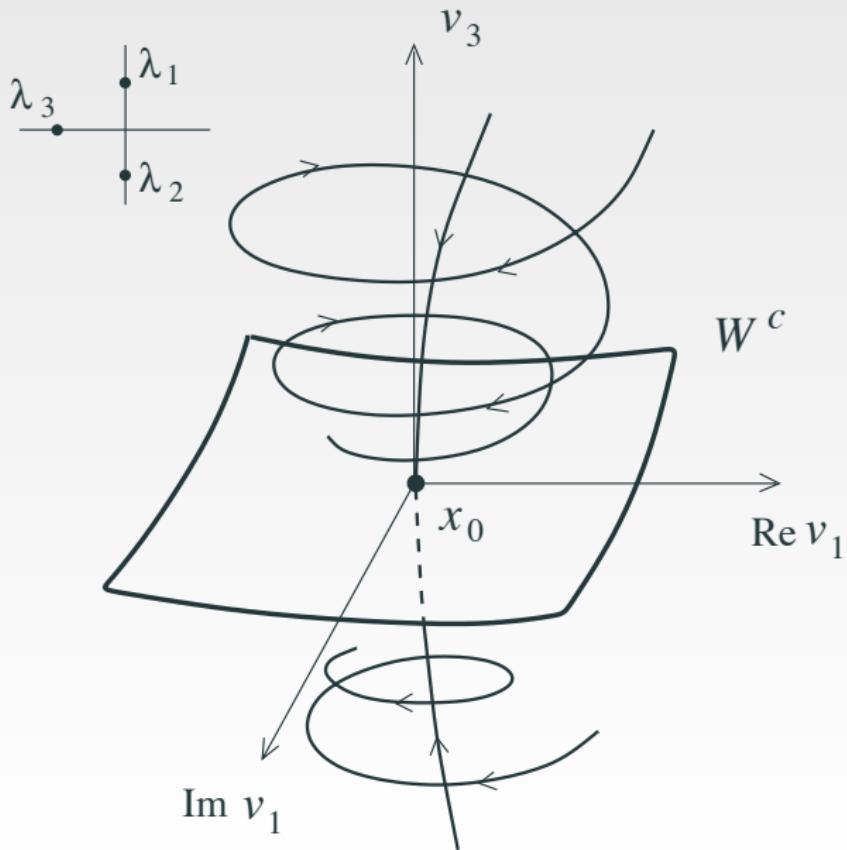
Theorem 2/2

- 2 $\mathcal{M}(\mu)$ contains the set of **bounded solutions** of (1) staying in \mathcal{O}_x for all $t \in \mathbb{R}$, i.e. if x is a solution of (1) satisfying for all $t \in \mathbb{R}$, $x(t) \in \mathcal{O}_x$, then $x(0) \in \mathcal{M}(\mu)$.
- 3 (Parabolic case) if $n_+ = 0$, then $\mathcal{M}(\mu)$ is **locally attracting**, i.e. if x is a solution of (1) with $x(0) \in \mathcal{O}_x$ and $x(t) \in \mathcal{O}_x$ for all $t > 0$, then there exists $v(0) \in \mathcal{M}(\mu) \cap \mathcal{O}_c$ and $\tilde{\gamma} > 0$ such that

$$x(t) = v(t) + O(e^{-\tilde{\gamma}t}) \text{ as } t \rightarrow \infty$$

where v is a solution of (1) with initial condition $v(0)$.

Center manifold



Center manifold: Additional properties

- ① The Center manifold is not unique. Uniqueness can be achieved if \mathbf{R} is Lipschitzian with sufficiently small Lipschitz constant.
- ② If $x_c(0) \in \mathcal{M}(\mu)$, then

$$\dot{x}_c = \mathbf{L}_c x_c + P_c \mathbf{R}(x_c + \Psi(x_c, \mu), \mu) \equiv f(x_c, \mu)$$

where P_c is the projector on \mathcal{X}_c .

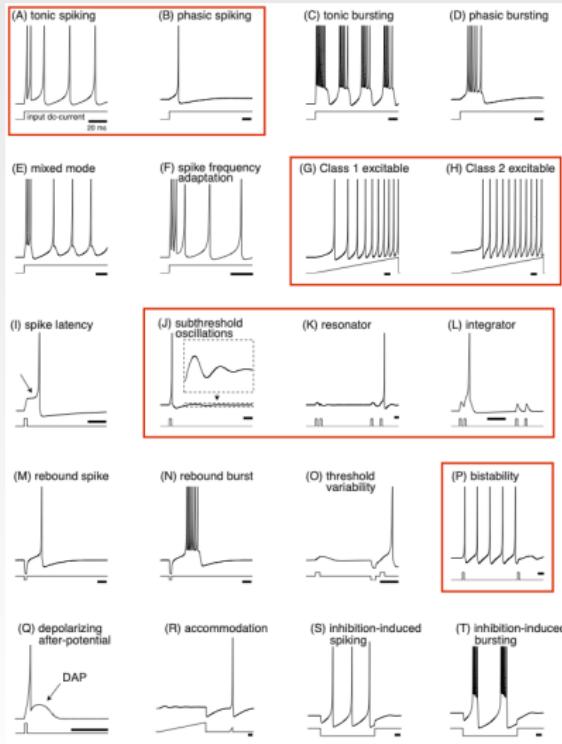
- ③ The local coordinates function satisfies

$$d\Psi(x_c, \mu) \cdot f(x_c) = P_h \mathbf{L} \cdot \Psi(x_c, \mu) + P_h \mathbf{R}(x_c + \Psi(x_c, \mu))$$

- ④ There are extensions for non-autonomous systems, with symmetries...
- ⑤ Extensions to Banach spaces possible in some cases
- ⑥ Taylor expansion of Ψ is uniquely determined.

Application to Neurosciences

Goal of this lecture: excitability



⇒ Have a geometric understanding of excitability

Reminder on the $I_{Na,p} + I_K$ model

This is a 2-dimensional simplification of the Hodgkin-Huxley model where we assume

- a leak current
- a persistent Na -current with instantaneous kinetics
- a persistent K -current with slower kinetics

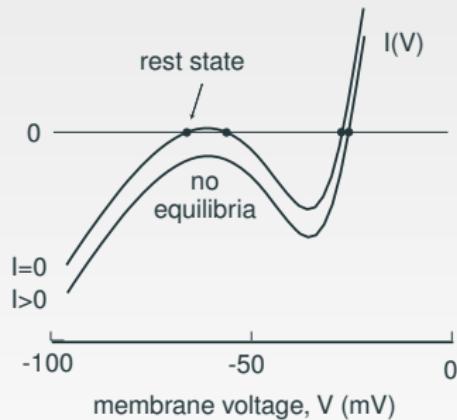
$$C \frac{dV}{dt} = I - \overbrace{\bar{g}_K n(V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m_\infty(V)(V - E_{Na})}^{I_{Na}} - g_L(V - E_L) \equiv I(V)$$
$$\frac{dn}{dt} = \frac{n_\infty(V) - n}{\tau(V)}$$

where

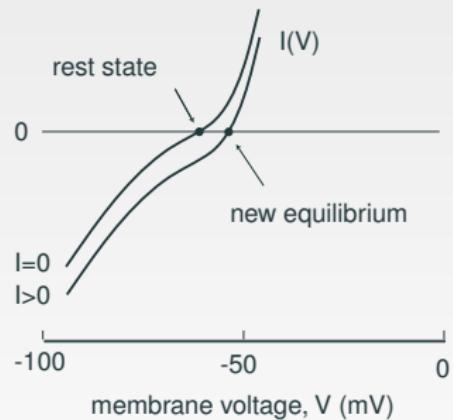
$$(m, n)_\infty = \frac{1}{1 + \exp[(V_{1/2} - V)/k]}$$

I-V curves in the $I_{Na,p} + I_K$ model

saddle-node bifurcation

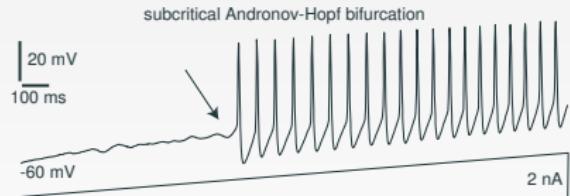
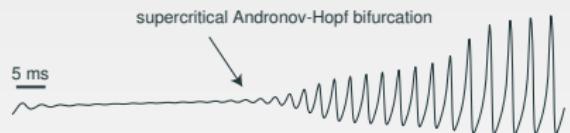
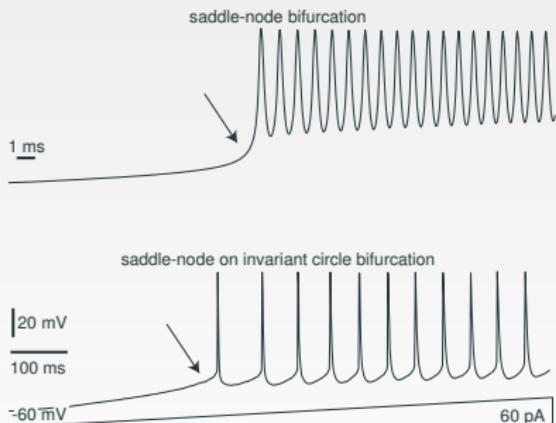


Andronov-Hopf bifurcation



$$C \frac{dV}{dt} = I - \overbrace{\bar{g}_K n(V - E_K)}^{I_K} - \overbrace{\bar{g}_{Na} m_\infty(V)(V - E_{Na})}^{I_{Na}} - g_L(V - E_L) \equiv I(V)$$

4 examples of codim 1 bifurcations (Figures from Izhikevich)



Last 2 are in-vitro recordings

Codim 1 bifurcations of equilibria

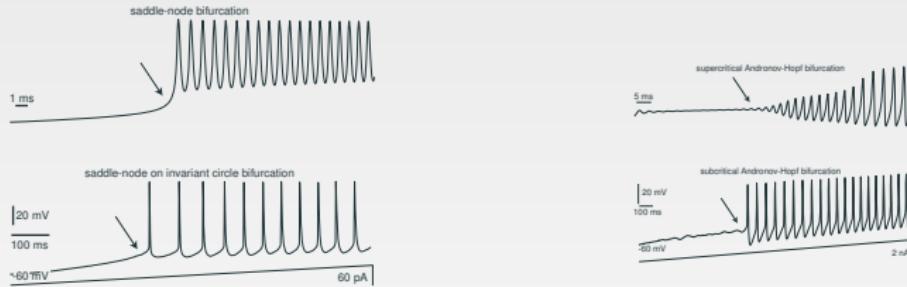
Outline

- 1 Center manifold
- 2 Application to Neurosciences
- 3 Codim 1 bifurcations of equilibria
 - Around the Saddle-Node bifurcation
 - Around the Hopf bifurcation
- 4 Excitability
- 5 Summary

Transitions from resting to spiking.

