

Centre for
Data Analytics



Introduction to Constraint Programming

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CRT-AI CP Week 2025



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Acknowledgments

This publication has emanated from research conducted with the financial support of Science Foundation Ireland under Grant number 12/RC/2289-P2 at Insight the SFI Research Centre for Data Analytics at UCC, which is co-funded under the European Regional Development Fund.

A version of this material was developed as part of the ECLiPSe ELearning course:

<https://eclipseclp.org/ELearning/index.html>.

Support from Cisco Systems and the Silicon Valley Community Foundation is gratefully acknowledged.

Objectives

- Overview of Core Constraint Programming
- Three Main Concepts
 - Constraint Propagation
 - Global Constraints
 - Customizing Search
- Topics will be treated in more detail in later parts of the school
- Based on Examples, not Formal Description

Outline

- Why Constraint Programming?
- Constraint Propagation
- Global Constraints
- Customizing Search

Tutorial Based on ECLiPSe ELearning Course

- Self-study course in constraint programming
- Supported by Cisco Systems and Silicon Valley Community Foundation
- Multi-media format, video lectures, slides, handout etc
- <https://eclipseclp.org/ELearning/index.html>

Also Part of CRT-AI Constraint Week

- Annual one week course on CP and Optimization in Ireland
- Part of national training program for PhD students in AI
- <https://www.crt-ai.ie/>

Constraint Programming - in a nutshell

- Declarative description of problems with
 - *Variables* which range over (finite) sets of values
 - *Constraints* over subsets of variables which restrict possible value combinations
 - A *solution* is a value assignment which satisfies all constraints
- Constraint propagation/reasoning
 - Removing inconsistent values for variables
 - Detect failure if constraint can not be satisfied
 - Interaction of constraints via shared variables
 - Incomplete
- Search
 - User controlled assignment of values to variables
 - Each step triggers constraint propagation
- Different domains require/allow different methods

Constraint Programming is Different

- Declarative Programming
 - Concentrate on what you want
 - Not how to get there
 - Program \neq Algorithm
 - Program = Model
- Applied to Combinatorial Problems
 - No complete polynomial algorithms known (exist?)
 - CP less ad-hoc than heuristics
 - Models can evolve

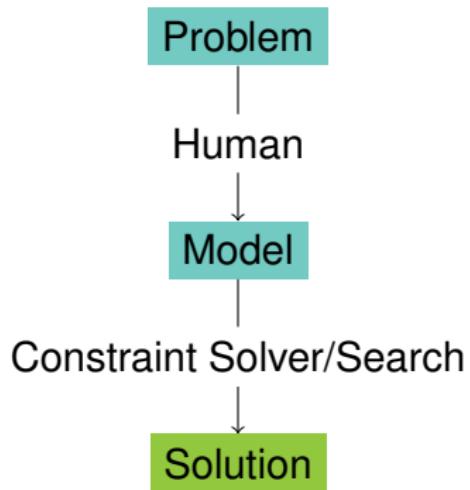
A Subtractive Process



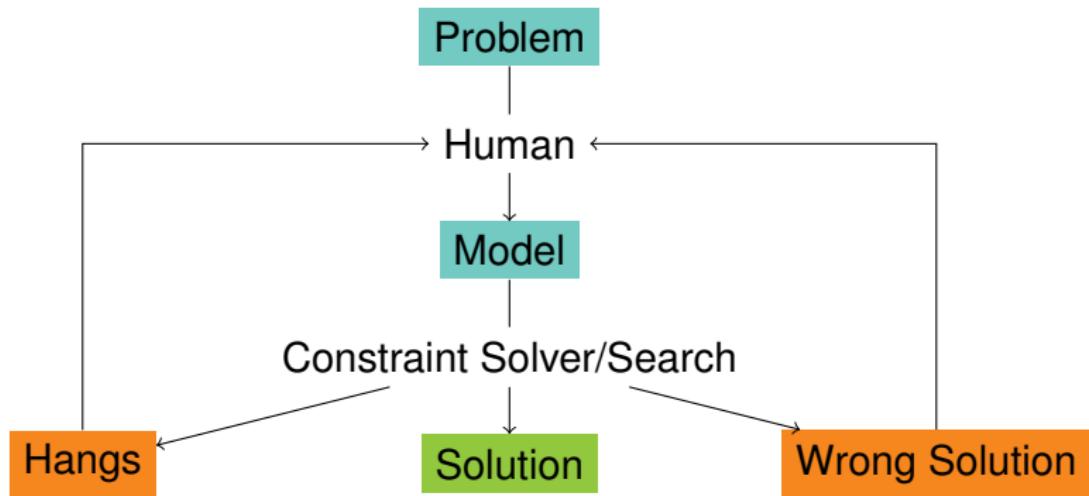
“Oh, bosh, as Mr. Ruskin says. Sculpture, per se, is the simplest thing in the world. All you have to do is to take a big chunk of marble and a hammer and chisel, make up your mind what you are about to create and chip off all the marble you don’t want.” -Paris Gaulois.

Source: <https://quoteinvestigator.com/2014/06/22/chip-away/>

Basic Process



More Realistic



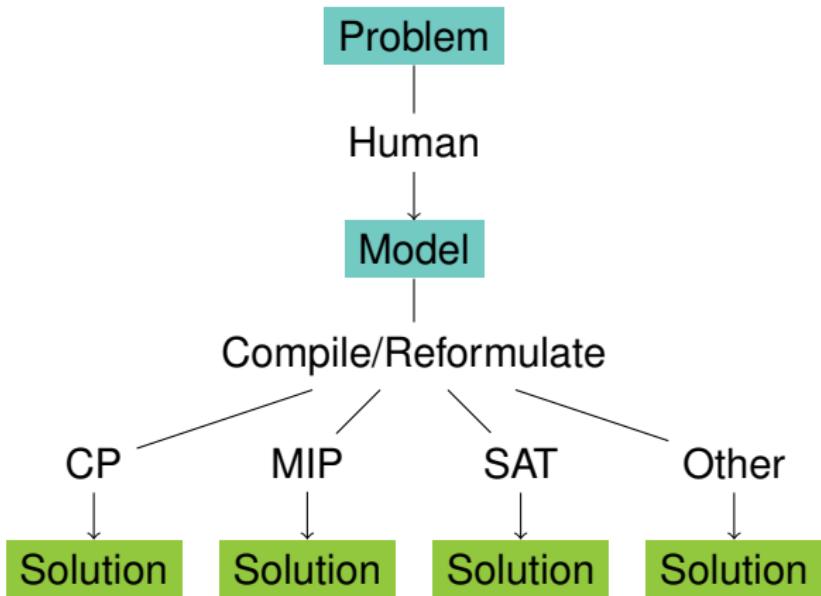
Dual Role of Model

- Allows Human to Express Problem
 - Close to Problem Domain
 - Constraints as Abstractions
- Allows Solver to Execute
 - Variables as Communication Mechanism
 - Constraints as Algorithms

Modelling Frameworks

- MiniZinc (NICTA, Monash University, Australia)
- NumberJack (Insight, Ireland)
- EssencePrime/SavilleRow (UK)
- CPMpy (KU Leuven)
- Allow use of multiple back-end solvers
- Compile model into variants for each solver
- A priori solver independent model(CP, MIP, SAT)

Framework Process



Why use Puzzles as Examples?

- Easy to understand the problem
- Solvable by hand without specialized knowledge
- Possible to compare automated to manual solving process

The puzzle, though inanimate, is presented as a solvable problem without lasting negative consequences, a very low-risk low-reward situation. By being a puzzle, the object is attempting to convince the user that it must be completed.

Source: Every Day Rhetoric

Part I

Basic Constraint Propagation

Example 1: SEND+MORE=MONEY

- Example of Finite Domain Constraint Problem
- Models and Programs
- Constraint Propagation and Search
- Some Basic Constraints: linear arithmetic, alldifferent, disequality
- A Built-in search
- Visualizers for variables, constraints and search

Outline

Problem

Program

Constraint Setup

Search

Points to Remember

Problem Definition

A Crypt-Arithmetic Puzzle

We begin with the definition of the SEND+MORE=MONEY puzzle. It is often shown in the form of a hand-written addition:

$$\begin{array}{r} \text{S} \quad \text{E} \quad \text{N} \quad \text{D} \\ + \text{M} \quad \text{O} \quad \text{R} \quad \text{E} \\ \hline \text{M} \quad \text{O} \quad \text{N} \quad \text{E} \quad \text{Y} \end{array}$$

The puzzle was first proposed by Henry Dudeney in the Strand Magazine from 1924.

Rules

- Each character stands for a digit from 0 to 9.
- Numbers are built from digits in the usual, positional notation.
- Repeated occurrence of the same character denote the same digit.
- Different characters denote different digits.
- Numbers do not start with a zero.
- The equation must hold.

$$\begin{array}{r} & S & E & N & D \\ + & M & O & R & E \\ \hline M & O & N & E & Y \end{array}$$

Outline

Problem

Program

Constraint Setup

Search

Points to Remember

Model

- Each character is a variable, which ranges over the values 0 to 9.
- An *alldifferent* constraint between all variables, which states that two different variables must have different values. This is a very common constraint, which we will encounter in many other problems later on.
- Two *disequality constraints* (variable X must be different from value V) stating that the variables at the beginning of a number can not take the value 0.
- An arithmetic *equality constraint* linking all variables with the proper coefficients and stating that the equation must hold.

SEND+MORE=MONEY Models

- ECLiPSe [▶ Show](#)
- MiniZinc [▶ Show](#)
- NumberJack [▶ Show](#)
- CPMPy [▶ Show](#)
- Choco-solver [▶ Show](#)

ECLiPSe Model

```
: - lib(ic).  
  
sendmore(Digits) :-  
    Digits = [S,E,N,D,M,O,R,Y],  
    Digits :: [0..9],  
    alldifferent(Digits),  
    S #\= 0,  
    M #\= 0,  
    1000*S + 100*E + 10*N + D  
        + 1000*M + 100*O + 10*R + E  
    #= 10000*M + 1000*O + 100*N + 10*E + Y,  
    labeling(Digits).
```

► Continue

MiniZinc Model

```
include "alldifferent.mzn";
var 0..9: S;
var 0..9: E;
var 0..9: N;
var 0..9: D;
var 0..9: M;
var 0..9: O;
var 0..9: R;
var 0..9: Y;
constraint S != 0;
constraint M != 0;
constraint
    1000 * S + 100 * E + 10 * N + D
    + 1000 * M + 100 * O + 10 * R + E
    = 10000 * M + 1000 * O + 100 * N + 10 * E + Y;
constraint alldifferent([S,E,N,D,M,O,R,Y]);
solve satisfy;
```

▶ Continue

NumberJack Model

(from <https://github.com/eomahony/Numberjack/>)

```
from Numberjack import *

def get_model():
    model = Model()
    s, m = VarArray(2, 1, 9)
    e, n, d, o, r, y = VarArray(6, 0, 9)
    model.add(      s*1000 + e*100 + n*10 + d +
                    m*1000 + o*100 + r*10 + e ==
                    m*10000 + o*1000 + n*100 + e*10 + y)
    model.add(AllDiff((s, e, n, d, m, o, r, y)))
    return s, e, n, d, m, o, r, y, model

def solve(param):
    s, e, n, d, m, o, r, y, model = get_model()
    solver = model.load(param['solver'])
    solver.setVerbosity(param['verbose'])
    solver.solve()
```

▶ Continue

CPMpy Model (from <https://github.com/CPMpy/>)

```
from cpmpy import *
import numpy as np

s,e,n,d,m,o,r,y = intvar(0,9, shape=8)
model = Model(
    AllDifferent([s,e,n,d,m,o,r,y]),
    (    sum([s,e,n,d] * np.array([      1000, 100, 10, 1])) ) \
    + sum([m,o,r,e] * np.array([      1000, 100, 10, 1])) ) \
    == sum([m,o,n,e,y] * np.array([10000, 1000, 100, 10, 1])), 
    s > 0,
    m > 0,
)
model.solve()
```

▶ Continue

Choco-solver Model (from <https://choco-solver.org/>)

```
Model model = new Model("SEND+MORE=MONEY");
IntVar S = model.intVar("S", 1, 9, false);
IntVar E = model.intVar("E", 0, 9, false);
IntVar N = model.intVar("N", 0, 9, false);
IntVar D = model.intVar("D", 0, 9, false);
IntVar M = model.intVar("M", 1, 9, false);
IntVar O = model.intVar("O", 0, 9, false);
IntVar R = model.intVar("R", 0, 9, false);
IntVar Y = model.intVar("Y", 0, 9, false);

model.allDifferent(new IntVar[]{S, E, N, D, M, O, R, Y}).post();

IntVar[] ALL = new IntVar[]{
    S, E, N, D,
    M, O, R, E,
    M, O, N, E, Y};
int[] COEFFS = new int[]{
    1000, 100, 10, 1,
    1000, 100, 10, 1,
    -10000, -1000, -100, -10, -1};
model.scalar(ALL, COEFFS, "=" , 0).post();

Solver solver = model.getSolver();
solver.showStatistics();
solver.showSolutions();
solver.findSolution();
```

▶ Continue

A Note on Syntax

- Some formulations may seem simpler than others
- This largely is an artifact of a very simple problem
- In most models, you do not write down constraints one by one
- You create constraints based on data
- Ease of integration becomes more important than syntax
- Debugging tools for those who need a debugger :-)

Choice of Model

- This is *one* model, not *the* model of the problem
- Many possible alternatives
- Choice often depends on your constraint system
 - Constraints available
 - Reasoning attached to constraints
- Not always clear which is the *best* model
- Often: Not clear what is the *problem*

Running the Program (Minizinc IDE)

sendmore.mzn — Untitled Project*

File Edit Minizinc View Help

New model Open Save Copy Cut Paste Undo Redo Shift left Shift right Run Solver configuration: Gecode 6.3.1 (built-in)

Show configuration editor Show project explorer

sendmore.mzn * queen_binary.mzn

```
1 include "alldifferent.mzn";
2 var 0..9: S;
3 var 0..9: E;
4 var 0..9: N;
5 var 0..9: D;
6 var 0..9: M;
7 var 0..9: O;
8 var 0..9: R;
9 var 0..9: Y;
10 constraint S != 0;
11 constraint M != 0;
12 constraint 1000 * S + 100 * E + 10 * N + D
13     + 1000 * M + 100 * O + 10 * R + E
14     = 10000 * M + 1000 * O + 100 * N + 10 * E + Y;
15 constraint alldifferent([S,E,N,D,M,O,R,Y]);
16 |
17 solve satisfy;
18
```

Output

```
Compiling sendmore.mzn
Running sendmore.mzn
S = 9;
E = 5;
N = 6;
D = 7;
M = 1;
O = 0;
R = 8;
Y = 2;
-----
Finished in 3s 103ms
```

Line: 16, Col: 1

3s 103ms

Question

- But how did the program come up with this solution?
- We show solution with ECLiPSe, other solvers vary slightly

Outline

Problem

Program

Constraint Setup

Domain Definition

AllDifferent Constraint

Disequality Constraints

Equality Constraint

Search

Points to Remember

Domain Definition

```
var 0..9: S;  
var 0..9: E;  
var 0..9: N;  
var 0..9: D;  
var 0..9: M;  
var 0..9: O;  
var 0..9: R;  
var 0..9: Y;
```

Domain Visualization

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Domain Visualization

Rows =
Variables

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Domain Visualization

Columns = Values

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Domain Visualization

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M			Cells= State							
O										
R										
Y										

Alldifferent Constraint

```
include "alldifferent.mzn";  
  
constraint alldifferent([S,E,N,D,M,O,R,Y]);
```

- Built-in alldifferent predicate included
- No initial propagation possible
- *Suspends*, waits until variables are changed
- When variable is fixed, remove value from domain of other variables
- *Forward checking*

Alldifferent Visualization

Uses the same representation as the domain visualizer

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Disequality Constraints

```
constraint S != 0;  
constraint M != 0;
```

Remove value from domain

$$S \in \{1..9\}, M \in \{1..9\}$$

Constraints solved, can be removed

Domains after Disequality

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Equality Constraint

- Normalization of linear terms
 - Single occurrence of variable
 - Positive coefficients
- Propagation

Normalization

1000*S+	100*E+	10*N+	D
+1000*M+	100*O+	10*R+	E
10000*M+	1000*O+	100*N+	10*E+ Y

Normalization

1000*S+	100*E+	10*N+	D
+1000*M+	100*O+	10*R+	E
10000*M+	1000*O+	100*N+	10*E+

Normalization

1000*S+	100*E+	10*N+	D
+	100*O+	10*R+	E
9000*M+	1000*O+	100*N+	10*E+ Y

Normalization

1000*S+	100*E+	10*N+	D
+	100*O+	10*R+	E
9000*M+ 1000*O+	100*N+	10*E+	Y

Normalization

1000*S+	100*E+	10*N+	D
	+ 10*R+	E	
9000*M+	900*O+	100*N+	10*E+ Y

Normalization

1000*S+	100*E+	10*N+	D
		+ 10*R+	E
9000*M+	900*O+	100*N+	10*E+ Y

Normalization

$$\begin{array}{r} 1000^*S+ \quad 100^*E+ \quad D \\ \quad \quad \quad + \quad 10^*R+ \quad E \\ \hline 9000^*M+ \quad 900^*O+ \quad \mathbf{90^*N+} \quad 10^*E+ \quad Y \end{array}$$

Normalization

$$\begin{array}{r} 1000^*S+ \quad \mathbf{100^*E+} \quad \quad \quad D \\ \quad \quad \quad + \quad 10^*R+ \quad \mathbf{E} \\ \hline 9000^*M+ \quad 900^*O+ \quad 90^*N+ \quad \mathbf{10^*E+} \quad Y \end{array}$$

Normalization

$$\begin{array}{r} 1000^*S+ \quad 91^*E+ \quad \quad \quad D \\ \quad \quad \quad + \quad 10^*R \\ \hline 9000^*M+ \quad 900^*O+ \quad 90^*N+ \quad \quad \quad Y \end{array}$$

Simplified Equation

$$1000 * S + 91 * E + 10 * R + D = 9000 * M + 900 * O + 90 * N + Y$$

Propagation

$$1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9} = \\ 9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}$$

Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{1000..9918} =$$
$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..89919}$$

Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} =$$
$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} =$$
$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

Deduction:

$$M = 1, S = 9, O \in \{0..1\}$$

Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} =$$
$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

Deduction:

$$M = 1, S = 9, O \in \{0..1\}$$

Why? 

Consider lower bound for S

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Lower bound of equation is 9000
- Rest of lhs (left hand side) ($91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}$) is atmost 918
- S must be greater or equal to $\frac{9000 - 918}{1000} = 8.082$
 - otherwise lower bound of equation not reached by lhs
- S is integer, therefore $S \geq \lceil \frac{9000 - 918}{1000} \rceil = 9$
- S has upper bound of 9, so $S = 9$

Consider upper bound of M

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}$ is at least 0
- M must be smaller or equal to $\frac{9918-0}{9000} = 1.102$
- M must be integer, therefore $M \leq \lfloor \frac{9918-0}{9000} \rfloor = 1$
- M has lower bound of 1, so $M = 1$

Consider upper bound of O

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $9000 * 1 + 90 * N^{0..9} + Y^{0..9}$ is at least 9000
- O must be smaller or equal to $\frac{9918 - 9000}{900} = 1.02$
- O must be integer, therefore $O \leq \lfloor \frac{9918 - 9000}{900} \rfloor = 1$
- O has lower bound of 0, so $O \in \{0..1\}$

Propagation of equality: Result

	0	1	2	3	4	5	6	7	8	9
S	-	-	-	-	-	-	-	-	-	*
E										
N										
D										
M	*	*	-	-	-	-	-	-	-	-
O			*	*	*	*	*	*	*	*
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S	-	-	-	-	-	-	-	-	-	*
E										
N										
D										
M	*	-	-	-	-	-	-	-	-	-
O		*	*	*	*	*	*	*	*	*
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										*
E										
N										
D										
M		*								
O										
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										█
E		█								
N		█								
D		█								
M	█	█	█							
O		█								
R		█								
Y		█								

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										█
E										
N										
D										
M			█							
O	★									
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										█
E	█									
N	█									
D	█									
M			█							
O	█	*								
R	█									
Y	█									

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										█
E										
N										
D										
M		█								
O	█									
R										
Y										

$$O = 0, [E, R, D, N, Y] \in \{2..8\}$$

Waking the equality constraint

- Triggered by assignment of variables
- or update of lower or upper bound

Removal of constants

$$1000 * 9 + 91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = \\ 9000 * 1 + 900 * 0 + 90 * N^{2..8} + Y^{2..8}$$

Removal of constants

$$\begin{aligned}1000 * 9 + 91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = \\9000 * 1 + 900 * 0 + 90 * N^{2..8} + Y^{2..8}\end{aligned}$$

Removal of constants

$$91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = 90 * N^{2..8} + Y^{2..8}$$

Propagation of equality (Iteration 1)

$$\frac{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8}}{204..816} = \frac{90 * N^{2..8} + Y^{2..8}}{182..728}$$

Propagation of equality (Iteration 1)

$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = 90 * N^{2..8} + Y^{2..8}}_{204..728}$$

Propagation of equality (Iteration 1)

$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8}}_{204..728} = 90 * N^{2..8} + Y^{2..8}$$

$$N \geq 3 = \lceil \frac{204 - 8}{90} \rceil, E \leq 7 = \lfloor \frac{728 - 22}{91} \rfloor$$

Propagation of equality (Iteration 2)

$$91 * E^{2..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}$$

Propagation of equality (Iteration 2)

$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8}}_{204..725} = \underbrace{90 * N^{3..8} + Y^{2..8}}_{272..728}$$

Propagation of equality (Iteration 2)

$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8}}_{272..725} = 90 * N^{3..8} + Y^{2..8}$$

Propagation of equality (Iteration 2)

$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8}}_{272..725} = 90 * N^{3..8} + Y^{2..8}$$
$$E \geq 3 = \lceil \frac{272 - 88}{91} \rceil$$

Propagation of equality (Iteration 3)

$$91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}$$

Propagation of equality (Iteration 3)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = \underbrace{90 * N^{3..8} + Y^{2..8}}_{272..728}$$

Propagation of equality (Iteration 3)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = 90 * N^{3..8} + Y^{2..8}$$

Propagation of equality (Iteration 3)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = 90 * N^{3..8} + Y^{2..8}$$
$$N \geq 4 = \lceil \frac{295 - 8}{90} \rceil$$

Propagation of equality (Iteration 4)

$$91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}$$

Propagation of equality (Iteration 4)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = \underbrace{90 * N^{4..8} + Y^{2..8}}_{362..728}$$

Propagation of equality (Iteration 4)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{362..725} = 90 * N^{4..8} + Y^{2..8}$$

Propagation of equality (Iteration 4)

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{362..725} = 90 * N^{4..8} + Y^{2..8}$$
$$E \geq 4 = \lceil \frac{362 - 88}{91} \rceil$$

Propagation of equality (Iteration 5)

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}$$

Propagation of equality (Iteration 5)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = \underbrace{90 * N^{4..8} + Y^{2..8}}_{362..728}$$

Propagation of equality (Iteration 5)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = 90 * N^{4..8} + Y^{2..8}$$

Propagation of equality (Iteration 5)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = 90 * N^{4..8} + Y^{2..8}$$
$$N \geq 5 = \lceil \frac{386 - 8}{90} \rceil$$

Propagation of equality (Iteration 6)

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$

Propagation of equality (Iteration 6)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = \underbrace{90 * N^{5..8} + Y^{2..8}}_{452..728}$$

Propagation of equality (Iteration 6)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{452..725} = 90 * N^{5..8} + Y^{2..8}$$

Propagation of equality (Iteration 6)

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{452..725} = 90 * N^{5..8} + Y^{2..8}$$

$$N \geq 5 = \lceil \frac{452 - 8}{90} \rceil, E \geq 4 = \lceil \frac{452 - 88}{91} \rceil$$

No further propagation at this point

Domains after setup

	0	1	2	3	4	5	6	7	8	9
S										■
E										
N										
D										
M		■								
O	■									
R										
Y										

Outline

Problem

Program

Constraint Setup

Search

Step 1

Step 2

Further Steps

Solution

Points to Remember

Search

solve satisfy;

- Try to find a feasible solution, choice left to solver
- Naive search strategy shown here
 - Try variable in order given
 - Try values starting from smallest value in domain
 - When failing, backtrack to last open choice
 - *Chronological Backtracking*
 - *Depth First search*

Search Tree Step 1

S
9
E

Variable S already fixed

Step 2, Alternative $E = 4$

Variable $E \in \{4..7\}$, first value tested is 4



Assignment $E = 4$

	0	1	2	3	4	5	6	7	8	9
S										
E					★	-	-	-		
N										
D										
M										
O										
R										
Y										

Propagation of $E = 4$, equality constraint

$$91 * 4 + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$

Propagation of $E = 4$, equality constraint

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8}}_{386..452} = \underbrace{90 * N^{5..8} + Y^{2..8}}_{452..728}$$

Propagation of $E = 4$, equality constraint

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8}}_{452} = 90 * N^{5..8} + Y^{2..8}$$

Propagation of $E = 4$, equality constraint

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8}}_{452} = 90 * N^{5..8} + Y^{2..8}$$

$$N = 5, Y = 2, R = 8, D = 8$$

Result of equality propagation

	0	1	2	3	4	5	6	7	8	9
S										█
E					█					
N						★	-	-	-	
D			-	-	-	-	-	-	★	
M		█								
O	█									
R			-	-	-	-	-	-	★	
Y			★	-	-	-	-	-	-	

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D			-	-	-	-	-	-		*
M			*							
O	*									
R			-	-	-	-	-	-	*	
Y			*	-	-	-	-	-	-	

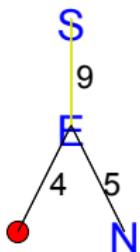
Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N						*	-	-		
D			-	-	-	-	-	-	*	
M										
O	*									
R			-	-	-	-	-	-	*	
Y			*	-	-	-	-	-		

Alldifferent fails!

Step 2, Alternative $E = 5$

Return to last open choice, E , and test next value



Assignment $E = 5$

	0	1	2	3	4	5	6	7	8	9
S										
E					-	*	-	-		
N										
D										
M										
O										
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E					-	*	-	-		
N										
D										
M										
O										
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E						*				
N										
D										
M										
O										
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										█
E						█				
N										
D										
M		█								
O	█									
R										
Y										

$$N \neq 5, N \geq 6$$

Propagation of equality

$$91 * 5 + 10 * R^{2..8} + D^{2..8} = 90 * N^{6..8} + Y^{2..8}$$

Propagation of equality

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8}}_{477..543} = \underbrace{90 * N^{6..8} + Y^{2..8}}_{542..728}$$

Propagation of equality

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8}}_{542..543} = 90 * N^{6..8} + Y^{2..8}$$

Propagation of equality

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8}}_{542..543} = 90 * N^{6..8} + Y^{2..8}$$

$$N = 6, Y \in \{2, 3\}, R = 8, D \in \{7..8\}$$

Result of equality propagation

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D			✗	✗	✗		✗			
M										
O										
R			-	-	-		-	-		✗
Y					✗		✗	✗	✗	

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N							*	-	-	
D		*	*	*		*				
M										
O	*									
R			-	-	-		-	-	*	
Y					*		*	*	*	

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

$$D = 7$$

Propagation of equality

$$91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}$$

Propagation of equality

$$\underbrace{91 * 5 + 10 * 8 + 7}_{542} = \underbrace{90 * 6 + Y^{2..3}}_{542..543}$$

Propagation of equality

$$\underbrace{91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}}_{542}$$

Propagation of equality

$$\underbrace{91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}}_{542}$$
$$Y = 2$$

Last propagation step

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

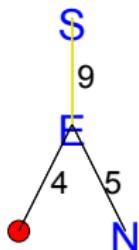
The table shows the state of variables S through Y across 10 time steps (0 to 9). Cells are colored according to their state:

- Grey: Unassigned or initial state.
- Maroon: Assigned value.
- Orange with a black asterisk (*): Assigned value with a propagation mark.
- Grey with a minus sign (-): Invalid or removed value.

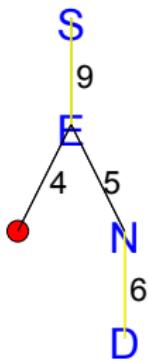
Specific values observed:

- S: Unassigned (Grey).
- E: Unassigned (Grey).
- N: Unassigned (Grey).
- D: Unassigned (Grey).
- M: Unassigned (Grey).
- O: Unassigned (Grey).
- R: Unassigned (Grey).
- Y: Unassigned (Grey).
- Step 0: No assignments.
- Step 1: No assignments.
- Step 2: No assignments.
- Step 3: No assignments.
- Step 4: No assignments.
- Step 5: E is maroon (value assigned).
- Step 6: N is maroon (value assigned).
- Step 7: D is maroon (value assigned).
- Step 8: R is maroon (value assigned).
- Step 9: S is maroon (value assigned).
- Step 10: Y is orange with an asterisk (*), indicating a propagated value.

