Problem 1 (20%). How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 = 21$ with

1. $x_i \geq 0$.
2. $x_i > 0$.
3. $0 \leq x_i \leq 12$.

Problem 2 (20%). Prove the following identities **using path-walking argument**.

1. For any $n, r \in \mathbb{Z}^{\geq 0}$,

$$\sum_{0 \leq k \leq r} \binom{n+k}{k} = \binom{n+r+1}{r}.$$

2. For any $m, n, r \in \mathbb{Z}^{\geq 0}$ with $0 \leq r \leq m+n$,

$$\sum_{0 \leq k \leq r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

Problem 3 (20%). Let $\mathcal{F}$ be a set family on the ground set X and $d(x)$ be the degree of any $x \in X$, i.e., the number of sets in $\mathcal{F}$ that contains x. Use the double counting principle to prove the following two identities.

$$\sum_{x \in Y} d(x) = \sum_{A \in \mathcal{F}} |Y \cap A| \text{ for any } Y \subseteq X.$$

$$\sum_{x \in X} d(x)^2 = \sum_{A \in \mathcal{F}} \sum_{x \in A} d(x) = \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} |A \cap B|.$$

Problem 4 (20%). Let H be a 2α-dense 0-1 matrix. Prove that at least an $\alpha/(1-\alpha)$ fraction of its rows must be α-dense.

Problem 5 (20%). Let $\mathcal{F}$ be a family of subsets defined on an n-element ground set X. Suppose that $\mathcal{F}$ satisfies the following two properties:

1. $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$.
2. For any $A \subsetneq X$, $A \notin \mathcal{F}$, there always exists $B \in \mathcal{F}$ such that $A \cap B = \emptyset$.

Prove that

$$2^{n-1} - 1 \leq |\mathcal{F}| \leq 2^{n-1}.$$

Hint: Consider any set $A \subseteq X$ and its complement $\overline{A}$. Apply the conditions given above and prove the two inequalities "$\leq$" and "$\geq$" separately.

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1-1

Let us map this problem to a equivalent problem.
Suppose there are $N=21$ the same blue balls and $M=4$ the same red balls.

Q: How many ways to permutate them?

We can assign X_1 = # of blue balls which are to the left of the leftmost red ball,

X_2 = # of blue balls which are between the leftmost red ball and the second leftmost red ball,

X_3 = # of blue balls which are between the second leftmost red ball and the third leftmost red ball, and

X_4 = # of blue balls which are to the right of the rightmost red ball.

Since such mapping function is 1-1 and onto, these two problems have the same answer.

Hence, # of integer solution (X_1, X_2, X_3, X_4)

$$\ni ① \sum_{i=1}^{4} X_i = 21 \quad ② X_i \geq 0$$

$$= \frac{24!}{3!\,21!} = \frac{24 \times 23 \times 22}{6} = 2024$$

1.2 Notice that the condition "$x_i > 0 \wedge x_i \in \mathbb{Z}$" is equivalent to "$x_i \geq 1 \wedge x_i \in \mathbb{Z}$".

Moreover, # of integer solution (x_1, x_2, x_3, x_4)

$\ni$ ① $\sum_{i=1}^{4} x_i = 21$ ② $x_i \geq 1$ is equal to

# of integer solution (y_1, y_2, y_3, y_4)

$\ni$ ① $\sum_{i=1}^{4} y_i = 21 - 4 = 17$ ② $y_i \geq 0$

by constructing a 1-1 $\wedge$ onto mapping

$$f(y_1, y_2, y_3, y_4) = (y_1+1, y_2+1, y_3+1, y_4+1) =: (x_1, x_2, x_3, x_4).$$

Hence, we only need to calculate

# of integer solution (y_1, y_2, y_3, y_4)

$\ni$ ① $\sum_{i=1}^{4} y_i = 17$ ② $y_i \geq 0$

$$= \binom{17+(4-1)}{4-1} = \binom{20}{3} = \frac{20 \times 19 \times 18}{6} = 1140 \quad \blacksquare$$

