A three-dimensional continuum model for coupled size segregation and flow in bidisperse granular mixtures

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Scan Poster

Granular Matter

Summary

Dense, bidisperse granular mixtures tend to segregate based on size during flow. The two main driving forces of size segregation are

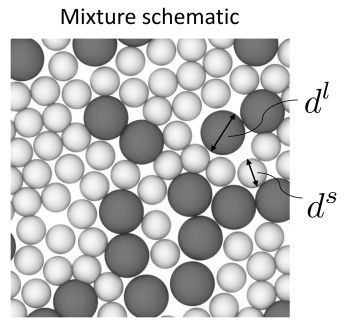
- **Pressure gradients (typically due to gravity) :** Savage and Lun, *JFM* (1988); Gray et al., *PRSA* (2005)
- **Shear-strain-rate gradients :** Fan and Hill, *PRL* (2008); *PRE* (2010)

In this work, we propose three-dimensional constitutive equations for the diffusion and segregation fluxes based on discrete element method (DEM) simulations. When the segregation constitutive equations are coupled with the nonlocal granular fluidity (NGF) model (a nonlocal continuum model for dense granular flow), the coupled model can predict the segregation dynamics and flow fields across several complex flow configurations, including annular shear flow with gravity and split bottom flow.

Continuum theory

Dense, bidisperse granular mixtures (3D spheres)

Mixture schematic



	Large grains	Small grains	Mixture
Solid volume fraction	ϕ^l	ϕ^s	$\phi = \phi^l + \phi^s$
Concentration	$c^l = \phi^l/\phi$	$c^s = \phi^s/\phi$	$c^l + c^s = 1$
Grain size	d^l	d^s	$\bar{d} = c^l d^l + c^s d^s$
Velocity	v_i^l	v_i^s	$v_i = c^l v_i^l + c^s v_i^s$
Relative volume flux	$w_i^l = c^l(v_i^l - v_i)$	$w_i^s = c^s(v_i^s - v_i)$	$w_i^l + w_i^s = 0_i$

Constant solid fraction assumption

Mixture theory approach:
Gray et al., *PRSA* (2005), *JFM* (2006)
Fan and Hill, *NJoP* (2011)
Umbanhowar et al., *ARCBE* (2019)

Granular rheology

Local inertial rheology: MiDi, *EPJE* (2004); da Cruz et al., *PRE* (2005)

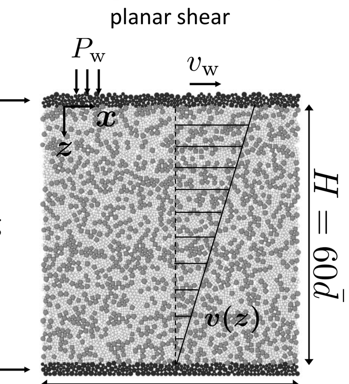
Stress ratio $\mu = \frac{\tau}{P},\ \tau = (\sigma'_{ij}\sigma'_{ij}/2)$

Inertial number $I = \dot{\gamma}\bar{d}\sqrt{\frac{\rho_s}{P}},\ \dot{\gamma} = (2D_{ij}D_{ij})^{1/2}$

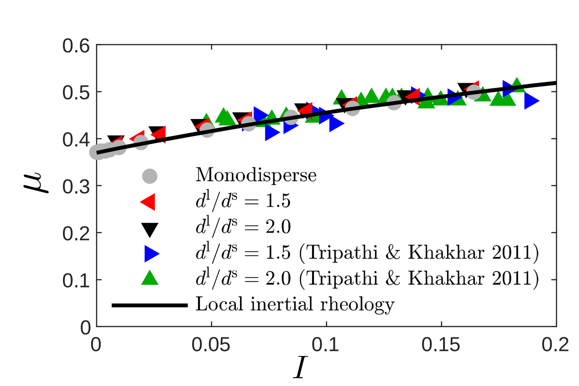
Local inertial rheology $\mu_{loc}(I) = \mu_s + \frac{\mu_2 - \mu_s}{I_0/I + 1}$