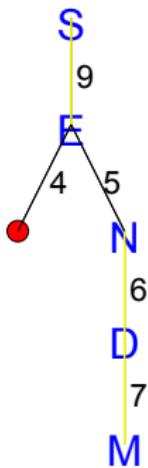
Further Steps: Nothing more to do



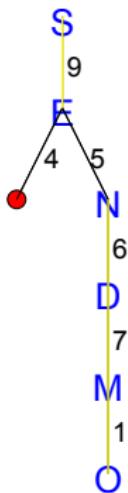
Further Steps: Nothing more to do



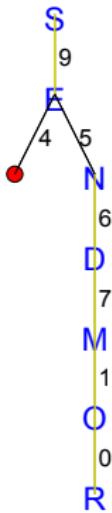
Further Steps: Nothing more to do



Further Steps: Nothing more to do



Further Steps: Nothing more to do



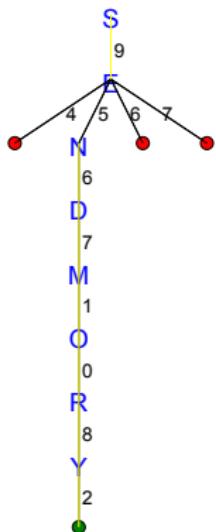
Further Steps: Nothing more to do



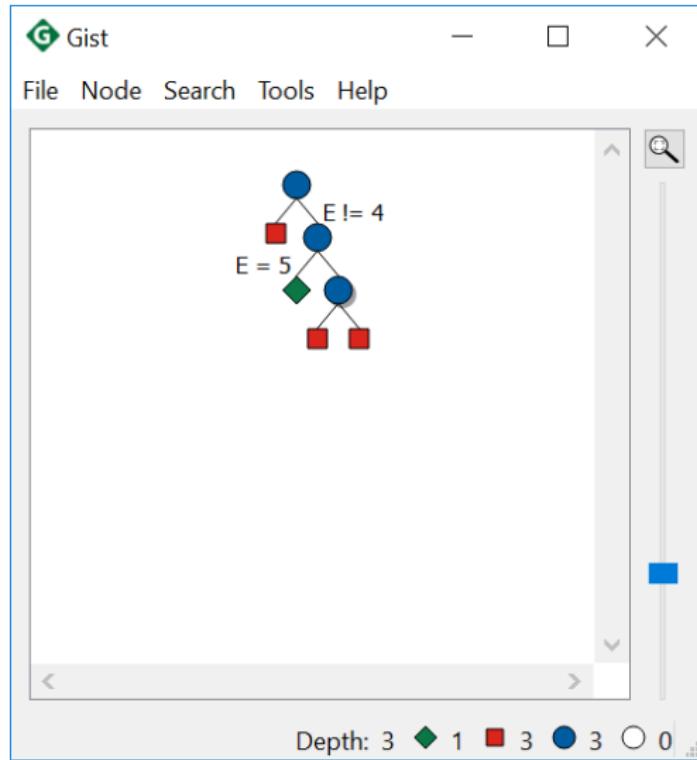
Further Steps: Nothing more to do



Complete Search Tree



Search Tree with Gecode/GIST



Some Differences

- Uses binary branching
 - var equal value, var not equal value
- Solutions in green, failure leafs in red, internal nodes in blue
- By default, shows all failed sub trees collapsed
- By default, uses different search strategy

Solution

$$\begin{array}{r} 9 & 5 & 6 & 7 \\ + & 1 & 0 & 8 & 5 \\ \hline 1 & 0 & 6 & 5 & 2 \end{array}$$

Outline

Problem

Program

Constraint Setup

Search

Points to Remember

Points to Remember

- Constraint models are expressed by variables and constraints.
- Problems can have many different models, which can behave quite differently. Choosing the best model is an art.
- Constraints can take many different forms.
- Propagation deals with the interaction of variables and constraints.
- It removes some values that are inconsistent with a constraint from the domain of a variable.
- Constraints only communicate via shared variables.

Points to Remember

- Propagation usually is not sufficient, search may be required to find a solution.
- Propagation is data driven, and can be quite complex even for small examples.
- The default search uses chronological depth-first backtracking, systematically exploring the complete search space.
- The search choices and propagation are interleaved, after every choice some more propagation may further reduce the problem.

For Puzzle Purists Only

- We did not follow the puzzler ethos!
- We should solve the puzzle without making choices
- Even a case analysis should be avoided
- The puzzle has a single solution, we should be able to deduce the solution
- In the process shown, we are limited by the underlying assumptions
 - Treat each constraint on its own, they interact only by domains of variables
 - We only use the constraints that we stated in the model
- Can we do better than that?

▶ Skip

Better Reasoning

- Three possible approaches (possibly many more)
 - Full domain reasoning for arithmetic, not just bound reasoning
 - Interaction of *sum* and *alldifferent* constraints
 - Deduced implied constraint

Looking at more than just bounds

- We only considered the smallest and largest values that can be achieved in the sum constraint
- We can do more
 - Can any of the values between be expressed as the sum of the terms
 - Consider holes in the domains, and in the range of possible values for LHS and RHS
- Usually not done in actual solvers for arithmetic constraints
- Easy to do with Dynamic Programming

Consider the interaction of multiple constraints

- Usually ignored, as only interaction is via domains of shared variables
- Here: Sum and *alldifferent* interact
 - When considering the bounds, we cannot assume that each variable takes its smallest/largest value independently
 - Find feasible assignment that minimizes/maximizes the total weight
 - To do this properly, we need some non-trivial reasoning
- Do we do this combined reasoning automatically, or only when prompted by the modeler?

Deduced Implied Constraints

- Look at the partially solved puzzle

$$\begin{array}{r} 9\text{END} \\ +10\text{RE} \\ \hline \end{array}$$

- 10NEY
- In the hundreds position, we have $E + 0 + C_{10} = N + 10 * C_{100}$, with C_{10} the 0/1 carry from the tens position
 - NB: No carry C_{100} into the thousands, $C_{100} = 0$
 - N must be equal to $E + 1$ with $C_{10} = 1$
 - If $C_{10} = 0$, then $N = E$, not possible
 - We can substitute $N = E + 1$ into our main equation, but keep $N = E + 1$ as well

Expert Mode Reasoning

Starting with

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$

Expert Mode Reasoning

we get

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * E^{4..7} + 90 + Y^{2..8}$$

Expert Mode Reasoning

Eliminating duplicate occurrences of E

$$\underbrace{E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{26..95} = \underbrace{90 + Y^{2..8}}_{92..98}$$

shared range 92..95

Expert Mode Reasoning

To reach 92, R must be equal to 8, therefore N, E, D, Y must be less than 8

As $N = E + 1$, E must be less than 7

$$E^{4..6} + 10 * 8 + D^{2..7} = 90 + Y^{2..7}$$

Expert Mode Reasoning

Simplification yields

$$\underbrace{E^{4..6} + D^{2..7}}_{6..13} = \underbrace{10 + Y^{2..7}}_{12..17}$$

shared range 12..13

Expert Mode Reasoning

To reach 12 on LHS, E must be greater than 4, D must be greater than 5

To reach 13 in RHS, Y must be smaller than 4

$$E^{5..6} + D^{6..7} = 10 + Y^{2..3}$$

Expert Mode Reasoning

As both N and D must be in 6..7, no other variable can use those values

NB: This requires better reasoning than *forward checking* on the *alldifferent* constraint

If we remove 6 from the domain of E, E must be $E = 5$, and thus $N = 6$, due to $N = E + 1$

Alldifferent forces $D = 7$, leaving

$$5 + 7 = 10 + Y^{2..4}$$

Expert Mode Reasoning

This only leaves $Y = 2$

$$5 + 7 = 10 + 2$$

That is a solution, the constraint is satisfied

Expert Mode Summary

- Often there is more propagation that can be done
- Can be difficult/expensive to do
- Balancing
 - How much work it done at each step of search?
 - How many steps of search you need?
- For hard problems, doing all possible propagation may be exponential
- Not aware that any CP system does the full reasoning shown here

This is how people solve the puzzle by hand



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- When writing the first version of this puzzle for CHIP (in 1986), we wanted to mimic the way we solve the puzzle by hand

Part II

Global Constraints

Example 2: Sudoku

- Global Constraints
 - Powerful modelling abstractions
 - Non-trivial propagation
 - Different consistency levels
- Example: Sudoku puzzle

Outline

Problem

Initial Propagation (Forward Checking)

Improved Reasoning

Search

Other Global Constraints

Problem Definition

Sudoku

Fill in numbers from 1 to 9 so that each row, column and 3x3 block contain each number exactly once

4	1 2 3	8	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
1 2 3	2 3 4	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	4 5 6	1 7	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	4 5 6	8	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	6	4 5 6	4 5 6	8	2	5	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	3	7	6	4	5	6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	9	4 5 6	4 5 6
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
2	7	1 2	4 5	4 5	5	4 5 6	4 5 6	4 5 6
4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	7 8 9	7 8 9	7 8 9
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9

Problem Definition

Sudoku

Fill in numbers from 1 to 9 so that each row, column and 3x3 block contain each number exactly once

4	5	3	8	2	6	7	9	1
1	2	3	2	3	1	2	3	4
4	5	6	5	6	4	5	6	7
7	8	9	8	9	7	8	9	1
1	2	3	2	3	1	2	3	4
4	5	6	5	6	4	5	6	7
7	8	9	8	9	7	8	9	1
1	2	3	2	3	1	2	3	4
4	5	6	5	6	4	5	6	7

4	2	8	5	6	3	1	7	9
3	5	9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	3	9	8	2	5	7
5	9	2	7	4	1	3	8	6
8	3	7	6	2	5	9	4	1
2	7	4	9	5	6	8	1	3
6	8	3	2	1	4	7	9	5
9	1	5	8	3	7	6	2	4

Model

- A variable for each cell, ranging from 1 to 9
- A 9×9 matrix of variables describing the problem
- Preassigned integers for the given hints
- all different constraints for each row, column and 3×3 block

Sudoku Models

- ECLiPSe [▶ Show](#)
- MiniZinc [▶ Show](#)
- NumberJack [▶ Show](#)
- CPMPy [▶ Show](#)
- Choco-solver [▶ Show](#)

ECLIPSe Sudoku Model (from <https://eclipseclp.org/>)

```
:- lib(ic).
:- import alldifferent/1 from ic_global.

top :-  
    problem(Board),  
    print_board(Board),  
    sudoku(Board),  
    labeling(Board),  
    print_board(Board).

sudoku(Board) :-  
    dim(Board, [N,N]),  
    Board :: 1..N,  
    ( for(I,1,N), param(Board) do  
        alldifferent(Board[I,*]),  
        alldifferent(Board[*,I])  
    ),  
    NN is integer(sqrt(N)),  
    ( multifor([I,J],1,N,NN), param(Board,NN) do  
        alldifferent(concat(Board[I..I+NN-1, J..J+NN-1]))  
    ).  
  
print_board(Board) :-  
    dim(Board, [N,N]),  
    ( for(I,1,N), param(Board,N) do  
        ( for(J,1,N), param(Board,I) do  
            X is Board[I,J],  
            ( var(X) -> write(" _") ; printf(" %2d", [X]) )  
        ), nl  
    ), nl.
```

ECLiPSe Data Definition

```
problem([](
```

- [] (4, _, 8, _, _, _, _, _, _, _),
- [] (_, _, _, 1, 7, _, _, _, _, _),
- [] (_, _, _, _, 8, _, _, _, 3, 2),
- [] (_, _, 6, _, _, 8, 2, 5, _),
- [] (_, 9, _, _, _, _, _, 8, _),
- [] (_, 3, 7, 6, _, _, 9, _, _),
- [] (2, 7, _, _, 5, _, _, _, _),
- [] (_, _, _, _, 1, 4, _, _, _),
- [] (_, _, _, _, _, _, 6, _, 4))).

► Continue

MiniZinc Sudoku Model

```
int: s;
int: n=s*s;
array[1..n,1..n] of var 1..n: puzzle;

include "sudoku.dzn";

predicate alldifferent(array[int] of var int: x) =
  forall(i,j in index_set(x) where i < j)
    (x[i] != x[j]);

constraint forall(i in 1..n)
  (alldifferent([puzzle[i,j] | j in 1..n]));
constraint forall(j in 1..n)
  (alldifferent([ puzzle[i,j] | i in 1..n]));
constraint forall(i,j in 1..s)
  (alldifferent([puzzle[s*(i-1)+p, s*(j-1)+q] |
    p,q in 1..s]));
solve satisfy;
```

MiniZinc Output

```
output [ "sudoku:\n" ] ++
[ show(puzzle[i,j]) ++
if j = n then
  if i mod s = 0 /\ i < n then "\n\n"
  else "\n"
  endif
else
  if j mod s = 0 then " "
  else " "
  endif
endif
| i,j in 1..n];
```

MiniZinc Data File (sudoku.dzn)

```
s=3;  
puzzle=[|  
    4, _, 8, _, _, _, _, _, _, _ |  
    _, _, _, 1, 7, _, _, _, _, _ |  
    _, _, _, _, 8, _, _, _, 3, 2 |  
    _, _, 6, _, _, 8, 2, 5, _ |  
    _, 9, _, _, _, _, _, _, 8, _ |  
    _, 3, 7, 6, _, _, 9, _, _ |  
    2, 7, _, _, 5, _, _, _, _ |  
    _, _, _, _, 1, 4, _, _, _, _ |  
    _, _, _, _, _, 6, _, 4, _ |  
];
```

▶ Continue

NumberJack Sudoku Model

```
from Numberjack import *

def get_model(N, clues):
    grid = Matrix(N*N, N*N, 1, N*N)

    sudoku = Model([
        AllDiff(row) for row in grid.row],
        [AllDiff(col) for col in grid.col],
        [AllDiff(grid[x:x + N, y:y + N]) for x in range(0, N*N, N)
         for y in range(0, N * N, N)],
        [(x == int(v)) for x, v in
         zip(grid.flat, "".join(open(clues)).split()) if v != '*']
    )
    return grid, sudoku

def solve(param):
    N = param['N']
    clues = param['file']

    grid, sudoku = get_model(N, clues)

    solver = sudoku.load(param['solver'])
    solver.setVerbosity(param['verbose'])
    solver.setTimeLimit(param['tcutoff'])

    solver.solve()
```

NumberJack Data File

```
4 * 8 * * * * *
* * * 1 7 * * * *
* * * * 8 * * 3 2
* * 6 * * 8 2 5 *
* 9 * * * * 8 *
* 3 7 6 * * 9 * *
2 7 * * 5 * * * *
* * * * 1 4 * * *
* * * * * 6 * 4
```

▶ Continue

CPMpy Sudoku Model (from <https://github.com/CPMpy/>)

```
import numpy as np
from cpmpy import *

# Variables
puzzle = intvar(1,9, shape=given.shape, name="puzzle")

model = Model(
    # Constraints on values (cells that are not empty)
    puzzle[given!=e] == given[given!=e], # numpy's indexing, vectorized equality
    # Constraints on rows and columns
    [AllDifferent(row) for row in puzzle],
    [AllDifferent(col) for col in puzzle.T], # numpy's Transpose
)

# Constraints on blocks
for i in range(0,9, 3):
    for j in range(0,9, 3):
        model += AllDifferent(puzzle[i:i+3, j:j+3]) # python's indexing

model.solve()
```

CPMy Data Definition

```
e = 0 # value for empty cells
given = np.array([
    [4, e, 8, e, e, e, e, e, e],
    [e, e, e, 1, 7, e, e, e, e],
    [e, e, e, 8, e, e, 3, 2],
    [e, e, 6, e, e, 8, 2, 5, e],
    [e, 9, e, e, e, e, e, 8, e],
    [e, 3, 7, 6, e, e, 9, e, e],
    [2, 7, e, e, 5, e, e, e, e],
    [e, e, e, e, 1, 4, e, e, e],
    [e, e, e, e, e, e, 6, e, 4]
])

```

▶ Continue

Choco-solver Sudoku Model

```
Model model = new Model("Sudoku");
int blockSize = 3;
int m = blockSize*blockSize;

IntVar[][] vars = new IntVar[m][m];
for(int i=0;i<m;i++) {
    for(int j=0;j<m;j++) {
        vars[i][j] = model.intVar("X"+i+"_"+j, 1, m);
        if (data[i][j]>0) {
            model.arithm(vars[i][j], "=", data[i][j]).post();
        }
    }
}
for(int i=0;i<m;i++) {
    model.allDifferent(row(i,m,vars)).post();
    model.allDifferent(column(i,m,vars)).post();
}
for(int i=0;i<m;i+=blockSize) {
    for(int j=0;j<m;j+=blockSize) {
        model.allDifferent(block(i,j,blockSize,vars)).post();
    }
}
Solver solver = model.getSolver();
solver.solve();
```

Choco-solver Data

```
int[][] data = new int[m][m] {  
    {4, 0, 8, 0, 0, 0, 0, 0, 0},  
    {0, 0, 0, 1, 7, 0, 0, 0, 0},  
    {0, 0, 0, 0, 8, 0, 0, 3, 2},  
    {0, 0, 6, 0, 0, 8, 2, 5, 0},  
    {0, 9, 0, 0, 0, 0, 0, 8, 0},  
    {0, 3, 7, 6, 0, 0, 9, 0, 0},  
    {2, 7, 0, 0, 5, 0, 0, 0, 0},  
    {0, 0, 0, 0, 1, 4, 0, 0, 0},  
    {0, 0, 0, 0, 0, 0, 6, 0, 4}  
};
```

Choco-solver Utilities

```
IntVar[] row(int row, int size, IntVar[][] array){  
    return array[row];  
}  
  
IntVar[] column(int col,int size,IntVar[][] array){  
    IntVar[] column = new IntVar[size];  
    for(int i=0; i<size; i++){  
        column[i] = array[i][col];  
    }  
    return column;  
}  
  
IntVar[] block(int x,int y,int blockSize,IntVar[][] array){  
    IntVar[] block = new IntVar[blockSize*blockSize];  
    int k=0;  
    for(int i=0;i<blockSize;i++){  
        for(int j=0;j<blockSize;j++){  
            block[k++] = array[x+i][y+j];  
        }  
    }  
    return block;  
}
```

» Continue

Domain Visualizer

- Problem shown as matrix
 - Each cell corresponds to a variable
 - Instantiated: Shows integer value (large)
 - Uninstantiated: Shows values in domain

A 9x9 grid containing a 4x4 Sudoku puzzle. The grid has red numbers (4, 8, 1, 7, 6, 8, 9, 2) and green numbers (4, 5, 6, 7, 8, 9, 7, 8, 9). The red numbers are located in the top-left 4x4 subgrid, while the green numbers are scattered throughout the rest of the grid.

Outline

Problem

Initial Propagation (Forward Checking)

Improved Reasoning

Search

Other Global Constraints

Initial State (Forward Checking)

4	1 2 3	8	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4	5 6	8	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	6	1 2 3	1 2 3	8	2	5	1 2 3	4 5 6
4 5 6	4 5 6	7 8 9	4 5 6	4 5 6	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	9	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	4 5 6
4 5 6	4 5 6	7 8 9	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	3	7	6	1 2 3	1 2 3	9	1 2 3	1 2 3
4 5 6	4 5 6	7 8 9	4 5 6	4 5 6	7 8 9	7 8 9	7 8 9	7 8 9	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
2	7	1 2 3	1 2 3	5	1 2 3	1 2 3	1 2 3	1 2 3	4 5 6
4 5 6	4 5 6	7 8 9	4 5 6	4 5 6	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	4 5 6
4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	6	1 2 3	4 5 6
4 5 6	4 5 6	7 8 9	4 5 6	4 5 6	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9

Propagation Steps (Forward Checking)

	1	2	3		1	2	3	1	2	3	1	2	3	1	2	3				
4	4	5	6		4	5	6	4	5	6	4	5	6	4	5	6				
7	8	9			7	8	9	7	8	9	7	8	9	7	8	9				
1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3			
4	5	6	4	5	6	4	5	6	4	5	6	4	5	6	4	5	6			
7	8	9	7	8	9	7	8	9	1	7	1	2	3	4	5	6				
1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	7	8	9			
4	5	6	4	5	6	4	5	6	8	1	2	3	4	5	6	4	5	6		
7	8	9	7	8	9	7	8	9	7	8	9	3	2	1	2	3	2			
1	2	3	1	2	3	1	2	3	1	2	3	8	2	5	1	2	3			
4	5	6	4	5	6	4	5	6	4	5	6	8	2	5	4	5	6			
7	8	9	7	8	9	7	8	9	6	4	5	6	4	5	6	7	8	9		
1	2	3	1	2	3	1	2	3	1	2	3	8	2	5	1	2	3			
4	5	6	4	5	6	4	5	6	4	5	6	8	2	5	4	5	6			
7	8	9	7	8	9	7	8	9	9	4	5	6	4	5	6	7	8	9		
1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3			
4	5	6	4	5	6	4	5	6	4	5	6	9	4	5	6	4	5	6		
7	8	9	7	8	9	7	8	9	3	7	6	4	5	6	7	8	9			
1	2	3	1	2	3	1	2	3	1	2	3	9	1	2	3	1	2	3		
4	5	6	4	5	6	4	5	6	4	5	6	9	4	5	6	4	5	6		
7	8	9	7	8	9	7	8	9	3	7	6	4	5	6	7	8	9			
2	7		1	2	3	1	2	3	5	1	2	3	1	2	3	1	2	3		
			4	5	6	4	5	6	7	4	5	6	4	5	6	4	5	6		
			7	8	9	7	8	9	7	8	9	5	4	5	6	7	8	9		
1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3			
4	5	6	4	5	6	4	5	6	1	4	4	5	6	4	5	6	4	5	6	
7	8	9	7	8	9	7	8	9	7	8	9	6	4	5	6	7	8	9		
1	2	3	1	2	3	1	2	3	1	2	3	6	4	5	6	1	2	3		
4	5	6	4	5	6	4	5	6	7	8	9	4	5	6	4	5	6	4	5	6
7	8	9	7	8	9	7	8	9	7	8	9	4	5	6	7	8	9			
1	2	3	1	2	3	1	2	3	1	2	3	4	5	6	4	5	6	4	5	6
4	5	6	4	5	6	4	5	6	7	8	9	4	5	6	4	5	6	4	5	6
7	8	9	7	8	9	7	8	9	7	8	9	4	5	6	7	8	9			

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Propagation Steps (Forward Checking)

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Propagation Steps (Forward Checking)

4	1 2 3 5 6 7 9	8	1 2 3 5 6 7 9	1 2 3 5 6 9 7	1 2 3 5 6 9 9			
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 7	1 2 3 4 5 6 7 8 9					
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	2		
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	4 5 6 7 8 9		
1 2 3 4 5 6 7 8 9	3 7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	4 5 6 7 8 9		
2 7	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9					
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 4	1 2 3 4 5 6 7 8 9					
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	4 5 6 7 8 9	4	

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Propagation Steps (Forward Checking)

4	1 2 3	5 6	8	1 2 3	5 6	5 6	5 6	5 6	5 6	5 6	5 6
7	9	7	9	7	9	7	9	7	9	7	9
1	3	1 2 3	1 2 3	1	7	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
5	6	4 5 6	4 5 6	5	6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7	8 9	7 8 9	7 8 9	8	1	7	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
1	3	1 2 3	1 2 3	1 2 3	8	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
5	6	4 5 6	4 5 6	4 5 6	8	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7	8 9	7 8 9	7 8 9	7	8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1	3	1 2 3	1 2 3	6	1 2 3	1 2 3	8	2	5	1 2 3	1 2 3
5	6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	8	2	5	4 5 6	4 5 6
7	8 9	7 8 9	7 8 9	7	8 9	7 8 9	7	8 9	7	8 9	7 8 9
1	3	9	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	8	1 2 3	1 2 3
5	6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	8	4 5 6	4 5 6
7	8 9	7 8 9	7 8 9	7	8 9	7 8 9	7	8 9	7	8 9	7 8 9
1	3	3	7	6	1 2 3	1 2 3	9	1 2 3	1 2 3	1 2 3	1 2 3
5	6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	9	4 5 6	4 5 6	4 5 6	4 5 6
7	8 9	7 8 9	7 8 9	7	8 9	7 8 9	7	8 9	7	8 9	7 8 9
2	7	1 2 3	1 2 3	5	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4	5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7	8 9	7 8 9	7 8 9	7	8 9	7 8 9	7	8 9	7	8 9	7 8 9
1	3	1 2 3	1 2 3	1 2 3	1	4	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
5	6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7	8 9	7 8 9	7 8 9	7	8 9	7 8 9	7	8 9	7	8 9	7 8 9
1	3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	6	1 2 3	1 2 3	4	4 5 6
5	6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	6	4 5 6	4 5 6	4	5 6
7	8 9	7 8 9	7 8 9	7	8 9	7 8 9	7	8 9	7	8 9	7 8 9

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Propagation Steps (Forward Checking)

4	1 2 3 5 6	8	1 2 3 5 6 6 5	1 2 3 6 5 5 6	1 2 3 5 6 5 6	1 2 3 5 6 5 6	1 2 3 5 6 5 6
7	9		7 9 7 9 7 9	9 7 9 7 9 7			
3	2 3 4 5 6 8 9	1	7	2 3 4 5 6 8 9			
5 6	4 5 6 8 9	7	8	1 2 3 4 5 6 7 8 9			
1	3 1 2 3 5 6 4 5 6 7 8 9 7 8 9	8	3	2	3	2	3
1	3 1 2 3 5 6 4 5 6 7 8 9 7 8 9	6	8	2	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 1 2 3 5 6 4 5 6 7 8 9 7 8 9	9	8	2	5	8	1 2 3 4 5 6 7 8 9
1	3 1 2 3 5 6 4 5 6 7 8 9 7 8 9	3 7	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9
2	7	5	1 2 3 4 5 6 7 8 9				
1	3 1 2 3 5 6 4 5 6 7 8 9 7 8 9	1	4	1 2 3 4 5 6 7 8 9			
1	3 1 2 3 5 6 4 5 6 7 8 9 7 8 9	6	4	1 2 3 4 5 6 7 8 9			

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	1 2 3 5 6 9 7					
3	2	2 3	1	7	2 3	2 3	2 3	2 3
5 6	4 5 6	4 5 6	8	9	4 5 6	4 5 6	4 5 6	4 5 6
8 9	8	8 9	1	3	1 2 3	1 2 3	1 2 3	1 2 3
1	3	1 2	1 2 3	1 2 3	8	4 5 6	4 5 6	3
5 6	4 5 6	4 5 6	7	8 9	7 8 9	7 8 9	7 8 9	2
7 8 9	8	7 8 9	7 8 9	7 8 9	8	2	5	
1	3	1 2	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	
5 6	4 5 6	6	4 5 6	4 5 6	8	2	5	4 5 6
7 8 9	8	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	
1	3	9	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	
5 6	4 5 6	4 5 6	7	8 9	7 8 9	7 8 9	7 8 9	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	
1	3	3	7	6	1 2 3	1 2 3	1 2 3	1 2 3
5 6	4 5 6	4 5 6	7	8 9	7 8 9	7 8 9	7 8 9	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	9	7 8 9	7 8 9	
2	7	2	7	5	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	4 5 6	7	8 9	7 8 9	7 8 9	7 8 9	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	
1	3	1 2	1 2 3	1 2 3	1	4	1 2 3	1 2 3
5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4	5 6	4 5 6
7 8 9	8	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	
1	3	1 2	1 2 3	1 2 3	1 2 3	6	1 2 3	4
5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4	5 6	
7 8 9	8	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	1 2 3 5 6 5 6					
3	2 5 6 8 9	1	2 3 4 5 6 8 9	7 9 7 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1	1 5 6 7 9	8	1 4 5 6 7 9	3 4 5 6 7 8 9	2 4 5 6 7 8 9			
1	3 1 2 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8 2 5	2 5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 5 6 7 8 9	3 7 6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9 7 8 9	4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	5	1 2 3 4 5 6 7 8 9					
1	3 1 2 5 6 7 8 9	1	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 1 2 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6 7 8 9	4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4 7 8 9

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	1 2 3 5 6 7	1 2 3 5 6 9 7	1 2 3 5 6 9			
3	2 5 6 8 9	2 3 4 5 8	1 7	2 3 4 5 6 8 9	2 3 4 5 6 8 9			
1	1 5 6 7 9	1 4 5 6 9 7 9	1 8	1 4 5 6 7 9 7	1 4 5 6 9 7 9	3	2	
1	3 1 2 5 6 7 8 9	1 2 3 4 5 6 8	1 2 3 4 5 6 7 8 9					
1	3 5 6 7 8 9	9 4 5 9 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	4 5 6 7 8 9	
1	3 5 6 7 8 9	3 7 6 9 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	4 5 6 7 8 9	
2	7	1 2 3 4 5 9 7 8 9	5	1 2 3 4 5 6 7 8 9				
1	3 1 2 5 6 7 8 9	1 2 3 4 5 6 8	1 4	1 2 3 4 5 6 7 8 9				
1	3 1 2 5 6 7 8 9	1 2 3 4 5 6 8	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4		

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	1 2 3 5 6 7 9	1 2 3 5 6 9 7	1 2 3 5 6 9 9			
3	2 5 6 8 9	2 3 4 5 8	1 7	2 3 4 5 6 8 9				
1	1 5 6 7 9	1 4 5 6 9 7 9	1 8	1 4 5 6 7 9 7 9	1 4 5 6 7 9 7 9	1 4 5 6 7 9 7 9	3 2	1 3 4
1	3 1 7 9	6 4	1 3 1 3 4 4	8 7 9 7 9	2 8 2 5	5 7 9	1 3 4 7 9	1 3 4 7 9
1	3 5 6 7 8 9	9 4 5 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	8 7 8 9	1 2 3 4 5 6 7 8 9
1	3 5 6 7 8 9	3 7 6 4 5 9 7 8 9	1 2 3 4 5 6 7 8 9					
2	7	5 4 5 9 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9 8	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9 8	6 4	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	4 7 8 9	1 2 3 4 5 6 7 8 9

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5	1 2 3 5 6	1 2 3 5 6			
3	2 5 6 4 5 8 9	2 3 4 5 9	1 7	2 3 4 5 6 8 9	2 3 4 5 6 8 9			
1	1 5 6 4 5 6 7 9	1 4 5 9 7 9	8	1 4 5 6 7 9 7 9	1 4 5 6 7 9 7 9	3	2	
1	3 1 4 7 9	6	3 1 4 4 7 9 7 9	8 2 5	1 3 4 7 9			
1	3 5 6 7 8 9	9	1 2 3 4 5 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	8	4 5 6 7 8 9	
1	3 5 6 7 8 9	3 7 6	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	9	4 5 6 7 8 9	
2	7	1 2 3 4 5 9 7 8 9	5	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 9 7 8 9	1 4	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 9 7 8 9	6	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	4		

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 5 6 5 6 5 6 5 6 5 6	1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3	1 2 3 2 3 2 3 4 5 6 4 5 6 4 5 6 4 5 6	1 2 3 2 3 2 3 4 5 6 4 5 6 4 5 6 4 5 6	1 2 3 2 3 2 3 4 5 6 4 5 6 4 5 6 4 5 6
3	2 5 6 4 5 6 8 9	2 3 1 9	7 9 7 9 7 9	1 2 3 5 6 4 5 6 8 9	2 3 4 5 6 8 9 8 9 8 9 8 9	2 3 4 5 6 4 5 6 8 9 8 9 8 9	2 3 4 5 6 4 5 6 8 9 8 9 8 9
1	1 5 6 4 5 6 7 9	1 8	1 4 5 9 7 9	1 4 5 6 7 9 7	1 4 5 6 7 9 7 9 7 9	1 4 5 6 4 5 6 8	1 3 4 7 9
1	3 1 4 7 9	6	3 1 4 7 9 7	3 1 3 4 5 4 5 7 9 7	8 2 8 2 8	2 5 4 7 9	1 3 4 5 6 7
1	3 5 6 7	9	1 2 3 4 5 7	2 3 1 2 3 4 5 6 4 5 6 7 7 7	1 2 3 1 2 3 4 5 6 4 5 6 7 7 7	1 2 3 4 5 6 8	1 2 3 4 5 6 7
1	3 5 6 7 8 9	3 7	6	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 9 7 8 9	2 3 4 5 7 8 9	5	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 9 7 8 9	2 3 4 5 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 6	1 2 3 4 5 6 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 9 7 8 9	2 3 4 5 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 6	1 2 3 4 5 6 7 8 9