Neural properties of bifurcations

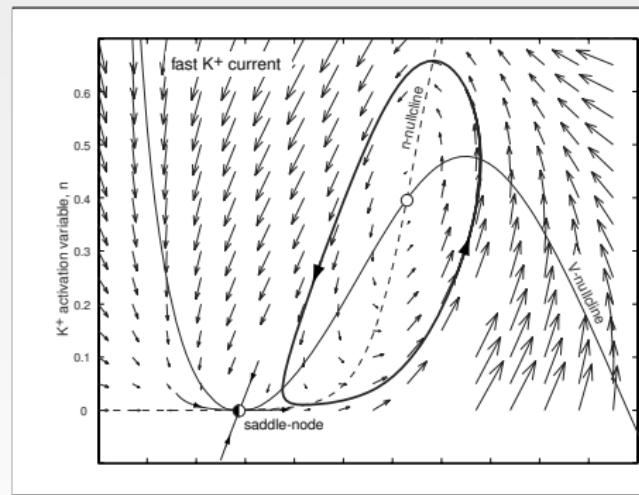
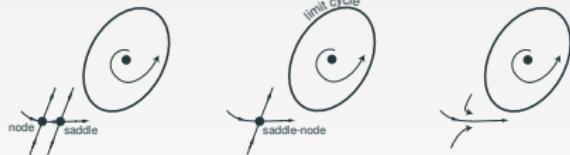
Quasi-static change in I , bifurcations?



Bifurcation of an equilibrium	Fast subthreshold oscillations	Amplitude of spikes	Frequency of spikes
saddle-node	no	nonzero	nonzero
saddle-node on invariant circle	no	nonzero	$A\sqrt{I-I_b} \rightarrow 0$
supercritical Andronov-Hopf	yes	$A\sqrt{I-I_b} \rightarrow 0$	nonzero
subcritical Andronov-Hopf	yes	nonzero	nonzero

Saddle-node bifurcation 1/2

Figures from Izhikevitch, $I_{Na,p} + I_K$ model



Saddle-node bifurcation 2/2

- the bifurcation occurs at $I_p = 4.51, (V_p, n_p) = (-61, 0.0007)$
- We can compute the center manifold

⇒ Useful to understand when SN occurs and codim 2 bifurcations

Saddle-node bifurcation 2/2

- the bifurcation occurs at $I_p = 4.51, (V_p, n_p) = (-61, 0.0007)$
- We can compute the center manifold
- we check that

$$a = \frac{1}{2} \partial_V^2 f(V_p, I_p) \neq 0 \quad c = \partial_I f(V_p, I_p) \neq 0$$

⇒ Useful to understand when SN occurs and codim 2 bifurcations

Saddle-node bifurcation 2/2

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$$a = \frac{1}{2} \partial_V^2 f(V_p, I_p) \neq 0 \quad c = \partial_I f(V_p, I_p) \neq 0$$

- We have the “normal form”

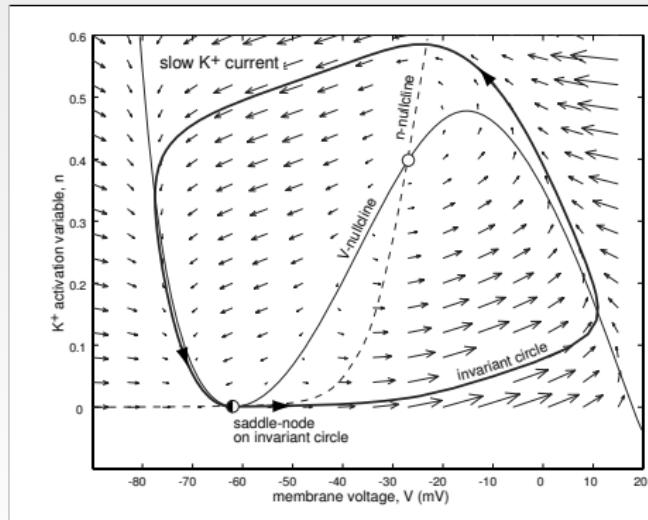
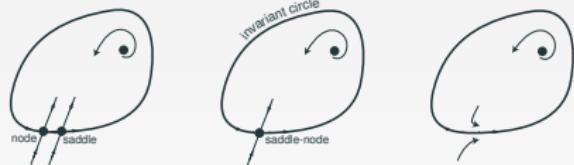
$$\dot{V} = c(I - I_p) + a(V - V_p)^2 + \dots \quad a = 0.1887, c = 1$$

⇒ Useful to understand when SN occurs and codim 2 bifurcations

Saddle-node bifurcation on invariant cycle (SNIC) 1/2

(Figures from Izhikevitch, $I_{Na,p} + I_K$ model with high threshold, $\tau(V) = 1$)

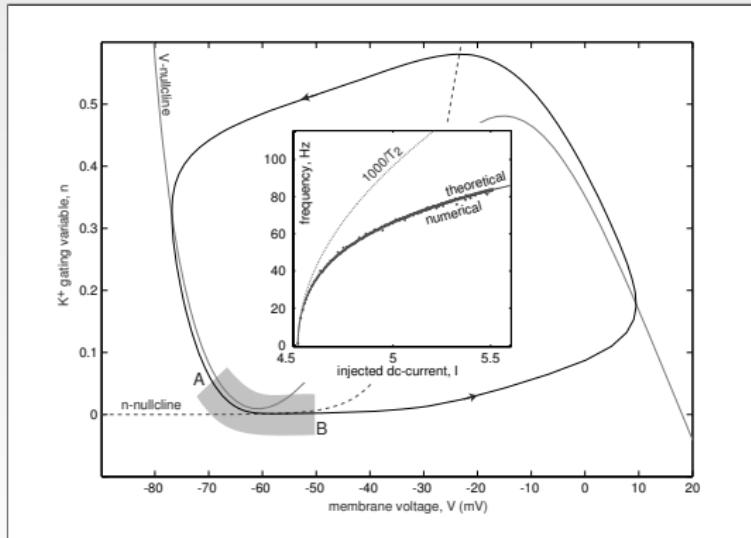
Also called **Saddle-node homoclinic bifurcation**



Saddle-node bifurcation on invariant cycle (SNIC) 2/2

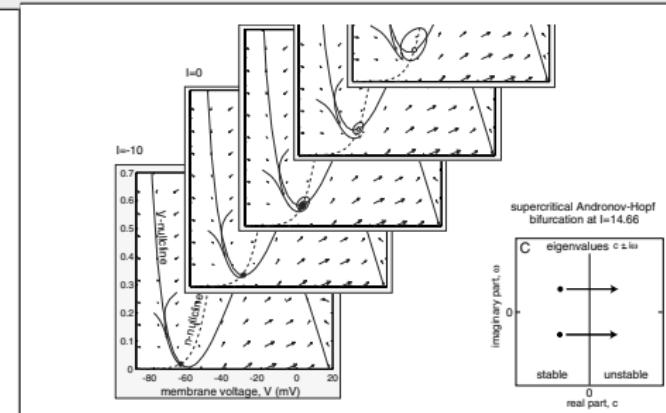
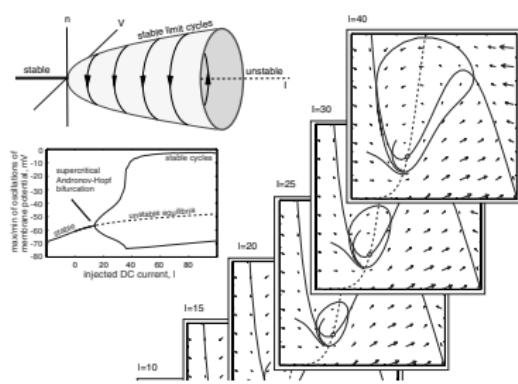
(Saddle-node homoclinic bifurcation)

- Time from A to B
$$T_2 \approx \frac{\pi}{\sqrt{ac(I - I_p)}}$$
- Action potential duration
$$T_1 \approx 4.7ms$$
- Firing frequency
$$\omega = \frac{1000}{T_1 + T_2}$$



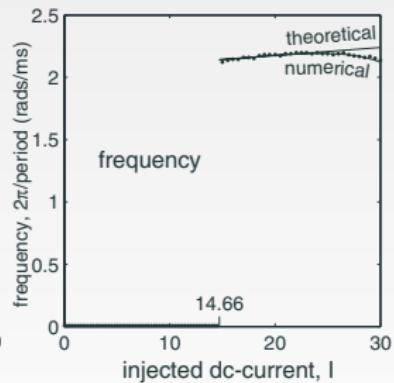
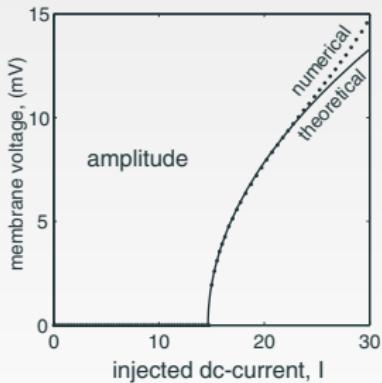
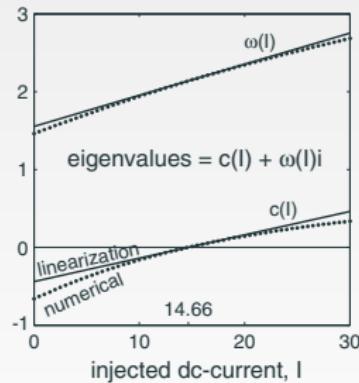
⇒ Exercise with θ -model or SN normal form (Now!)

