Centre for
Data Analytics

Insight

Chapter 4: Basic Constraint Reasoning (SEND+MORE=MONEY)

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CRT-AI CP Week 2024

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Acknowledgments

This publication has emanated from research conducted with the financial support of Science Foundation Ireland under Grant number 12/RC/2289-P2 at Insight the SFI Research Centre for Data Analytics at UCC, which is co-funded under the European Regional Development Fund.
A version of this material was developed as part of the ECLiPSe ELearning course:
`https://eclipseclp.org/ELearning/index.html`.
Support from Cisco Systems and the Silicon Valley Community Foundation is gratefully acknowledged.

Example 1: SEND+MORE=MONEY

- Example of Finite Domain Constraint Problem
- Models and Programs
- Constraint Propagation and Search
- Some Basic Constraints: linear arithmetic, alldifferent, disequality
- A Built-in search
- Visualizers for variables, constraints and search

Problem Definition

A Crypt-Arithmetic Puzzle

We begin with the definition of the SEND+MORE=MONEY puzzle. It is often shown in the form of a hand-written addition:

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline & M & O & N & E & Y \end{array}$$

The puzzle was first proposed by Henry Dudeney in the Strand Magazine from 1924.

Rules

- Each character stands for a digit from 0 to 9.
- Numbers are built from digits in the usual, positional notation.
- Repeated occurrence of the same character denote the same digit.
- Different characters denote different digits.
- Numbers do not start with a zero.
- The equation must hold.

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline & M & O & N & E & Y \end{array}$$

Model

- Each character is a variable, which ranges over the values 0 to 9.
- An *alldifferent* constraint between all variables, which states that two different variables must have different values. This is a very common constraint, which we will encounter in many other problems later on.
- Two *disequality constraints* (variable X must be different from value V) stating that the variables at the beginning of a number can not take the value 0.
- An arithmetic *equality constraint* linking all variables with the proper coefficients and stating that the equation must hold.

SEND+MORE=MONEY Models

- ECLiPSe ▸ Show
- MiniZinc ▸ Show
- NumberJack ▸ Show
- CPMpy ▸ Show
- Choco-solver ▸ Show

ECLiPSe Model

```
:- lib(ic).

sendmore(Digits) :-
    Digits = [S,E,N,D,M,O,R,Y],
    Digits :: [0..9],
    alldifferent(Digits),
    S #\= 0,
    M #\= 0,
                 1000*S + 100*E + 10*N + D
               + 1000*M + 100*O + 10*R + E
    #= 10000*M + 1000*O + 100*N + 10*E + Y,
    labeling(Digits).
```

Continue

MiniZinc Model

```
include "alldifferent.mzn";
var 0..9: S;
var 0..9: E;
var 0..9: N;
var 0..9: D;
var 0..9: M;
var 0..9: O;
var 0..9: R;
var 0..9: Y;
constraint S != 0;
constraint M != 0;
constraint           1000 * S + 100 * E + 10 * N + D
                   + 1000 * M + 100 * O + 10 * R + E
       = 10000 * M + 1000 * O + 100 * N + 10 * E + Y;
constraint alldifferent([S,E,N,D,M,O,R,Y]);

solve satisfy;
```

Continue

NumberJack Model (from https://github.com/eomahony/Numberjack/)

```
from Numberjack import *

def get_model():
    model = Model()
    s, m = VarArray(2, 1, 9)
    e, n, d, o, r, y = VarArray(6, 0, 9)
    model.add(      s*1000 + e*100 + n*10 + d +
                    m*1000 + o*100 + r*10 + e ==
          m*10000 + o*1000 + n*100 + e*10 + y)
    model.add(AllDiff((s, e, n, d, m, o, r, y)))
    return s, e, n, d, m, o, r, y, model

def solve(param):
    s, e, n, d, m, o, r, y, model = get_model()
    solver = model.load(param['solver'])
    solver.setVerbosity(param['verbose'])
    solver.solve()
```

