Lecture 02
Gauss Elimination Method

MAT 121 - Introduction to Linear Algebra

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Recall

- Systems of linear equations
- Homogeneous and non-homogeneous system
- Solution
- Consistent and Inconsistent System

True or False = Homogeneous system always inconsistent. ?

Outline of lecture

- Row-equivalent Form
- Gauss Elimination Method
- Row-Echelon Form (REF)
- Reduced Row-Echelon Form (RREF)

Carl Friedrich Gauss (1777-1855)
A German mathematician and physicist
Mathematics is the queen of the sciences.

System of linear equations

A system of m linear equations in n variables has the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \quad \vdots \quad \vdots \ \vdots \ \vdots \ \vdots &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned} \tag{1}$$

Here, a_{ij}'s, $\quad 1 \le i \le m,\ 1 \le j \le n$ are the coefficients;
$x_1, x_2, \cdots, x_n$ are the unknown variables,
and $[b_1, b_2, , \cdots b_n]^t$ is the right hand side vector.

Gauss Elimination Method

Gauss Elimination is a standard method for solving linear systems of the form $\boxed{AX = B}$, where

$$\underset{(A)}{A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}} \qquad \underset{(X)}{X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} \qquad \underset{(B)}{B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}$$

- A is the coefficient matrix;
- X is a column vector of unknown variables,
- B is the right hand side vector,

Augmented matrix

We illustrate the Gauss-elimination method for some examples before formalizing the method. Since the linear system is completely determined by its augmented matrix, defined below, the Gauss elimination process can be down by merely considering the matrices.
The matrix $\tilde{A}$ obtained by augmenting A by column $\mathbf{b}$, written as

$$[A:b] = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}, \text{ or } \tilde{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & \vdots & b_m \end{pmatrix}$$

is called the **augmented matrix**.

Basic Strategy

Replace the system of linear equations with an equivalent system (one with the same solution set) which is easier to solve.

$$[A:b] = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix} \overset{\text{Convert}}{\Longrightarrow} \begin{pmatrix} a'_{11} & a'_{12} & \dots & a'_{1n} & b'_1 \\ 0 & a'_{22} & \dots & a'_{2n} & b'_2 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & a'_{mn} & b'_m \end{pmatrix}$$

Using by Row-operations

Basic Strategy

Replace the system of linear equations with an equivalent system (one with the same solution set) which is easier to solve.

In order to solve the above system of linear equations, Gauss proposed 3 kinds of operations each of which gives a new linear system, (hopefully) simpler and *equivalent* to the original one.

- Interchanging two different equations, say equations (i) and (j).
- Multiplying an equation by a scalar and adding it to some <u>other</u> equation of the system.
- Multiplying an equation by a *non-zero* number.

How to obtain equivalent systems?

We state the **elementary row operations** for matrices and equations which yield **row-equivalent** linear systems.

	Elementary Row Operations for Matrices	Elementary Row Operations for Eqn
1. (Swap)	Interchange two rows ($R_i \leftrightarrow R_j$) interchange ith and jth rows	Interchange two equations
2 (Replacement)	Addition of a constant multiple of one row to another row $R_i \to R_i + cR_j \ (j \neq i)$	Addition of a constant multiple of one equation to another equation
3 (Scale)	Multiplication of a row by a non-zero constant $c\,(\neq 0)$ $R_i \to cR_i$	Multiplication of an equation by a non-zero constant $c\,(\neq 0)$

Definition (Row-equivalent systems)

A linear system S_1 is row-equivalent to a linear system S_2 $(S_1 \sim S_2)$, if S_2 can be obtained from S_1 by finitely many row operations.

Note that the solutions of the two row -equivalent systems are identical (same set of solution).

Example 1

Solve

$$
\begin{aligned}
-x_1 + x_2 + 2x_3 &= 2 \\
3x_1 - x_2 + x_3 &= 6 \\
-x_1 + 3x_2 + 4x_3 &= 4.
\end{aligned}
$$

Solution : The linear system of equations can be expressed as

$$
\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}
$$

The augmented matrix is given by

$$\widetilde{A} = \left[\begin{array}{ccc:c} -1 & 1 & 2 & 2 \\ \boxed{3} & -1 & 1 & 6 \\ \boxed{-1} & 3 & 4 & 4 \end{array}\right] \qquad \begin{cases} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - R_1 \end{cases}$$

[1]

$$\sim \left[\begin{array}{ccc:c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & \boxed{2} & 2 & 2 \end{array}\right] \qquad \begin{cases} R_3 \to R_3 - R_2 \end{cases}$$

$$\sim \left[\begin{array}{ccc:c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 0 & -5 & -10 \end{array}\right]$$

(ROW - ECHELON FORM)

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[1] The circled element is the pivot element at each stage and the elements in the boxes need to be eliminated.

