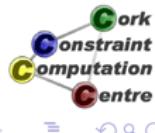


Chapter 4: Basic Constraint Reasoning (SEND+MORE=MONEY)

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ECLiPSe ELearning Overview

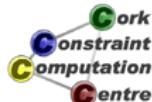


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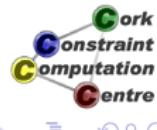
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Outline

- 1 Problem
- 2 Program
- 3 Constraint Setup
- 4 Search
- 5 Lessons Learned



What we want to introduce

- Finite Domain Solver in ECLiPSe
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, alldifferent, disequality
- Built-in search: Labeling
- Visualizers for variables, constraints and search



Outline

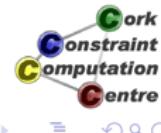
1 Problem

2 Program

3 Constraint Setup

4 Search

5 Lessons Learned

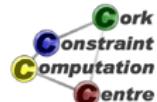


Problem Definition

A Crypt-Arithmetic Puzzle

We begin with the definition of the SEND+MORE=MONEY puzzle. It is often shown in the form of a hand-written addition:

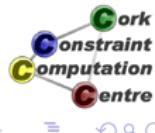
$$\begin{array}{r} \text{S} \quad \text{E} \quad \text{N} \quad \text{D} \\ + \text{M} \quad \text{O} \quad \text{R} \quad \text{E} \\ \hline \text{M} \quad \text{O} \quad \text{N} \quad \text{E} \quad \text{Y} \end{array}$$



Rules

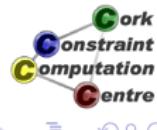
- Each character stands for a digit from 0 to 9.
- Numbers are built from digits in the usual, positional notation.
- Repeated occurrence of the same character denote the same digit.
- Different characters denote different digits.
- Numbers do not start with a zero.
- The equation must hold.

$$\begin{array}{r} & S & E & N & D \\ + & M & O & R & E \\ \hline M & O & N & E & Y \end{array}$$



Outline

- 1 Problem
- 2 Program
- 3 Constraint Setup
- 4 Search
- 5 Lessons Learned



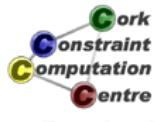
Model

- Each character is a variable, which ranges over the values 0 to 9.
- An *alldifferent* constraint between all variables, which states that two different variables must have different values. This is a very common constraint, which we will encounter in many other problems later on.
- Two *disequality constraints* (variable X must be different from value V) stating that the variables at the beginning of a number can not take the value 0.
- An arithmetic *equality constraint* linking all variables with the proper coefficients and stating that the equation must hold.

Program Sendmory

```
:– module(sendmory).  $\Rightarrow$  Define Module
:– export(sendmory/1).
:– lib(ic).

sendmory(L) :–
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).
```

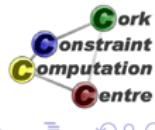


Program Sendmory

```
:– module(sendmory).  
:– export(sendmory/1).  
:– lib(ic).  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y],  
    L :: 0..9,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L).
```

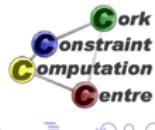
Program Sendmory

```
:– module(sendmory).  
:– export(sendmory/1).  
:– lib(ic).⇒ Use ic library  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y],  
    L :: 0..9,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L).
```



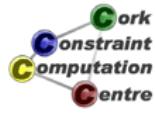
Program Sendmory

```
:– module(sendmory) .  
:– export(sendmory/1) .  
:– lib(ic) .  
sendmory(L) :- Predicate definition  
    L = [S,E,N,D,M,O,R,Y],  
    L :: 0..9,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L) .
```



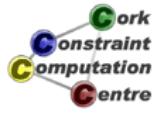
Program Sendmory

```
:– module(sendmory) .  
:– export(sendmory/1) .  
:– lib(ic) .  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y],  $\Rightarrow$  Define list  
    L :: 0..9,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L) .
```



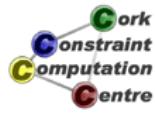
Program Sendmory

```
:– module(sendmory) .  
:– export(sendmory/1) .  
:– lib(ic) .  
  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y] ,  
    L :: 0..9,  
    Define integer domain 0..9  
    alldifferent(L) ,  
    S #\= 0, M #\= 0 ,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y ,  
    labeling(L) .
```



Program Sendmory

```
:– module(sendmory) .  
:– export(sendmory/1) .  
:– lib(ic) .  
  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y] ,  
    L :: 0..9 ,  
    alldifferent(L) , ⇒ Digits must be different  
    S #\= 0 , M #\= 0 ,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y ,  
    labeling(L) .
```



Program Sendmory

```

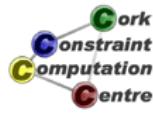
:- module(sendlmory).
:- export(sendlmory/1).
:- lib(ic).

sendlmory(L) :-
    L = [S,E,N,D,M,O,R,Y],
    L :: 0..9,
    alldifferent(L),
    S #\= 0, M #\= 0, ⇒ Numbers don't start with 0
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L).

```

	S	E	N	D
+	M	O	R	E
	M	O	N	E
				Y

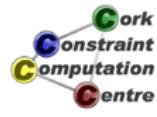




Program Sendmory

```
:– module(sendmory).  
:– export(sendmory/1).  
:– lib(ic).  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y],  
    L :: 0..9,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L).
```

$$\begin{array}{r} & S & E & N & D \\ + & M & O & R & E \\ \hline M & O & N & E & Y \end{array}$$



Program Sendmory

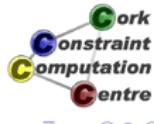
```
:– module(sendmory).  
:– export(sendmory/1).  
:– lib(ic).  
  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y],  
    L :: 0..9,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L).  $\Rightarrow$  built-in search routine
```

Program Sendmory

```
:– module(sendmory).  
:– export(sendmory/1).  
:– lib(ic).  
  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y],  
    L :: 0..9,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L).
```

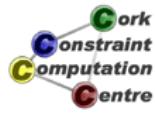
Program Sendmory

```
:- module(sendmory).  
:- export(sendmory/1).  
:- lib(ic).  
sendmory(L) :-  
    for predicate definition  
        L = [S,E,N,D,M,O,R,Y],  
        L :: 0..9,  
        alldifferent(L),  
        S #\= 0, M #\= 0,  
        1000*S + 100*E + 10*N + D +  
        1000*M + 100*O + 10*R + E #=  
        10000*M + 1000*O + 100*N + 10*E + Y,  
        labeling(L).
```



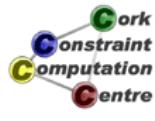
Program Sendmory

```
:– module(sendmory) .  
:– export(sendmory/1) .  
:– lib(ic) .  
  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y] ,  
    L :: 0..9 ,  
    alldifferent(L) ,  
    S #\= 0 , M #\= 0 , ⇒ Special symbol for ic  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y ,  
    labeling(L) .
```



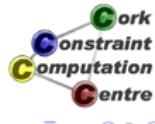
Program Sendmory

```
:– module(sendmory).  
:– export(sendmory/1).  
:– lib(ic).  
  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y],  $\Rightarrow$  Confusing name!  
    L :: 0..9,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L).
```



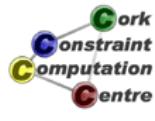
Program Sendmory

```
:– module(sendmory) .  
:– export(sendmory/1) .  
:– lib(ic) .  
  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y] ,  
    L :: 0..9 ,  
    alldifferent(L) ,  
    S #\= 0 , M #\= 0 ,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y ,  
    labeling(L) .
```



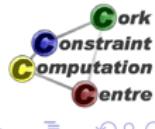
General Program Structure

```
:– module(sendmory).  
:– export(sendmory/1).  
:– lib(ic).  
  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y], ⇒ Variables  
    L :: 0..9,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L).
```



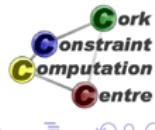
General Program Structure

```
:– module(sendmory).  
:– export(sendmory/1).  
:– lib(ic).  
  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y],  
    L :: 0..9,  
    alldifferent(L),  $\Rightarrow$  Constraints  
    S #\= 0, M #\= 0,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L).
```



General Program Structure

```
:– module(sendmory).  
:– export(sendmory/1).  
:– lib(ic).  
  
sendmory(L) :-  
    L = [S,E,N,D,M,O,R,Y],  
    L :: 0..9,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L).  $\Rightarrow$  Search
```

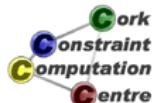


Choice of Model

- This is *one* model, not *the* model of the problem
- Many possible alternatives
- Choice often depends on your constraint system
 - Constraints available
 - Reasoning attached to constraints
- Not always clear which is the *best* model
- Often: Not clear what is the *problem*

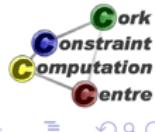
▶ Alternative 1

▶ Alternative 2



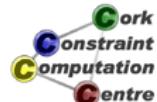
Running the program

- To run the program, we have to enter the query
 - `sendmory:sendmory(L).`
- Result
 - `L = [9, 5, 6, 7, 1, 0, 8, 2]`
 - yes (0.00s cpu, solution 1, maybe more)



Question

- But how did the program come up with this solution?



Outline

1 Problem

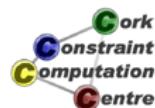
2 Program

3 Constraint Setup

- Domain Definition
- Alldifferent Constraint
- Disequality Constraints
- Equality Constraint

4 Search

5 Lessons Learned



Domain Definition

```
L = [S,E,N,D,M,O,R,Y],  
L :: 0..9,
```

$$[S, E, N, D, M, O, R, Y] \in \{0..9\}$$



Domain Visualization

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Domain Visualization

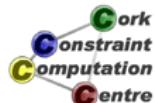
Rows =
Variables

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Domain Visualization

Columns = Values

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										



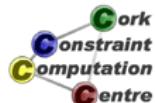
Domain Visualization

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M			Cells = State							
O										
R										
Y										

Alldifferent Constraint

```
alldifferent(L),
```

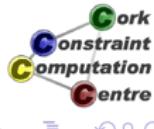
- Built-in of `ic` library
- No initial propagation possible
- *Suspends*, waits until variables are changed
- When variable is fixed, remove value from domain of other variables
- *Forward checking*



Alldifferent Visualization

Uses the same representation as the domain visualizer

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										



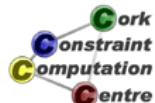
Disequality Constraints

$S \# \backslash = 0, M \# \backslash = 0,$

Remove value from domain

$$S \in \{1..9\}, M \in \{1..9\}$$

Constraints solved, can be removed

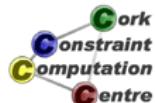


Domains after Disequality

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Equality Constraint

- Normalization of linear terms
 - Single occurrence of variable
 - Positive coefficients
- Propagation



Normalization

1000*S+	100*E+	10*N+	D
+1000*M+	100*O+	10*R+	E
10000*M+	1000*O+	100*N+	10*E+ Y

Normalization

1000*S+	100*E+	10*N+	D
+1000*M+	100*O+	10*R+	E
10000*M+	1000*O+	100*N+	10*E+ Y

Normalization

$$\begin{array}{r} 1000^*S+ \quad 100^*E+ \quad 10^*N+ \quad D \\ + \quad 100^*O+ \quad 10^*R+ \quad E \\ \hline \mathbf{9000^*M+} \quad 1000^*O+ \quad 100^*N+ \quad 10^*E+ \quad Y \end{array}$$



Normalization

$$\begin{array}{r} 1000^*S+ \quad 100^*E+ \quad 10^*N+ \quad D \\ + \quad \mathbf{100^*O+} \quad 10^*R+ \quad E \\ \hline 9000^*M+ \quad \mathbf{1000^*O+} \quad 100^*N+ \quad 10^*E+ \quad Y \end{array}$$

Normalization

$$\begin{array}{r} 1000^*S+ \quad 100^*E+ \quad 10^*N+ \quad D \\ \quad \quad \quad + \quad 10^*R+ \quad E \\ \hline 9000^*M+ \quad \mathbf{900^*O+} \quad 100^*N+ \quad 10^*E+ \quad Y \end{array}$$

Normalization

$$\begin{array}{r} 1000^*S+ \quad 100^*E+ \quad \mathbf{10^*N+} \quad D \\ \qquad \qquad + \quad 10^*R+ \quad E \\ \hline 9000^*M+ \quad 900^*O+ \quad \mathbf{100^*N+} \quad 10^*E+ \quad Y \end{array}$$



Normalization

$$\begin{array}{r} 1000^*S+ \quad 100^*E+ \quad & D \\ & + \quad 10^*R+ \quad E \\ \hline 9000^*M+ \quad 900^*O+ \quad \mathbf{90^*N+} \quad 10^*E+ \quad Y \end{array}$$

Normalization

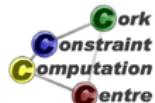
$$\begin{array}{r} 1000^*S+ \quad \mathbf{100^*E+} \quad & D \\ & + \quad 10^*R+ \quad E \\ \hline 9000^*M+ \quad 900^*O+ \quad 90^*N+ \quad \mathbf{10^*E+} \quad Y \end{array}$$

Normalization

$$\begin{array}{r} 1000^*S+ \quad 91^*E+ \quad \quad \quad D \\ \quad \quad \quad + \quad 10^*R \\ \hline 9000^*M+ \quad 900^*O+ \quad 90^*N+ \quad \quad \quad Y \end{array}$$

Simplified Equation

$$1000 * S + 91 * E + 10 * R + D = 9000 * M + 900 * O + 90 * N + Y$$

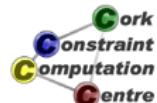


Propagation

$$1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9} = \\ 9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}$$

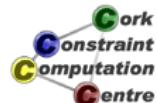
Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{1000..9918} =$$
$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..89919}$$



Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} =$$
$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

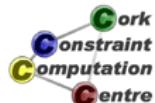


Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \\ \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

Deduction:

$$M = 1, S = 9, O \in \{0..1\}$$



Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} =$$

$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

Deduction:

$$M=1, S=9, O \in \{0..1\}$$

Why? ▶ Skip



Consider lower bound for S

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Lower bound of equation is 9000
- Rest of lhs (left hand side) ($91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}$) is atmost 918
- S must be greater or equal to $\frac{9000 - 918}{1000} = 8.082$
 - otherwise lower bound of equation not reached by lhs
- S is integer, therefore $S \geq \lceil \frac{9000 - 918}{1000} \rceil = 9$
- S has upper bound of 9, so $S = 9$

Consider upper bound of M

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}$ is at least 0
- M must be smaller or equal to $\frac{9918 - 0}{9000} = 1.102$
- M must be integer, therefore $M \leq \lfloor \frac{9918 - 0}{9000} \rfloor = 1$
- M has lower bound of 1, so $M = 1$

Consider upper bound of O

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $9000 * 1 + 90 * N^{0..9} + Y^{0..9}$ is at least 9000
- O must be smaller or equal to $\frac{9918 - 9000}{900} = 1.02$
- O must be integer, therefore $O \leq \lfloor \frac{9918 - 9000}{900} \rfloor = 1$
- O has lower bound of 0, so $O \in \{0..1\}$

Propagation of equality: Result

	0	1	2	3	4	5	6	7	8	9
S	-	-	-	-	-	-	-	-	-	*
E										
N										
D										
M	*	*	-	-	-	-	-	-	-	-
O			*	*	*	*	*	*	*	*
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S	-	-	-	-	-	-	-	-	-	*
E										
N										
D										
M	*	*	-	-	-	-	-	-	-	-
O			*	*	*	*	*	*	*	*
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										*
E										/
N										/
D										/
M		*								
O										
R										/
Y										/

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M		*								
O										
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O	*									
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O		*								
R										
Y										