Continue

CPMpy Model (from https://github.com/CPMpy/)

```
from cpmpy import *
import numpy as np

s,e,n,d,m,o,r,y = intvar(0,9, shape=8)
model = Model(
    AllDifferent([s,e,n,d,m,o,r,y]),
    (     sum(   [s,e,n,d] * np.array([       1000, 100, 10, 1]) ) \
       + sum(   [m,o,r,e] * np.array([       1000, 100, 10, 1]) ) \
      == sum( [m,o,n,e,y] * np.array([10000, 1000, 100, 10, 1]) ) ),
    s > 0,
    m > 0,
)

model.solve()
```

Continue

Choco-solver Model (from https://choco-solver.org/)

```
Model model = new Model("SEND+MORE=MONEY");
IntVar S = model.intVar("S", 1, 9, false);
IntVar E = model.intVar("E", 0, 9, false);
IntVar N = model.intVar("N", 0, 9, false);
IntVar D = model.intVar("D", 0, 9, false);
IntVar M = model.intVar("M", 1, 9, false);
IntVar O = model.intVar("O", 0, 9, false);
IntVar R = model.intVar("R", 0, 9, false);
IntVar Y = model.intVar("Y", 0, 9, false);

model.allDifferent(new IntVar[]{S, E, N, D, M, O, R, Y}).post();

IntVar[] ALL = new IntVar[]{
    S, E, N, D,
    M, O, R, E,
    M, O, N, E, Y};
int[] COEFFS = new int[]{
    1000, 100, 10, 1,
    1000, 100, 10, 1,
    -10000, -1000, -100, -10, -1};
model.scalar(ALL, COEFFS, "=", 0).post();

Solver solver = model.getSolver();
solver.showStatistics();
solver.showSolutions();
solver.findSolution();
```

Continue

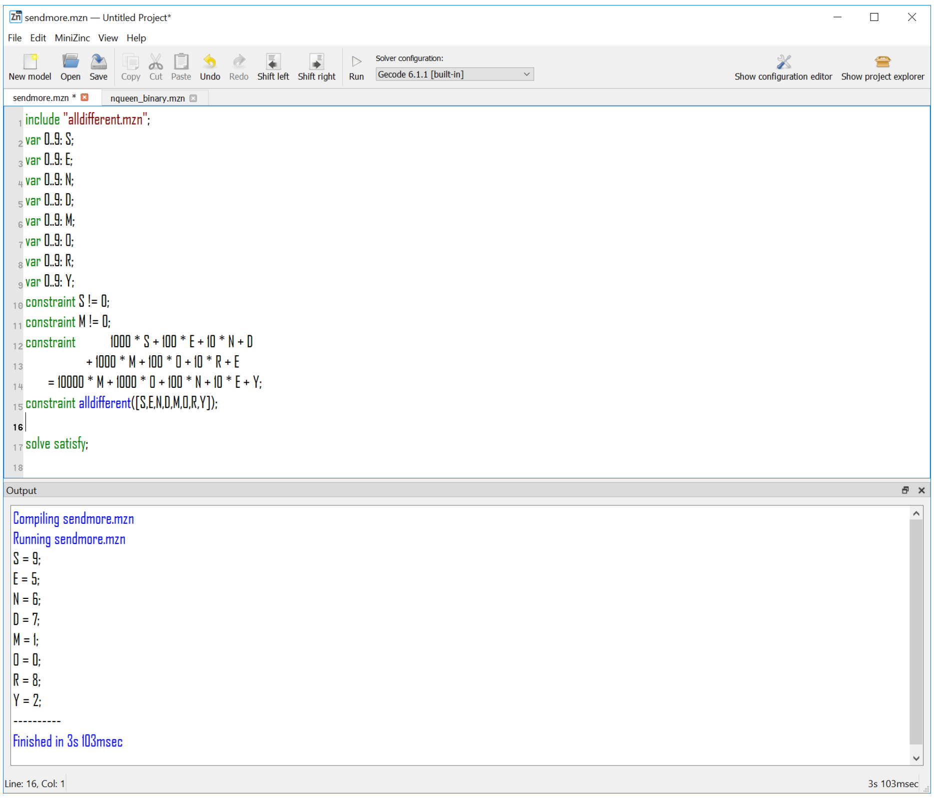
A Note on Syntax

- Some formulations may seem simpler than others
- This largely is an artifact of a very simple problem
- In most models, you do not write down constraints one by one
- You create constraints based on data
- Ease of integration becomes more important than syntax
- Debugging tools for those who need a debugger :-)

