Centre for
Data Analytics

Insight

Chapter 6: Search Strategies (N-Queens)

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Acknowledgments

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What we want to introduce

- Importance of search strategy, constraints alone are not enough
- Two schools of thought
  - Black-box solver, solver decides by itself
  - Human control over process
- Dynamic variable ordering exploits information from propagation
- Variable and value choice
- Hard to find strategy which works all the time
- Different way of improving stability of search routine

Example Problem

- N-Queens puzzle
- Rather weak constraint propagation
- Many solutions, limited number of symmetries
- Easy to scale problem size

Outline

Problem

Program

Naive Search

Improvements

Problem Definition

8-Queens

Place 8 queens on an 8×8 chessboard so that no queen attacks another. A queen attacks all cells in horizontal, vertical and diagonal direction. Generalizes to boards of size $N \times N$.

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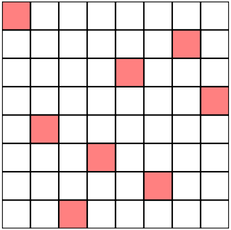

Solution for board size 8×8

Outline

Problem

Program

- Model

Naive Search

Improvements

Basic Model

- Cell based Model
  - A 0/1 variable for each cell to say if it is occupied or not
  - Constraints on rows, columns and diagonals to enforce no-attack
  - N^2 variables, $6N - 2$ constraints
- Column (Row) based Model
  - A 1..N variable for each column, stating position of queen in the column
  - Based on observation that each column must contain exactly one queen
  - N variables, $N^2/2$ binary constraints

Model

$$\text{assign} \quad [X_1, X_2, \ldots X_N]$$

s.t.

$$\forall 1 \leq i \leq N: \quad X_i \in 1..N$$
$$\forall 1 \leq i < j \leq N: \quad X_i \neq X_j$$
$$\forall 1 \leq i < j \leq N: \quad X_i + j \neq X_j + i$$
$$\forall 1 \leq i < j \leq N: \quad X_i + i \neq X_j + j$$

Outline

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Nqueens Models

- ECLiPSe ▸ Show
- MiniZinc ▸ Show
- NumberJack ▸ Show
- CPMpy ▸ Show
- Choco-solver ▸ Show

ECLiPSe N-Queens Model

```
:- lib(lists).
:- lib(ic).
:- lib(ic_search).

top:-
    queens(8,Board),
    search(Board, 0, input_order, indomain, complete).

queens(N, Board) :-
    length(Board, N),
    Board :: 1..N,
    ( fromto(Board, [Q1|Cols], Cols, []) do
        ( foreach(Q2, Cols), param(Q1), count(Dist,1,_) do
            noattack(Q1, Q2, Dist)
        )
    ).

noattack(Q1,Q2,Dist) :-
    Q2 #\= Q1,
    Q2 - Q1 #\= Dist,
    Q1 - Q2 #\= Dist.
```

▸ Continue

MiniZinc N-Queens Model

```
int: n=8;
array[1..n] of var 1..n: queens;
constraint
    forall(i, j in 1..n where i < j) (
         queens[i] != queens[j] /\
         queens[i] + i != queens[j] + j /\
         queens[i] - i != queens[j] - j
    )
 ;
solve :: int_search(
        queens,
        input_order,
        indomain_min)
        satisfy;
```

Continue

NumberJack N-Queens Model

```
from Numberjack import *

def get_model(N):
    queens = VarArray(N, N)
    model = Model(
        AllDiff(queens),
        AllDiff([queens[i] + i for i in range(N)]),
        AllDiff([queens[i] - i for i in range(N)])
    )
    return queens, model

def solve(param):
    queens, model = get_model(param['N'])
    solver = model.load(param['solver'])
    solver.setHeuristic(param['heuristic'], param['value'])
    solver.setVerbosity(param['verbose'])
    solver.setTimeLimit(param['tcutoff'])
    solver.solve()
```

Continue

CPMpy N-Queens Model

```
def nqueens_naive(n=8):
    queens = IntVar(1,n, shape=n)

    model = Model()
    for i in range(n):
        for j in range(i):
            model += [queens[i] != queens[j],
                      queens[i] + i != queens[j] + j,
                      queens[i] - i != queens[j] - j,
                      ]
```

▸ Continue

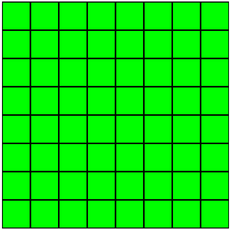
Choco-solver N-Queens Program

```
int n = 8;
Model model = new Model(n + "-queens problem");
IntVar[] vars = new IntVar[n];
for(int q = 0; q < n; q++){
    vars[q] = model.intVar("Q_"+q, 1, n);
}
for(int i  = 0; i < n-1; i++){
    for(int j = i + 1; j < n; j++){
        model.arithm(vars[i], "!=",vars[j]).post();
        model.arithm(vars[i], "!=", vars[j], "-", j - i).post();
        model.arithm(vars[i], "!=", vars[j], "+", j - i).post();
    }
}
Solution solution = model.getSolver().findSolution();
if(solution != null){
    System.out.println(solution.toString());
}
```

Continue

Default Strategy

1



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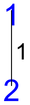

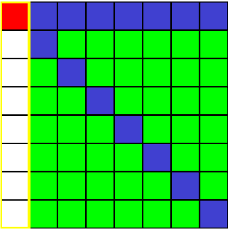

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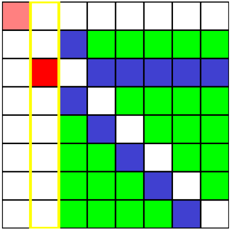

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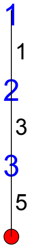

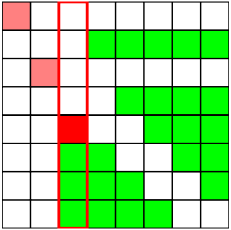

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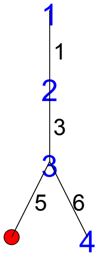

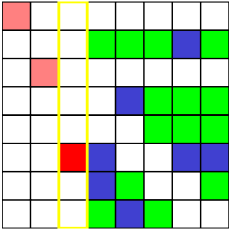

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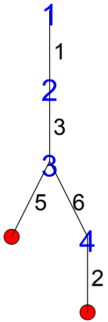

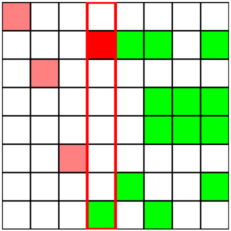

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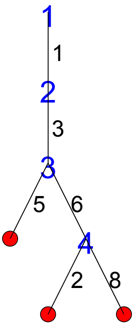