1.3 Define $E_i = \{(x_1, x_2, x_3, x_4) : \sum_{i=1}^{4} x_i = 21 \text{ and } x_i \geq 13 \wedge x_j \geq 0 \text{ for } j = 1,2,3,4\}$ for $i = 1,2,3,4$.

$$|E_1 \cup E_2 \cup E_3 \cup E_4|$$

$$= |E_1| + |E_2| + |E_3| + |E_4| + \sum_{|I|=2,\, I \subseteq \{1,2,3,4\}} (-1)^{2+1} |A_I| + \sum_{|I|=3\, I \subseteq \{1,2,3,4\}} (-1)^{3+1} |A_I| + \sum_{|I|=4\, I \subseteq \{1,2,3,4\}} (-1)^{4+1} |A_I| \quad \text{by Prop 1.14}$$

$= 4|E_1|$ since $|A_I| = 0$ $\forall I \subseteq \{1,2,3,4\}$ with its size ≥ 2 and the symmetric of E_i.

Similar to Problem 1.2, $|E_1| = \binom{21-13+(4-1)}{(4-1}) = \binom{11}{3} = \frac{11 \times 10 \times 9}{6} = 165$

Hence, $|E_1 \cup E_2 \cup E_3 \cup E_4| = 4|E_1|$

$= 4 \times 165$

$= 660$ - ⊛

We only need to calculate

\# of integer solution (x_1, x_2, x_3, x_4)

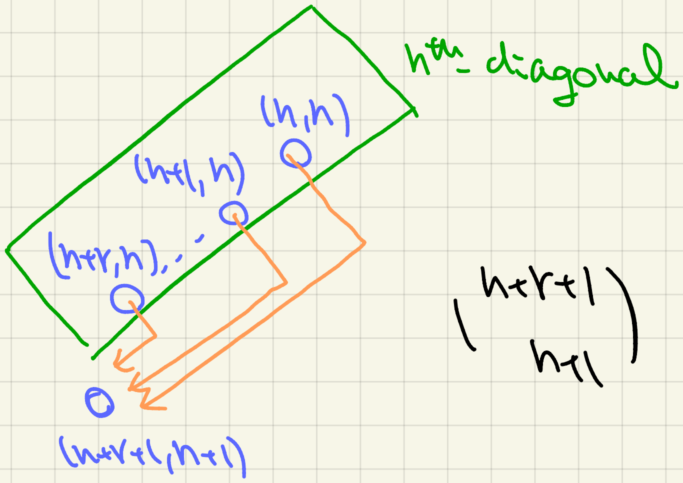
$\ni$ ① $\sum_{i=1}^{4} x_i = 21$ ② $x_i \geq 1$ minus to $|E_1 \cup E_2 \cup E_3 \cup E_4|$

$= \binom{24}{3} - 660$ by Problem 1.1 and ⊛

$= 2024 - 660$

$= 1364$ ∎

2.1



$$\binom{n+r+1}{n+1}$$

Consider where is the last 'R' when we walk to $(n+r+1, n+1)$.

There are $\sum_{0\le k\le r}\binom{n+k}{n} = \binom{n+r+1}{n+1}$ such paths.

By double counting principle, $\sum_{0\le k\le r}\binom{n+k}{n} = \binom{n+r+1}{n+1}$. $(*)$

By symmetric of walk, $\binom{n+k}{n} = \binom{n+k}{k}$ and $\binom{n+r+1}{n+1} = \binom{n+r+1}{r}$. $(**)$

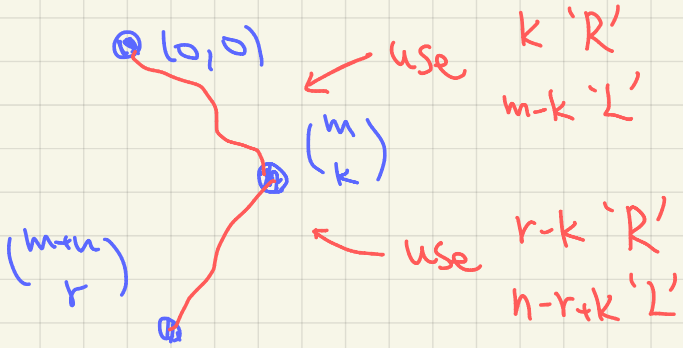
Hence, $\sum_{0\le k\le r}\binom{n+k}{k} = \sum_{0\le k\le r}\binom{n+k}{n}$ by $(**)$