Supercritical Andronov-Hopf bifurcation 1/2

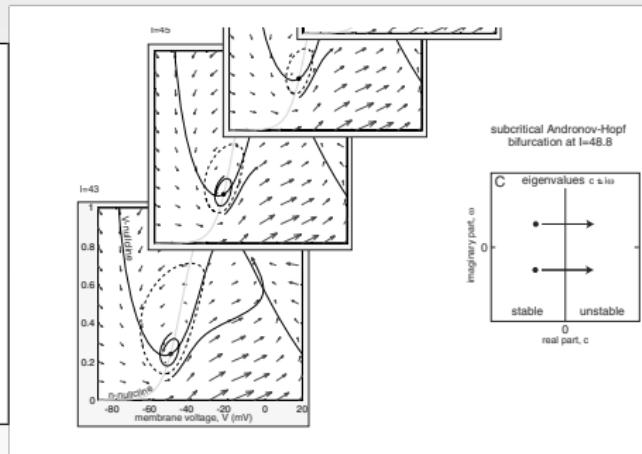
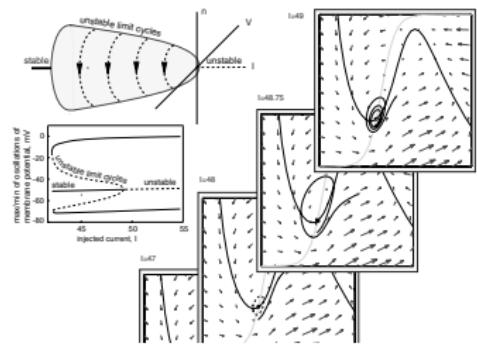


Supercritical Andronov-Hopf bifurcation 2/2

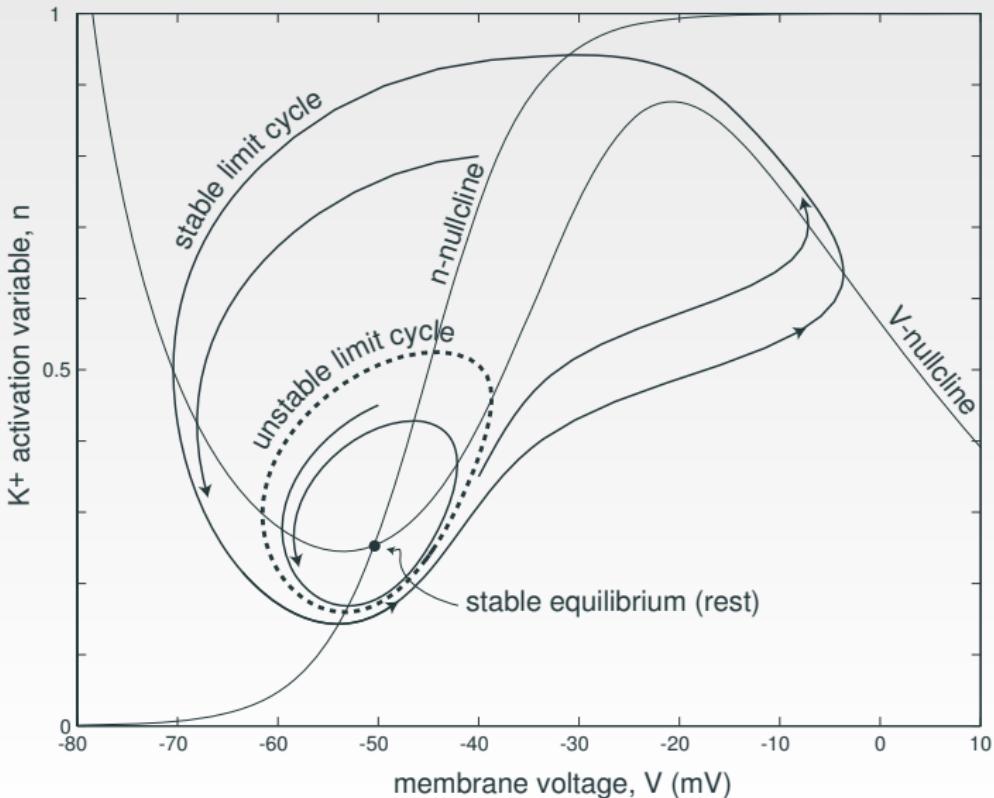
$I_{Na,p} + I_K$ model with low threshold



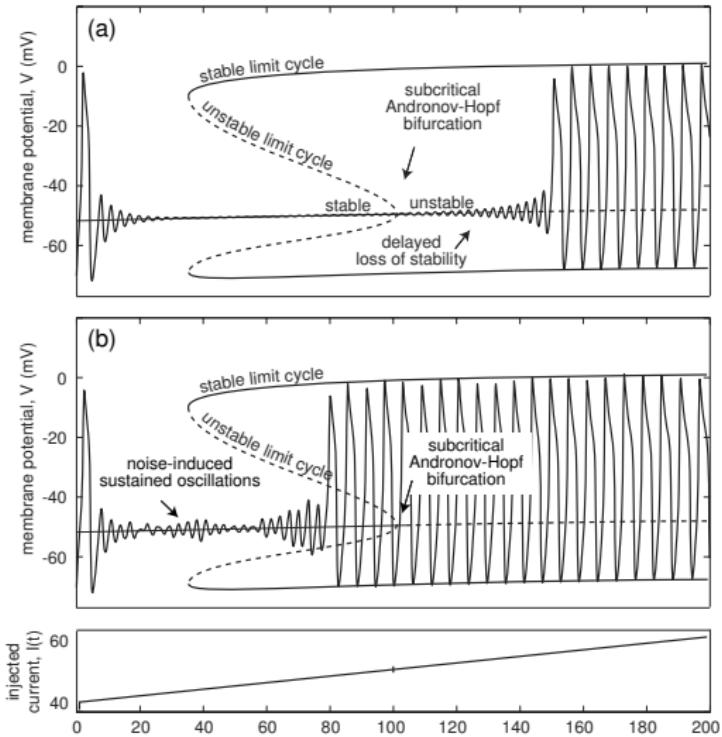
Subcritical Andronov-Hopf bifurcation



Bistability in the $I_{Na,p} + I_K$ model (To be explained later)

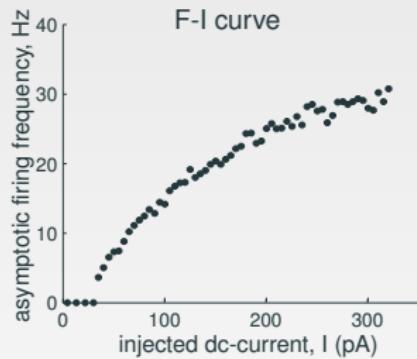
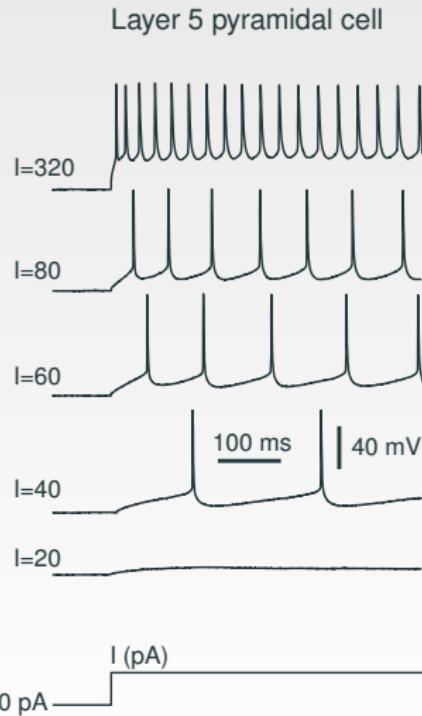


Delayed loss of stability (To be analyzed later (also) ...)



