Modeling dynamics of the string in the presence of doubly curved obstacle

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Abstract

The motion of a string in the presence of doubly curved obstacle has been studied. In particular, the effect of the obstacle on the dynamics of the string, coupling between the modes and stability of motions has been investigated.

Motivation

- Indian stringed musical instruments

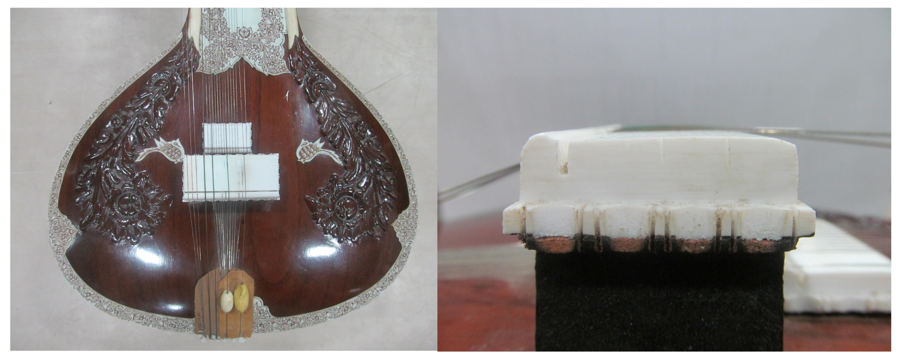
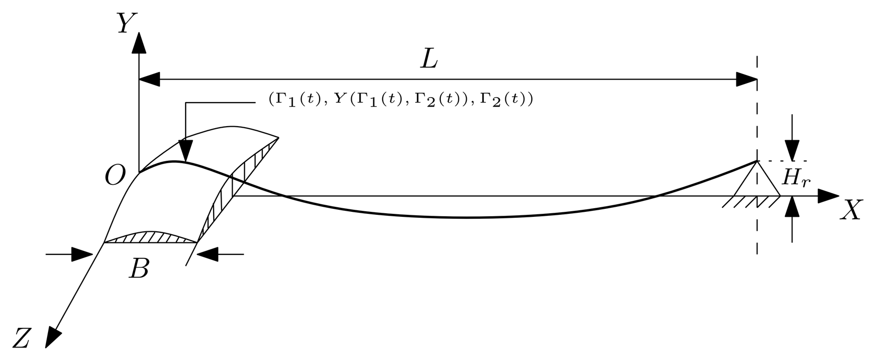


Figure 1: Bridge of Indian musical instrument called *sitar*

Mathematical model

The schematic representation of the system under consideration is



The surface of the obstacle is defined as

$$Y_B(X,Z) = AX(B-X) + C\left(\frac{D}{2}-Z\right)\left(\frac{D}{2}+Z\right).$$

The assumptions made in the model are the following

- The string perfectly wraps and unwraps over the bridge
- The string remains tangent at the point of separation

The Lagrangian of the system, taking into consideration the constraint imposed by the obstacle and non-uniform tension, can be written as

$$\Lambda = \int_0^{\Gamma_1(t)_-}\Big(\frac{\rho}{2}(Y_t^2+Z_t^2) - T_0(Y_X^2+Z_X^2) - \frac{EA}{8}(Y_X^2+Z_X^2)^2 - \lambda(Y - AX(B-X) + C(\frac{D}{2}-Z)(\frac{D}{2}+Z))\Big)dx\,dt + \int_{\Gamma_1(t)_+}^{L}\Big(\frac{\rho}{2}(Y_t^2+Z_t^2) - T_0(Y_X^2+Z_X^2) - \frac{EA}{8}(Y_X^2+Z_X^2)^2\Big)dx\,dt \quad (1)$$

