In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
import math
from timeit import timeit
from scipy.stats import binom
from collections import deque
```

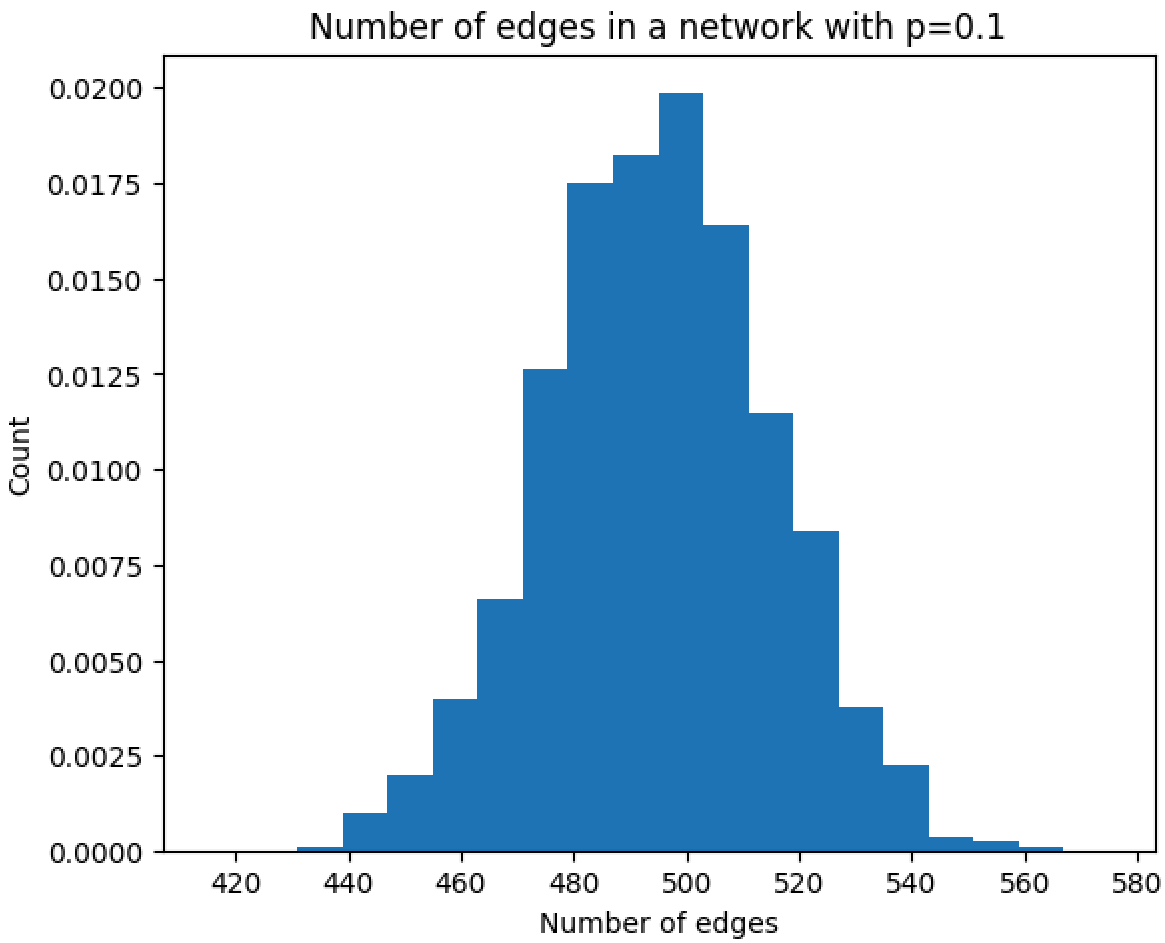
Q1

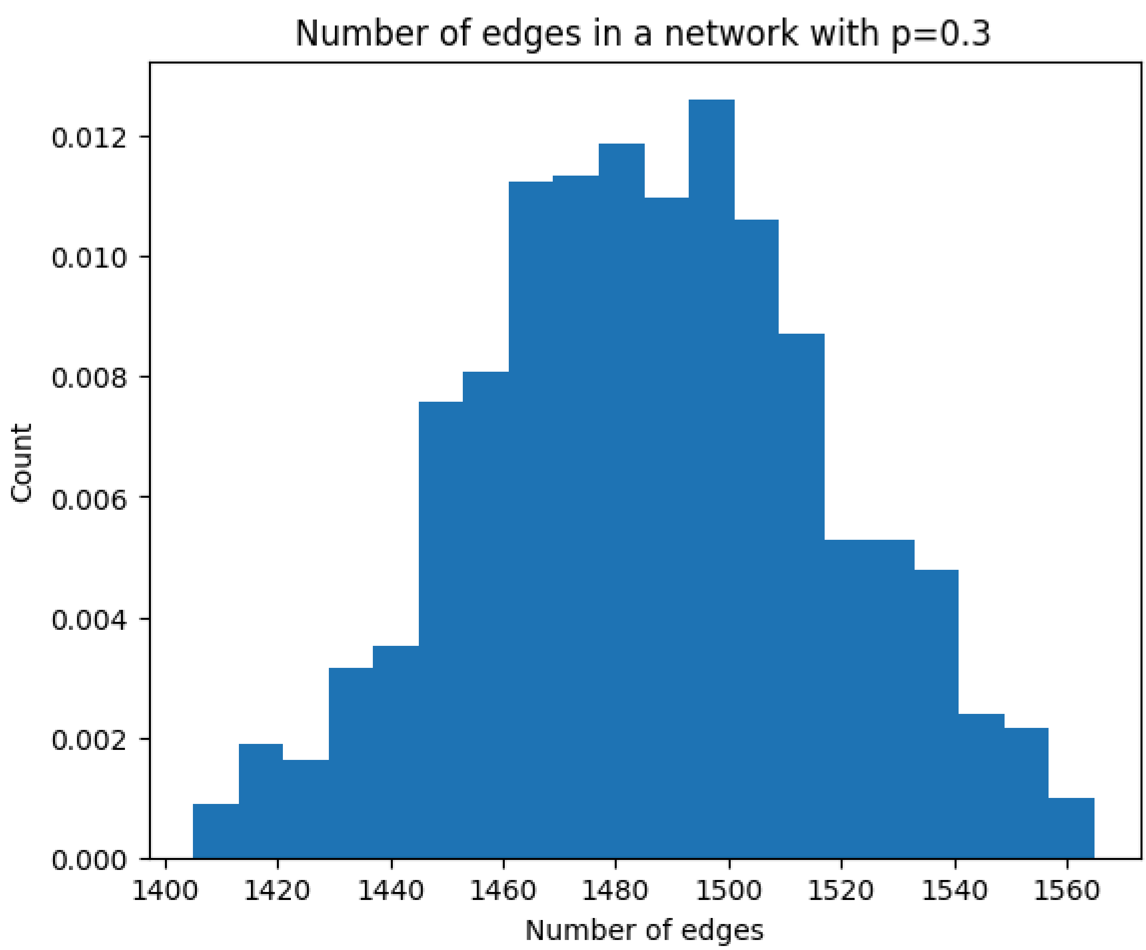
In [ ]:
```
class Network():
    def __init__(self,num_nodes):
        self.adj = {i:set() for i in range(num_nodes)}
        self.num_edge = 0

    def add_edge(self,i,j):
        self.adj[i].add(j)
        self.adj[j].add(i)
        self.num_edge+=1

    def neighbors (self,i):
        return self.adj[i]

    def edge_list(self):
        return [(i,j) for i in self.adj for j in self.adj[i] if i<j]
```

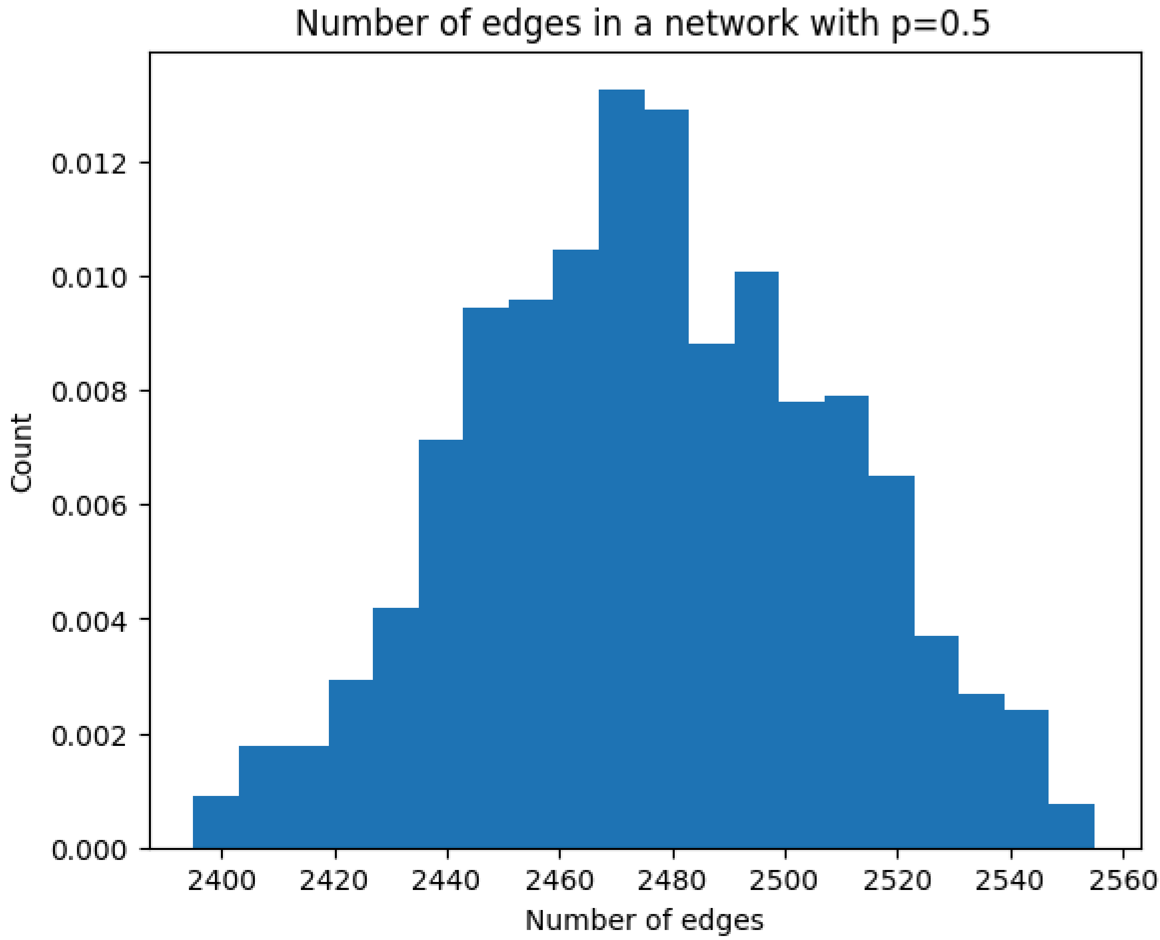
In [ ]:
```
class Bernoulli_Network(Network):