$= \binom{n+r+1}{n+1}$ by $(*)$

$= \binom{n+r+1}{r}$ by $(**)$

Thus, $\sum_{0\le k\le r}\binom{n+k}{k} = \binom{n+r+1}{r}$. $\blacksquare$

2.2



Consider we reach m^{th} row using k 'R' first,

and finally, we reach $(m+n, r)$.

There are $\sum_{0\le k\le r}\binom{m}{k}\binom{n}{r-k}$ path.

By double counting principle, $\sum_{0\le k\le r}\binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$. $\blacksquare$

3.1 $\left(\sum_{x\in Y} d(x) = \sum_{A\in F} |Y \cap A| \quad \forall Y \subseteq X\right)$

Let $Y \subseteq X$ be a set.

Consider $|X| \times |F|$ incidence cube $M = (m_{x,A})$,

where $m_{xA} = \begin{cases} 1 & \text{if } x\in Y \wedge x \in A \\ 0 & \text{else.} \end{cases}$

Then $d(x)$ for each $x \in Y$ is the # of 1s in x-th row and

$|Y \cap A|$ for each $A \in F$ is the # of 1s in A-th column.

By the double counting principle, they must be the same.

Hence, $\sum_{x\in Y} d(x) = \sum_{A\in F} |Y \cap A|$.

Since $Y \subseteq X$ is arbitrary chosen, $\sum_{x\in Y} d(x) = \sum_{A \in F} |Y \cap A| \quad \forall Y \subseteq X$.

3.2 $\left(\sum_{x\in X} d(x)^2 = \sum_{A\in F} \sum_{x\in A} d(x) = \sum_{A\in F}\sum_{B\in F} |A \cap B|\right)$

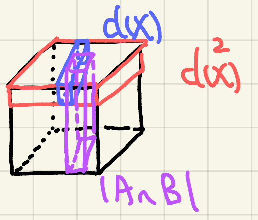
Consider $|X| \times |F| \times |F|$ incidence matrix $M = (m_{x,A,B})$,

where $m_{xA} = \begin{cases} 1 & \text{if } x\in A \wedge x \in B \\ 0 & \text{else.} \end{cases}$

Then $d^2(x)$ for each $x \in X$ is the # of 1s in x-th YZ-plane,

$d(x)$ for each $A \in F$, $x \in A$ is the # of 1s in $(x, A, \cdot)$ straight line, and

$|A \cap B|$ for each $A \in F, B \in F$ is the # of 1s in $(\ , A, B)$ straight line.



Hence, $\sum_{x\in X} d(x)^2$, $\sum_{A\in F}\sum_{x\in F} d(x)$, $\sum_{A\in F}\sum_{B\in F} |A\cap B|$ is the # of 1s in the cube.

By the double counting principle, they must be the same.

Hence, $\sum_{x\in X} d(x)^2 = \sum_{A\in F}\sum_{x\in A} d(x) = \sum_{A\in F}\sum_{B\in F} |A \cap B|$.

4. By definition of dense, $2\alpha \le 1$, i.e. $\alpha \le 0.5$.
If "at least an $\alpha/(1-\alpha)$ fraction of its row must be α-dense" is false, it means that at most or equal an $\alpha/(1-\alpha)$ fraction of its row are α-dense.

