Lecture 03
Rank of Matrix & Elementary Metrics

MAT 121 - Introduction to Linear Algebra

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April 29, 2022

Recall

- Systems of linear equations
- Homogeneous and non-homogeneous system
- Solution
- Consistent and Inconsistent System
- Gauss Elimination Method
- Elementary Row operations
- Row Echelon form

Outline of lecture

- Reduced Row Echelon form
- Rank of Matrix
- Elementary matrix

REF

Any matrix after a sequence of elementary row operations gets reduced to what is known as a *Row Echelon form*.

> **[Row-Echelon Form (REF)]** A matrix is said to be in a row echelon form (or to be a row echelon matrix) if it has a staircase-like pattern characterized by the following properties:
> (a) The all-zero rows (if any) are at the bottom.
> (b) The first nonzero entry in a nonzero row is called a pivot. The pivot in any row is farther to the right than the pivots in rows above.
> (c) All entries in a column below a leading entry (pivot) is zero.

The first non-zero entry in the j^{th} row is known as the j^{th} *pivot*.

The matrix $\begin{bmatrix} \boxed{3} & 2 & 1 & 3 \\ 0 & \boxed{1} & -1 & 6 \\ 0 & 0 & 0 & \boxed{12} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in REF. The three pivots are indicated.

The matrix $\begin{bmatrix} \boxed{3} & 2 & 1 & 3 \\ 0 & \boxed{1} & -1 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{12} \end{bmatrix}$ is NOT in REF.

Reduced REF

- A matrix in REF can be further row-operated upon to ensure that (i) each pivot becomes 1
  and (ii) all the entries above each pivot become 0.
  This is the *reduced* REF and it is unique.
- Reduced REF is mainly of theoretical interest only.

1. $$\begin{bmatrix} 2 & 3 & -2 & 4 \\ 0 & -9 & 7 & -8 \\ 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is in REF, whereas
2. $$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 20 \end{bmatrix}$$ is in RREF.

Reduced REF

- The all-zero rows (if any) are at the bottom.
- The first nonzero entry in a nonzero row is called a pivot and each pivot becomes 1. The pivot in any row is farther to the right than the pivots in rows above.
- All entries in a column below and above of the pivot is zero

Note that: Reduced REF is mainly of theoretical interest only.

Ex. Reduce following matrix in RREF

1. $\begin{bmatrix} 5 & 3 & 1 & 4 \\ 0 & -1 & 4 & -8 \\ 3 & 2 & 1 & 20 \end{bmatrix}$ is in RREF.

More Examples - REF, RREF

Determine whether the following matrices in REF, RREF :

1. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$ REF not RREF.

2. $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$ is in RREF. Is it in REF?? YES.

3. The matrix $\begin{bmatrix} 2 & -1 & 2 & 1 & 5 \\ 0 & 1 & 1 & -3 & 3 \\ 0 & 2 & 0 & 0 & 5 \\ 0 & 0 & 0 & 3 & 2 \end{bmatrix}$ is NOT in REF.

4. $\begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 1 & -1/3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$ is in RREF and hence in REF.

Note that the left side of a matrix in RREF need not be identity matrix.

Remarks

1. Any reduced row-echelon matrix is also a row-echelon matrix.
2. Any nonzero matrix may be row reduced into more than one matrix in echelon form. But the reduced row echelon form that one obtains from a matrix is unique.
3. A pivot is a nonzero number in a pivot position that is used as needed to create zeros with the help of row operations.
4. Different sequences of row operations might involve a different set of pivots.
5. Given a matrix A, its REF is NOT unique. However, the position of each of its pivots is unalterable.

A quick quiz!

Determine whether the following statements are true or false.

1. If a matrix is in row echelon form, then the leading entry of each nonzero row must be 1.
2. If a matrix is in reduced row echelon form, then the leading entry of each nonzero row is 1.
3. Every matrix can be transformed into one in reduced row echelon form by a sequence of elementary row operations.
4. If the reduced row echelon form of the augmented matrix of a system of linear equations contains a zero row, then the system is consistent.
5. If the only nonzero entry in some row of an augmented matrix of a system of linear equations lies in the last column, then the system is inconsistent.
6. If the reduced row echelon form of the augmented matrix of a consistent system of m linear equations in n variables contains r nonzero rows, then its general solution contains r basic variables. (number of free variables $= n - r$).

