

Chapter 4: Basic Constraint Reasoning (SEND+MORE=MONEY)

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ECLiPSe ELearning [Overview](#)



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Outline

- 1 Problem
- 2 Program
- 3 Constraint Setup
- 4 Search
- 5 Lessons Learned



What we want to introduce

- Finite Domain Solver in ECLiPSe
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, alldifferent, disequality
- Built-in search: Labeling
- Visualizers for variables, constraints and search



Problem Definition

A Crypt-Arithmetic Puzzle

We begin with the definition of the SEND+MORE=MONEY puzzle. It is often shown in the form of a hand-written addition:

$$\begin{array}{r}
 \text{S E N D} \\
 + \text{M O R E} \\
 \hline
 \text{M O N E Y}
 \end{array}$$



Rules

- Each character stands for a digit from 0 to 9.
- Numbers are built from digits in the usual, positional notation.
- Repeated occurrence of the same character denote the same digit.
- Different characters denote different digits.
- Numbers do not start with a zero.
- The equation must hold.

$$\begin{array}{r}
 \text{S E N D} \\
 + \text{M O R E} \\
 \hline
 \text{M O N E Y}
 \end{array}$$



Model

- Each character is a variable, which ranges over the values 0 to 9.
- An *alldifferent* constraint between all variables, which states that two different variables must have different values. This is a very common constraint, which we will encounter in many other problems later on.
- Two *disequality constraints* (variable X must be different from value V) stating that the variables at the beginning of a number can not take the value 0.
- An arithmetic *equality constraint* linking all variables with the proper coefficients and stating that the equation must hold.



General Program Structure

```
:- module(sendmory) .
:- export(sendmory/1) .
:- lib(ic) .
sendmory(L) :-
    L = [S,E,N,D,M,O,R,Y], ⇨ Variables
    L :: 0..9,
    alldifferent(L), ⇨ Constraints
    S #\= 0, M #\= 0,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(L). ⇨ Search
```



Choice of Model

- This is *one* model, not *the* model of the problem
- Many possible alternatives
- Choice often depends on your constraint system
 - Constraints available
 - Reasoning attached to constraints
- Not always clear which is the *best* model
- Often: Not clear what is the *problem*

► Alternative 1

► Alternative 2



Running the program

- To run the program, we have to enter the query
 - `sendmory:sendmory(L) .`
- Result
 - `L = [9, 5, 6, 7, 1, 0, 8, 2]`
 - `yes (0.00s cpu, solution 1, maybe more)`



Question

- But how did the program come up with this solution?



Domain Definition

$L = [S, E, N, D, M, O, R, Y],$

$L :: 0..9,$

$[S, E, N, D, M, O, R, Y] \in \{0..9\}$

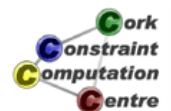


Domain Visualization

Columns = Values

Rows = Variables

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M			Cells= State							
O										
R										
Y										



Alldifferent Constraint

`alldifferent (L) ,`

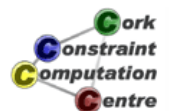
- Built-in of `ic` library
- No initial propagation possible
- *Suspends*, waits until variables are changed
- When variable is fixed, remove value from domain of other variables
- *Forward checking*



Alldifferent Visualization

Uses the same representation as the domain visualizer

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										



Disequality Constraints

$$S \neq 0, M \neq 0,$$

Remove value from domain

$$S \in \{1..9\}, M \in \{1..9\}$$

Constraints solved, can be removed



Domains after Disequality

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										



Equality Constraint

- Normalization of linear terms
 - Single occurrence of variable
 - Positive coefficients
- Propagation



Normalization

$$\begin{array}{rclcl}
 & 1000 * S + & 100 * E + & 10 * N + & D \\
 & + 1000 * M + & 100 * O + & 10 * R + & E \\
 \hline
 10000 * M + & 1000 * O + & 100 * N + & 10 * E + & Y
 \end{array}$$

is transformed into

$$\begin{array}{rclcl}
 & 1000 * S + & 91 * E + & & D \\
 & & + & 10 * R & \\
 \hline
 9000 * M + & 900 * O + & 90 * N + & & Y
 \end{array}$$



Simplified Equation

$$1000 * S + 91 * E + 10 * R + D = 9000 * M + 900 * O + 90 * N + Y$$



Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{1000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..89919}$$

Deduction:

$$M = 1, S = 9, O \in \{0..1\}$$

Why? [Skip](#)