Choice of Model

- This is *one* model, not *the* model of the problem
- Many possible alternatives
- Choice often depends on your constraint system
  - Constraints available
  - Reasoning attached to constraints
- Not always clear which is the *best* model
- Often: Not clear what is the *problem*

Running the Program (MiniZinc IDE)



Question

- But how did the program come up with this solution?
- We show solution with ECLiPse, other solvers vary slightly

Domain Definition

```
var 0..9: S;
var 0..9: E;
var 0..9: N;
var 0..9: D;
var 0..9: M;
var 0..9: O;
var 0..9: R;
var 0..9: Y;
```

Domain Visualization

Columns = Values

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Rows = Variables

Cells= State

Alldifferent Constraint

```
include "alldifferent.mzn";

constraint alldifferent([S,E,N,D,M,O,R,Y]);
```

- Built-in alldifferent predicate included
- No initial propagation possible
- *Suspends*, waits until variables are changed
- When variable is fixed, remove value from domain of other variables
- *Forward checking*

Alldifferent Visualization

Uses the same representation as the domain visualizer

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Disequality Constraints

```
constraint S != 0;
constraint M != 0;
```

Remove value from domain

$$S \in \{1..9\}, M \in \{1..9\}$$

Constraints solved, can be removed

Domains after Disequality

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

Equality Constraint

- Normalization of linear terms
  - Single occurence of variable
  - Positive coefficients
- Propagation

Normalization

	1000*S+	100*E+	10*N+	D
	+1000*M+	100*O+	10*R+	E
10000*M+	1000*O+	100*N+	10*E+	Y

is transformed into

	1000*S+	91*E+		D
		+	10*R	
9000*M+	900*O+	90*N+		Y

Simplified Equation

$$1000 * S + 91 * E + 10 * R + D = 9000 * M + 900 * O + 90 * N + Y$$

Propagation

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{1000..9918} =$$

$$\underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..89919}$$

Deduction:

$$M = 1, S = 9, O \in \{0..1\}$$

Why? ▸ Skip

Consider lower bound for S

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Lower bound of equation is 9000
- Rest of lhs (left hand side) $(91 * E^{0..9} + 10 * R^{0..9} + D^{0..9})$ is atmost 918
- S must be greater or equal to $\frac{9000-918}{1000} = 8.082$
  - otherwise lower bound of equation not reached by lhs
- S is integer, therefore $S \geq \lceil \frac{9000-918}{1000} \rceil = 9$
- S has upper bound of 9, so $S = 9$

Consider upper bound of M

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}$ is at least 0
- M must be smaller or equal to $\frac{9918-0}{9000} = 1.102$
- M must be integer, therefore $M \leq \lfloor \frac{9918-0}{9000} \rfloor = 1$
- M has lower bound of 1, so $M = 1$

Consider upper bound of O

$$\underbrace{1000 * S^{1..9} + 91 * E^{0..9} + 10 * R^{0..9} + D^{0..9}}_{9000..9918} = \underbrace{9000 * M^{1..9} + 900 * O^{0..9} + 90 * N^{0..9} + Y^{0..9}}_{9000..9918}$$

- Upper bound of equation is 9918
- Rest of rhs (right hand side) $9000 * 1 + 90 * N^{0..9} + Y^{0..9}$ is at least 9000
- O must be smaller or equal to $\frac{9918-9000}{900} = 1.02$
- O must be integer, therefore $O \leq \lfloor \frac{9918-9000}{900} \rfloor = 1$
- O has lower bound of 0, so $O \in \{0..1\}$

Propagation of equality: Result

	0	1	2	3	4	5	6	7	8	9
S		-	-	-	-	-	-	-	-	✹
E										
N										
D										
M		✹	-	-	-	-	-	-	-	-
O			✖	✖	✖	✖	✖	✖	✖	✖
R										
Y										

Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

$$O = 0, [E, R, D, N, Y] \in \{2..8\}$$

Waking the equality constraint

- Triggered by assignment of variables
- *or* update of lower or upper bound

Removal of constants

$$\begin{aligned}1000 * 9 + 91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} &= \\ 9000 * 1 + 900 * 0 + 90 * N^{2..8} + Y^{2..8}\end{aligned}$$

$$\begin{aligned}\mathbf{1000} * \mathbf{9} + 91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} &= \\ \mathbf{9000} * \mathbf{1} + \mathbf{900} * \mathbf{0} + 90 * N^{2..8} + Y^{2..8}\end{aligned}$$

$$91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = 90 * N^{2..8} + Y^{2..8}$$