$\rho_s \longrightarrow$ grain material density

DEM configuration for bidisperse planar shear



Local inertial rheology fit to DEM data



3 local material parameters:
$\{\mu_s = 0.37,\ \mu_2 = 0.95,\ I_0 = 0.58\}$

Nonlocal granular fluidity (NGF) model: Kamrin and Koval, *PRL* (2012); Henann and Kamrin, *PNAS* (2013), *IJP* (2014)

- ***Granular fluidity:*** The model utilizes a positive, scalar, kinematic (Zhang and Kamrin, *PRL* (2017)) field quantity, g.
- ***Flow rule:*** Constitutively, the granular fluidity relates the stress ratio μ to the consequent shear strain rate $\dot{\gamma}$: $\dot{\gamma} = g\mu$.
- ***Nonlocal rheology:*** The granular fluidity is governed by a nonlocal relation with the steady-state form:

$$\boxed{g = g_{loc}(\mu, P) + \xi(\mu)^2 \frac{\partial^2 g}{\partial x_i \partial x_i}}$$

Local fluidity $g_{loc}(\mu, P) = \begin{cases} I_0\sqrt{\frac{P}{\rho_s \bar{d}^2}}\frac{(\mu - \mu_s)}{\mu(\mu_2 - \mu)} & \text{if } \mu > \mu_s \\ 0 & \text{if } \mu \le \mu_s \end{cases}$

Cooperativity length $\xi(\mu) = A\sqrt{\frac{\mu_2 - \mu}{(\mu_2 - \mu_s)|\mu - \mu_s|}}\bar{d}$

Nonlocal amplitude for frictional spheres: $\{A\} = 0.43$

Generalized segregation model

Conservation of mass $\dot{c}^l + \frac{\partial w_i^l}{\partial x_i} = 0$

Decomposition of the volume flux $w_i^l = w_i^{diff} + w_i^{seg}$

Diffusion flux $w_i^{diff} = -C_{diff}\bar{d}^2\dot{\gamma}\frac{\partial c^l}{\partial x_i}$

Segregation flux $w_i^{seg} = \boxed{C_S\bar{d}^2 c^l(1-c^l)\frac{\partial \dot{\gamma}}{\partial x_i}}$ (Strain rate gradient driven)

Pressure gradient driven $\longrightarrow \boxed{-C_P\frac{\bar{d}^2\dot{\gamma}}{P}c^l(1-c^l)(1-\alpha+\alpha c^l)\frac{\partial P}{\partial x_i}}$

Segregation dynamics equation (evolution equation for c^l)

$$\dot{c}^l + \frac{\partial}{\partial x_i}\left(-C_{diff}\bar{d}^2\dot{\gamma}\frac{\partial c^l}{\partial x_i} - C_P\frac{\bar{d}^2\dot{\gamma}}{P}c^l(1-c^l)(1-\alpha+\alpha c^l)\frac{\partial P}{\partial x_i} + C_S\bar{d}^2 c^l(1-c^l)\frac{\partial\dot{\gamma}}{\partial x_i}\right) = 0$$

Parameters: $\boxed{\{\alpha = 0.4,\ C_S = 0.08\}}$

Model parameters

Generalized diffusion coefficient is given as: $\hat{C}_{diff} = \hat{C}_{diff}^{anti} + 2k^c\left(\hat{C}_{diff}^{in} - \hat{C}_{diff}^{anti}\right)$

where $k^c = n_i^c N_{ij} N_{jk} n_k^c,\quad n_i^c = \frac{\partial c^l/\partial x_i}{|\partial c^l/\partial \mathbf{x}|},\quad N_{ij} = \frac{D_{ij}}{|\mathbf{D}|}$

and $\hat{C}_{diff}^{anti} = \left(\frac{C_0 + C_\infty}{2}\right) + \left(\frac{C_\infty - C_0}{2}\right)\tanh\left(\frac{\log(I) - B_1}{B_2}\right)$

Complete list of parameters to characterize diffusion:

$\boxed{\{C_{diff}^{in} = 0.045, C_0 = 0.06, C_\infty = 0.1, B_1 = -4.8, B_2 = 0.63\}}$

Similarly, the pressure gradient driven segregation coefficient is defined as:

$\hat{C}_P = \hat{C}_P^{anti} + 2k^P\left(C_P^{in} - \hat{C}_P^{anti}\right)$

where $k^P = n_i^P N_{ij} N_{jk} n_k^P,\quad n_i^P = \frac{\partial P/\partial x_i}{|\partial P/\partial \mathbf{x}|}$

The anti-plane segregation parameter is given as

$\hat{C}_P^{anti} = CJ^n\left(1 + \frac{C_\infty - C_0}{C_\infty + C_0}\right)\tanh\left(\frac{\log(I) - B_1}{B_2}\right)$ where $J = \frac{P}{\bar{d}|dP/d\mathbf{x}|}$

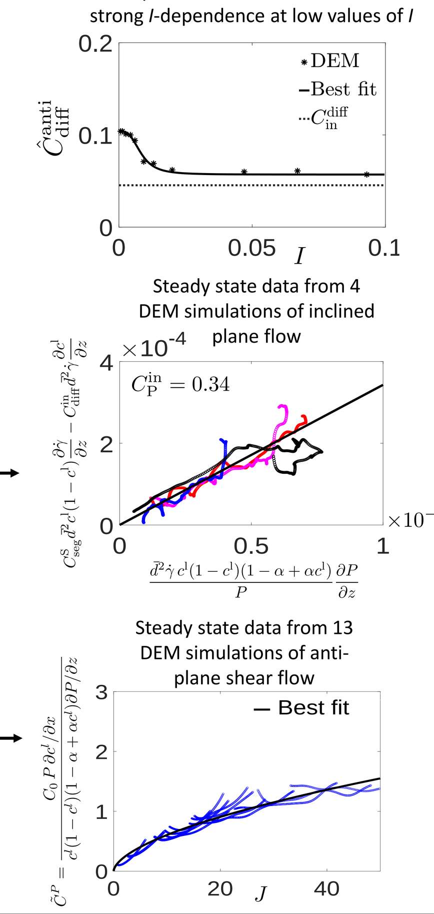
Complete list of parameters to characterize pressure gradient driven segregation:

$\boxed{\{C_P^{in} = 0.34,\ C = 0.16,\ n = 0.58\}}$

Anti-plane mode of diffusion shows strong I-dependence at low values of I

In-plane flux balance

Anti-plane flux balance



Model validation

Inclined plane flow

- Strain-rate-gradient and pressure gradient driven fluxes act against one another.
- Flow is dominated by the local rheology.

Strain rate tensor reduces to

$$[\mathbf{D}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & \partial v_x/\partial z \\ 0 & 0 & 0 \\ \partial v_x/\partial z & 0 & 0 \end{bmatrix}$$

and unit vectors

$[\mathbf{n}^c] = [\mathbf{n}^P] = [0\ \ 0\ \ 1]^\top$

Scalar invariants become

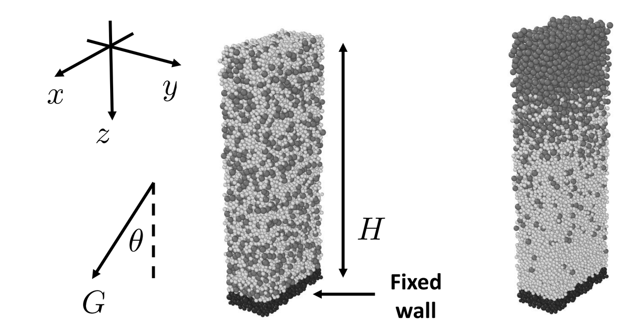
$k^c = n_i^c N_{ij} N_{jk} n_k^c = \frac{1}{2}$

