Lecture 01
Matrices: Systems of linear equations

MAT 121 - Introduction to Linear Algebra

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Outline of lecture

- Introduction
- Systems of linear equations
- Solution

Solve

$$x + 2y = 3 \tag{1}$$

$$4x + 5y = 6 \tag{2}$$

Eq. 1 $\times 4$ − Eq 2

$$\begin{array}{r} 4x + 8y = 12 \\ -\ 4x + 5y = 6 \\ \hline 3y = 6 \end{array} \Rightarrow y = 2$$

Put $y = 2$ in Eq^n 1

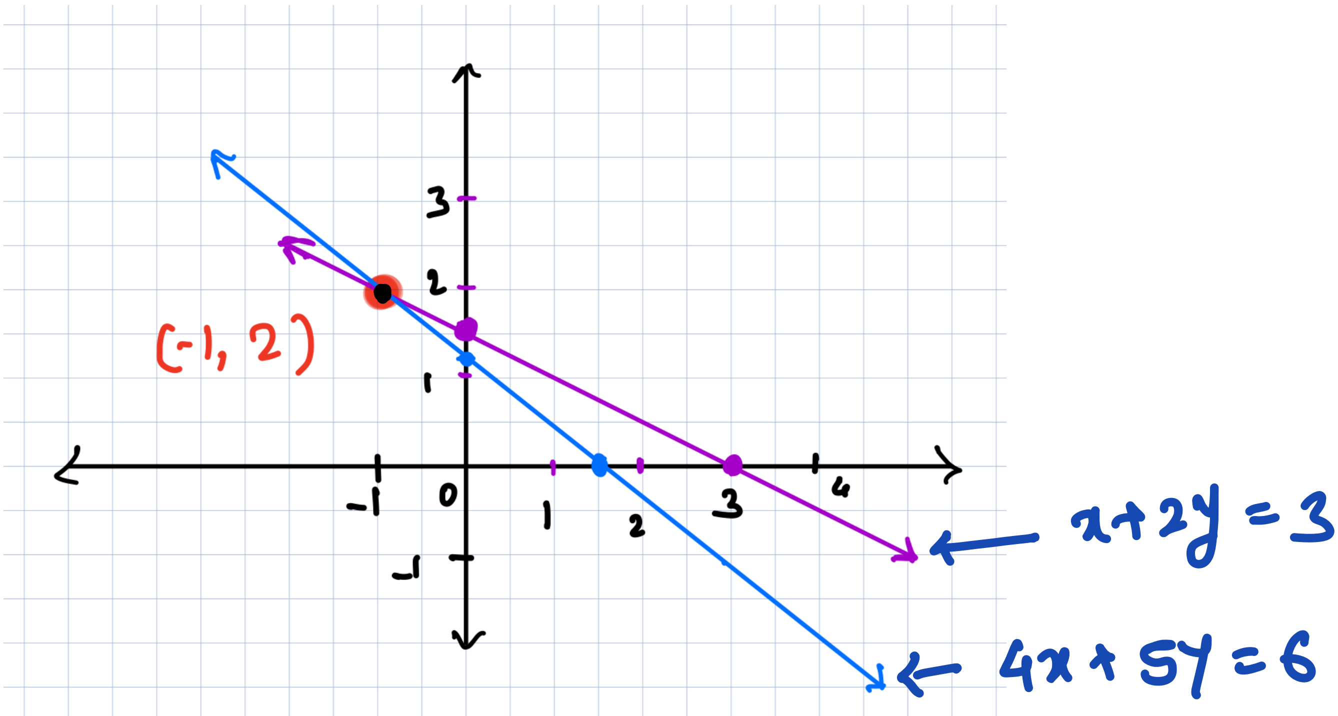
$$x = 3 - 2y$$
$$= 3 - 4 = -1$$

$x = -1, \quad y = 2 \Rightarrow (x, y) = (-1, 2)$

Geometry of the linear equations

$$x + 2y = 3 \tag{3}$$

$$4x + 5y = 6 \tag{4}$$



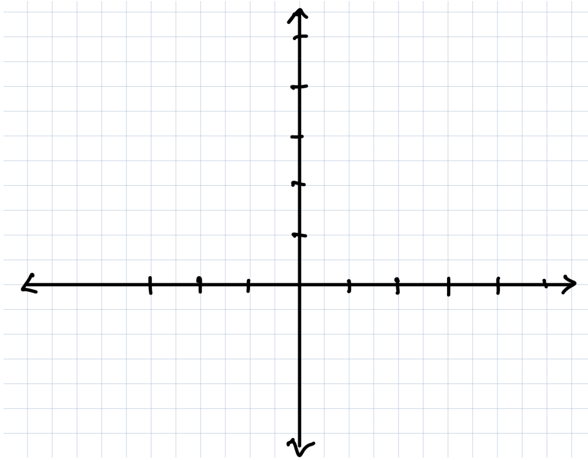
Row and column pictures

Matrix form

$$\left.\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}\right\}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Line