Here, λ is the Langrange multiplier which determines the constraint force. Using Hamilton's principle, the governing equation for the wrapped portion in the non-dimensionalized form can be written as

$$2cz\frac{\partial^2 z}{\partial\tau^2} + 2c\left(\frac{\partial z}{\partial\tau}\right)^2 - 2cz\frac{\partial^2 z}{\partial x^2} - 2c\left(\frac{\partial z}{\partial x}\right) + \sigma\frac{\partial}{\partial x}\left((ab-2ax-2czz_x)\Big((ab-2ax-2czz_x)^2+z_x^2\Big)\right) - 2a + \lambda = 0, \quad (2)$$

$$-\frac{\partial^2 z}{\partial\tau^2} + \frac{\partial^2 z}{\partial x^2} + \sigma\frac{\partial}{\partial x}\left(z_x\Big((ab-2ax-2czz_x)^2+z_x^2\Big)\right) + \lambda(2cz) = 0. \quad (3)$$

And for the free portion of the string

$$-\frac{\partial^2 y}{\partial\tau^2} + \frac{\partial^2 y}{\partial x^2} + \sigma\frac{\partial}{\partial x}\Big(y_x(y_x^2+z_x^2)\Big) = 0, \quad (4)$$

$$-\frac{\partial^2 z}{\partial\tau^2} + \frac{\partial^2 z}{\partial x^2} + \sigma\frac{\partial}{\partial x}\Big(z_x(y_x^2+z_x^2)\Big) = 0. \quad (5)$$

The boundary conditions for this set of governing equations are given as

$$y|_{\gamma_1} = a\gamma_1(b-\gamma_1) + c\left(\frac{d}{2}-\gamma_2\right)\left(\frac{d}{2}+\gamma_2\right), \quad y|_1 = 0, \quad (6)$$

$$z|_0 = 0 \text{ and } z|_1 = 0 \quad (7)$$

In addition to this, slope continuity at the point of separation yields

$$\left.\frac{\partial y}{\partial x}\right|_{\gamma_1} = a(b-2\gamma_1) - 2c\gamma_2\left.\frac{\partial z}{\partial x}\right|_{\gamma_1}. \quad (8)$$

Equations (3)-(8) complete the mathematical formulation for our system.

Model order reduction

The assumed transverse displacements are of the form

$$z = \alpha(t)\sin(\pi x), \qquad 0 \le x \le 1, \quad (9)$$

$$y = \frac{a\gamma_1(b-\gamma_1) + c\left(\frac{d}{2}-\gamma_2\right)\left(\frac{d}{2}+\gamma_2\right)}{1-\gamma_1}(1-x) + \beta(t)\sin\left(\frac{\pi(x-\gamma_1)}{1-\gamma_1}\right), \qquad \gamma_1^+ \le x \le 1, \quad (10)$$

Next, we eliminate λ using (2) and (3) followed by substituting (9) to get residual term denoted by R_{z1}. Again, we substitute (9)-(10) into (4) and (5) to obtain residual forms denoted by R_y and R_{z2}, respectively. In order to minimize to the error, we use the Galerkin projection approach in the following manner

$$\int_0^{\gamma_1} R_{z1}\sin(\pi x) + \int_{\gamma_1}^{1} R_{z2}\sin(\pi x) = 0, \quad (11)$$

$$\int_{\gamma_1}^{1} R_y \sin\left(\frac{\pi(x-\gamma_1)}{1-\gamma_1}\right) = 0. \quad (12)$$

Thus, Galerkin projection reduces the non-linear PDEs to a set of nonlinear ODEs. Also, using (10), the slope continuity equation given by (8) can be written as

$$a\gamma_1^2 - 2a\gamma_1 + ab + \frac{cd^2}{4} - c\gamma_2^2 - 2c\pi\gamma_2\alpha(t)\cos(\pi\gamma_1) + 2c\pi\gamma_2\gamma_1\alpha(t)\cos(\pi\gamma_1) - \pi\beta(t) = 0. \quad (13)$$

where

$$\gamma_2 = \alpha(t)\sin(\pi\gamma_1) \quad (14)$$

Equation (13) is a nonlinear algebraic equation with feasible roots lying between 0 and b which can be solved numerically.