Observation from last lectures Examples = system $\longrightarrow$ RE form

$[A|B] \to$ Row Echelon forms were (suppose diagonal x are some non-zero entries)

$$\left[\begin{array}{ccc|c} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{array}\right] , \quad \left[\begin{array}{ccc|c} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & 0 & 0 \end{array}\right] , \quad \left[\begin{array}{ccc|c} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & 0 & x \neq 0 \end{array}\right]$$

$\Downarrow$ Unique soln

$\Downarrow$ Infinitely many soln

No solution

REF

$$A_{m\times n} X_{n\times 1} = B_{m\times 1}$$

The $m \times n$ coefficient matrix A of the linear system $A\mathbf{x} = \mathbf{b}$ is thus reduced to an $(m \times n)$ row echelon matrix U and the augmented matrix $[A|\mathbf{b}]$ is reduced to

$$[U|\mathbf{c}] = \left[\begin{array}{cccccccccccc|c}
0 & \dots & p_1 & * & * & * & * & * & * & * & \dots & * & c_1 \\
0 & \dots & 0 & \dots & p_2 & * & * & * & * & * & \dots & * & c_2 \\
0 & \dots & 0 & 0 & 0 & \dots & p_3 & * & * & * & \dots & * & c_3 \\
\vdots & & \vdots & \vdots & \vdots & 0 & 0 & \dots & \vdots & \vdots & & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & p_r & * & \dots & * & c_r \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & c_{r+1} \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & c_m
\end{array}\right].$$

The entries denoted by $*$ and the c_i's are real numbers; they may or may not be zero. The p_i's denote the pivots; they are non-zero. Note that there is exactly one pivot in each of the first r rows of U and that any column of U has at most one pivot. Hence $r \leq m$ and $r \leq n$.

Consistent & Inconsistent Systems

If $r < m$ (the number of non-zero rows is less than the number of equations) and $c_{r+k} \neq 0$ for some $k \geq 1$, then the $(r+k)$th row corresponds to the self-contradictory equation $0 = c_{r+k}$ and so the system has **no solutions** (inconsistent system).

If (i) $r = m$ or (ii) $r < m$ and $c_{r+k} = 0$ for all $k \geq 1$, then there exists a solution of the system (consistent system).

Definition (Basic & Free Variables)

If the jth column of U contains a pivot, then x_j is called a **basic variable**; otherwise x_j is called a **free variable**. (non-pivotal)

In fact, there are $n - r$ free variables, where n is the number of columns (unknowns) of A (and hence of U).

Rank of matrix

Definition (Rank)

Let A be an $m \times n$ matrix and $\widehat{A}$ be any of its row echelon forms. The rank of A is the number of pivots in $\widehat{A}$.

This definition is ad-hoc and not rigorously justified since REF is not unique.

The justification will be given in due course. We will denote the rank of A by $\text{rank}(A)$. Immediate to observe that

$$\text{rank}(A) \leq m \text{ and } \text{rank}(A) \leq n.$$

Observation from last Lecture's Examples = system $\longrightarrow$ RE form

$[A|B] \rightarrow$ Row Echelon forms were (suppose . diagonal $\times$ are some non-zero entries)

$$\left[\begin{array}{ccc|c} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{array}\right], \quad \left[\begin{array}{ccc|c} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & 0 & \underset{\neq 0}{\times} \end{array}\right]$$