    def __init__(self, num_nodes,prob):
        super().__init__(num_nodes)

        for i in range(num_nodes):
            for j in range(i+1,num_nodes):

                if np.random.binomial(1,prob):
                    self.add_edge(i,j)
```

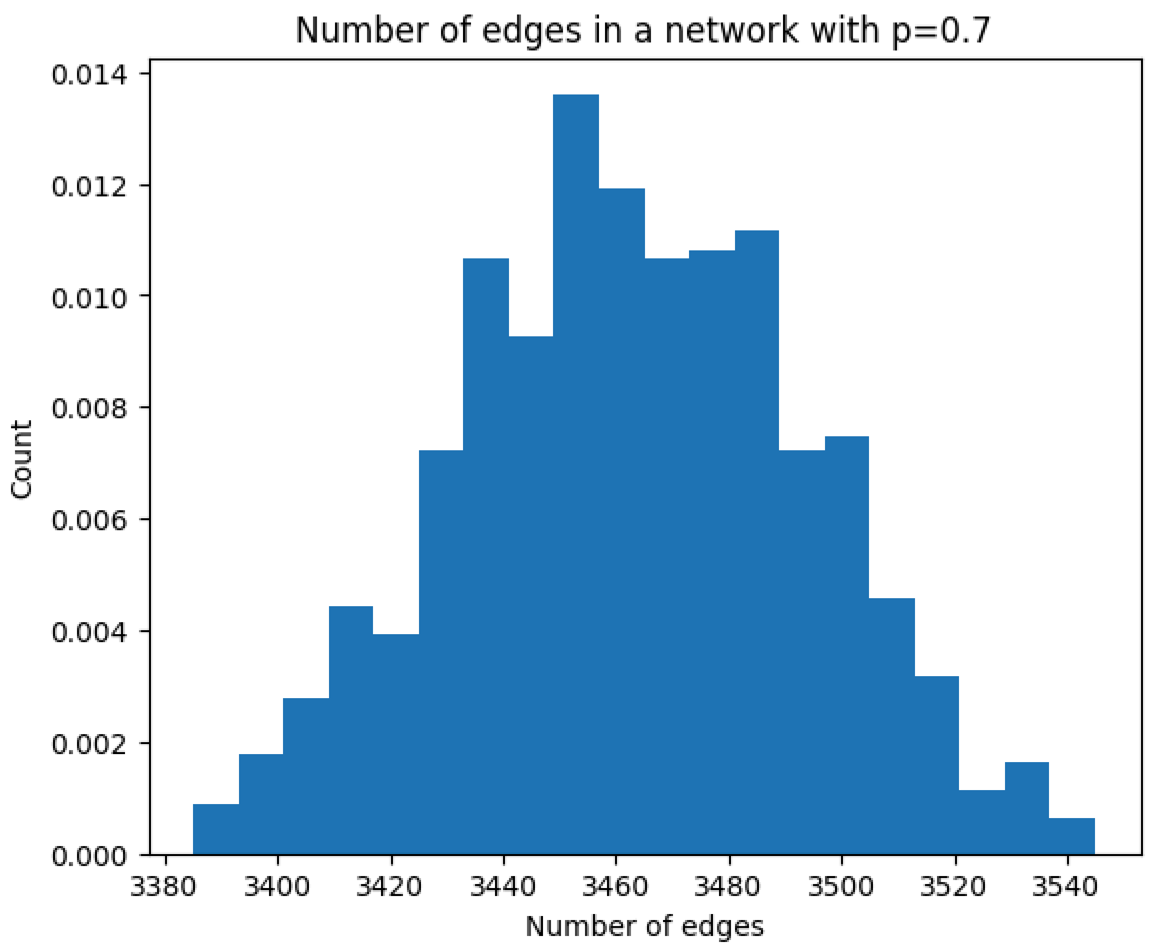
In [ ]:
```
np.random.seed(1)

n = 100 #number of nodes
probs = [0.1,0.3,0.5,0.7,0.9]
sample_size = 1000

fig, ax = plt.subplots(nrows=len(probs),ncols=1)
fig.set_figheight(30)

m_arrays=[] #for Q2

for i,prob in enumerate(probs):
    m_array=[]

    for _ in range(sample_size):
```

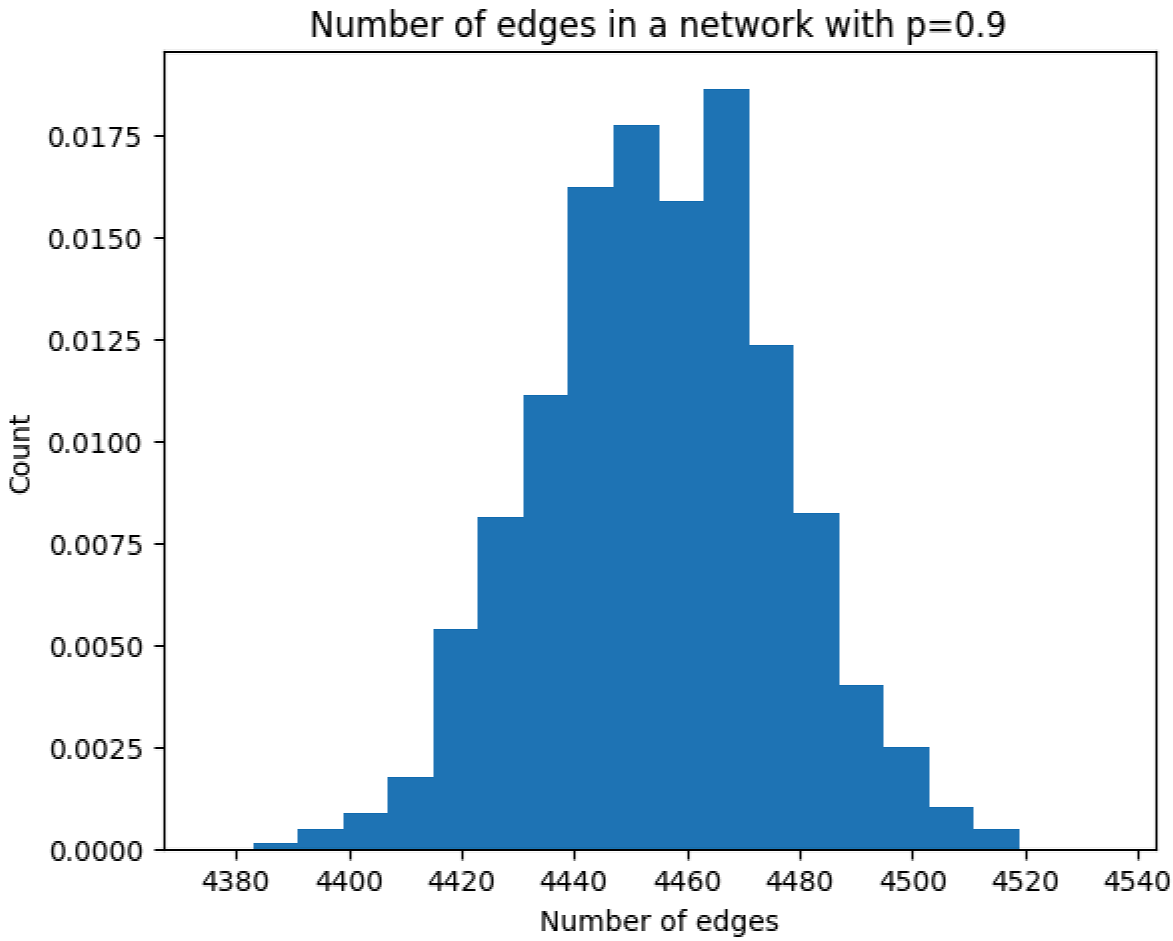
```
        network = Bernoulli_Network(n,prob)
        m_array.append(network.num_edge)