Consider lower bound for S

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Lower bound of equation is 9000
- Rest of lhs (left hand side) ($91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}$) is atmost 918
- S must be greater or equal to $\frac{9000-918}{1000} = 8.082$
 - otherwise lower bound of equation not reached by lhs
- S is integer, therefore $S \geq \lceil \frac{9000-918}{1000} \rceil = 9$
- S has upper bound of 9, so $S = 9$



Consider upper bound of M

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}$ is at least 0
- M must be smaller or equal to $\frac{9918-0}{9000} = 1.102$
- M must be integer, therefore $M \leq \lfloor \frac{9918-0}{9000} \rfloor = 1$
- M has lower bound of 1, so $M = 1$



Consider upper bound of O

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $9000 * 1 + 90 * N^{0..9} + Y^{0..9}$ is at least 9000
- O must be smaller or equal to $\frac{9918-9000}{900} = 1.02$
- O must be integer, therefore $O \leq \lfloor \frac{9918-9000}{900} \rfloor = 1$
- O has lower bound of 0, so $O \in \{0..1\}$



Propagation of equality: Result

	0	1	2	3	4	5	6	7	8	9
S		-	-	-	-	-	-	-	-	✱
E										
N										
D										
M		✱	-	-	-	-	-	-	-	-
O			✕	✕	✕	✕	✕	✕	✕	✕
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

$$O = 0, [E, R, D, N, Y] \in \{2..8\}$$

Waking the equality constraint

- Triggered by assignment of variables
- *or* update of lower or upper bound



Removal of constants

$$1000 * 9 + 91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = 9000 * 1 + 900 * 0 + 90 * N^{2..8} + Y^{2..8}$$

$$\mathbf{1000 * 9 + 91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = 9000 * 1 + 900 * 0 + 90 * N^{2..8} + Y^{2..8}}$$

$$91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = 90 * N^{2..8} + Y^{2..8}$$



Propagation of equality (Iteration 1)

$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8}}_{204..816} = \underbrace{90 * N^{2..8} + Y^{2..8}}_{182..728}$$

$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8}}_{204..728} = 90 * N^{2..8} + Y^{2..8}$$

$$N \geq 3 = \lceil \frac{204 - 8}{90} \rceil, E \leq 7 = \lfloor \frac{728 - 22}{91} \rfloor$$



Propagation of equality (Iteration 2)

$$91 * E^{2..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}$$

$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8}}_{204..725} = \underbrace{90 * N^{3..8} + Y^{2..8}}_{272..728}$$

$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8}}_{272..725} = 90 * N^{3..8} + Y^{2..8}$$

$$E \geq 3 = \lceil \frac{272 - 88}{91} \rceil$$



Propagation of equality (Iteration 3)

$$91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}$$

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = \underbrace{90 * N^{3..8} + Y^{2..8}}_{272..728}$$

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = 90 * N^{3..8} + Y^{2..8}$$

$$N \geq 4 = \left\lceil \frac{295 - 8}{90} \right\rceil$$



Propagation of equality (Iteration 4)

$$91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}$$

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = \underbrace{90 * N^{4..8} + Y^{2..8}}_{362..728}$$

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{362..725} = 90 * N^{4..8} + Y^{2..8}$$

$$E \geq 4 = \left\lceil \frac{362 - 88}{91} \right\rceil$$



Propagation of equality (Iteration 5)

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}$$

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = \underbrace{90 * N^{4..8} + Y^{2..8}}_{362..728}$$

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = 90 * N^{4..8} + Y^{2..8}$$

$$N \geq 5 = \left\lceil \frac{386 - 8}{90} \right\rceil$$



Propagation of equality (Iteration 6)

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = \underbrace{90 * N^{5..8} + Y^{2..8}}_{452..728}$$

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{452..725} = 90 * N^{5..8} + Y^{2..8}$$

$$N \geq 5 = \left\lceil \frac{452 - 8}{90} \right\rceil, E \geq 4 = \left\lceil \frac{452 - 88}{91} \right\rceil$$

No further propagation at this point



Domains after setup

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										



labeling built-in

```
labeling([S,E,N,D,M,O,R,Y])
```

- Try variable is order given
- Try values starting from smallest value in domain
- When failing, backtrack to last open choice
- *Chronological Backtracking*
- *Depth First search*



