Conditioned & ill Conditioned problems

STABILITY :- small error in input

$$\Rightarrow$$ Result in small error in output

$\Rightarrow$ stable otherwise unstable

Well-Conditioned :-
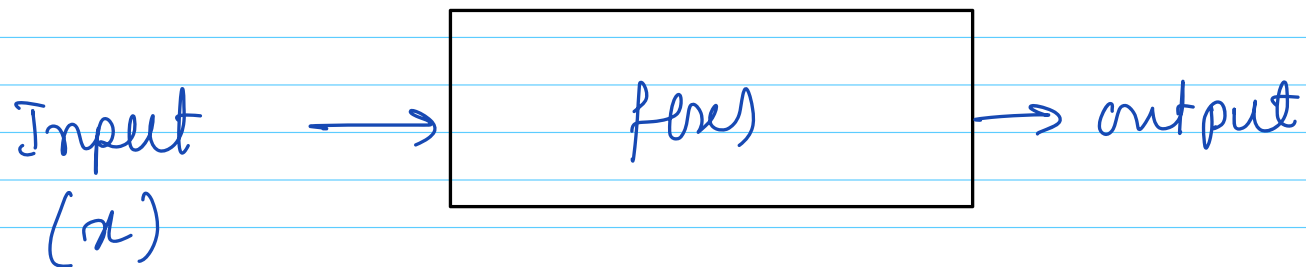
small change in input $\Rightarrow$ small change in output

ill-Conditioned :-

small change input $\longrightarrow$ Large variation in output

functions

Input $\longrightarrow$ | $f(x)$ | $\longrightarrow$ output
(x)

Ex $f(x) = x$ $x = 0.001$ $x_A = 0.0012$

$$|x - x_A| \leq 0.0002 = 2 \times 10^{-4}$$

$$\Rightarrow |f(x) - f(x_A)| \leq 2 \times 10^{-4}$$

$$f(x) = \tan x$$

$$x_1 = \frac{\pi}{2} - 0.001 \qquad \& \quad x_2 = \frac{\pi}{2} - 0.002$$

Absolute Error $|x_1 - x_2| = 0.001$ $\&$ $|F(x_1) - F(x_2)| \leq 500$

Relative Error.

$$|\tan x_1 - \tan x_2|$$

500

So small Change in x- values leads to

Large difference in f. values.

(why?)

Why? bound RA f in RA of x.

Taylor's series $\dfrac{|x - x_A|}{|x|}$

$$f(x + \Delta x) \simeq f(x) + \Delta x \, f'(x)$$

$$f(x + \Delta x) - f(x) = \Delta x \, f'(x)$$

$$\frac{\left| f(x + \Delta x) - f(x) \right|}{|f(x)|} \leq \left| \Delta x \, \frac{f'(x)}{f(x)} \right|$$

$x, x + \Delta x$

RA

RA

$f(x)$

$$\frac{|f(x + \Delta x) - f(x)|}{|f(x)|} \leq \frac{x f'(x)}{f(x)} \left| \frac{\Delta x}{x} \right|$$

RA (x)

$$\boxed{\left| \frac{f(x + \Delta x) - f(x)}{f(x)} \right| \simeq \left| \frac{x f'(x)}{f(x)} \right| \left| \frac{\Delta x}{x} \right|}$$

$$E(f) = 10^5 \times 0.001$$

$$= -$$

$$R\overset{\lor}{\theta}(f) \leq C \ RA(x)^{-K+L(x)} \ \checkmark$$

Ex. $f(x) = \dfrac{10}{1-x^2}$

for which values of
'x' f is ill-conditioned
$=$
why ?
$\checkmark$

$$\Rightarrow \frac{x f'(x)}{f(x)} = \frac{2x}{1-x^2} \longrightarrow \infty$$

$$x = \pm 1 \qquad x \rightarrow \pm 1$$

Roots of Polynomials

EX 1 = $x^3 - 21x^2 + 120x - 100 = 0$

Roots = 1, 10, 10

② $0.99 \, x^3 - 21x^2 + 120x - 100 = 0$ (1 % change in coefficient
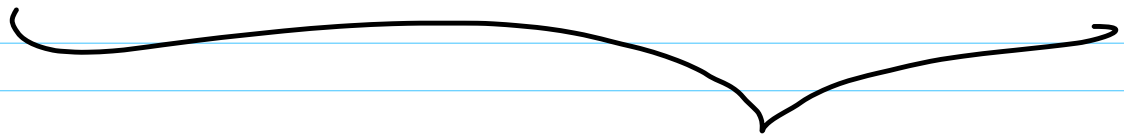
Roots = 1, 11.17, 9.041

