Non-local granular fluidity(NGF) model

Equations of motion

$$\boxed{\begin{aligned}&\frac{\partial \sigma_{ij}}{\partial x_j} + \phi\rho_{\mathrm{s}} G_i = \phi\rho_{\mathrm{s}}\dot{v}_i\\ &\text{with}\quad \sigma_{ij} = -P\delta_{ij} + 2\frac{P}{g}D_{ij} \quad \text{and} \quad D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\end{aligned}}$$

Nonlocal rheology

Granular fluidity $\longrightarrow$ $g = g^{\mathrm{loc}}(\mu, P) + \xi^2(\mu)\dfrac{\partial^2 g}{\partial x_j \partial x_j}$

Local rheology

$$g^{\mathrm{loc}}(\mu, P) = \begin{cases} \dfrac{1}{\mu}\sqrt{\dfrac{P}{\rho_s d^2}}\left(\dfrac{\mu-\mu_s}{b}\right)^{1/\alpha} & \mu > \mu_s \\ 0 & \mu \le \mu_s \end{cases}$$

Cooperativity length

$$\xi(\mu) = \begin{cases} \dfrac{A}{\sqrt{\alpha(\mu-\mu_s)}}d & \mu > \mu_s \\ \dfrac{A}{\sqrt{(\mu-\mu_s)}}d & \mu \le \mu_s \end{cases}$$

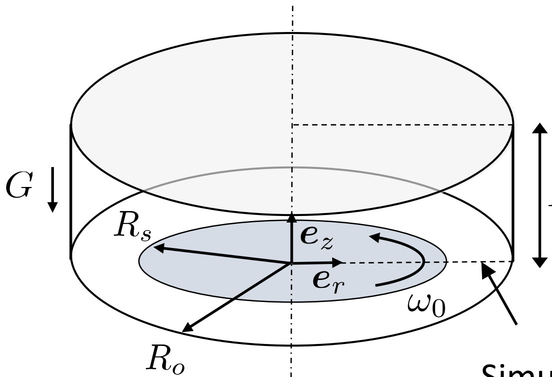
with $\tau = (\sigma'_{ij}\sigma'_{ij}/2)^{1/2}$, $\mu = \tau/P$, and $\dot{\gamma} = (2D_{ij}D_{ij})^{1/2}$

- Model paramaters[†] are $\{\underbrace{\mu_s, b, \alpha}_{\text{Local parameters}}, A\}$ ← Non-local parameter

† Frictional glass beads:
$\{\mu_s = 0.37, b = 1.0, \alpha = 0.7, A = 0.48\}$

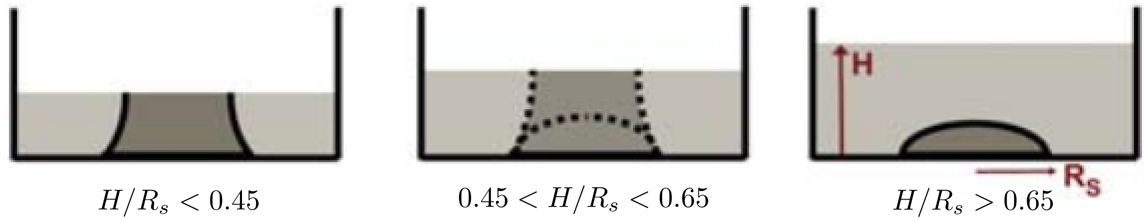
Polyacrylamide hydrogels:
$\rightarrow$ **Ongoing research**

Split-Bottom shear cell



Simulated plane

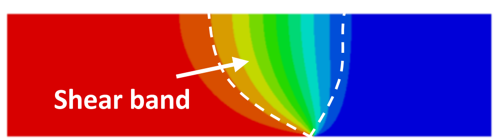
Fenistein and Van Hecke, Nature(2003)
Fenistein et.al, PRL(2004), PRL(2006)



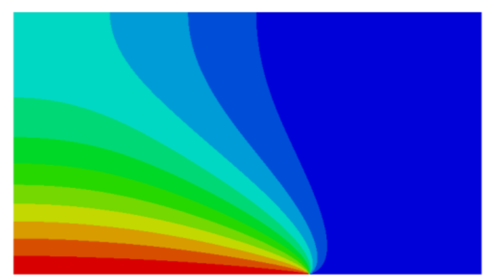
Hard frictional grains: Joshua and Van Hecke, Soft Matter(2010)