Propagation of equality (Iteration 1)

$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8}}_{204..816} = \underbrace{90 * N^{2..8} + Y^{2..8}}_{182..728}$$

$$\underbrace{91 * E^{2..8} + 10 * R^{2..8} + D^{2..8} = 90 * N^{2..8} + Y^{2..8}}_{204..728}$$

$$N \geq 3 = \lceil \frac{204 - 8}{90} \rceil, E \leq 7 = \lfloor \frac{728 - 22}{91} \rfloor$$

Propagation of equality (Iteration 2)

$$91 * E^{2..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}$$

$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8}}_{204..725} = \underbrace{90 * N^{3..8} + Y^{2..8}}_{272..728}$$

$$\underbrace{91 * E^{2..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}}_{272..725}$$

$$E \geq 3 = \lceil \frac{272 - 88}{91} \rceil$$

Propagation of equality (Iteration 3)

$$91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}$$

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = \underbrace{90 * N^{3..8} + Y^{2..8}}_{272..728}$$

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{3..8} + Y^{2..8}}_{295..725}$$

$$N \geq 4 = \lceil \frac{295 - 8}{90} \rceil$$

Propagation of equality (Iteration 4)

$$91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}$$

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8}}_{295..725} = \underbrace{90 * N^{4..8} + Y^{2..8}}_{362..728}$$

$$\underbrace{91 * E^{3..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}}_{362..725}$$

$$E \geq 4 = \lceil \frac{362 - 88}{91} \rceil$$

Propagation of equality (Iteration 5)

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}$$

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = \underbrace{90 * N^{4..8} + Y^{2..8}}_{362..728}$$

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{4..8} + Y^{2..8}}_{386..725}$$

$$N \geq 5 = \lceil \frac{386 - 8}{90} \rceil$$

Propagation of equality (Iteration 6)

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$

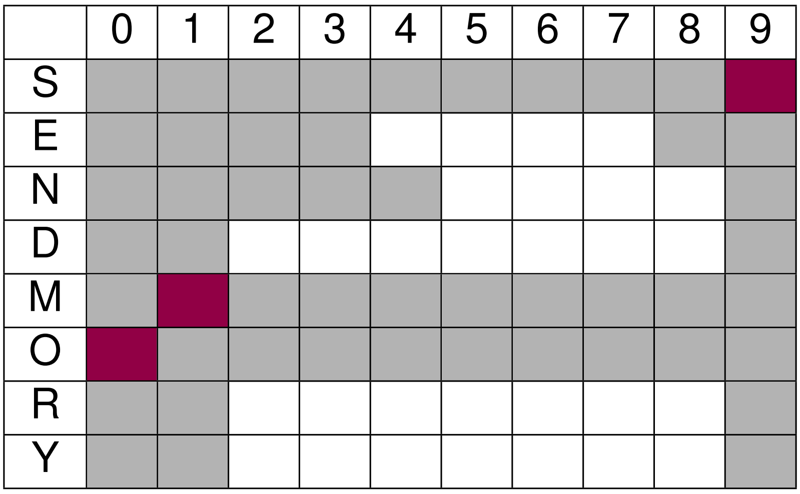
$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{386..725} = \underbrace{90 * N^{5..8} + Y^{2..8}}_{452..728}$$

$$\underbrace{91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}}_{452..725}$$

$$N \geq 5 = \lceil \frac{452 - 8}{90} \rceil, E \geq 4 = \lceil \frac{452 - 88}{91} \rceil$$

No further propagation at this point

Domains after setup



Search

solve satisfy;

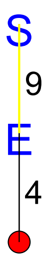
- Try to find a feasible solution, choice left to solver
- Naive search strategy shown here
  - Try variable in order given
  - Try values starting from smallest value in domain
  - When failing, backtrack to last open choice
  - *Chronological Backtracking*
  - *Depth First search*