Column vectors & linear combinations

Matrix form

$$\left.\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}\right\} \quad \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

It can be written as

find x, y such that

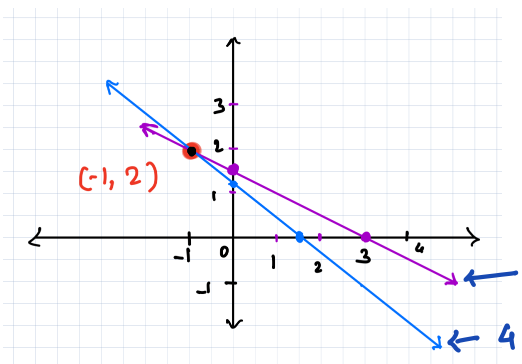
$$\underbrace{x \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\text{column I}} + \underbrace{y \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{Column II}} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

column I , Column II

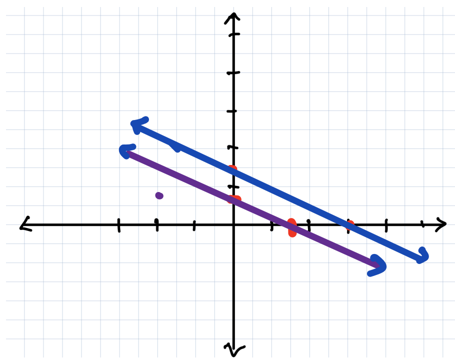
RHS $= \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ is linear combination column I & column II.

Singular Cases

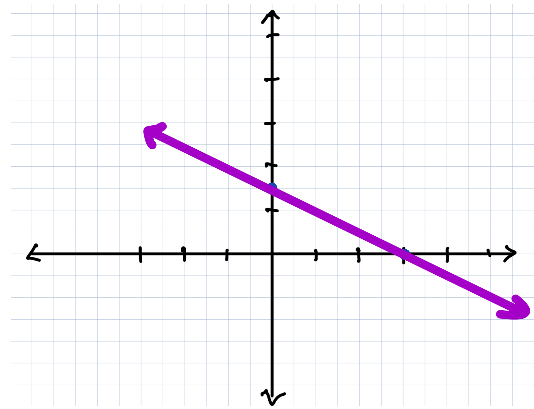
(I)	(II)	(III)
$x + 2y = 3$	$x + 2y = 3$	$x + 2y = 3$
$4x + 5y = 6$	$4x + 8y = 6$	$4x + 8y = 12$



$(x, y) = (-1, 2)$



No solution



Infinitely Many solution

Geometry: three planes

Solve

$$\begin{aligned} 2x + y + z &= 5 \\ 4x - 6y &= -2 \\ -2x + 7y + 2z &= 9 \end{aligned}$$

Geometry: three planes

Solve

$$\text{I} \left[\begin{array}{l} 2x+y+z=5 \quad \text{—} (1) \\ 4x-6y=-2 \quad \text{—} (2) \\ -2x+7y+2z=9 \quad \text{—} (3) \end{array} \right.$$

Eq 2 − 2x Eq (1)
Eq 3 + Eq^n (1) $\Rightarrow$

$$\left. \begin{array}{l} 2x+y+z=5 \quad \text{—} (4) \\ -8y-2z=-12 \quad \text{—} (5) \\ 8y+3z=14 \quad \text{—} (6) \end{array} \right] \text{II}$$

Eq 6 − Eq (5) $\Rightarrow$

$$\left. \begin{array}{l} 2x+y+z=5 \quad \text{—} (7) \\ -8y-2z=-12 \quad \text{—} (8) \\ z=2 \quad \text{—} (9) \end{array} \right\} \text{III}$$