The two main factors that dictate the flow are

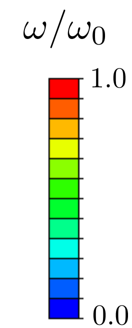
(1) H/R_s



$H/R_s = 0.42$

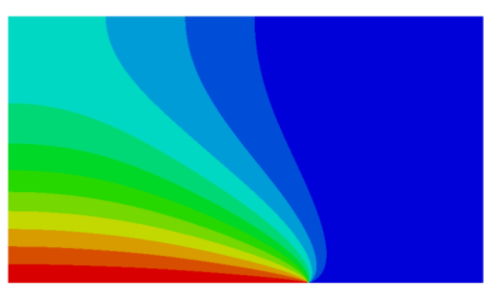


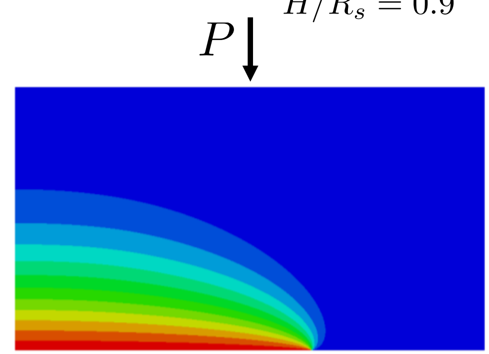
$H/R_s = 0.9$



(2) Confined pressure† P

Free surface





† Recent experimental observations in soft polyacrylamide hydrogels

Finite deformation plasticity framework for dense granular flows

Free energy imbalance: $$\dot{\psi} - \underbrace{\frac{1}{2}T^e : \dot{C}^e}_{\text{Elastic power}} - \underbrace{M^e : L^P}_{\text{Plastic power}} \leq 0$$

where $T^e = JF^{e-1}TF^{e-T}$ is the second Piola stress and $M^e = C^eT^e$ is the Mandel stress

Consequence of isotropy of elastic response

C^e and T^e have same eigenvalues $\Rightarrow M^e = JR^{eT}TR^e \rightarrow$ symmetric

Free energy and elastic response

$$\psi^{\dagger} = G|E^e_o|^2 + \frac{1}{2}K(trE^e)^2 \quad \text{where } E^e = \sum_{i=1}^{3} ln\lambda^e_i r^e_i \otimes r^e_i \longleftarrow$$ Hencky strain

Mandel stress $\Rightarrow M^e = 2GE^e_0 + K(trE^e)I$ and $\bar{p} = -\frac{1}{3}trM^e$

Define equivalent shear stress $\bar{\tau}$ and stress ratio μ

$$\bar{\tau} = \frac{1}{\sqrt{2}}|M^e_0| \quad \text{and} \quad \mu = \frac{\bar{\tau}}{\bar{p}}$$

Yield conditon

$$\mu = \mu_s \ \Rightarrow \begin{cases} \mu \geq \mu_s & \text{Plastic} \\ \mu < \mu_s & \text{Elastic} \end{cases}$$

† Since we are going to study the well developed granular flow which is the outcome of perfectly plastic deformations . Thus the elastic response does not really matter, so we adopt the simple quadratic expression for the free energy in terms of Hencky strain.

Flow rule

Evolution equation $\dot{F}^p = D^p F^p$

Define unit flow tensor $N^p = \dfrac{D^p}{\mid D^p \mid}$ and equivalent plastic shear $\dot{\gamma}^p = \sqrt{2} \mid D^p \mid$

Assume co-directionality $N^p = \dfrac{M_0^e}{\mid M_0^e \mid} \Rightarrow D^p = \dfrac{1}{2}\dot{\gamma}^p \dfrac{M_0^e}{\bar{\tau}}$

Rate-dependence $\dot{\gamma}^p = g\mu$

Granular fluidity

- Plastic response is rate-dependent and perfectly plastic with no hardening
- Local agitations can cause agitation in the far away material even if it's below yield

Local fluidity

$$g^{\mathrm{loc}}(\bar{p},\mu) = \begin{cases} \dfrac{1}{\mu}\sqrt{\dfrac{\bar{p}}{\rho_s d^2}}\left(\dfrac{\mu-\mu_s}{b}\right)^{1/\alpha} & \mu > \mu_s \\ 0 & \mu \le \mu_s \end{cases}$$

Coperativity length

$$\xi(\mu) = \begin{cases} \dfrac{A}{\sqrt{\alpha(\mu-\mu_s)}}d & \mu > \mu_s \\ \dfrac{A}{\sqrt{(\mu-\mu_s)}}d & \mu \le \mu_s \end{cases}$$

- **NGF model can be implemented using user defined elements(UEL) in Abaqus**

Constitutive update: Time integration procedure

Given F_n^P at time t_n and $\{F_{n+1}, g_{n+1}\}$ at time t_{n+1}

Determine $\{T_{n+1}, F_{n+1}^P, g_{n+1}^{loc}, \xi_{n+1}\}$

Begin by integrating the evolution equation for F^P via exponential map

$$F_{n+1}^p = \exp(\Delta t\, D_{n+1}^p) F_n^p \Rightarrow F_{n+1}^{p-1} = F_n^{p-1} \exp(-\Delta t\, D_{n+1}^p)$$

Recall Kroner-Lee decomposition $F_{n+1} = F_{n+1}^e F_{n+1}^p$

$$\Rightarrow F_{n+1}^e = F_{n+1} F_{n+1}^{p-1} \Rightarrow F_{n+1}^e = \underbrace{F_{n+1} F_n^{p-1}}_{F_{tr}^e} \exp(-\Delta t D_{n+1}^p) \qquad (1)$$