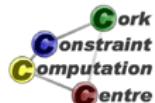
Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

$$O = 0, [E, R, D, N, Y] \in \{2..8\}$$

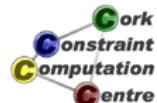
Waking the equality constraint

- Triggered by assignment of variables
- or* update of lower or upper bound



Removal of constants

$$1000 * 9 + 91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = \\ 9000 * 1 + 900 * 0 + 90 * N^{2..8} + Y^{2..8}$$

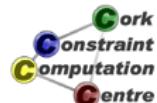


Removal of constants

$$\begin{aligned} \mathbf{1000 * 9 + 91 * E^{2..8} + 10 * R^{2..8} + D^{2..8}} &= \\ \mathbf{9000 * 1 + 900 * 0 + 90 * N^{2..8} + Y^{2..8}} \end{aligned}$$

Removal of constants

$$91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = 90 * N^{2..8} + Y^{2..8}$$



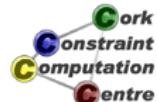
Propagation of equality (Iteration 1)

$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8}}_{204..816} = \underbrace{90 * N^{2..8} + Y^{2..8}}_{182..728}$$



Propagation of equality (Iteration 1)

$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8}}_{204..728} = 90 * N^{2..8} + Y^{2..8}$$



Propagation of equality (Iteration 1)

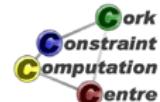
$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8}}_{204..728} = 90 * N^{2..8} + Y^{2..8}$$

$$N \geq 3 = \lceil \frac{204 - 8}{90} \rceil, E \leq 7 = \lfloor \frac{728 - 22}{91} \rfloor$$



Propagation of equality (Iteration 2)

$$91 * E^{2..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}$$

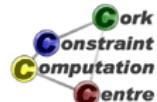


Propagation of equality (Iteration 2)

$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8}}_{204..725} = \underbrace{90 * N^{3..8} + Y^{2..8}}_{272..728}$$

Propagation of equality (Iteration 2)

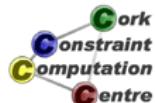
$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8}}_{272..725} = 90 * N^{3..8} + Y^{2..8}$$



Propagation of equality (Iteration 2)

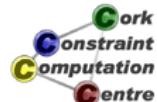
$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8}}_{272..725} = 90 * N^{3..8} + Y^{2..8}$$

$$E \geq 3 = \lceil \frac{272 - 88}{91} \rceil$$



Propagation of equality (Iteration 3)

$$91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}$$

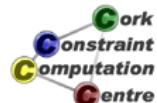


Propagation of equality (Iteration 3)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = \underbrace{90 * N^{3..8} + Y^{2..8}}_{272..728}$$

Propagation of equality (Iteration 3)

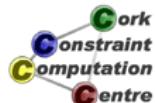
$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = 90 * N^{3..8} + Y^{2..8}$$



Propagation of equality (Iteration 3)

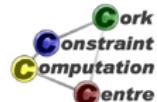
$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = 90 * N^{3..8} + Y^{2..8}$$

$$N \geq 4 = \lceil \frac{295 - 8}{90} \rceil$$



Propagation of equality (Iteration 4)

$$91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}$$

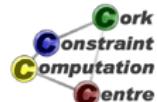


Propagation of equality (Iteration 4)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = \underbrace{90 * N^{4..8} + Y^{2..8}}_{362..728}$$

Propagation of equality (Iteration 4)

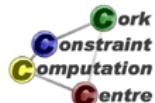
$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{362..725} = 90 * N^{4..8} + Y^{2..8}$$



Propagation of equality (Iteration 4)

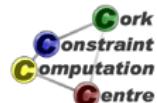
$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{362..725} = 90 * N^{4..8} + Y^{2..8}$$

$$E \geq 4 = \lceil \frac{362 - 88}{91} \rceil$$



Propagation of equality (Iteration 5)

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}$$

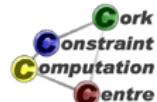


Propagation of equality (Iteration 5)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = \underbrace{90 * N^{4..8} + Y^{2..8}}_{362..728}$$

Propagation of equality (Iteration 5)

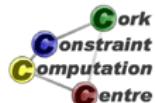
$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = 90 * N^{4..8} + Y^{2..8}$$



Propagation of equality (Iteration 5)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = 90 * N^{4..8} + Y^{2..8}$$

$$N \geq 5 = \lceil \frac{386 - 8}{90} \rceil$$



Propagation of equality (Iteration 6)

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$

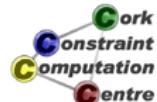


Propagation of equality (Iteration 6)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = \underbrace{90 * N^{5..8} + Y^{2..8}}_{452..728}$$

Propagation of equality (Iteration 6)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{452..725} = 90 * N^{5..8} + Y^{2..8}$$



Propagation of equality (Iteration 6)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{452..725} = 90 * N^{5..8} + Y^{2..8}$$

$$N \geq 5 = \lceil \frac{452 - 8}{90} \rceil, E \geq 4 = \lceil \frac{452 - 88}{91} \rceil$$

No further propagation at this point

Domains after setup

	0	1	2	3	4	5	6	7	8	9
S										█
E										
N										
D										
M		█								
O	█									
R										
Y										

Outline

- 1 Problem
- 2 Program
- 3 Constraint Setup
- 4 Search
 - Step 1
 - Step 2
 - Further Steps
 - Solution
- 5 Lessons Learned

labeling built-in

```
labeling([S,E,N,D,M,O,R,Y])
```

- Try variable in order given
- Try values starting from smallest value in domain
- When failing, backtrack to last open choice
- *Chronological Backtracking*
- *Depth First search*



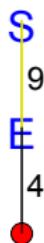
Search Tree Step 1

S
|
9
E

Variable S already fixed

Step 2, Alternative $E = 4$

Variable $E \in \{4..7\}$, first value tested is 4

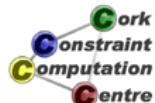


Assignment $E = 4$

	0	1	2	3	4	5	6	7	8	9
S										█
E					★	-	-	-		
N										
D										
M		█								
O	█									
R										
Y										

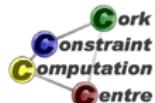
Propagation of $E = 4$, equality constraint

$$91 * 4 + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$



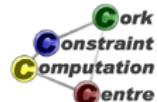
Propagation of $E = 4$, equality constraint

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8}}_{386..452} = \underbrace{90 * N^{5..8} + Y^{2..8}}_{452..728}$$



Propagation of $E = 4$, equality constraint

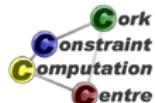
$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8}}_{452} = 90 * N^{5..8} + Y^{2..8}$$



Propagation of $E = 4$, equality constraint

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8}}_{452} = 90 * N^{5..8} + Y^{2..8}$$

$$N = 5, Y = 2, R = 8, D = 8$$



Result of equality propagation

	0	1	2	3	4	5	6	7	8	9
S										Red
E					Red					
N						Red	-	-	-	
D			-	-	-	-	-	-	Red	
M		Red								
O	Red									
R		-	-	-	-	-	-	-	Red	
Y			Red	-	-	-	-	-	-	

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N						*	-	-	-	
D			-	-	-	-	-	-	*	
M										
O	*									
R			-	-	-	-	-	-	*	
Y				*	-	-	-	-	-	

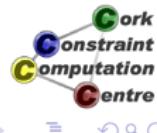
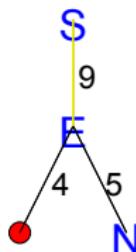
Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N						*	-	-		
D			-	-	-	-	-	-	*	
M										
O		*								
R			-	-	-	-	-	-	*	
Y			*	-	-	-	-	-		

Alldifferent fails!

Step 2, Alternative $E = 5$

Return to last open choice, E , and test next value



Assignment $E = 5$

	0	1	2	3	4	5	6	7	8	9
S										██████
E				-	████	-	-			
N										
D										
M		████								
O	████									
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E					-	*	-	-		
N										
D										
M										
O										
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E						*				
N										
D										
M										
O	*									
R										
Y										

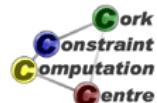
Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										█████
E						████				
N										
D										
M		████								
O	████									
R										
Y										

$$N \neq 5, N \geq 6$$

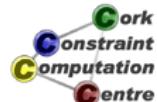
Propagation of equality

$$91 * 5 + 10 * R^{2..8} + D^{2..8} = 90 * N^{6..8} + Y^{2..8}$$



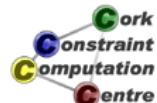
Propagation of equality

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8}}_{477..543} = \underbrace{90 * N^{6..8} + Y^{2..8}}_{542..728}$$



Propagation of equality

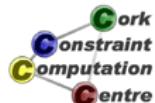
$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8}}_{542..543} = 90 * N^{6..8} + Y^{2..8}$$



Propagation of equality

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8}}_{542..543} = 90 * N^{6..8} + Y^{2..8}$$

$$N = 6, Y \in \{2, 3\}, R = 8, D \in \{7..8\}$$



Result of equality propagation

	0	1	2	3	4	5	6	7	8	9
S										█
E						█				
N							★	-	-	
D			✗	✗	✗		✗			
M		█								
O	█									
R			-	-	-		-	-	★	
Y					✗		✗	✗	✗	

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N							*	-	-	
D			*	*	*		*			
M										
O										
R			-	-	-		-	-	*	
Y					*		*	*	*	

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										█
E						█				
N							█			
D									█	
M		█								
O	█									
R								█	★	
Y				█	█					

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

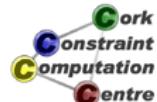
Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										█
E						█				
N							█			
D								█		
M		█								
O	█									
R								█		
Y				█	█					

$$D = 7$$

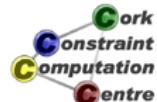
Propagation of equality

$$91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}$$



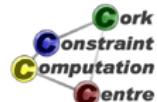
Propagation of equality

$$\underbrace{91 * 5 + 10 * 8 + 7}_{542} = \underbrace{90 * 6 + Y^{2..3}}_{542..543}$$



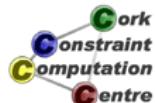
Propagation of equality

$$\underbrace{91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}}_{542}$$



Propagation of equality

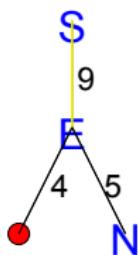
$$\underbrace{91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}}_{542}$$
$$Y = 2$$



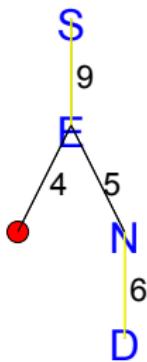
Last propagation step

	0	1	2	3	4	5	6	7	8	9
S										█
E						█				
N							█			
D								█		
M			█							
O	█									
R		█						█		
Y				█	*	-				

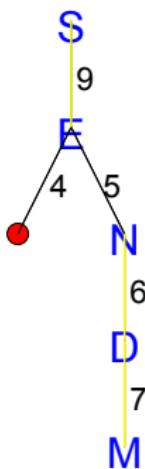
Further Steps: Nothing more to do



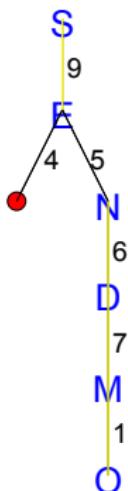
Further Steps: Nothing more to do



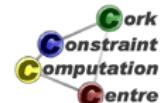
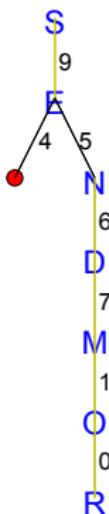
Further Steps: Nothing more to do



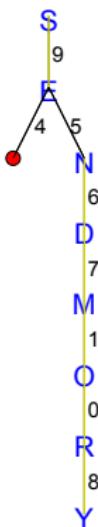
Further Steps: Nothing more to do



Further Steps: Nothing more to do



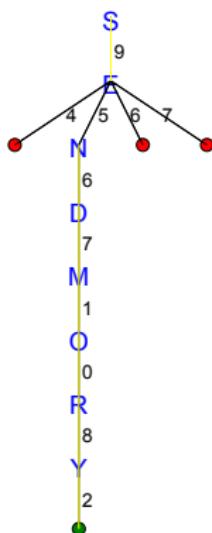
Further Steps: Nothing more to do



Further Steps: Nothing more to do



Complete Search Tree



Solution

$$\begin{array}{r} 9 & 5 & 6 & 7 \\ + & 1 & 0 & 8 & 5 \\ \hline 1 & 0 & 6 & 5 & 2 \end{array}$$

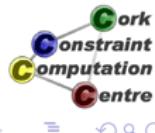
Outline

- 1 Problem
- 2 Program
- 3 Constraint Setup
- 4 Search
- 5 Lessons Learned



Topics introduced

- Finite Domain Solver in ECLiPSe, `ic` library
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, `allDifferent`, `disequality`
- Built-in search: labeling
- Visualizers for variables, constraints and search



Lessons Learned

- Constraint models are expressed by variables and constraints.
- Problems can have many different models, which can behave quite differently. Choosing the best model is an art.
- Constraints can take many different forms.
- Propagation deals with the interaction of variables and constraints.
- It removes some values that are inconsistent with a constraint from the domain of a variable.
- Constraints only communicate via shared variables.



Lessons Learned

- Propagation usually is not sufficient, search may be required to find a solution.
- Propagation is data driven, and can be quite complex even for small examples.
- The default search uses chronological depth-first backtracking, systematically exploring the complete search space.
- The search choices and propagation are interleaved, after every choice some more propagation may further reduce the problem.

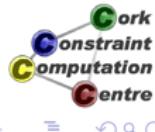


Outline

6 Alternative Models

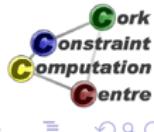
- Model without Disequality
- Multiple Equations

7 Exercises



Alternative 1

- Do we need the constraint “Numbers do not begin with a zero”?
- This is not given explicitly in the problem statement
- Remove disequality constraints from program
- Previous solution is still a solution
- Does it change propagation?
- Does it have more solutions?



