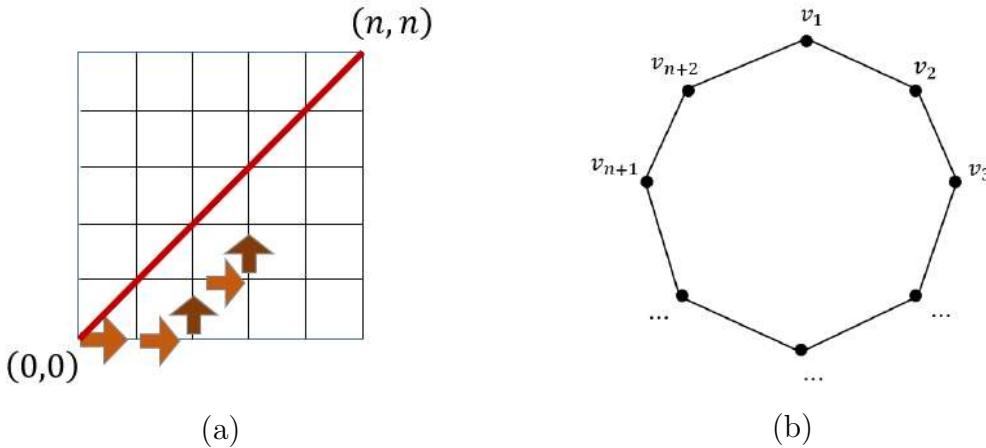


Problem 1 (20%). Let X, Y be discrete random variables. The variance of a random variable X is defined as $\text{Var}[X] := E[(X - E[X])^2]$. Prove that

1. $E[aX + bY] = a \cdot E[X] + b \cdot E[Y]$ for any constant a, b .
2. If X and Y are independent, then $E[X \cdot Y] = E[X] \cdot E[Y]$ and $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$.
3. $\text{Var}[X] = E[X^2] - E[X]^2$. Hint: Use the fact that $E[X \cdot E[X]] = E[X]^2$.

Problem 2 (20%). Consider the slides #2. Prove that the graphs H_i defined in the proof of Theorem 3 are bicliques.

Problem 3 (20%). For any integer $n \geq 1$, consider the grid points (r, c) with $1 \leq r, c \leq n$. Let C_n be the number of possible paths from $(0, 0)$ to (n, n) that use only \rightarrow and \uparrow and that never cross the diagonal $r = c$. See also the Figure (a) below. For convenience, define $C_0 := 1$.



For any integer $n \geq 2$, consider the convex $(n+2)$ -gon with vertices labeled with v_1, v_2, \dots, v_{n+2} . Let P_n denote the number of possible ways to triangulate the polygon. It follows that $P_2 = 2$, $P_3 = 5$, etc. For convenience, also define $P_0 := 1$ and $P_1 := 1$.

1. Prove that for any $n \geq 2$, P_n satisfies the recurrence

$$P_n = \sum_{0 \leq k < n} P_k \cdot P_{n-k-1}.$$

2. Prove that for any $n \geq 2$, C_n satisfies the same recurrence

$$C_n = \sum_{0 \leq k < n} C_k \cdot C_{n-k-1}.$$

Note that this proves that P_n also equals the n^{th} -Catalan number.

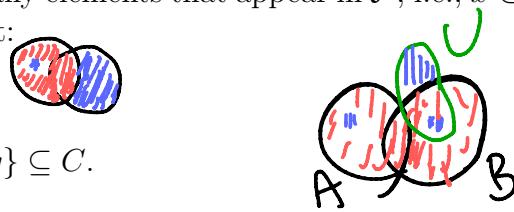


Problem 4 (20%). Let \mathcal{F} be a family of subsets, where

$$|A| \geq 3 \text{ for any } A \in \mathcal{F} \quad \text{and} \quad |A \cap B| = 1 \text{ for any } A, B \in \mathcal{F}, A \neq B.$$

Suppose that \mathcal{F} is not 2-colorable. Let x, y be any elements that appear in \mathcal{F} , i.e., $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$ for some $A, B \in \mathcal{F}$. Prove that:

- 1. x belongs to at least two members of \mathcal{F} .
- 2. There exists some $C \in \mathcal{F}$ such that $\{x, y\} \subseteq C$.