    m_arrays.append(m_array)

    ax[i].hist(m_array,bins=20,range=(prob*math.comb(n,2)-80,prob*math.comb(n,2)+80
    ax[i].set_xlabel('Number of edges')
    ax[i].set_ylabel('Count')
    ax[i].set_title('Number of edges in a network with p={}'.format(prob))

plt.show()
```


Number of edges in a network with p=0.1
Count
Number of edges
0.0000
0.0025
0.0050
0.0075
0.0100
0.0125
0.0150
0.0175
0.0200
420
440
460
480
500
520
540
560
580


Number of edges in a network with p=0.3
Count
Number of edges
0.000
0.002
0.004
0.006
0.008
0.010
0.012
1400
1420
1440
1460
1480
1500
1520
1540
1560


Number of edges in a network with p=0.5
Count
Number of edges
0.000
0.002
0.004
0.006
0.008
0.010
0.012
2400
2420
2440
2460
2480
2500
2520
2540
2560


Number of edges in a network with p=0.7
Count
Number of edges
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0.014
3380
3400
3420
3440
3460
3480
3500
3520
3540



Q2

For probability p and number of edge 0<=m<=(nC2). Plot probability P(m)

In [ ]:
```
def pmf_num_edge (num_nodes,prob):

    total_edge = math.comb(num_nodes,2)
    m_range = [m for m in range(total_edge+1)]

    return binom.pmf(m_range,total_edge,prob)
```

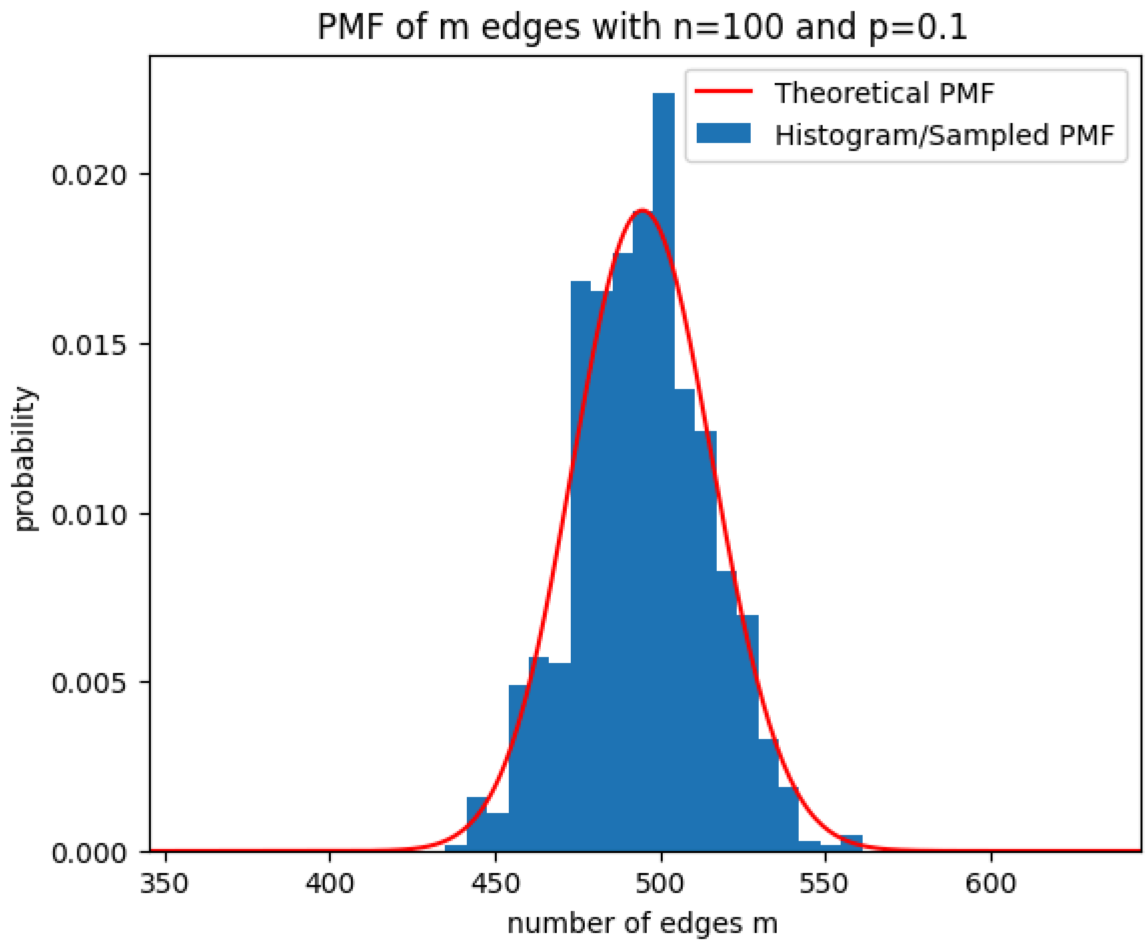
In [ ]:
```
def mean(X,P_X):
    '''Mean for discrete random variable X with PDF P_X'''

    return sum([x*p_x for x,p_x in zip(X,P_X)])
```

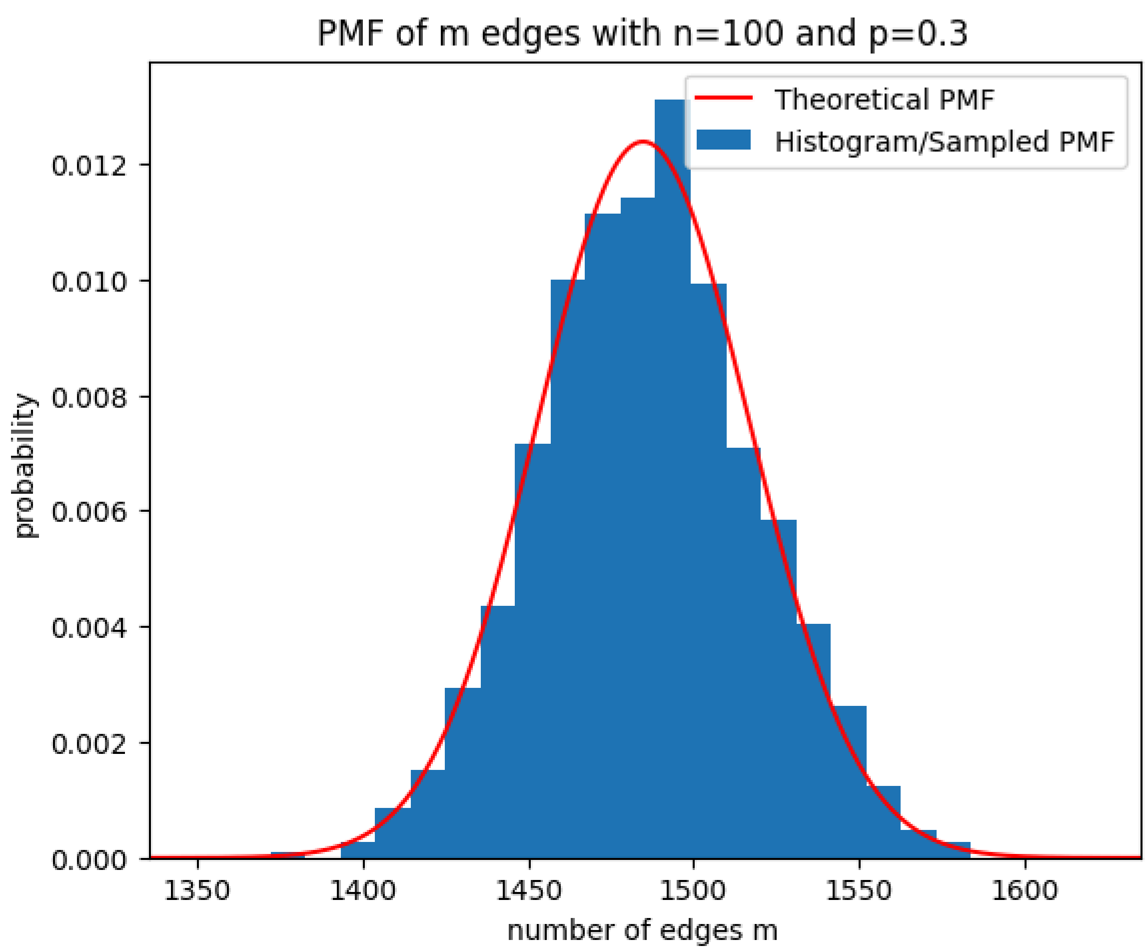
In [ ]:
```
def variance(X,P_X,mean):
    '''Variance for discrete random var X with PDF P_X'''