Program without Disequality

Listing 1: Alternative 1

```
:-module(alternative1).  
:-export(sendmory/1).  
:-lib(ic).  
  
sendmory(L):-  
    L = [S,E,N,D,M,O,R,Y] ,  
    L :: 0..9 ,  
    alldifferent(L) ,  
    1000*S + 100*E + 10*N + D +  
    1000*M + 100*O + 10*R + E #=  
    10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(L).
```

After Setup without Disequality

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Setup Comparison

original

	0	1	2	3	4	5	6	7	8	9
S										■
E				■						
N										
D		■	■	■	■	■	■	■	■	
M		■								
O	■									
R										
Y										

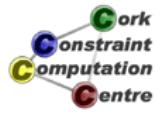
alternative 1

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Search Tree: Many Solutions

S

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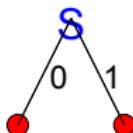


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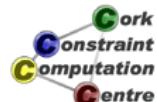
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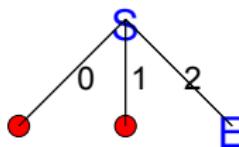
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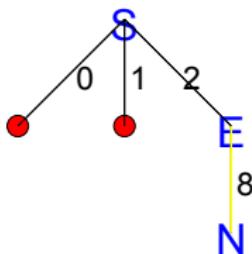
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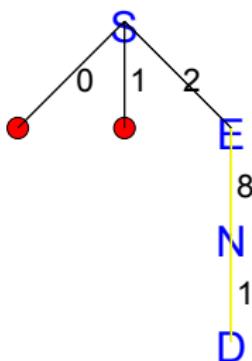


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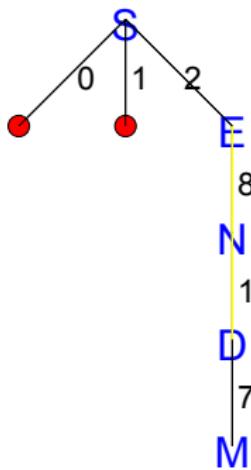


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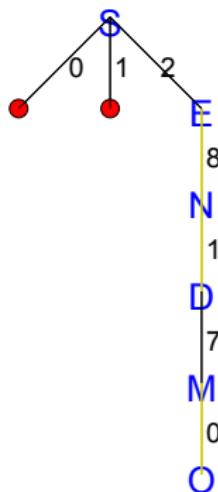
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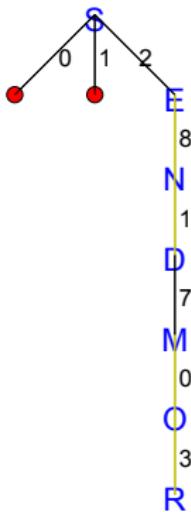


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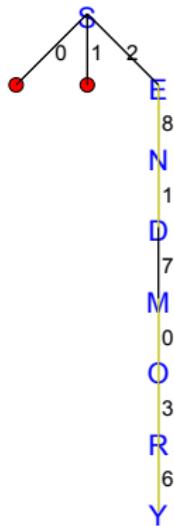
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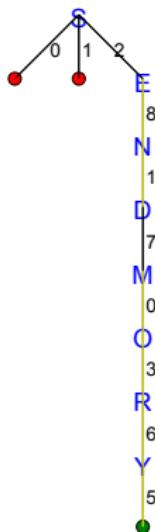


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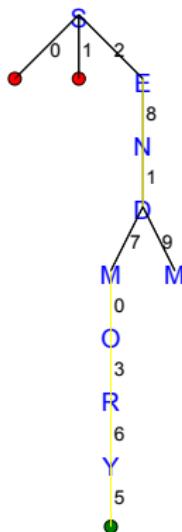


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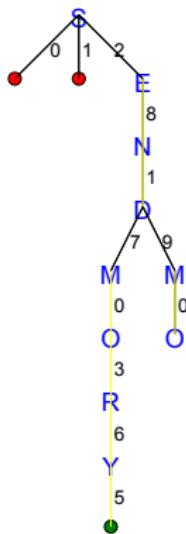


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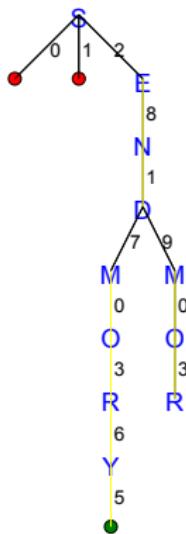
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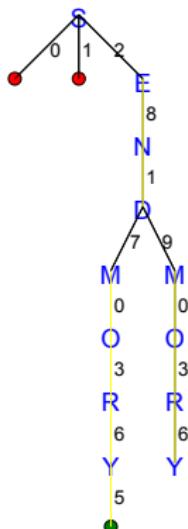


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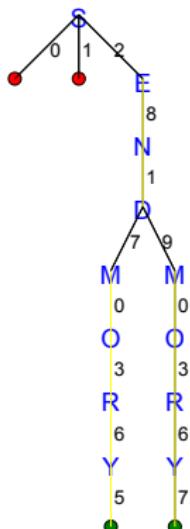
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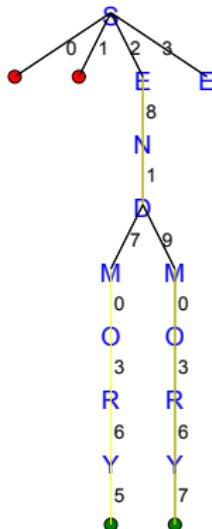
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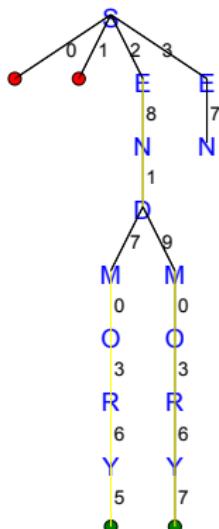
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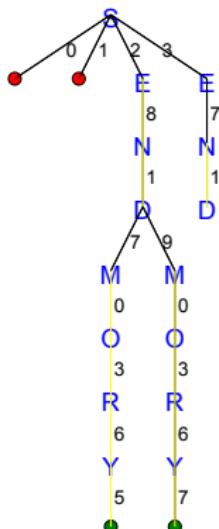
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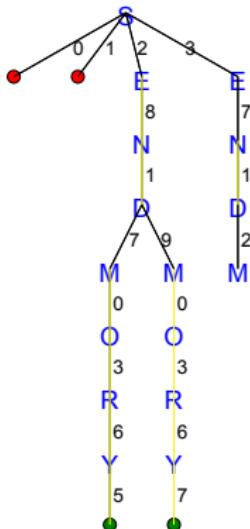
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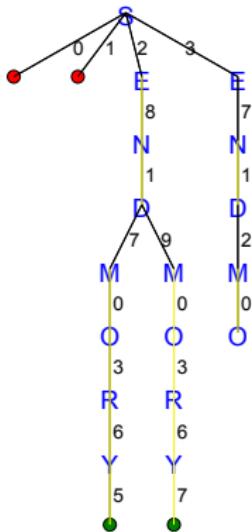


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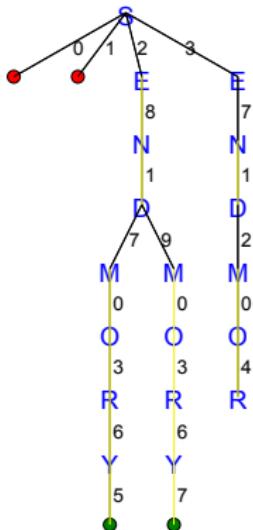
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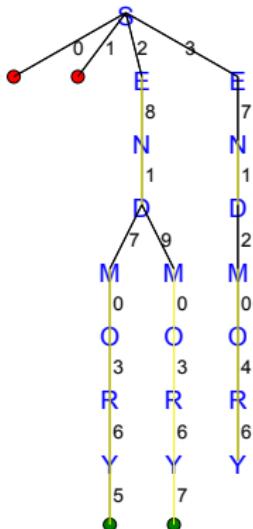
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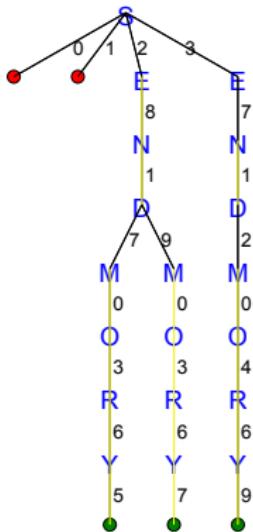
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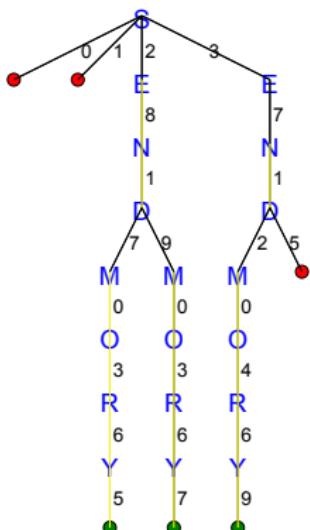
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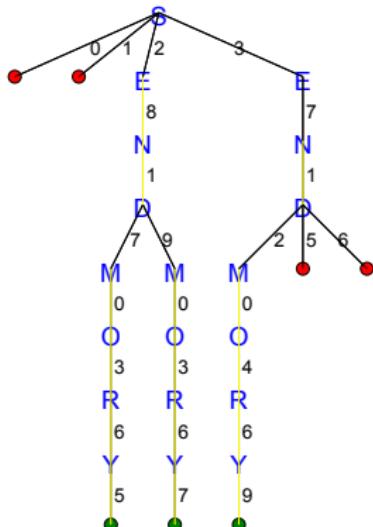
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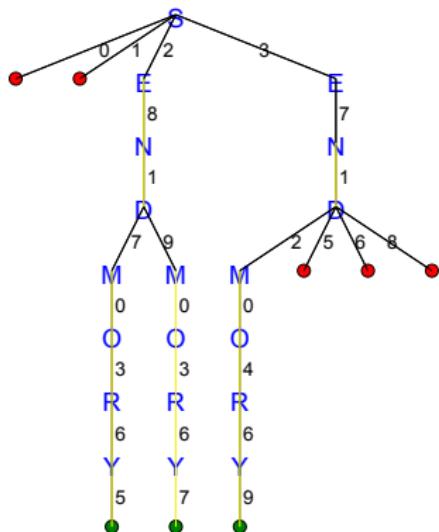
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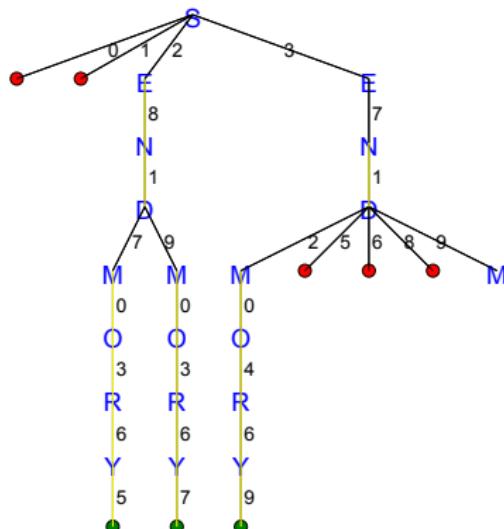
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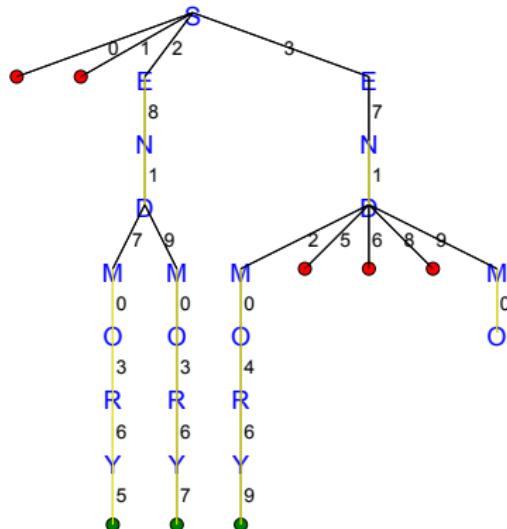
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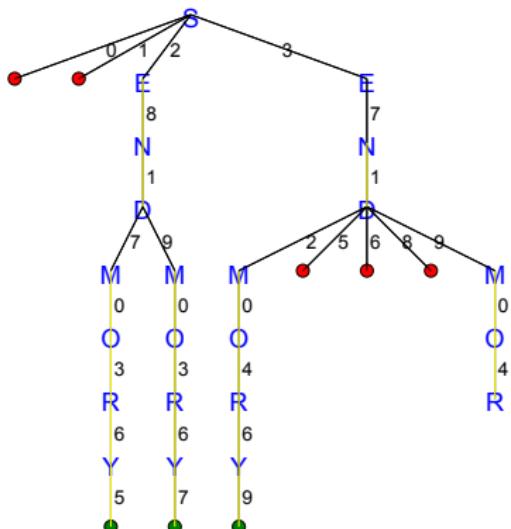
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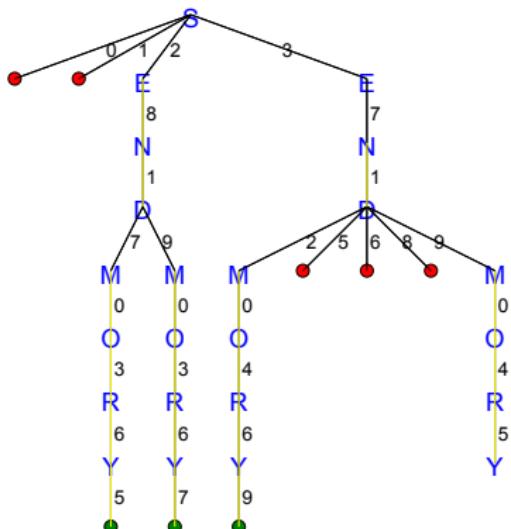
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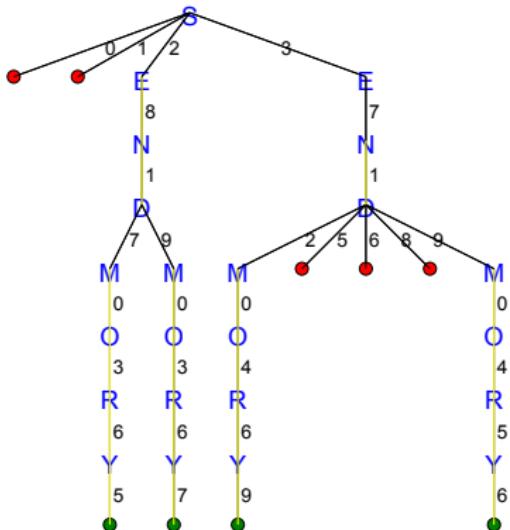
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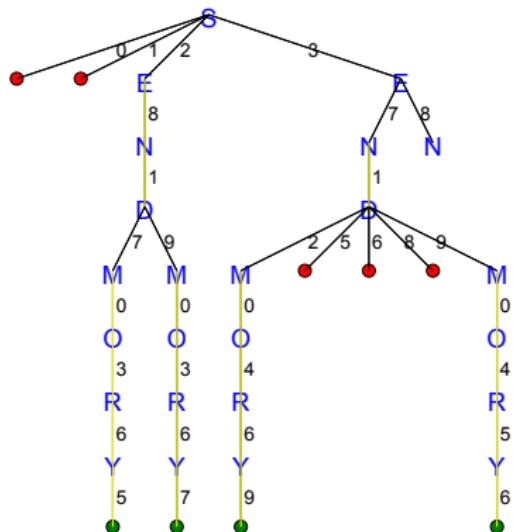
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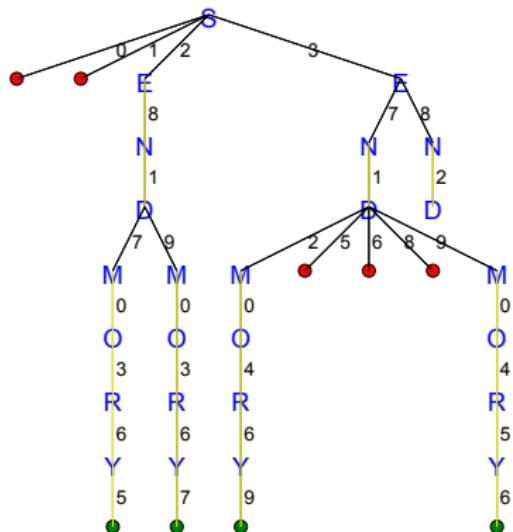
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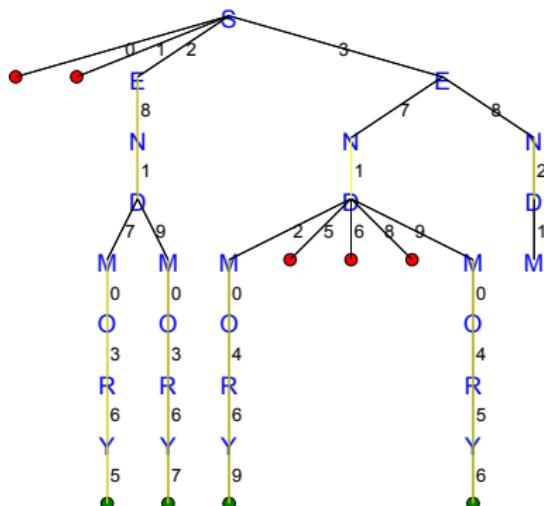
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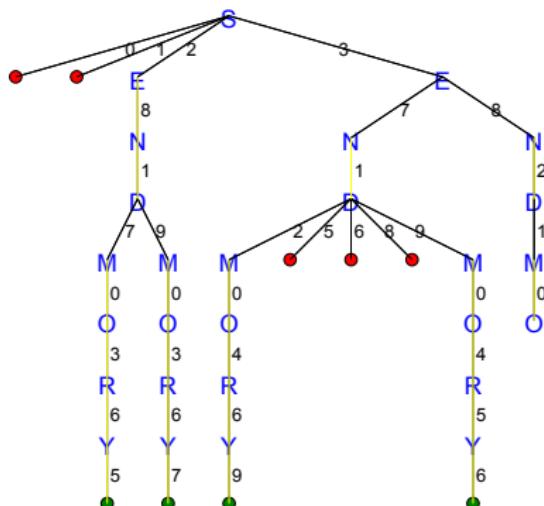
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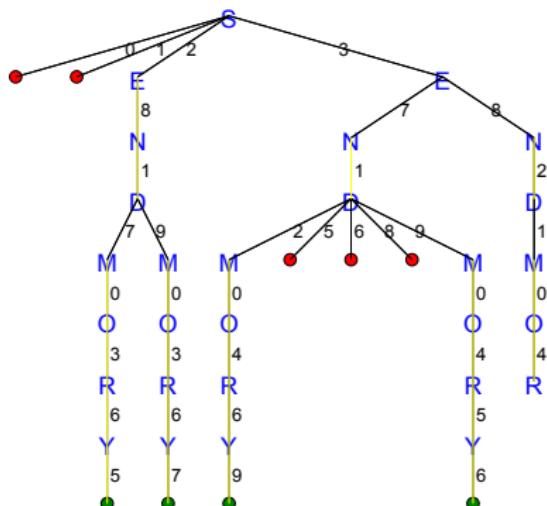
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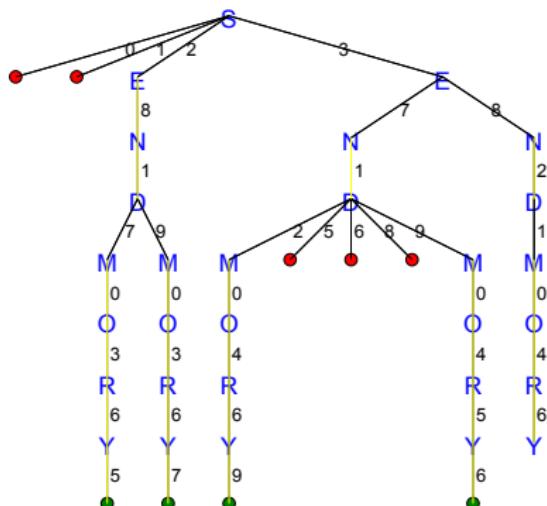
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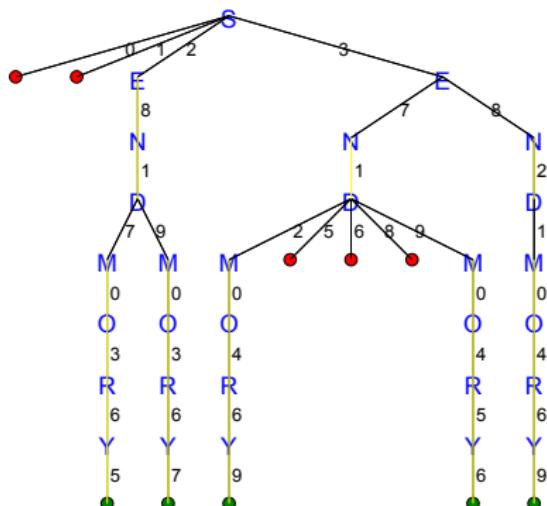
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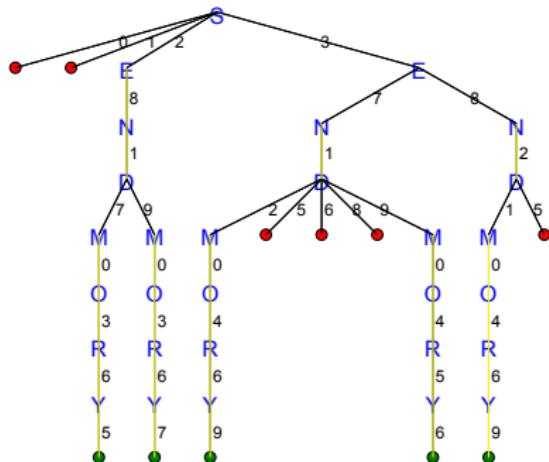
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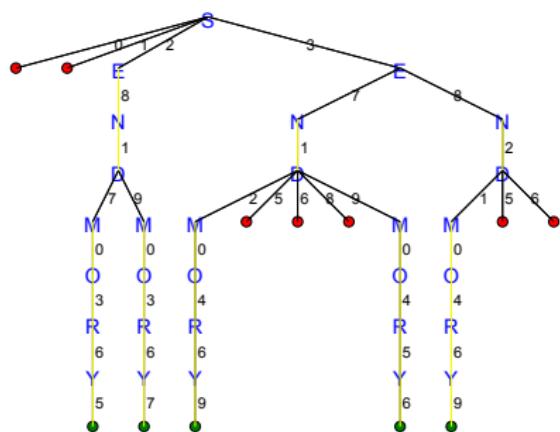
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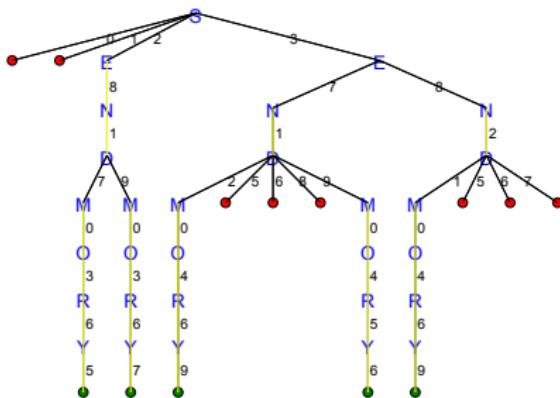