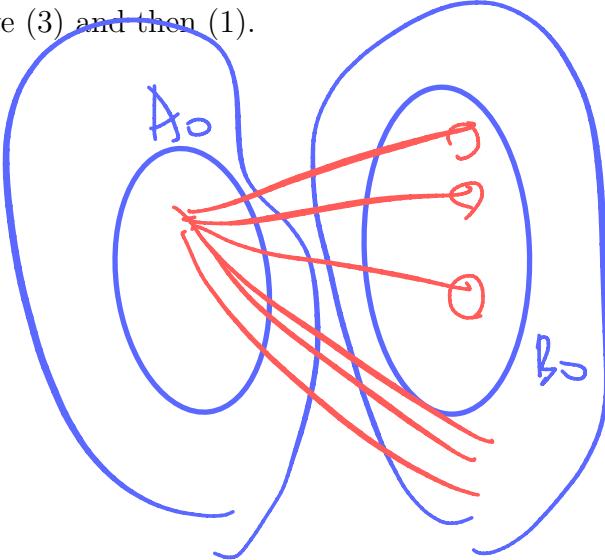
Hint: Construct proper coloring to prove the properties. For (1), consider a particular A with $x \in A \in \mathcal{F}$. Color $A \setminus \{x\}$ red and the remaining blue. Show that this leads to the conclusion of (1). For (2), consider particular A, B with $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$. Color $(A \cup B) \setminus \{x, y\}$ red and the remaining blue. Prove that it leads to (2).

Problem 5 (20%). Let $G = (A \cup B, E)$ be a bipartite graph, d be the minimum degree of vertices in A and D the maximum degree of vertices in B . Assume that $|A|d \geq |B|D$.

Show that, for every subset $A_0 \subseteq A$ with the density α defined as $\alpha := |A_0|/|A|$, there exists a subset $B_0 \subseteq B$ such that:

1. $|B_0| \geq \alpha \cdot |B|/2$,
2. every vertex of B_0 has at least $\alpha D/2$ neighbors in A_0 , and
3. at least half of the edges leaving A_0 go to B_0 .

Hint: Let B_0 consist of all vertices in B that have at least $\alpha D/2$ neighbors in A_0 . First prove (3) and then (1).



P1 Since X and Y are discrete random variables,

X and Y have finite outcome.

Let the possible outcome of X is $S_X = \{x_1, \dots, x_n\}$ and

$$\text{''} \quad Y \text{ is } S_Y = \{y_1, \dots, y_m\}.$$

$$1. (1) E[aX+bY] = \sum_{\substack{x \in S_X \\ y \in S_Y}} (ax+by) P(X=x, Y=y) \quad (\text{by the def. of expectation})$$

$$= \sum_{\substack{x \in S_X \\ y \in S_Y}} ax P(X=x, Y=y) + \sum_{\substack{x \in S_X \\ y \in S_Y}} by P(X=x, Y=y) \quad (\text{by the linearity of } \Sigma)$$

$$= \sum_{x \in S_X} ax P(X=x) + \sum_{y \in S_Y} by P(Y=y) \quad (\text{since } \sum_{y \in S_Y} P(X=x, Y=y) = P(X=x))$$

$$= a \sum_{x \in S_X} x P(X=x) + b \sum_{y \in S_Y} y P(Y=y) \quad (\text{by the linearity of } \Sigma)$$

$$= aE[X] + bE[Y]. \quad (\text{by the def. of expectation}) \quad \text{①}$$

$$(2) E[XY] = \sum_{\substack{x \in S_X \\ y \in S_Y}} (xy) P(X=x, Y=y) \quad (\text{by the def. of expectation})$$

$$= \sum_{\substack{x \in S_X \\ y \in S_Y}} (xy) P(X=x) P(Y=y) \quad (\text{since } X \text{ and } Y \text{ are independent,} \\ P(X=x, Y=y) = P(X=x)P(Y=y))$$

$$= \sum_{x \in S_X} x P(X=x) \sum_{y \in S_Y} y P(Y=y) \quad (\text{by the linearity of } \Sigma)$$

$$= E[X] + E[Y] \quad (\text{by the def. of expectation})$$

$$2) \text{Var}(X+Y) = E[(X+Y) - (E[X+Y])]^2 \quad (\text{by the def. of variance})$$

$$= E[(X+Y) - (E[X] + E[Y])]^2$$

$$(\text{since } X \text{ and } Y \text{ are independent, } E[X+Y] = E[X] + E[Y]) \quad (\text{P1-2})$$

$$= E[(X-E[X]) + (Y-E[Y])]^2$$

$$= E[(X-E[X])^2] + 2E[(X-E[X])(Y-E[Y])] + E[(Y-E[Y])^2]$$

$$(\text{by the linearity of expectation, i.e. P1-1})$$

$$\begin{aligned}
 &= \text{Var}[X] + 2\{E[X]E[Y - E[Y]] - \underbrace{E[X]}_{E[X]}E[Y - E[Y]]\} + \text{Var}[Y] \\
 &\quad (\text{by P1-1 and P1-2}) \\
 &= \text{Var}[X] + \text{Var}[Y]
 \end{aligned}$$

$$\begin{aligned}
 (3) \text{Var}[X] &= E[(X - E[X])^2] \quad (\text{by the def. of variance}) \\
 &= E[X^2 - 2XE[X] + (E[X])^2] \\
 &= E[X^2] - 2E[XE[X]] + E[X]^2 \\
 &= E[X^2] - E[X]^2 \quad (\text{since } E[XE[X]] = E[X]^2)
 \end{aligned}$$