Search Tree Step 1

S
9
E

Variable S already fixed



Step 2, Alternative $E = 4$

Variable $E \in \{4..7\}$, first value tested is 4

S
9
E
4



Assignment $E = 4$

	0	1	2	3	4	5	6	7	8	9
S										
E					☀	-	-	-		
N										
D										
M										
O										
R										
Y										



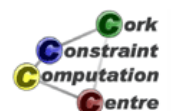
Propagation of $E = 4$, equality constraint

$$91 * 4 + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8}}_{386..452} = \underbrace{90 * N^{5..8} + Y^{2..8}}_{452..728}$$

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8}}_{452} = 90 * N^{5..8} + Y^{2..8}$$

$$N = 5, Y = 2, R = 8, D = 8$$



Result of equality propagation

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

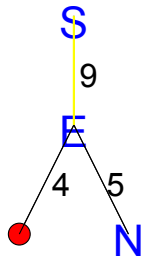
Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Alldifferent fails!

Step 2, Alternative $E = 5$

Return to last open choice, E , and test next value



Assignment $E = 5$

	0	1	2	3	4	5	6	7	8	9
S										
E					-	✱	-	-		
N										
D										
M										
O										
R										
Y										



Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

$$N \neq 5, N \geq 6$$



Propagation of equality

$$91 * 5 + 10 * R^{2..8} + D^{2..8} = 90 * N^{6..8} + Y^{2..8}$$

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8}}_{477..543} = \underbrace{90 * N^{6..8} + Y^{2..8}}_{542..728}$$

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8}}_{542..543} = 90 * N^{6..8} + Y^{2..8}$$

$$N = 6, Y \in \{2, 3\}, R = 8, D \in \{7..8\}$$



Result of equality propagation

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

$$D = 7$$

Propagation of equality

$$91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}$$

$$\underbrace{91 * 5 + 10 * 8 + 7}_{542} = \underbrace{90 * 6 + Y^{2..3}}_{542..543}$$

$$\underbrace{91 * 5 + 10 * 8 + 7}_{542} = 90 * 6 + Y^{2..3}$$

$$Y = 2$$

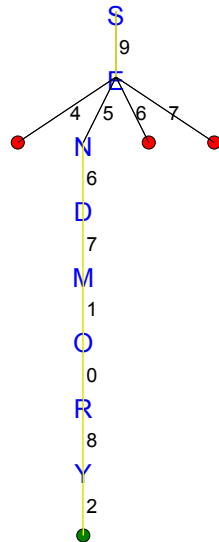


Last propagation step

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

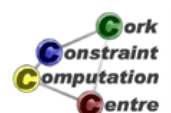


Complete Search Tree



Solution

$$\begin{array}{r}
 9 \ 5 \ 6 \ 7 \\
 + \ 1 \ 0 \ 8 \ 5 \\
 \hline
 1 \ 0 \ 6 \ 5 \ 2
 \end{array}$$



Topics introduced

- Finite Domain Solver in ECLiPSe, `ic` library
- Models and Programs
- Constraint Propagation and Search
- Basic constraints: linear arithmetic, `alldifferent`, disequality
- Built-in search: `labeling`
- Visualizers for variables, constraints and search



Lessons Learned

- Constraint models are expressed by variables and constraints.
- Problems can have many different models, which can behave quite differently. Choosing the best model is an art.
- Constraints can take many different forms.
- Propagation deals with the interaction of variables and constraints.
- It removes some values that are inconsistent with a constraint from the domain of a variable.
- Constraints only communicate via shared variables.



Lessons Learned

- Propagation usually is not sufficient, search may be required to find a solution.
- Propagation is data driven, and can be quite complex even for small examples.
- The default search uses chronological depth-first backtracking, systematically exploring the complete search space.
- The search choices and propagation are interleaved, after every choice some more propagation may further reduce the problem.



Alternative 1

- Do we need the constraint “Numbers do not begin with a zero”?
- This is not given explicitly in the problem statement
- Remove disequality constraints from program
- Previous solution is still a solution
- Does it change propagation?
- Does it have more solutions?



Program without Disequality

Listing 1: Alternative 1

```

:-module( alternative1 ).
:-export( sendmory / 1 ).
:-lib( ic ).

sendmory(L):-
    L = [S,E,N,D,M,O,R,Y] ,
    L :: 0..9 ,
    alldifferent(L) ,
    1000*S + 100*E + 10*N + D +
    1000*M + 100*O + 10*R + E #=
    10000*M + 1000*O + 100*N + 10*E + Y ,
    labeling(L).