$\Downarrow$ Unique solun

$\Downarrow$ Infinitely many solun

No solution

Rank $A = 3$

Rank $[A|b] = 3$

Rank $A = 2$

Rank $[A|b] = 2$

$\Downarrow$ Consistent

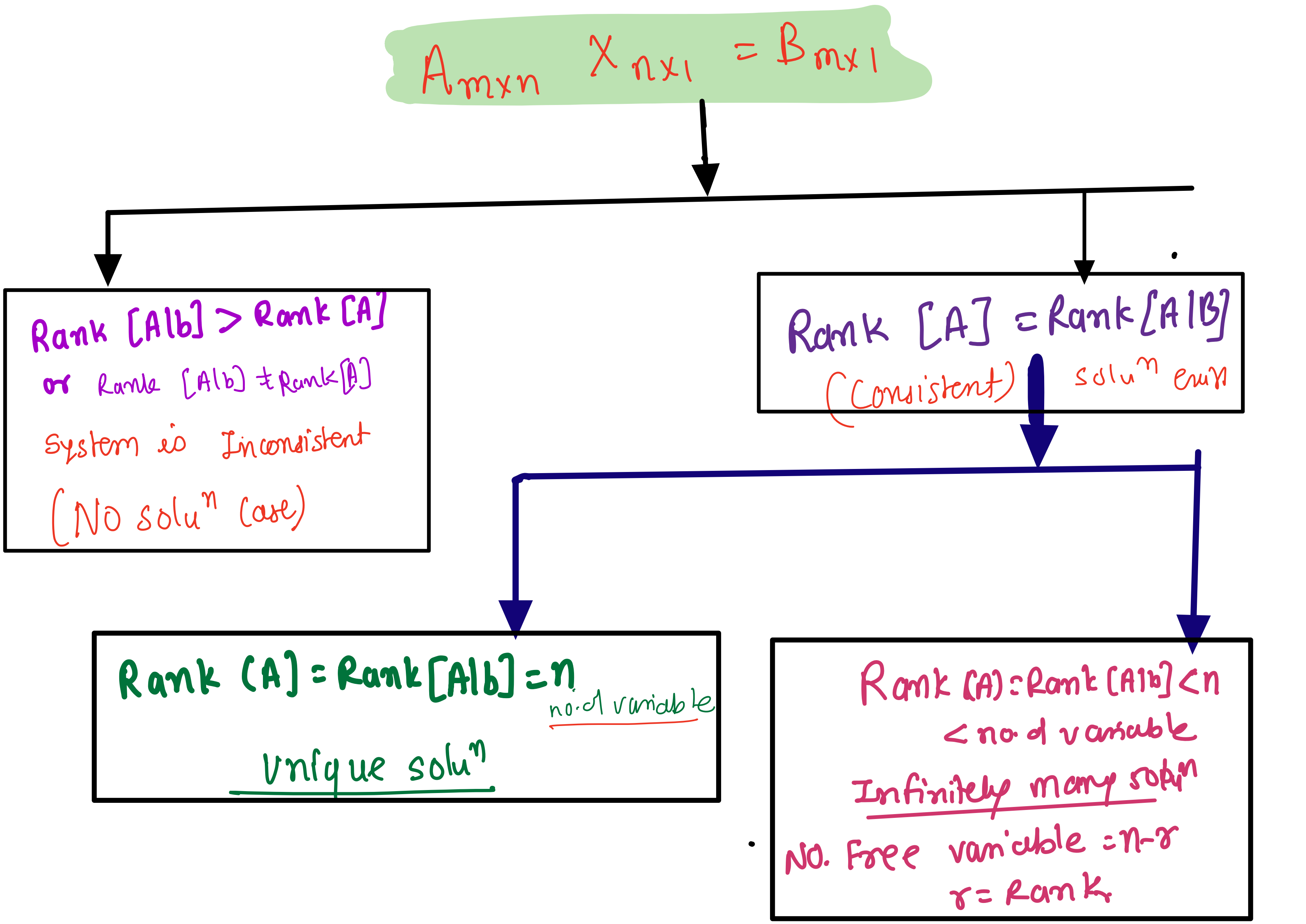
Rank $A = 2$

Rank $[A|b] = 3$

F

$\Downarrow$

Inconsistent.


$A_{m\times n} X_{n\times 1} = B_{m\times 1}$
Rank [A|b] > Rank [A]
or Rank [A|b] ≠ Rank [A]
System is Inconsistent
(No solu^n case)
Rank [A] = Rank [A|B]
(Consistent) solu^n exist
Rank (A] = Rank [A|b] = n
no. of variable
Unique solu^n
Rank (A) = Rank [A|b] < n
< no. of variable
Infinitely many solu^n
No. Free variable = n − r
r = Rank

Homo. System

$A_{m\times n} X_{n\times 1} = 0$

Rank A = Rank $[A|0]$

(always consistent)

Rank $A = n$

Unique solu^n

$\Rightarrow$ trivial solu^n

Rank $A < n$

Infinitely many solu^n.