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9
3	2 5 6 4 5 6 8 9	2 3 4 5 8	1 7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1	1 5 6 4 5 6 7 9	1 4 5 9 7 9	8	1 4 5 6 7 9 7 9	1 4 5 6 7 9 7 9	3	2
1	3 1 4 7 9	6	3 4 7 9	3 4 9	8 2	5	1 3 4 7 9
1	3 5 6 7 9	9	1 2 3 4 5 7	2 3 4 6 4 5 6 7 7	1 2 3 4 5 6 7 7	8	1 2 3 4 5 6 7 7
1	3 5 6 7 8 9	3 7 6	1 2 3 4 5 9 7 8 9	2 3 4 6 4 5 6 9 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9
2	7	6	1 2 3 4 5 9 7 8 9	5	1 2 3 4 5 6 7 8 9	7 8 9	1 2 3 4 5 6 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9	8	1 2 3 4 5 9 7 8 9	1 4	1 2 3 4 5 6 7 8 9	7 8 9	1 2 3 4 5 6 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9	8	1 2 3 4 5 9 7 8 9	2 3 4 6 4 5 6 9 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9
							4

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7
3	2 5 6 8 9	2 3 4 5 8	1 7 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1	1 5 6 7 9	1 4 5 9 7 9	1 8 7 9	1 4 5 6 7 9 7	1 4 5 6 7 9 7	3	2	
1	3 1 4 7 9	1 6 4 5 7 9	3 4 9	3 8 9	2 8 2 7 7	5	1 3 4 7 9	
1	3 5 6 7	1 2 3 4 5 7	2 3 4 5 7	2 3 4 6 4 5 6 7 7	1 2 3 4 5 6 7	8	1 2 3 4 5 6 7	
1	5 8	3 7 4 5 9 7 8 9	7 6 4 8	2 1 2 4 5 8	9	1 2 4 5 8	1 2 4 5 8	
2	7	1 2 3 4 5 9 7 8 9	2 3 4 5 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 6 9 7 8 9	2 3 4 5 9 7 8 9	1 2 3 4 6 4 5 6 9 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 6 9 7 8 9	2 3 4 5 9 7 8 9	1 2 3 4 6 4 5 6 9 7 8 9	6	1 2 3 4 5 6 7 8 9	4	

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7
3	2 5 6 8 9	2 3 4 5 8	1 7 9	2 3 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1	1 5 6 7 9	1 4 5 9 7 9	1 8 7 9	1 5 6 7 9	1 4 5 6 9 7 9	3	2	
1	3 1 4 7 9	6	3 4 4 7 9	3 4 6 9	8 5 6 7 7	2 4 5 6 8	5 4 7 9	1 3 4 7 9
1	3 5 6 7	9	1 2 3 4 5 7	2 3 4 6 7	1 2 3 5 6 7	1 2 3 4 5 6 8	1 2 3 4 5 6 7	
1	5 8	3 7 6	2 4	2 5	1 2 9	1 2 4 5 8	1 2 4 5 8	
2	7	1 2 3 4 5 9 7 8 9	2 3 4 5 8 9	5	1 2 3 5 6 7 9	1 2 3 4 5 6 8 9	1 2 3 4 5 6 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 6 8	2 3 4 5 9 7 8 9	1 4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 6 8	2 3 4 5 9 7 8 9	6	1 2 3 4 5 6 9 7 9	1 2 3 4 5 6 7 8 9	4	

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	1 2 3 5 6 9	1 2 3 5 6 9	1 2 3 5 6 9	1 2 3 5 6 9
3	2 5 6 8 9	2 3 4 5 8	1 7	7	2 3 5 6 4 5 6 9 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1	1 5 6 7 9	1 4 5 9 7 9	1 8	1 7	1 5 6 4 5 6 9 7 9	1 3	2 4 5 6 7 9	2 4 5 6 7 9
1	3 1 4	6	3 4 7 9	3 4 9	8 2 9	2 5	1 3 4 7 9	1 3 4 7 9
1	3 5 6 7	9	1 2 3 4 5 7	2 3 4 6 7	1 2 3 5 6 4 5 6 7 7	1 2 3 8	1 2 3 4 5 6 7	1 2 3 4 5 6 7
1	5 8	3 7	6	4	2 1 2 5	9	1 2 4 5 8	1 2 4 5 8
2	7	1 3 4 9	3 4 8 9	5	1 3 1 6 4 9	3 1 6 4 8 9	3 1 6 4 8 9	1 3 6 4 8 9
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 6 8	2 3 4 5 9 7 8 9	1 4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 6 8	2 3 4 5 9 7 8 9	2 3 4 6 9 7 9	1 2 3 5 6 6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	1 2 3 5 6 9	1 5 7 9	3 7	1 2 3 5 6 9	1 2 3 5 6 9
3 5 6 8 9	2 4 5 8	2 3 4 5 9	1 7	2 3 5 6 9	3 4 5 8	2 3 4 5 6 8 9			
1 5 6 7 9	1 4 5 6 9	1 4 5 9	8	1 5 6 4 5 7 9 7	1 5 6 4 5 9 7	3 4 5 3	2 4 5 2	1 3 4 7 9	1 3 4 5 6 7
1 5 6 7 9	3 1 4	6	3 4 4 7 9	3 8	2 8 2	5 9	2 5 4 5 7 9	1 3 4 7 9	1 3 4 5 6 7
1 5 6 7	3 4 5 7	9	1 2 3 4 5 7	2 3 4 6 7	1 2 3 5 6 4 5 7 7	1 3 4 5 8	1 2 3 4 5 6 7	1 2 3 4 5 6 7	1 2 3 4 5 6 7
1 5 8	3 5	7	6	2 4	1 2 5	9	1 2 4 5 8	1 2 4 5 8	1 2 4 5 8
2 7	1 3 4	7	3 4	5	1 3 1 3 6 4 9 8	1 3 1 3 4 6 4 6 8 9 8 9	1 3 1 3 4 6 4 6 8 9 8 9	1 3 1 3 4 6 4 6 8 9 8 9	1 3 1 3 4 6 4 6 8 9 8 9
1 5 6 7 8 9	3 1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	1 4	1 2 3 4 5 7 8	1 2 3 4 5 6 7 8 9			
1 5 6 7 8 9	3 1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 5 7 8 9	2 3 4 6 9 7 9	1 2 3 5 6 7 9	6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 5 7	1 2 3 5 6 7 9	1 2 3 5 6 9 7 9
3	2 5 6 8 9	2 3 4 5 9	1 7	2 3 5 6 9 8	3 4 5 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1	1 5 6 7 9	1 4 5 9 7 9	8	1 5 6 7 9 7	1 4 5 7	3	2	1
1	3 1 4 7 9	6	3 4 4 7 9 9	3 4 6 9	8 5 6 7 7	2 4 5 7	5 4 7	1 3 4 7 9
1	3 5 6 7	9	1 2 3 4 5 7	2 3 4 6 7	1 2 3 5 6 7	1 4 5 7	3 4 5 6 7	1 2 3 4 5 6 7
1	5 8	3 7 9	7 6 4	2 5	1 2 5	9 4 5 8	1 2 4 5 8	1 2 4 5 8
2	7	1 3 4 9	3 4 8 9	5	1 3 1 3 1 3 1 3	6 4 4 6 4 6 8 9	1 2 3 5 6 5 6 8 9	1 2 3 5 6 5 6 8 9
3	2 5 6 7 8 9	2 3 5 8	2 3 5 9 7 8 9	1 4	2 3 1 2 3	5 7 8	2 3 5 6 7 8 9 7 8 9	2 3 5 6 7 8 9 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 8	2 3 4 5 9 7 8 9	2 3 1 2 3 4 6 5 6 9 7 9	1 2 3 4 5 6 7 8 9	6 7 8 9	1 2 3 4 5 6 7 8 9	4

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	1 2 3 5 6 9	1 5	3 7	1 2 6 7	1 2 3 5 6 9
3 5 6 8 9	2 4 5 8	2 3 4 5 9	1 7	2 3 5 6 9	3 4 5 8	2 4 6 9	2 3 4 5 6 8 9		
1 5 6 7 9	1 4 5 6 9	1 4 5 9	8	1 5 6 4 5 7 9 7	1 5 6 4 5 7 9 7	3 4 5 9	2 4 5 6 7 9		
1 5 6 7 9	3 1 4	6	3 4 4 7 9	3 8	2 8 2 5	5 8	1 3 4 7 7 9		
1 5 6 7	3 4 5	9	1 2 3 4 5 7	2 3 4 6 7	1 2 3 5 6 4 5 7 7	3 8	1 2 3 4 5 6 7		
1 5 8	3 4	7	6	2 4	1 2 5	9	1 2 4 5 8		
2	7	1 3 4 9	3 4 8 9	5	1 3 1 3 6 4 9 8	1 4 9	1 3 4 6 4 6 9 8 9		
3 5 6 7 8 9	2 5 6 8	2 3 5 9	2 3 5 9	1 4	3 5 7 8	2 6 7	2 3 6 5 6 7 8 9		
1 5 6 7 8 9	3 1 2 4 5 6 8	1 2 3 4 5 9	2 3 4 6 9 7 8 9	2 3 4 6 9 7 9	1 2 3 5 6 9	1 2 4 6 7 9	2 3 6 5 6 7 8 9		

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 5	3 7	1 2 6 7 9	1 2 3 5 6 7 9
3	2 5 6 4 5 6 8 9	2 3 4 5 9	1 7	2 3 5 6 9 8	3 4 5 9	2 4 6 9	2 3 4 5 6 8 9		
1	1 5 6 4 5 6 7 9	1 4 5 9 7 9	8	1 5 6 4 5 7 9 7	1 3 4 5 7	3 2	2 4 6 9	1 2 3 4 7 9	
1	3 1 4 7 9	6	3 4 5 7 9	3 4 6 9	8 5 6 4 5 7 7	2 5	1 3 4 7 9	1 2 3 4 5 6 7	
1	3 5 6 7	9	1 2 3 4 5 7	2 3 4 6 7	1 2 3 5 6 4 5 7 7	3 8	1 2 3 4 5 6 7	1 2 3 4 5 6 7	
1	5 8	3 7 6	4	2 1 2 5	9	1 2 4	1 2 4 5 8	1 2 4 5 8	
2	7	1 3 4 9	3 4 8 9	5	1 3 1 3 6 4 9 8	1 4 9	1 3 4 6 4 6 9 8 9	1 3 4 6 4 6 9 8 9	
3	2 5 6 7 8 9	2 3 5 9	2 3 5 7 8 9	1 2 3 9 7 9	3 5 7 8	2 6 7 9	2 3 5 6 7 8 9	2 3 5 6 7 8 9	
1	3 1 2 5 5 7 8 9	1 2 3 5 9	2 3 5 7 8 9	1 2 3 5 9 7 9	6	1 2 7 9	4		

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	1 2 3 5 6 9	1 3 5 7	1 2 6 7	1 3 5 9
3	2 5 6 4 5 8 9	2 3 4 5 8 9	1 7	2 3 5 6 9 8	3 4 5 9	2 4 6 9	3 5 6 8 9	
1	1 5 6 4 5 6 7 9	1 4 5 9 7 9	8	1 5 6 4 5 7 9 7	1 3 3	2	1 3 7 9	
1	3 1 4 7 9	6	3 4 5 7 9	3 4 6 9	8 5 6 7	2 4 5 9	5 1 3 7 9	
1	3 5 6 7	9	1 2 3 4 5 7	2 3 4 6 7	1 2 3 5 6 4 5 7 7	3 8	1 3 5 6 7	
1	5 8	3 7	7 6	2 4	1 2 5	9	1 2 4 5 8	
2	7	1 4 9	3 4 8 9	5	1 6 4 9 8	3 4 6 9	1 3 6 8 9	
3	2 5 6 7 8 9	2 3 5 8	2 3 5 9 7 8 9	1 2 3 9 7 9	2 3 5 7 8	3 6 7 9	2 5 6 8 9	
1	3 5 7 8 9	1 2 5 8	1 2 3 5 9 7 8 9	2 3 5 9 7 9	1 2 3 5 9	6 7 9	1 2 4	

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	1 2 3 5 6 9	1 5	3 7	1 2 6 9	1 3 5 6
3 5 6 9	2 5 6 9	1	7	2 3 5 6 4 5 9 8	3 4 6 9	2 5 6 8 9			
1 5 6 7 9	1 5 6 9	8	1 4 5 7 9	1 5 6 4 5 7 9 7	3 4 5 9 7	2 3 2 7 9			
1 3 1 4 7 9	6	2 3 4 5 7 9	3 4 5 9	8	2 3 2 7 9	5 7 9	1 3 5 6 7		
1 3 5 6 7	9	1 2 3 4 5 7	2 3 4 5 6	2 3 4 6 7	1 2 3 5 6 4 5 7 7	1 3 8 7	1 3 5 6 7		
1 5 8 7	3 7 6	2 3 4 5 9	1 2 5 4	9	1 2 4 5	1 2 4 5	1 3 5 8		
2 7	5	1 3 4 9	3 4 8 9	5	1 3 6 4 9 8	1 3 4 6 9	1 3 6 8 9		
3 5 6 7 8 9	2 5 6 8	2 3 5 9	2 3 8 9	1 2 3 9 7 9	3 5 7 8	2 6 7 9	3 5 6 7 8 9		
1 5 7 8 9	1 2 5 8 9	1 2 3 5 9	1 2 3 5 7 8 9	6	1 2 7 9	1 2 4	1 3 5 8		

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	2 3 5 6	1 5	3 7	1 2 6 9	1 3 5 6
3 5 6 9	2 5 6 9	2 3 5 9	1 7	2 3 5 6	4 5 9 8	3 4 6	2 4 9	2 5 6 8 9	3
1 5 6 7 9	1 5 6 9	1 5 4 5 9	8	1 5 6 4 5 9 7	1 3	3 2 5	2 1 3	2 1 3	2
1 3 4 7 9	3 1 4	6	3 4 3 7 9 9	8	2 5	1 3 7 9	1 3 7 9	1 3 5 6 7	3
1 3 5 6 7	9	1 2 3 4 5 7	2 3 4 5 6	2 3 1 2 3 4 6 5 6 7 7	1 3 4 5 7	8	1 2 4 5 7	1 3 5 6 7	3
1 5 8	3 7	7 6	4 2 1 2 5	9	1 2 4 9	1 2 4 9	1 2 4 9	1 2 4 9	1 2 5 8
2	7	1 3 4 9	3 4 8 9	5	1 3 1 3 6 4 9 8	1 3 4 6 9 9	1 3 4 6 9 9	1 3 4 6 9 9	1 3 6 6 8 9
3 5 6 7 8 9	2 5 6 8	2 3 5 9	2 3 5 7 8 9	1 2 3 1 2 3 5 9 7 9	3 5 7 8	2 6 7 9	2 6 7 9	2 6 7 9	3
1 5 7 8 9	3 5 8	1 2 5 9	1 2 3 5 7 8 9	2 3 1 2 3 5 9 7 9	6 1 2 7 9	4 2 9	4	4	4

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	2 3 5 6	1 5	1 7	1 9	1 6 5 6 9
3 5 6 9	2 5 6 9	1 7	2 3 5 6 9	4 5 8	4 5 9 8	4 9	6 9	5 6 8 9	
1 5 6 7 9	1 5 6 9	8	1 5 6 9 7	4 5 9	3 2				
1 5 6 7 9	4 7 9	6 8	3 4 9	3 4 5 9	8 2	5 1 3 7 9			
5 9	2 4 5 7	2 3 4 6 7	2 3 4 6 7	1 2 3 5 6 7	1 3 4 5 7	8 1 2	1 3 7	5 6 8	
8 3	7 1 3 4 9	6 3 8 9	2 4 5	1 2 5	9 4	1 2 4	1 3 5 8		
2 7	5 4 9	1 3 4 8 9	5 1	3 1 3 6 4 9 8	1 3 4 6 9	1 2 4 6 9	1 3 6 6 8 9		
3 5 6 7 8 9	2 5 6 8	2 3 5 9	2 3 7 8 9	1 2 3 9 7 8 9	3 5 7 8	2 7 9	3 6 7 8 9	5 6 8 9	
1 5 7 8 9	3 5 8	1 2 5 9	1 2 3 5 9 7 8 9	2 3 1 2 3 5 9 7 9	6 1 2 7 9	1 2 7 9	1 2 7 9	3 6 5 6 8 9	

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	2 3 5 6	1 5	1 7	1 9	1 6 5 6	9
3 6 9	2 5 6 9	1 7	2 3 5 6 9	4 5 8	4 5 9	4 6	4 9	5 6 8 9		
1 6 7	1 5 6 9	8		1 5 6 9	1 4 5 7	3	2			
1 5	4 9	6 2	3 7 9	3 9	8 2	5				3
8	3	7	6	2 4	1 2 5	9	1 2 4	1 4	1 5	
2	7	1 3 4 9	3 4 8 9	5	1 3 6 4 9 8	9	1 2 4 9	1 3 6 8 9		
3 7	2 6 9	5 8	3 5 9 7 8 9	1 2 3	2 3 1 2 3 9 7 9	5 7 8	3 7 9	2 6 5 6 7 8 9		3
3 7	1 2 5 9	1 3 5 8	2 3 5 9 7 8 9	1 2 3 5 9 7 9	6	1 2 7 9	1 2 7 9	4		

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	2 3 5 6	1 5	1 7	1 9	6 5 6 9 7 9
3 6 9	2 5 6 9	1 7	2 3 5 6 9 8	4 5 9 8	4 6	6 9	5 6 8 9		
1 6 7	1 5 6 9	8	1 5 6 9 7	4 5 9 7	3	2			
1 5	4 9	6 2	3 7 9	3 9	8 2	5	7 9	3 1 3 8 7	6
8 2	3 7	7 6	2 5	5 9	9 4	4 1			
3 7	2 6 9	1 3 4 9	3 4 8 9	3 7	1 3 1 3 6 4 9 8	1 3 1 3 4 6 9 9	1 3 6 8 9		
3 7	2 6 9	1 3 5 9	2 3 5 7 8 9	2 3 1 2 3 5 9 7 9	3 5 7 8	2 6 7 9	3 5 6 7 8 9		
3 7	1 2 5 9	1 3 5 9	2 3 5 7 8 9	2 3 1 2 3 5 9 7 9	6	1 2 7 9	4		

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	3 6 9	2 3 6 9 7	1 5	1 7	6 9 7	5 6 9 9
3 6 9	2 5 6 9	1 7	2 3 6 4 5 9 8			6	5 6 9 8 9		
1 6 7	1 5 6 9	8		1 6 4 5 9 7	3	2			
1 5	4 9	6 2	3 7 9	3 1 3	8 7 7	2 5	5 7		3
8 2	3 7	7 6	2 5	5 9	9 4	4 1			
3 7	2 6 9	1 3 4 9	3 4 8 9	3 5	1 3 1 3 6 4 9 8	1 6	3 6 9	8 9	
3 7	2 6 9	1 3 4 9	2 3 5 8 9	1 4	3 5 7 8	2 6	2 7 9	5 6 7 8 9	3
3 7	1 2 5 9	1 3 5 5	2 3 5 9	3 1 2 3 9 7 9	6 9	1 2 7 9			4

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	3 6 9	2 3 6 9 7	1 5	1 7	6 9 7	5 6 9 9
3 6 9	2 5 6 9	1 7	2 3 6 4 5 9 8	6 4 5 9 7	6 4 5 9 7	6 5 6 9 8 9			
1 6 7	1 5 6 9	8	1 6 4 5 9 7	3 2	3 2	3 2			
1 5	4 9	6 2	3 4 7	3 1 7	2 5 7	5 8 7	3 2	3 6 7	3 6 7
8 2	3 7	7 6	2 4 9	5 3 7	9 5 7	4 9 8	1 2 3	3 6 9	1 6 9
3 6 9	5 6 8	3 9 7 8 9	2 3 5 8 9	1 4 7 8	4 5 7 8	3 6 7 9	2 5 6 7 8 9	3 6 9	3 6 9
3 6 9	1 5 8	1 5 9	3 5 7 8 9	2 3 5 8 9	6 7 9	1 2 7 9	2 5 6 7 8 9	3 6 9	4 6 9

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	3 6 9	2 3 6 9 7	1 5	1 7	6 9 7	5 6 9 9
3 6 9	2 5 6	3 5 9	1 7	2 3 6 4 5 9 8		6	5 6 9 9		
1 6 7	1 5 6	1 5 9	4 5 9	8	1 6 4 5 9 7	3	2		
1 5	4 9	6 2	3 7 9	3 9	8 2	5		3 7	
8 2	3 7	7 6	2 5	2 5	9 4	1			
3 6 9	1 3 4 9	3 8 9	5	3 1 6 4 9 8	3 1 6 9	6 6	8 9		
3 6 9	5 6 8	5 9 7 8 9	1 4	3 2 3 7 8	5 7 8	3 2 7	6 9 7 8 9		
3 6 9	1 5 8	1 3 5 9 7 8 9	2 3	3 9 7 9	6 7 9	1 2 7	4		

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Propagation Steps (Forward Checking)

4	1 2 5 6	8	2 3 5 9	3 6 9	2 3 6 9 7	1 5	1 7	6 9 7	5 6 9 9
3 6 9	2 5 6	3 5 9	1 7	2 3 6 4 5 9 8		6	5 6 9 9	8 9	
1 6 7	1 5 6	1 5 9	4 5 9	8	1 6 4 5 9 7	3	2		
1 5	4 9	6 2	3 7 9	3 9	8 2	5		3 7	
8 2	3 7	7 6	2 5	2 5	9 4	4	1		
3 6 9	1 5 6	3 5	3 8 9	5	3 1 6 9 8	3 1			3 8 9
3 6 9	1 5 6	3 5	2 3 7 8 9	1 4	5 7 8	3 2		5 7 8 9	
3 6 9	1 5 6	1 5	3 9 7 8 9	3 9 7 9	6	1 2 7 9		4	

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After Setup (Forward Checking)

4	1 2 5 6	8	2 3 5 9	3 6 9	2 3 6 7	1 5	1 7	6 9	5 6 7 9
3 6 9	2 5 6 5 9	3 9	1 7	2 3 6 4 5 9 8			6 9	5 6 8 9	
6 7	1 5 6 5 9	1 9	8	1 6 4 5 9 7	3 3	2 7			3
1	4	6	3 7 9	3 9	8 3 1 3	2 3	5 7		
5	9	2	3 4 7	3 4	7 7	7 7	8 3		6 7
8	3	7	6 7 9	2 8 9	5 5	9 6 9	4 8	1 9	
2	7	1 3 4	3 9	3 8 9	5 5	3 1 6 9	3 1 8		3 8 9
3 6 9	5 6 8	5 9 7 8 9	2 3 9 7 8 9	1 9 7 9	4 7 8	5 7 8	2 7	3 9 7 8 9	5
3 9	1 5 8	1 3 5 9 7 8 9	2 3 9 7 8 9	3 9 7 9	2 3 6 7 8	1 2 6 7 9	4 9		

Outline

Problem

Initial Propagation (Forward Checking)

Improved Reasoning

Bounds Consistency

Domain Consistency

Comparison

Search

Other Global Constraints

Can we do better?

- The alldifferent constraint is missing propagation
 - How can we do more propagation?
 - Do we know when we derive all possible information from the constraint?
- Constraints only interact by changing domains of variables

A Simpler Example

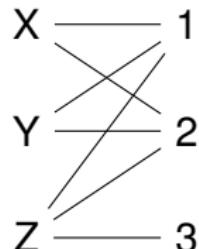
```
include "alldifferent.mzn";  
  
var 1..2:X;  
var 1..2:Y;  
var 1..3:Z;  
  
constraint alldifferent([X, Y, Z]);  
  
solve satisfy;
```

Using Forward Checking

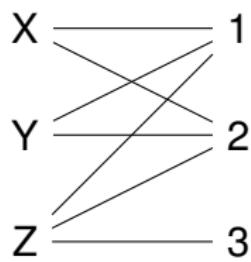
- No variable is assigned
- No reduction of domains
- But, values 1 and 2 can be removed from Z
- This means that Z is assigned to 3

Visualization of alldifferent as Graph

- Show problem as graph with two types of nodes
 - Variables on the left
 - Values on the right
- If value is in domain of variable, show link between them
- This is called a *bipartite* graph



A Simpler Example



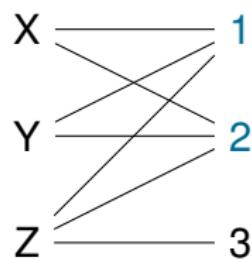
Value Graph for

`var 1..2:X;`

`var 1..2:Y;`

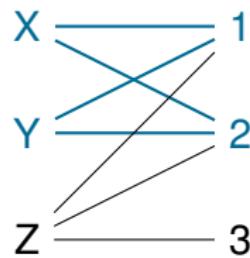
`var 1..3:Z;`

A Simpler Example



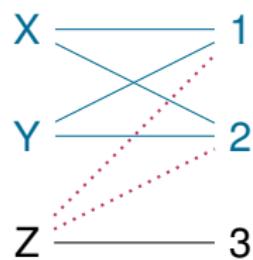
Check interval [1,2]

A Simpler Example



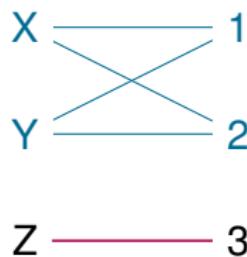
- Find variables completely contained in interval
- There are two: X and Y
- This uses up the capacity of the interval

A Simpler Example



No other variable can use that interval

A Simpler Example



Only one value left in domain of Z,
this can be assigned

Idea (Hall Intervals)

- Take each interval of possible values, say size N
- Find all K variables whose domain is completely contained in interval
- If $K > N$ then the constraint is infeasible
- If $K = N$ then no other variable can use that interval
- Remove values from such variables if their bounds change
- If $K < N$ do nothing
- Re-check whenever domain bounds change

Implementation

- Problem: Too many intervals ($O(n^2)$) to consider
- Solution:
 - Check only those intervals which update bounds
 - Enumerate intervals incrementally
 - Starting from lowest(highest) value
 - Using sorted list of variables
- Complexity: $O(n \log(n))$ in standard implementations
- Important: Only looks at min/max bounds of variables

Bounds Consistency

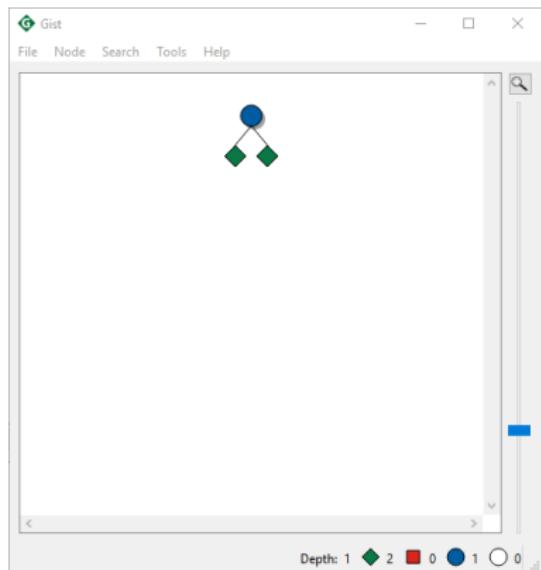
Definition

A constraint achieves *bounds consistency*, if for the lower and upper bound of every variable, it is possible to find values for all other variables between their lower and upper bounds which satisfy the constraint.

Annotation: :: bounds

```
include "alldifferent.mzn";  
  
var 1..2:X;  
var 1..2:Y;  
var 1..3:Z;  
  
constraint alldifferent([X,Y,Z]) :: bounds;  
solve satisfy;
```

Running with Gecode Gist



All Solutions

A screenshot of the Gist Console window titled "Gist Console: Gecode/FlatZinc". It shows the following FlatZinc code:

```
X = [1..2];
Y = [1..2];
Z = 3;
```

Node Inspector (Root)

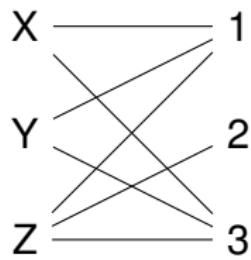
Can we do even better?