```



After Setup without Disequality

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										



Setup Comparison

original

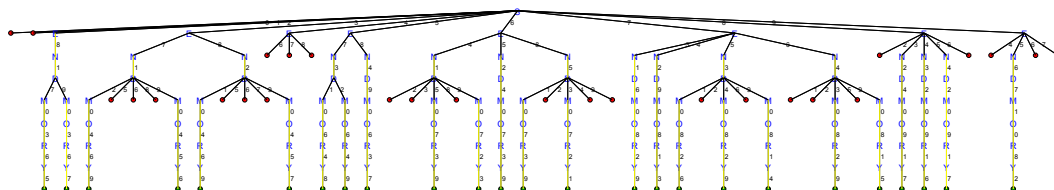
	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

alternative 1

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										



Search Tree: Many Solutions



Note:

- Not just a different model, solving a different problem!
- Often we can choose which problem we want to solve
 - Which constraints to include
 - What to ignore
- In this case not acceptable

◀ Choice of Model



Alternative 2

- Large equality difficult to understand by humans
- Replace with multiple, simpler equations
- Linked by carry variables (0/1)
- Should produce same solutions
- Does it give same propagation?

$$\begin{array}{rcccccc}
 & & S & E & N & D \\
 + & M & O & R & E & \\
 +C5 & C4 & C3 & C2 & & \\
 \hline
 M & O & N & E & Y &
 \end{array}$$



Carry Variables with Multiple Equations

```
:-module(alternative2), export(sendmory/1), lib(ic).
sendmory(L) :-  $\Rightarrow$  same as before
    L=[S,E,N,D,M,O,R,Y], L :: 0..9,
    [C2,C3,C4,C5] :: 0..1,  $\Rightarrow$  new
    alldifferent(L),
    S #\= 0, M #\= 0,
    M #= C5,
    S+M+C4 #= 10*C5+O,
    E+O+C3 #= 10*C4+N,
    N+R+C2 #= 10*C3+E,
    D+E #= 10*C2+Y,
    labeling(L).
```

$$\begin{array}{rcccccc}
 & & S & E & N & D \\
 + & M & O & R & E & \\
 +C5 & C4 & C3 & C2 & & \\
 \hline
 M & O & N & E & Y &
 \end{array}$$



With Carry Variables: After Setup

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										



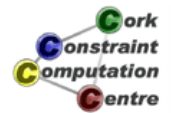
Setup Comparison

original

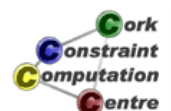
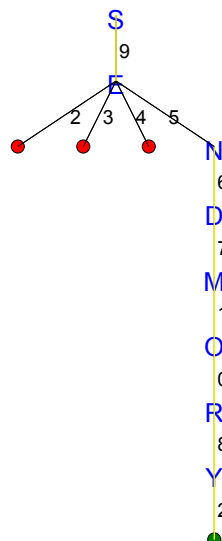
	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

alternative2

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

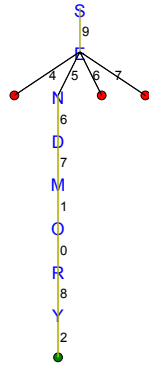


Search Tree: First Solution

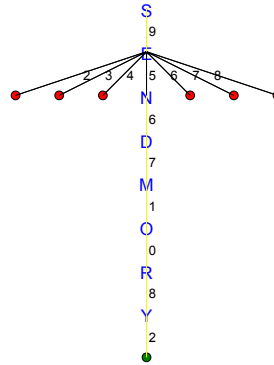


Comparison

Single Equation



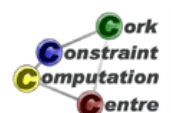
Multiple Equations



Observations

- This is solving the original problem
- Search tree slightly bigger
- Caused here by missing interaction of equations
- And repeated variables
- But: Introducing auxiliary variables not always bad!

◀ Choice of Model



More Information



Henry Dudeney.

Send+More=Money.

Strand Magazine, Volume 68:pages 97 and 214, July 1924.



Henry Dudeney.

Amusements in Mathematics.

Project Gutenberg, 1917.

<http://www.gutenberg.org/etext/16713>.



Exercises

- 1 Does the reasoning for the equality constraints that we have presented remove all inconsistent values? Consider the constraint $Y=2*X$.
- 2 Why is it important to remove multiple occurrences of the same variable from an equality constraint? Give an example!
- 3 Solve the puzzle $DONALD+GERALD=ROBERT$. What is the state of the variables before the search, after the initial constraint propagation?
- 4 Solve the puzzle $Y*WORRY = DOOOOD$. What is different?
- 5 (extra credit) How would you design a program that finds new crypt-arithmetic puzzles? What makes a good puzzle?