Solvability of a linear system

Theorem

Let $A\mathbf{x} = \mathbf{b}$ be a given system of linear equations. Let $A^+ = [A|\mathbf{b}]$ denote the augmented matrix.

1. *(Existence) The solution set is non-empty if and only if* $\operatorname{rank}(A) = \operatorname{rank}(A^+)$.
2. *(Uniqueness) The system has a unique solution if and only if* $\operatorname{rank}(A) = \operatorname{rank}(A^+) = n$.
3. *(Non-uniqueness) The system has infinitely many solutions if and only if* $\operatorname{rank}(A) = \operatorname{rank}(A^+) < n$.
4. *(Gauss elimination or Completeness) If* $\operatorname{rank}(A) = \operatorname{rank}(A^+)$, *Gauss elimination method gives the complete set of solutions.*

Proof of the theorem

Let $A^+ = [A|\mathbf{b}]$ be the augmented matrix and $\widehat{A^+}$ be a row-echelon form of A^+. Then $\widehat{A^+} = [\widehat{A}|\widehat{\mathbf{b}}]$ where perforce, $\widehat{A}$ is a REF of A.

1. (Existence): If $\text{rank}(A^+) = \text{rank}(A)$, then there can be no pivot in the last column (augmented part) and we can solve the system by back substitutions.

Conversely, if $\text{rank}(A^+) \neq \text{rank}(A)$, then perforce $\text{rank}(A^+) = \text{rank}(A) + 1$ and the last pivot is in the augmented column. The corresponding equation will read

$$0 = \text{last pivot} \neq 0$$

and hence the system is inconsistent.

Proof of the theorem contd.

2. (Uniqueness): If $\text{rank}(A) = \text{rank}(A^+) = n$, then since the number of columns in A is n, the reduced REF will look like $\widehat{A^+} = \left[\begin{array}{c|c} \mathbf{I}_n & \widehat{\mathbf{b}} \\ \hline [\mathbf{0}] & \mathbf{0} \end{array}\right]$ and the unique solution is $\mathbf{x} = \widehat{\mathbf{b}}$. In the obvious notations, $x_j = \widehat{b}_j$.

3. (Non-uniqueness): If $\text{rank}(A) = \text{rank}(A^+) = r < n$, $n - r$ variables out of $x_1, \ldots, x_n$ will be free to take any values, thereby giving infinitely many solutions (non-uniqueness).

The $(n - r)$ free variables correspond to the pivot-free columns.

4. (Gauss elimination or completeness): Since each row opretation is reversible, the system in REF is equivalent to the original. Hence we get all the solutions by GEM.

Solvability

Let $A\mathbf{x} = \mathbf{b}$ be a given system of linear equations. Let $A^+ = [A|\mathbf{b}]$ denote the $m \times (n+1)$ augmented matrix.

Rank A	Rank $[A|b]$	Cases	Solution set
r	$r+1$		Empty
r	r	$r < n$	Infinite set
r	r	$r = n$	Singleton set
r	r	$r > n$	????

[1.5]

Elementary Matrices

An $m \times m$ elementary matrix is a matrix obtained from the $m \times m$ identity matrix I_m by one of the elementary operations; namely,

1. interchange of two rows,
2. multiplying a row by a non-zero constant,
3. adding a constant multiple of a row to another row.

That is, an elementary matrix is a matrix which differs from the identity matrix by one single elementary row operation.

Ex. Elementary Matrix

(1) $E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xleftarrow{R_{23}} \quad \xleftarrow[R_2 \& R_3]{\text{Interchange}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(2) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow \xleftarrow[\text{one element}]{7 \times R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(3) $E = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow \xleftarrow{R_2 + 5R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

1. Row Switching Operation

Interchange of two rows - $\boxed{R_i \leftrightarrow R_j}$.