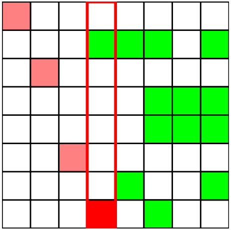

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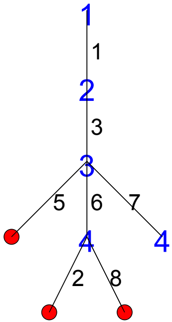

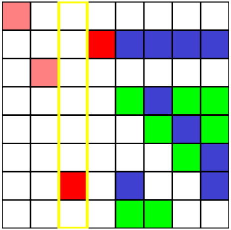

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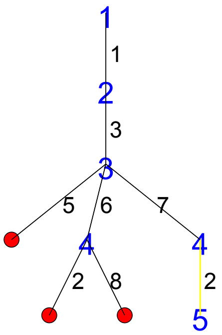

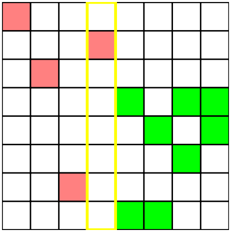

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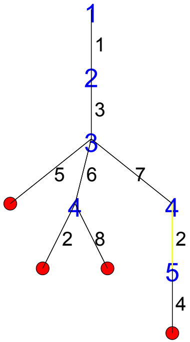

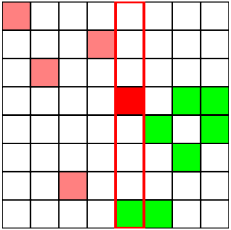

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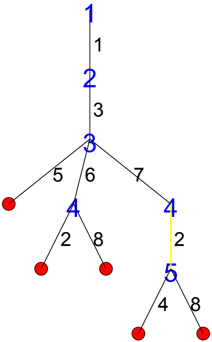

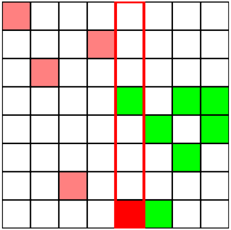

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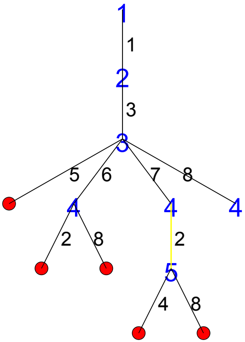

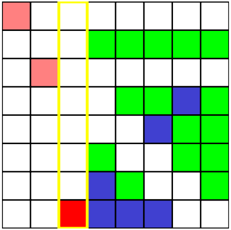

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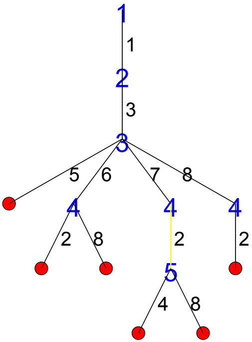

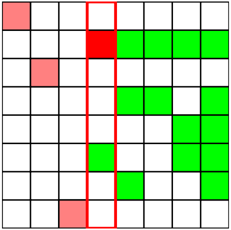

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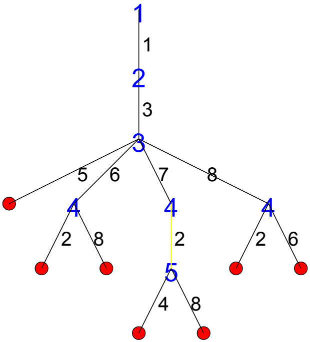

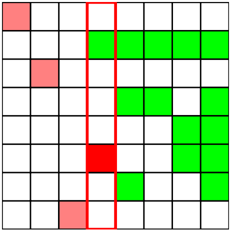

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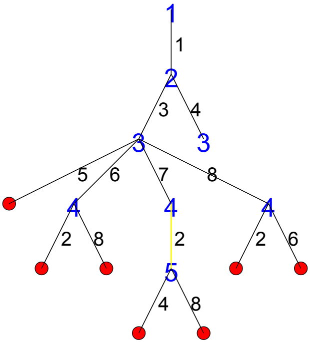

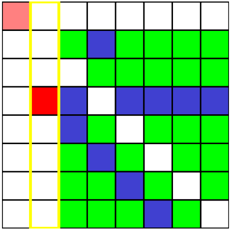

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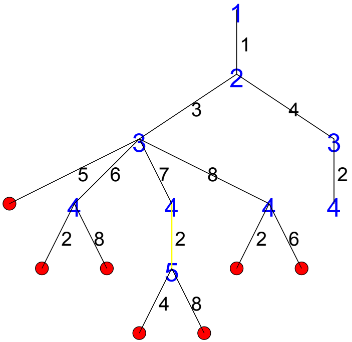

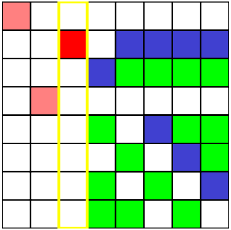

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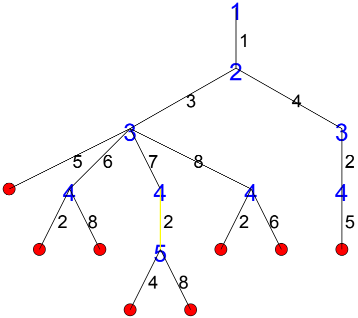

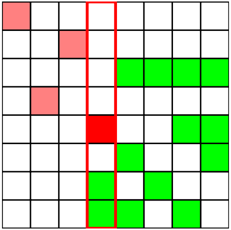

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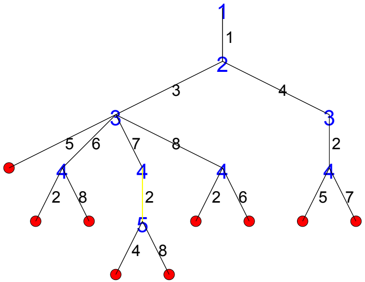

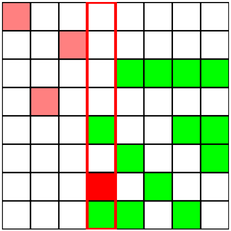

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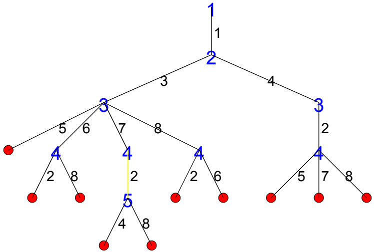

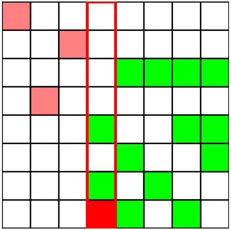

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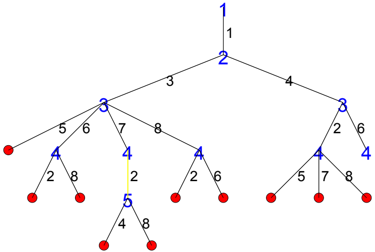