Excitability

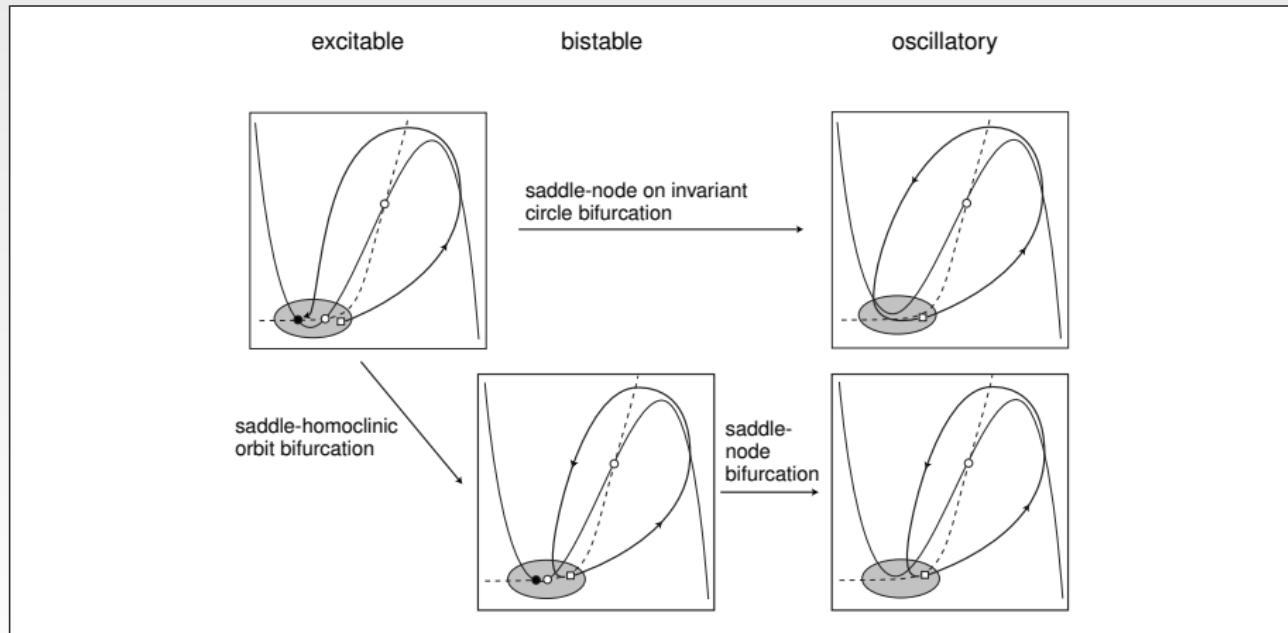
Hodgkin's Classification, Class I (Figures from Izhikevich)



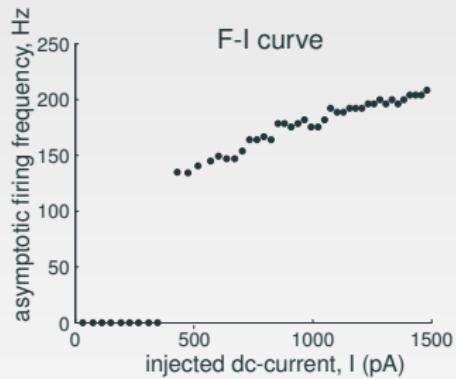
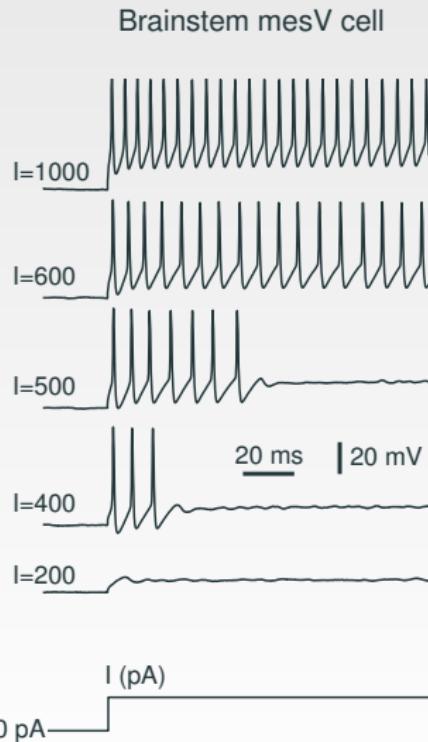
- Action potentials can be generated with arbitrarily low frequency, depending on the strength of the applied current.
⇒ encodes stim. strength

Hodgkin's Classification: Class I

Transition of excitable systems to oscillatory ones. **Period varies.**



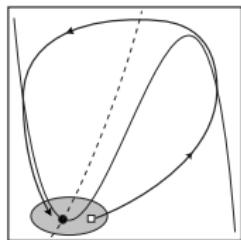
Hodgkin's Classification, Class II (Figures from Izhikevich)



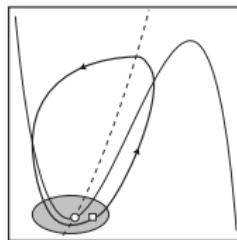
- Action potentials are generated in a certain frequency band that is relatively insensitive to changes in the strength of the applied current.
⇒ encodes threshold.

Hodgkin's Classification: Class II

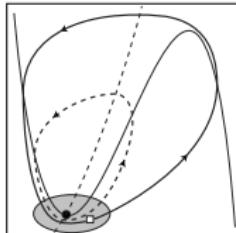
Period varies weakly.



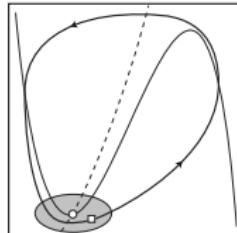
supercritical Andronov-Hopf bifurcation



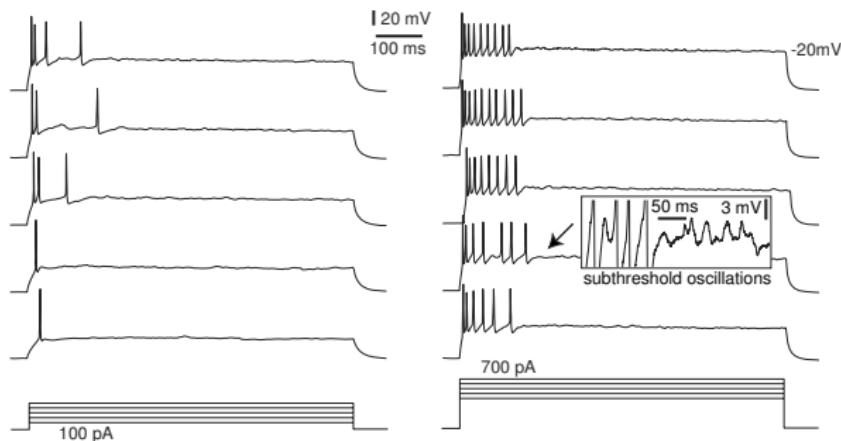
fold limit cycle bifurcation



subcritical Andronov-Hopf bifurcation



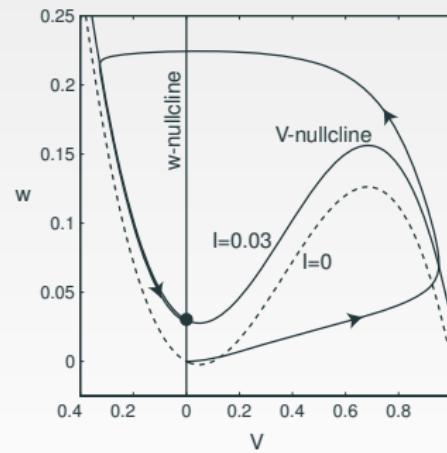
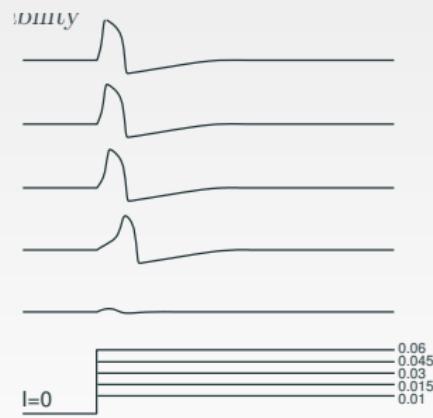
Hodgkin's Classification, Class III (Figures from Izhikevich)



- L5 Pyramidal neuron in rat visual cortex. One spike is generated in response to a current step. Repetitive (tonic) spiking can be generated only for extremely strong injected currents or not at all.

Hodgkin's Classification, Class III (Figures from Izhikevich)

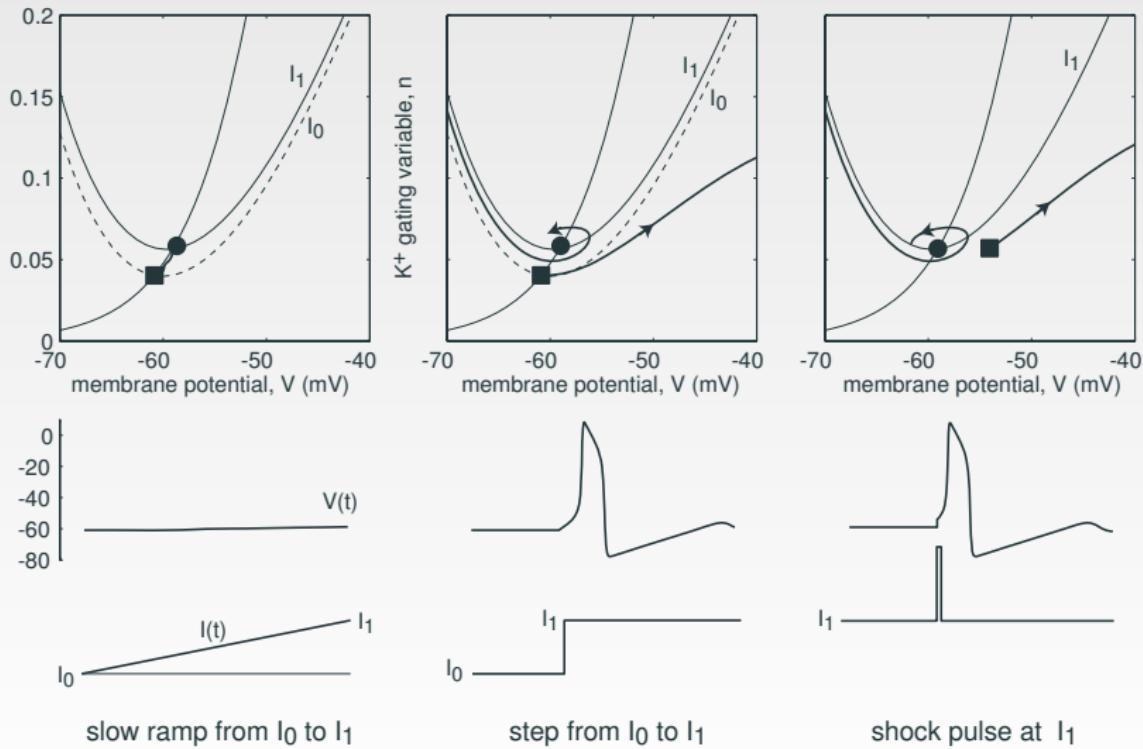
Class 3 neural excitability occurs when the resting state remains stable for any fixed I in a biophysically relevant range