- Bounds consistency only considers min/max bounds
- Ignores “holes” in domain
- Sometimes we can improve propagation looking at those holes

Another Simple Example

```
include "alldifferent.mzn";  
  
var {1,3}:X; % note enumerated domain  
var {1,3}:Y;  
var 1..3:Z; % note domain as interval  
  
% annotated constraint  
constraint alldifferent([X,Y,Z]) :: bounds;  
solve satisfy;
```

Another Simple Example



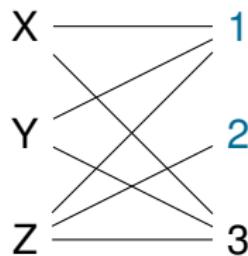
Value Graph for

`var {1, 3}:X;`

`var {1, 3}:Y;`

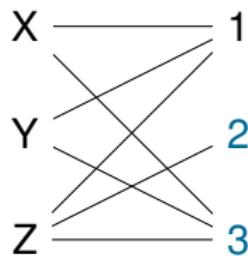
`var 1..3:Z;`

Another Simple Example



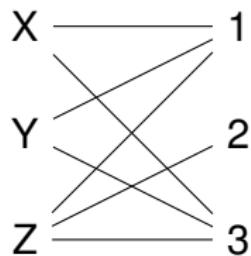
- Check interval [1,2]
- No domain of a variable completely contained in interval
- No propagation

Another Simple Example



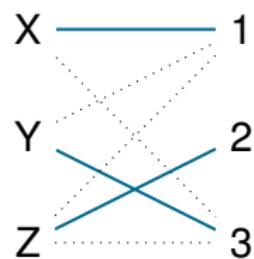
- Check interval [2,3]
- No domain of a variable completely contained in interval
- No propagation

Another Simple Example



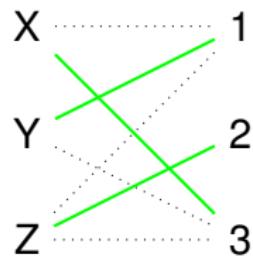
But, more propagation is possible,
there are only two solutions

Another Simple Example



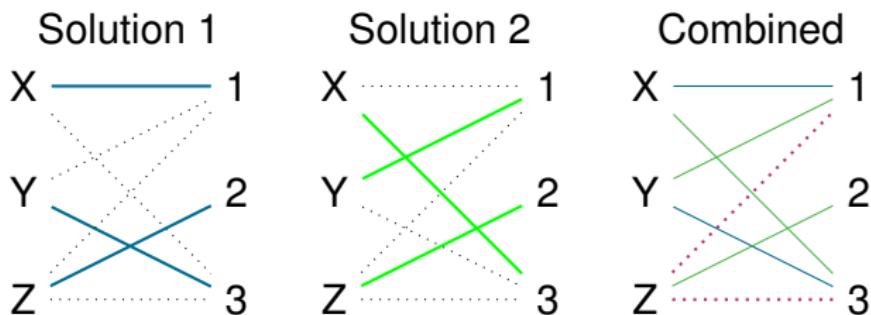
Solution 1: assignment in blue

Another Simple Example



Solution 2: assignment in green

Another Simple Example

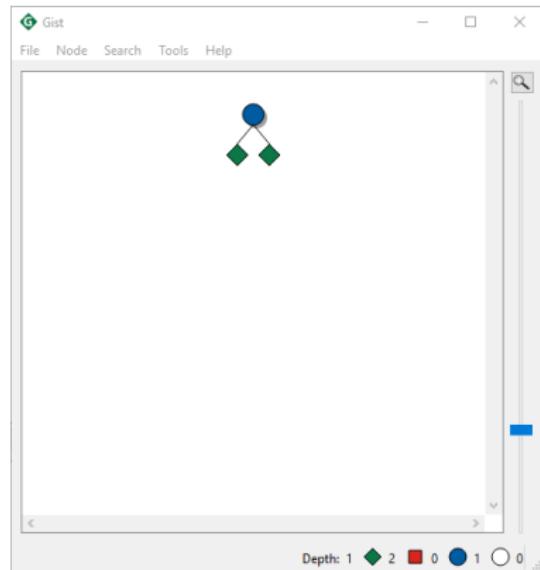


Combining solutions shows that $Z=1$ and $Z=3$ are not possible.

Another Simple Example

Can we deduce this without enumerating solutions?

Bounds Consistency with Gecode Gist: No Propagation



All Solutions

The screenshot shows the Gist Console window titled "Gist Console: Gecode/FlatZinc". It contains the following text:
Clear Stay on top
X = {1..3};
Y = {1..3};
Z = [1..3];

Node Inspector (Root)

Solutions and Maximal Matchings

- A *Matching* is subset of edges which do not coincide in any node
- No matching can have more edges than number of variables
- Every solution corresponds to a *maximal matching* and vice versa
- If a link does not belong to some maximal matching, then it can be removed

Implementation

- Possible to compute all links which belong to some matching
 - Without enumerating all of them!
- Enough to compute **one** maximal matching
- Requires algorithm for *strongly connected components*
- Extra work required if more values than variables
- All links (values in domains) which are not supported can be removed
- Complexity: $O(n^{1.5}d)$

Domain Consistency

Definition

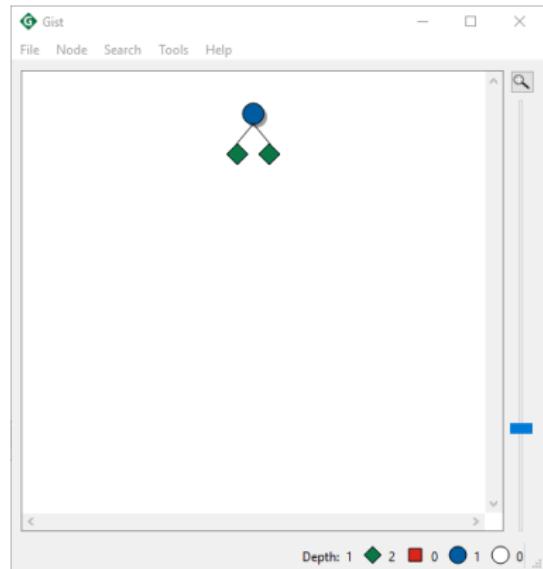
A constraint achieves *domain consistency*, if for every variable and for every value in its domain, it is possible to find values in the domains of all other variables which satisfy the constraint.

- Also called *generalized arc consistency (GAC)*
- or *hyper arc consistency*

Simple Example Revisited

```
include "alldifferent.mzn";  
  
var {1,3}:X; % note enumerated domain  
var {1,3}:Y;  
var 1..3:Z; % note domain as interval  
  
% note different annotation  
constraint alldifferent([X,Y,Z]) :: domain;  
solve satisfy;
```

Domain Consistency with Gecode Gist: Propagation



All Solutions

A screenshot of the Gist Console window titled "Gist Console: Gecode/FlatZinc". It contains the following text:

```
X = {1, 3};  
Y = {1, 3};  
Z = 2;
```

Node Inspector (Root)

Can we still do better?

- NO! This extracts all information from this one constraint
- We could perhaps improve speed, but not propagation
- But possible to use different model
- Or model interaction of multiple constraints

Should all constraints achieve domain consistency?

- Domain consistency is usually more expensive than bounds consistency
 - Overkill for simple problems
 - Nice to have choices
- For some constraints achieving domain consistency is NP-hard
 - We have to live with more restricted propagation

Modified MiniZinc Program

```
int: s;
int: n=s*s;
array[1..n,1..n] of var 1..n: puzzle;

include "sudoku.dzn";

include "alldifferent.mzn";

constraint forall(i in 1..n)
    (alldifferent([puzzle[i,j] | j in 1..n])::domain);
constraint forall(j in 1..n)
    (alldifferent([ puzzle[i,j] | i in 1..n])::domain);
constraint forall(i,j in 1..s)
    (alldifferent([puzzle[s*(i-1)+p, s*(j-1)+q] |
        p,q in 1..s])::domain);

solve satisfy;
```

Modified Choco-solver Sudoku Model

```
Model model = new Model("Sudoku");
int blockSize = 3;
int m = blockSize*blockSize;

IntVar[][] vars = new IntVar[m][m];
for(int i=0;i<m;i++){
    for(int j=0;j<m;j++) {
        vars[i][j] = model.intVar("X"+i+"_"+j, 1, m);
        if (data[i][j]>0)
            model.arithm(vars[i][j], "=", data[i][j]).post();
    }
}
// Consistency level AC: domain consistency, BC: bounds consistency, default: mix
for(int i=0;i<m;i++){
    model.allDifferent(row(i,m,vars),AC).post();
    model.allDifferent(column(i,m,vars),AC).post();
}
for(int i=0;i<m;i+=blockSize){
    for(int j=0;j<m;j+=blockSize){
        model.allDifferent(block(i, j, blockSize, vars),AC).post();
    }
}
Solver solver = model.getSolver();
solver.solve();
```

Initial State (Domain Consistency)

4	1 2 3	8	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	7 8 9	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3
4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	6	1 2 3	1 2 3	8	2	5	1 2 3	4 5 6
4 5 6	4 5 6	7 8 9	4 5 6	4 5 6	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	9	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	4 5 6
4 5 6	4 5 6	7 8 9	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	3	7	6	1 2 3	1 2 3	9	1 2 3	1 2 3
4 5 6	4 5 6	7 8 9	4 5 6	4 5 6	7 8 9	7 8 9	7 8 9	7 8 9	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
2	7	1 2 3	1 2 3	5	1 2 3	1 2 3	1 2 3	1 2 3	4 5 6
4 5 6	4 5 6	7 8 9	4 5 6	4 5 6	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	4 5 6
4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6	4 5 6
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	1 2 3	6	1 2 3	4 5 6
4 5 6	4 5 6	7 8 9	4 5 6	4 5 6	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9
7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9	7 8 9

Propagation Steps (Domain Consistency)

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Propagation Steps (Domain Consistency)

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Propagation Steps (Domain Consistency)

4	1 2 3 5 6 7 9	8	1 2 3 5 6 7 9	1 2 3 5 6 9 7	1 2 3 5 6 9 9			
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1	7	1 2 3 4 5 6 7 8 9				
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	2		
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9	
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	4 5 6 7 8 9		
1 2 3 4 5 6 7 8 9	3	7	6	1 2 3 4 5 6 7 8 9	9	4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	
2	7			5	1 2 3 4 5 6 7 8 9			
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9		1	4	1 2 3 4 5 6 7 8 9			
1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9			6	1 2 3 4 5 6 7 8 9	4		

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Propagation Steps (Domain Consistency)

4	1 2 3 5 6	8	1 2 3 5 6					
7	9	7	9	7	9	7	9	7
1	3 1 2 3 5 6 4 5 6 4 5 6	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	1	7	1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 4 5 6 4 5 6 4 5 6
7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9
1	3 1 2 3 5 6 4 5 6 4 5 6	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	8	1 2 3 1 2 3 7 8 9 7 8 9 7 8 9	3	2	1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 7 8 9 7 8 9 7 8 9
7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9
1	3 1 2 3 5 6 4 5 6 4 5 6	6	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	8	2	5	1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 7 8 9 7 8 9 7 8 9
7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9
1	3 1 2 3 5 6 4 5 6 4 5 6	9	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	8	1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 7 8 9 7 8 9 7 8 9	1 2 3 4 5 6 4 5 6 4 5 6
7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9
1	3 1 2 3 5 6 4 5 6 4 5 6	3	7	6	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	9	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 1 2 3 7 8 9 7 8 9 7 8 9
7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9
2	7	5	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	5	1 2 3 1 2 3 7 8 9 7 8 9 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6
7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9
1	3 1 2 3 5 6 4 5 6 4 5 6	1 2 3 1 2 3 7 8 9 7 8 9 7 8 9	1	4	1 2 3 1 2 3 7 8 9 7 8 9 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6
7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9
1	3 1 2 3 5 6 4 5 6 4 5 6	1 2 3 1 2 3 7 8 9 7 8 9 7 8 9	1	2 3 1 2 3 4 5 6 4 5 6 4 5 6	6	1 2 3 1 2 3 4 5 6 4 5 6 4 5 6	1 2 3 1 2 3 7 8 9 7 8 9 7 8 9	4
7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7	8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9	7 8 9 7 8 9 7 8 9

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Propagation Steps (Domain Consistency)

4	1 2 3 5 6	8	1 2 3 5 6 6 5	1 2 3 6 5 5 6	1 2 3 5 6 5 6	1 2 3 5 6 5 6	1 2 3 5 6 5 6
3 5 6 8 9	2 3 4 5 6 8 9	1 7	2 3 4 5 6 8 9				
1 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	2	
1 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	8	2	5	1 2 3 4 5 6 7 8 9
1 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	4 5 6 7 8 9	
1 5 6 7 8 9	3 7 1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	
2 1 3 5 6 7 8 9	7 1 2 3 4 5 6 7 8 9	7	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8 9	1 2 3 4 5 6 7 8 9	1 4	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 5 6 7 8 9	1 2 3 4 5 6 7 8 9	7 8 9	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4	

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	1 2 3 5 6 7	1 2 3 5 6 9 7	1 2 3 5 6 9			
3	2 5 6 8 9	2 3 4 5 6 8 9	1 7	2 3 4 5 6 8 9	2 3 4 5 6 8 9			
1 5 6 7 8 9	3 1 2 4 5 6 8	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	3	2	
1 5 6 7 8 9	3 1 2 4 5 6 8	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	8 2 5	1 2 3 4 5 6 7 8 9		
1 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8	1 2 3 4 5 6 7 8 9	
1 5 6 7 8 9	3 7	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	
2	7	1 2 3 4 5 6 7 8 9	5	1 2 3 4 5 6 7 8 9				
1 5 6 7 8 9	3 1 2 4 5 6 8	1 2 3 4 5 6 7 8 9	1 4	1 2 3 4 5 6 7 8 9				
1 5 6 7 8 9	3 1 2 4 5 6 8	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	4	

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	1 2 3 5 6 7 9	1 2 3 5 6 9 7	1 2 3 5 6 9 9			
3	2 5 6 8 9	1	2 3 4 5 6 8 9	7 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1	1 5 6 7 9	8	1 4 5 6 7 9	3 4 5 6 7 8 9	2 4 5 6 7 8 9			
1	3 1 2 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8 2 5 7 8 9	2 5 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 5 6 7 8 9	9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 5 6 7 8 9	3 7 6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	5	1 2 3 4 5 6 7 8 9					
1	3 1 2 5 6 7 8 9	1	1 2 3 4 5 6 7 8 9	4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 1 2 5 6 7 8 9	6	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6 7 8 9	1 2 3 4 5 6 7 8 9	4 5 6 7 8 9	4 7 8 9

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	1 2 3 5 6 7	1 2 3 5 6 9 7	1 2 3 5 6 9			
3	2 5 6 8 9	2 3 4 5 8	1 7	2 3 4 5 6 8 9				
1	1 5 6 7 9	1 4 5 6 9 7 9	1 8	1 4 5 6 7 9 7	1 4 5 6 9 7 9	3 4 5 6 8 9	2 4 5 6 7 8 9	2 4 5 6 7 8 9
1	3 1 2 5 6 7 8 9	1 2 3 4 5 6 8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 5 6 7 8 9	9 4 5 9 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	8 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 5 6 7 8 9	3 7 4 5 9 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	9 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	2 7 4 5 9 7 8 9	1 2 3 4 5 6 5	1 2 3 4 5 6 7 8 9				
1	3 1 2 5 6 7 8 9	1 2 3 4 5 6 8	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 7 8 9				
1	3 1 2 5 6 7 8 9	1 2 3 4 5 6 8	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	6 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	1 2 3 5 6 7	1 2 3 5 6 9	1 2 3 5 6 9	1 2 3 5 6 9	1 2 3 5 6 9	1 2 3 5 6 9	1 2 3 5 6 9
3	2 5 6 8 9	2 3 4 5 9	1 7	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1	1 5 6 7 9	1 4 5 6 9 7 9	1 8	1 4 5 6 7 9 7 9	1 4 5 6 7 9 7 9	1 4 5 6 7 9 7 9	3 4 5 6 7 8 9	2 4 5 6 7 8 9	2 4 5 6 7 8 9
1	3 1 4 7 9	6 4 7 9 7 9	1 3 1 3 4 4 8 2 5	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 3 4 4 7 8 9	2 5 4 4 7 8 9	2 5 4 4 7 8 9
1	3 5 6 7 8 9	9 4 5 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9 7 8 9	8 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 5 6 7 8 9	3 4 5 9 7 8 9	7 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	9 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
2	7	2 7 4 5 9 7 8 9	1 2 3 4 5 6 5	1 2 3 4 5 6 5	1 2 3 4 5 6 5	1 2 3 4 5 6 5	2 7 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9 8	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 1	1 2 3 4 5 6 1	1 2 3 4 5 6 1	1 2 3 4 5 6 1	4 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9 8	1 2 3 4 5 6 9 7 8 9	1 2 3 4 5 6 9	1 2 3 4 5 6 9	1 2 3 4 5 6 9	1 2 3 4 5 6 9	6 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4 4 5 6 7 8 9

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5	1 2 3 5 6	1 2 3 5 6			
3	2 5 6 4 5 8 9	2 3 4 5 8	1 7	2 3 4 5 6 8 9	2 3 4 5 6 8 9			
1	1 5 6 4 5 6 7 9	1 4 5 9 7 9	8	1 4 5 6 7 9 7 9	1 4 5 6 7 9 7 9	3	2	
1	3 1 4	6	3 1 4 4 7 9 7 9	8	2 5	1 3 4		
1	3 5 6 7 8 9	9	1 2 3 4 5 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	8	4 5 6 7 8 9	
1	3 5 6 7 8 9	3 7 6	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	9	1 2 3 4 5 6 7 8 9 7 8 9		
2	7	1 2 3 4 5 9 7 8 9	5	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 9 7 8 9	1 4	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 9 7 8 9	6	1 2 3 4 5 6 7 8 9 7 8 9	1 2 3 4 5 6 7 8 9 7 8 9	4		

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 5 6 5 6 5 6 5 6 5 6	1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3 1 2 3	1 2 3 2 3 2 3 2 3 2 3 2 3 2 3	1 2 3 2 3 2 3 2 3 2 3 2 3 2 3	1 2 3 2 3 2 3 2 3 2 3 2 3 2 3
3	2 5 6 4 5 6 8 9	2 3 1 7 9	1 7 4 5 6 8 9	2 3 2 3 4 5 6 8 9	2 3 2 3 4 5 6 8 9	2 3 2 3 4 5 6 8 9	2 3 2 3 4 5 6 8 9
1	1 5 6 4 5 6 7 9	1 8 9	1 8 4 5 9 7 9	1 1 4 5 6 7 9 7 9	1 1 4 5 6 7 9 7 9	3	2
1	3 1 4 7 9	1 6 4 7 9 7 9	3 1 3 4 4 8 2 5 9 7 8 9	1 2 3 8 2 5 4 5 6 7 8 9 7 8 9	1 2 3 8 2 5 4 5 6 7 8 9 7 8 9	1 3 4 4 7 9	1 3 4 4 7 9
1	3 5 6 7	9	1 2 3 4 5 7	2 3 1 2 3 1 2 3 4 5 6 4 5 6 4 5 6 7 7 7 7	1 2 3 1 2 3 1 2 3 4 5 6 4 5 6 4 5 6 7 7 7 7	8	4 5 6 7
1	3 5 6 7 8 9	3 7 6	1 2 3 1 2 3 1 2 3 4 5 6 4 5 6 4 5 6 7 8 9 7 8 9	9	1 2 3 1 2 3 1 2 3 4 5 6 4 5 6 4 5 6 7 8 9 7 8 9	9	1 2 3 4 5 6 7 8 9
2	7	1 2 3 4 5 9 7 8 9	2 3 4 5 7 8 9	5	1 2 3 1 2 3 1 2 3 4 5 6 4 5 6 4 5 6 7 8 9 7 8 9	5	1 2 3 4 5 6 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 8	2 3 4 5 9 7 8 9	1 4	1 2 3 1 2 3 1 2 3 4 5 6 4 5 6 4 5 6 7 8 9 7 8 9	1 4	1 2 3 4 5 6 7 8 9
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 8	2 3 4 5 9 7 8 9	6	1 2 3 1 2 3 1 2 3 4 5 6 4 5 6 4 5 6 7 8 9 7 8 9	6	1 2 3 4 5 6 7 8 9

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9
3	2 5 6 8 9	2 3 4 5 8	1 7 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1	1 5 6 7 9	1 4 5 9 7 9	1 8 7 9	1 4 5 6 7 9 7 9	1 4 5 6 7 9 7 9	3	2	
1	3 1 4 7 9	6 4 7 9	3 4 9	3 8 9	2 8 9	5 2 7 9	1 3 4 7 9	
1	3 5 6 7 8 9	9 4 5 7	1 2 3 4 5 9	2 3 4 7	1 2 3 6 7	1 2 3 4 5 6 8	1 2 3 4 5 6 7	
1	3 5 6 7 8 9	3 7 9	1 2 3 4 5 9 7 8 9	2 3 4 9 7 8 9	1 2 3 6 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	
2	7 4 5 9 7 8 9	6 4 5 9 7 8 9	1 2 3 4 5 9 7 8 9	5 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 8	1 2 3 4 5 9 7 8 9	1 4 9	1 2 3 6 9 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 8	1 2 3 4 5 9 7 8 9	2 3 4 9 7 8 9	1 2 3 6 9 7 8 9	6 1 2 3 4 5 6 7 8 9	4 1 2 3 4 5 6 7 8 9	

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7
3	2 5 6 8 9	2 3 4 5 8	1 7 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9			
1	1 5 6 7 9	1 4 5 6 9 7 9	1 8 7 9	1 4 5 6 7 9 7	1 4 5 6 7 9 7	3	2	
1	3 1 4 7 9	6	3 4 7 9	3 4 5 6 9	8 2	5	1 3 4 7 9	
1	3 5 6 7	9	1 2 3 4 5 7	2 3 1 2 3 4 6 4 5 6 7 7	1 2 3 4 5 6 8	1 2 3 4 5 6 7		
1	5 8	3 7 6	4 4 5 9 7 8 9	2 1 2 4 5 8	9	1 2 4 5 8	1 2 4 5 8	
2	7	1 2 3 4 5 9 7 8 9	2 3 4 5 5	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 8	2 3 4 5 9 7 8 9	1 2 3 1 2 3 4 6 4 5 6 9 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	1 2 3 1 2 3 4 5 6 4 5 6 7 8 9 7 8 9	
1	3 1 2 5 6 4 5 6 7 8 9	1 2 3 4 5 8	2 3 4 5 9 7 8 9	2 3 1 2 3 4 6 4 5 6 9 7 8 9	6	1 2 3 4 5 6 7 8 9	4	

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7
3 5 6 8 9	2 4 5 8	2 3 4 5 9	1 7	2 3 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 9	1 4 5 9	8	1 5 6 7	1 4 5 6 9	3	2	
1 5 6 7 9	3 1 4	1 6	3 4 7 9	3 4 9	8 2	2 5	1 3 4 7 9	
1 5 6 7	3 4 5 7	1 2 3 4 5 7	2 3 4 6 7	1 2 3 5 6 7	1 2 3 4 5 6 7	8	4 5 6 7	
1 5 8	3 7	7 6	4	2 5	1 2 9	1 2 4 5 8	1 2 4 5 8	
2 7	1 2 3 4 5 9 7 8 9	2 3 4 5 7 8 9	5	1 2 3 5 6 7	1 2 3 4 5 6 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	
1 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9 7 8 9	1 4	1 2 3 4 6 9 7 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	
1 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9 7 8 9	2 3 4 9 7 9	1 2 3 5 6 9 7 9	6	1 2 3 4 5 6 7 8 9	4	

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7	1 2 3 5 6 9 7 9
3 5 6 8 9	2 4 5 8	2 3 4 5 9	1 7	2 3 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 9	1 4 5 9	8	1 5 6 4 5 6 7 9 7 9	1 3	1 3	2
1 5 6 7 9	3 1 4	6	3 4 7 9	3 8 9	2 8 2 9	5 4 5 7 9	1 3 4 7 9
1 5 6 7	3 4 5 7	1 2 3 4 5 7	2 3 4 6 7	1 2 3 5 6 4 5 6 7 7	1 2 3 8	1 2 3 4 5 6 7	1 2 3 4 5 6 7
1 5 8	3 7	7	6 4	2 1 2 5	9 4 5 8	1 2 4 5 8	1 2 4 5 8
2 7	1 3 4 9	1 3 4 8 9	3 5	1 3 1 6 4 9	3 1 6 4 8 9	3 1 6 4 8 9	1 3 6 4 8 9
1 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9 7 8 9	2 3 4 6 9 7 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 5 6 7 8 9	3 1 2 4 5 6 8	1 2 3 4 5 9 7 8 9	2 3 4 6 9 7 9	1 2 3 5 6 9	6 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 5	3 7	1 2 3 5 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 8	2 3 4 5 9	1 7	2 3 5 6 9 8	3 4 5 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 9	1 4 5 9 7 9	8	1 5 6 4 5 7 9 7	1 3 7	3 8 9	2 5	2 5	2 5
1 5 6 7 9	3 1 4	1 6	3 4 4 7 9	3 8 9	2 8 2	5 2 5	1 4 7 9	1 3 4 7 9	1 3 4 7 9
1 5 6 7	3 9	1 2 3 4 5 7	2 3 4 5 9	2 3 4 6 7	1 2 3 5 6 4 5 7 7	1 3 8	1 2 3 4 5 6 7	1 2 3 4 5 6 7	1 2 3 4 5 6 7
1 5 6 8	3 7	3 6	4	2 5	1 2 9	1 2 9	1 2 4 5 8	1 2 4 5 8	1 2 4 5 8
2 7	1 3 4 9	1 3 4 8 9	3 5	1 6 4 9 8	3 6 4 8	1 2 4 6 8 9	1 3 4 6 8 9	1 3 4 6 8 9	1 3 4 6 8 9
1 5 6 7 8 9	1 2 4 5 6 8	1 2 3 4 5 9 7 8 9	2 3 4 5 9 7 8 9	2 3 4 6 9 7 9	1 2 3 5 6 4 5 7 8 7 8 9	1 4 5 7 8	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9
1 5 6 7 8 9	3 1 2 4 5 6 8	1 2 3 4 5 9 7 8 9	2 3 4 5 9 7 8 9	2 3 4 6 9 7 9	1 2 3 5 6 4 5 7 8 7 8 9	6 7 8 9	1 2 3 4 5 6 7 8 9	4 7 8 9	4 7 8 9

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 5 7	1 2 3 5 6 7 9	1 2 3 5 6 9 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1 7	2 3 5 6 9 8	3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 8 9
1 5 6 7 9	1 4 5 6 9	1 4 5 7 9	8	1 5 6 7 9	1 4 5 7	3 4	2 4	2 4
1 5 6 7 9	3 1 4	6	3 4 7 9	3 4 9	8 2	2 5	1 3 4	1 3 4
1 5 6 7	3 4 5	1 2 3 4 5 7	2 3 4 5 6	2 3 5 6 7	1 2 3 4 5 7	8 7	1 2 3 4 5 6 7	1 2 3 4 5 6 7
1 5 6 8	3 4	7	6	2 4	1 2 5	9 4 5 8	1 2 4 5 8	1 2 4 5 8
2	7	1 3 4 9	3 4 8 9	5	1 3 6 4 9 8	1 3 4 6 4 8 9	1 3 4 6 4 8 9	1 3 4 6 4 8 9
3 5 6 7 8 9	2 5 6 8	2 3 5 9 7 8 9	2 3 5 9 7 8 9	1 4	2 3 5 6 7 8 9	3 5 6 7 8 9	2 3 5 6 7 8 9	2 3 5 6 7 8 9
1 5 6 7 8 9	3 1 2 4 5 6 8	1 2 3 4 5 9 7 8 9	2 3 4 6 9 7 9	1 2 3 5 6 9 7 9	6 4 5 6 7 8 9	1 2 3 4 5 6 7 8 9	4 7 8 9	4 7 8 9

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 5	3 7	1 2 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1 7	2 3 5 6 9 8	3 4 5 9	2 4 6 9	2 3 4 5 6 8 9		
1 5 6 7 9	1 4 5 6 9	1 4 5 7 9	8	1 5 6 4 5 7 9 7	1 3 4 5 7 9 7	3 8	2 3 2 5 7 9	2 3 4 5 6 7 9	
1 5 6 7 9	3 1 4	6	3 4 4 7 9	3 4 6 5 6 9 7 7	8 5 6 4 5 9 7 7	2 3 4 5 7 9 7	5 1 3 4 7 9	1 2 3 4 5 6 7	
1 5 6 7 9	3 4 5	9	1 2 3 4 5 7	2 3 1 2 3 4 6 5 6 4 5 7 7 7	1 3 3 4 5 7 7	8 8	1 2 4 5 6 7	1 2 3 4 5 6 7	
1 5 8	3 4	7	6	2 4 5	1 2 5 9	9 9	1 2 4 4 5 8	1 2 4 5 8	
2	7	1 3 4 9	3 4 8 9	5	1 3 1 3 6 4 9 8	1 4 9	1 3 4 6 4 6 9 8 9	1 3 4 6 4 6 9 8 9	
3 5 6 7 8 9	2 5 6 8	2 3 5 9 7 8 9	2 3 5 9 7 8 9	1 4 9 7 9	3 5 7 8	2 6 7 9	2 3 6 5 6 7 8 9	2 3 6 5 6 7 8 9	
1 5 6 7 8 9	3 1 2 4 5 6 8	1 2 3 4 5 9 7 8 9	2 3 4 6 5 6 9 7 9	2 3 1 2 3 4 6 5 6 9 7 9	1 2 4 6 7 9	4 6 7 9	1 2 4 6 7 9	4 6 7 9	4

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 5	3 7	1 2 6 7 9	1 2 3 5 6 7 9
3 5 6 8 9	2 4 5 6 8	2 3 4 5 9	1 7	2 3 5 6 9 8	3 4 5 9	2 4 6 9	2 3 4 5 6 8 9		
1 5 6 7 9	1 4 5 6 9	1 4 5 7 9	8	1 5 6 4 5 7 9 7	1 5 6 4 5 7 9 7	3 4 5 9	2 4 5 7 9		
1 5 6 7 9	3 1 4	1 6	3 4 4 7 9	3 4 6 9	8 5 6 4 5 7 9 7	2 4 5 9	5 7 9	1 3 4 7 9	
1 5 6 7	3 4 5	1 2 3 4 5 7	2 3 4 5 7	2 3 1 2 3 4 6 5 6 4 5 7 7 7	1 3 5 6 4 5 7 7 7	8 7	1 2 3 4 5 6 7		
1 5 8	3 7	7 6	4	2 1 2 5	9 6 4 9 8	1 2 4 9	1 2 4 5 8	1 2 4 5 8	
2	7	1 3 4 9	3 4 8 9	5	1 3 1 3 6 4 9 8	1 4 9	1 3 4 6 4 6 9 8 9	1 3 4 6 4 6 9 8 9	
3 5 6 7 8 9	2 5 6 8	2 3 5 9 7 8 9	2 3 5 9 7 8 9	2 3 1 2 3 5 9 7 9	3 5 7 8	2 6 7 9	2 3 5 6 7 8 9		
1 5 7 8 9	3 1 2 5 5 8	1 2 3 5 9 7 8 9	2 3 5 9 7 8 9	2 3 1 2 3 5 9 7 9	6 7 8	1 2 7 9	4 7 9		