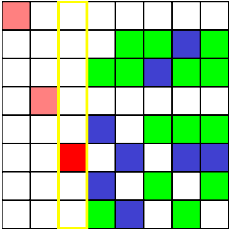

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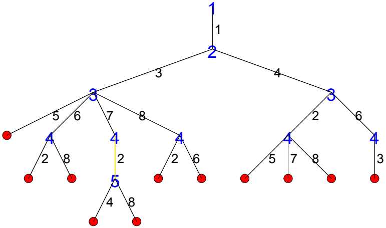

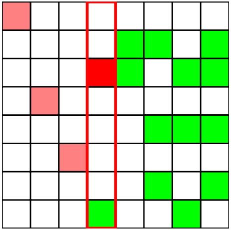

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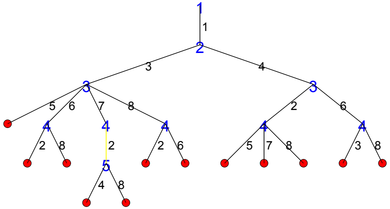

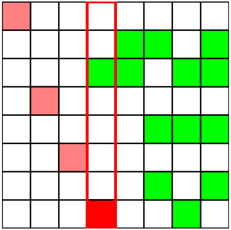

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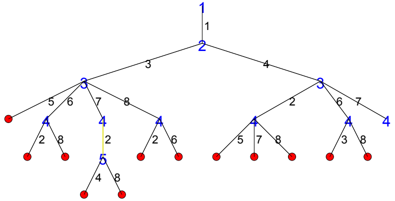

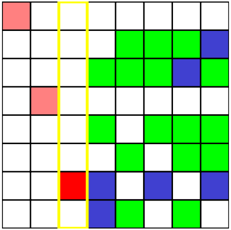

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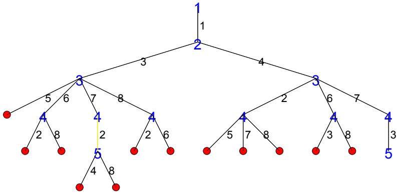

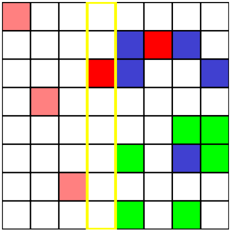

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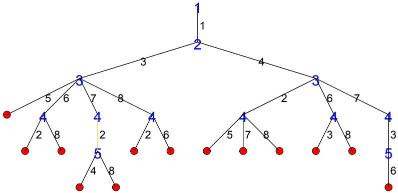

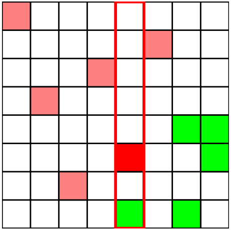

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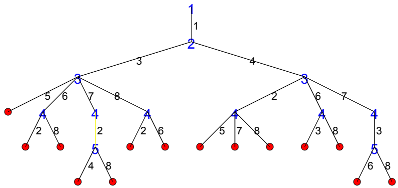

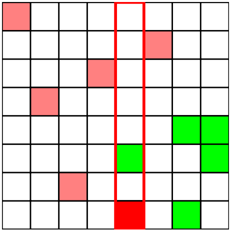

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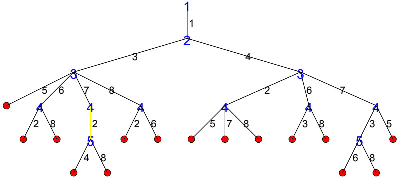

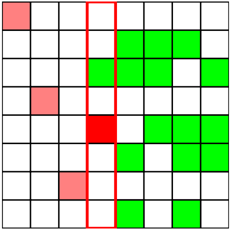

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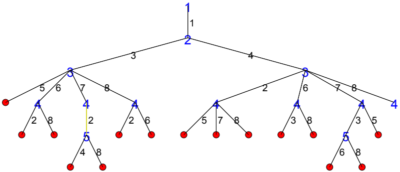

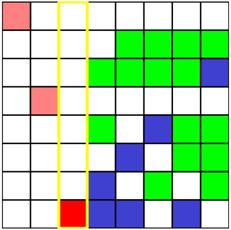

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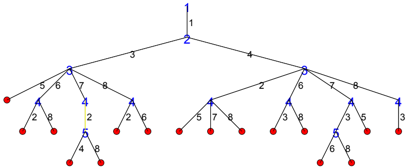
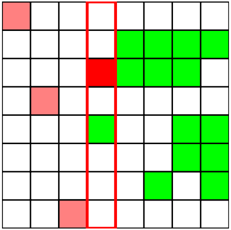
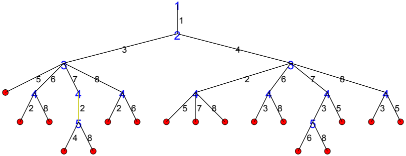
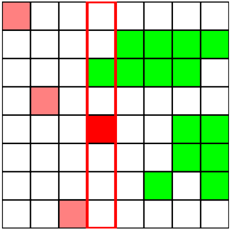