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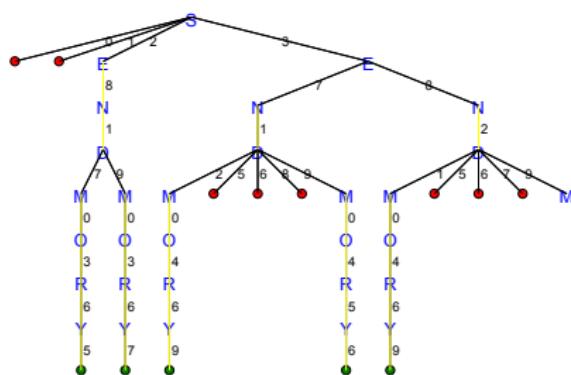
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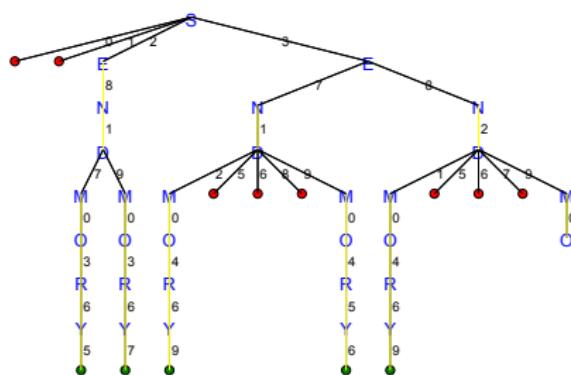


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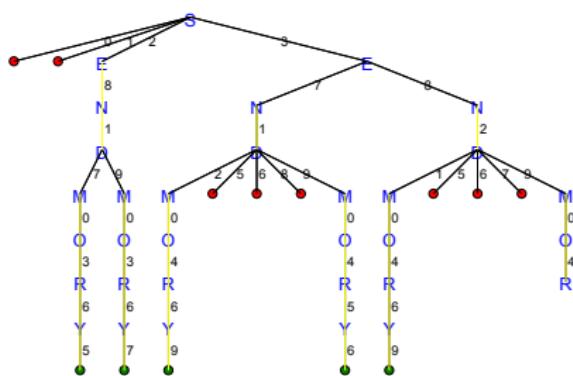


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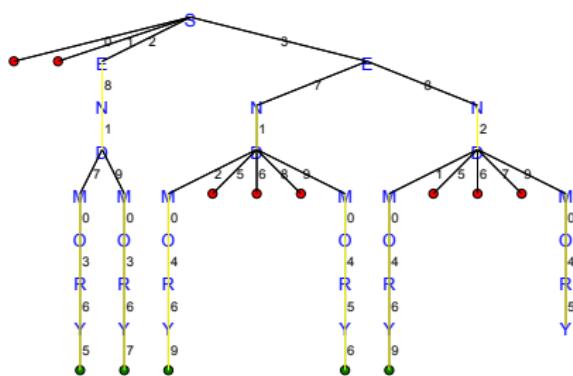


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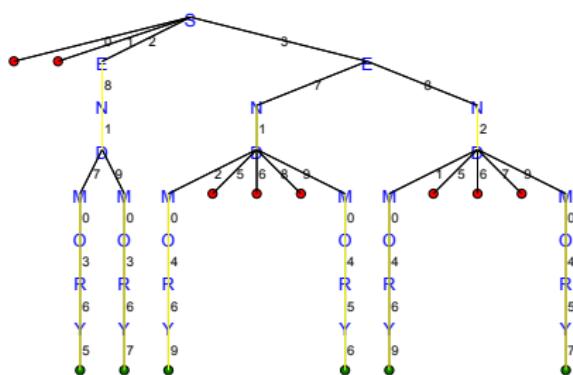
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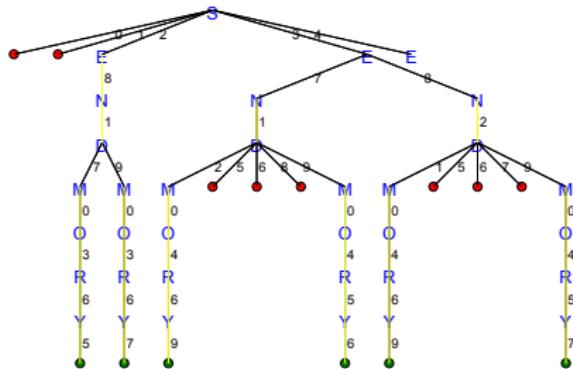
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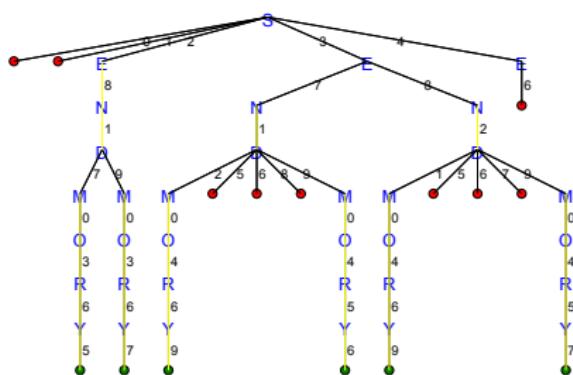
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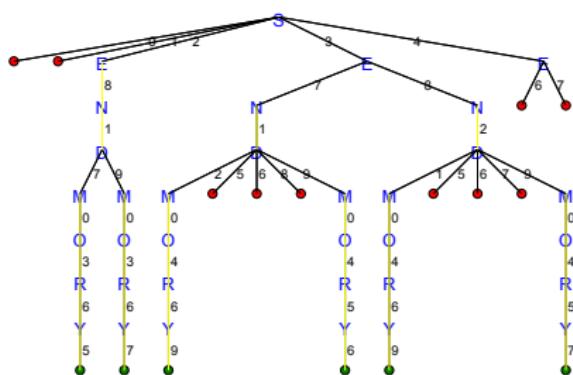
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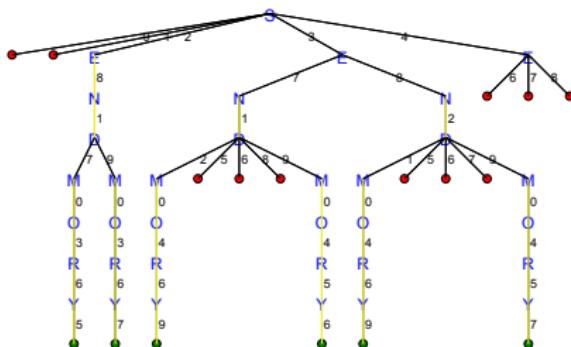


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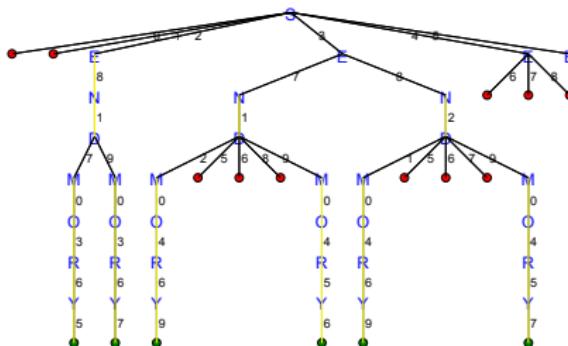
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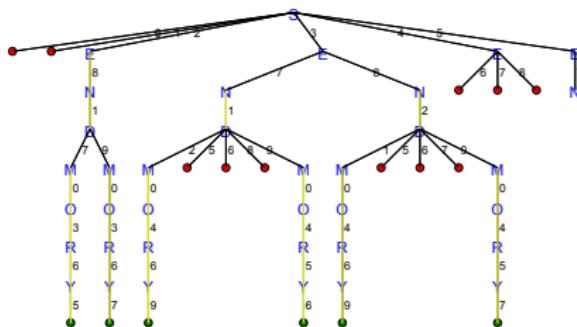
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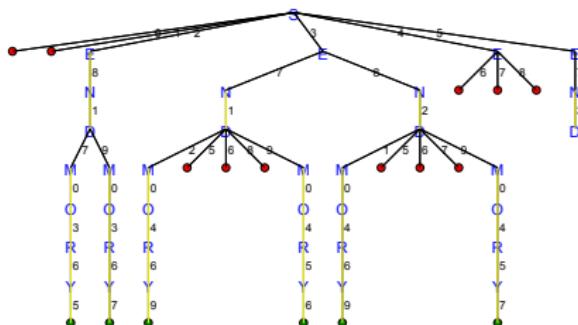
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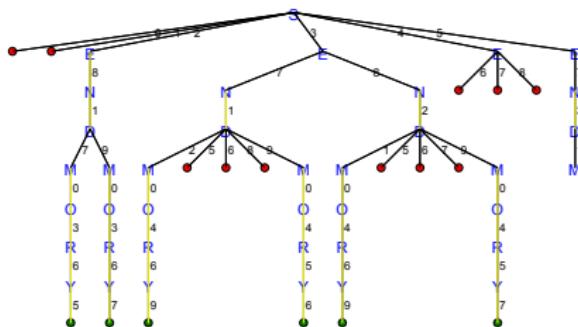
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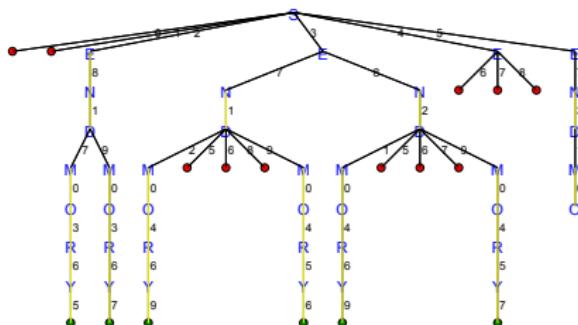