2. Let $i \in \{1, \dots, m\}$.

Let $X_i = \{v \in K_n : \text{the } i^{\text{th}}\text{-coordinate of } v = 1\}$ and

$Y_i = \{v \in K_n : \text{the } i^{\text{th}}\text{-coordinate of } v = 0\}$.

Claim1: $H_i \cong$ a bipartite graph

By the construction of H_i , $\nexists x, y \in X_i$ or $(x, y) \in Y_i$,
since the i^{th} -coordinate is the same.

Hence, the Claim1 holds. \square

Claim2: $\forall u \in X_i, \forall v \in Y_i, \exists e = (u, v) \in E(H_i)$

Let $u \in X_i$ and $v \in Y_i$.

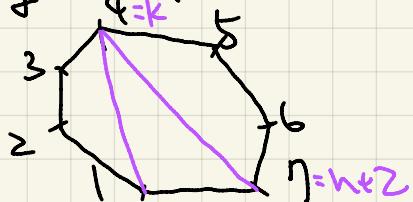
Since $u \in X_i$ and $v \in Y_i$, since the i^{th} -coordinate of u and v are differ, $\exists (u, v) \in E(H_i)$.

Since $u \in X_i$ and $v \in Y_i$ are arbitrary, the Claim2 holds. \square

By Claim1 and Claim2, since i is arbitrary,
 $H_i \cong$ a bipartite for all $i = 1, 2, \dots, m$. \square

3.1. Note that if we fixed the side $(1, h+2)$,
 and let $k \in \{2, 3, \dots, h+1\}$ be the partition point.
 There are a convex k -gon on the RHS and convex
 $h+2-k+1 = (h+3-k)$ -gon on the LHS.

A toy example is below:



Moreover, there are P_{k-2} ways to triangulate
 a convex $(k-2)$ -gon and P_{h+1-k} ways to triangulate
 a convex $(h+3-k)$ -gon.

$$\text{Hence, } P_n = \sum_{k=2}^{h+1} P_{k-2} P_{h+1-k} = \sum_{k=0}^{h-1} P_k P_{h-k-1}. \quad \text{④}$$

3.2. Let $k \in \mathbb{N}$ be the coordinate of 1^{st} at the diagonal.
 It is (k, k) .

Clearly, when 1^{st} step, we can only move " \rightarrow ";
 when $(2k-1)^{\text{th}}$ step, we can only move " \uparrow ";
 and # of " \rightarrow " \geq # of " \uparrow " during \geq 2^{nd} step to $(2k-2)^{\text{th}}$ step.

Therefore, there are h_{k-1} ways from $(0, 0)$ to (k, k) .

Similarly, when $(2k+1)^{\text{th}}$ step, we can only move " \rightarrow ";
 when $(2h-1)^{\text{th}}$ step, we can only move " \uparrow ";
 and # of " \rightarrow " \geq # of " \uparrow " during $(2k+1)^{\text{th}}$ step to
 $(2h-2)^{\text{th}}$ step.

Therefore, there are h_{n-k} ways from (k, k) to (h, h) .

$$\text{Hence, } C_n = \sum_{k=1}^n C_{k-1} C_{n-k} = \sum_{k=0}^{h-1} C_k C_{h-k-1}. \quad \text{⑤}$$

4.1 Suppose "Statement 1" is false.

(We want to show that F is 2-colorable.)

Since "Statement 1" is false, $\exists! A \in F \nexists X \in A$.

Thus, $\forall B \in F \setminus \{A\}, X \in B$. $\text{---} \otimes$

Since $|A \cap B| = 1$ and \otimes , $\exists Y \in \text{ground set } N \ni Y \neq X$ and $\{Y\} = A \cap B$. $\text{---} \otimes$

Claim F is 2-colorable.

Consider the coloring of the element in ground set N :

Color $A \setminus \{X\}$ red and the remaining blue.

(Case1: A is not monochromatic)

By apply this coloring, $g(x) = B$, where $X \in A$

Since $|A| \geq 3$, $\exists Z \in A \setminus \{X\}$ with $g(Z) = R$.

Hence, A is not monochromatic.

(Case2: $\forall B \in F \setminus \{A\}$, B is not monochromatic)

If $F \setminus \{A\} = \emptyset$, the statement holds clearly.