◂ Back to Start ⏩ Skip Animation

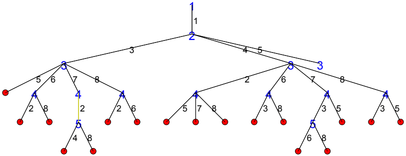
Default Strategy

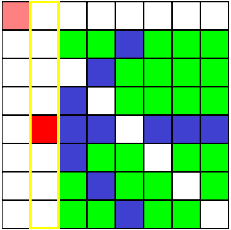




Back to Start
Skip Animation

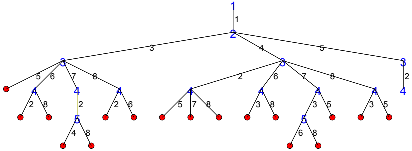
Default Strategy

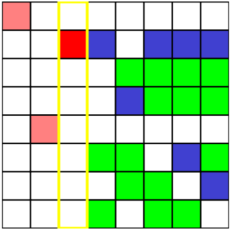




◂ Back to Start ⏩ Skip Animation

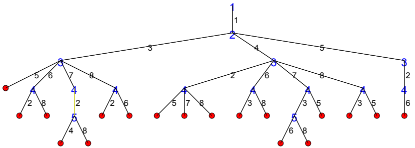
Default Strategy

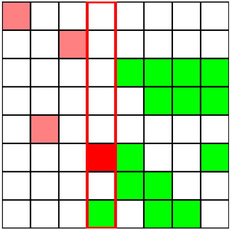




Back to Start Skip Animation

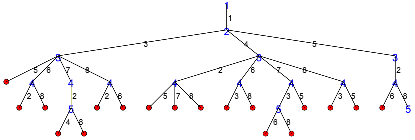
Default Strategy

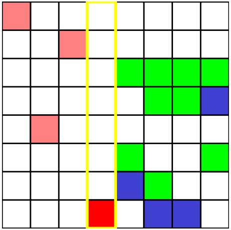




◂ Back to Start ⏩ Skip Animation

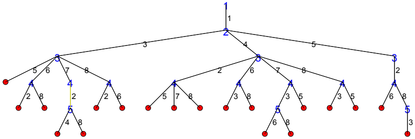
Default Strategy

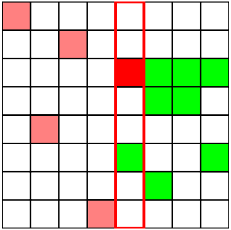




Back to Start
Skip Animation

Default Strategy





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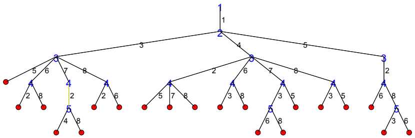
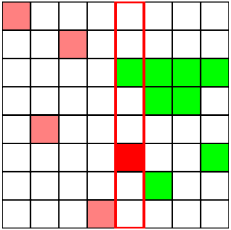
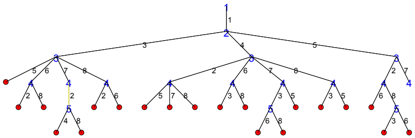
Default Strategy