$k^P = n_i^P N_{ij} N_{jk} n_k^P = \frac{1}{2}$

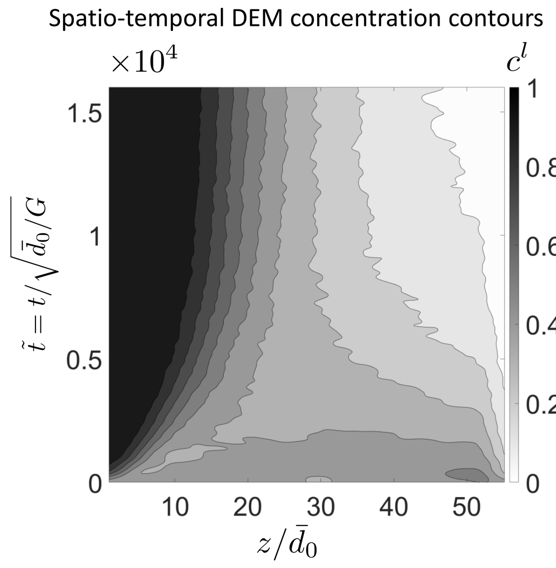
Therefore

$\hat{C}_P = \hat{C}_P^{in}$

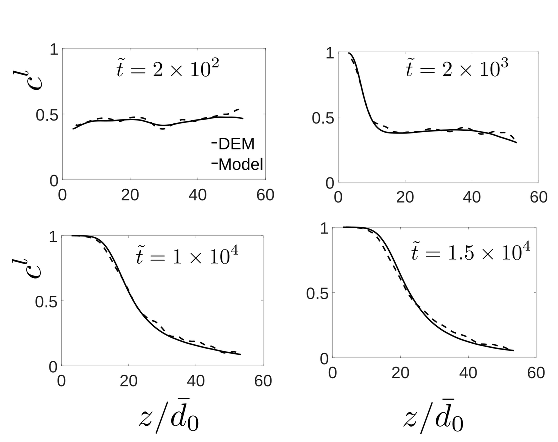
$\hat{C}_{diff} = \hat{C}_{diff}^{in}$



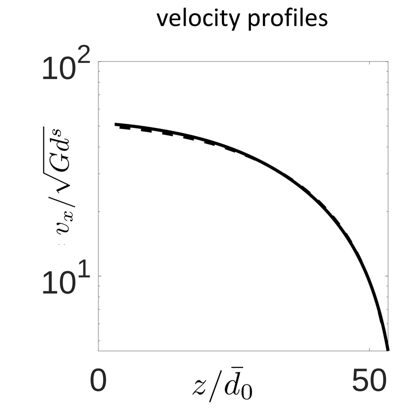
Spatio-temporal DEM concentration contours



Comparison of concentration profiles at sample times



Comparison of steady state velocity profiles



Flow field is a Bagnold profile.

Important Parameters: $\{H/\bar{d}_0 = 50, \theta = 26°, c_0^l = 0.50, d^l/d^s = 1.5\}$

Anti-plane shear flow

- Uniform strain rate and unidirectional pressure gradient that drives segregation.

Deformation tensor reduces to

$$[\mathbf{D}] = \frac{1}{2}\begin{bmatrix} 0 & \partial v_x/\partial y & 0 \\ \partial v_x/\partial y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$[\mathbf{n}^c] = [\mathbf{n}^P] = [0\ \ 0\ \ 1]^\top$