    return sum([p_x*(x-mean)**2 for x,p_x in zip(X,P_X)])
```

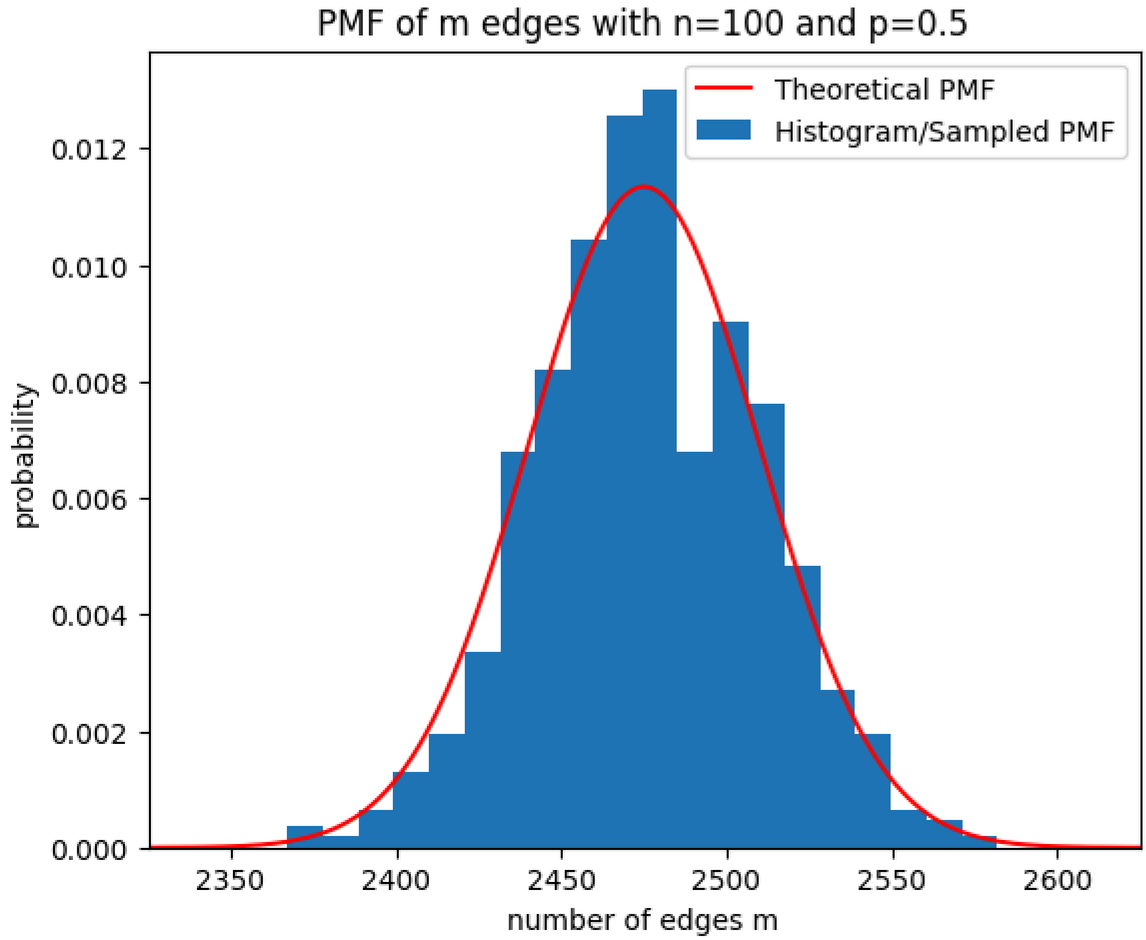
In [ ]:
```
n = 100
probs = [0.1,0.3,0.5,0.7,0.9]
total_edge = math.comb(n,2)

fig, ax = plt.subplots(nrows=len(probs),ncols=1)
```

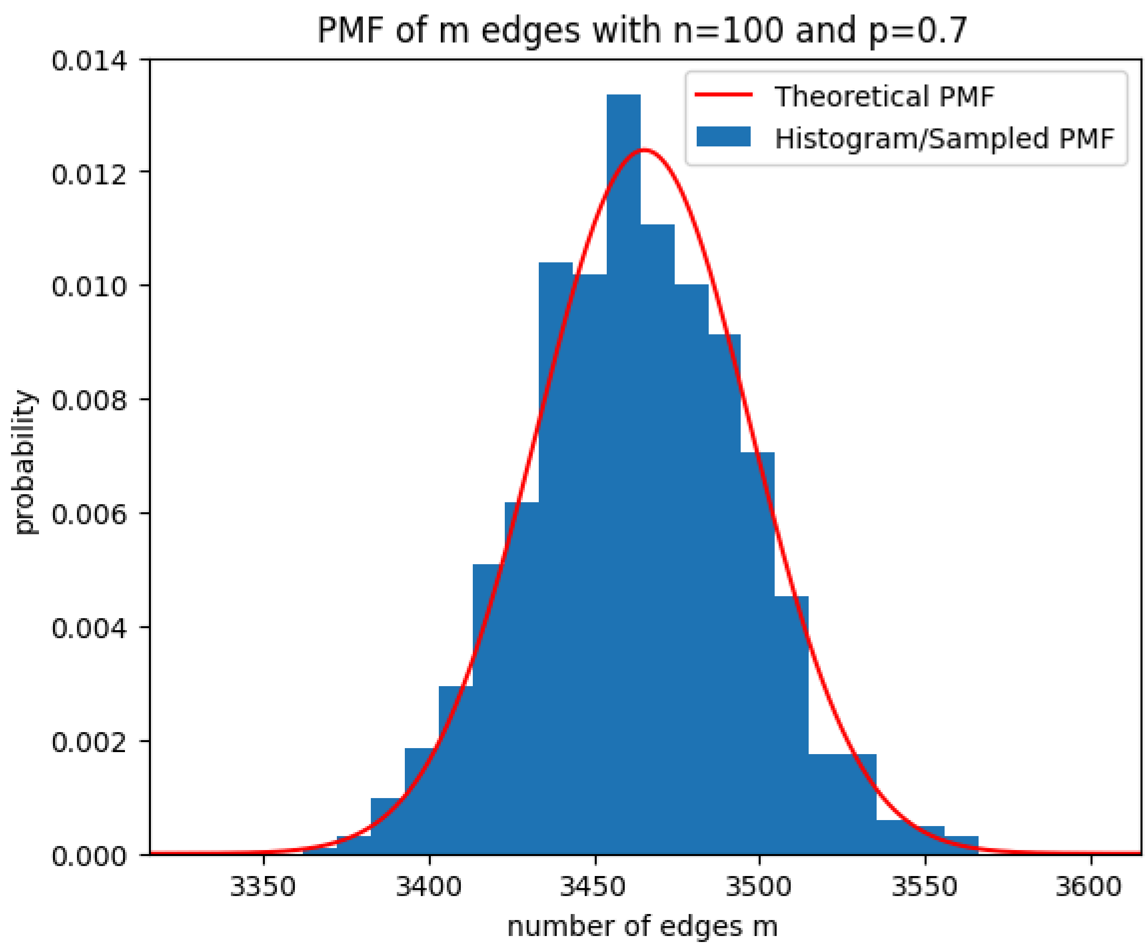
```
fig.set_figheight(30)

for i,prob in enumerate(probs):

    m_range = [m for m in range(total_edge+1)]
    p_m = pmf_num_edge(n,prob)

    ax[i].plot(m_range,p_m,color='r',label='Theoretical PMF')
    ax[i].hist(m_arrays[i],bins=20,density=True, label='Histogram/Sampled PMF')
    ax[i].set_xlabel('number of edges m')
    ax[i].set_ylabel('probability')
    ax[i].set_title('PMF of m edges with n={} and p={}'.format(n,prob))
    ax[i].set_xlim([prob*math.comb(n,2)-150,prob*math.comb(n,2)+150])
    ax[i].legend(loc='upper right')

    mn = mean(m_range,p_m)
    s_mn = np.mean(m_arrays[i])
    var = variance(m_range,p_m,mn)
    s_var = np.var(m_arrays[i])