Trial value of F^e assuming plastic flow is frozen

Polar Decomposition

$$F^e_{n+1} = R^e_{n+1} U^e_{n+1} \quad \text{and} \quad F^e_{tr} = R^e_{tr} U^e_{tr}$$

Using (1)

$$R^e_{n+1} U^e_{n+1} \exp(\Delta t D^p_{n+1}) = R^e_{tr} U^e_{tr}$$

$$\Rightarrow \begin{cases} R^e_{n+1} = R^e_{tr} \\ \\ U^e_{n+1} = U^e_{tr} \exp(-\Delta t\, D^p_{n+1}) \end{cases}$$

Taking log

$$E^e_{n+1} = E^e_{tr} - \Delta t\, D^p_{n+1} \qquad \text{with} \quad E^e_{tr} = \ln(U^e_{tr})$$

Using stress strain relation

$$M^e_{n+1} = M^e_{tr} - 2G\Delta t D^P_{n+1} \quad \text{with} \quad M^e_{tr} = \mathbb{C}[E^e_{tr}] \tag{2}$$

with $\dot{\gamma}^p_{n+1} = \sqrt{2} \mid D^p_{n+1} \mid \quad N^p_{n+1} = D^p_{n+1} / \mid D^p_{n+1} \mid$, we write $D^p_{n+1} = \dfrac{1}{\sqrt{2}} \dot{\gamma}^p_{n+1} N^p_{n+1}$

Also, $\dot{\gamma}^p_{n+1} = g_{n+1}\mu_{n+1}$ and $N^p_{n+1} = \dfrac{M^e_{0,n+1}}{\sqrt{2}\bar{\tau}_{n+1}}$ (3)

where, $\bar{\tau}_{n+1} = \dfrac{1}{\sqrt{2}} \mid M^e_{0,n+1} \mid$, $\bar{p}_{n+1} = -\dfrac{1}{3} tr M^e_{n+1}$, and $\mu_{n+1} = \dfrac{\bar{\tau}_{n+1}}{\bar{p}_{n+1}}$ (4)

Using (2)

$$M^e_{n+1} = M^e_{tr} - \sqrt{2} G(\Delta t \dot{\gamma}^p_{n+1}) N^p_{n+1} \quad \Rightarrow \left[\begin{array}{l} M^e_{0,n+1} = M^e_{0,tr} - \sqrt{2} G(\Delta t \dot{\gamma}^p_{n+1}) N^p_{n+1} \\ \bar{p}_{n+1} = \bar{p}_{tr} \end{array} \right.$$

In terms of equivalent shear stress

$$\left(\bar{\tau}_{n+1} + G(\Delta t \dot{\gamma}^p_{n+1})\right) N^p_{n+1} = \bar{\tau}_{tr} N^p_{tr} \quad \Rightarrow \left[\begin{array}{l} \bar{\tau}_{n+1} + G(\Delta t \dot{\gamma}^p_{n+1}) = \bar{\tau}_{tr} \\ N^p_{n+1} = N^p_{tr} \end{array} \right.$$

Using $(3)_1$ and $(4)_3$

$$\bar{\tau}_{n+1} = \frac{\tau_{tr} \bar{p}_{tr}}{\bar{p}_{tr} + G \Delta t g_{n+1}}$$

Updating Mandel stress

$$M^e_{n+1} = M^e_{tr} - \sqrt{2} \left(\tau_{tr} - \tau_{n+1}\right) N^p_{tr} \quad \Rightarrow \quad T_{n+1} = J^{-1}_{n+1} R^e_{tr} M^e_{n+1} R^{eT}_{tr}$$

Updating $\{\dot{\gamma}^p,\, D^p,\, F^p\}$

$$\dot{\gamma}^p_{n+1} = g_{n+1}\left(\frac{\bar{\tau}_{n+1}}{\bar{p}_{tr}}\right)$$

$$D^p_{n+1} = \frac{1}{\sqrt{2}}\dot{\gamma}^p_{n+1} N^p_{tr}$$

$$F^p_{n+1} = \exp\left(\Delta t D^p_{n+1}\right) F^p_n$$

Updating g^{loc} and ξ

$$g^{\mathrm{loc}}_{n+1} = \begin{cases} \sqrt{\dfrac{\bar{p}_{tr}}{\rho_s d^2}}\left(\dfrac{\mu_{n+1}-\mu_s}{b\,\mu_{n+1}}\right)^{1/\alpha} & \mu_{n+1} > \mu_s \\ 0 & \mu_{n+1} \leq \mu_s \end{cases}$$

$$\xi_{n+1} = \begin{cases} \dfrac{A\,d}{\sqrt{\alpha \mid \mu_{n+1}-\mu_s \mid}} & \mu_{n+1} > \mu_s \\ \dfrac{A\,d}{\sqrt{\mid \mu_{n+1}-\mu_s \mid}} & \mu_{n+1} \leq \mu_s \end{cases}$$