Back substution $\Rightarrow$ $z=2$

put $z=2$ in (8) $\Rightarrow$ $-8y = -12+2z = -12+2\times2$

$y=1$

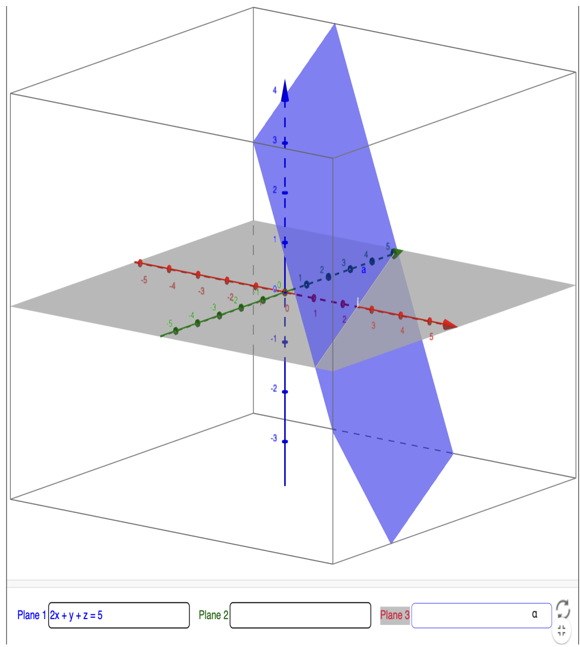
put $y=1$, $z=2$ in eq^n (7) $\Rightarrow$ $x=1$

Solution $(x, y, z) = (1, 1, 2)$

Note that :- system of eq^n I, II & III are equivalent, they obtain from some operations $(\text{I} \rightarrow \text{II} \rightarrow \text{III})$ & solution $(1, 1, 2)$ satisfy all I, II, III.

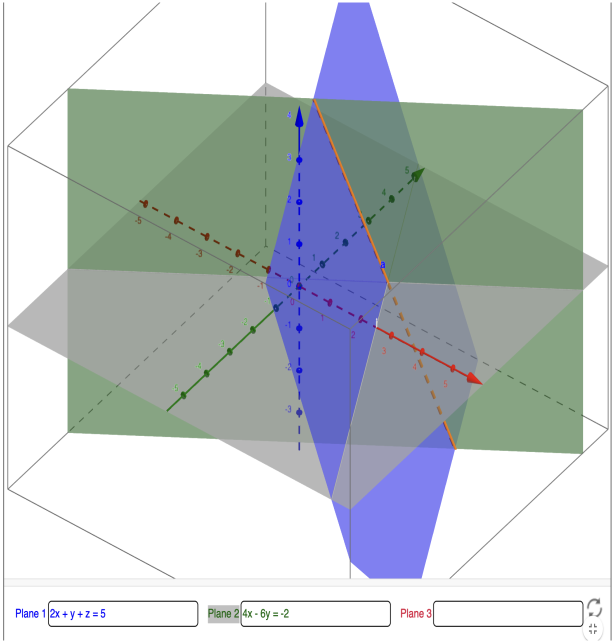
Geometry: three planes

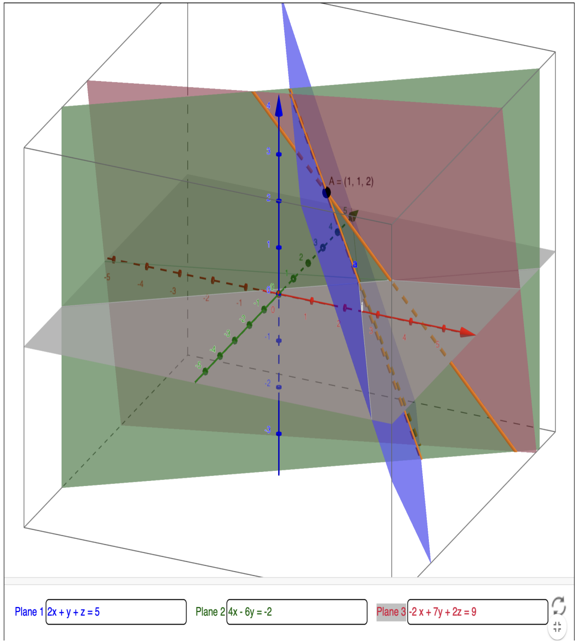
$$
\begin{aligned}
2x + y + z &= 5 \\
4x - 6y &= -2 \\
-2x + 7y + 2z &= 9
\end{aligned}
$$