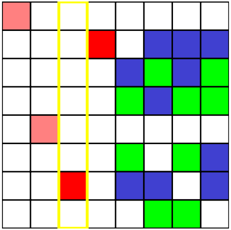




Back to Start Skip Animation

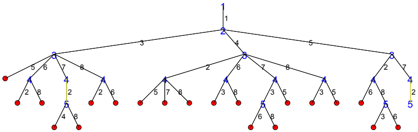
Default Strategy

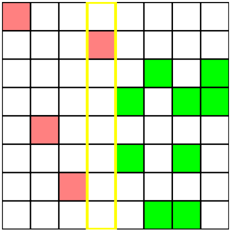




◂ Back to Start ⏩ Skip Animation

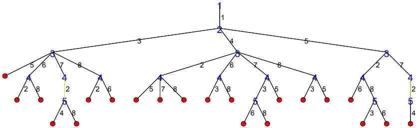
Default Strategy

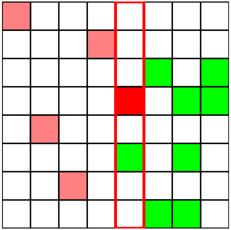




◂ Back to Start ⏩ Skip Animation

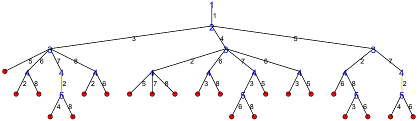
Default Strategy

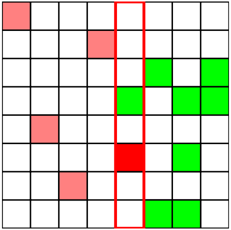




◂ Back to Start ▸▸ Skip Animation

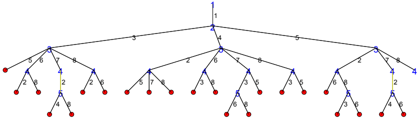
Default Strategy

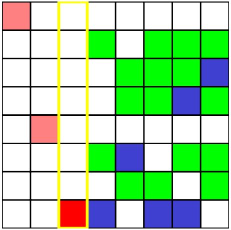




Back to Start
Skip Animation

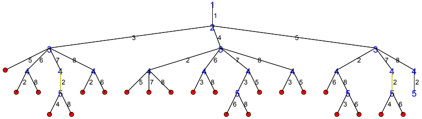
Default Strategy

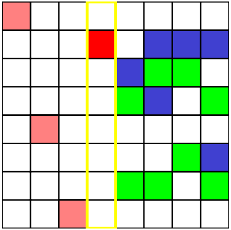




◂ Back to Start ⏩ Skip Animation

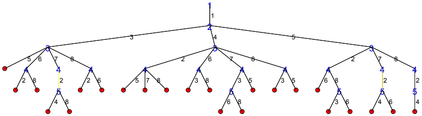
Default Strategy

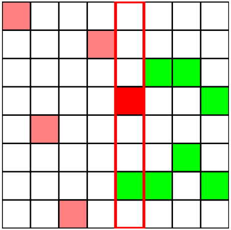




Back to Start
Skip Animation

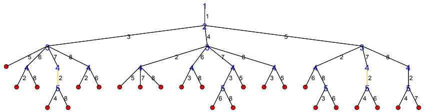
Default Strategy

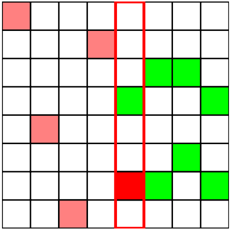




◂ Back to Start ⏩ Skip Animation

Default Strategy

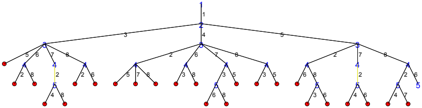


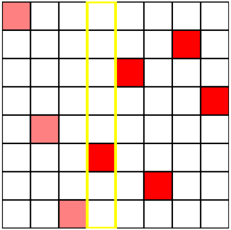


Back to Start

Skip Animation

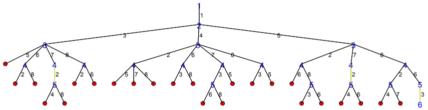
Default Strategy

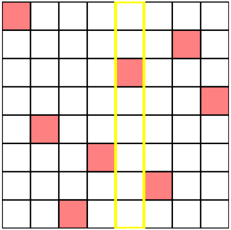




Back to Start Skip Animation

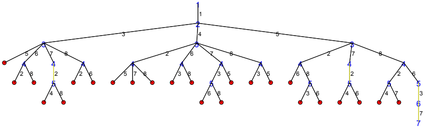
Default Strategy

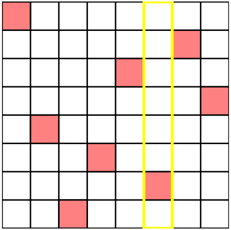




◂ Back to Start ⏩ Skip Animation

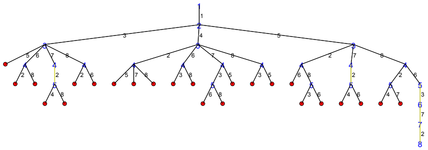
Default Strategy

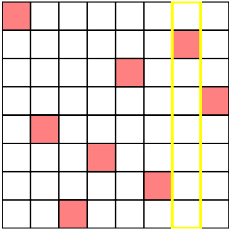




Back to Start   Skip Animation

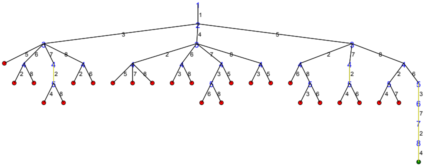
Default Strategy

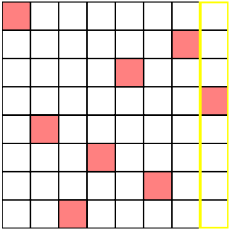




Back to Start
Skip Animation

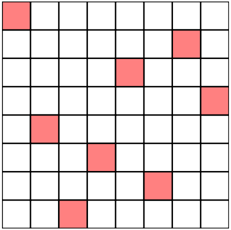
Default Strategy

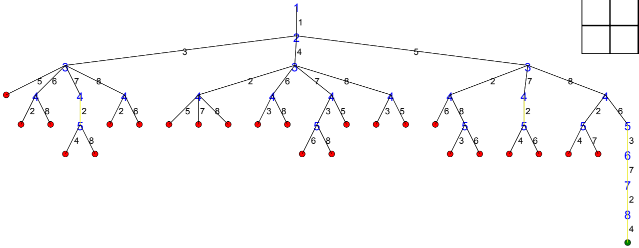




Back to Start

First Solution



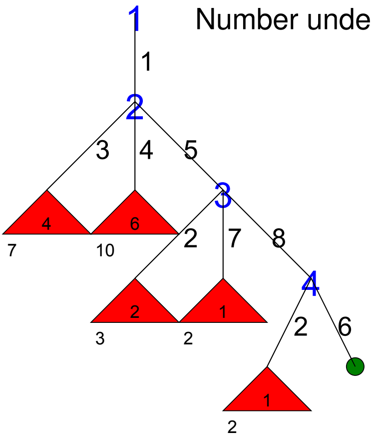


Observations

- Even for small problem size, tree can become large
- Not interested in all details
- Ignore all automatically fixed variables
- For more compact representation abstract failed sub-trees

Compact Representation

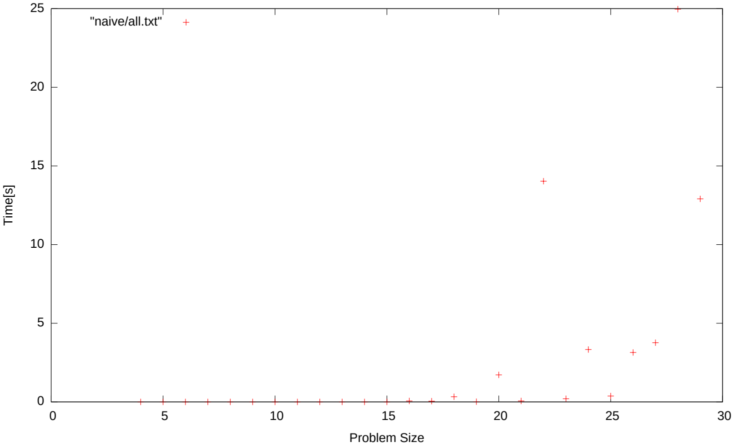
Number inside triangle: Number of choices
Number under triangle: Number of failures



Exploring other board sizes

- How stable is the model?
- Try all sizes from 4 to 100
- Timeout of 100 seconds

Naive Stategy, Problem Sizes 4-100



Observations

- Time very reasonable up to size 20
- Sizes 20-30 times very variable
- Not just linked to problem size
- No size greater than 30 solved within timeout

Outline

Problem

Program

Naive Search

Improvements

- Dynamic Variable Choice
- Weighted Degree
- Improved Heuristics
- Making Search More Stable

Possible Improvements

- Better constraint reasoning
  - Remodelling problem with 3 `alldifferent` constraints
  - Global reasoning as described before
- Better control of search
  - Static vs. dynamic variable ordering
  - Better value choice
  - Not using complete depth-first chronological backtracking

Static vs. Dynamic Variable Ordering

- Heuristic Static Ordering
  - Sort variables before search based on heuristic
  - Most important decisions
  - Smallest initial domain
- Dynamic variable ordering
  - Use information from constraint propagation
  - Different orders in different parts of search tree
  - Use all information available

First Fail strategy

- Dynamic variable ordering
- At each step, select variable with smallest domain
- Idea: If there is a solution, better chance of finding it
- Idea: If there is no solution, smaller number of alternatives
- Needs tie-breaking method

Search Stategy Choices

- Minizinc ▸ Show
- Choco-solver ▸ Show

Modified MiniZinc Program

```
int: n=8;
array[1..n] of var 1..n: queens;
constraint
    forall(i, j in 1..n where i < j) (
         queens[i] != queens[j] /\
         queens[i] + i != queens[j] + j /\
         queens[i] - i != queens[j] - j
    )
 ;
solve :: int_search(
        queens,
        first_fail,
        indomain_min)
        satisfy;
```

Variable Choice (MiniZinc)

- Determines the order in which variables are assigned
- `input_order` assign variables in static order given
- `smallest` assign variable with smallest value in domain first
- `first_fail` select variable with smallest domain first
- `dom_w_deg` consider ratio of domain size and failure count
- Others, including programmed selection for specific solvers

Value Choice (MiniZinc)

- Determines the order in which values are tested for selected variables
- `indomain_min` Start with smallest value, on backtracking try next larger value
- `indomain_median` Start with value closest to middle of domain
- `indomain_random` Choose values in random order
- `indomain_split` Split domain into two intervals

Continue

Modified Choco-solver Model

```
int n = 8;

Model model = new Model(n + "-queens problem");
IntVar[] vars = model.intVarArray("Q", n, 1, n, false);
IntVar[] diag1 = IntStream.range(0, n).
                           mapToObj(i -> vars[i].sub(i).intVar()).
                           toArray(IntVar[]::new);
IntVar[] diag2 = IntStream.range(0, n).
                           mapToObj(i -> vars[i].add(i).intVar()).
                           toArray(IntVar[]::new);

model.post(
    model.allDifferent(vars),
    model.allDifferent(diag1),
    model.allDifferent(diag2)
);

Solver solver = model.getSolver();
solver.showStatistics();
solver.setSearch(Search.domOverWDegSearch(vars));
Solution solution = solver.findSolution();

if (solution != null) {
    System.out.println(solution.toString());
}
```

VariableSelector Choice (Choco-solver)

- Determines the order in which variables are assigned
- `InputOrder` assign variables in static order given
- `Smallest` assign variable with smallest value in domain first
- `FirstFail` select variable with smallest domain first
- `DomOverWDeg` consider ratio of domain size and failure count
- `ActivityBased` dynamic, based on dynamic observed behaviour
- `ImpactBased` dynamic, based on dynamic observed behaviour

IntValueSelector Choice (Choco-solver)