Example of the Fitzhugh-Nagumo model

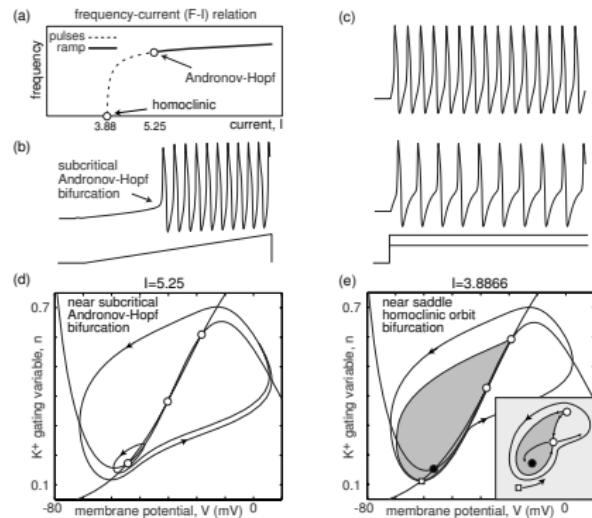
$$\dot{V} = V(a - V)(V - 1) - w + I, \quad \dot{w} = bV - cw, \quad a = 0.1, \quad b = 0.01$$

Ramps, Steps, and Shocks



The $I_{Na,p} + I_K$ model, Class I or II?

Bistability equilibrium - periodic orbit.

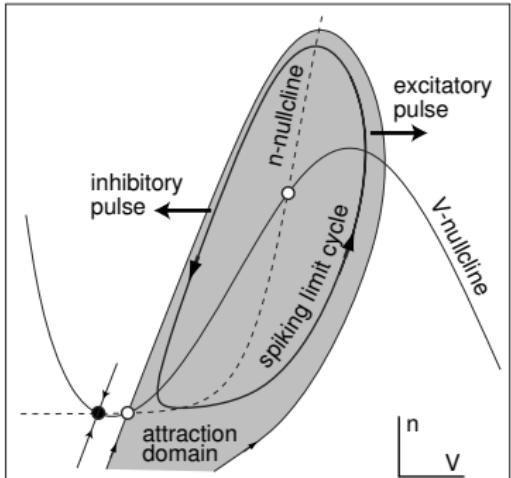


- Class II for ramps, I for Steps
- subcr. AH at $I = 5.25$ for ramps
- Saddle homoclinic for $I \approx 3.89$ for steps
- near BT

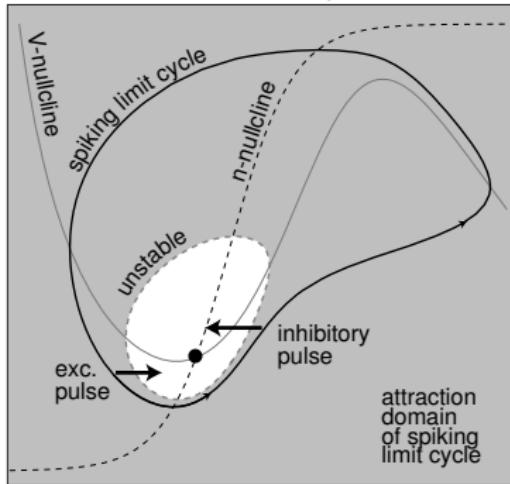
Gray: attraction domain for resting state.

Bistability

saddle-node bifurcation



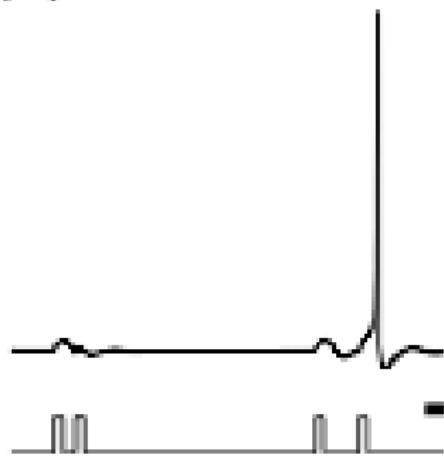
subcritical Andronov-Hopf bifurcation



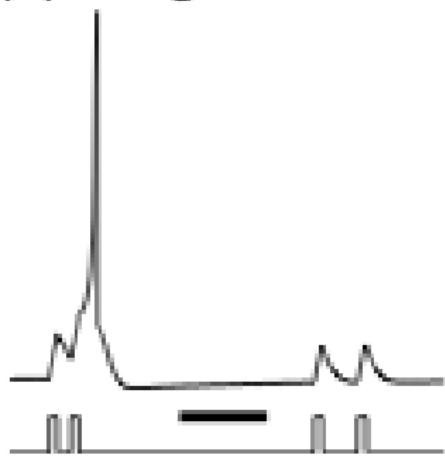
Excitatory / inhibitory pulses can shift the neuron from its resting state to repetitive firing.

Integrators vs. Resonators 1/2

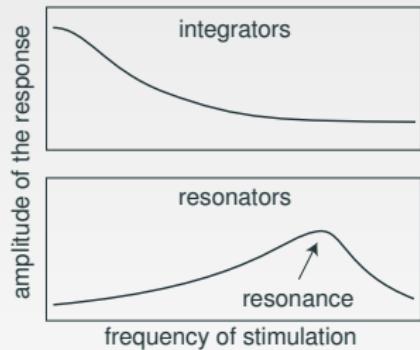
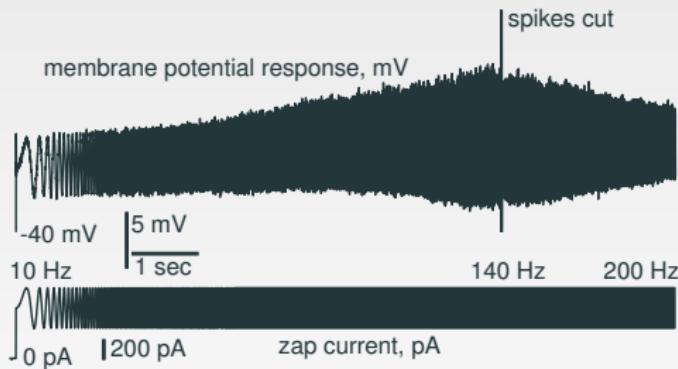
(K) resonator



(L) integrator

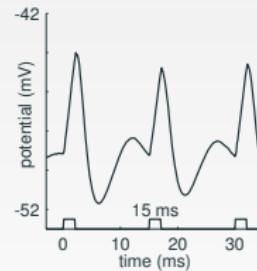
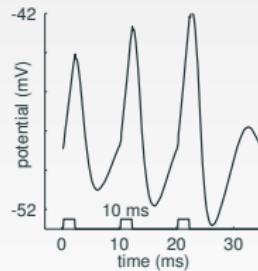
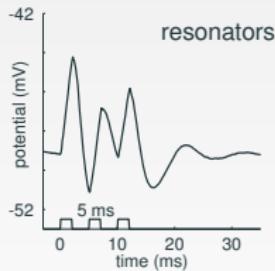
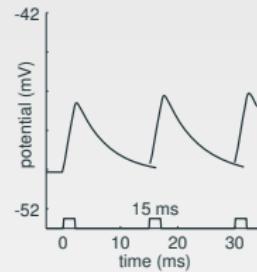
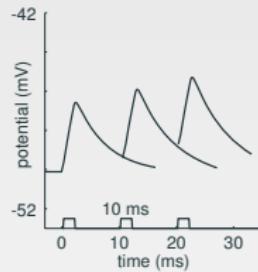
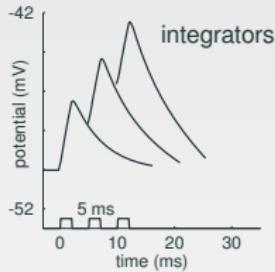


Integrators vs. Resonators 1/2



- ① Presence of sub-threshold oscillations
- ② Exp. mechanism to test their presence

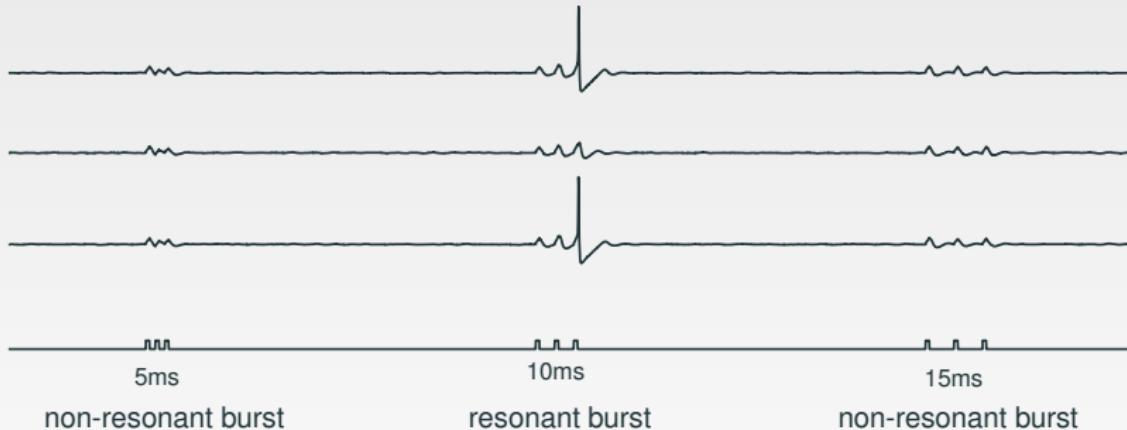
Integrators vs. Resonators 2/2



Integrators prefer high frequencies: detect coincidences.