Scalar invariants become

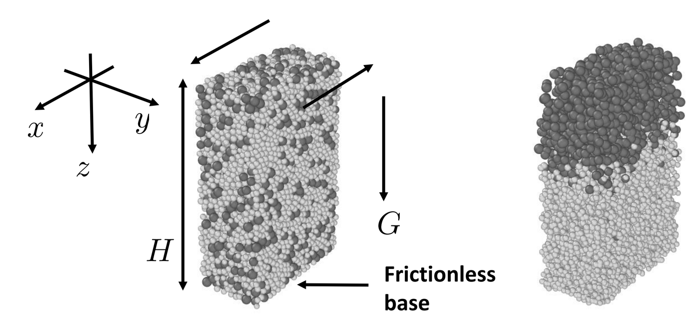
$k^P = n_i^P N_{ij} N_{jk} n_k^P = 0$

$k^c = n_i^c N_{ij} N_{jk} n_k^c = 0$

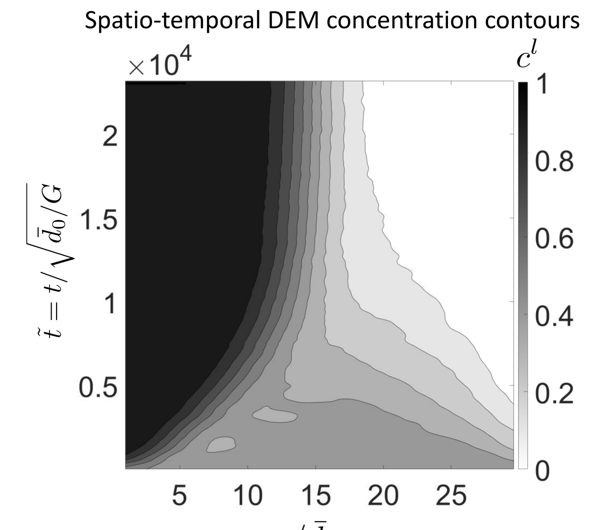
Thus

$\hat{C}_P = \hat{C}_P^{anti}$

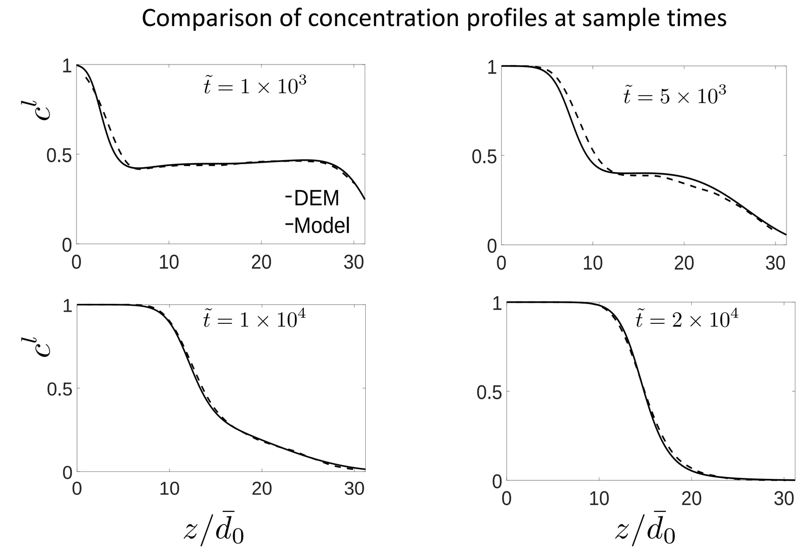
$\hat{C}_{diff} = \hat{C}_{diff}^{anti}$



Spatio-temporal DEM concentration contours



Comparison of concentration profiles at sample times



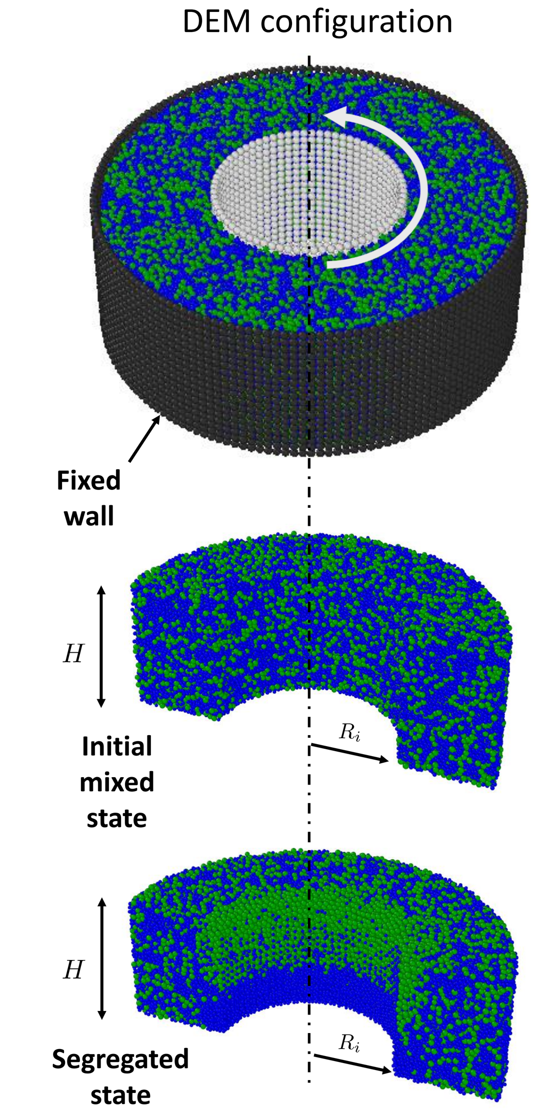
Important Parameters: $\{H/\bar{d}_0 = 30, \dot{\gamma}\sqrt{\bar{d}_0/g} = 0.064, c_0^l = 0.50, d^l/d^s = 1.5\}$