Search Tree Step 1

Variable S already fixed

Step 2, Alternative $E = 4$

Variable $E \in \{4..7\}$, first value tested is 4



Assignment $E = 4$

	0	1	2	3	4	5	6	7	8	9
S										
E					✸	-	-	-		
N										
D										
M										
O										
R										
Y										

Propagation of $E = 4$, equality constraint

$$91 * 4 + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8}}_{386..452} = \underbrace{90 * N^{5..8} + Y^{2..8}}_{452..728}$$

$$\underbrace{91 * 4 + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}}_{452}$$

$$N = 5, Y = 2, R = 8, D = 8$$

Result of equality propagation

	0	1	2	3	4	5	6	7	8	9
S										
E										
N						✹	-	-	-	
D			-	-	-	-	-	-	✹	
M										
O										
R			-	-	-	-	-	-	✹	
Y			✹	-	-	-	-	-	-	

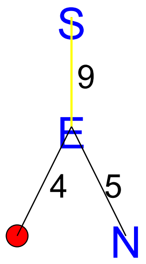
Propagation of alldifferent

	0	1	2	3	4	5	6	7	8	9
S										
E										
N						✹	-	-		
D			-	-	-	-	-	-	✹	
M										
O										
R			-	-	-	-	-	-	✹	
Y			✹	-	-	-	-	-		

Alldifferent fails!

Step 2, Alternative $E = 5$

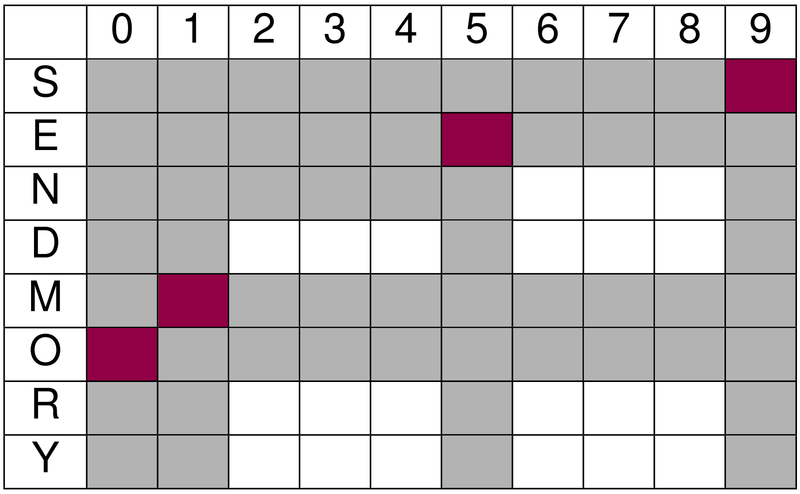
Return to last open choice, E, and test next value



Assignment $E = 5$

	0	1	2	3	4	5	6	7	8	9
S										
E					-	✹	-	-		
N										
D										
M										
O										
R										
Y										

Propagation of alldifferent



$$N \neq 5, N \geq 6$$

Propagation of equality

$$91 * 5 + 10 * R^{2..8} + D^{2..8} = 90 * N^{6..8} + Y^{2..8}$$

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8}}_{477..543} = \underbrace{90 * N^{6..8} + Y^{2..8}}_{542..728}$$

$$\underbrace{91 * 5 + 10 * R^{2..8} + D^{2..8} = 90 * N^{6..8} + Y^{2..8}}_{542..543}$$

$$N = 6, Y \in \{2, 3\}, R = 8, D \in \{7..8\}$$

Result of equality propagation

	0	1	2	3	4	5	6	7	8	9
S										
E										
N							✹	-	-	
D			✖	✖	✖		✖			
M										
O										
R			-	-	-		-	-	✹	
Y					✖		✖	✖	✖	

Propagation of `alldifferent`

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y										

$D = 7$

Propagation of equality

$$91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}$$

$$\underbrace{91 * 5 + 10 * 8 + 7}_{542} = \underbrace{90 * 6 + Y^{2..3}}_{542..543}$$