plot= https://www.geogebra.org/

$$2x + y + z = 5$$
$$4x - 6y = -2$$
$$-2x + 7y + 2z = 9$$





Column vectors & linear combinations

$$
\begin{aligned}
2x + y + z &= 5 \quad \text{—— } ① \\
4x - 6y &= -2 \quad \text{—— } ② \\
-2x + 7y + 2z &= 9 \quad \text{—— } ③
\end{aligned}
$$

matrix form

$$
\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}
$$

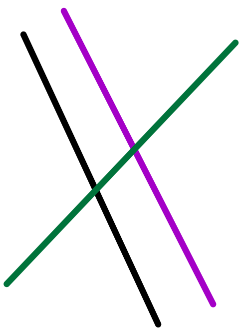
Column form

$$
x \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + y \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}
$$

sol^n $(x, y$

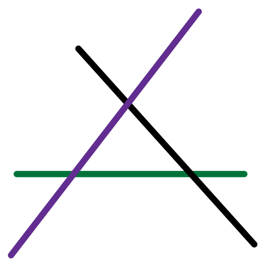
Singular Cases

Possible cases for planes



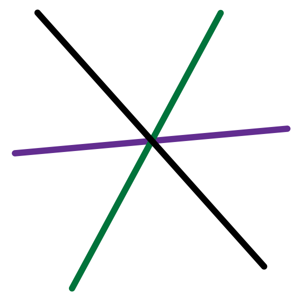
Two planes are Parallel

(a)



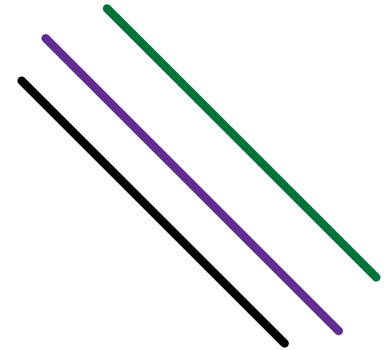
No intersection

(b)



line of Intersection

(c)



all planes parallel

(d)

a, b, d = No solution

(c) = Infinite no. of sol^n.

Summary

- Solving linear systems there are three cases: Unique solution, no solution, infinitely many solution.
- How to find solution to the linear system with more number of unknowns and equations?
- Gauss Elimination Method

Before going to Gauss Elimination Method, lets see some definitions....

System of linear equations

A system of m linear equations in n variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{5}$$

Here, a_{ij}'s, $\quad 1 \leq i \leq m,\ 1 \leq j \leq n$ are the coefficients;
$x_1, x_2, \cdots, x_n$ are the unknown variables,
and $[b_1,\, b_2,\, ,\cdots b_n]^t$ is the right hand side vector.

Matrix form

The linear system (5) can be written as $\boxed{AX = B}$, where

$$\begin{array}{c} (A) \\ A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \end{array} \qquad \begin{array}{c} (X) \\ X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \end{array} \qquad \begin{array}{c} (B) \\ B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \end{array}$$

- A is the coefficient matrix;
- X is a column vector of unknown variables,
- B is the right hand side vector,

Matrix

A **matrix** is a rectangular array of numbers (or functions) enclosed in brackets with the numbers (or functions) called as entries or elements of the matrix.
Matrices are denoted by capital letters A, B, C, $\cdots$.
For integers $m,n \geq 1$, an $m \times n$ matrix is an array given by

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Denoting the matrix above as $A = (a_{ij})_{1 \leq i \leq m,\, 1 \leq j \leq n}$, $(a_{ij}),\ i = 1, \dots, m$ correspond to the m **rows** of the matrix and and $(a_{ij}),\ j = 1, \dots, n$ correspond to the n **columns** of the matrix.
a_{ij} is called as the ij-th **entry** (or **component**) of the matrix.

Row and Column vectors

A matrix consisting of a single row is called a **row vector**.
A **row vector** $(x_1, \ldots, x_n)$ is a $1 \times n$ matrix.
A matrix consisting of a single column is called a **column vector**.
A **column vector** $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is an $n \times 1$ matrix.
Row and column vectors are denoted using lower case letters such as $\mathbf{a}$, $\mathbf{b}$, and so on.
A **square matrix** is a matrix with the same number of rows and columns.
For $n \geq 1$, an $n \times n$ matrix is called a **square** matrix of order n.
A matrix (a_{ij}) is called the **zero matrix** if all its entries are zeroes and is, denoted by $\mathbf{O}$.
The size of the zero matrix should be clear from the context.