Back Substitution

The given system has the same solution as the system

$$\begin{aligned} -x_1 + x_2 + 2x_3 &= 2 \\ 2x_2 + 7x_3 &= 12 \\ -5x_3 &= -10. \end{aligned}$$

and this system can be solved easily as

$$\begin{aligned} -5x_3 = -10 \quad &\Longrightarrow \quad x_3 = 2; \\ 2x_2 + 7x_3 = 12 \quad &\Longrightarrow \quad 2x_2 + 14 = 12 \Longrightarrow x_2 = -1; \\ -x_1 + x_2 + 2x_3 = 2 \quad &\Longrightarrow \quad -x_1 - 1 + 4 = 2 \Longrightarrow x_1 = 1. \end{aligned}$$

Back Substitution

The given system has the same solution as the system

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and this system can be solved easily as

$$\begin{aligned} -5x_3 = -10 &\implies x_3 = 2; \\ 2x_2 + 7x_3 = 12 &\implies 2x_2 + 14 = 12 \implies x_2 = -1; \\ -x_1 + x_2 + 2x_3 = 2 &\implies -x_1 - 1 + 4 = 2 \implies x_1 = 1. \end{aligned}$$

Hence, $\boxed{x_1 = 1,\ x_2 = -1,\ x_3 = 2.}$

THIS IS AN EXAMPLE OF A CONSISTENT SYSTEM WITH A UNIQUE SOLUTION (DETERMINED SYSTEM).

Example 2 (Over Determined System)

More no. of eq^n than unknown

Solve :

$$\begin{aligned} x_1 + 2x_2 \quad &= 2 \\ 3x_1 + 6x_2 - x_3 \quad &= 8 \\ x_1 + 2x_2 + x_3 \quad &= 0 \\ 2x_1 + 5x_2 - 2x_3 \quad &= 9. \end{aligned}$$

Solution : The linear system of equations can be expressed as

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & -1 \\ 1 & 2 & 1 \\ 2 & 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 0 \\ 9 \end{bmatrix}$$

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Echelon Form using Row Operations

The augmented matrix is given by

$$\widetilde{A} = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ \boxed{3} & 6 & -1 & 8 \\ \boxed{1} & 2 & 1 & 0 \\ \boxed{2} & 5 & -2 & 9 \end{array}\right] \qquad \begin{cases} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1 \end{cases}$$

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$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & -2 & 5 \end{array}\right]$$

no x_2 in second equation;
interchange with 4th equation
as 3rd also doesn't have x_2
$R_2 \leftrightarrow R_4$: PARTIAL PIVOTING

Contd...

$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & 2 \\ 0 & 1 & -2 & \vdots & 5 \\ 0 & \boxed{0} & 1 & \vdots & -2 \\ 0 & \boxed{0} & -1 & \vdots & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & 2 \\ 0 & 1 & -2 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & -2 \\ 0 & 0 & \boxed{-1} & \vdots & 2 \end{bmatrix} \left\{ R_4 \to R_4 + R_3 \right.$$

Contd..

$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & 2 \\ 0 & 1 & -2 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & -2 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \Big\{ \textbf{ROW-ECHELON FORM}$$

Back Substitution

The given system has the same solution as the system

$$\begin{aligned} x_1 + 2x_2 \quad &= 2 \\ x_2 - 2x_3 \quad &= 5 \\ x_3 \quad &= -2. \end{aligned}$$

and this system can be solved easily as

$$\begin{aligned} x_3 &= -2; \\ x_2 - 2x_3 = 5 \quad &\Longrightarrow \quad x_2 = 5 + 2 \times -2 \Longrightarrow x_2 = 1; \\ x_1 + 2x_2 = 2 \quad &\Longrightarrow \quad x_1 = 2 - 2 \Longrightarrow x_1 = 0. \end{aligned}$$

Hence, $\boxed{x_1 = 0,\ x_2 = 1,\ x_3 = -2.}$

THIS IS AN EXAMPLE OF A CONSISTENT OVER DETERMINED SYSTEM.

Back Substitution

The given system has the same solution as the system

$$\begin{aligned} x_1 + 2x_2 \quad &= 2 \\ x_2 - 2x_3 \quad &= 5 \\ x_3 \quad &= -2. \end{aligned}$$

and this system can be solved easily as

$$\begin{aligned} x_3 &= -2; \\ x_2 - 2x_3 &= 5 \quad \Longrightarrow \quad x_2 = 5 + 2 \times -2 \Longrightarrow x_2 = 1; \\ x_1 + 2x_2 &= 2 \quad \Longrightarrow \quad x_1 = 2 - 2 \Longrightarrow x_1 = 0. \end{aligned}$$

Hence, $\boxed{x_1 = 0,\ x_2 = 1,\ x_3 = -2.}$

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REF

Any matrix after a sequence of elementary row operations gets reduced to what is known as a *Row Echelon form*.