Annular shear with gravity

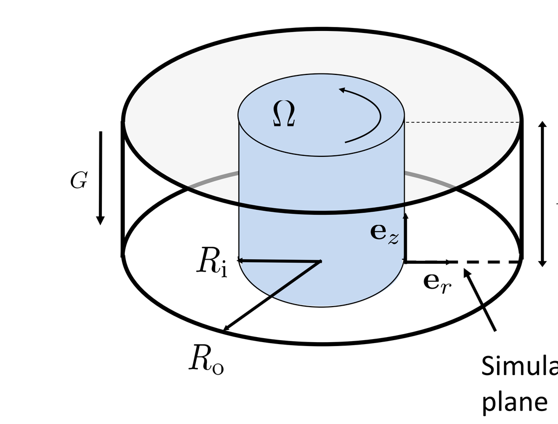
- Strain-rate-gradient and pressure gradient driven fluxes act perpendicular to one another, therefore, driving large grains towards the top corner near the inner wall.
- This flow configuration involves both in-plane and anti-plane modes of diffusion and segregation.

Important parameters:
$\left[\begin{array}{l} R_i/\bar{d}_0 = 20,\ R_o/\bar{d}_0 = 40 \\ H/\bar{d}_0 = 30,\ R_i\Omega/\sqrt{G\bar{d}_0} = 1 \end{array}\right]$