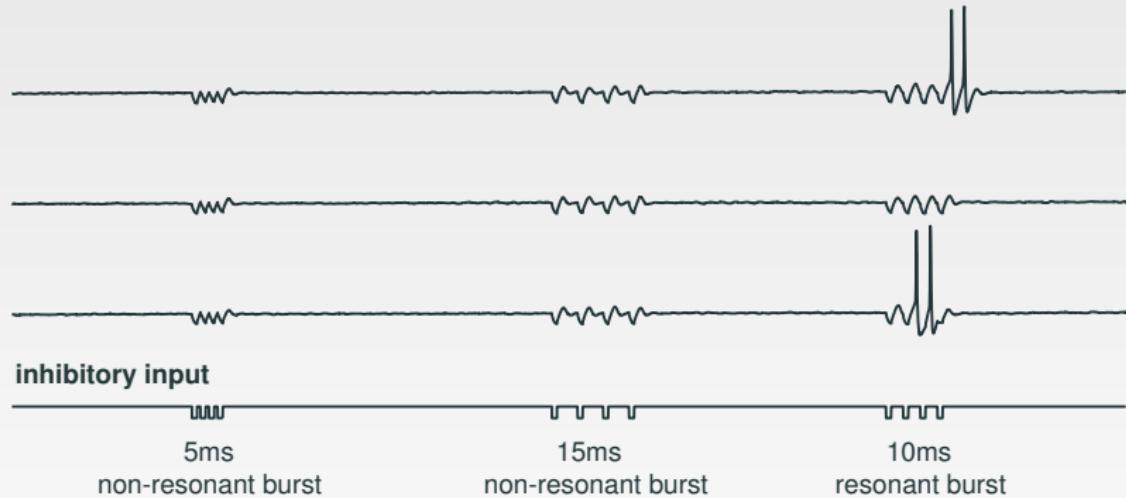
Resonators are selective in a frequency band.

Selective response: excitation or inhibition



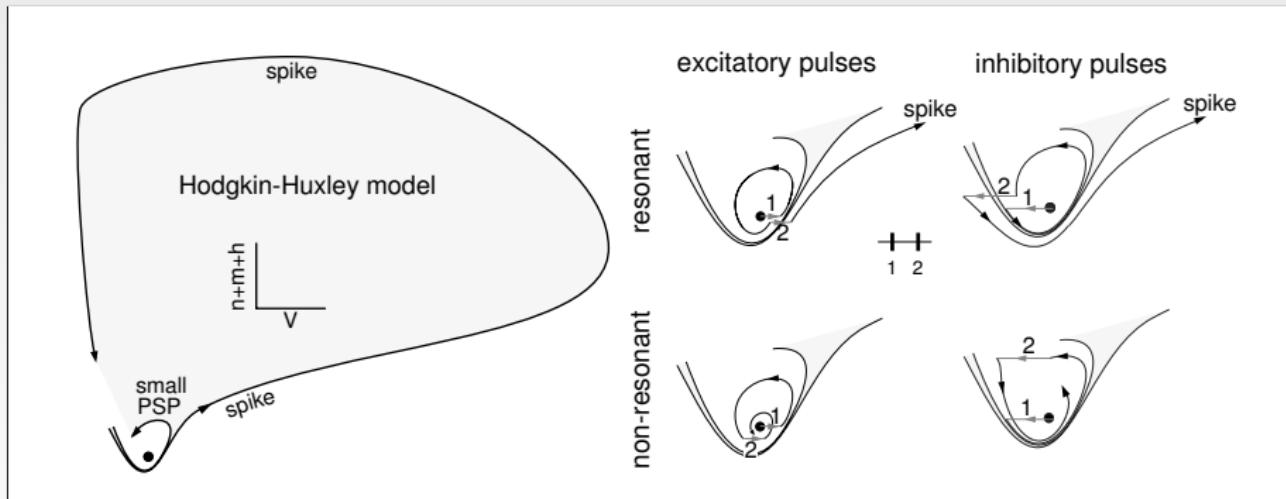
Mesencephalic V neurons in brainstem having subthreshold membrane oscillations with a natural period around 9 ms. Three consecutive voltage traces are shown to demonstrate some variability of the result

Selective response: excitation or inhibition



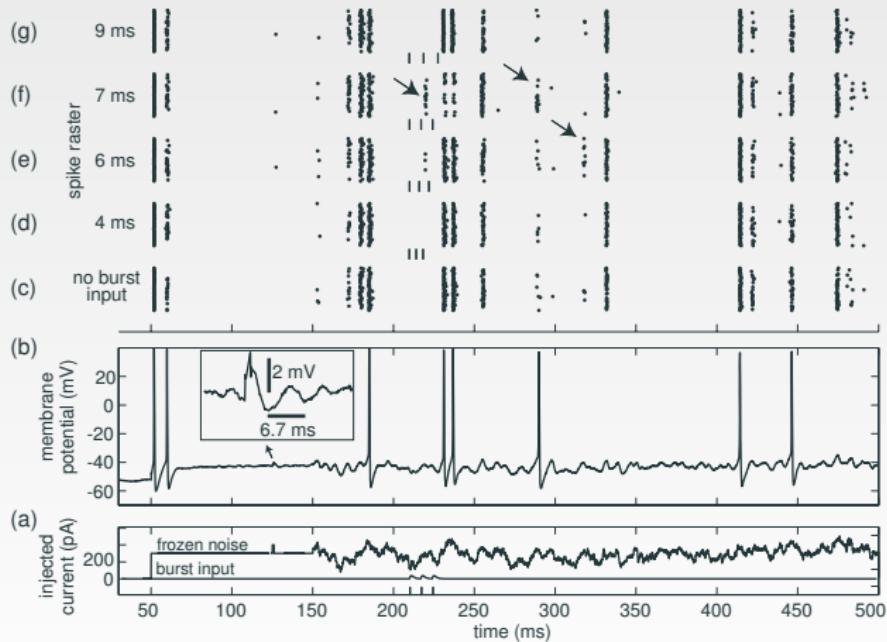
Experimental observations of selective response to inhibitory resonant burst in mesencephalic V neurons in brainstem having oscillatory potentials with the natural period around 9 ms. (Modified from Izhikevich et al. 2003).

Selective response: excitation or inhibition, Geometrical explanation



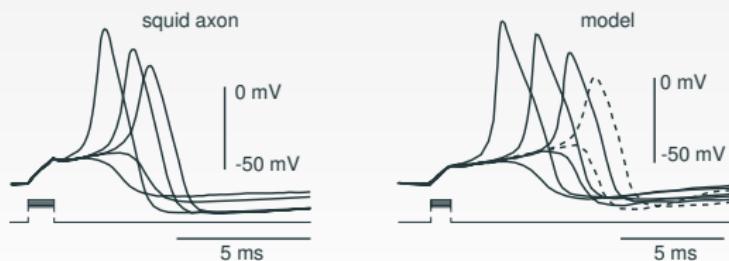
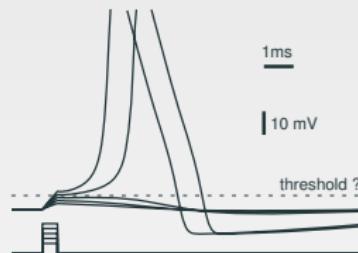
Important temporal coherence of pulses.

Influence of synaptic bombardment, Frozen noise experiment



In vitro exp., sub-th. oscillations.