Dynamics of string without obstacle

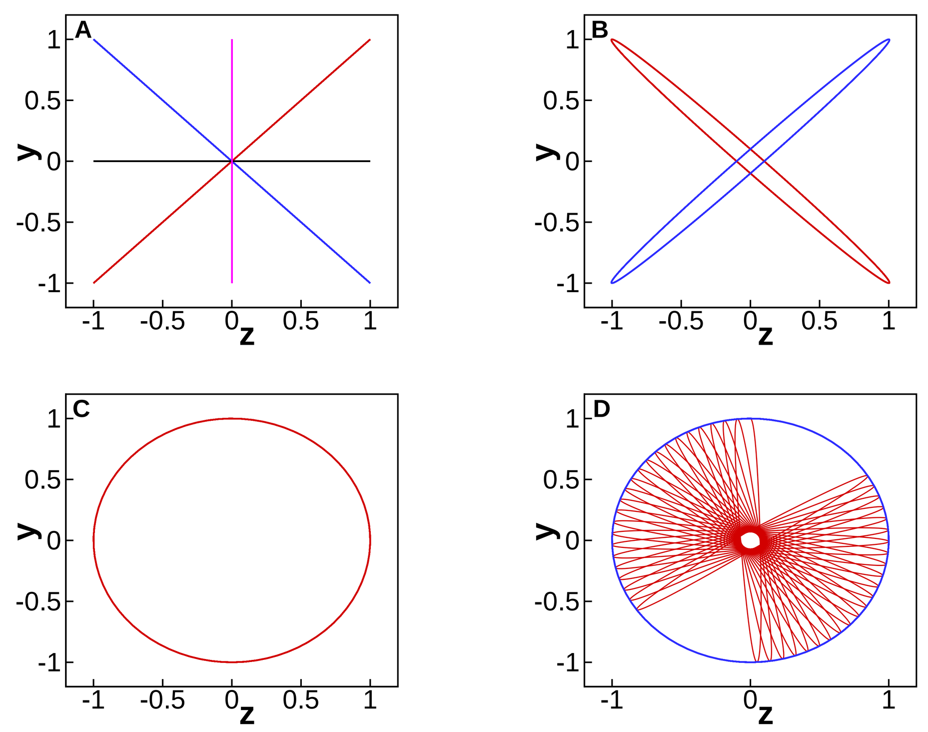


Figure 2: All possible trajectories in the absence of obstacle

- Infinitely many planar, elliptical, circular trajectories
- Precessional motion is the outcome of unstable planar motion

Dynamics of string with obstacle

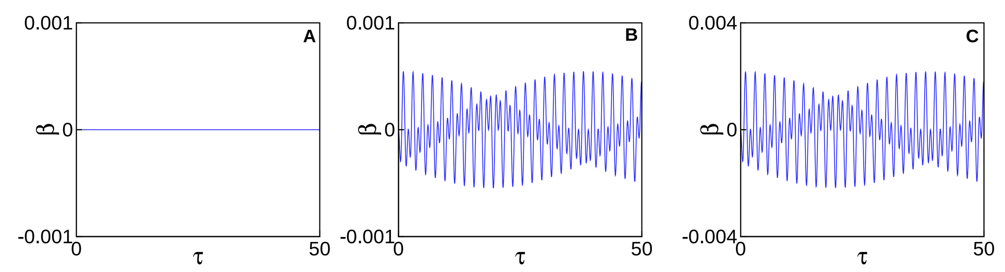


Figure 3: Representation of coupling between the perpendicular modes. Figures A, B and C show the β variation with non-dimensional time for $c = 0$, $c = 0.1$ and $d = 2$, $c = 0.4$ and $d = 1$, respectively. The other parameters are $a = 1440$ and $b = 0.05$.

- Coupling exists even in the absence of stretching nonlinearity
- Strength of coupling increases with increase in curvature along z direction