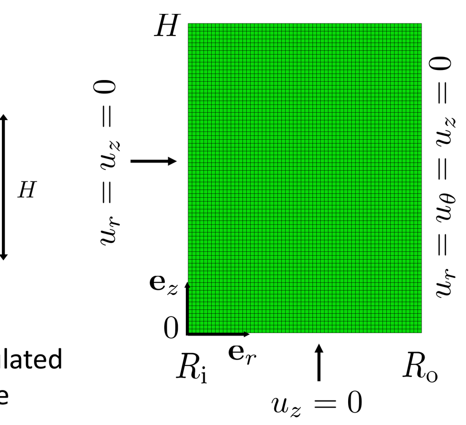
DEM configuration

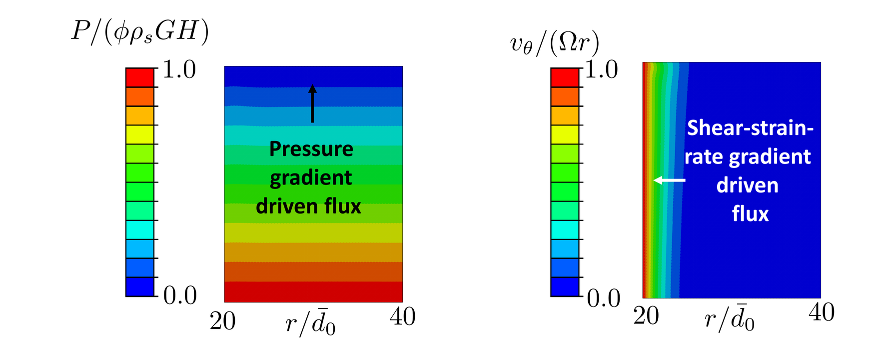


Schematic

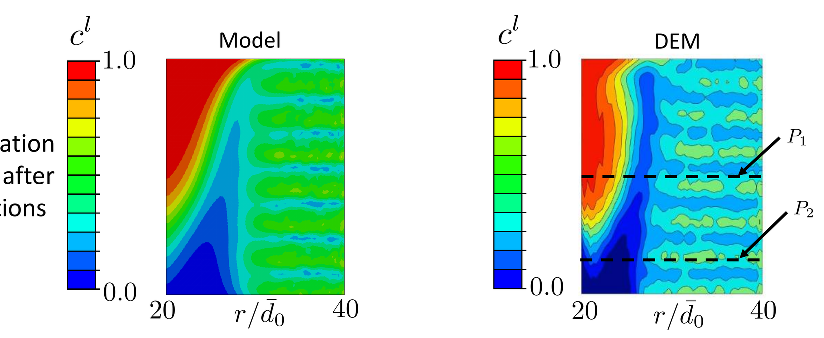


Simulated plane



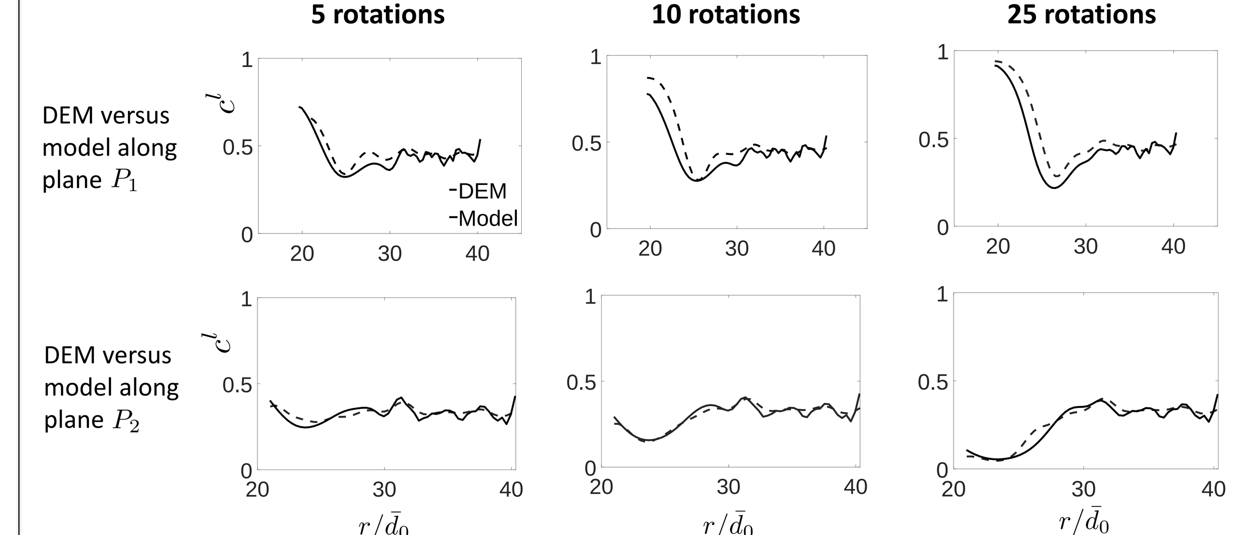


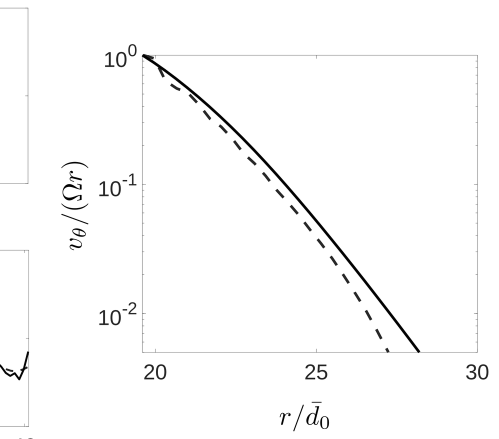
Concentration contours after 20 rotations