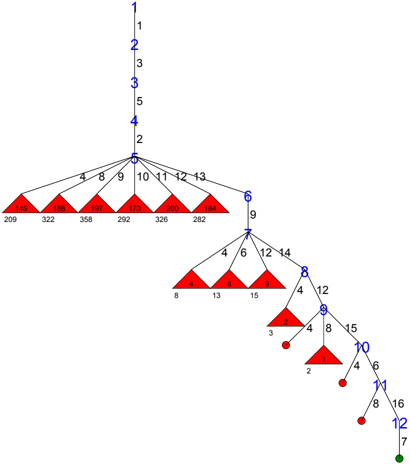
- Determines the order in which values are tested for selected variables
- `IntDomainMin` Start with smallest value, on backtracking try next larger value
- `IntDomainMiddle` Start with value closest to middle of domain
- `IntDomainRandom` Choose values in random order
- `IntDomainRandomBound` Randomly choose between smallest and largest value

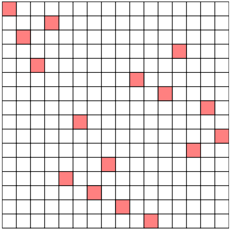
Continue

Comparison

- Board size 16x16
- Naive (Input Order) Strategy
- First Fail variable selection

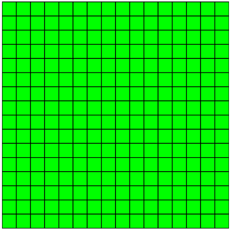
Naive (Input Order) Strategy (Size 16)





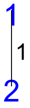
FirstFail Strategy (Size 16)

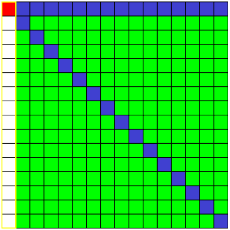
1



Skip Animation

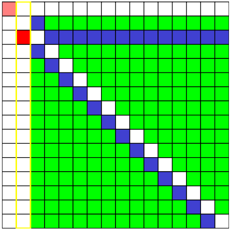
FirstFail Strategy (Size 16)





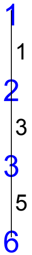
◂ Back to Start ⏩ Skip Animation

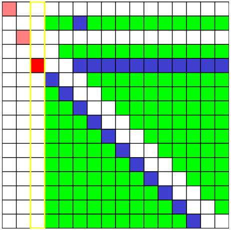
FirstFail Strategy (Size 16)



Back to Start
Skip Animation

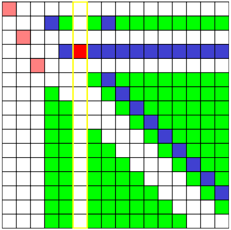
FirstFail Strategy (Size 16)





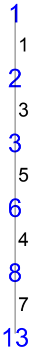
Back to Start
Skip Animation

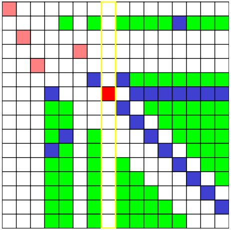
FirstFail Strategy (Size 16)



◂ Back to Start ⏩ Skip Animation

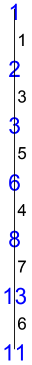
FirstFail Strategy (Size 16)

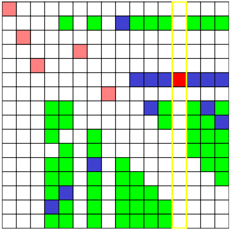




Back to Start   Skip Animation

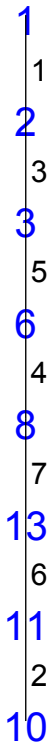
FirstFail Strategy (Size 16)

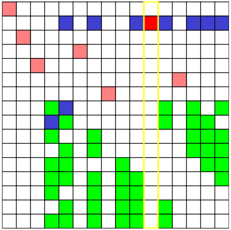




Back to Start Skip Animation

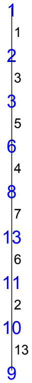
FirstFail Strategy (Size 16)

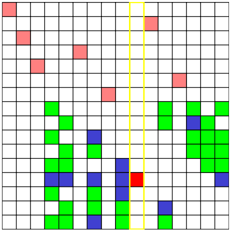




Back to Start
Skip Animation

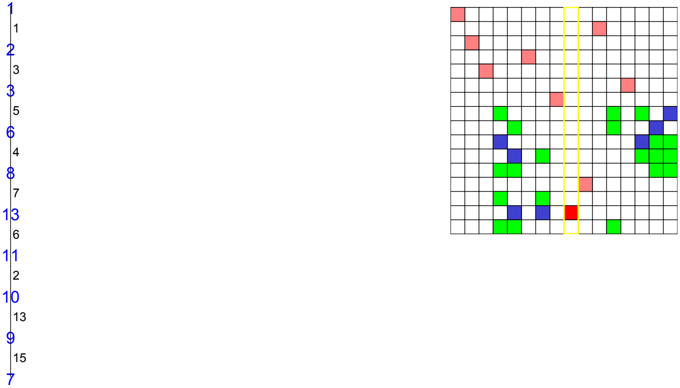
FirstFail Strategy (Size 16)





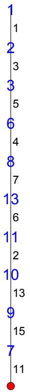
◂ Back to Start ▸▸ Skip Animation

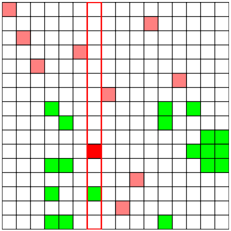
FirstFail Strategy (Size 16)



◂ Back to Start ⏩ Skip Animation

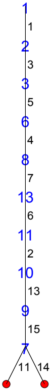
FirstFail Strategy (Size 16)

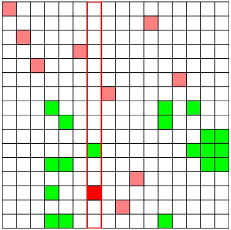




◂ Back to Start ⏩ Skip Animation

FirstFail Strategy (Size 16)





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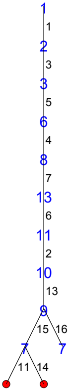
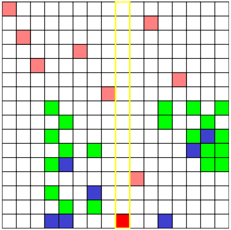
FirstFail Strategy (Size 16)





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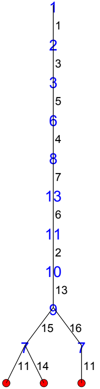
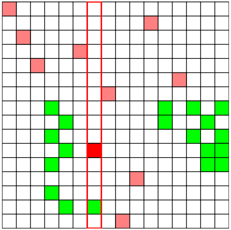
FirstFail Strategy (Size 16)