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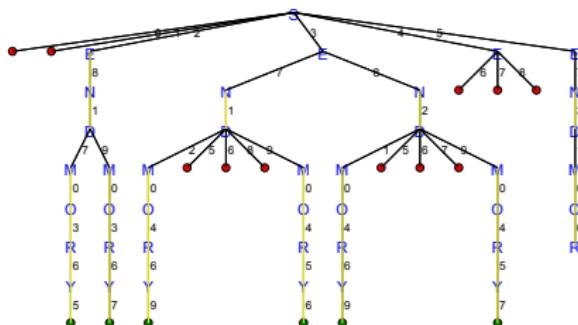
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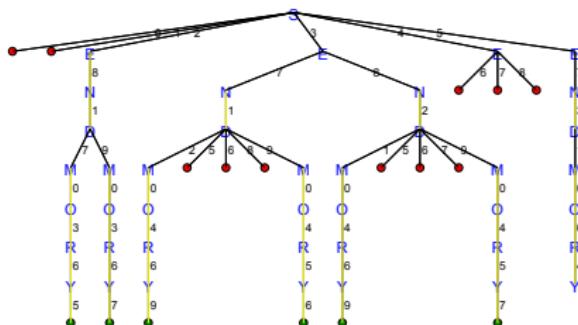
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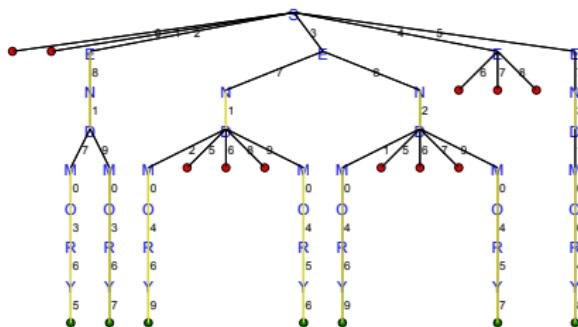


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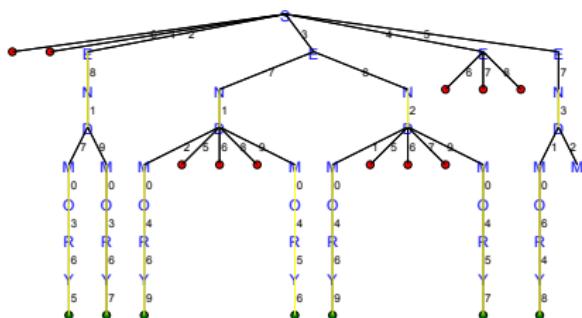
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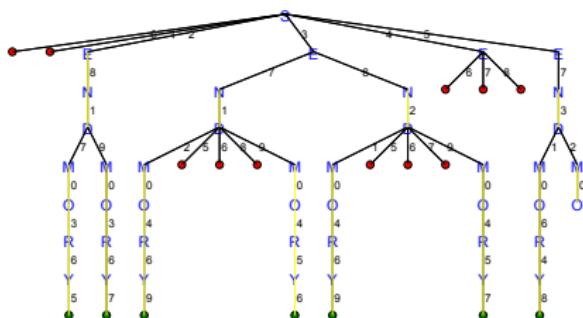
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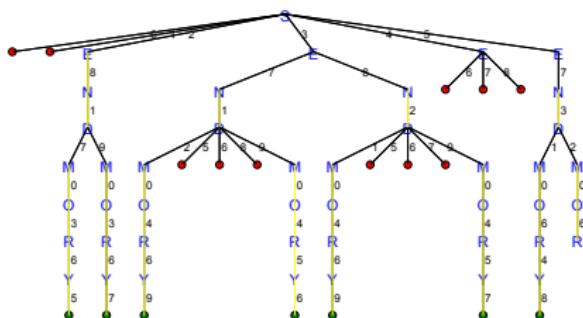


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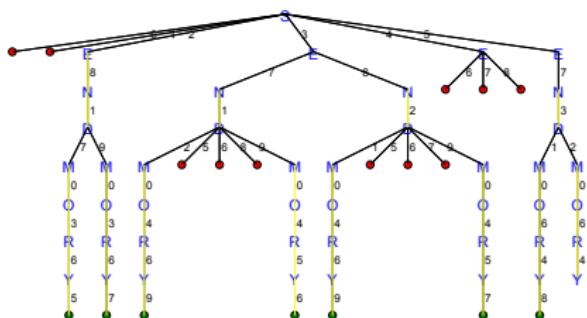
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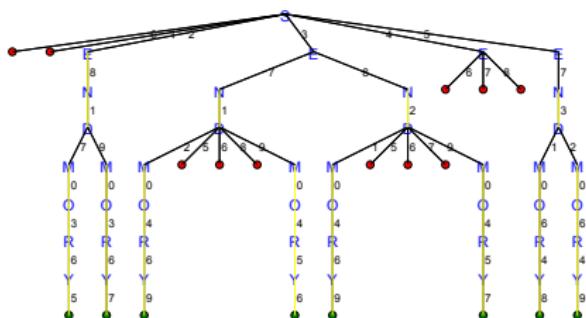
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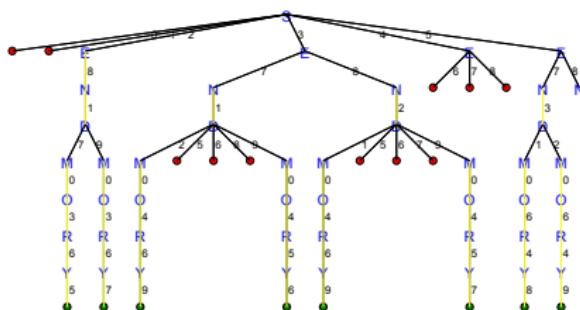
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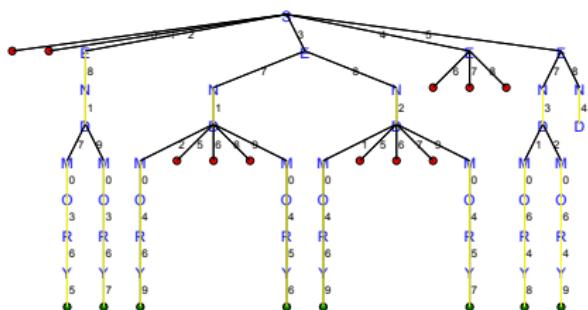
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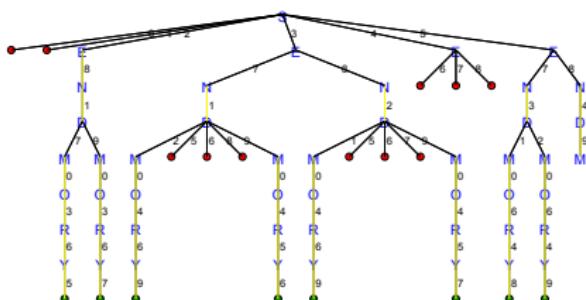
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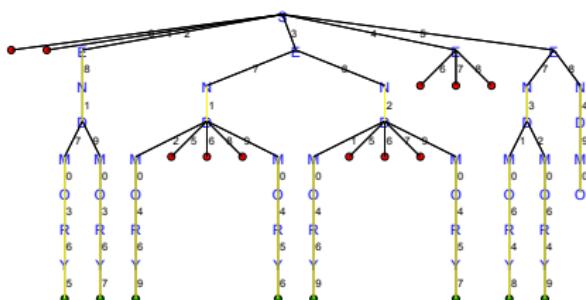
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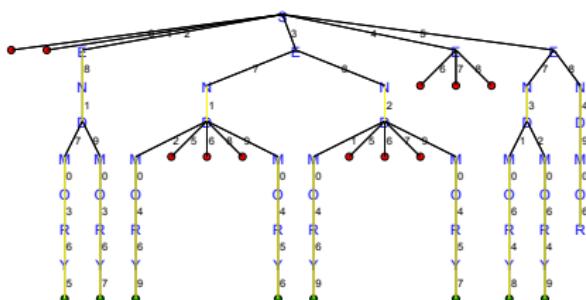


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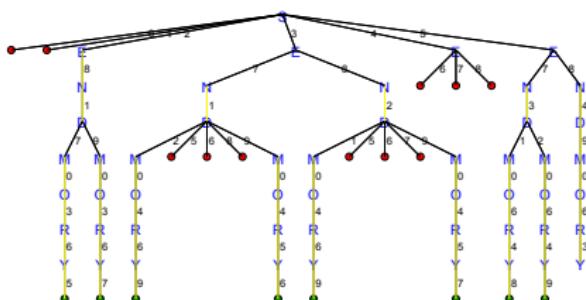
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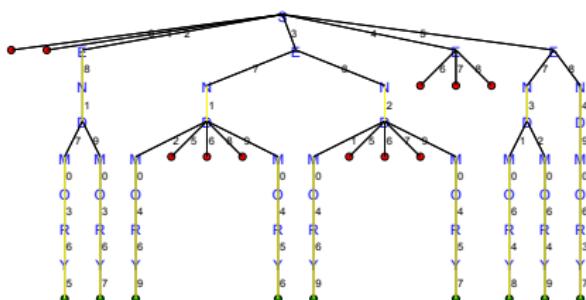
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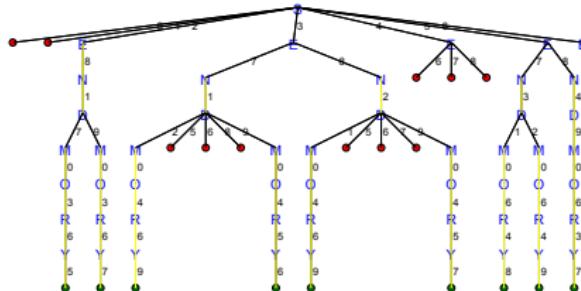


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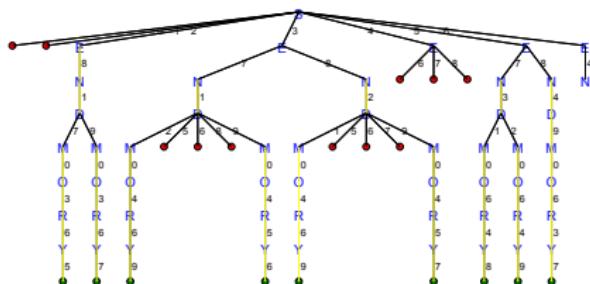
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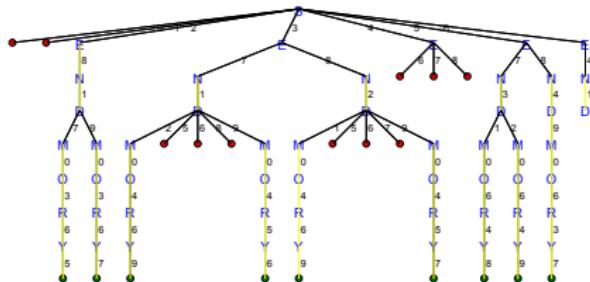
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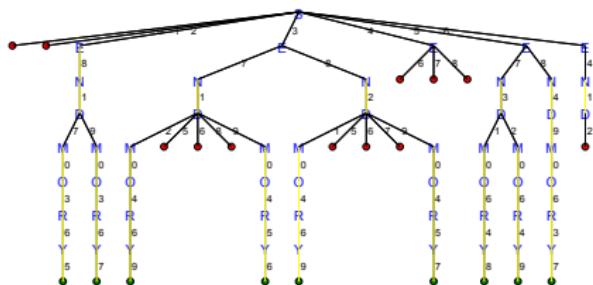
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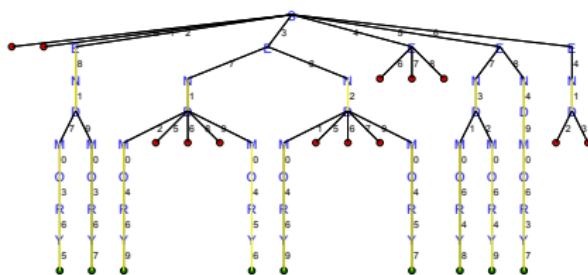


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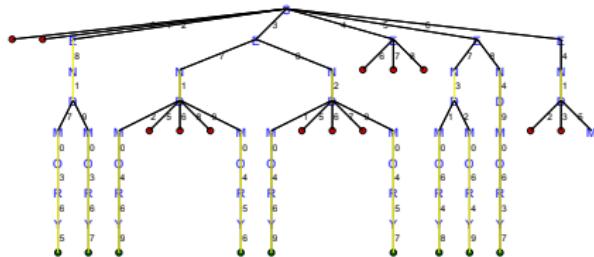


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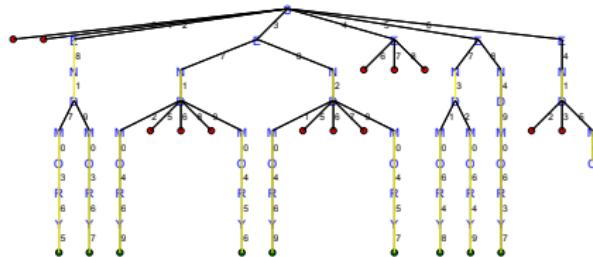


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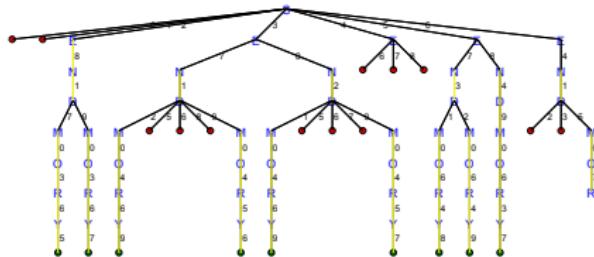


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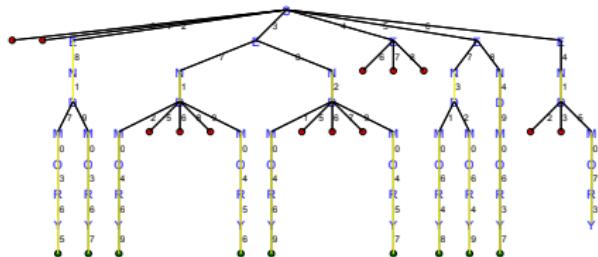


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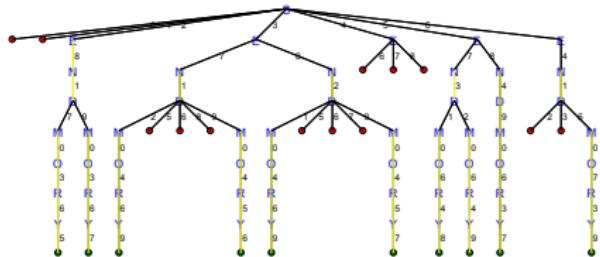


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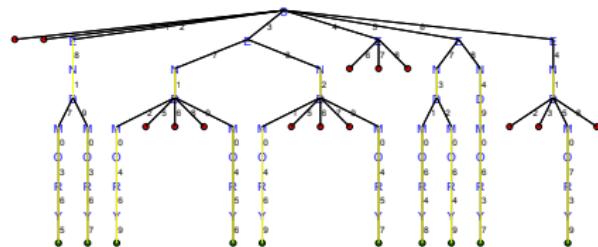


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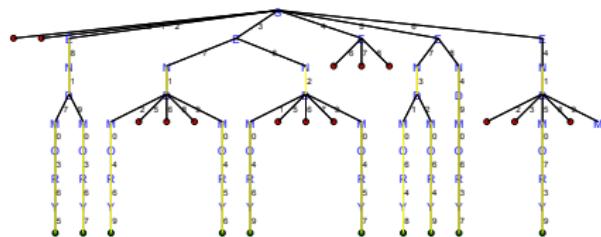


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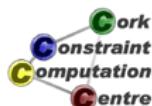