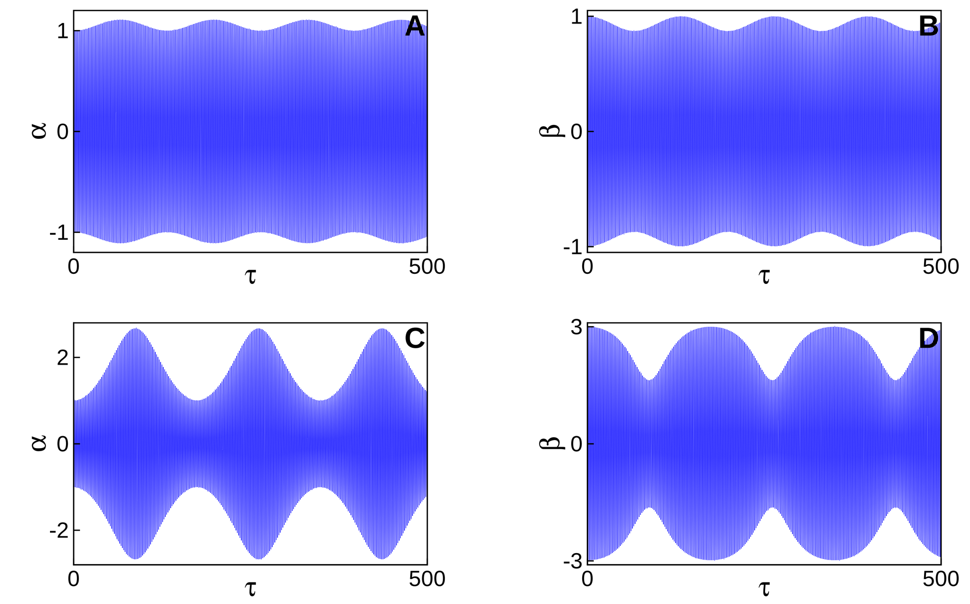


Figure 4: Figure A, B and figure C,D correspond to $\beta(0) = 1$ and $\beta(0) = 3$, respectively($\alpha(0) = 1$ in all figures). The obstacle parameter values are $a = 1440$, $b = 0.05$, $c = 0.1$ and $d = 2$.

- Inter modal exchange of energy is observed
- Modulation becomes significant beyond certain amplitude
- Maximum value attained by α remains same irrespective of smallness of pertubation

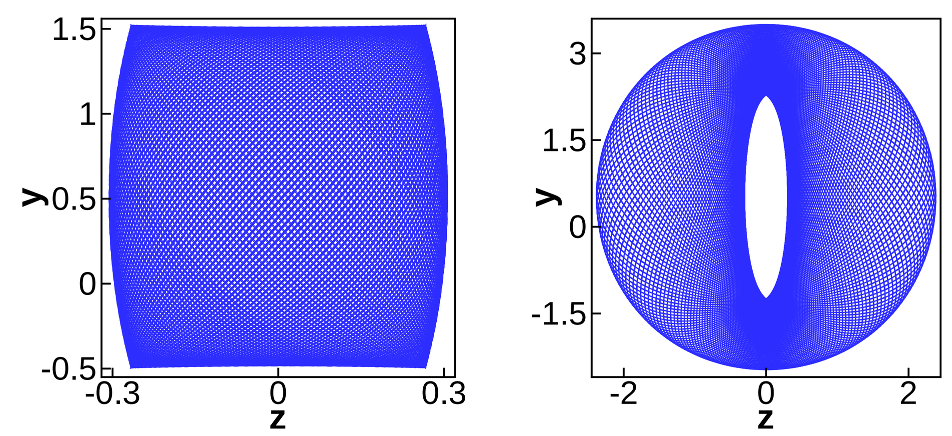


Figure 5: Trajectory of a typical point in the presence of obstacle. Figure A corresponds to $\dot{\alpha}(0) = 1$ and $\beta(0) = 1$, and figure B corresponds to $\dot{\alpha}(0) = 1$ and $\beta(0) = 3$. The obstacle parameter values are $a = 1440$, $b = 0.05$, $c = 0.1$ and $d = 2$.

- Oscillating elliptical trajectories for smaller values of β
- Whirling motions beyond certain critical amplitude