    print('For p = {}: theoretical mean = {:.2f}, theoretical variance = {:.2f}, sa

plt.show()
```

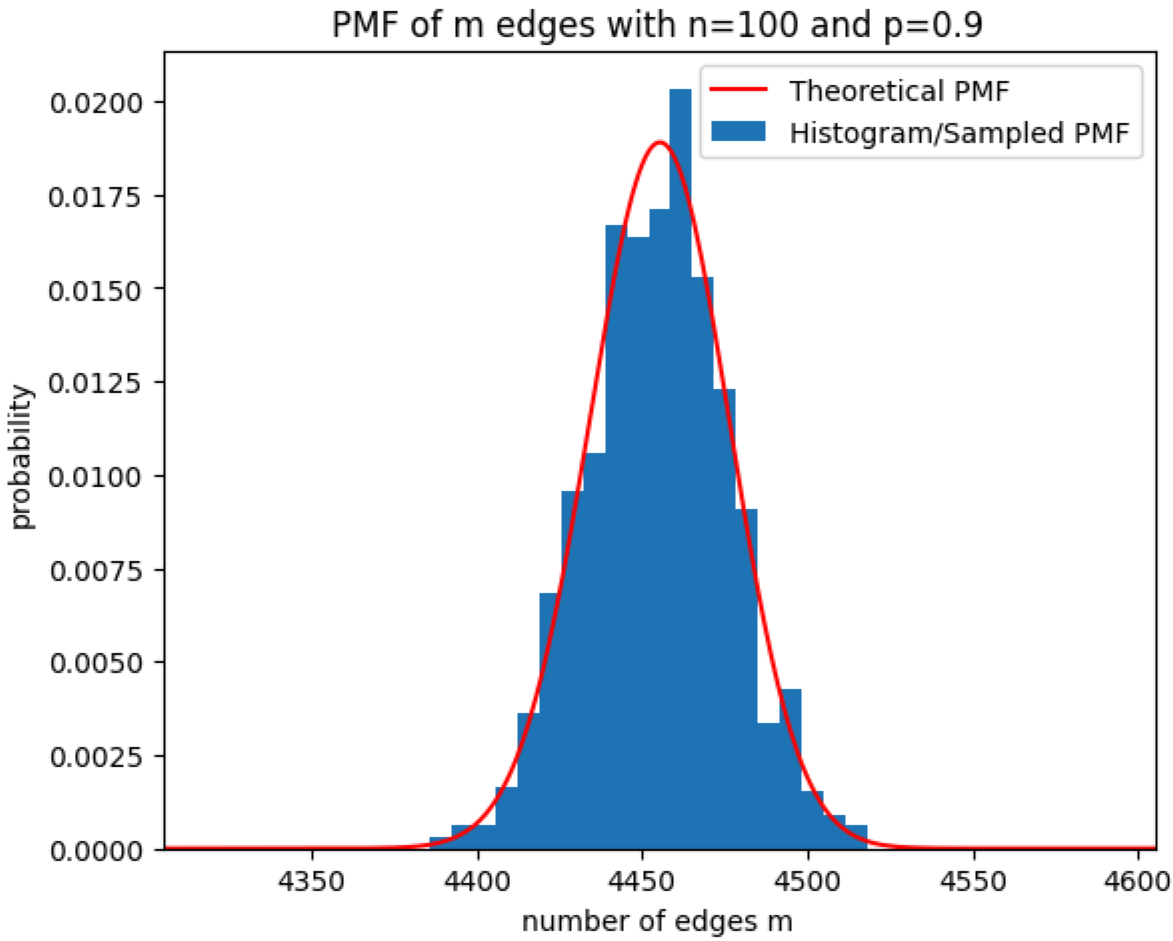
```
For p = 0.1: theoretical mean = 495.00, theoretical variance = 445.50, sample mean = 494.51, sample variance = 396.51
For p = 0.3: theoretical mean = 1485.00, theoretical variance = 1039.50, sample mean = 1486.02, sample variance = 1053.66
For p = 0.5: theoretical mean = 2475.00, theoretical variance = 1237.50, sample mean = 2476.08, sample variance = 1161.90
For p = 0.7: theoretical mean = 3465.00, theoretical variance = 1039.50, sample mean = 3462.69, sample variance = 1058.74
For p = 0.9: theoretical mean = 4455.00, theoretical variance = 445.50, sample mean = 4454.70, sample variance = 456.17
```


PMF of m edges with n=100 and p=0.1
Theoretical PMF
Histogram/Sampled PMF
probability
0.000
0.005
0.010
0.015
0.020
350
400
450
500
550
600
number of edges m


PMF of m edges with n=100 and p=0.3
Theoretical PMF
Histogram/Sampled PMF
probability
0.000
0.002
0.004
0.006
0.008
0.010
0.012
1350
1400
1450
1500
1550
1600
number of edges m


PMF of m edges with n=100 and p=0.5
Theoretical PMF
Histogram/Sampled PMF
probability
0.000
0.002
0.004
0.006
0.008
0.010
0.012
2350
2400
2450
2500
2550
2600
number of edges m


PMF of m edges with n=100 and p=0.7
Theoretical PMF
Histogram/Sampled PMF
probability
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0.014
3350
3400
3450
3500
3550
3600
number of edges m



Q3 Computing degree distribution P(k) where k is the degree of a node.

For a graph of G(n,p), 0<=k<=n-1

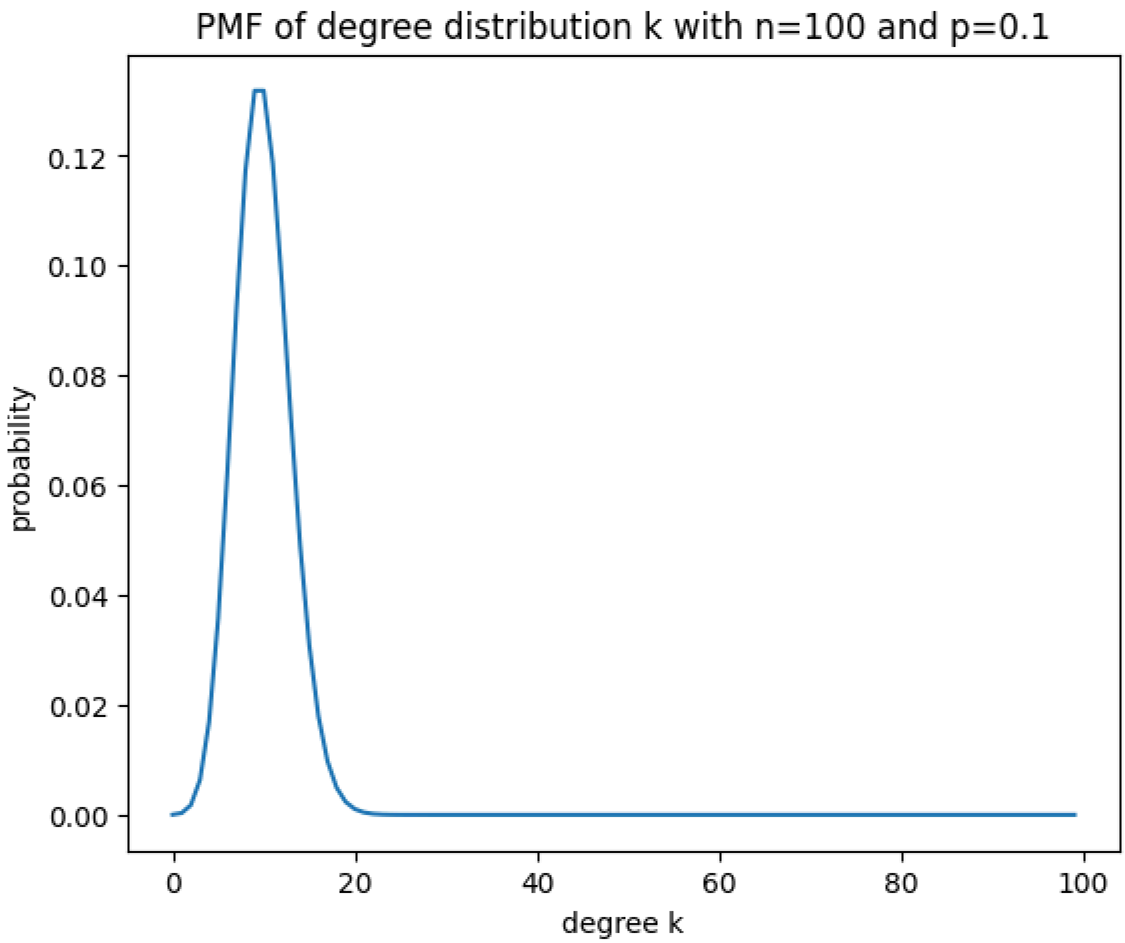
$$P(k) = (n-1)C_k * p^k * (1-p)^{n-1-k}$$

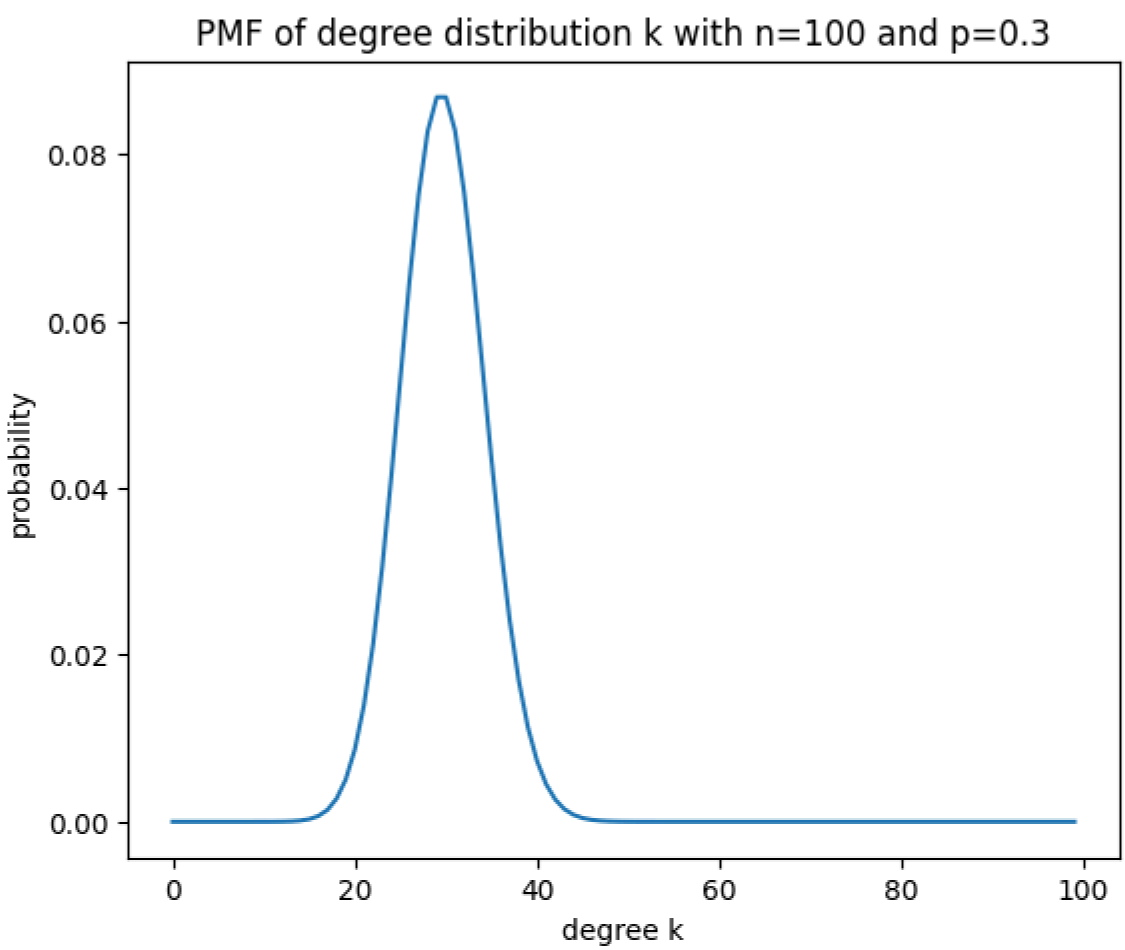
In [ ]:
```
def degree_distribution(prob,num_nodes):