Hence, # of 1s in H = # of 1s in α-dense row + # of 1s in non-α-dense row

$$< \underbrace{\left(\frac{\alpha}{1-\alpha}m\right)n}_{\text{by hypothesis}} + \underbrace{\left(1-\frac{\alpha}{1-\alpha}m\right)\alpha n}_{\text{the other rows}}$$

by hypothesis: at most $\frac{\alpha}{1-\alpha}m$ rows are all 1s

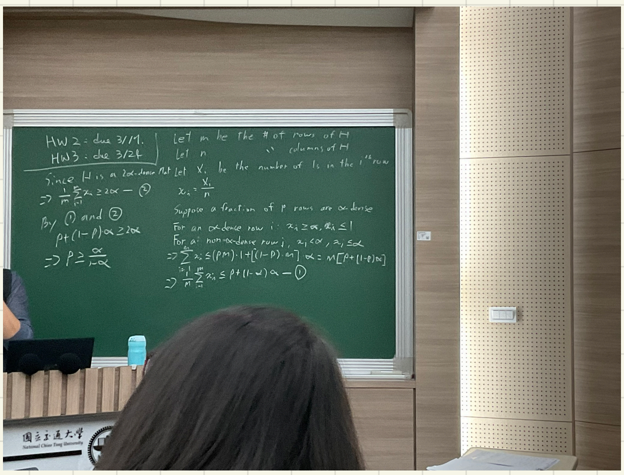
the other rows are not α-dense (strictly larger than 2nd term in LHS)

$$= mn\left(\frac{\alpha}{1-\alpha} + \frac{\alpha - 2\alpha^2}{1-\alpha}\right)$$

$$= \frac{2\alpha - 2\alpha^2}{1-\alpha} mn = 2\alpha mn \quad (*)$$

Moreover, since H is 2α-dense, # of 1s in H $\ge 2\alpha mn$ $(**)$

Hence, $2\alpha mn \overset{(*)}{>}$ # of 1s in H $\overset{(**)}{\ge} 2\alpha mn$. ※
Thus, our hypothesis is false.
Therefore, at least an $\alpha/(1-\alpha)$ fraction of its row must be α-dense. ∎



5.1 ($|F| \le 2^{n-1}$)

If not, $|F| > 2^n$.

Let $F = \{f_1, \dots, f_m\}$ with distinct elements and $m > 2^{n-1}$. - $\circledast$

Consider $S = F \cup \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m\}$.

Note that F and $\{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m\}$ must overlap.

(Otherwise, $|S| = |F| + m = 2m > 2^n$ by $\circledast$ $*$)

Since $f_i \neq f_j$ and $\bar{f}_i \neq \bar{f}_j$ for all $i \neq j$, $\exists p, q \in \{1, \dots, m\} \ni f_i = \bar{f}_j$.

And $f_i \cap f_j = \bar{f}_j \cap f_j = \emptyset$. ($*$ to Condition 1)

Hence, $|F| \le 2^{n-1}$.

5.2 ($2^{n-1} - 1 \le |F|$)

If not, $2^{n-1} - 1 > |F|$.

Let $F = \{f_1, \dots, f_m\}$ with distinct elements and $m < 2^{n-1} - 1$ - $\circledast$

Consider $S = F \cup \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m\} \cup \emptyset \cup \{X\}$.

Note that $|S| \le |F| + m + 1 + 1 = 2m + 2 < 2^n$ by $\circledast$.

Hence, $\exists A \subsetneq X,\ A \neq \emptyset \ni A \notin S$.

Since $F \subseteq S$ and $A \notin S$, $A \notin F$.

Claim 1 $\bar{A} \notin F$

If not, $\bar{\bar{A}} = A \in \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m\} \subseteq S$. $*$

Claim 2 $\nexists B \in F \ni A \cap B = \emptyset$.

If not, $\exists B \in F \ni A \cap B = \emptyset$.

Thus, $B = \bar{A}$ or $\emptyset$.

By claim 1, $B = \emptyset$.

Thus, $\exists B = B = \emptyset \in F \ni B \cap B = \emptyset$. ($*$ to Condition 1)

Hence, $2^{n-1} - 1 \le |F|$

By 5.1 and 5.2, $2^{n-1} - 1 \le |F| \le 2^{n-1}$. $\square$