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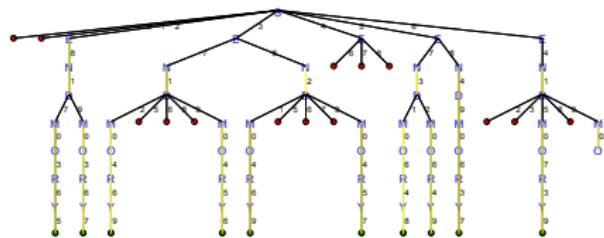


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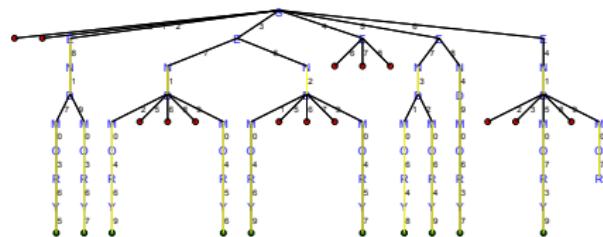
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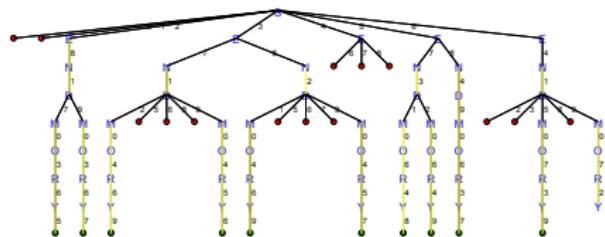
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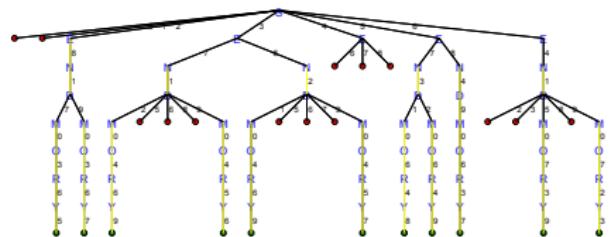
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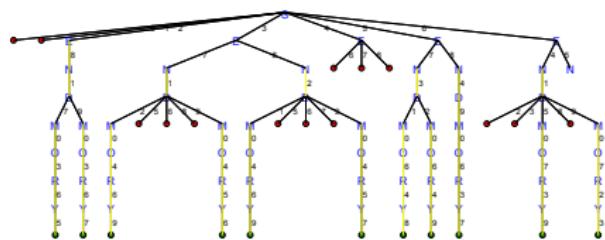


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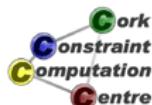


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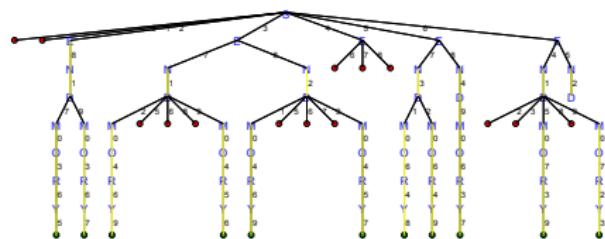


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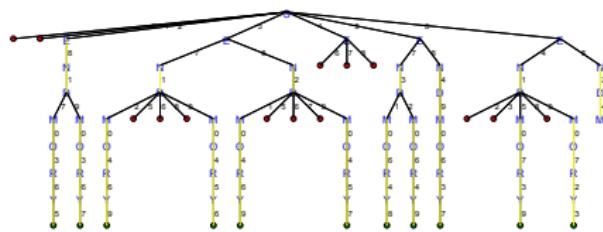
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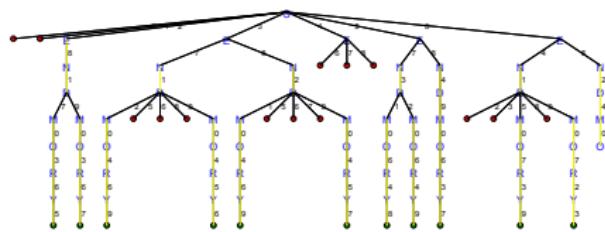


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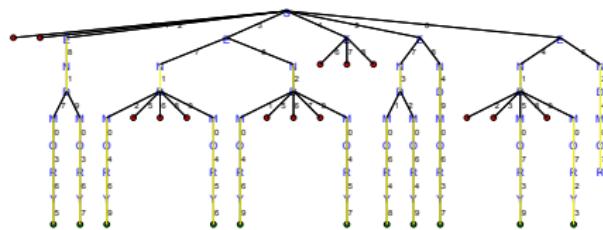


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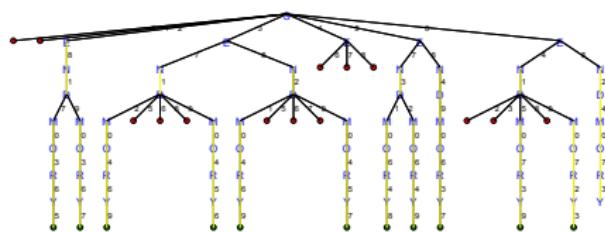


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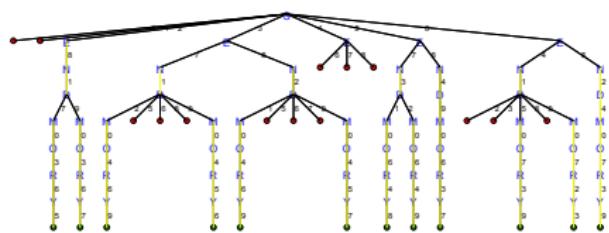
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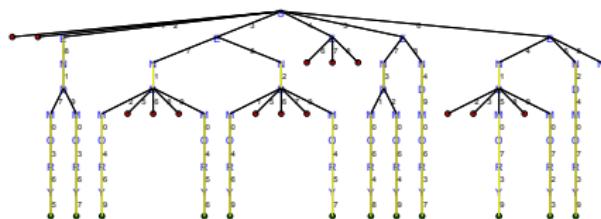
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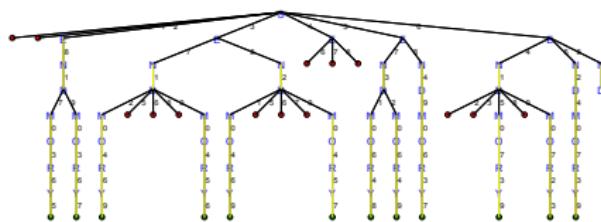
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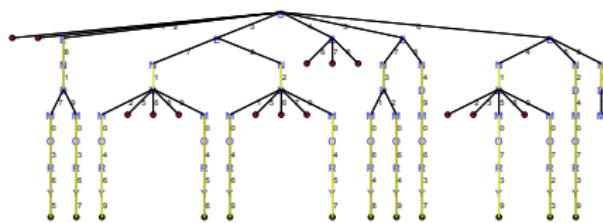


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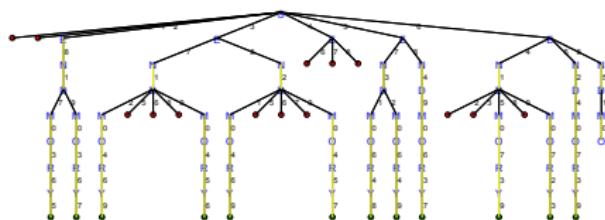


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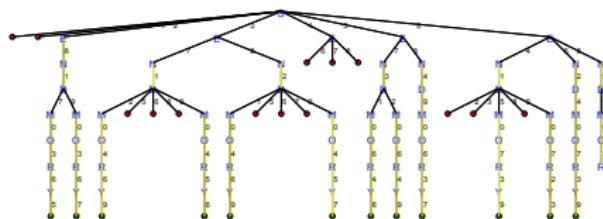


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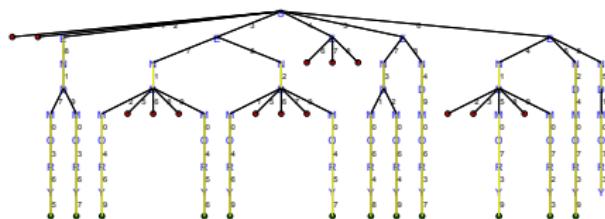


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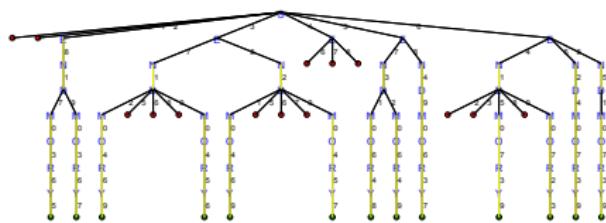


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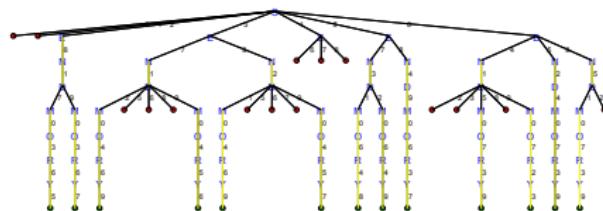


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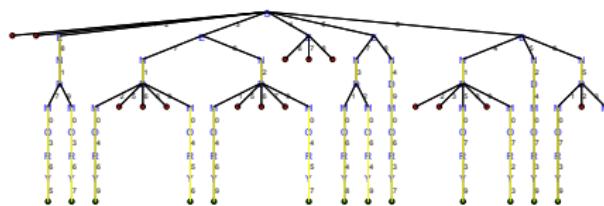


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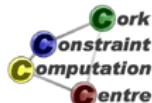


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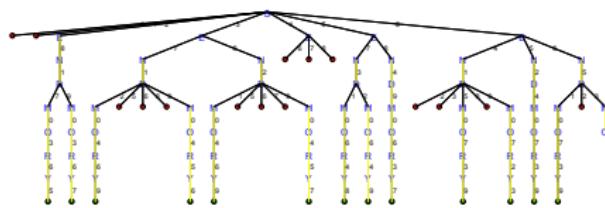


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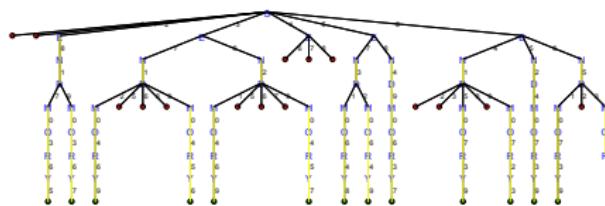
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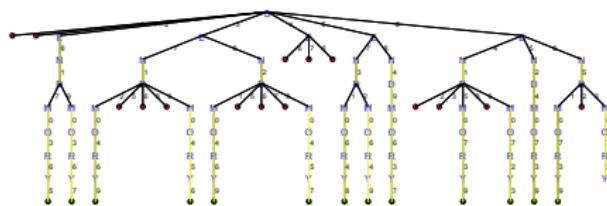


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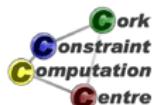
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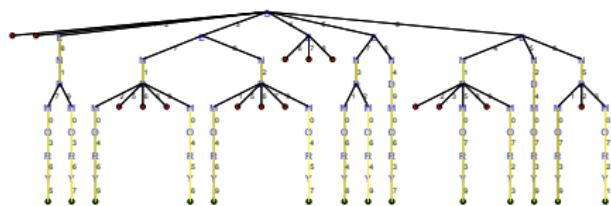
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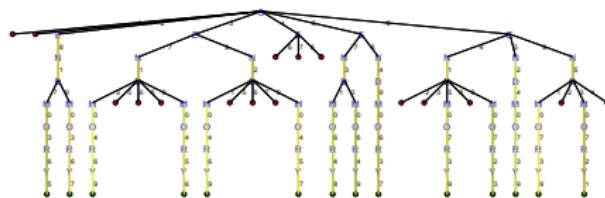


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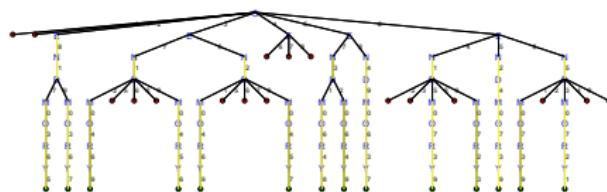


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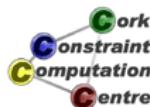


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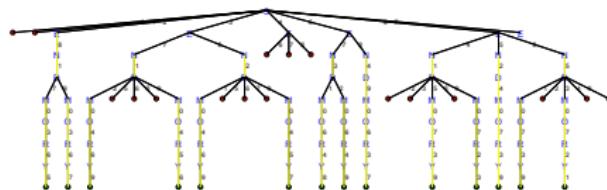


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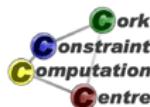
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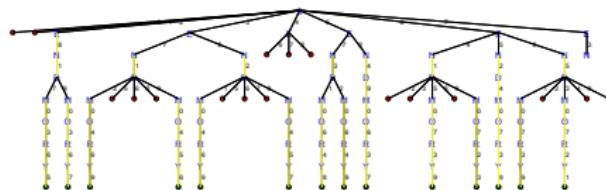
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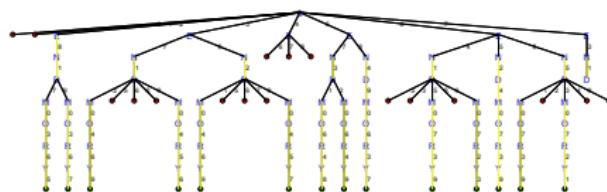


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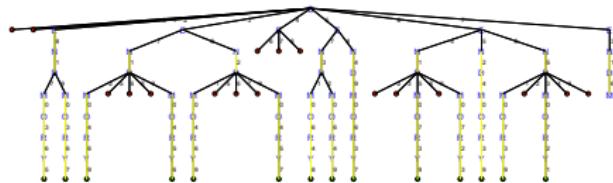
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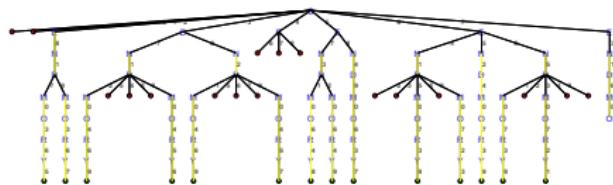


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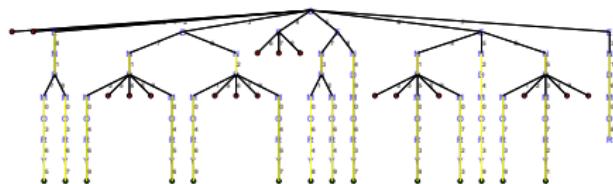


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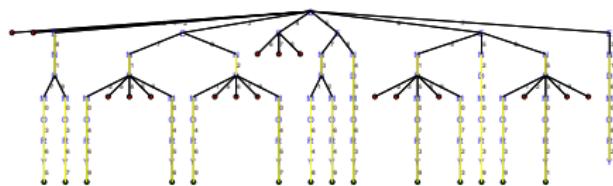


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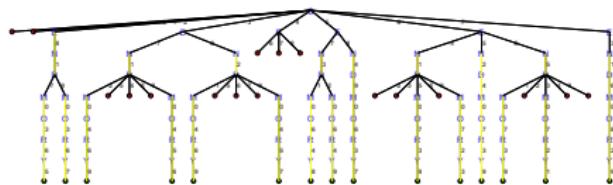
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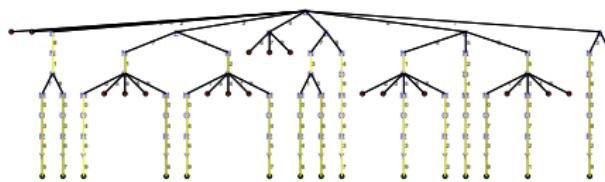
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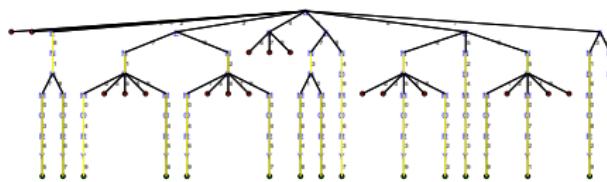