> **[Row-Echelon Form (REF)]** A matrix is said to be in a row echelon form (or to be a row echelon matrix) if it has a staircase-like pattern characterized by the following properties:
> (a) The all-zero rows (if any) are at the bottom.
> (b) The first nonzero entry in a nonzero row is called a pivot. The pivot in any row is farther to the right than the pivots in rows above.
> (c) All entries in a column below a leading entry (pivot) is zero.

The first non-zero entry in the j^{th} row is known as the j^{th} *pivot*.

The matrix $\begin{bmatrix} \boxed{3} & 2 & 1 & 3 \\ 0 & \boxed{1} & -1 & 6 \\ 0 & 0 & 0 & \boxed{12} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in REF. The three pivots are indicated.

The matrix $\begin{bmatrix} \boxed{3} & 2 & 1 & 3 \\ 0 & \boxed{1} & -1 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{12} \end{bmatrix}$ is NOT in REF.

Row echelon form ctd.

In this form each row, except perhaps the first, starts with a string of zeroes. Each row

$\boxed{\text{starts with strictly more number of leading zeroes than the previous row.}}$

- The j^{th} pivot is below and strictly to the right of $(j-1)^{th}$ pivot. All the entries *below* a pivot are zeroes.
- The no. of pivots in a REF of A $\leq$ the no. of rows in A.

Some More Examples of Row-Echelon form

(1) $\begin{bmatrix} 0 & 2 & 1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 5 & 6 & 7 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

Ex. 3 (Determined, Inconsistent System)

Solve

$$\begin{aligned} -x_2 + 3x_3 &= -1 \\ x_1 + 2x_2 - x_3 &= -8 \\ x_1 + x_2 + 2x_3 &= -1 \end{aligned}$$

Solution : The linear system of equations can be expressed as

$$\begin{bmatrix} 0 & -1 & 3 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \\ -1 \end{bmatrix}$$

Echelon Form using Row Operations

The augmented matrix is given by

$$\widetilde{A} = \begin{bmatrix} 0 & -1 & 3 & \vdots & -1 \\ 1 & 2 & -1 & \vdots & -8 \\ 1 & 1 & 2 & \vdots & -1 \end{bmatrix} \left\{ \begin{array}{l} \text{no } x_1 \text{ in the first equation;} \\ \text{interchange with 2nd equation} \\ R_2 \leftrightarrow R_1 : \boxed{\text{PARTIAL PIVOTING}} \end{array} \right.$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & -8 \\ \boxed{0} & -1 & 3 & \vdots & -1 \\ \boxed{1} & 1 & 2 & \vdots & -1 \end{bmatrix} \left\{ R_3 \rightarrow R_3 - R_1 \right.$$

Echelon Form using Row Operations

The augmented matrix is given by

$$\widetilde{A} = \begin{bmatrix} 0 & -1 & 3 & \vdots & -1 \\ 1 & 2 & -1 & \vdots & -8 \\ 1 & 1 & 2 & \vdots & -1 \end{bmatrix} \left\{ \begin{array}{l} \text{no } x_1 \text{ in the first equation;} \\ \text{interchange with 2nd equation} \\ R_2 \leftrightarrow R_1 : \boxed{\text{PARTIAL PIVOTING}} \end{array} \right.$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & -8 \\ \boxed{0} & -1 & 3 & \vdots & -1 \\ \boxed{1} & 1 & 2 & \vdots & -1 \end{bmatrix} \left\{ R_3 \to R_3 - R_1 \right.$$

Contd...

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & -8 \\ 0 & -1 & 3 & \vdots & -1 \\ 0 & \boxed{-1} & 3 & \vdots & 7 \end{bmatrix} \Bigg\{ R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & \vdots & -8 \\ 0 & -1 & 3 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 8 \end{bmatrix} \Bigg\{ \text{3rd equation} \Longrightarrow \text{INCONSISTENCY!}$$

NO SOLUTION.

Ex. 4 : Consistent (Infinitely many solu.)