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	1 2 3 5 6 9	1 3 5 7	1 2 6 7	1 3 5 9
3 5 6 8 9	2 3 4 5 8	1 7	2 3 5 6 9 8	3 4 5 9	2 3 4 6 9	2 3 5 6 8 9	2 3 4 6 9	2 3 5 6 8 9
7	1 1 4 5 6 9	4 5 7 9	8	1 1 5 6 4 5 7 9 7	3	2	3	2
1 3 7 9	4 6 7 9	4 5 7 9	8	2 5 7 9	1 3 7 9	1 3 7 9	1 3 7 9	1 3 7 9
1 3 5 6 7	9	1 2 3 4 5 7	2 3 4 6	1 2 3 5 6 4 5 7 7	8	1 2 5 4 5 7	1 3 5 6 7	1 3 5 6 7
1 5 8	3 7 6	4 5 9 8 9	2 4	1 2 5 4 5	9	1 2 4 4 5 7	1 3 5 8	1 3 5 8
2 7	1 3 3 4 4 9	5	1 3 3 6 4 9	1 3 3 6 4 8	1 3 1 6 6 4 6 9			
3 5 6 7 8 9	2 2 5 5 6 9	2 3 5 8 9	1 7 8 9	2 3 1 2 3 5 7 9	3 5 6 7 8 9	2 6 6 7 9 7 8 9	3 6 6 7 8 9	3 6 6 7 8 9
1 3 5 7 8 9	1 2 5 5 5 9	1 2 3 5 2 3 9 7 8 9	2 3 9 7 9	1 2 3 5 5 9 7 9	6	1 2 7 7 9	1 2 7 7 9	4

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 7 9	2 3 6 9 7	1 2 3 5 6 9 7	1 5	3 7	1 2 6 7 9	1 3 5 6 9 7 9
3 5 6 9	2 5 6 9	1	7	2 3 5 6 4 5 9 8	3 4 6 9	2 4 6 9	3 5 6 8 9	2 4 6 9	3 5 6 8 9
7	1 5 6 9	4	8	1 5 6 4 5 9	1 5 6 9	3 5 6 9	2 4 6 9	3 5 6 9	2 4 6 9
1 9	4 6	7	9	3 6 9	8 2 5	1 3 7 9	5 2 8	2 5 7	1 3 7 9
1 5 6	9	1 2 3 4 5 7	2 3 4 5 6	2 3 4 6 7	1 2 3 5 6 4 5 7 7	1 3 7	8 4 6 7	1 3 5 6 7	1 3 5 6 7
1 5 8	3 7	6	4	2 1 2 5	9 4 5	1 2 4 9	1 2 4 9	1 5 8	1 5 8
2	7	1 3 4 9	3 8 9	5 8 9	1 3 6 4 9 8	1 3 4 9	1 6 4 9	1 6 6 9	1 3 6 6 8 9
3 5 6 8 9	2 5 6 8	2 3 5 9 7 8 9	2 3 5 9 7 8 9	1 2 3 5 9 7 9	4 1 2 3 5 7 8	3 5 7 8	2 6 7 9	3 5 6 7 8 9	3 5 6 7 8 9
1 5 8	3 5 8	1 2 5 9 7 8 9	1 2 3 5 9 7 8 9	2 3 1 2 3 5 9 7 9	6 1 2 3 5 9 7 9	1 2 7 9	1 2 7 9	1 4	1 4

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► Skip Animation

Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	2 3 5 6	1 5	3	1 2 6 7	1 3 5 9
3 5 6 9	2 5 6	2 3 5 9	1 7	2 3 5 6	4 5 9 8	3 5	2 4 6	2 3 5 6 8 9	
7	1 5 6	1 5 9	4 8		1 5 9	3 5	3 2 5	1 3 7 9	2
1 9	4 6		3 7 9	3 9	8 2	2 5		1 3 7 9	
1 3 5 6	9 4 5	1 2 3 5	2 3 4	2 3 6	1 2 3 5 6	1 3 4 5 7	8 4 5	1 3 5 6 7	
1 5 8	3 7	7 9	6 8 9	2 5	1 2 5	9 8	1 2 4	1 3 5 8	
2	7 4	1 3 9	3 8 9	5 7 8 9	1 7 8	3 5	1 4	1 3 6 9	6 8 9
3 5 6 8 9	2 5 6 8	2 3 5	2 3 7 8 9	2 3 9 7 9	1 2 3 5	3 7 8	2 7	2 3 6 7 8 9	
1 5 8	3 5 8	1 2 5	1 2 3 5	2 3 9 7 8 9	1 2 3 5	6 9	1 2 7	1 2 4	

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Propagation Steps (Domain Consistency)

4	1 2 5 6	8	2 3 5 9	2 3 6 9	2 3 5 6	1 5	1 7	1 9	1 6 5 6 9
3 5 6 9	2 5 6 9	2 3 5 9	1 7	2 3 5 6	4 5 9 8	4 6	4 9	5 6 8 9	
7	1 5 6 9	1 5	4 9	8	5 6 9	1 5	3 7	2 9	
1 4	6 7	3 9	3 9	8	2 7	5 7	1 3 9		
5 9	2 7	2 3 5 4	2 3 6 7	1 2 3 5 6 7	1 4 5 7	3 8	1 5 6 7		
8 3	7 6	4 8	2 5	1 2 5	9 4	1 2 4	1 5 8		
2 7	4 8	3 8 9	5 9	1 6 9	3 4 8	1 4 9	1 6 9	3 6 8 9	
3 5 6 8 9	2 5 6 8	2 3 5 7	2 3 8 9	1 2 3 9 7	3 5 7 8	2 6 7 9	3 6 7 8 9		
1 5 8	3 5 9	1 2 5 9	1 2 3 5 7 8 9	2 3 1 2 3 9 7 9	6 5	1 2 7 9	4 9		

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Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	1 7	1 9	5 6 9
3 6 9	5	3 5 9	1	7	2	4	6	8	
7	6	1	4	8	9	5	3	2	
1	4	6	7 9	3 9	3 9	8	2	5	3 7 9
5	9	2	7 3	4	1	1 4 7	3	8	6
8	3	7	6	2	5	9	4	1	
2	7	4	8 9	5	6	8	1	1 3 6 8 9	
6	8	2	1	4	5 7 8	3	2	6 7 9	5
3 9	1	5	8	2 3 9	7	6	2	4	

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Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	7	9	7	9
3	5	3	1	7	2	4	6	8		
9	9									
7	6	1	4	8	9	5	3	2		
1	4	6	7	9	9	8	2	5	7	9
5	9	2	7	3	4	1	7	3	8	6
8	3	7	6	2	5	9	4	1		
2	7	4	3	9	5	6	8	1		3
6	8	3	2	1	4	7	3	7	9	5
3	1	5	8	3	7	6	2	4		
9			9							

◀ Back to Start

► Skip Animation

Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	7	9	7	9
3	5	3	1	7	2	4	6	8		
9	9									
7	6	1	4	8	9	5	3	2		
1	4	6	7	9	9	8	2	5	7	3
5	9	2	7	3	4	1	7	3	8	6
8	3	7	6	2	5	9	4	1		
2	7	4	3	9	5	6	8	1	9	3
6	8	2	1	4	7	3	7	9	5	
3	1	5	8	9	7	6	2	4		

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Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	7	9	7	9
3	5	3	1	7	2	4	6	8		
9	9									
7	6	1	4	8	9	5	3	2		
1	4	6	7	9	9	8	2	5	7	3
5	9	2	7	3	4	1	7	3	8	6
8	3	7	6	2	5	9	4	1		
2	7	4	3	9	5	6	8	1	9	3
6	8	2	1	4	7	3	7	9	5	
3	1	5	8	9	7	6	2	4		

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Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	7	9	7	9
3	5	3	1	7	2	4	6	8		
9	9									
7	6	1	4	8	9	5	3	2		
1	4	6	7	9	9	8	2	5		3
5	9	2	3	4	1	7	3	8	6	
8	3	7	6	2	5	9	4	1		
2	7	4	3	5	6	8	1		3	9
6	8	3	2	1	4	7	3	7	9	5
3	1	5	8	3	7	6	2	4		
9			9							

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Propagation Steps (Domain Consistency)

4	2	8	5	6	3	1	7	9	7	9
3 9	5	3 9	1	7	2	4	6	8		
7	6	1	4	8	9	5	3	2		
1	4	6	7 9	3 9	3 9	8	2	5	7	3
5	9	2	7 9	4	1	7 9	8	6		
8	3	7	6	2	5	9	4	1		
2	7	4	3 9	5	6	8	1	3 9		
6	8	3 9	2	1	4	7 9	3 9	5		
3 9	1	5	8	3 9	7	6	2	4		

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After Setup (Domain Consistency)

4	2	8	5	6	3	1			
3 9	5	3 9	1	7	2	4	6	8	
7	6	1	4	8	9	5	3	2	
1	4	6	7 9	3 9	3	8	2	5	3 7
5	9	2	7 9	4	1	7	3	8	6
8	3	7	6	2	5	9	4	1	
2	7	4	3 9	5	6	8	1		3 9
6	8	3 9	2	1	4	7 9	7 9	5	
3 9	1	5	8	3 9	7	6	2	4	

Comparison

Forward Checking

4	1 2	8	2 3	3	2 3	1	1	6	5 6
3	2	5 8	3	9	6	5	7	6	7 8
6	5 8	5	1	7	2 3	6	4 5	8	5 6
9	8	9	9	8	9	8	9	8 9	8 9
1	6	5 6	5	4 5	8	1	6 5	3	2
7	9	9	9	9	7	9	7	7	6
1	4	6	7	9	8	2	5	7	3
5	9	2	4	7	3	3	5	3	3
8	3	7	6	2	5	9	4	1	3
2	7	4	3	3	5	6	1	3	3
3	6	5 8	5	2 3	8	8	2	5	3
6	5 8	5	7	8 9	1	4	7	8	7 8
9	8	8	7	8 9	9	9	7	8 9	8 9
3	1	5	5	2 3	2	2 3	1	2	4
9	8	9	9	9	9	6	7	8	4

Bounds Consistency

4	1 2	8	5	6	2 3	1	1		
3	2	5	3	9	6	4 5	8	6	5
6	5	8	1	7	2 3	4 5	8	3	8 9
9	8	9	9	8	9	8	9	8 9	8 9
7	6	5	4	8	9	1	5	3	2
1	4	6	7	9	8	2	5	7	3
5	9	2	4	7	3	1	8	6	3
8	3	7	6	2	5	9	4	1	3
2	7	4	3	3	5	6	1	3	3
6	5	8	5	2 3	8	8	3	5	3
9	8	9	7	8 9	1	4	7	8	7 8 9
3	1	5	5	2 3	2	2 3	1	2	4
9	8	9	9	9	9	7	6	8	4

Domain Consistency

4	2	8	5	6	3	1	7	9	9
3	5	1	7	2	4	6	8		
7	6	1	4	8	9	5	3	2	
1	4	6	7	8	9	2	5	7	
5	9	2	4	1	7	8	6		
8	3	7	6	2	5	9	4	1	
2	7	4	3	5	6	8	1	9	
6	8	2	1	4	7	7	9	5	
9	1	5	8	7	6	2	4		

Typical?

- This does not always happen
- Sometimes, two methods produce same amount of propagation
- Possible to predict in certain special cases
- In general, tradeoff between speed and propagation
- Not always fastest to remove inconsistent values early
- But often required to find a solution at all

Outline

Problem

Initial Propagation (Forward Checking)

Improved Reasoning

Search

Other Global Constraints

Simple search routine

- Enumerate variables in given order
- Try values starting from smallest one in domain
- Complete, chronological backtracking
- Advantage: Results can be compared with each other
- Disadvantage: Usually not a very good strategy

Asking for Naive Search in MiniZinc

```
solve :: int_search(  
    puzzle,  
    input_order,  
    indomain_min)  
satisfy;
```

Search Tree (Forward Checking)

2

4	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
3	2	5	6	8	9	5	6	7	8	2	3	4	5	6	7	8	9	5
6	5	6	5	9	9	1	7	8	9	6	4	5	6	7	8	9	8	9
9	7	8	9	1	2	3	4	5	6	7	8	9	4	5	6	7	2	3
1	6	5	6	1	2	3	4	5	8	1	2	3	4	5	6	7	3	2
7	9	8	9	9	9	9	9	9	9	7	8	9	7	8	9	7	8	9
1	4	6	8	9	7	8	9	2	5	1	4	6	7	8	9	2	5	7
5	9	2	8	7	8	9	3	4	5	7	8	9	7	8	9	7	8	6
8	3	7	2	1	2	3	4	5	6	3	4	5	6	7	8	9	4	1
2	7	4	5	3	4	5	6	7	8	1	2	3	4	5	6	7	8	3
9	8	9	9	9	8	9	9	8	9	9	8	9	8	9	9	8	9	5
3	6	5	6	5	3	2	3	4	5	3	1	2	3	4	5	6	7	3
6	9	8	9	9	9	7	8	9	1	4	5	6	7	8	9	7	8	9
9	8	9	9	9	9	7	8	9	3	2	3	4	5	6	7	8	9	4
3	1	5	5	1	2	3	4	5	6	3	2	3	4	5	6	7	8	1
6	5	5	5	9	9	7	8	9	9	7	8	9	7	8	9	7	8	9
9	8	9	9	9	9	7	8	9	9	7	8	9	7	8	9	7	8	4

Search Tree (Forward Checking)

2
1
4

4	1	8	2 3	3	2 3	1	1	6	5 6
3	2	3	5	6	6	5	7	7	9 7
6	5 6	5	9	1	7	2 3	4 5	6	5 6
9		9				6	9 8	9	8 9
1	6	1	4 5	8	6	4 5	9 7	3	2
6	5 6	5	9		3	3			3
9	9	9	9		3	1	3	3	
1	4	6	7 8	8	2	5	7		
5	9	2	4 7	7	7	7	7	8	7
8	3	7	6	2	5	9	4	1	
2	7	1 4	3 9	5	3 1	3 1			3
3	5 6	5	8 9	5	6	8	9	9	8 9
6	9 8	9	7 8 9	1	4	3	2	5	
9	8	9	7 8 9	9	7	7	8 9	6	7 9 4

Search Tree (Forward Checking)

2
1
4
2
5

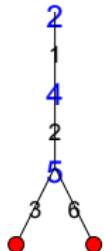
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3	2	3		6	6	6	5	6	4	5	6	8	6	5	9
6	5	6	5	9	1	7		6	4	5	6	8	9	5	6
9			9				8	6	4	5	9	7	3	2	3
6	5	6	5	4	5	8	3	3	1	3	3	7	2	5	7
7	9		9	9	9		7	8	9	7	7	7	8	7	6
1	4	6					8	2	5						3
5	9	2	4	4	4		7	9	3	1	3	7	7	8	7
8	3	7	6	2	5	9	4	1	3	1	3	1	8	9	3
2	7	1	3	3	3	5	3	1	3	1	8	9	6	7	3
		4	9	8	9		9	6	6	6	8	9	8	9	8
3		3	2	3			5	5	5	5	7	8	7	9	3
6	5	6	5	9	7	8	9	1	4	1	2	1	2	5	5
9	8		9	7	8	9	8	7	9	6	7	9	7	8	9
3		1	3	2	3		3	2	3	1	2	1	2	5	5
5		5					6			6					4
9	8		9	7	8	9	9	7	9	6	7	9	7	8	9

Search Tree (Forward Checking)

2
1
4
2
5
3

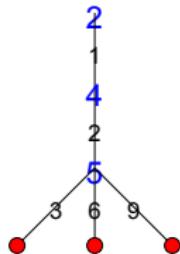
4	1	8	2	3	3	3	5	6	5	6	5	6
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6	5	6	5	9	1	7	6	4	5	6	5	6
9	8	9	7	8	8	9	8	9	8	9	8	9
7	9	5	6	5	9	4	5	9	7	1	4	5
6	5	6	5	9	8	8	9	7	7	3	2	3
1	4	6	7	8	9	8	2	5	7	7	6	3
5	9	2	4	3	4	3	1	3	3	8	7	6
8	3	7	6	2	5	9	4	1	3	1	3	3
2	7	1	3	3	4	5	6	7	8	9	8	9
3	6	5	6	5	9	8	9	7	8	7	9	8
6	5	6	5	9	7	8	9	1	4	3	2	3
9	8	9	7	8	9	8	7	9	7	8	9	8
3	5	6	5	9	7	8	9	8	7	9	6	7
6	5	6	5	9	7	8	9	9	7	9	4	5
9	8	9	7	8	9	8	7	9	7	9	8	9

Search Tree (Forward Checking)



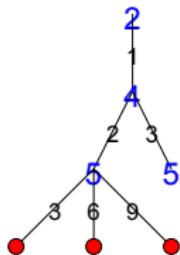
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6	5	6	5	9	1	7	6	4	5	6	5
9			9				9	8	9	8	9
7	6	5	6	5	4	5	8	6	4	5	3
9				9	9	9	7	7	3	2	
1	4	6			3	3	8	2	5		7
5	9	2	4	4	3	3	1	3	3	8	6
7	7	7	7	7	7	7	7	7	7	7	
8	3	7	6	2	5	9	4	1			3
2	7	1	3	3	3	3	1	3	1		
		4	9	8	9	5	6	8	9		
3		3	3	8	9	5	7	8	7	9	8
6	5	6	5	9	7	8	9	7	8	7	9
9	8			1	4	5	2	1	2	5	3
3		1	3	3	3	3	2	3	1	2	
6	5	6	5	9	7	8	9	8	7	9	4
9	8			9	7	8	9	9	7	9	

Search Tree (Forward Checking)



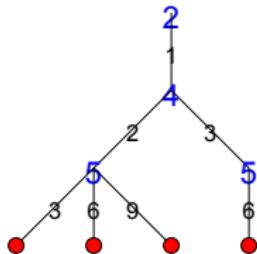
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6	5	6	5	1	7	6	4	5	6	5	6
9	9	9	9	8	8	9	8	9	8	9	8
7	9	6	5	6	5	4	5	9	7	3	2
1	4	6	7	8	9	8	2	5	7	3	3
5	9	2	4	4	3	3	1	3	3	8	7
8	3	7	6	2	5	9	4	1	3	3	3
2	7	1	3	3	3	3	1	3	1	9	8
9	8	4	9	8	9	5	6	8	9	7	8
3	6	5	3	3	1	4	5	2	5	7	3
9	8	8	9	7	8	9	7	8	7	9	8
3	5	1	3	3	3	3	2	3	1	2	4
9	8	5	9	7	8	9	8	7	9	7	9

Search Tree (Forward Checking)



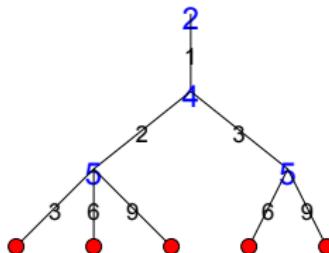
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6	5	6	5	6	4	5	6	4	5	6	4	5
9	9	9	9	9	9	7	8	9	8	9	9	8
7	6	5	6	5	4	5	8	6	4	5	3	2
9	7	9	8	7	8	9	7	8	7	8	7	6
1	4	6	7	8	9	3	3	1	3	3	1	3
5	9	2	4	4	4	3	3	1	3	3	1	3
8	3	7	6	2	5	9	4	1	3	1	3	1
2	7	1	3	3	3	3	1	3	1	3	1	3
9	8	4	9	8	9	5	6	9	8	9	9	8
3	6	5	6	5	3	2	3	1	3	2	3	1
9	8	9	7	8	9	1	4	5	6	7	8	9
3	5	6	5	6	7	8	9	8	7	9	7	8
9	8	9	7	8	9	8	7	9	8	7	9	4

Search Tree (Forward Checking)



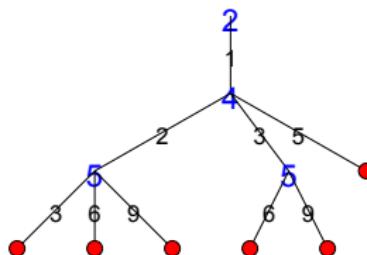
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3	2	3	9	9	7	9	7	9	7	9
6	5	6	5	9	1	7	6	4	5	6
9	9	9	9	9	9	9	8	9	8	9
7	6	5	6	5	4	5	8	6	4	5
9	9	9	9	9	9	9	7	9	7	9
1	4	6	7	9	8	2	5	7	3	3
5	9	2	4	4	3	1	3	3	8	7
8	3	7	6	2	5	9	4	1	3	3
2	7	1	3	4	3	1	3	1	9	8
9	8	7	9	8	9	8	9	8	9	8
3	6	5	6	5	3	2	5	3	2	3
9	8	9	7	8	9	1	4	7	8	7
3	5	6	5	9	7	8	9	8	7	9
9	8	9	7	8	9	9	7	9	7	9
6	6	5	5	9	7	8	9	8	7	9
9	8	9	9	7	8	9	9	7	9	4

Search Tree (Forward Checking)



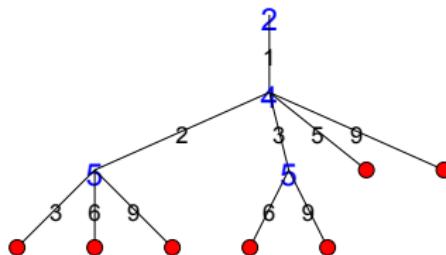
4	1	8	3	9	2	6	5	6	5	6	5	6
3	2	3			2							
6	5	6	5	1	7	6	4	5	6	5	6	5
9		9				9	8	9	8	9	8	9
7	6	5	6	5	4	5	8	6	4	5	3	2
9					7	9	8	7	9	7	7	7
1	4	6			8	2	5					
5	9	2	4	4	3	1	3	3	3	3	3	3
8	3	7	6	2	5	9	4	1				
2	7	1	3		5	3	1	3	1			
		4		9	8	6	6	8	9	8	9	8
3	6	5	6	5	2	1	4	3	2	5	3	3
9	8		9	7	8	9	1	4	7	8	7	8
3	5	6	5	2		3	2	3	1	2	5	4
9	8		9	7	8	9	8	7	9	7	9	4

Search Tree (Forward Checking)



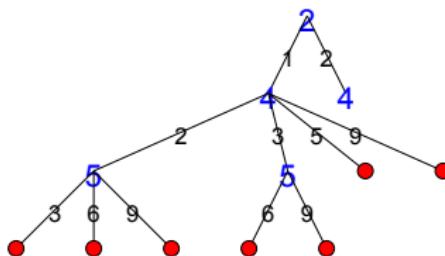
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3	2	9	6	6	9	1	7	2 3	6	4 5	6	5 6	8
6	5 6	5	9	9	9	7	8	6	4 5	9	8	9	8 9
9	9	9	9	9	9	8	8	1	1	3	3	3	3
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5	9	2	4	3	1	3	8	7	7	7	7	7	6
8	3	7	6	2	5	9	4	1	3	1	3	1	3
2	7	1	4	3	3	5	6	9	8	9	8	9	8 9
3	6	5 6	5	9	8 9	1	4	3	2	5	5	6	3
9	8	9	9	7 8 9	9	7	8	7	8	7	9	7 8 9	3
3	5	1	3	2 3	3	2 3	6	7	1 2	5	6	7	4
9	8	9	7 8 9	9	7	9	8	7	9	7	9	8	4

Search Tree (Forward Checking)



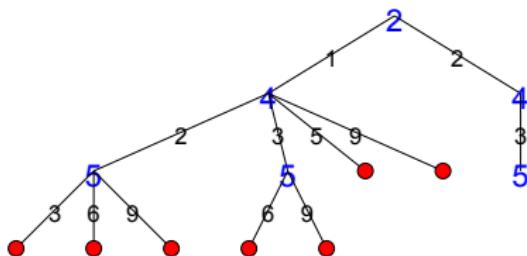
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3	2	3			2 3 6 9	4 5 8	6	5 6 8
6	5 6	5			6 4 5 9 7	8	9	5 6 8
9		9						
7	6	5 6	5 9	4 5 9	8	6 4 5 9 7	3	2
9								
1	4	6			3	3	8	2 5 7
5	9	2	4 7	4	3 1 3	3	8	3 6
8	3	7	6	2	5	9 4	1	
2	7	1 3 4	3	5	3 1 3 1	6		3
3	6	5 6	5 9	8 9	9 8	8	9	2 3 8 9
8								
3	5 6	5	3 2 3	1	4	3 2	5	
9			9 7 8 9			7 8 7 9	7 8 9	
8	5	5	1 3 2 3	3 2 3		1 2	6	4
9	8	9	9 7 8 9	9 7 9		7 9	7	

Search Tree (Forward Checking)



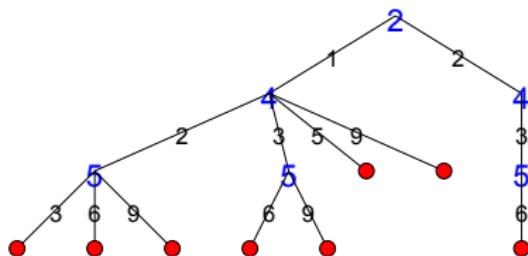
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9	9	9	9	9	9	9	9	9	9	8	8	8	8	8	8	8	8	8	8	8
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4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
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Search Tree (Forward Checking)



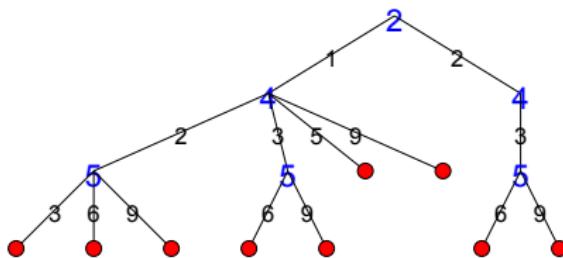
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1	5	6	5	4	5	6	4	5	6	4	5	6	4	5	6	4
6	9	9	9	8	9	7	8	7	9	8	7	9	8	7	9	8
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6	9	8	9	7	8	9	7	8	9	1	3	2	5	7	8	9
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Search Tree (Forward Checking)



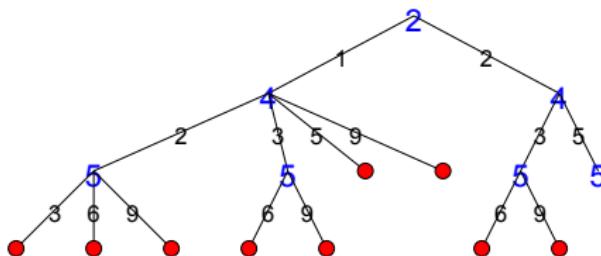
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9	9	9	9	9	9	7	7	7	7	7	3
1	5	1	5	4	5	8	1	5	3	2	7
6	5	6	5	9	9	3	1	3	3	3	3
7	9	7	9	7	9	7	7	7	7	7	7
1	4	6	7	9	8	2	5	7	8	9	1
5	9	2	4	4	3	1	3	3	8	7	6
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6	5	6	5	7	8	1	4	5	2	5	3
9	8	9	9	8	9	7	8	7	9	7	8
3	1	5	1	3	2	3	2	3	1	2	4
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9	8	9	9	7	8	9	9	7	9	7	9

Search Tree (Forward Checking)



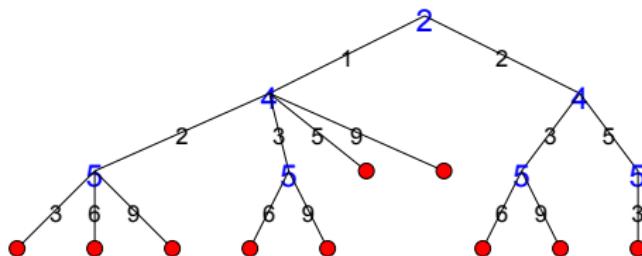
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9	9	9	9	9	8	8	6	4	5	3	2
1	5	6	1	4	8	1	5	3	2	5	7
6	5	6	5	9	3	1	3	3	3	3	3
7	9	9	9	9	7	7	7	7	7	7	7
1	4	6	7	9	8	2	5	7	8	9	6
5	9	2	4	4	3	1	3	3	8	7	6
8	3	7	6	2	5	9	4	1	3	3	1
2	7	1	3	4	5	6	1	3	1	8	9
3	6	5	6	5	9	8	9	5	6	7	8
9	8	8	9	9	7	8	9	1	4	7	6
3	1	5	1	3	2	3	2	3	3	2	3
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Search Tree (Forward Checking)



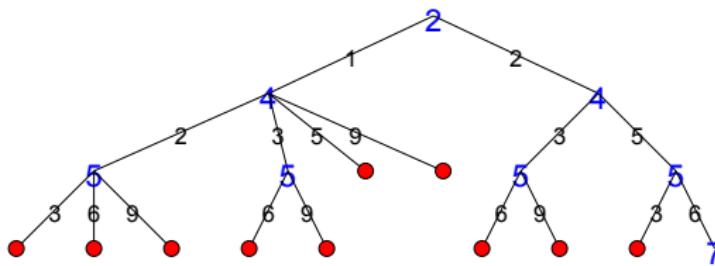
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6	9	8	9	9	9	9	9	9	9	9	9	9	9	9	9	9
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1	5	5	9	9	7	8	9	9	7	9	7	9	7	9	7	9
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Search Tree (Forward Checking)



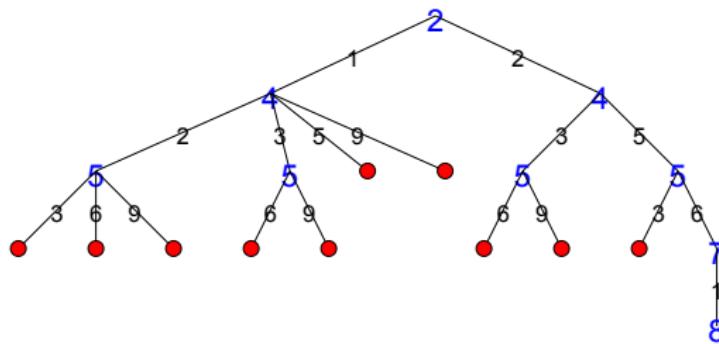
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2	7	1	3	3	3	1	3	1	3
4	9	8	9	8	9	6	6	6	6
3	6	5	3	2	3	9	8	7	8
8	8	9	7	8	9	1	4	5	5
3	1	5	1	3	2	3	3	2	3
9	8	9	7	8	9	8	7	9	8
6	6	6	6	6	6	6	6	6	6
7	8	7	9	7	8	7	9	7	8
9	7	9	8	9	9	7	9	7	8
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Search Tree (Forward Checking)