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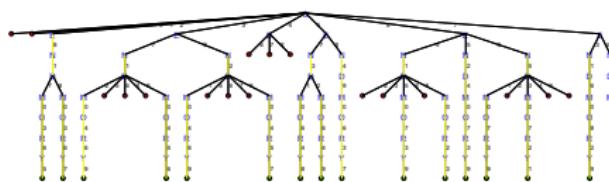


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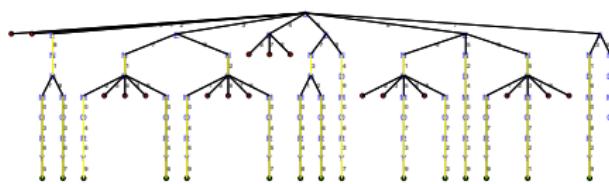
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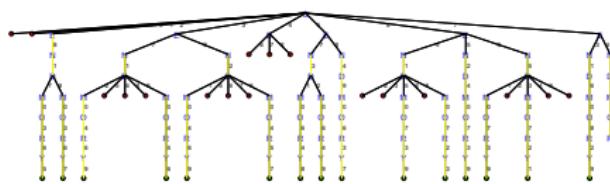


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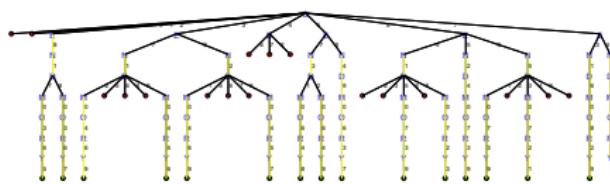


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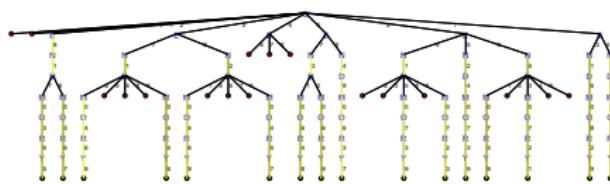


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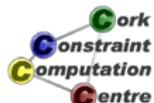
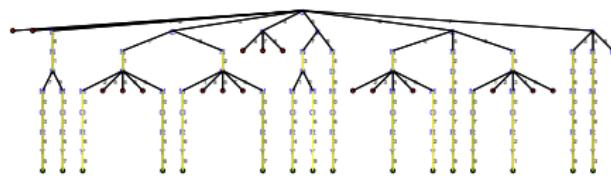


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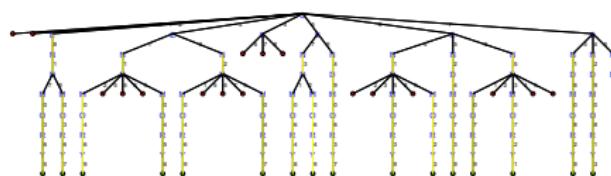


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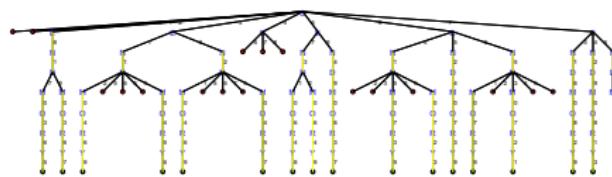


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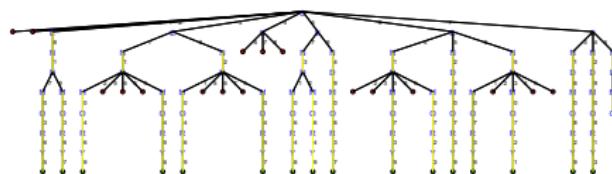
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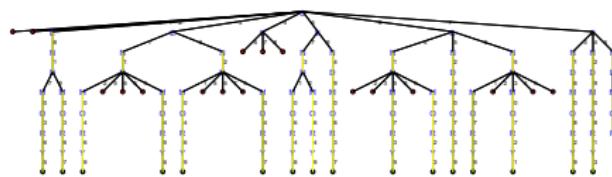
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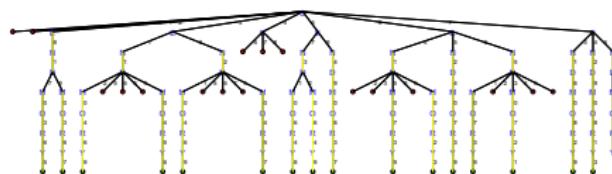


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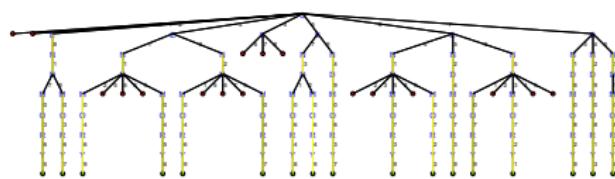


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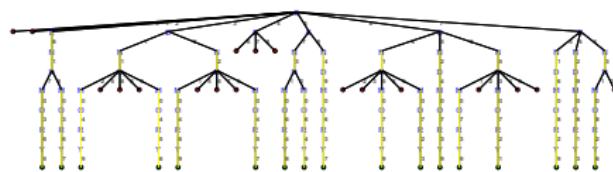
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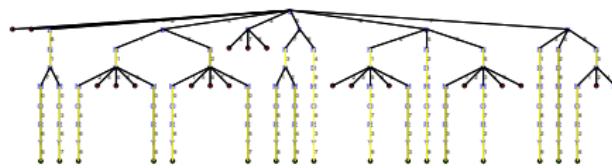
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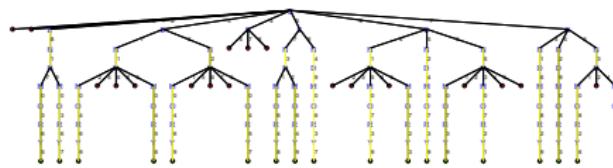


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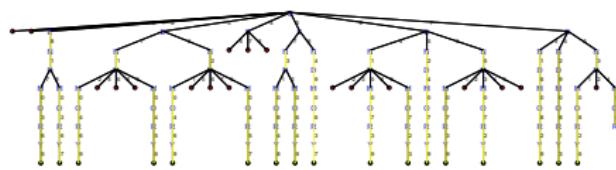


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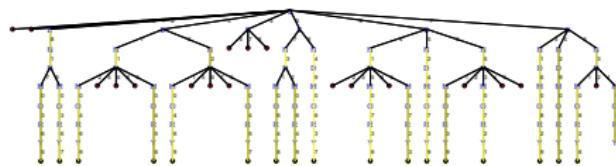
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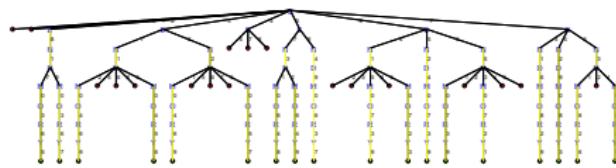
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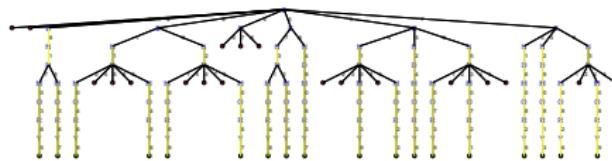
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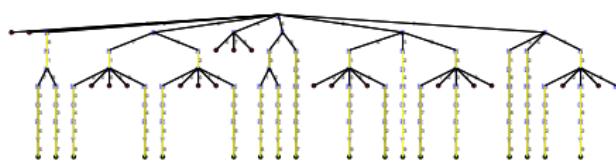


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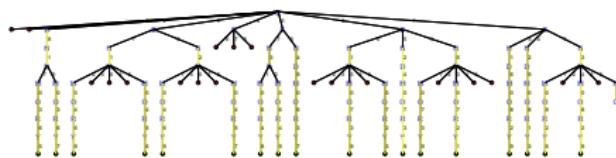


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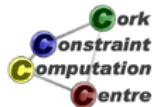


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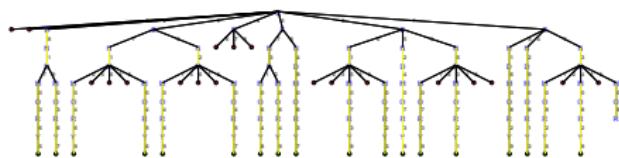


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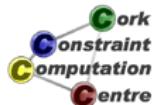


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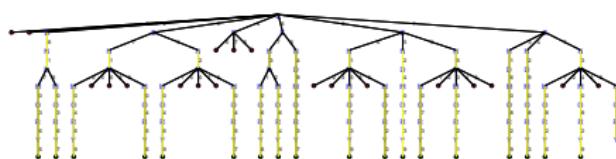


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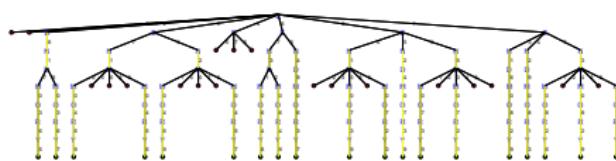
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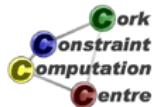


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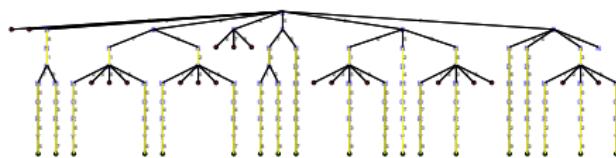


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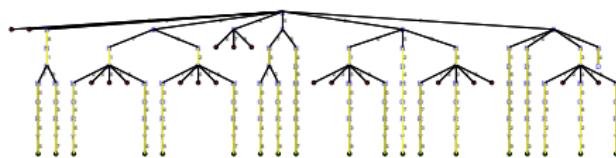


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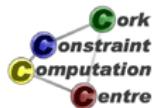


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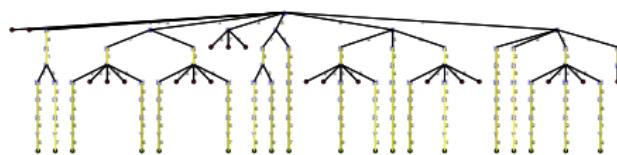


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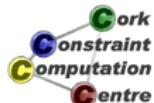
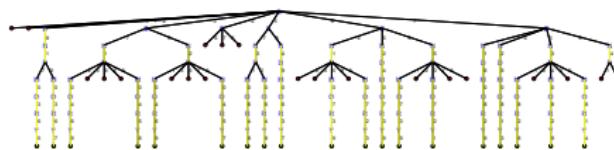
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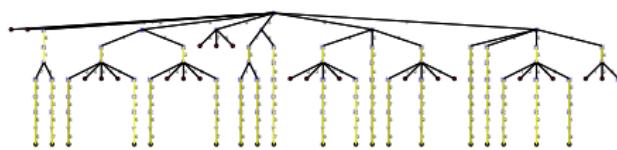
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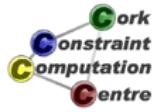
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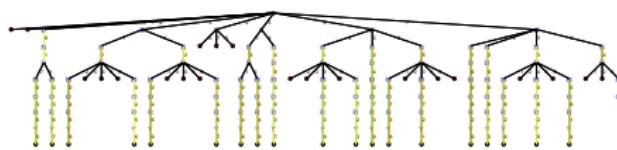
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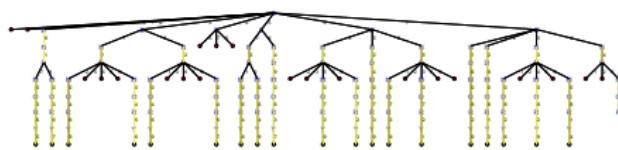
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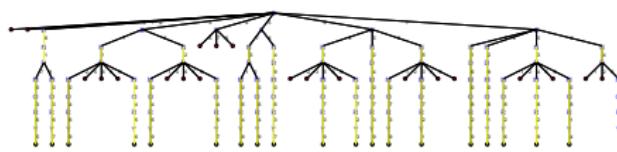
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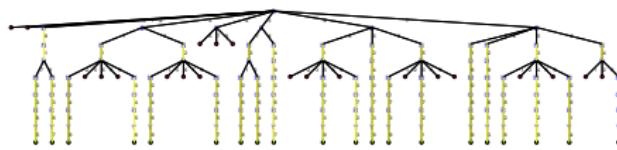


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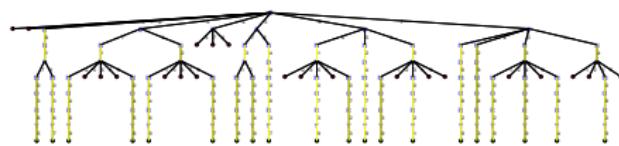
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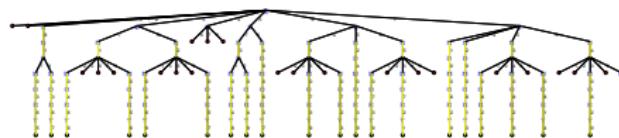
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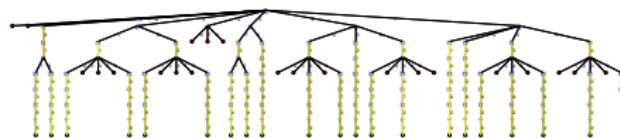


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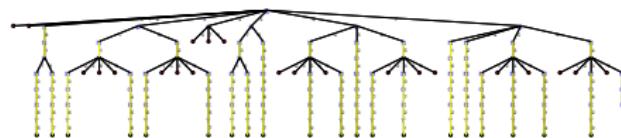
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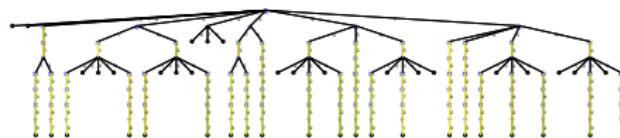
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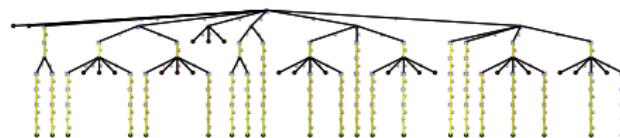
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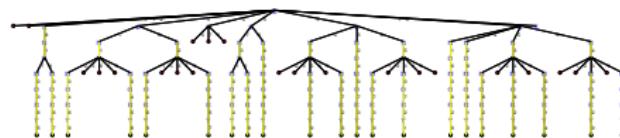
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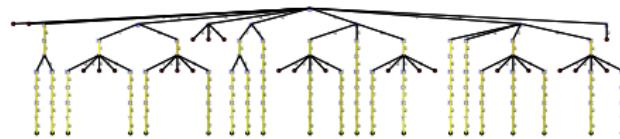


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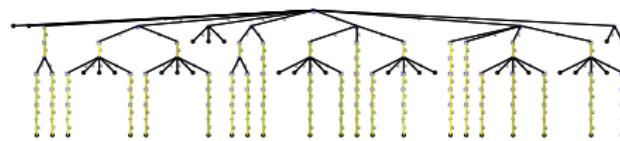


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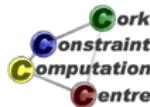


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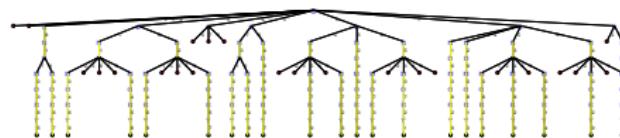


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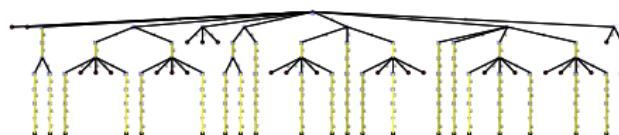