    k_range = [k for k in range(num_nodes)]

    return binom.pmf(k_range,num_nodes-1,prob)
```

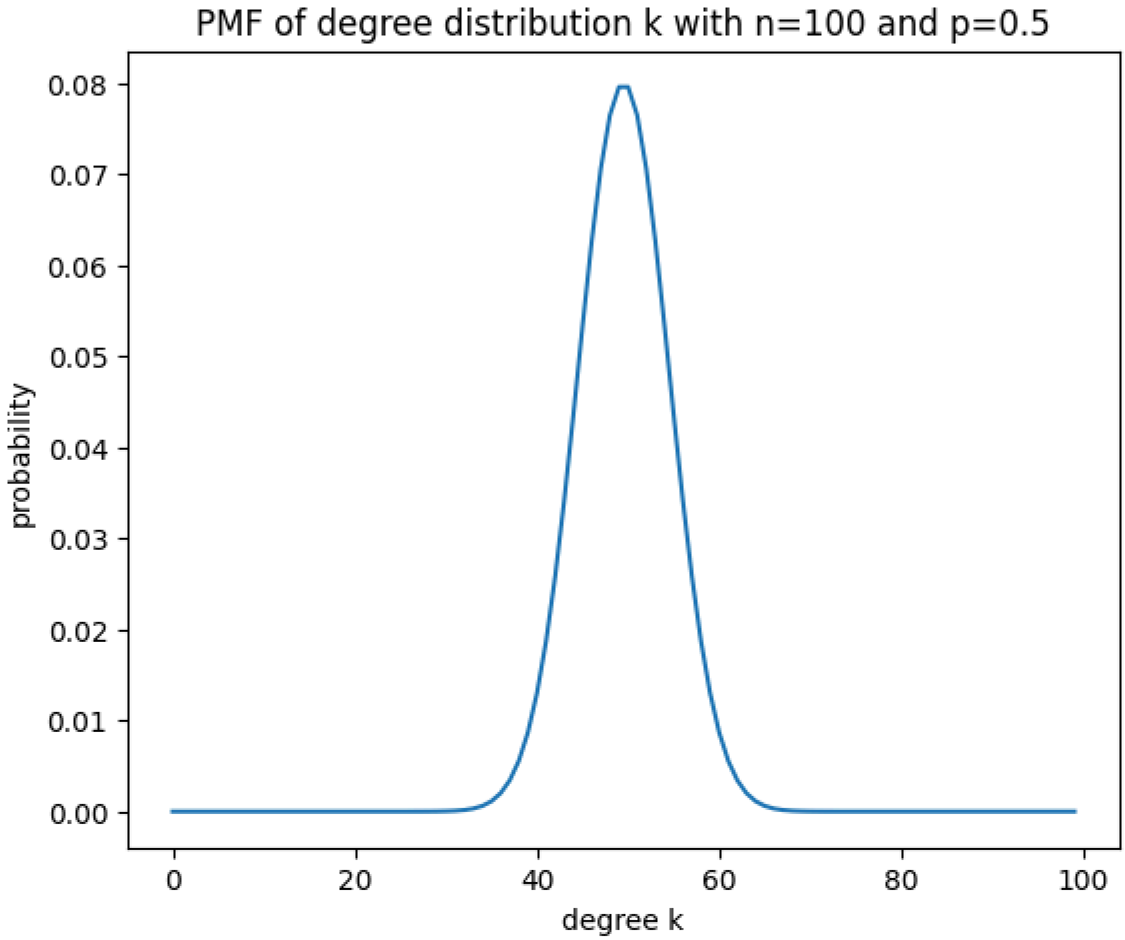
In [ ]:
```
probs = [0.1,0.3,0.5,0.7,0.9]
n = 100

fig, ax = plt.subplots(nrows=len(probs),ncols=1)
fig.set_figheight(30)

for i,prob in enumerate(probs):

    k_range = [k for k in range(n)]
    p_k = degree_distribution(prob,n)

    ax[i].plot(k_range,p_k)
    ax[i].set_xlabel('degree k')
    ax[i].set_ylabel('probability')
```

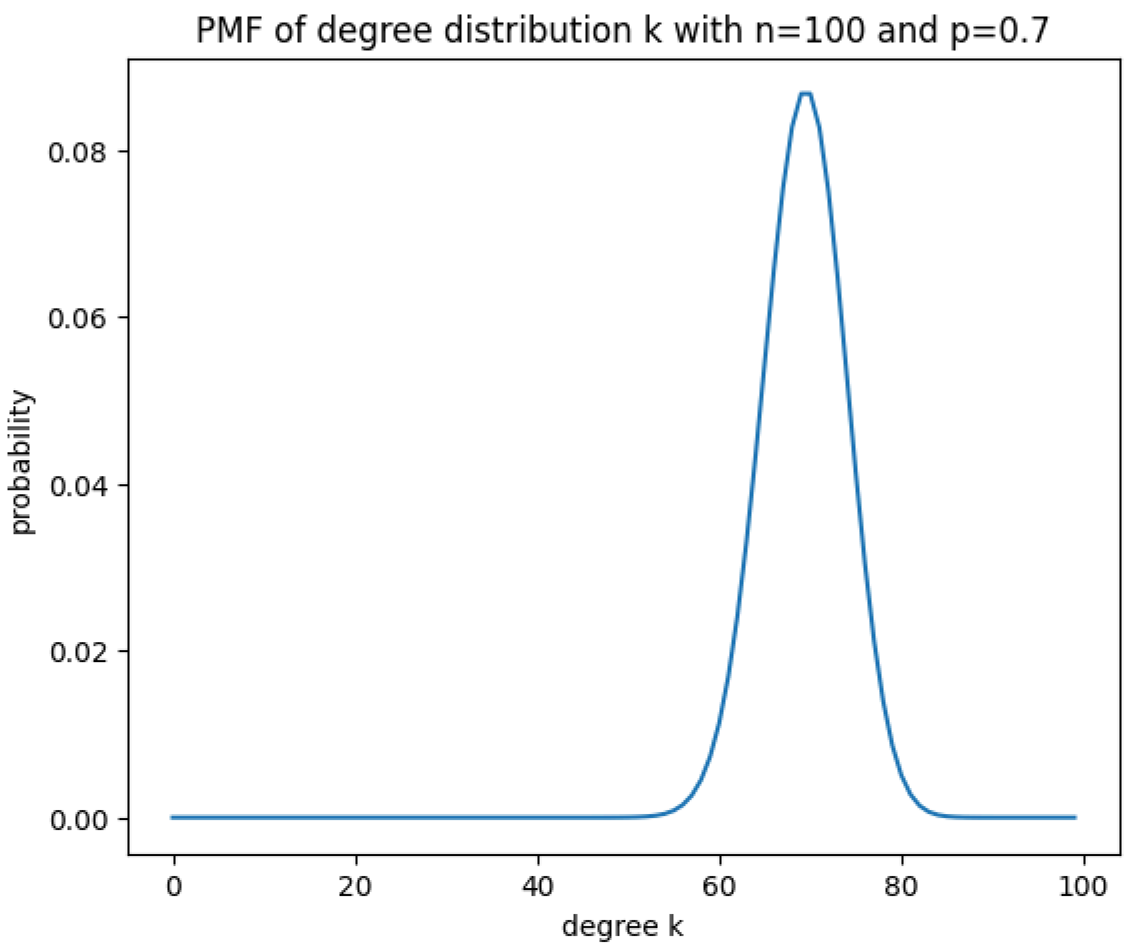
```
    ax[i].set_title('PMF of degree distribution k with n={} and p={}'.format(n,prob

    mn = mean(k_range,p_k)
    var = variance(k_range,p_k,mn)
    print('For p = {}: mean = {:.2f}, variance = {:.2f}'.format(prob,mn,var))