Some definitions

- *[Homogeneous & Non-homogeneous Systems]* The linear system (5) is said to be **homogeneous** if $b_i = 0 \ \forall i = 1, 2, \cdots, m$.
  If at least one $b_i \neq 0$ for $1 \leq i \leq m$, then the system is called a **non-homogeneous system.**

- *[Solution]* A **solution** is a set of numbers $x_1, x_2, \cdots, x_n$ that satisfies all the m equations of the system.
- *[Solution Vector]* A solution vector is a vector $\mathbf{x}$ whose components constitute a solution of the system.

System of linear equations

A homogeneous system of m linear equations in n variables :

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \quad \vdots \quad \vdots \; \vdots \; \vdots \; \vdots &= 0 \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0
\end{aligned}$$

has at least the **trivial solution** $x_1 = 0, x_2 = 0, \cdots x_n = 0$. Any other solution, if it exists, is called a **non-zero** or **non-trivial** solution.

Consistent & Inconsistent Systems

The linear system (5) may have

- ◇ NO SOLUTION,
- ■ PRECISELY ONE SOLUTION (Unique Solution)
- ► or INFINITELY MANY SOLUTIONS.

A linear system is

- ■ CONSISTENT, if it has at least one solution;
- ■ INCONSISTENT if it has no solution.

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Singular Cases

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$$x + 2y = 3$$
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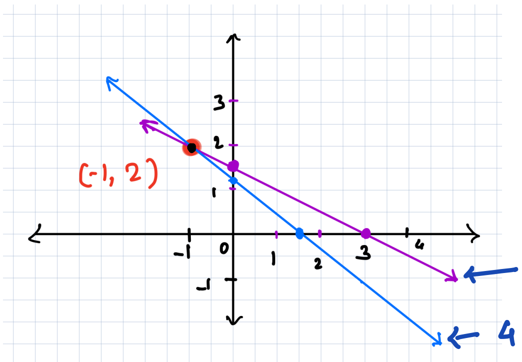
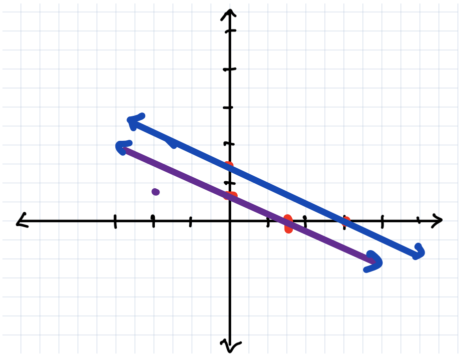


unique solu^n

Consistent

(II)

$$x + 2y = 3$$
$$4x + 8y = 6$$

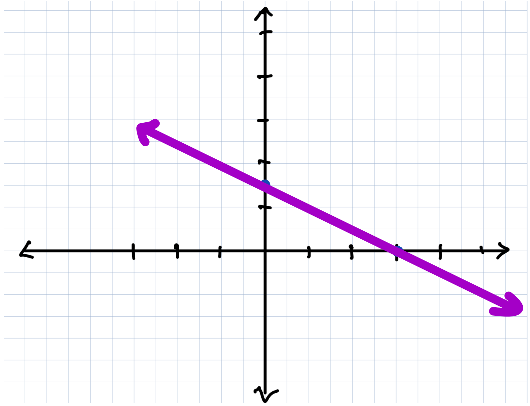


No solu^n

Inconsistent

(III)

$$x + 2y = 3$$
$$4x + 8y = 12$$



Infinitely soln^n.

Consistent

Summary of lecture

- Solving linear systems there are three cases: Unique solution, no solution, infinitely many solution.
- Consistent and Inconsistent linear system
- Homogeneous and Non-homogeneous linear systems
- How to find solution to the linear system with more number of unknowns and equations?
- Gauss Elimination Method

Self-study

- Matrix and Properties
- Matrix Multiplications
- Inverse of Matrix

Solve. Let A, B, C be matrices(equal size) , then

- True or false $AB = BA$, if false, give contour example
- True or false $AB = 0 \implies A = 0 \; or \; B = 0$, if false, give contour example
- True or false $AC = AD \; \& \; A \neq 0 \implies C = D$, if false, give contour example