DEM versus model along plane P_1

DEM versus model along plane P_2





Comparison of steady state velocity profiles along plane P_1

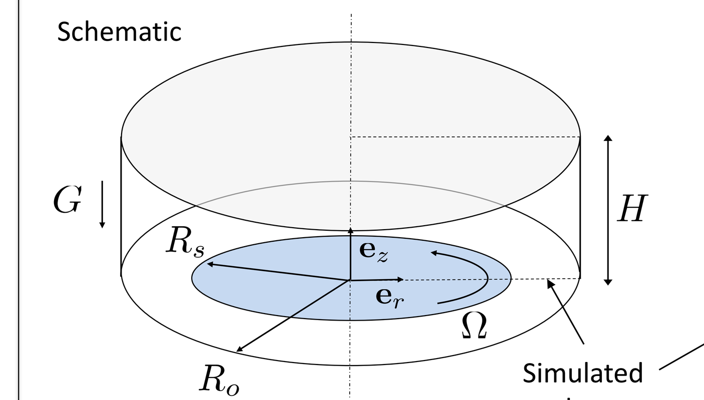
Comparison of concentration fields near the bottom and along the midplane at different times

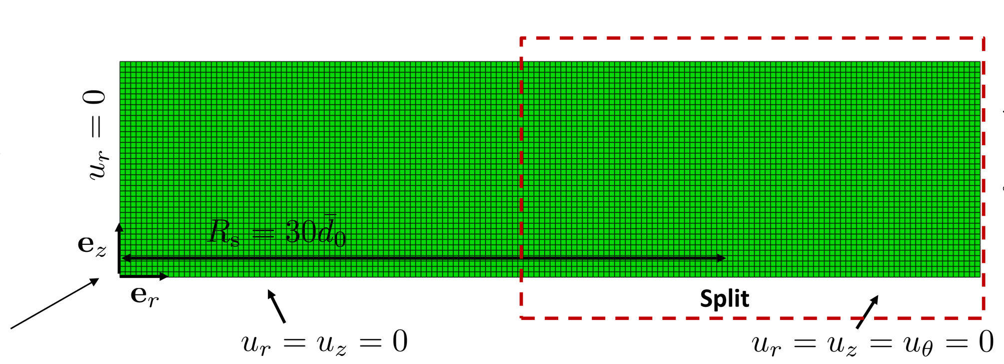
Split bottom flow

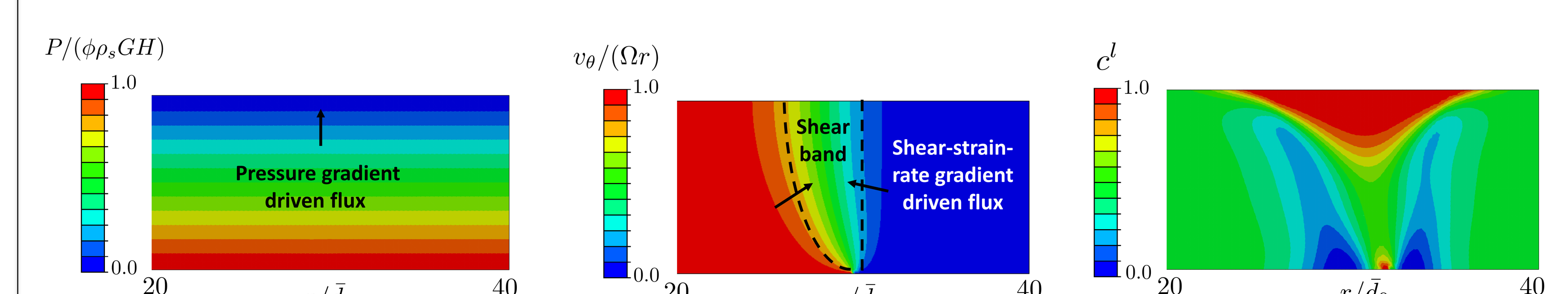
- Introduced by Fenistein and van Hecke (Nature, 2003), flows in the split-bottom cell exhibit wide shear bands. The coupled model predicts complex segregation dynamics wherein large grains segregate towards the top and inside the shear band.

$\left[\begin{array}{l} R_s/\bar{d}_0 = 30,\ R_o/\bar{d}_0 = 40 \\ H/\bar{d}_0 = 10,\ R_s\Omega/\sqrt{G\bar{d}_0} = 10^{-2} \end{array}\right]$

Schematic







Pressure field is hydrostatic

Shear band originates near the split and grows in width away from the bottom

Large grains move towards the top and the shear band region

The coupled continuum model can predict both the segregation dynamics and flow fields in dense bidisperse granular flows.

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