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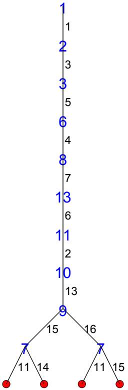
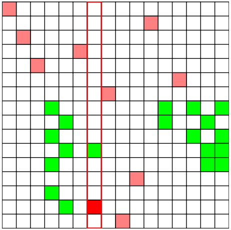
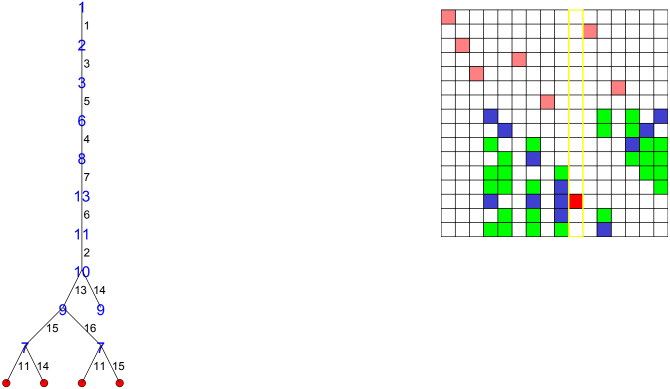
FirstFail Strategy (Size 16)





Back to Start Skip Animation

FirstFail Strategy (Size 16)



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Skip Animation

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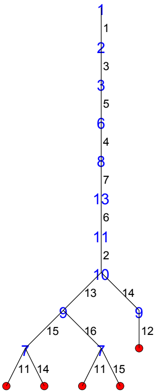
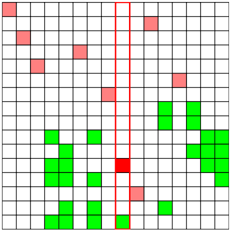
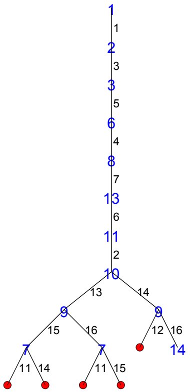
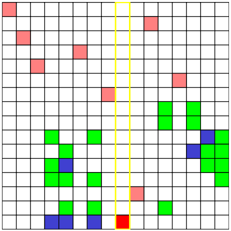




Back to Start
Skip Animation

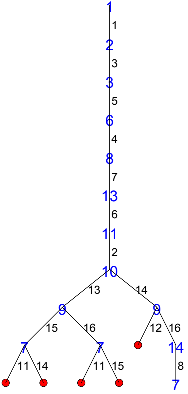
FirstFail Strategy (Size 16)

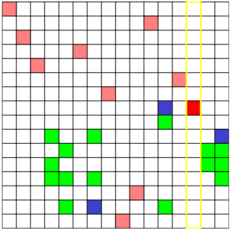




Back to Start
Skip Animation

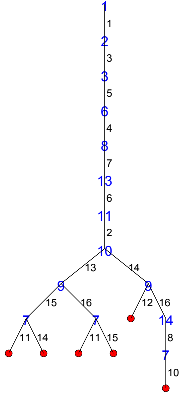
FirstFail Strategy (Size 16)

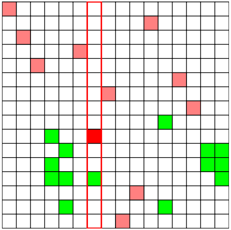




◂ Back to Start ⏩ Skip Animation

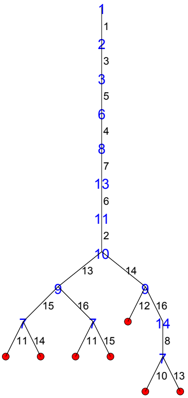
FirstFail Strategy (Size 16)

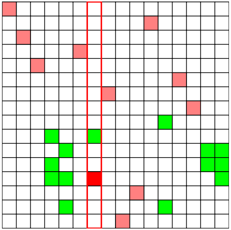




Back to Start
Skip Animation

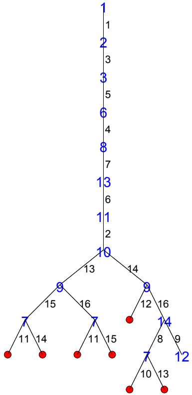
FirstFail Strategy (Size 16)

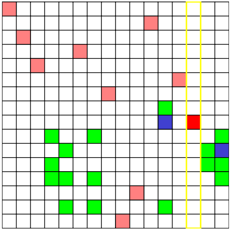




◂ Back to Start
⏩ Skip Animation

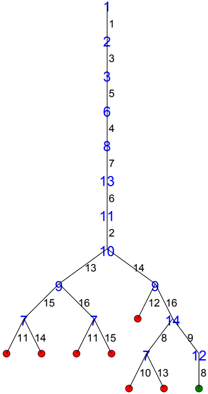
FirstFail Strategy (Size 16)

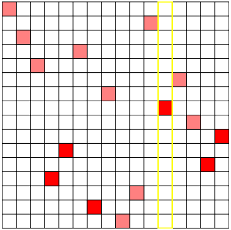




◂ Back to Start ⏩ Skip Animation

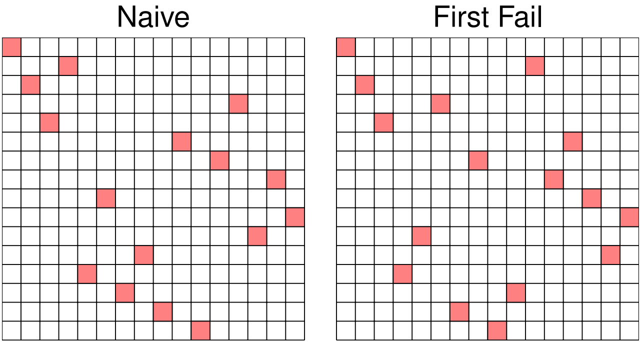
FirstFail Strategy (Size 16)





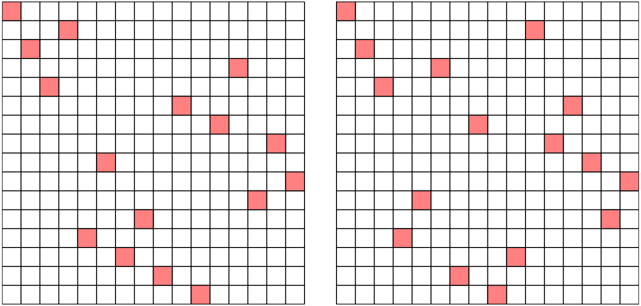
Back to Start

Comparing Solutions



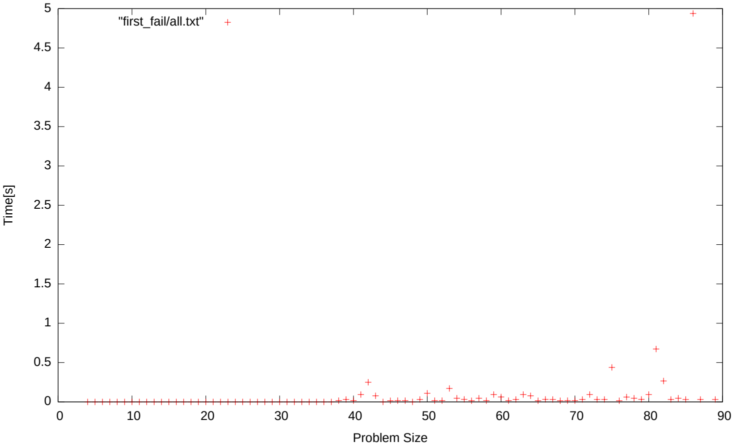
Comparing Solutions

Naive　　First Fail



Solutions are different!