plt.show()
```

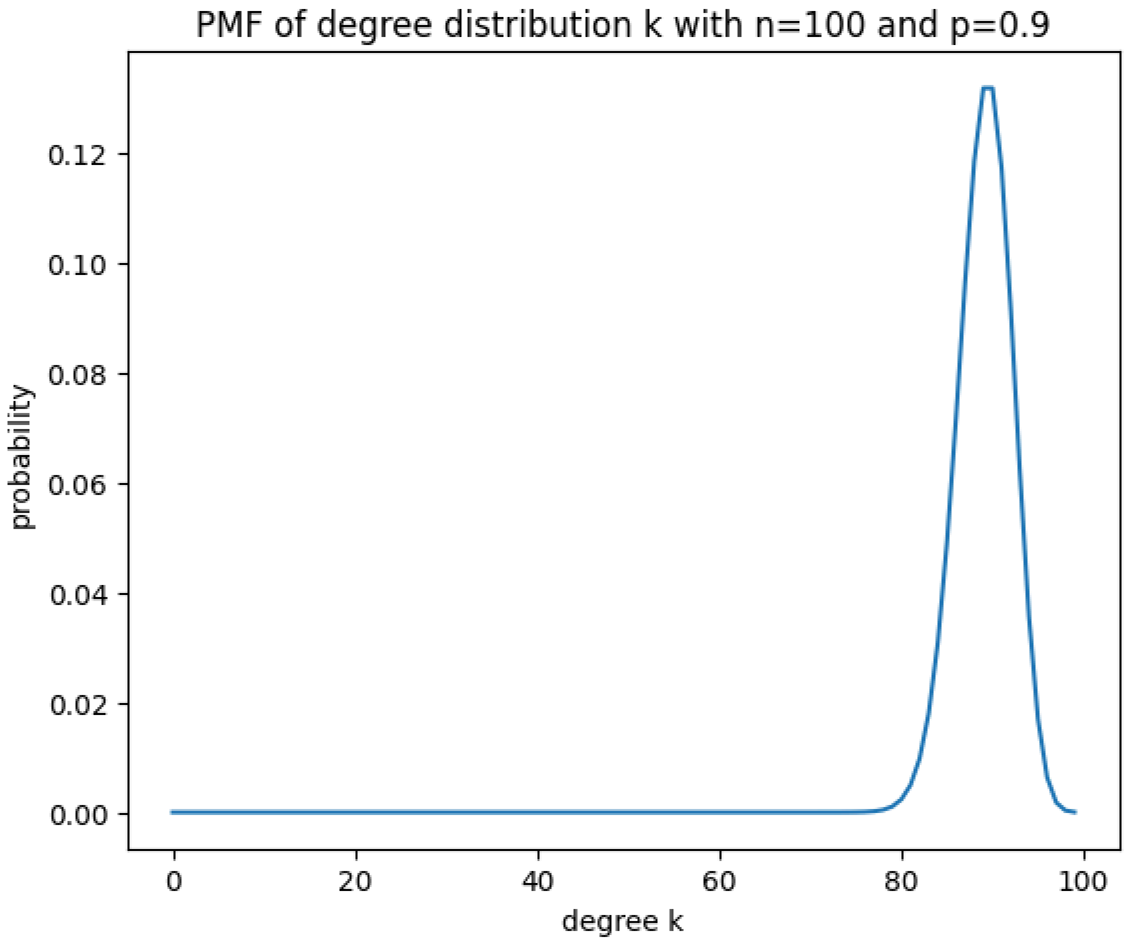
```
For p = 0.1: mean = 9.90, variance = 8.91
For p = 0.3: mean = 29.70, variance = 20.79
For p = 0.5: mean = 49.50, variance = 24.75
For p = 0.7: mean = 69.30, variance = 20.79
For p = 0.9: mean = 89.10, variance = 8.91
```


PMF of degree distribution k with n=100 and p=0.1
probability
degree k
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0
20
40
60
80
100


PMF of degree distribution k with n=100 and p=0.3
probability
degree k
0.00
0.02
0.04
0.06
0.08
0
20
40
60
80
100


PMF of degree distribution k with n=100 and p=0.5
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
probability
0
20
40
60
80
100
degree k


PMF of degree distribution k with n=100 and p=0.7
0.08
0.06
0.04
0.02
0.00
probability
0
20
40
60
80
100
degree k



Q4

Now we let $p = \frac{\lambda}{n-1}$ i.e

$$G(n, \frac{\lambda}{n-1})$$

For some constant λ.

Investigate how the PMF, mean and variance changes as n increases.

Investigate what happens in the limit as

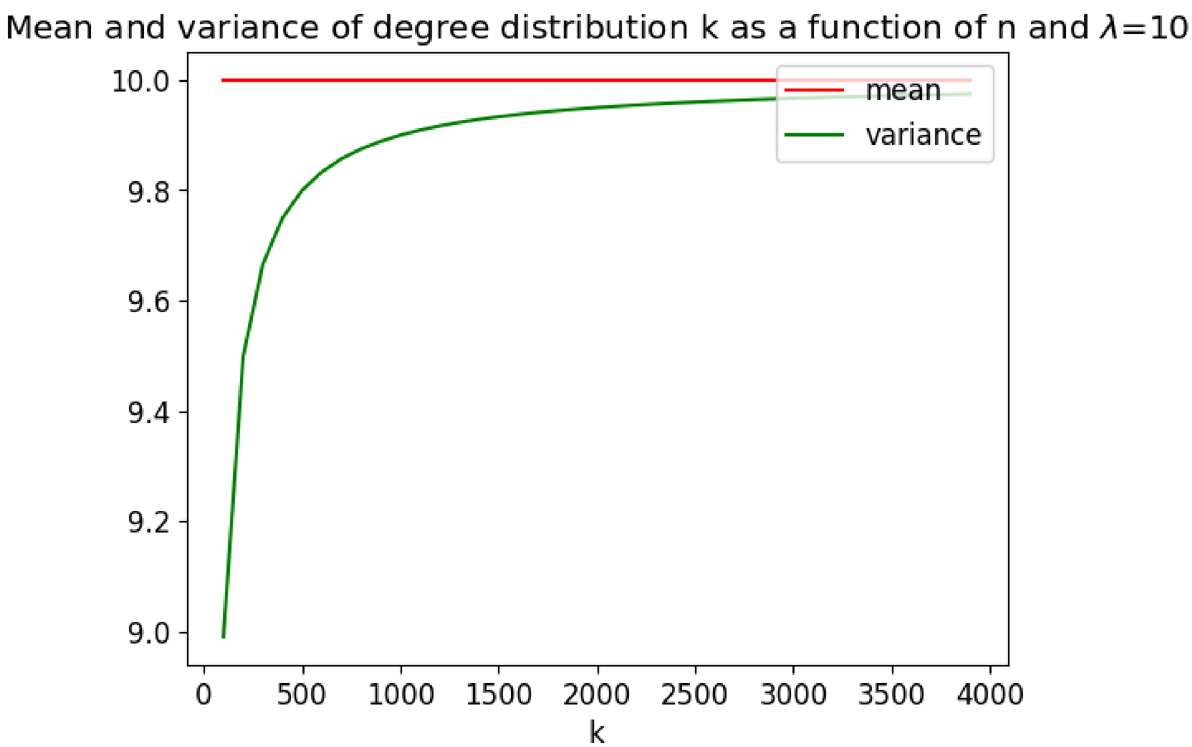
$$n\rarr\infty$$

Using expression in Q3, we obtain

$$P(k) = (n-1)C_k * p^k * (1-p)^{n-1-k}$$
$$= (n-1)C_k * \frac{\lambda}{n-1}^k * (1 - \frac{\lambda}{n-1})^{n-1-k}$$

In [ ]:
```
lmbda = 10
n_range = [n for n in range(100,4000,100)]
means = []
vars = []
```

```
for n in n_range:

    prob = lmbda/(n-1)

    k_range=[k for k in range(n)]
    p_k = degree_distribution(prob,n)

    mn = mean(k_range,p_k)
    means.append(mn)
    vars.append(variance(k_range,p_k,mn))

plt.rcParams.update({'font.size': 12})
plt.plot(n_range,means,label='mean',color='r')
plt.plot(n_range,vars,label='variance',color='g')
plt.xlabel('k')
plt.title(r'Mean and variance of degree distribution k as a function of n and $\lam
plt.legend(loc='upper right')
plt.show()
```



We observe that mean is always constant and equal to λ. However, as n increase, p decreases and variance increases but appears to reach an asymptote of λ.

Q5, two stage algorithm.

For a graph

$$G(n,p)$$

Stage 1:

Generate value m, number of edges according to PMF in Q3.

https://compphys.notes.dmaitre.phyip3.dur.ac.uk/lectures/lecture-5/probability-distributions/#:~:text=By%20inverting%20the%20cumulative%20function,making%20these%20v%20tau%20*%20numpy.

^ If interested in applying monte carlo rejection sampling etc.

Stage 2:

Add m edges uniformly to the network.

In [ ]:
```
class Two_Stage_Network(Network):

    def __init__(self, num_nodes,prob):
        super().__init__(num_nodes)

        #stage 1
        total_edge = math.comb(num_nodes,2)
        m = np.random.binomial(total_edge,prob)