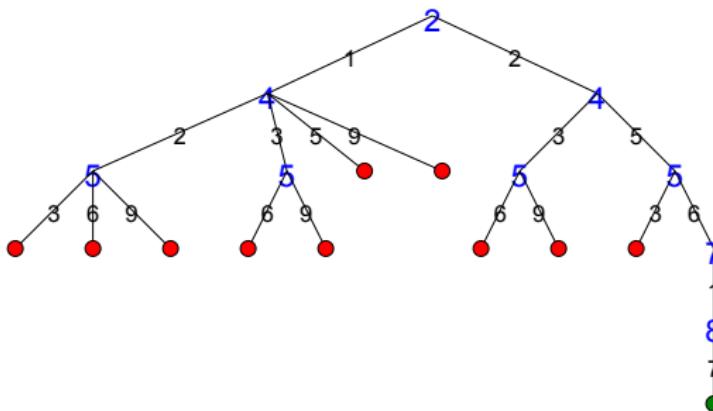
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9	9	9	9	9	7	7	7	9	9
1	5	6	1	4	8	9	1	5	3
6	5	6	5	9	4	8	9	3	2
7	9	9	9	9	7	7	7	7	3
1	4	6	7	8	9	8	2	5	7
5	9	2	4	5	6	1	3	8	6
8	3	7	6	2	5	9	4	1	3
2	7	1	3	3	5	6	1	3	3
4	9	8	9	8	9	8	9	9	8
3	5	6	5	3	2	3	5	2	3
6	8	9	9	7	8	9	1	4	5
9	8	9	9	7	8	9	7	8	9
3	1	1	3	2	3	3	1	2	3
5	5	5	5	7	8	9	7	6	5
9	8	9	9	7	8	9	7	9	4

Search Tree (Forward Checking)



4	2	8	5	6	3	1	7	9	7	9	7	9
3	6	5	6	5	9	1	7	2	4	5	6	8
6	1	5	6	1	5	4	8	9	1	5	3	2
7	6	5	6	5	9	3	3	3	7	8	7	3
1	4	6	7	8	9	8	2	5	7	3	6	1
5	9	2	7	3	4	1	7	8	7	6	8	7
8	3	7	6	2	5	9	4	1	3	1	3	3
2	7	1	4	3	3	5	6	7	8	9	5	3
3	6	5	6	5	9	8	9	1	4	7	8	9
9	8	5	9	8	9	3	2	7	6	1	2	3
3	1	5	6	5	9	8	9	7	6	1	2	4
9	8	5	9	8	9	8	9	7	6	1	2	3

Search Tree (Forward Checking)



4	2	8	5	6	3	1	7	9
3	5	9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	3	9	8	2	5	7
5	9	2	7	4	1	3	8	6
8	3	7	6	2	5	9	4	1
2	7	4	9	5	6	8	1	3
6	8	3	2	1	4	7	9	5
9	1	5	8	3	7	6	2	4

Search Tree (Bounds Consistency)

2

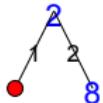
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3	2	3			2 3	4 5	6	5	9
	5		1	7		8	9	8	9
9		9							
7	6	4	8	9	1	5	3	2	
1	4	6	7	8	9	8	2	5	3
5	9	2	7	3	3	1	7	8	6
8	3	7	6	2	5	9	4	1	
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8	9	9	8	9	1	4	3	2	5
3	1	1	2	3	7	8	7	9	7 8 9
5	5	5	8	9	7	6	1	2	4
9	8								

Search Tree (Bounds Consistency)

2
1

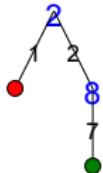
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9	5	9	7	8	7	8	9	8	9	9
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1	4	6	7	8	9	8	2	5	7	
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6	5	3	2	3	1	4	5	2	5	3
3	1	1	2	3	7	8	7	9	7	8
9	5	5	8	9	7	6	1	2	4	

Search Tree (Bounds Consistency)



4	2	8	5	6	3	1	1		
3 9	5	3 9	1	7	2	4	6	5 8 9	
7	6	1	4	8	9	5	3	2	
1	4	6	7 8 9	3 9	8	2	5	7 3	
5	9	2	7 8 9	3 4	1	7 8	8	6	
8	3	7	6	2	5	9	4	1	
2	7	4	8 9	3 5	6	8	1	7 8 9	
6	8	2	1	4	7 8 9	3 2	5	7 8 9	
3 9	1	5	8	7 9	6	2	4		

Search Tree (Bounds Consistency)



4	2	8	5	6	3	1	7	9
3	5	9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	3	9	8	2	5	7
5	9	2	7	4	1	3	8	6
8	3	7	6	2	5	9	4	1
2	7	4	9	5	6	8	1	3
6	8	3	2	1	4	7	9	5
9	1	5	8	3	7	6	2	4

Search Tree (Domain Consistency)

8

4	2	8	5	6	3	1			
3 9	5	3 9	1	7	2	4	6	8	
7	6	1	4	8	9	5	3	2	
1	4	6	7 8 9	3 9	8	2	5	7 3	
5	9	2	7	3 9	4	1	7 3	8	6
8	3	7	6	2	5	9	4	1	
2	7	4	3 9	5	6	8	1	7 9 3	
6	8	2	1	4	7	7 9	5		
3 9	1	5	8	3 9	7	6	2	4	

Search Tree (Domain Consistency)

8
7
6

4	2	8	5	6	3	1	7	9
3	5	9	1	7	2	4	6	8
7	6	1	4	8	9	5	3	2
1	4	6	3	9	8	2	5	7
5	9	2	7	4	1	3	8	6
8	3	7	6	2	5	9	4	1
2	7	4	9	5	6	8	1	3
6	8	3	2	1	4	7	9	5
9	1	5	8	3	7	6	2	4

Trading Propagation Against Search

- If we perform more propagation, search is more constrained
- Fewer values left, fewer alternatives to explore in search
- Best compromise is not obvious
- But can be learned from examples or during search
- Annotations are optional
 - Some MiniZinc back-end solvers do the search they want, not the one you specify
 - Some solvers simply do not work in a way that these search annotations apply

Outline

Problem

Initial Propagation (Forward Checking)

Improved Reasoning

Search

Other Global Constraints

Are there other Global Constraints?

- alldifferent is the most commonly used constraint
- Propagation methods can be explained
- But there are many more

Global Constraint Catalog

- <https://sofdem.github.io/gccat/>
- Description of 354 global constraints, 2800 pages
- Not all of them are widely used
- Detailed, meta-data description of constraints in Prolog

Families of Global Constraints

- Value Counting
 - alldifferent, global cardinality
- Scheduling
 - cumulative
- Properties of Sequences
 - sequence, no_valley
- Graph Properties
 - circuit, tree

Common Algorithmic Techniques

- Bi-Partite Matching
- Flow Based Algorithms
- Automata
- Task Intervals
- Reduced Cost Filtering
- Decomposition

Part III

Customizing Search

What we want to introduce

- Importance of search strategy, constraints alone are not enough
- Two schools of thought
 - Black-box solver, solver decides by itself
 - Human control over process
- Dynamic variable ordering exploits information from propagation
- Variable and value choice
- Hard to find strategy which works all the time
- Different way of improving stability of search routine

Example Problem

- N-Queens puzzle
- Rather weak constraint propagation
- Many solutions, limited number of symmetries
- Easy to scale problem size

Outline

Problem

Program

Naive Search

Improvements

Problem Definition

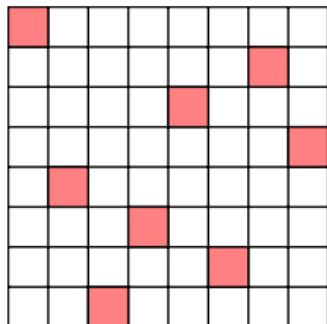
8-Queens

Place 8 queens on an 8×8 chessboard so that no queen attacks another. A queen attacks all cells in horizontal, vertical and diagonal direction. Generalizes to boards of size $N \times N$.

Problem Definition

8-Queens

Place 8 queens on an 8×8 chessboard so that no queen attacks another. A queen attacks all cells in horizontal, vertical and diagonal direction. Generalizes to boards of size $N \times N$.



Solution for board size 8×8

Outline

Problem

Program
Model

Naive Search

Improvements

Basic Model

- Cell based Model
 - A 0/1 variable for each cell to say if it is occupied or not
 - Constraints on rows, columns and diagonals to enforce no-attack
 - N^2 variables, $6N - 2$ constraints
- Column (Row) based Model
 - A 1..N variable for each column, stating position of queen in the column
 - Based on observation that each column must contain exactly one queen
 - N variables, $N^2/2$ binary constraints

Model

assign $[X_1, X_2, \dots, X_N]$

s.t.

$$\forall 1 \leq i \leq N : X_i \in 1..N$$

$$\forall 1 \leq i < j \leq N : X_i \neq X_j$$

$$\forall 1 \leq i < j \leq N : X_i + j \neq X_j + i$$

$$\forall 1 \leq i < j \leq N : X_i + i \neq X_j + j$$

Outline

Problem

Program

Naive Search

Improvements

Nqueens Models

- ECLiPSe [▶ Show](#)
- MiniZinc [▶ Show](#)
- NumberJack [▶ Show](#)
- CPMPy [▶ Show](#)
- Choco-solver [▶ Show](#)

ECLIPSe N-Queens Model

```
:- lib(lists).
:- lib(ic).

top:-
    queens(8,Board),
    search(Board, 0, input_order, indomain, complete, []),
    writeln(Board).

queens(N, Board) :-
    length(Board, N),
    Board :: 1..N,
    ( fromto(Board, [Q1|Cols], Cols, []) do
        ( foreach(Q2, Cols), param(Q1), count(Dist, 1,_) do
            noattack(Q1, Q2, Dist)
        )
    ) .

noattack(Q1,Q2,Dist) :-
    Q2 #\= Q1,
    Q2 - Q1 #\= Dist,
    Q1 - Q2 #\= Dist.
```

▶ Continue

MiniZinc N-Queens Model

```
int: n=8;
array[1..n] of var 1..n: queens;
constraint
  forall(i, j in 1..n where i < j) (
    queens[i] != queens[j] /\ 
    queens[i] + i != queens[j] + j /\ 
    queens[i] - i != queens[j] - j
  )
;
solve :: int_search(
  queens,
  input_order,
  indomain_min)
satisfy;
```

» Continue

NumberJack N-Queens Model

```
from Numberjack import *

def get_model(N):
    queens = VarArray(N, N)
    model = Model(
        AllDiff(queens),
        AllDiff([queens[i] + i for i in range(N)]),
        AllDiff([queens[i] - i for i in range(N)])
    )
    return queens, model

def solve(param):
    queens, model = get_model(param['N'])
    solver = model.load(param['solver'])
    solver.setHeuristic(param['heuristic'], param['value'])
    solver.setVerbosity(param['verbose'])
    solver.setTimeLimit(param['tcutoff'])
    solver.solve()
```

▶ Continue

CPMPy N-Queens Model

```
def nqueens_naive(n=8):
    queens = IntVar(1,n, shape=n)

    model = Model()
    for i in range(n):
        for j in range(i):
            model += [queens[i] != queens[j],
                      queens[i] + i != queens[j] + j,
                      queens[i] - i != queens[j] - j,
                      ]
```

► Continue

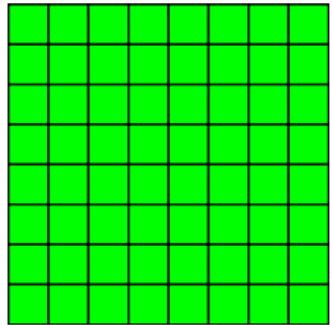
Choco-solver N-Queens Program

```
int n = 8;
Model model = new Model(n + "-queens problem");
IntVar[] vars = new IntVar[n];
for(int q = 0; q < n; q++){
    vars[q] = model.intVar("Q_"+q, 1, n);
}
for(int i = 0; i < n-1; i++){
    for(int j = i + 1; j < n; j++){
        model.arithm(vars[i], "!=" ,vars[j]).post();
        model.arithm(vars[i], "!=" , vars[j], "-", j - i).post();
        model.arithm(vars[i], "!=" , vars[j], "+", j - i).post();
    }
}
Solution solution = model.getSolver().findSolution();
if(solution != null){
    System.out.println(solution.toString());
}
```

► Continue

Default Strategy

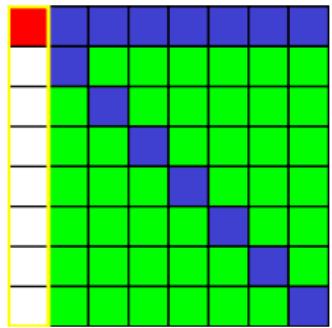
1



▶ Skip Animation

Default Strategy

1
1
2

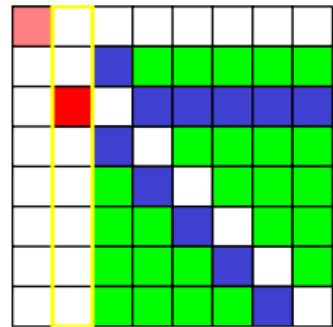


◀ Back to Start

► Skip Animation

Default Strategy

1
1
2
3
3

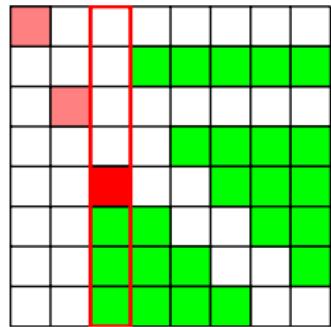


◀ Back to Start

► Skip Animation

Default Strategy

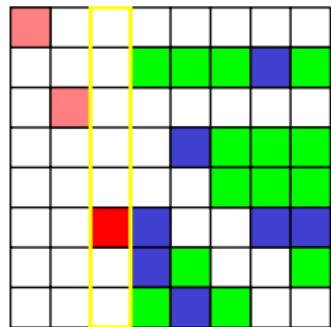
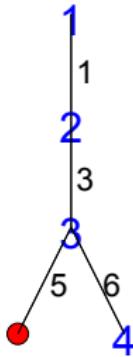
1
1
2
3
3
5
●



◀ Back to Start

▶ Skip Animation

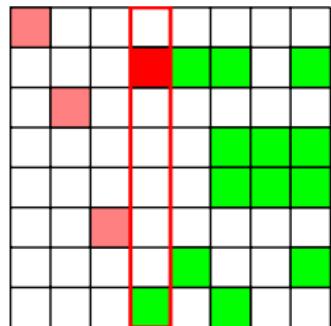
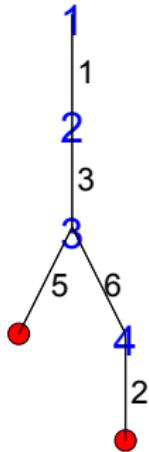
Default Strategy



◀ Back to Start

▶ Skip Animation

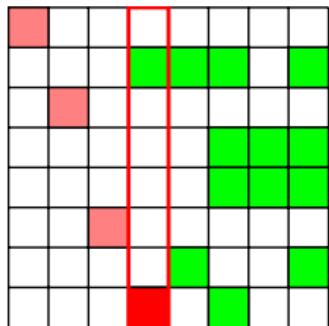
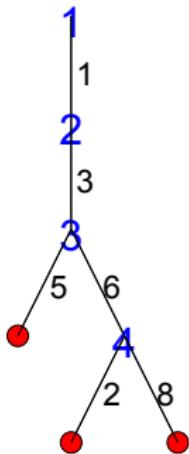
Default Strategy



◀ Back to Start

► Skip Animation

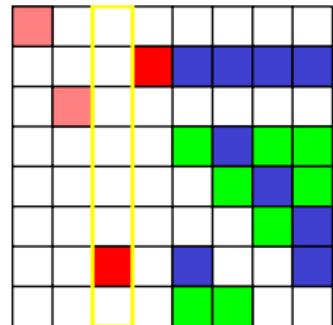
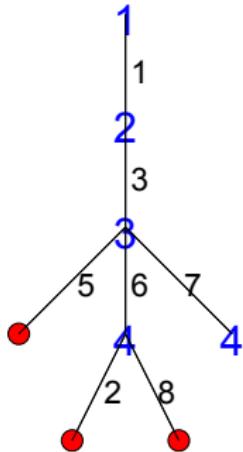
Default Strategy



◀ Back to Start

► Skip Animation

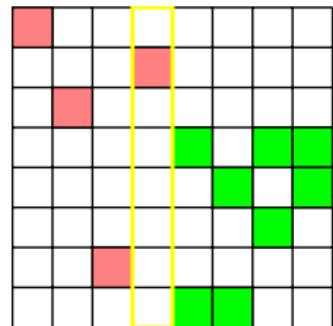
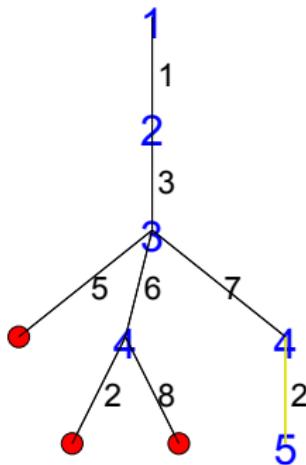
Default Strategy



◀ Back to Start

► Skip Animation

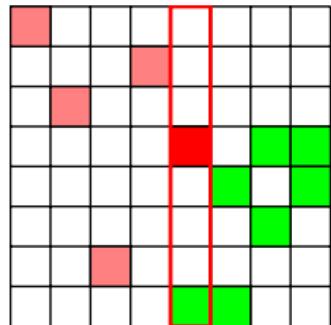
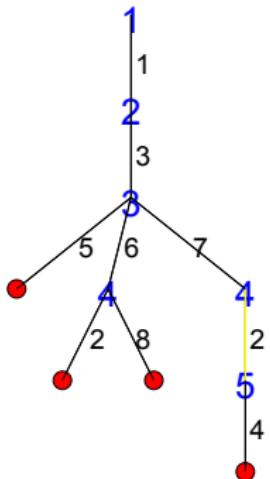
Default Strategy



◀ Back to Start

► Skip Animation

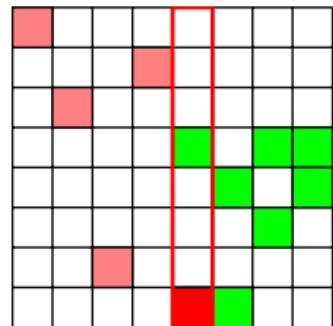
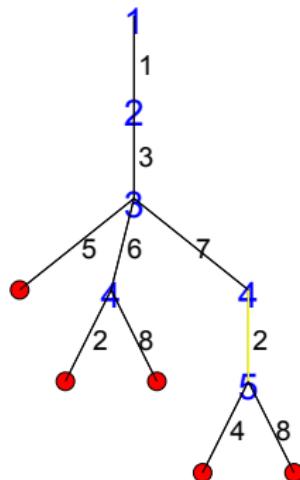
Default Strategy



◀ Back to Start

► Skip Animation

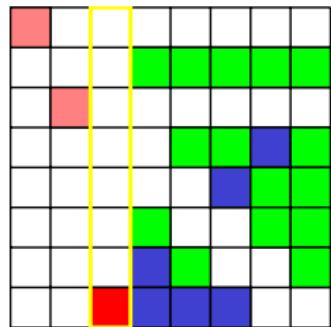
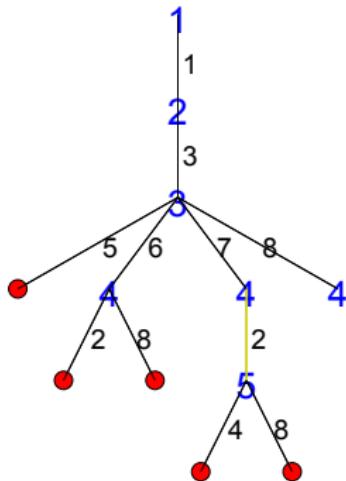
Default Strategy



◀ Back to Start

► Skip Animation

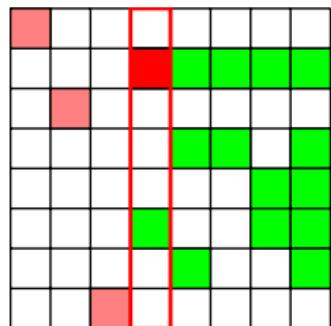
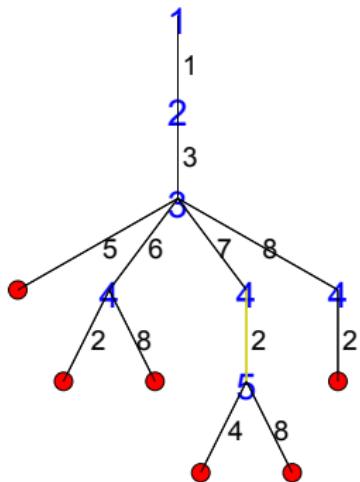
Default Strategy



◀ Back to Start

► Skip Animation

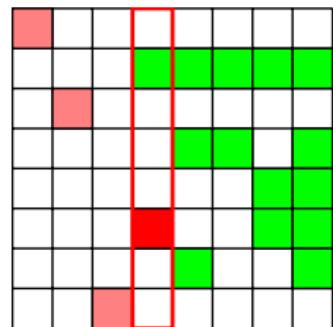
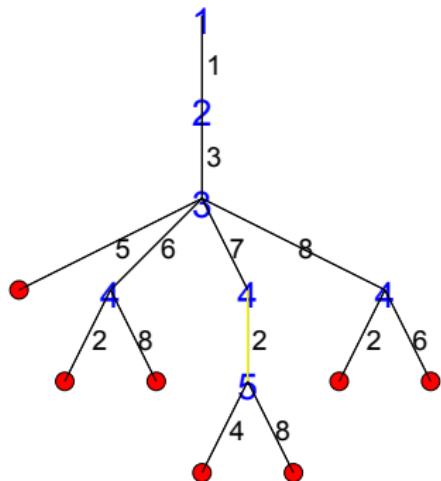
Default Strategy



[◀ Back to Start](#)

[▶ Skip Animation](#)

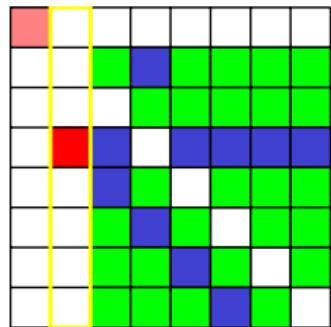
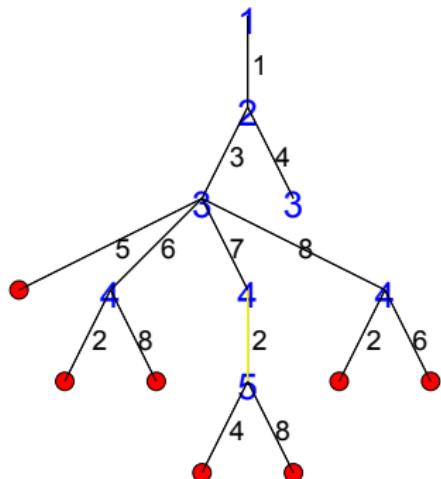
Default Strategy



◀ Back to Start

► Skip Animation

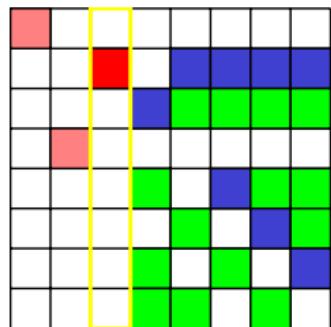
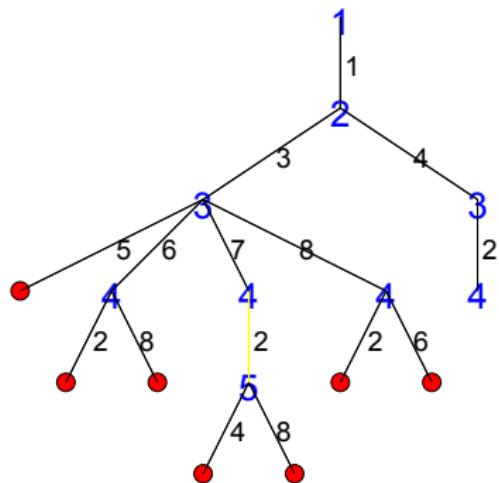
Default Strategy



◀ Back to Start

► Skip Animation

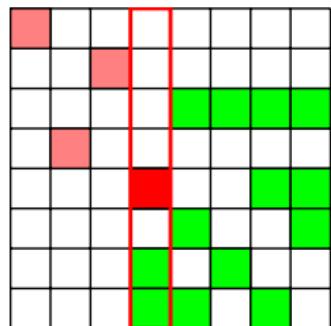
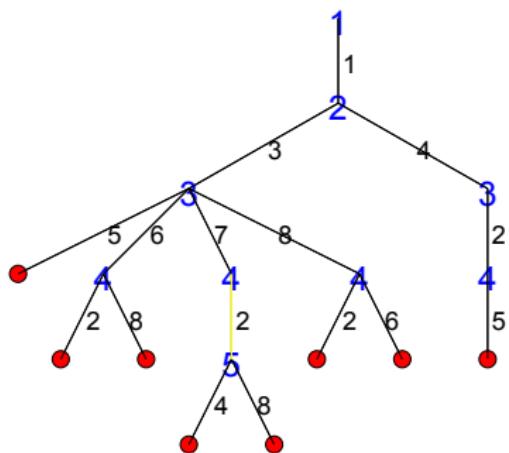
Default Strategy



◀ Back to Start

► Skip Animation

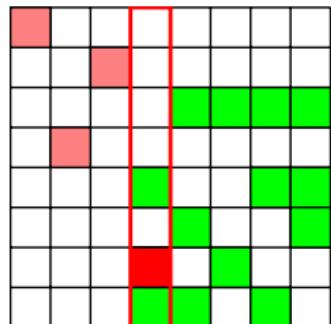
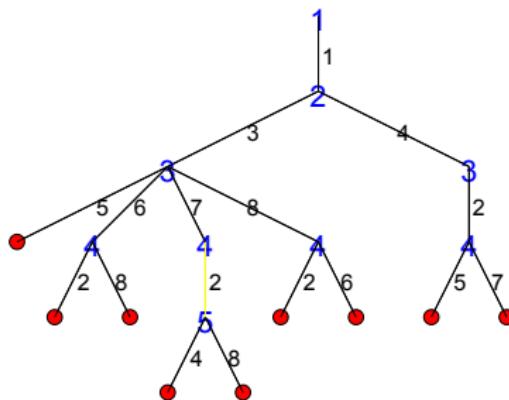
Default Strategy



◀ Back to Start

► Skip Animation

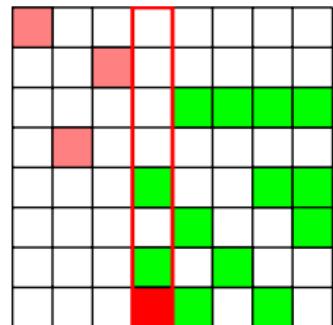
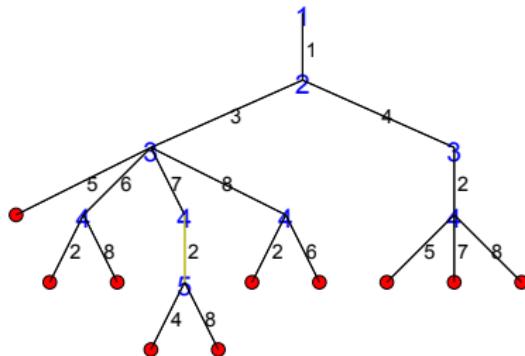
Default Strategy



◀ Back to Start

► Skip Animation

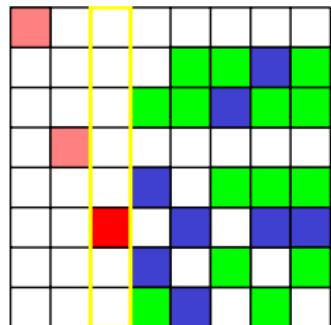
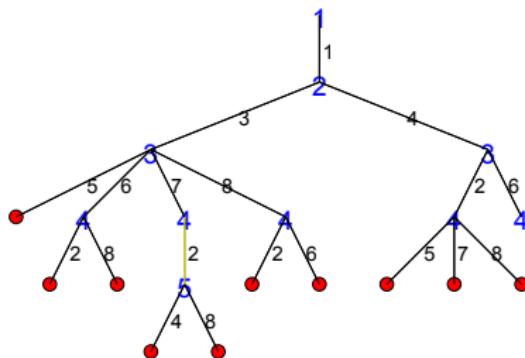
Default Strategy



◀ Back to Start

► Skip Animation

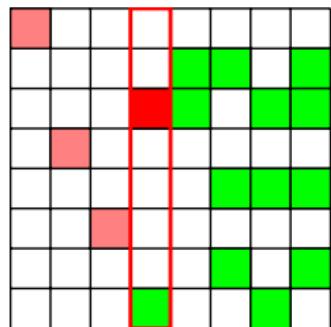
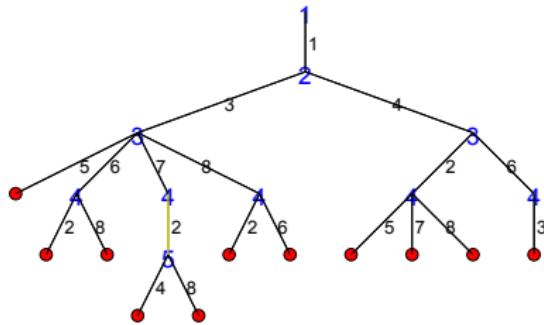
Default Strategy



◀ Back to Start

► Skip Animation

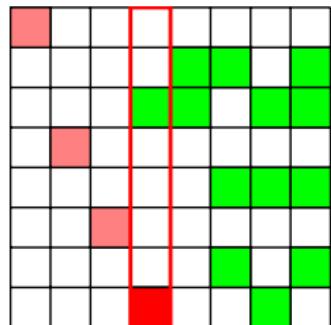
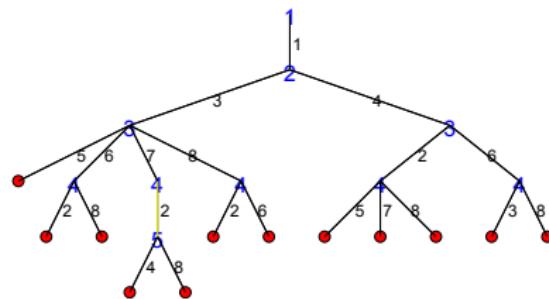
Default Strategy