Threshold definition: not obvious

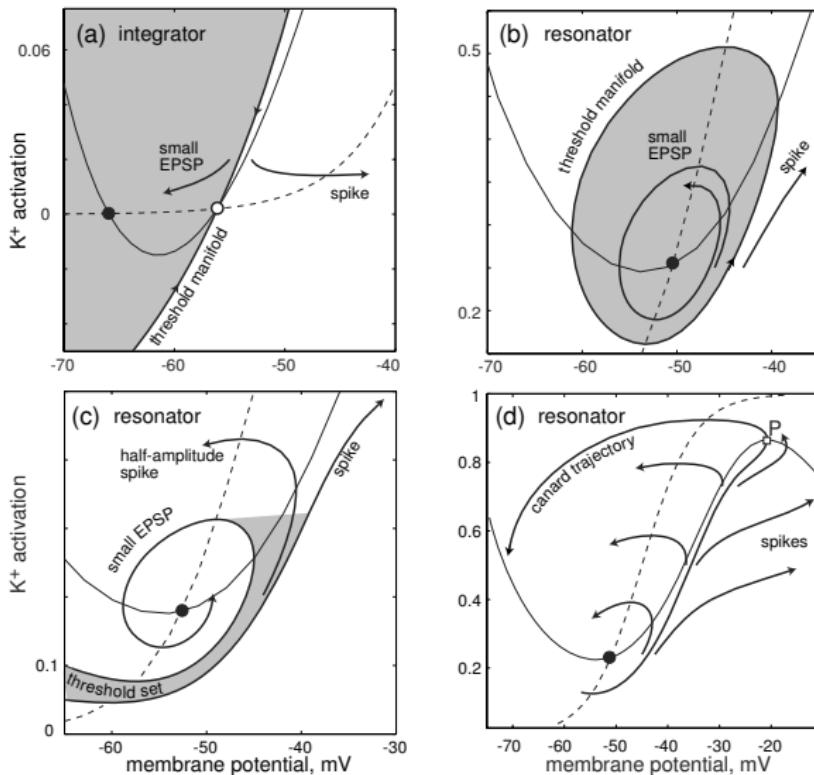


Threshold manifolds for the $I_{Na,p} + I_K$ model

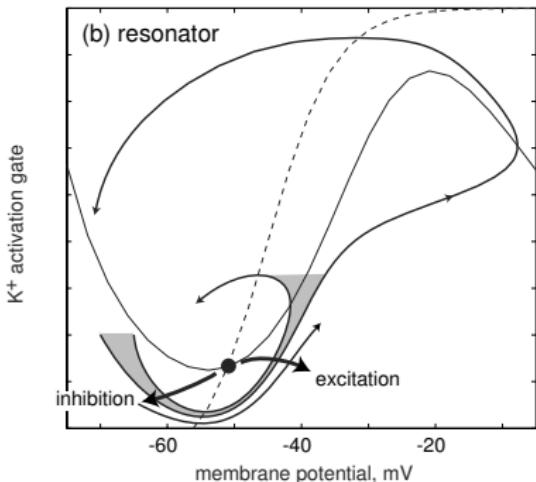
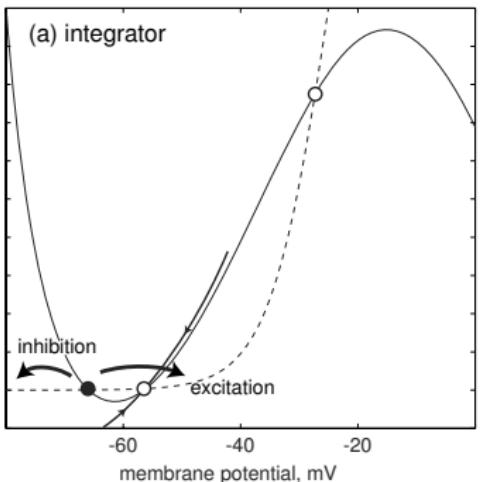
FitzHugh (1955) noticed that **thresholds, if they exist, are never numbers but manifolds**, e.g., curves in two-dimensional systems.

- Integrators (close to saddle-node bifurcation) have a threshold manifolds, given by stable manifold of fold point.
- For resonators, it depends on bifurcation type
 - Bistable regime: (close to sub. AH): the unstable LC is a th. manifold.
 - In all other cases, there is no well defined th. manifold.

Threshold manifolds for the $I_{Na,p} + I_K$ model

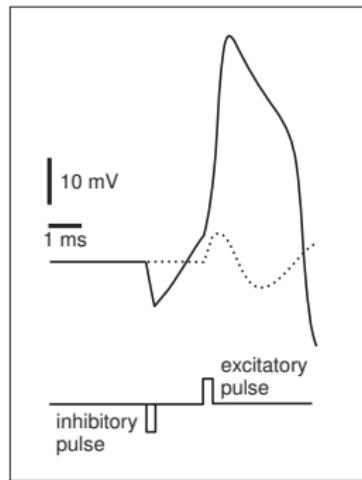
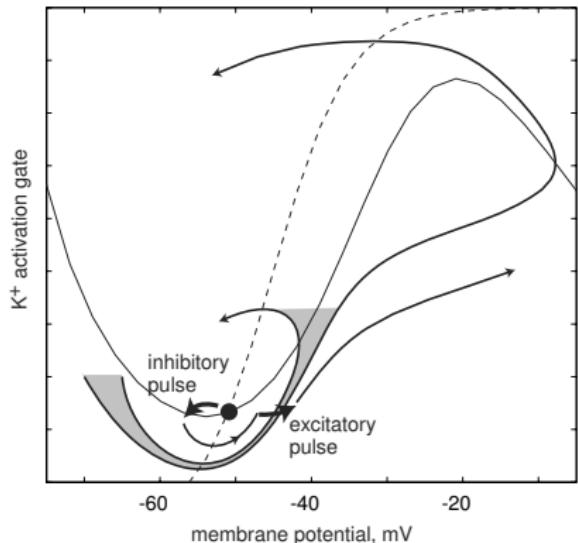


Spike emission: excitation or inhibition?



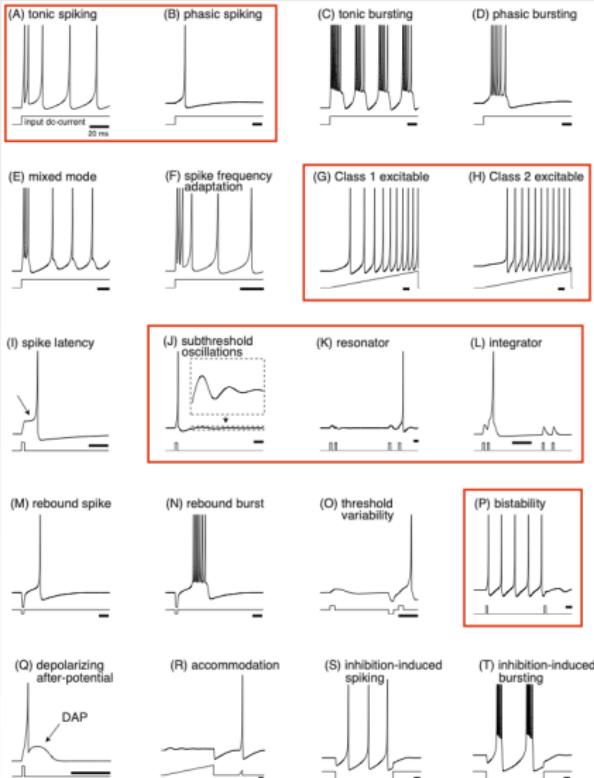
An integrator cannot spike in response to a hyper-polarizing current, a resonator can.

Spike emission: excitation or inhibition?

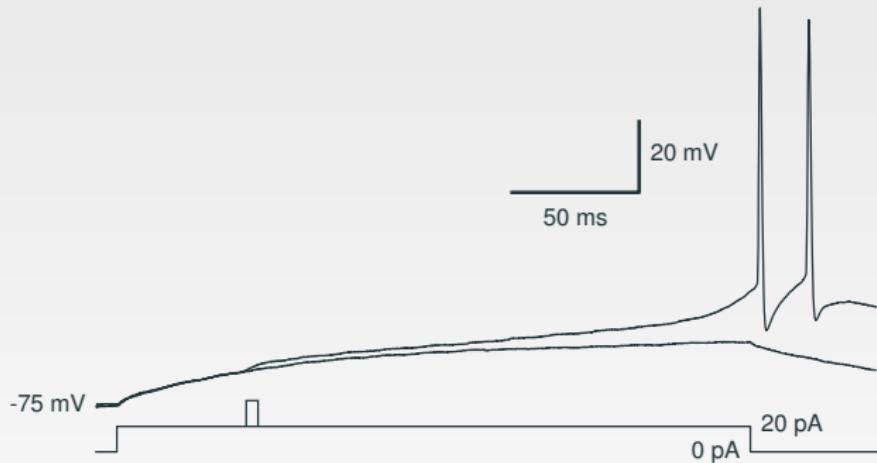


An hyper-polarizing current (inhib.) can enhance the effect of subsequent excitatory pulses.

Spiking behaviours



Latency to first spike (In Vitro)

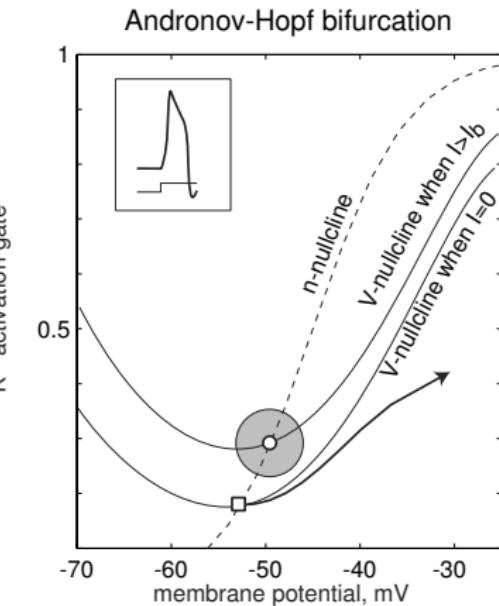
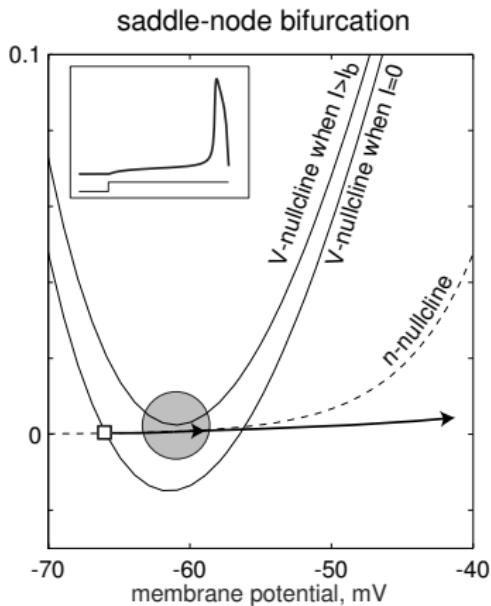