10%

③ $1.01 \, x^3 - 21x^2 + 120x - 100 = 0$

Roots are 1, $9.898 \pm 1.044j$

Google "Wilkinson's Polynomial"

Eigenvalue Problem

$$A = \begin{bmatrix} 10 & 100 & 0 & 0 \\ 0 & 10 & 100 & 0 \\ 0 & 0 & 10 & 100 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Eigenvalues $= 10, 10, 10, 10$.

$$A = \begin{bmatrix} 10 & 100 & 0 & 0 \\ 0 & 10 & 100 & 0 \\ 0 & 0 & 10 & 100 \\ 10^{-6} & 0 & 0 & 10 \end{bmatrix}$$

$= 11, \quad 10 \pm i^{\circ}, \quad 9$

clearly Eigenvalues are very sensitive to Perturbation of matrix entries

(1) $\quad I(a,b) = \int\limits_{-10}^{10} a e^x - b e^{-x}$

$I(1,1) = 0 \qquad \qquad \qquad f \; I(1, 1.01) \approx -220$

Integral sensitive is very sensitive to the coefficients

(2) $\quad x_0 = 1, \quad x_1 = \frac{1}{3} \; , \; x_n = \frac{13}{3} x_{n-1} - \frac{4}{3} x_{n-2} \qquad i = 2, 3, \ldots$

Exact $\text{solu}^n = \frac{1}{3^i}$

write an code & verify $\left(\text{like } n = 50 \right)$

it gives some growing solu^n.

$\left(\because x_0, x_1 \; \text{not represent properly in machine} \right)$

$$\mathcal{L}inear \ Systems.$$

solve

① $\begin{bmatrix} 99 & 98 \\ 100 & 99 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 197 \\ 199 \end{bmatrix}$ $\qquad x_1 = x_2 = 1$

$\Rightarrow \ ||A||_\infty =$

$||A^{-1}||_\infty$

Perturbed problem

② $\begin{bmatrix} 98.99 & 98 \\ 100 & 99 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 197 \\ 199 \end{bmatrix}$ $\qquad \begin{aligned} x_1 &= 100 \\ x_2 &= -99 \end{aligned}$

$\dfrac{|| \ x - \tilde{x} \ ||}{||x||} \lessgtr \ ?$

$$A \; X = B$$

① $$\begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 1 \end{bmatrix}$$

$= B$

$x_1 = 0$

$x_2 = 0.1$

$$\Rightarrow \begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.69 \\ 1.01 \end{bmatrix}$$

$\tilde{B}$

$x_1 = -0.17$

$x_2 = 0.22$

$$\frac{||x - \tilde{x}||_\infty}{||x||_\infty} \leq ||A||_\infty ||A^{-1}|| \frac{||b - \tilde{b}||_\infty}{||b||_\infty}$$

$\begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}_\infty$

$= 0.01$

final Condition

$$K(A) = \frac{\|A\|_\infty \|A^{-1}\|_\infty}{}$$

$$= 17 \times 17 = 289$$

$$\frac{\|x - \tilde{x}\|_\infty}{\|x\|_\infty} \leq 289 \times 0.01 \quad ①$$

① $A X = B$ ✓

② $(A+E) \tilde{X} = B$ $\| A - (A+E) \|$

Show $\dfrac{\|\tilde{X} - X\|}{\|X\|} \leq \| A \| \, \| A^{-1} \| \; \dfrac{\| E \|}{\| A \|}$

$R(\tilde{X}) \leq K(A) = \| A \| \, \| A^{-1} \|$ $R(A)$

① $-$ ②

$A X - A \tilde{X} - E \tilde{X} = 0$

$A (X - \tilde{X}) = E \tilde{X}$

$$(x - \tilde{\tilde{x}}) = A^{-1} E \tilde{x}$$

$$x - \tilde{x} = A^{-1} E x + A^{-1} E (\tilde{x} - x)$$

$$\| x - \tilde{x} \| \leq \| A^{-1} E x \|$$

$$\| x - \tilde{x} \|$$