FirstFail, Problem Sizes 4-100



Observations

- This is much better
- But some sizes are much harder
- Timeout for sizes 88, 91, 93, 97, 98, 99

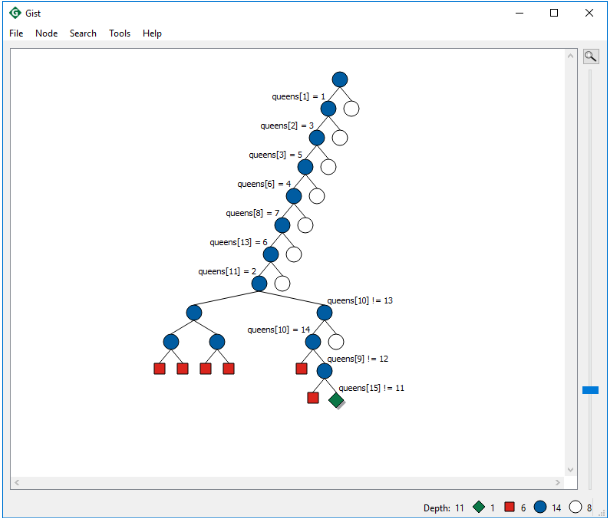
More Reactive Variable Selection

- Domain size is important, but other information is useful as well
- Dom/Weighted Degree: better results in many situations
- Weight Degree: count how often variable has been involved in failure
- Focus on more complicated part of problem
- Changes during search, learns from past performance
- Option **dom_w_deg**

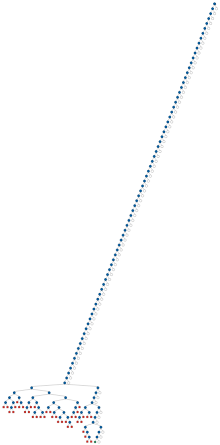
Weighted Degree Variable Selection

```
int: n=8;
array[1..n] of var 1..n: queens;
constraint
    forall(i, j in 1..n where i < j) (
         queens[i] != queens[j] /\
         queens[i] + i != queens[j] + j /\
         queens[i] - i != queens[j] - j
    )
 ;
solve :: int_search(
        queens,
        dom_w_deg,
        indomain_random)
        satisfy;
```

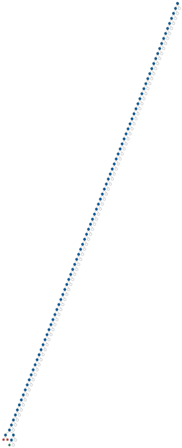
Result for size 16 with Gecode-Gist



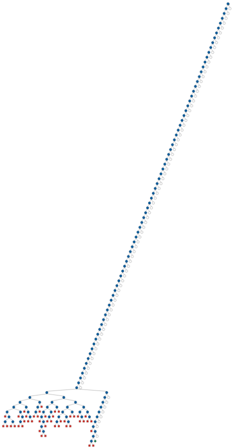
Sample Results for Larger Sizes



Size 93



Size 94



Size 95

Approach 1: Heuristic Portfolios

- Try multiple strategies for the same problem
- With multi-core CPUs, run them in parallel
- Only one needs to be successful for each problem

Approach 2: Restart with Randomization

- Only spend limited number of backtracks for a search attempt
- When this limit is exceeded, restart at beginning
- Requires randomization to explore new search branch
- Randomize variable choice by random tie break
- Randomize value choice by shuffling values
- Needs strategy when to restart

Random Variable Choice and Restarts

```
int: n=8;
array[1..n] of var 1..n: queens;
constraint
    forall(i, j in 1..n where i < j) (
         queens[i] != queens[j] /\
         queens[i] + i != queens[j] + j /\
         queens[i] - i != queens[j] - j
    )
 ;
solve :: int_search(
        queens,
        dom_w_deg,
        indomain_random)
        :: random_linear(100)
        satisfy;
```

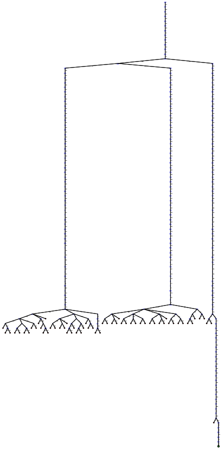
Approach 3: Partial Search

- Abandon depth-first, chronological backtracking
- Don’t get locked into a failed sub-tree
- A wrong decision at a level is not detected, and we have to explore the complete subtree below to undo that wrong choice
- Explore more of the search tree
- Spend time in promising parts of tree

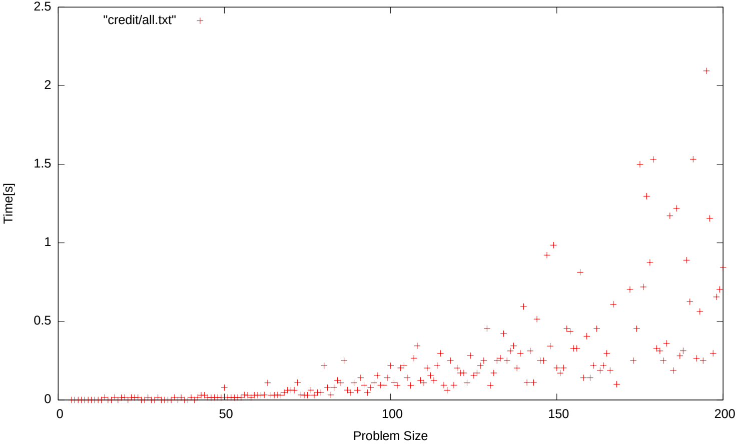
Example: Credit Search

- Not available in all solvers
- Explore top of tree completely, based on credit
- Start with fixed amount of credit
- Each node consumes one credit unit
- Split remaining credit amongst children
- When credit runs out, start bounded backtrack search
- Each branch can use only K backtracks
- If this limit is exceeded, jump to unexplored top of tree

Credit, Search Tree Problem Size 94



Credit, Problem Sizes 4-200



Points to Remember

- Choice of search can have huge impact on performance
- Dynamic variable selection can lead to large reduction of search space
- Packaged search can do a lot, but programming search adds even more
- Depth-first chronological backtracking not always best choice
- How to control this explosion of search alternatives?