        #stage 2
        while self.num_edge!=m:

            i, j = np.random.randint(0, num_nodes,2)

            if i!=j and j not in self.neighbors(i):
                self.add_edge(i,j)
```

Q6

For

$$n^k : k = 6, 7, \ldots, 10$$

Generate graphs

$$G(n, p)$$

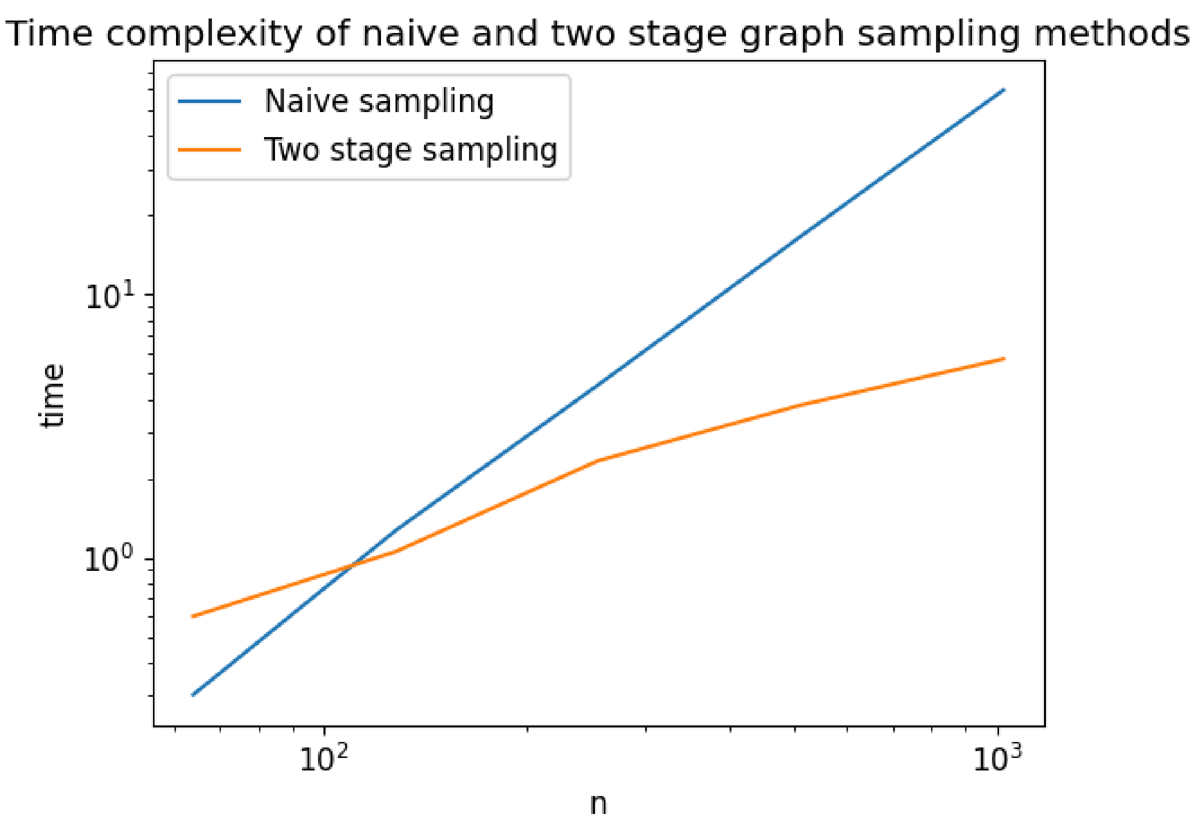
according to Q1, a naive algo and Q4, sampling m then generating graph$

Then time the two algorithms

In [ ]:
```
pows = 2**np.array(range(6,11))
lmbda = 10
timer_runs = 50
time_naive_list = []
time_two_stage_list = []
```

```
for n in pows:
    p = lmbda/(n-1)

    total_edges = math.comb(n,2)
    m_range = [m for m in range(total_edges+1)]

    #Naive generation
    time_naive = timeit(lambda: Bernoulli_Network(n,p), number=timer_runs)

    #Alternate generation
    time_two_stage = timeit(lambda: Two_Stage_Network(n,p),number=timer_runs)

    #taking logs of power2
    time_naive_list.append(time_naive)
    time_two_stage_list.append(time_two_stage)
```

In [ ]:
```
plt.loglog(pows,time_naive_list,label='Naive sampling')
plt.loglog(pows,time_two_stage_list, label='Two stage sampling')
plt.title('Time complexity of naive and two stage graph sampling methods')
plt.xlabel('n')
plt.ylabel('time')
plt.legend()
plt.show()
```



Q7

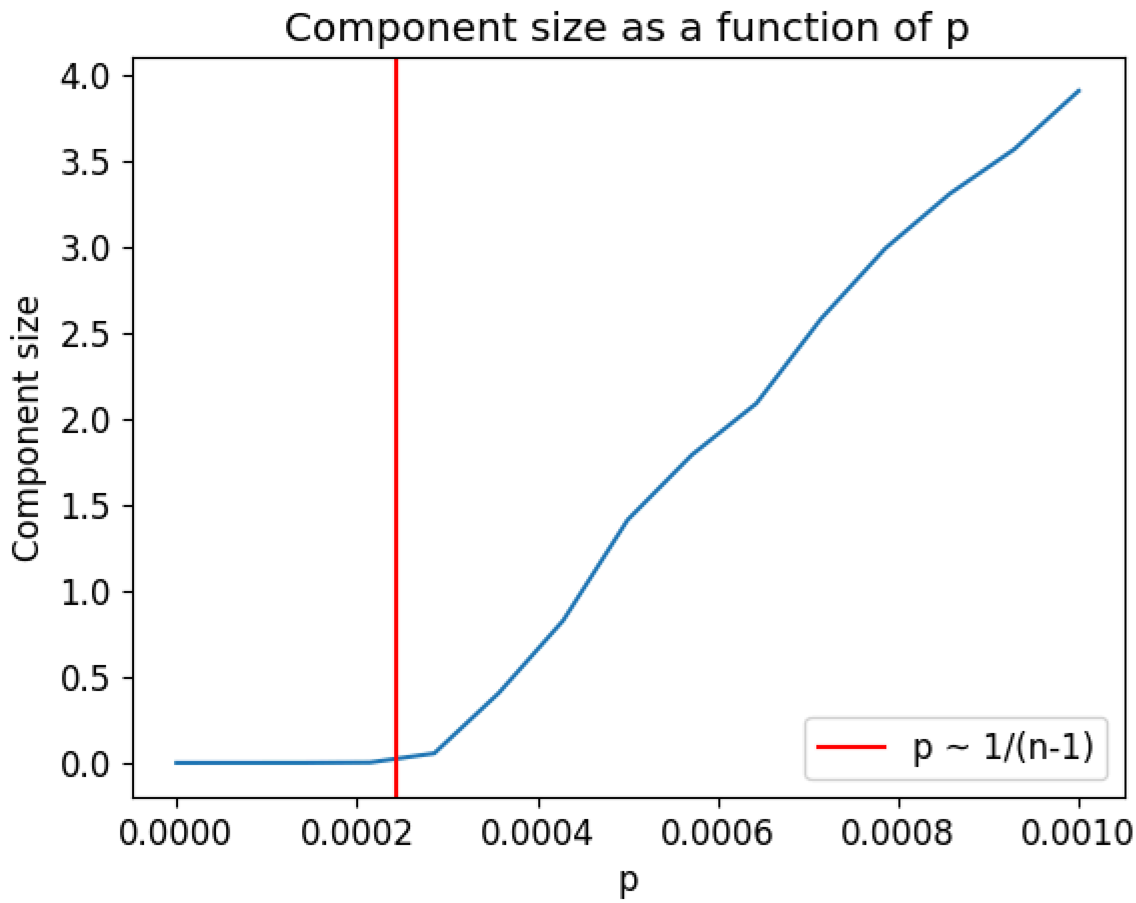
Investigating the size of component as a function of p.

Let $n = 4096$ and $p = [0, 0.001]$. Plot how the average size of the component containing node 1 changes as p varies, especially when $p \approx \frac{1}{n-1}$

1. First we generate sample according to alternate generation
2. Then carry our Breadth First Search to find all nodes reachable from node 1, which will be the size of the component containing node 1
3. Repeat step 1,2 by varying p and plot a graph of component size against p
4. Mark x axis where $p \approx \frac{1}{n-1}$

In [ ]:
```
n = 4096
probs = np.linspace(0,0.001,num=15)
runs = 50
comp_size_array_avg = []

for prob in probs:

    component_size_array=[]

    for _ in range(runs):

        graph = Two_Stage_Network(n,prob)

        component_size = 0
        visited=set()

        q=deque([0])

        while q:
            curr = q.popleft()
            visited.add(curr)

            for adj in graph.neighbors(curr):
                if adj not in visited:
                    component_size+=1
                    q.append(adj)

        component_size_array.append(component_size)

    comp_size_array_avg.append(np.mean(component_size_array)/n)
```

In [ ]:
```
plt.plot(probs,comp_size_array_avg)
plt.axvline(1/(n-1),color='r',label='p ~ 1/(n-1)')
plt.title('Component size as a function of p')
plt.ylabel('Component size')
plt.xlabel('p')
plt.legend()
plt.show()
```


Component size as a function of p
Component size
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
p ~ 1/(n-1)
0.0000
0.0002
0.0004
0.0006
0.0008
0.0010
p