Stability analysis

Linearizing and using single term harmonic balance

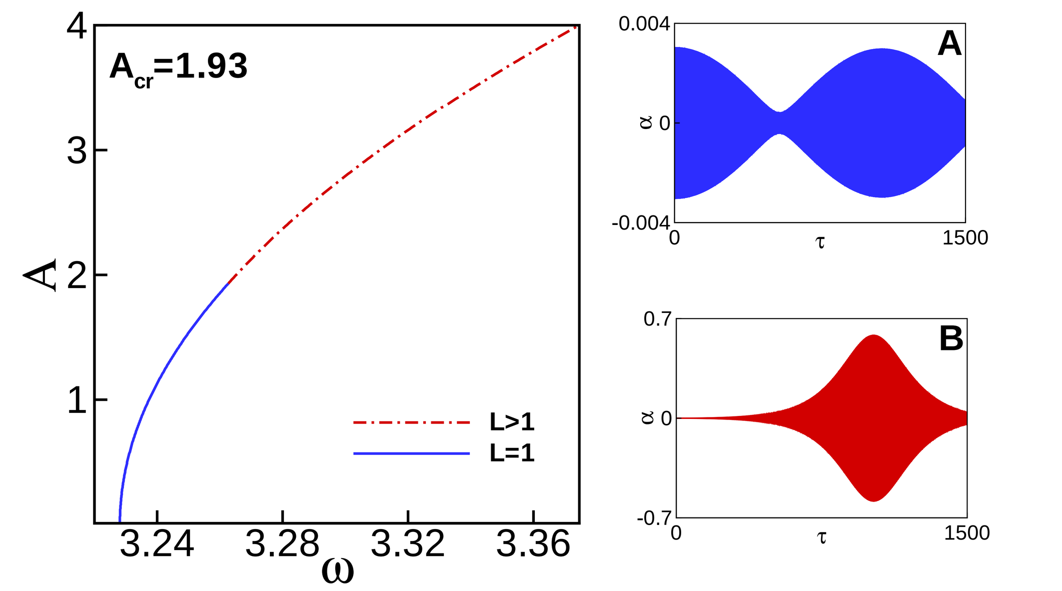


Figure 6: Left: L represents the magnitude of the dominant Floquet multiplier and A_{cr} represents the critical amplitude of planar motions. Right: Figures A and B correspond to $\dot{\alpha}(0) = 0.01$ with $\beta(0) = 1.9$ and $\beta(0) = 2.0$, respectively.

- Compute Floquet multipliers to determine stability
- The critical amplitude comes out to be **1.93**
- Transformation from oscillating to whirling motions beyond critical amplitude

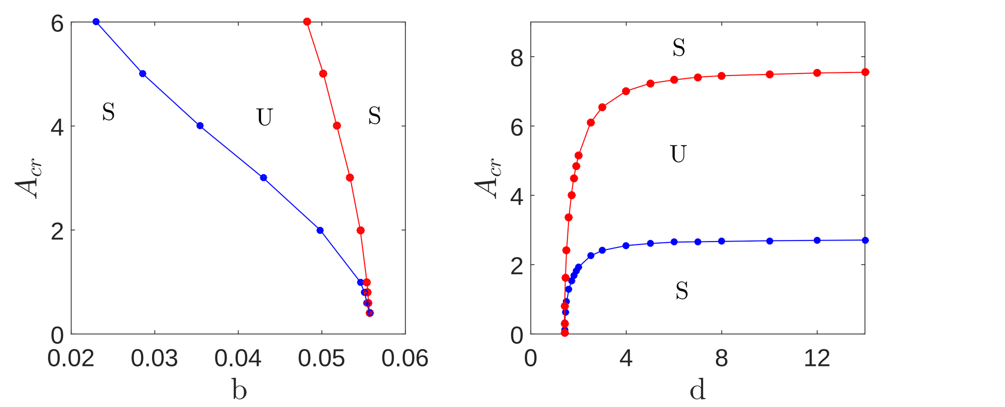


Figure 7: Variation of critical amplitude keeping height of the obstacle constant. Left: Effect of the obstacle width b (along the x direction) on the critical amplitude (A_{cr}) Right: Effect of the obstacle width d (along the z direction) on the critical amplitude (A_{cr}). Here, U and S denote the stable and unstable regime, respectively.

- Investigate stability in space spanned by amplitude and obstacle parameters
- Existence of two critical amplitudes for certain parameters values
- Stability branches merge at certain critical value of parameters

Conclusion

- Only one planar motion in the presence of doubly curved obstacle
- Obstacle has the stabilizing effect on the planar motions

Forthcoming research

- Multimodal analysis and interaction among the higher modes
- Modeling friction between the string and the obstacle

References

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