Assume $F \setminus \{A\} \neq \emptyset$.

Let $B \in F \setminus \{A\}$.

By $\text{---} \otimes$, $\exists Y \in N \ni Y \neq X$ and $\{Y\} = A \cap B$.

By apply this coloring, $g(Y) = R$ since $Y \in A \setminus \{X\}$.

Since $|B| \geq 3$, $\exists Z \in B \setminus \{Y\}$ with $g(Z) = B$.

Hence, B is not monochromatic.

Thus, $\forall B \in F \setminus \{A\}$, B is not monochromatic.

By Case1 and Case2, F is 2-colorable. * (F is not 2-colorable)

Thus, "Statement 1" is true. \blacksquare

4.2 Suppose "statement 2" is false. Let $A, B \subseteq F \setminus \{x, y\}$ and $y \in B$.
(We want to show that F is 2-colorable.)

Claim F is 2-colorable.

Consider the coloring of the element in ground set N :

Color $(A \cup B) \setminus \{x, y\}$ red and the remaining blue.

(Case1: A is not monochromatic)

By apply this coloring, $g(x) = B$, where $x \in A$.

Since $|A| \geq 3$, $\exists z \in A \setminus \{x\}$ with $g(z) = R$.

Hence, A is not monochromatic.

(Case2: $B \subseteq F$, B is not monochromatic)

If $B = A$ or $F = \{A\}$, the statement holds clearly.

Assume $B \neq A$.

Similar to case1, B is not monochromatic.

(Case3: $\forall C \subseteq F \setminus \{A, B\}$, C is not monochromatic)

If $F \setminus \{A \cup B\} = \emptyset$, the statement holds clearly.

Assume $F \setminus \{A \cup B\} \neq \emptyset$.

Suppose $C \subseteq F \setminus \{A \cup B\}$ is monochromatic.

By assumption since $C \neq A$, $\exists z_A \in N \setminus \{z_A\} = A \cap C$.

$C \neq B$, $\exists z_B \in N \setminus \{z_B\} = B \cap C$, where $z_A \neq z_B$.

($z_A = x$)

If not, $g(z_A) = R$ since $z_A \in (A \cup B) \setminus \{x, y\}$.

Since $|C| \geq 3$, $|C \cap B| = 1$, and $|C \cap A| = 1$, $\exists z \in N \setminus (A \cup B)$
 $\Rightarrow z \in C$.

Thus, $g(z) = B \neq C$ (C is monochromatic.)

Hence, $z_A = x$.

($Z_B = y$)

Similar to $Z_A = x$, we have $Z_B = y$.

Hence, $Z_A \in C$ and $Z_B \in C$. \nexists ("statement 2" is false)

Thus, $\forall C \in F \setminus \{A, B\}$, C is not monochromatic.

Thus, F is 2-colorable. \nexists (F is not 2-colorable)

Hence, "Statement 2" is true. \square

P5 Let $A_0 \subseteq A$ be a subset of A

and define $\alpha = \frac{|A_0|}{|A|}$.

Let B_0 consist of vertices in B that have at least $\alpha D/2$ neighbors in A_0 .

(Condition 2)

By the construction, B_0 satisfies the condition 2. \square

(Condition 3)

It is sufficient to proof "at most half of the edges leaving A_0 go to $B \setminus B_0$ ".

We denote $d(a) =$ the degree of vertex $a \in A \cup B$.

Then, $\frac{1}{2} \sum_{a \in A_0} d(a) \geq \frac{1}{2} \sum_{a \in A_0} d$ since $d \leq d(x) \forall x \in A$

$$= \frac{1}{2} |A_0| d$$

$$= \frac{1}{2} \alpha |A| d \quad \text{since } \alpha = \frac{|A_0|}{|A|}$$

$$\geq \frac{1}{2} \alpha |B| D \quad \text{since } |A|d \geq |B|D - \star$$

$$\geq \frac{1}{2} \alpha (|B| - |B_0|) D = \sum_{b \in B \setminus B_0} \frac{\alpha D}{2} \quad \text{since } |B_0| \geq 0$$

$$\geq \sum_{b \in B \setminus B_0} d(b) \quad \text{since the construct of } B_0$$

Hence, at most half of the edges leaving A_0 go to $B \setminus B_0$. \square

(Condition 1)

$$|B_0|D \geq \sum_{b \in B_0} d(b) \text{ since } D \geq d(x) \forall x \in B$$

$$\geq \frac{1}{2} \sum_{a \in A_0} d(a) \text{ by Condition 3}$$

$$\geq \frac{1}{2} d|B| D \text{ by } \oplus$$

Hence, $|B_0| \geq d|B|/2$. \blacksquare