◀ Back to Start

► Skip Animation

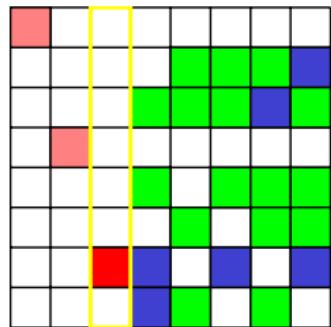
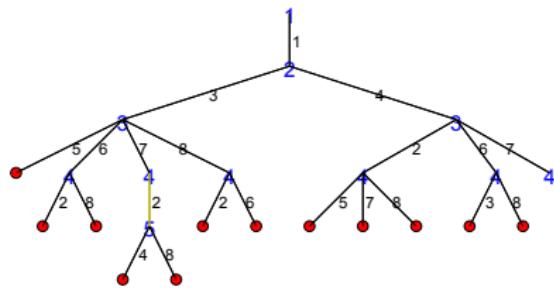
Default Strategy



◀ Back to Start

► Skip Animation

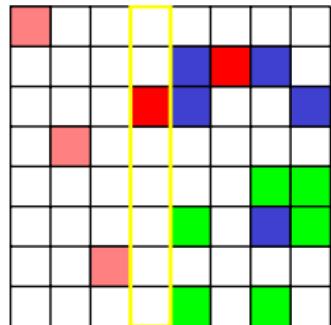
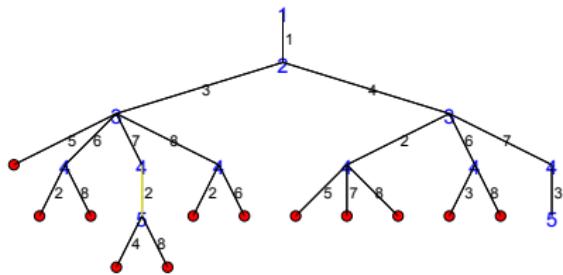
Default Strategy



◀ Back to Start

► Skip Animation

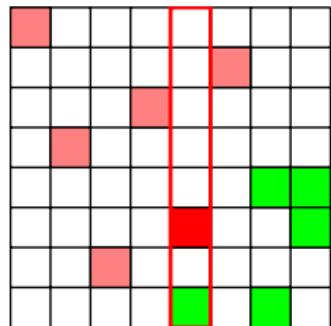
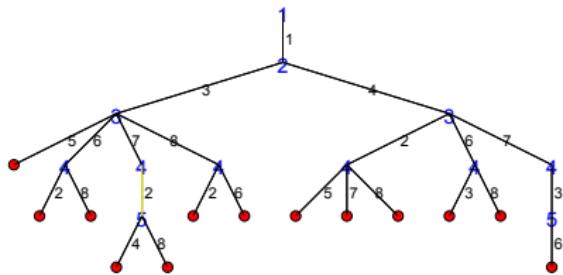
Default Strategy



◀ Back to Start

► Skip Animation

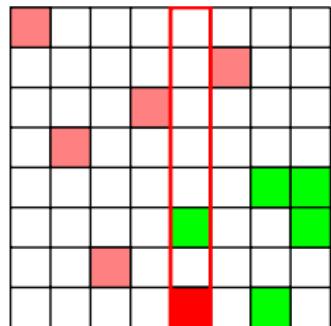
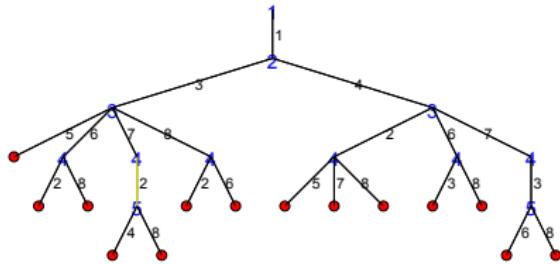
Default Strategy



◀ Back to Start

► Skip Animation

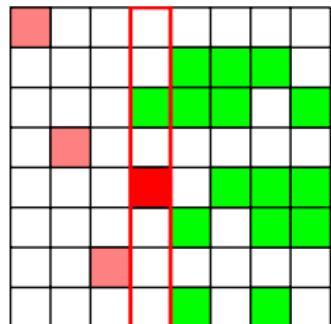
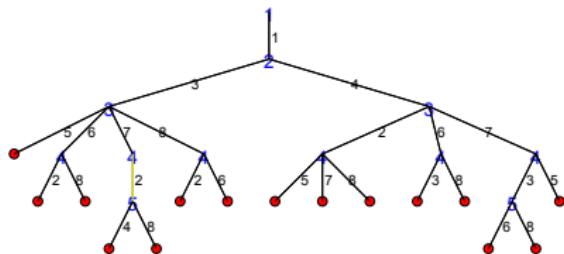
Default Strategy



◀ Back to Start

► Skip Animation

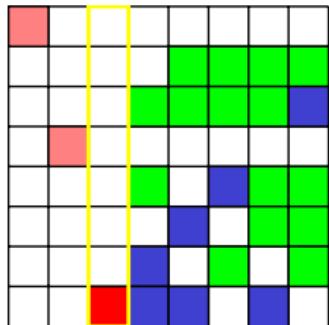
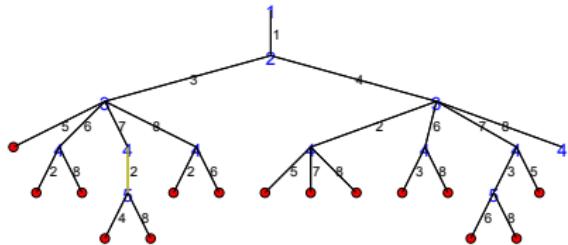
Default Strategy



◀ Back to Start

► Skip Animation

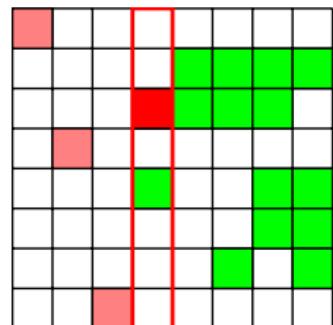
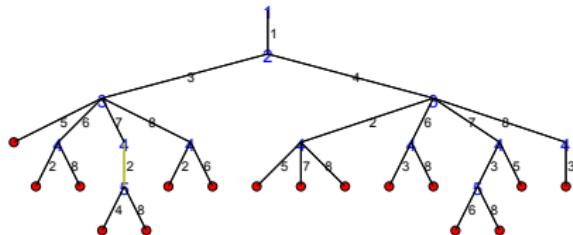
Default Strategy



◀ Back to Start

► Skip Animation

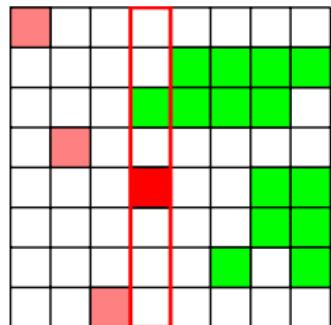
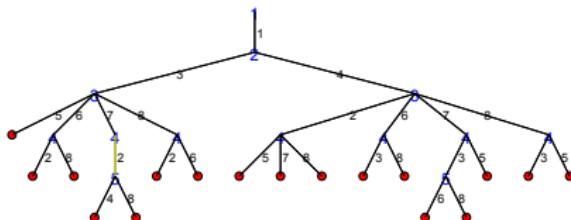
Default Strategy



◀ Back to Start

► Skip Animation

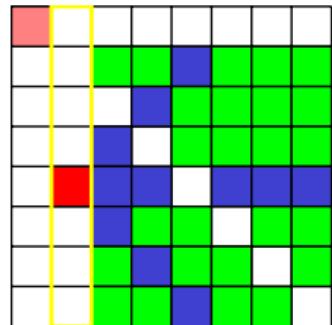
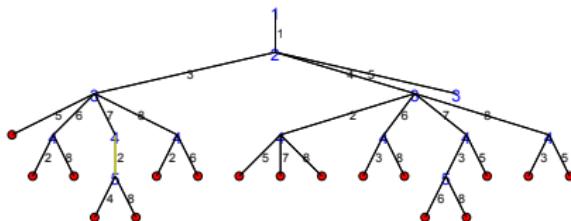
Default Strategy



◀ Back to Start

► Skip Animation

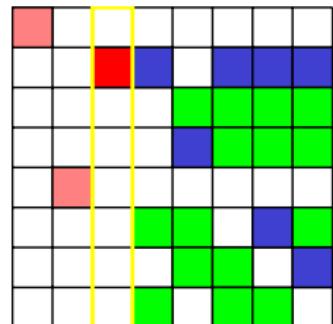
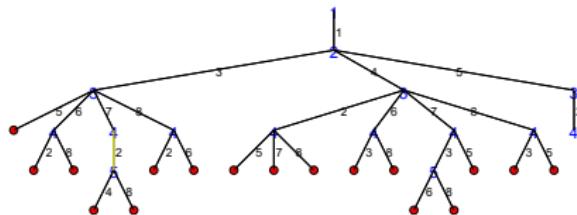
Default Strategy



◀ Back to Start

► Skip Animation

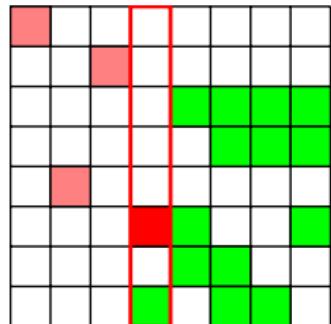
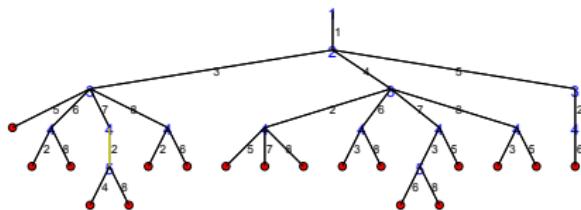
Default Strategy



◀ Back to Start

► Skip Animation

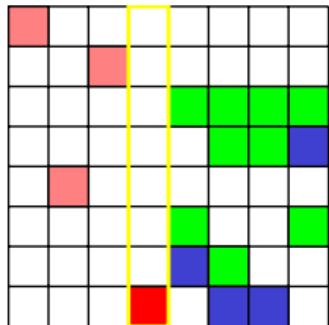
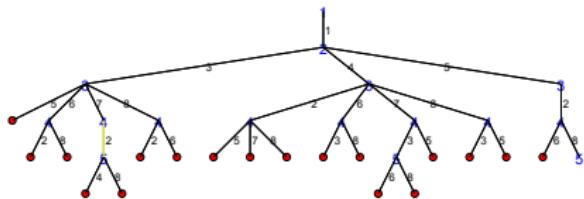
Default Strategy



◀ Back to Start

► Skip Animation

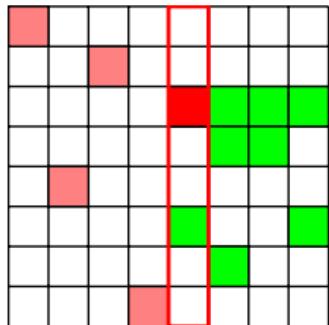
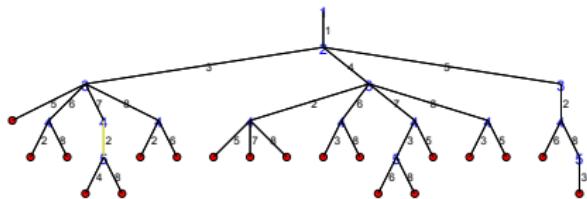
Default Strategy



◀ Back to Start

► Skip Animation

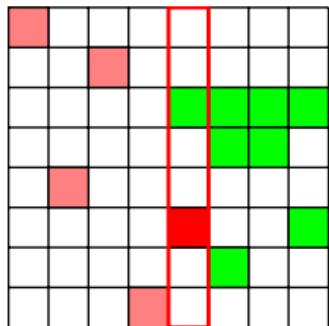
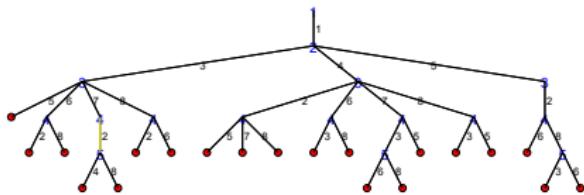
Default Strategy



◀ Back to Start

► Skip Animation

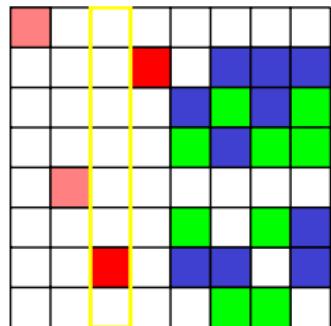
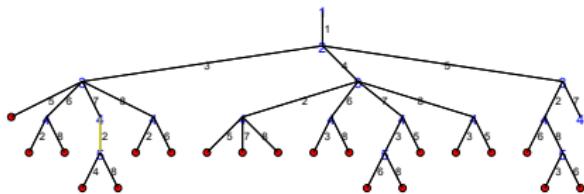
Default Strategy



◀ Back to Start

► Skip Animation

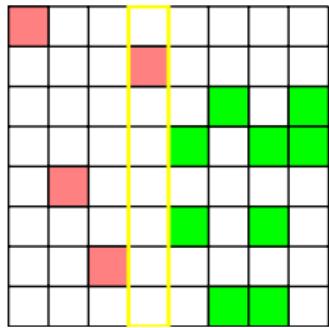
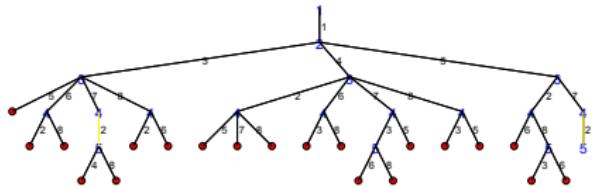
Default Strategy



◀ Back to Start

► Skip Animation

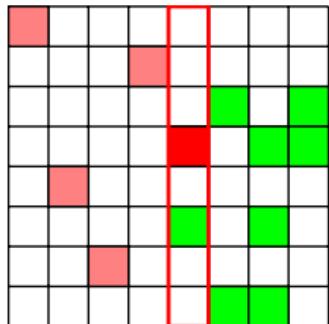
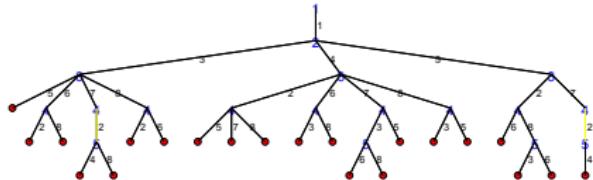
Default Strategy



◀ Back to Start

► Skip Animation

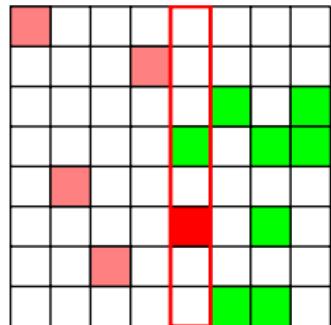
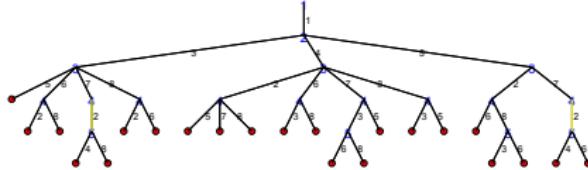
Default Strategy



◀ Back to Start

▶ Skip Animation

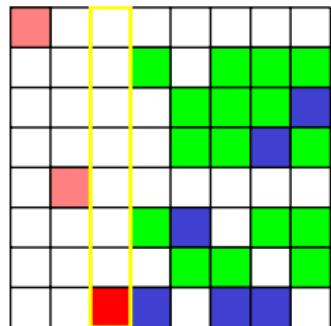
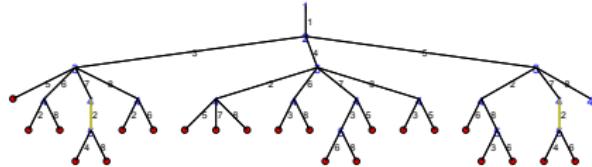
Default Strategy



◀ Back to Start

► Skip Animation

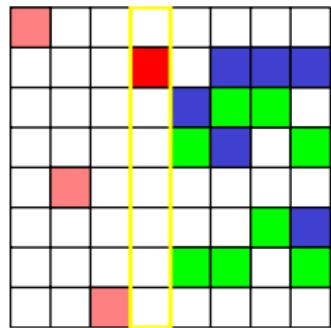
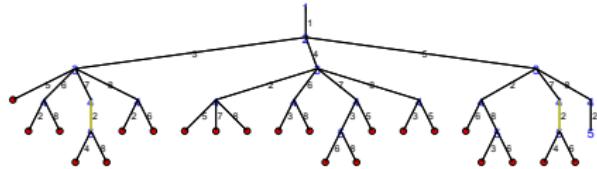
Default Strategy



◀ Back to Start

► Skip Animation

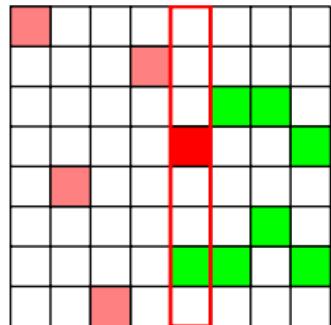
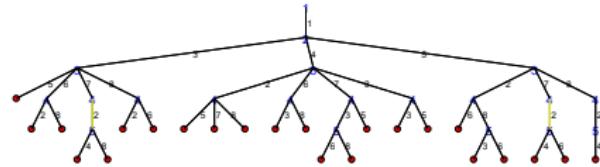
Default Strategy



◀ Back to Start

► Skip Animation

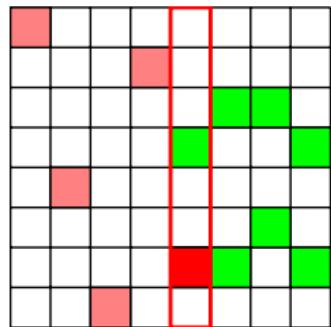
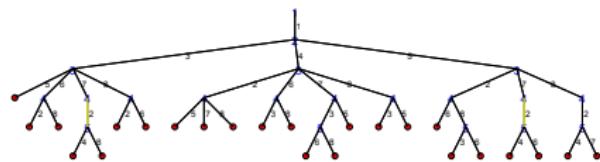
Default Strategy



◀ Back to Start

► Skip Animation

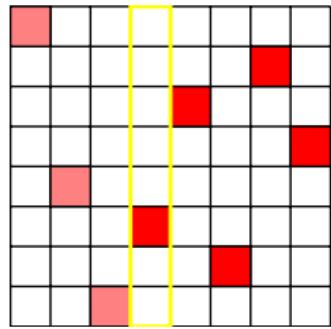
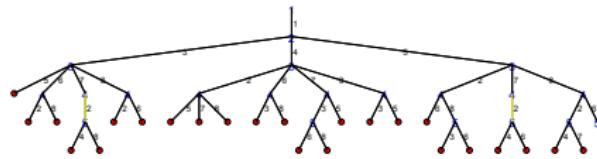
Default Strategy



◀ Back to Start

► Skip Animation

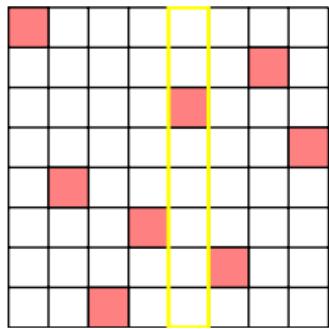
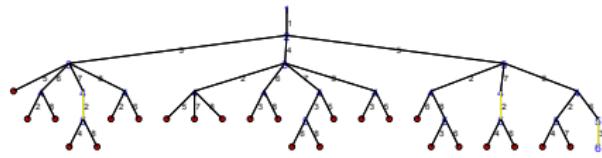
Default Strategy



Back to Start

► Skip Animation

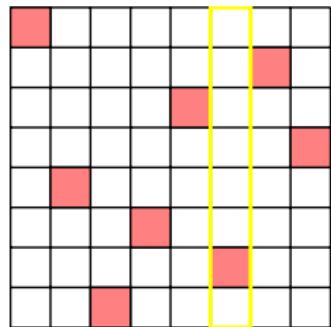
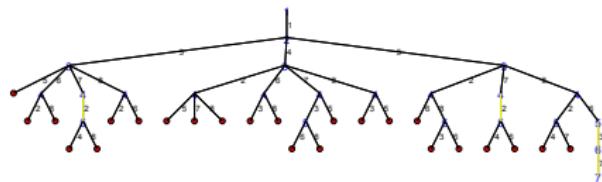
Default Strategy



◀ Back to Start

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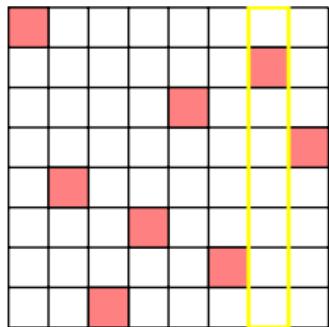
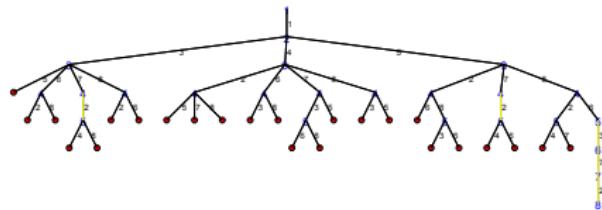
Default Strategy



◀ Back to Start

► Skip Animation

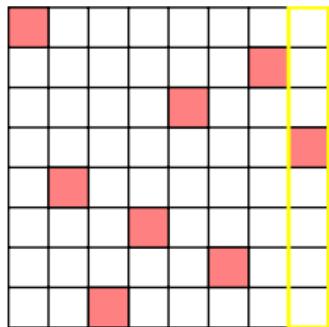
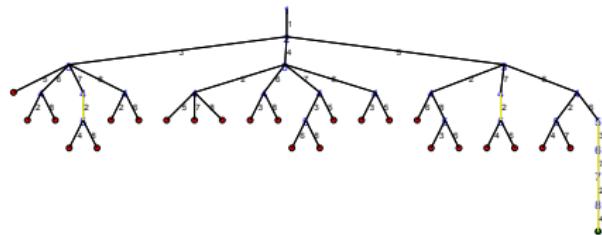
Default Strategy



◀ Back to Start

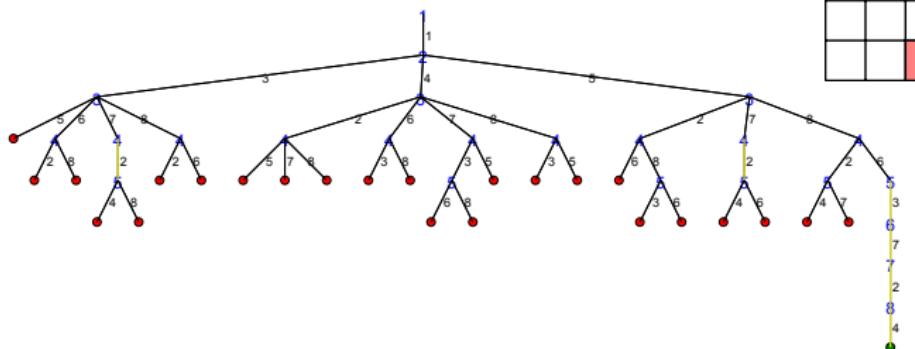
► Skip Animation

Default Strategy



[◀ Back to Start](#)

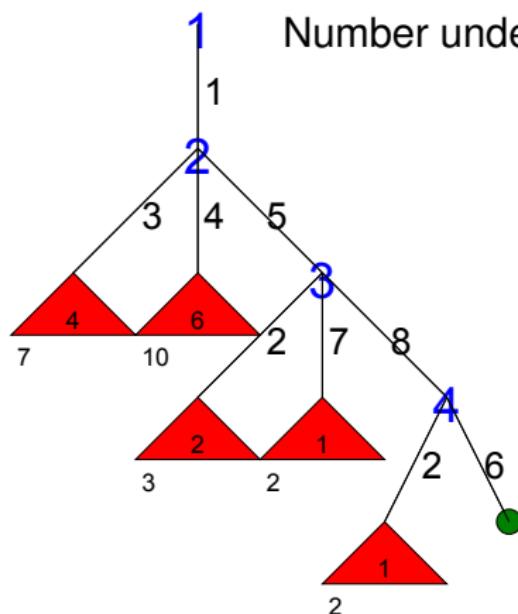
First Solution



Observations

- Even for small problem size, tree can become large
- Not interested in all details
- Ignore all automatically fixed variables
- For more compact representation abstract failed sub-trees

Compact Representation



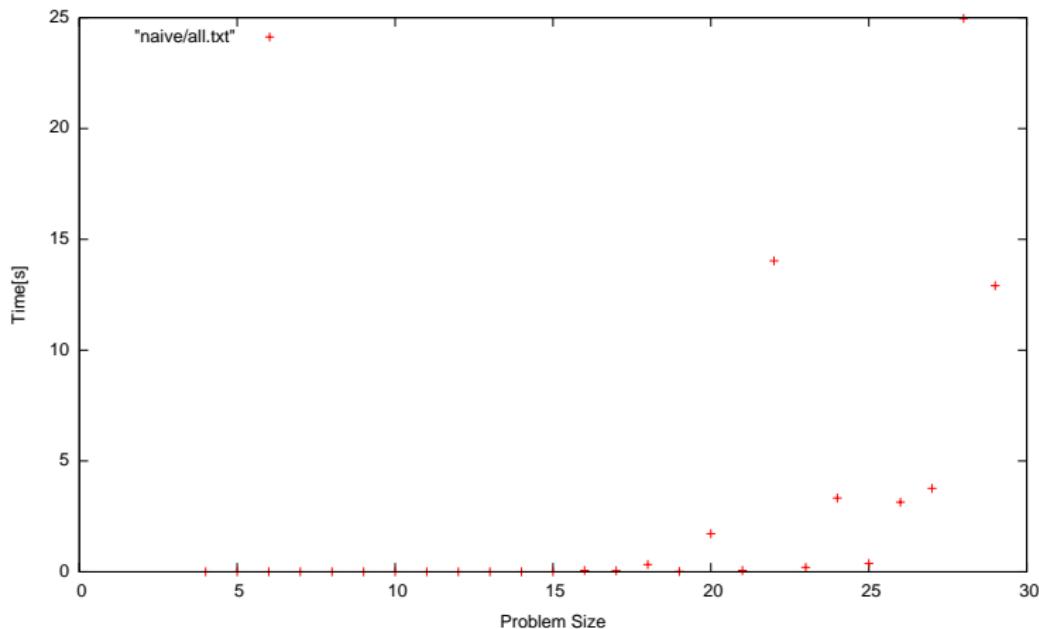
Number inside triangle:
Number under triangle:

Number of choices
Number of failures

Exploring other board sizes

- How stable is the model?
- Try all sizes from 4 to 100
- Timeout of 100 seconds

Naive Strategy, Problem Sizes 4-100



Observations

- Time very reasonable up to size 20
- Sizes 20-30 times very variable
- Not just linked to problem size
- No size greater than 30 solved within timeout

Outline

Problem

Program

Naive Search

Improvements

Dynamic Variable Choice

Weighted Degree

Improved Heuristics

Making Search More Stable

Possible Improvements

- Better constraint reasoning
 - Remodelling problem with 3 all different constraints
 - Global reasoning as described before
- Better control of search
 - Static vs. dynamic variable ordering
 - Better value choice
 - Not using complete depth-first chronological backtracking

Static vs. Dynamic Variable Ordering

- Heuristic Static Ordering
 - Sort variables before search based on heuristic
 - Most important decisions
 - Smallest initial domain
- Dynamic variable ordering
 - Use information from constraint propagation
 - Different orders in different parts of search tree
 - Use all information available

First Fail strategy

- Dynamic variable ordering
- At each step, select variable with smallest domain
- Idea: If there is a solution, better chance of finding it
- Idea: If there is no solution, smaller number of alternatives
- Needs tie-breaking method

Search Strategy Choices

- Minizinc [▶ Show](#)
- Choco-solver [▶ Show](#)

Modified MiniZinc Program

```
int: n=8;
array[1..n] of var 1..n: queens;
constraint
  forall(i, j in 1..n where i < j) (
    queens[i] != queens[j] /\ 
    queens[i] + i != queens[j] + j /\ 
    queens[i] - i != queens[j] - j
  )
;
solve :: int_search(
  queens,
  first_fail,
  indomain_min)
  satisfy;
```

Variable Choice (MiniZinc)

- Determines the order in which variables are assigned
- `input_order` assign variables in static order given
- `smallest` assign variable with smallest value in domain first
- `first_fail` select variable with smallest domain first
- `dom_w_deg` consider ratio of domain size and failure count
- Others, including programmed selection for specific solvers

Value Choice (MiniZinc)

- Determines the order in which values are tested for selected variables
- `indomain_min` Start with smallest value, on backtracking try next larger value
- `indomain_median` Start with value closest to middle of domain
- `indomain_random` Choose values in random order
- `indomain_split` Split domain into two intervals

▶ Continue

Modified Choco-solver Model

```
int n = 8;

Model model = new Model(n + "-queens problem");
IntVar[] vars = model.intVarArray("Q", n, 1, n, false);
IntVar[] diag1 = IntStream.range(0, n).
    mapToInt(i -> vars[i].sub(i).intVar()).
    toArray(IntVar[]::new);
IntVar[] diag2 = IntStream.range(0, n).
    mapToInt(i -> vars[i].add(i).intVar()).
    toArray(IntVar[]::new);

model.post(
    model.allDifferent(vars),
    model.allDifferent(diag1),
    model.allDifferent(diag2)
);

Solver solver = model.getSolver();
solver.showStatistics();
solver.setSearch(Search.domOverWDegSearch(vars));
Solution solution = solver.findSolution();

if (solution != null) {
    System.out.println(solution.toString());
}
```

VariableSelector Choice (Choco-solver)

- Determines the order in which variables are assigned
- InputOrder assign variables in static order given
- Smallest assign variable with smallest value in domain first
- FirstFail select variable with smallest domain first
- DomOverWDeg consider ratio of domain size and failure count
- ActivityBased dynamic, based on dynamic observed behaviour
- ImpactBased dynamic, based on dynamic observed behaviour

IntValueSelector Choice (Choco-solver)