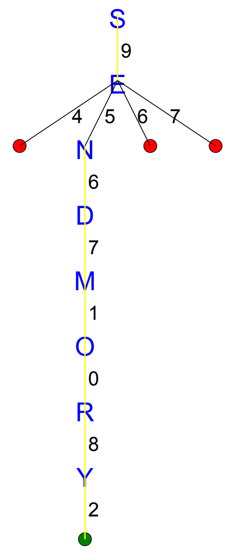
$$\underbrace{91 * 5 + 10 * 8 + 7 = 90 * 6 + Y^{2..3}}_{542}$$

$$Y = 2$$

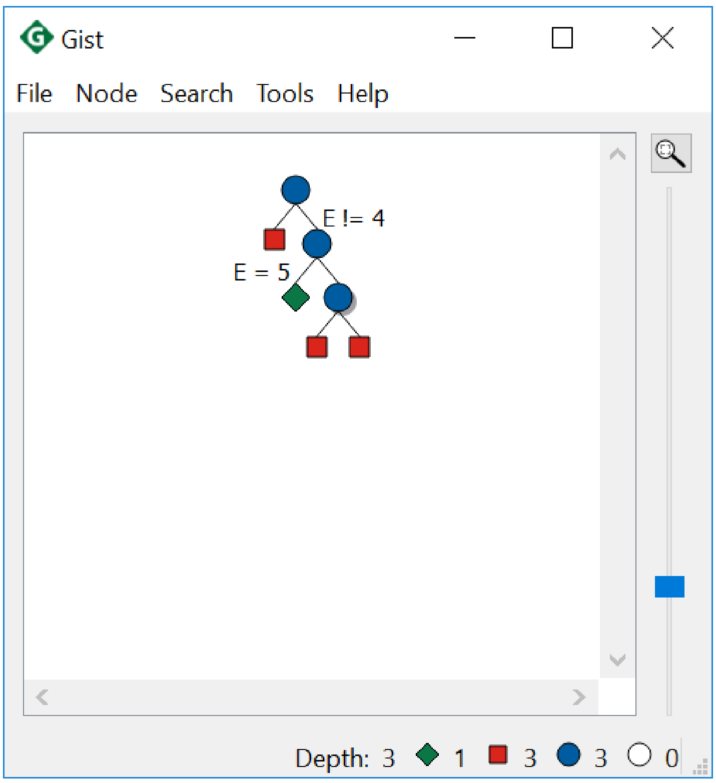
Last propagation step

	0	1	2	3	4	5	6	7	8	9
S										
E										
N										
D										
M										
O										
R										
Y			✹	-						

Complete Search Tree



Search Tree with Gecode/GIST



Some Differences

- Uses binary branching
  - var equal value, var not equal value
- Solutions in green, failure leafs in red, internal nodes in blue
- By default, shows all failed sub trees collapsed
- By default, uses different search strategy

Solution

$$
\begin{array}{ccccc}
 & 9 & 5 & 6 & 7 \\
+ & 1 & 0 & 8 & 5 \\
\hline
1 & 0 & 6 & 5 & 2
\end{array}
$$

Points to Remember

- Constraint models are expressed by variables and constraints.
- Problems can have many different models, which can behave quite differently. Choosing the best model is an art.
- Constraints can take many different forms.
- Propagation deals with the interaction of variables and constraints.
- It removes some values that are inconsistent with a constraint from the domain of a variable.
- Constraints only communicate via shared variables.

Points to Remember

- Propagation usually is not sufficient, search may be required to find a solution.
- Propagation is data driven, and can be quite complex even for small examples.
- The default search uses chronological depth-first backtracking, systematically exploring the complete search space.
- The search choices and propagation are interleaved, after every choice some more propagation may further reduce the problem.

For Puzzle Purists Only

- We did not follow the puzzler ethos!
- We should solve the puzzle without making choices
- Even a case analysis should be avoided
- The puzzle has a single solution, we should be able to deduce the solution
- In the process shown, we are limited by the underlying assumptions
    - Treat each constraint on its own, they interact only by domains of variables
    - We only use the constraints that we stated in the model
- Can we do better than that?