Solve :

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ 3x_1 + 6x_2 - x_3 &= 8 \\ x_1 + 2x_2 + x_3 &= 0 \end{aligned}$$

Solution : The linear system of equations can be expressed as

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix}$$

Echelon Form using Row Operations

The augmented matrix is given by

$$\widetilde{A} = \left[\begin{array}{ccc:c} 1 & 2 & 0 & 2 \\ \boxed{3} & 6 & -1 & 8 \\ \boxed{1} & 2 & 1 & 0 \end{array}\right] \quad \begin{cases} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \end{cases}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad \begin{cases} \text{Candidate for second pivot is 0} \\ \text{Go to next column} \end{cases}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & \boxed{1} & -2 \end{array}\right] \quad \begin{cases} R_3 \to R_3 + R_2 \end{cases}$$

Contd..

$$\sim \begin{bmatrix} 1 & 2 & 0 & \vdots & 2 \\ 0 & 0 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \Big\{ \textbf{ROW-ECHELON FORM}$$

Pivot Elements

Back Substitution

That is,

$$
\begin{aligned}
x_3 &= -2; \\
x_1 + 2x_2 = 2 \quad &\Longrightarrow \quad x_1 = 2 - 2x_2.
\end{aligned}
$$

Here, x_1 and x_3 are basic variables and x_2 is the free variable (no pivot).
If we choose $x_2 = t$, then $x_1 = 2 - 2t$.
The solution vector is

$$
\begin{bmatrix} 2-2t \\ t \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}
$$

THIS IS AN EXAMPLE OF A CONSISTENT DETERMINED SYSTEM WITH INFINITELY MANY SOLUTIONS.

Ex. 5 : Consistent, (Infinitely many solutions)

Solve :

$$\begin{aligned} 2x_1+3x_2-2x_3+4x_4 &= 2 \\ -6x_1+-9x_2+7x_3-8x_4 &= -3 \\ 4x_1+6x_2-x_3+20x_4 &= 13 \end{aligned}$$

Solution : The linear system of equations can be expressed as

$$\begin{bmatrix} 2 & 3 & -2 & 4 \\ -6 & -9 & 7 & -8 \\ 4 & 6 & -1 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 13 \end{bmatrix}$$

Echelon Form using Row Operations

The augmented matrix is given by

$$\widetilde{A} = \begin{bmatrix} 2 & 3 & -2 & 4 & \vdots & 2 \\ \boxed{-6} & -9 & 7 & -8 & \vdots & -3 \\ \boxed{4} & 6 & -1 & 20 & \vdots & 13 \end{bmatrix} \qquad \begin{cases} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{cases}$$

$$\sim \begin{bmatrix} 2 & 3 & -2 & 4 & \vdots & 2 \\ 0 & 0 & 1 & 4 & \vdots & 3 \\ 0 & 0 & \boxed{3} & 12 & \vdots & -2 \end{bmatrix}$$

{ Candidate for second pivot is 0,
x_2 is a free variableGo to next column and choose the pivot$R_3 \to R_3 - 3$

$$\sim \begin{bmatrix} 2 & 3 & -2 & 4 & \vdots & 2 \\ 0 & 0 & 1 & 4 & \vdots & 3 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

{ x_4 is also a free variable
ROW-ECHELON FORM

Back Substitution

That is, Here, $x_2 \& x_4$ are free variables and $x_1 \& x_3$ are the basic variables.
(Free variables do not occur at the beginning of any equation when the system has been reduced to echelon form).
If we choose $x_2 = t \& x_4 = s$, then $x_3 = 2 - 4s$ and

$$\begin{aligned} 2x_1 &= 2 - 3x_2 + 2x_3 - 4x_4 \\ &= 2 - 3t + 2(3 - 4s) - 4s \\ &= 8 - 3t - 12s \Longrightarrow x_1 = 4 - \frac{3}{2}t - 6s. \end{aligned}$$

Solution Vector

Hence, the solution vector is

$$\begin{bmatrix} 4-\frac{3}{2}t-6s \\ t \\ 3-4s \\ s \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -6 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

THIS IS AN EXAMPLE OF A CONSISTENT SYSTEM WITH INFINITELY MANY SOLUTIONS.

Remarks

1. Any reduced row-echelon matrix is also a row-echelon matrix.
2. Any nonzero matrix may be row reduced into more than one matrix in echelon form. But the reduced row echelon form that one obtains from a matrix is unique.
3. A pivot is a nonzero number in a pivot position that is used as needed to create zeros with the help of row operations.
4. Different sequences of row operations might involve a different set of pivots.
5. Given a matrix A, its REF is NOT unique. However, the position of each of its pivots is unalterable.

Solve using Gauss Elimination Method

Solve the following system of linear equations in the unknowns by GEM

(i)
$$\begin{array}{rrrrl} & 2x_3 & -2x_4 & +x_5 & = 2 \\ 2x_2 & -8x_3 & +14x_4 & -5x_5 & = 2 \\ x_2 & +3x_3 & & +x_5 & = 8 \end{array}$$

(ii)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & -x_4 & = 2 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$