Property of integrators

Allow coding of input strength in latency to first spike

Less sensitive to noise, since only prolonged inputs can cause spikes

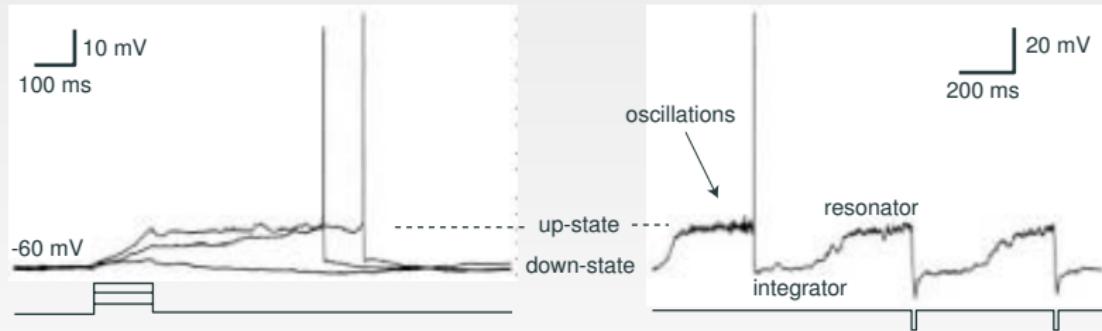
Latency to first spike



Flipping from an Integrator to a Resonator

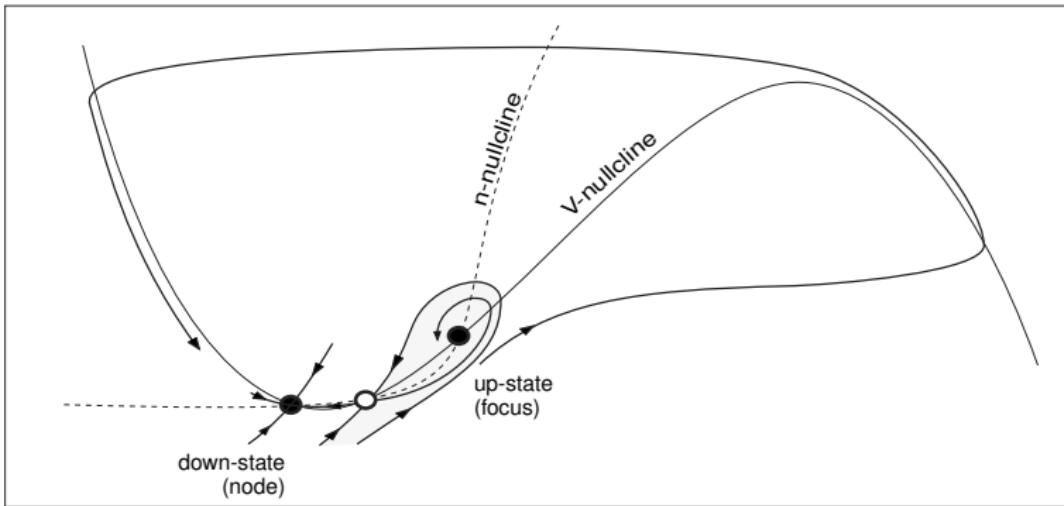
- Mitral cells can be switched from being integrators to being resonators by synaptic input.
- For resonators, it depends on bifurcation type

Integrator → Resonator 1/2



Multistability of equilibria.

Integrator → Resonator 2/2



Gray: attraction domain for UP state.

BT coming into play?

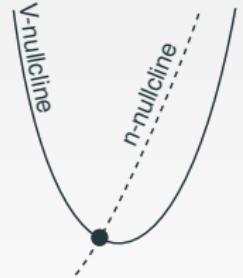
Need to combine in 2d a SN with a AH:

integrator
(saddle-node bifurcation)



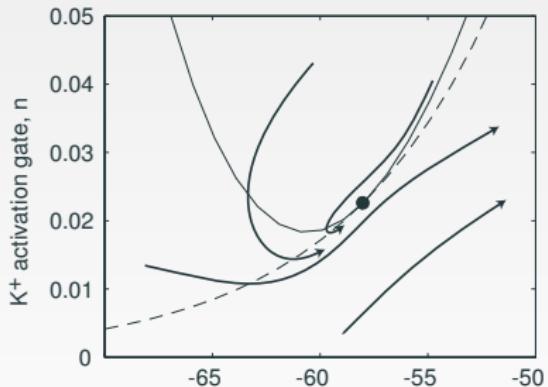
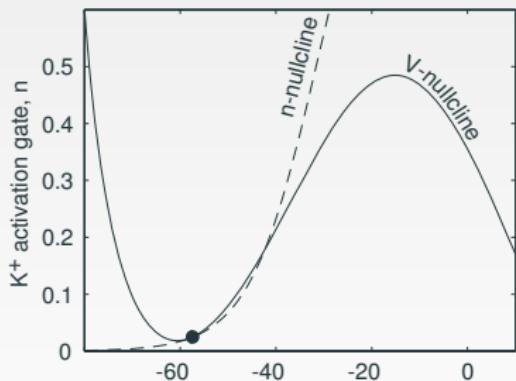
?

resonator
(Andronov-Hopf bifurcation)



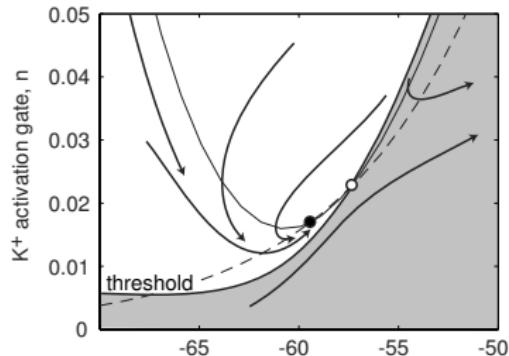
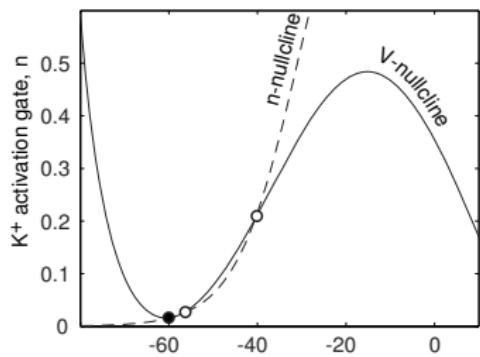
Bogdanov-Takens 1/3

In the $I_{Na,p} + I_K$ model (parameters E_L , $V_{1/2}$), the nullclines are parallel close to the left knee of the fast nullcline



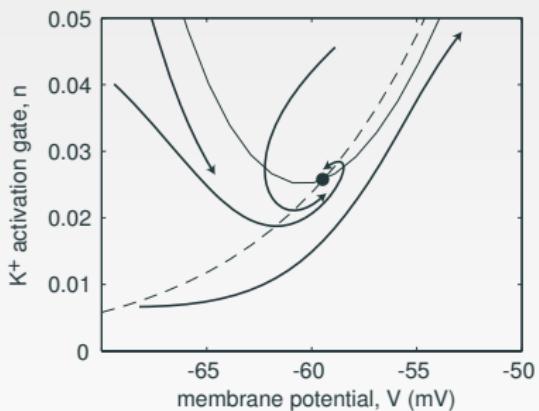
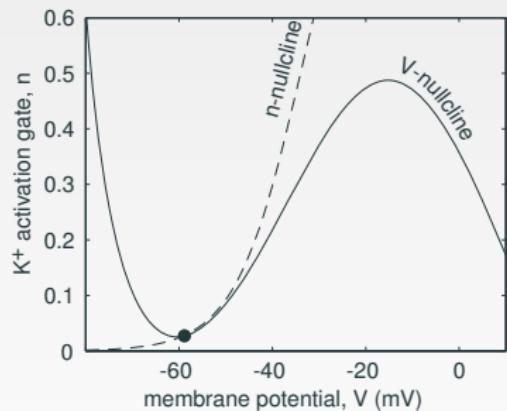
Bogdanov-Takens 2/3

A small change in the parameter $V_{1/2}$ can give a Saddle-Node: i.e. an integrator



Bogdanov-Takens 3/3

or a resonator via a stable focus.



Summary

Summary 1/2

		coexistence of resting and spiking states	
		YES (bistable)	NO (monostable)
subthreshold oscillations	NO (integrator)	saddle-node	saddle-node on invariant circle
	YES (resonator)	subcritical Andronov-Hopf	supercritical Andronov-Hopf

⇒ How does this impact network dynamics, ongoing research

Summary 2/2

properties	integrators		resonators	
bifurcation	saddle-node on invariant circle	saddle-node	subcritical Andronov-Hopf	supercritical Andronov-Hopf
excitability	class 1	class 2	class 2	class 2
oscillatory potentials		no		yes
frequency preference		no		yes
I-V relation at rest		non-monotone		monotone
spike latency		large		small
threshold and rheobase		well-defined		may not be defined
all-or-none action potentials		yes		no
co-existence of resting and spiking	no	yes	yes	no
post-inhibitory spike or facilitation (brief stimuli)		no		yes
inhibition-induced spiking		no		possible