The elementary matrix P_{ij} corresponding to this operation on I_m is obtained by swapping row i and row j of the identity matrix.

$$P_{ij} = \begin{array}{c} \\ R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_m \end{array} \begin{array}{c} \begin{array}{ccccccc} C_1 & \dots & C_i & \dots & C_j & \dots & C_m \end{array} \\ \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \dots & 1 & & \\ & & & \ddots & & & \\ & & 1 & \dots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

Example : Consider I_3. $R_1 \leftrightarrow R_2$ gives $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. Row Multiplying Transformation

Multiplying a row by a non-zero constant - $\boxed{R_i \longrightarrow kR_i}$. The elementary matrix $M_i(k)$ corresponding to this operation on I_m is obtained by multiplying row i of the identity matrix by a non-zero constant k.

$$M_i(k) = \begin{array}{c} \\ R_1 \\ R_2 \\ \vdots \\ R_i \\ \vdots \\ R_m \end{array} \begin{array}{c} \begin{array}{cccccc} C_1 & C_2 & \cdots & C_i & \cdots & C_m \end{array} \\ \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & k & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \end{array}$$

Example : Consider I_3. $R_3 \to 7R_3$ gives $M_3(7) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

3. Row Addition Transformation

$\boxed{R_i \longrightarrow R_i + kR_j}$: The elementary matrix $E_{ij}(k)$ corresponding to this operation on I_m is obtained by multiplying row j of the identity matrix by a non-zero constant k and adding with row i.

$$E_{ij}(k) = \begin{array}{c} \\ R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \\ R_m \end{array} \begin{array}{c} \begin{array}{ccccccc} C_1 & \dots & C_i & \dots & C_j & \dots & C_m \end{array} \\ \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & k & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \end{array}$$

Example : $R_2 \to R_2 + (-3)R_1$ for I_3 gives $E_{21}(-3) = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Proposition 1

Let A be an $m \times n$ matrix. If $\widetilde{A}$ is obtained from A by an elementary row operation, and $\mathcal{E}$ is the corresponding $m \times m$ elementary matrix, then $\mathcal{E}A = \widetilde{A}$.

Lets check for matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ l & m & n \end{bmatrix}$

Case I

$$A \xrightarrow[\text{Interchange of Rows}]{R_{23}} \begin{bmatrix} a & b & c \\ l & m & n \\ d & e & f \end{bmatrix} = \tilde{A},$$ then corresponding Elementary matrix is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ ← derived by change R_{23} in Idetn matrix

Now

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ l & m & n \end{bmatrix} = \begin{bmatrix} a & b & c \\ l & m & n \\ d & e & f \end{bmatrix} = \tilde{A}$$

Hence $EA = \tilde{A}$

Case II, multiply by $_{\wedge}$ constant no. to some row
non-zero.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ l & m & n \end{bmatrix} \xrightarrow{7R_2} \begin{bmatrix} a & b & c \\ 7d & 7e & 7f \\ l & m & n \end{bmatrix} = \tilde{A}$$

corresponding $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ — obtained by $I \xrightarrow{7 * R_{II}} E$

Now

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ l & m & n \end{bmatrix} = \begin{bmatrix} a & b & c \\ 7d & 7e & 7f \\ l & m & n \end{bmatrix}$$

$$= \tilde{A}.$$

Case III $\qquad A \xrightarrow{R_2 + 5R_1} \tilde{A}$

Find corresponding "E" elementary matrix & verify

$EA = \tilde{A}$.

Reduced Row Echelon form contd.

$[A]_{m \times n}$

Row operation I, II, III

Convert

B ← Row Echelon Form

corresponding Elementary matrix

A

E_1

E_2

E_3

$E \; m \times n$

$B =$

$B = E_3 E_2 E_1 A$

Exercise: Let A be an $m \times n$ matrix. There exist elementary matrices $E_1, E_2, \ldots, E_N$ of order m such that the product $E_N \cdots E_2 E_1 A$ is a row echelon form of A.

$= B$

$B = E_N E_{N-1} - - - E_2 E_1 A$

Summary

1.
2.
3.
4.