Skip

Better Reasoning

- Three possible approaches (possibly many more)
    - Full domain reasoning for arithmetic, not just bound reasoning
    - Interaction of *sum* and *alldifferent* constraints
    - Deduced implied constraint

Looking at more than just bounds

- We only considered the smallest and largest values that can be achieved in the sum constraint
- We can do more
  - Can any of the values between be expressed as the sum of the terms
  - Consider holes in the domains, and in the range of possible values for LHS and RHS
- Usually not done in actual solvers for arithmetic constraints
- Easy to do with Dynamic Programming

Consider the interaction of multiple constraints

- Usually ignored, as only interaction is via domains of shared variables
- Here: Sum and *alldifferent* interact
  - When considering the bounds, we cannot assume that each variable takes its smallest/largest value independently
  - Find feasible assignment that minimizes/maximizes the total weight
  - To do this properly, we need some non-trivial reasoning
- Do we do this combined reasoning automatically, or only when prompted by the modeler?

Deduced Implied Constraints

- Look at the partially solved puzzle

```
 9END
+10RE
  ---
```

- `10NEY`
- In the hundreds position, we have $E + 0 + C_{10} = N + 10 * C_{100}$, with C_{10} the 0/1 carry from the tens position
- NB: No carry C_{100} into the thousands, $C_{100} = 0$
- N must be equal to $E + 1$ with $C_{10} = 1$
- If $C_{10} = 0$, then $N = E$, not possible
- We can substitute $N = E + 1$ into our main equation, but keep $N = E + 1$ as well

Expert Mode Reasoning

Starting with

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * N^{5..8} + Y^{2..8}$$

we get

$$91 * E^{4..7} + 10 * R^{2..8} + D^{2..8} = 90 * E^{4..7} + 90 + Y^{2..8}$$

Eliminating duplicate occurrences of E

$$\underbrace{E^{4..7} + 10 * R^{2..8} + D^{2..8}}_{26..95} = \underbrace{90 + Y^{2..8}}_{92..98}$$

shared range 92..95 To reach 92, R must be equal to 8, therefore N, E, D, Y must be less than 8
As $N = E + 1$, E must be less than 7

$$E^{4..6} + 10 * 8 + D^{2..7} = 90 + Y^{2..7}$$

Simplification yields

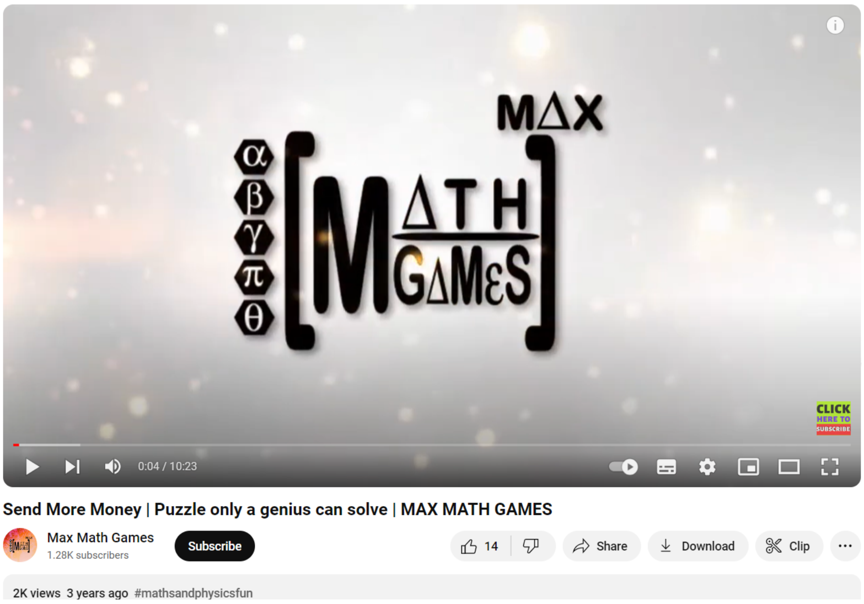
$$\underbrace{E^{4..6} + D^{2..7}}_{6..13} = \underbrace{10 + Y^{2..7}}_{12..17}$$

shared range 12..13 To reach 12 on LHS, E must be greater

Expert Mode Summary

- Often there is more propagation that can be done
- Can be difficult/expensive to do
- Balancing
  - How much work it done at each step of search?
  - How many steps of search you need?
- For hard problems, doing all possible propagation may be exponential
- Not aware that any CP system does the full reasoning shown here

This is how people solve the puzzle by hand



- When writing the first version of this puzzle for CHIP (in 1986), we wanted to mimic the way we solve the puzzle by hand