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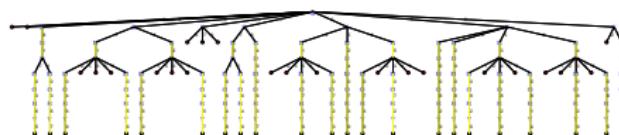


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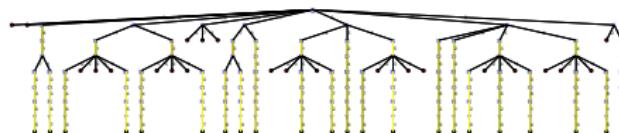


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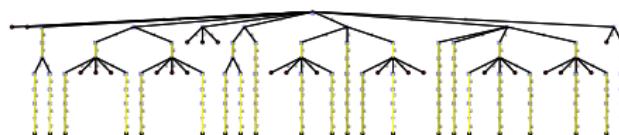


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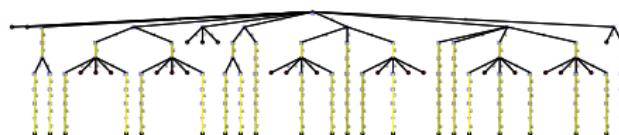
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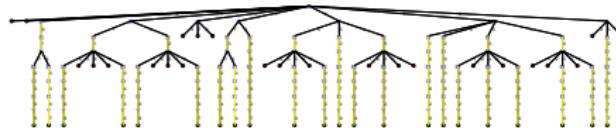


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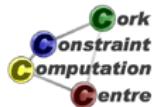
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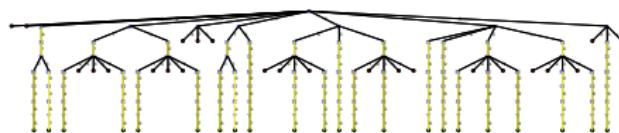
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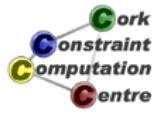


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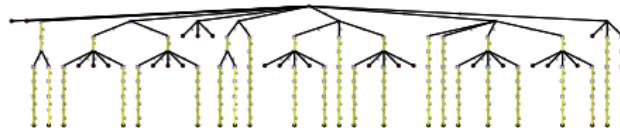


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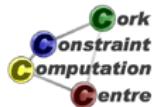


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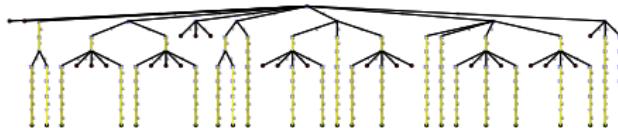


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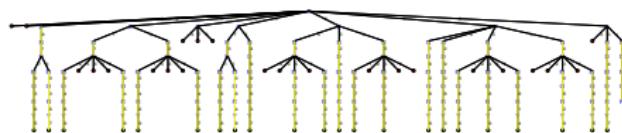
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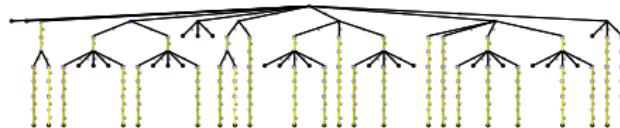
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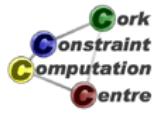


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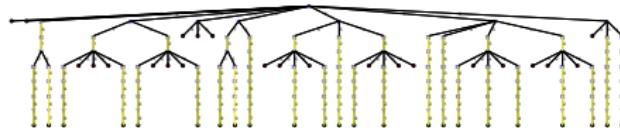


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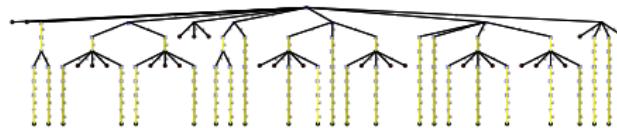
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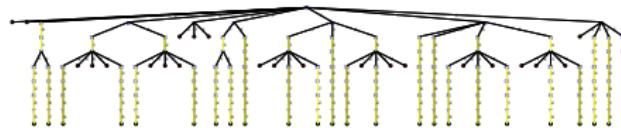


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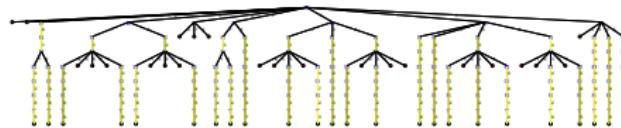


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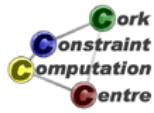


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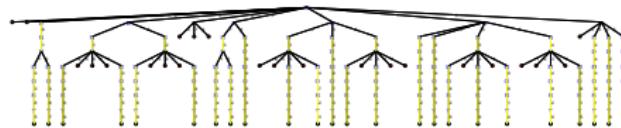


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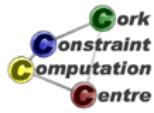


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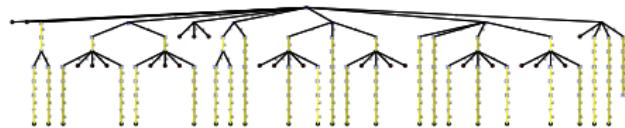


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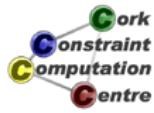


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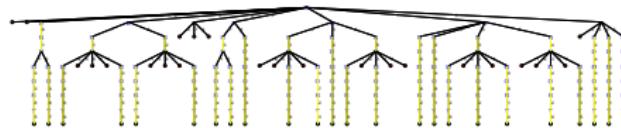


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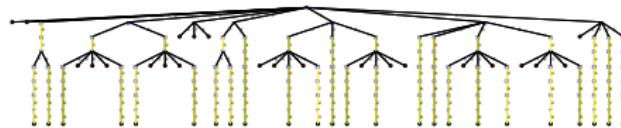
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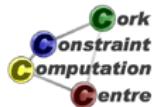


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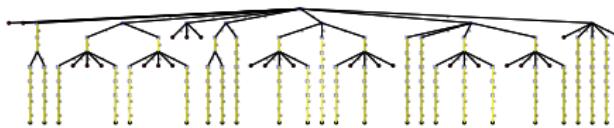


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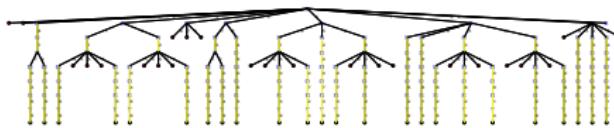


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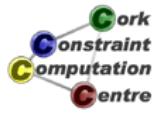


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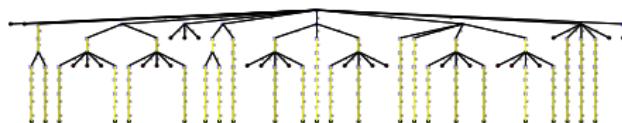


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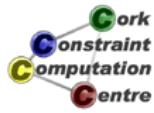


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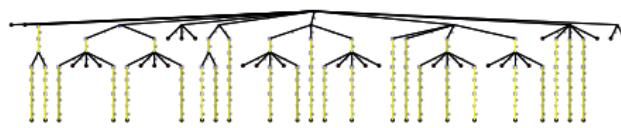


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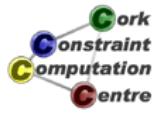
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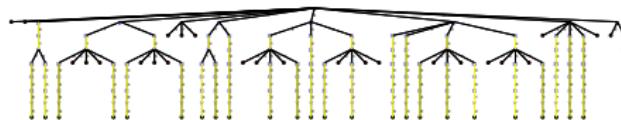
Search Tree: Many Solutions



◀ Back to Start ▶ Skip Animation



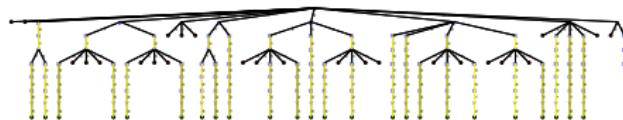
Search Tree: Many Solutions



◀ Back to Start ▶ Skip Animation



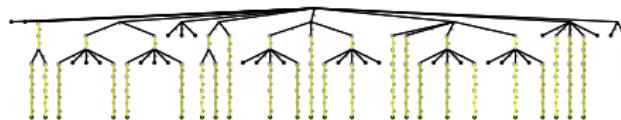
Search Tree: Many Solutions



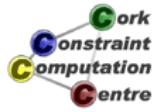
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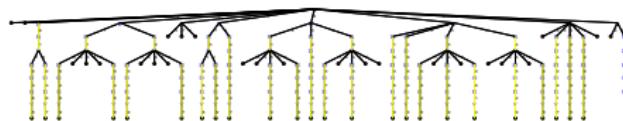
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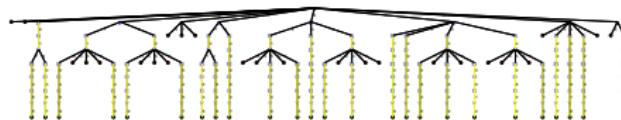
Search Tree: Many Solutions



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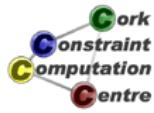


Search Tree: Many Solutions

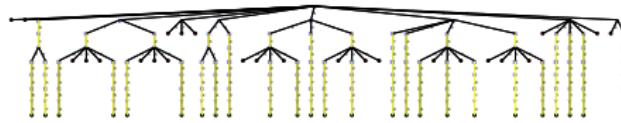


◀ Back to Start

▶ Skip Animation



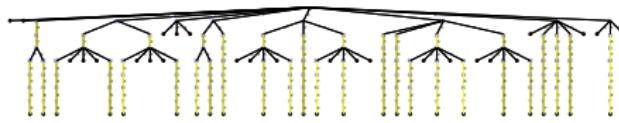
Search Tree: Many Solutions



◀ Back to Start ▶ Skip Animation



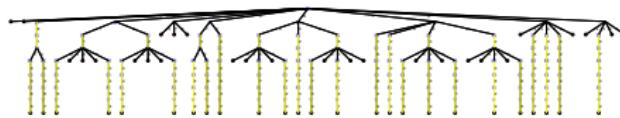
Search Tree: Many Solutions



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▶ Skip Animation

Search Tree: Many Solutions

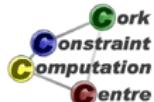


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Note:

- Not just a different model, solving a different problem!
- Often we can choose which problem we want to solve
 - Which constraints to include
 - What to ignore
- In this case not acceptable

◀ Choice of Model



Alternative 2

- Large equality difficult to understand by humans
- Replace with multiple, simpler equations
- Linked by carry variables (0/1)
- Should produce same solutions
- Does it give same propagation?

$$\begin{array}{r} & S & E & N & D \\ + & M & O & R & E \\ \hline & C_5 & C_4 & C_3 & C_2 \\ \hline M & O & N & E & Y \end{array}$$



Carry Variables with Multiple Equations

```
: -module(alternative2), export(senmory/1), lib(ic).  
senmory(L) :-  
    L = [S, E, N, D, M, O, R, Y], L :: 0..9,  
    [C2, C3, C4, C5] :: 0..1,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    M #= C5,  
    S+M+C4 #= 10*C5+O,  
    E+O+C3 #= 10*C4+N,  
    N+R+C2 #= 10*C3+E,  
    D+E #= 10*C2+Y,  
    labeling(L).
```

Carry Variables with Multiple Equations

```
: -module(alternative2), export(senmory/1), lib(ic).
senmory(L) :-same as before
    L = [S, E, N, D, M, O, R, Y], L :: 0..9,
    [C2, C3, C4, C5] :: 0..1,
    alldifferent(L),
    S #\= 0, M #\= 0,
    M #= C5,
    S+M+C4 #= 10*C5+O,
    E+O+C3 #= 10*C4+N,
    N+R+C2 #= 10*C3+E,
    D+E #= 10*C2+Y,
    labeling(L).
```

Carry Variables with Multiple Equations

```
: -module(alternative2), export(sendlmory/1), lib(ic).  
sendlmory(L) :-  
    L = [S, E, N, D, M, O, R, Y], L :: 0..9,  
    [C2, C3, C4, C5] :: 0..1,  $\Rightarrow$  new  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    M #= C5,  
    S+M+C4 #= 10*C5+O,  
    E+O+C3 #= 10*C4+N,  
    N+R+C2 #= 10*C3+E,  
    D+E #= 10*C2+Y,  
    labeling(L).
```

$$\begin{array}{r} & S & E & N & D \\ + & M & O & R & E \\ +C5 & C4 & C3 & C2 \\ \hline M & O & N & E & Y \end{array}$$



Carry Variables with Multiple Equations

```
: -module(alternative2), export(sendlmory/1), lib(ic).  
sendlmory(L) :-  
    L = [S, E, N, D, M, O, R, Y], L :: 0..9,  
    [C2, C3, C4, C5] :: 0..1,  
    alldifferent(L),  
    S #\= 0, M #\= 0,  
    M #= C5,  
    S+M+C4 #= 10*C5+O,  
    E+O+C3 #= 10*C4+N,  
    N+R+C2 #= 10*C3+E,  
    D+E #= 10*C2+Y,  
    labeling(L).
```

With Carry Variables: After Setup

	0	1	2	3	4	5	6	7	8	9
S										■
E										
N										
D										
M		■								
O	■									
R										
Y										

Setup Comparison

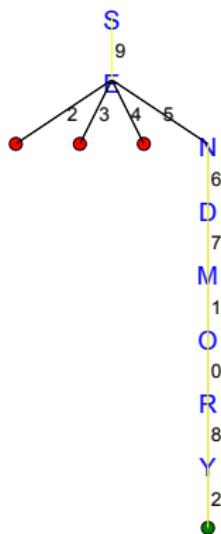
original

	0	1	2	3	4	5	6	7	8	9
S										red
E					white					
N										
D		white								
M		red								
O	red									
R										
Y										

alternative2

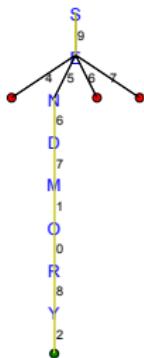
	0	1	2	3	4	5	6	7	8	9
S										red
E										
N										
D										
M		red								
O	red									
R										
Y										

Search Tree: First Solution

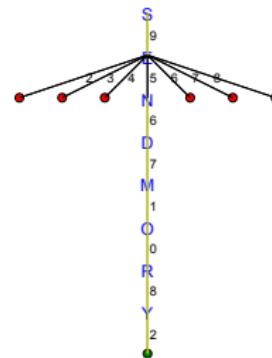


Comparison

Single Equation



Multiple Equations



Observations

- This is solving the original problem
- Search tree slightly bigger
- Caused here by missing interaction of equations
- And repeated variables
- But: Introducing auxiliary variables not always bad!

◀ Choice of Model



More Information

-  Henry Dudeney.
Send+More=Money.
Strand Magazine, Volume 68:pages 97 and 214, July 1924.
-  Henry Dudeney.
Amusements in Mathematics.
Project Gutenberg, 1917.
<http://www.gutenberg.org/etext/16713>.

Outline

6 Alternative Models

7 Exercises

Exercises

- ➊ Does the reasoning for the equality constraints that we have presented remove all inconsistent values? Consider the constraint $Y=2*X$.
- ➋ Why is it important to remove multiple occurrences of the same variable from an equality constraint? Give an example!
- ➌ Solve the puzzle DONALD+GERALD=ROBERT. What is the state of the variables before the search, after the initial constraint propagation?
- ➍ Solve the puzzle $Y*WORRY = DOOOOD$. What is different?
- ➎ (extra credit) How would you design a program that finds new crypt-arithmetic puzzles? What makes a good puzzle?