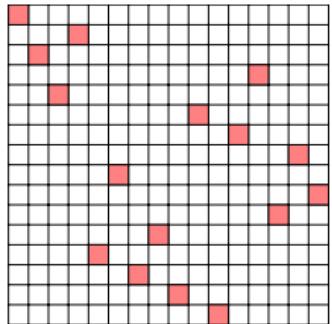
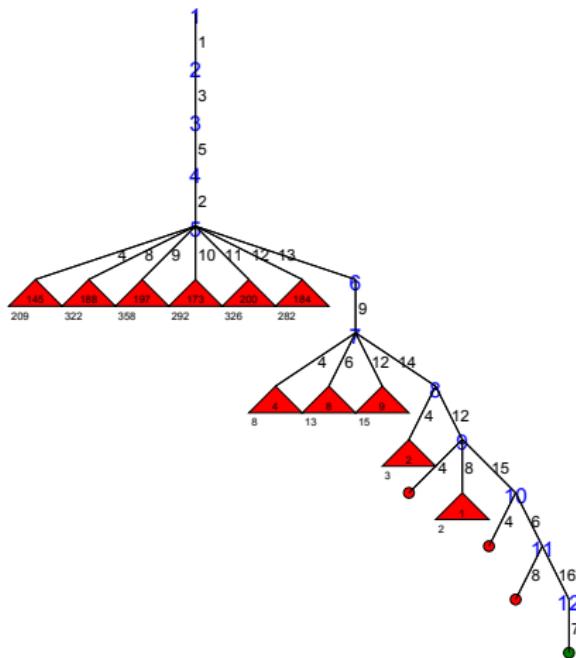
- Determines the order in which values are tested for selected variables
- IntDomainMin Start with smallest value, on backtracking try next larger value
- IntDomainMiddle Start with value closest to middle of domain
- IntDomainRandom Choose values in random order
- IntDomainRandomBound Randomly choose between smallest and largest value

▶ Continue

Comparison

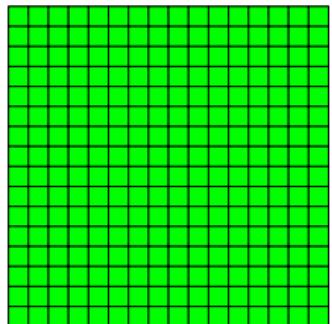
- Board size 16x16
- Naive (Input Order) Strategy
- First Fail variable selection

Naive (Input Order) Strategy (Size 16)



FirstFail Strategy (Size 16)

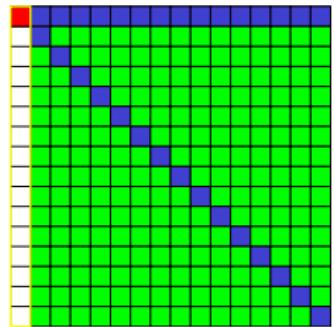
1



▶ Skip Animation

FirstFail Strategy (Size 16)

1
1
2

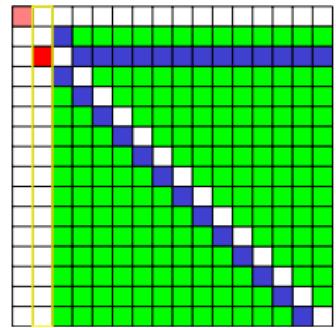


◀ Back to Start

► Skip Animation

FirstFail Strategy (Size 16)

1
1
2
3
3

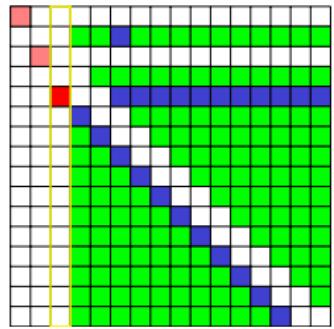


◀ Back to Start

► Skip Animation

FirstFail Strategy (Size 16)

1
1
2
3
3
5
6

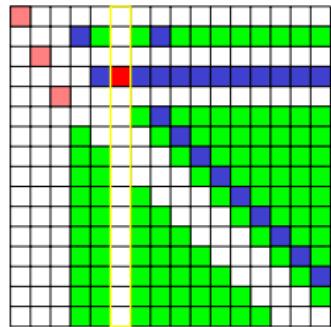


◀ Back to Start

► Skip Animation

FirstFail Strategy (Size 16)

1
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2
3
3
5
6
4
8

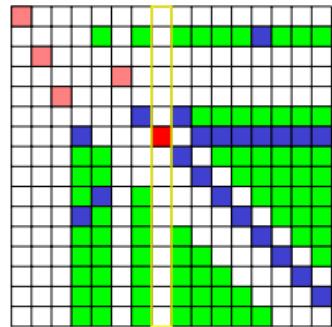


◀ Back to Start

▶ Skip Animation

FirstFail Strategy (Size 16)

1
1
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3
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6
4
8
7
13

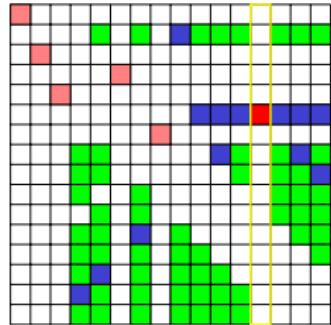


◀ Back to Start

► Skip Animation

FirstFail Strategy (Size 16)

1
1
2
3
3
5
6
4
8
7
13
6
11

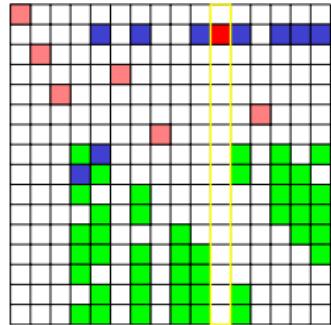


◀ Back to Start

► Skip Animation

FirstFail Strategy (Size 16)

1
1
2
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6
4
8
7
13
6
11
2
10

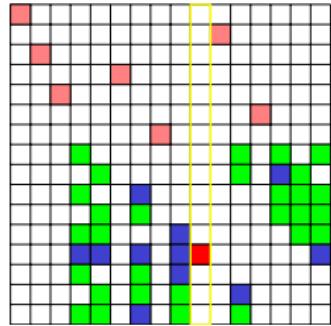


◀ Back to Start

► Skip Animation

FirstFail Strategy (Size 16)

1
1
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3
3
5
6
4
8
7
13
6
11
2
10
13
9

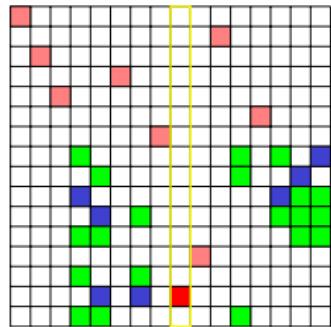


◀ Back to Start

► Skip Animation

FirstFail Strategy (Size 16)

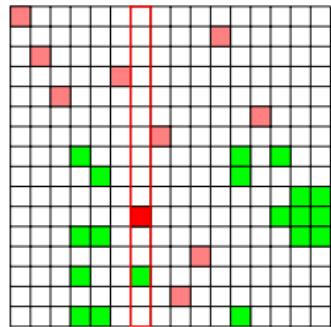
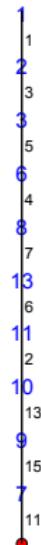
1
2
3
3
5
6
4
8
7
13
13
11
10
13
9
15
7



◀ Back to Start

▶ Skip Animation

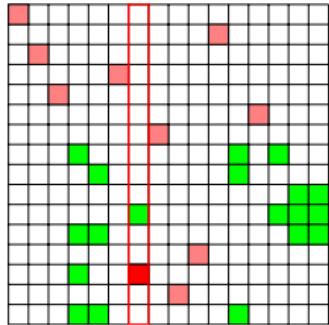
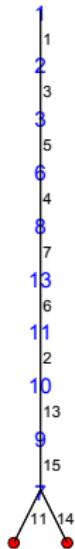
FirstFail Strategy (Size 16)



◀ Back to Start

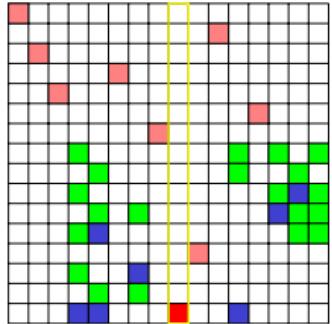
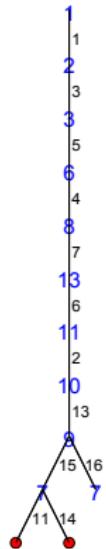
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FirstFail Strategy (Size 16)



► Skip Animation

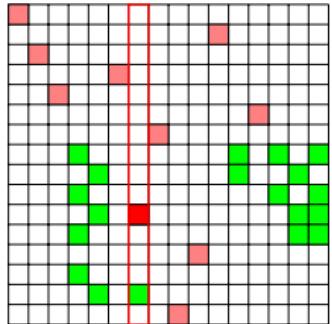
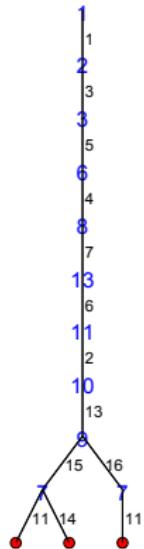
FirstFail Strategy (Size 16)



◀ Back to Start

► Skip Animation

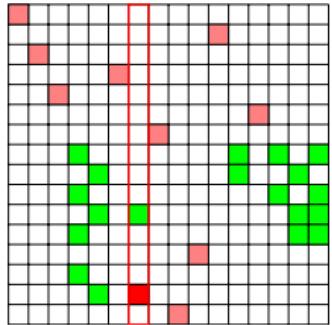
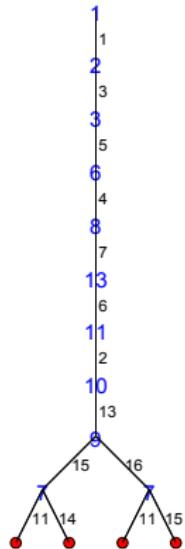
FirstFail Strategy (Size 16)



◀ Back to Start

► Skip Animation

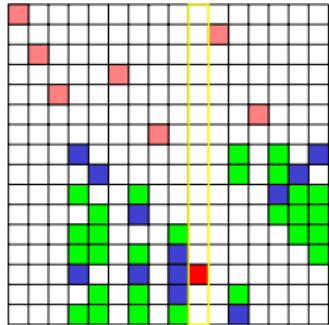
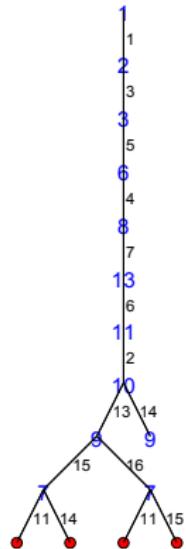
FirstFail Strategy (Size 16)



◀ Back to Start

► Skip Animation

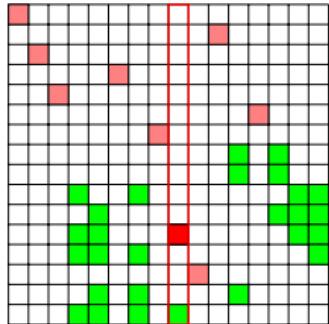
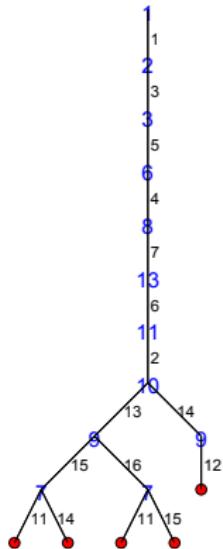
FirstFail Strategy (Size 16)



◀ Back to Start

► Skip Animation

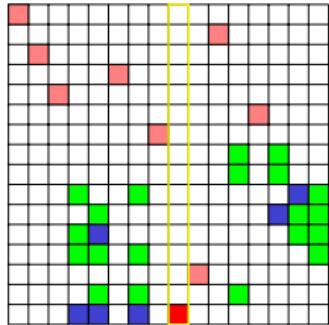
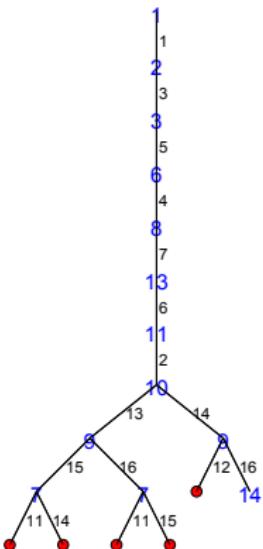
FirstFail Strategy (Size 16)



◀ Back to Start

► Skip Animation

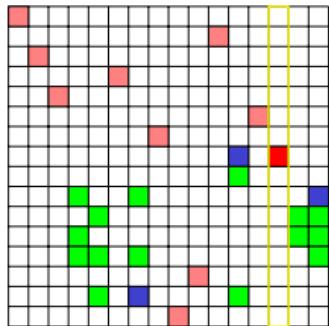
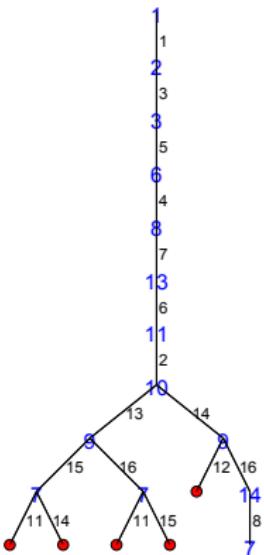
FirstFail Strategy (Size 16)



◀ Back to Start

► Skip Animation

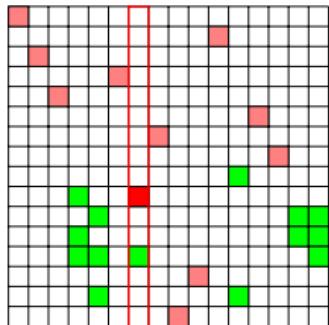
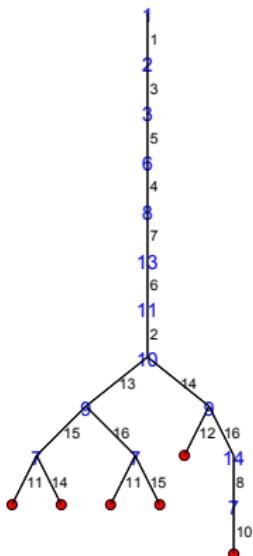
FirstFail Strategy (Size 16)



◀ Back to Start

► Skip Animation

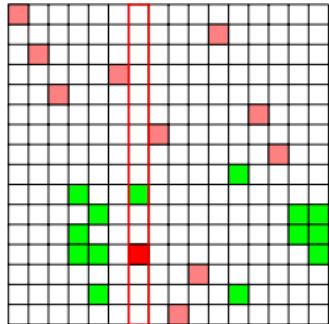
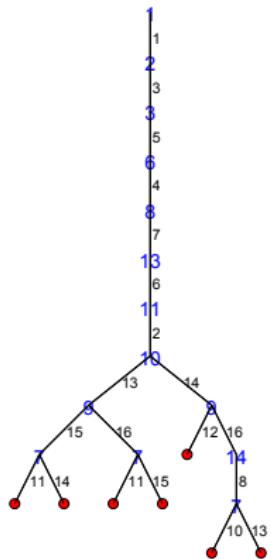
FirstFail Strategy (Size 16)



◀ Back to Start

► Skip Animation

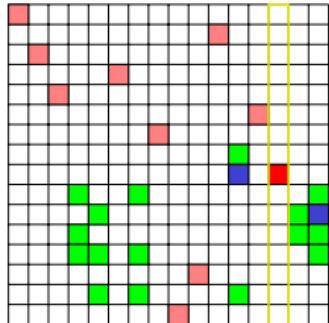
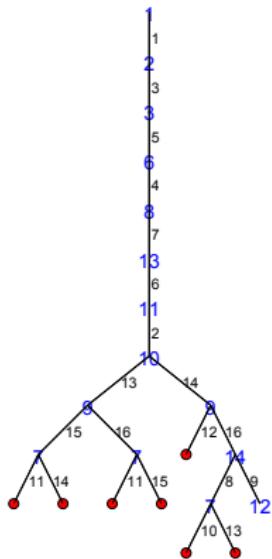
FirstFail Strategy (Size 16)



◀ Back to Start

► Skip Animation

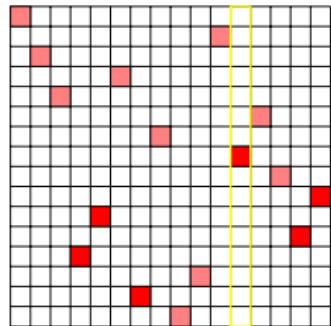
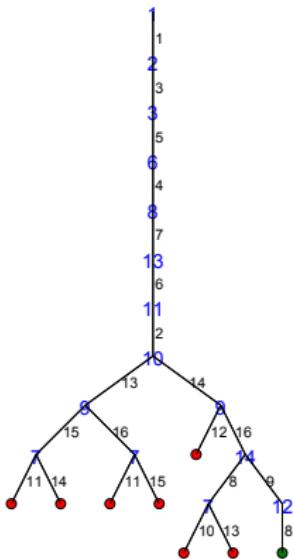
FirstFail Strategy (Size 16)



◀ Back to Start

▶ Skip Animation

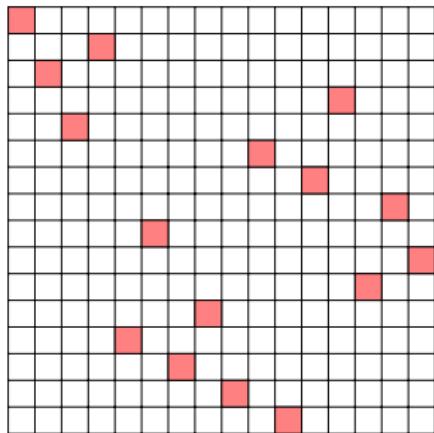
FirstFail Strategy (Size 16)



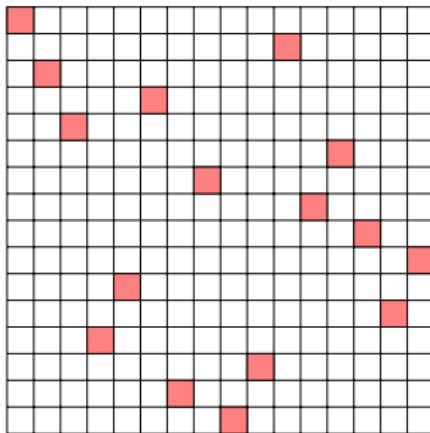
◀ Back to Start

Comparing Solutions

Naive

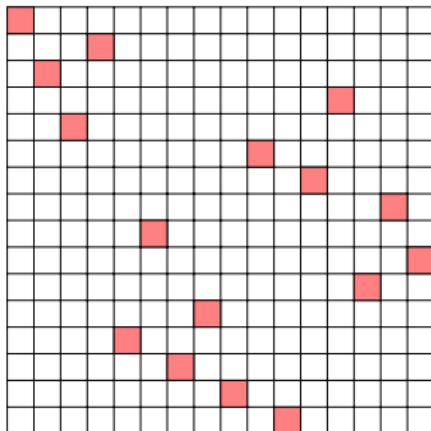


First Fail

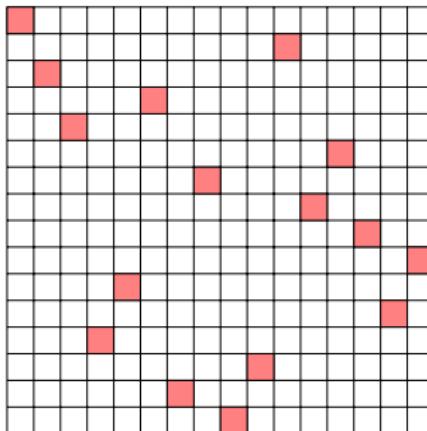


Comparing Solutions

Naive

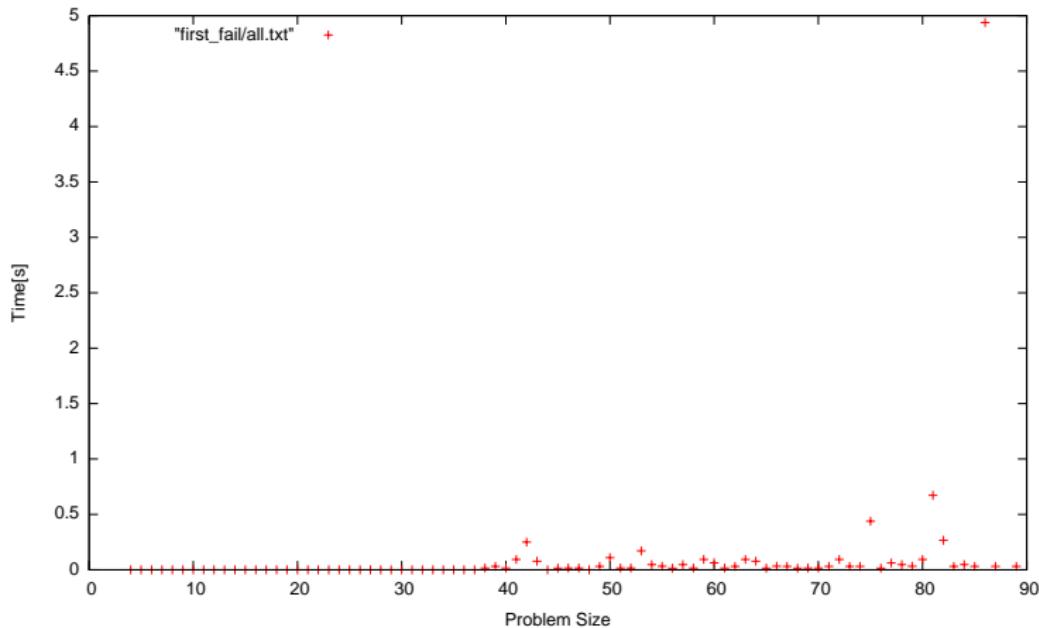


First Fail



Solutions are different!

FirstFail, Problem Sizes 4-100



Observations

- This is much better
- But some sizes are much harder
- Timeout for sizes 88, 91, 93, 97, 98, 99

More Reactive Variable Selection

- Domain size is important, but other information is useful as well
- Dom/Weighted Degree: better results in many situations
- Weight Degree: count how often variable has been involved in failure
- Focus on more complicated part of problem
- Changes during search, learns from past performance
- Option **dom_w_deg**

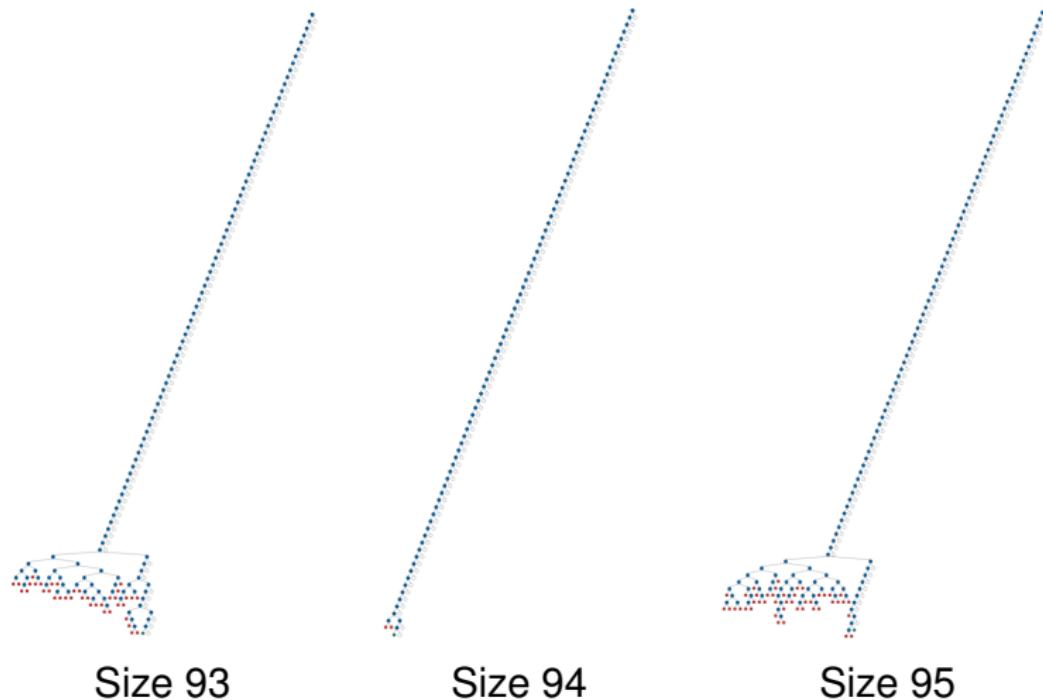
Weighted Degree Variable Selection

```
int: n=8;
array[1..n] of var 1..n: queens;
constraint
    forall(i, j in 1..n where i < j) (
        queens[i] != queens[j] /\ 
        queens[i] + i != queens[j] + j /\ 
        queens[i] - i != queens[j] - j
    )
;
solve :: int_search(
    queens,
    dom_w_deg,
    indomain_random)
    satisfy;
```

Result for size 16 with Gecode-Gist



Sample Results for Larger Sizes



Approach 1: Heuristic Portfolios

- Try multiple strategies for the same problem
- With multi-core CPUs, run them in parallel
- Only one needs to be successful for each problem

Approach 2: Restart with Randomization

- Only spend limited number of backtracks for a search attempt
- When this limit is exceeded, restart at beginning
- Requires randomization to explore new search branch
- Randomize variable choice by random tie break
- Randomize value choice by shuffling values
- Needs strategy when to restart

Random Variable Choice and Restarts

```
int: n=8;
array[1..n] of var 1..n: queens;
constraint
    forall(i, j in 1..n where i < j) (
        queens[i] != queens[j] /\ 
        queens[i] + i != queens[j] + j /\ 
        queens[i] - i != queens[j] - j
    )
;
solve :: int_search(
    queens,
    dom_w_deg,
    indomain_random)
    :: random_linear(100)
    satisfy;
```

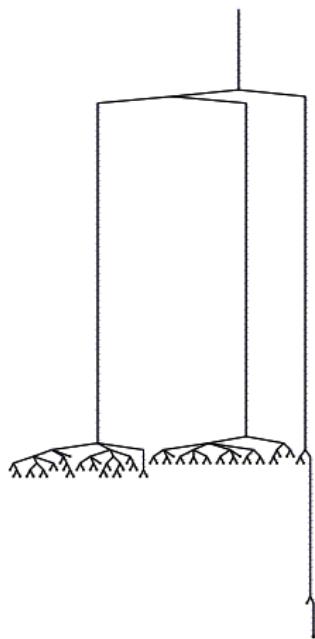
Approach 3: Partial Search

- Abandon depth-first, chronological backtracking
- Don't get locked into a failed sub-tree
- A wrong decision at a level is not detected, and we have to explore the complete subtree below to undo that wrong choice
- Explore more of the search tree
- Spend time in promising parts of tree

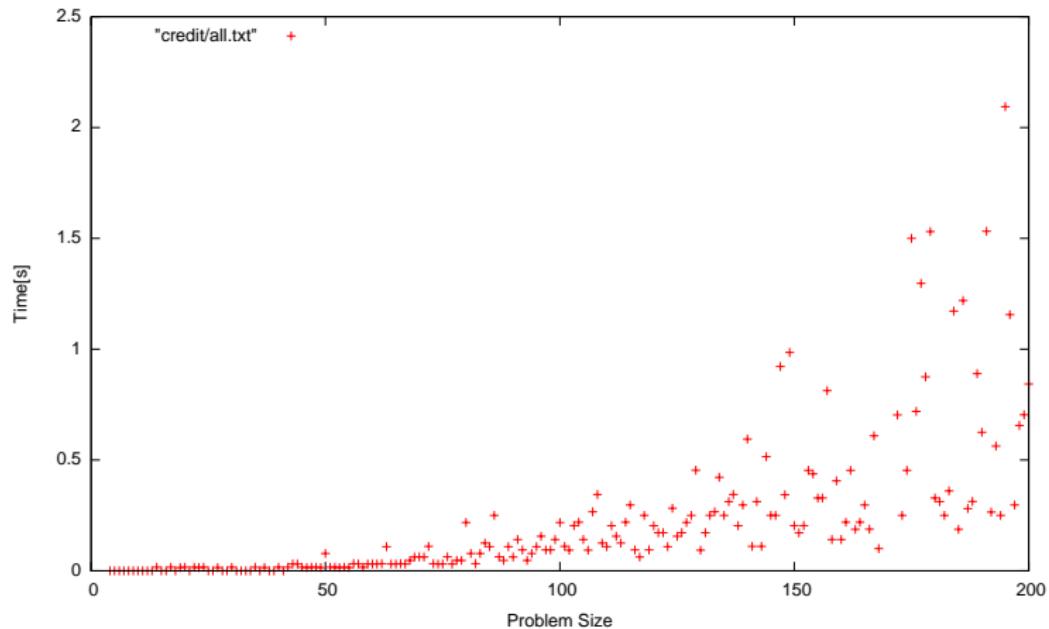
Example: Credit Search

- Not available in all solvers
- Explore top of tree completely, based on credit
- Start with fixed amount of credit
- Each node consumes one credit unit
- Split remaining credit amongst children
- When credit runs out, start bounded backtrack search
- Each branch can use only K backtracks
- If this limit is exceeded, jump to unexplored top of tree

Credit, Search Tree Problem Size 94



Credit, Problem Sizes 4-200



Points to Remember

- Choice of search can have huge impact on performance
- Dynamic variable selection can lead to large reduction of search space
- Packaged search can do a lot, but programming search adds even more
- Depth-first chronological backtracking not always best choice
- How to control this explosion of search alternatives?

Part IV

What is missing?

Many Specialized Topics

- How to design efficient core engine
- Hybrids with LP/MIP tools
- Hybrids with SAT
- Symmetry breaking
- Use of MDD/BDD to encode sets of solutions
- High level modelling tools
- Debugging/visualization

Reformulation

- Just because the user has modelled it this way, it doesn't mean we have to solve it that way
 - Replace some constraint(s) by other, equivalent constraints
 - Because we don't have that constraint in our system
 - For performance

Learning

- While solving the problem we can learn how to strengthen the model/search
 - Understand which constraints/method contribute to propagation and change schedule
 - Learn no-good constraints by explaining failure
 - Adapt search strategy based on search experience

More Learning Resources

- Survey of Methods, Resources, and Formats for Teaching Constraint Programming
 - by Tejas Santanam, Helmut Simonis
 - <https://doi.org/10.48550/arXiv.2403.12717>
 - Based on survey of community for WTCP 2023
 - <https://hsimonis.github.io/WTCP2023/>