Problem 1 (20%)**.** Prove that every set of $n+1$ distinct integers chosen from $\{1, 2, \ldots, 2n\}$ contains a pair of consecutive numbers and a pair whose sum is $2n+1$.

For each n, exhibit two sets of size n to show that the above results are the best possible, i.e., sets of size $n+1$ are necessary.

Hint: Use pigeonholes $(2i, 2i-1)$ and $(i, 2n-i+1)$ for $1 \le i \le n$.

Problem 2 (20%)**.** Let $G = (V, E)$ be a graph. Denote by $\chi(G)$ the minimum number of colors needed to color the vertices in V such that, no adjacent vertices are colored the same. Prove that, $\chi(G) \le \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices.

Hint: Order the vertices $v_1, v_2, \ldots, v_n$ and use greedy coloring. Show that it is possible to color the graph using $\Delta(G) + 1$ colors.

Problem 3 (20%)**.** Let $\alpha(G)$ be the *independence number* of a graph G, i.e., the maximum size of any independent set of G. Prove the following dual version of Turán's theorem:

If G is a graph with n vertices and $nk/2$ edges, where $k \ge 1$, then we have

$$\alpha(G) \ge n/(k+1).$$

Problem 4 (20%)**.** Consider the following two problems regarding Markov's and Chebyshev's inequalities.

- For any positive integer k, describe a non-negative random variable X such that

$$\Pr[\, X \ge k \cdot \mathrm{E}[X] \,] \;=\; \frac{1}{k}.$$

  Note that, this shows that Markov's inequality is as tight as it could possibly be.

- Can you provide an example that shows that Chebyshev's inequality is tight? If not, explain why not.

Problem 5 (20%)**.** Suppose that we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these random bits in any order. Let Y_i be the exclusive-or (XOR) of the i^{th} pair of bits, and let $Y := \sum_{1 \le i \le m} Y_i$.

- Show that $Y_i = 0$ and $Y_i = 1$ with probability $1/2$ each.
- Show that $\mathrm{E}[\, Y_i \cdot Y_j \,] = \mathrm{E}[Y_i] \cdot \mathrm{E}[Y_j]$ for any $1 \le i, j \le m$ and derive $\mathrm{Var}[Y]$.
- Use Chebyshev's inequality to derive a bound on $\Pr[\, |Y - \mathrm{E}[Y]| \ge n \,]$.

Problem 1.1.1 (consecutive numbers)

Consider $(2i, 2i-1)$ boxes for $i=1,\cdots,n$.

And we put $(n+1)$ numbers into such n boxes.

By pigeohole principle, there are at least 2 numbers in the same box.

Hence, every set of $n+1$ distinct integers choose from $\{1,2,\cdots,2n\}$ must contain a pair of consecutive numbers.

Problem 1.1.2 (sum $=2n+1$)

Consider $(i, 2n+1-i)$ boxes for $i=1,\cdots,n$.

And we put $(n+1)$ numbers into such n boxes.

By pigeohole principle, there are at least 2 numbers in the same box.

Hence, every set of $n+1$ distinct integers choose from $\{1,2,\cdots,2n\}$ must contain a pair whose sum is $2n+1$.

Problem 1.1.3 (consecutive numbers, $n+1$ is tight)

If not, every set of $n+1$ distinct integers choose from $\{1,2,\cdots,2n\}$ must contain a pair of consecutive numbers.

Consider choose $S=\{1,3,\cdots,2n-1\}$ with $|S|=n$. *

Hence, the above result are the best possible.

Problem 1.1.4 (sum $=2n+1$)

If not, every set of $n+1$ distinct integers choose from $\{1,2,\cdots,2n\}$ must contain a pair whose sum is $2n+1$.

Consider choose $S=\{1,2,\cdots,n\}$ with $|S|=n$. *

Hence, the above result are the best possible.

Problem 2.

Take $\Delta(G)+1$ distinct colors.

We random pick a uncolored node p.

Then we choose a color which does not use to color any neighbor of p.

(Since $d(p) \leq \Delta(G)$ and we have $\Delta(G)+1$ colors, the above color always exists.)

This process will until all nodes are colored.

Hence, $\chi(G) \leq \Delta(G)+1$.

Problem 3.

Idea 1: If we show that the complete graph of G has a $\lceil n/(k+1) \rceil$ clique, then we $\alpha(G) \geq \lceil n/(k+1) \rceil \geq n/(k+1)$ by picking such clique in the original graph G as a independent set.

Idea 2: If $E^c = \{(u,v) : (u,v) \notin E\}$ with $|E^c| > \left(1 - \frac{1}{\lceil n/(k+1) \rceil - 1}\right) \frac{n^2}{2}$,

then by Turán Thm, $G^c = (V, E^c)$ has a $\lceil n/(k+1) \rceil$-clique.

Since $|E| = \frac{nk}{2}$, $|E^c| = \frac{n(n-1)}{2} - \frac{nk}{2}$

$$= \frac{n^2}{2}\left(\frac{n-1}{n} - \frac{k}{n}\right) = \frac{n^2}{2}\left(1 - \frac{1+k}{n}\right)$$

$$= \frac{n^2}{2}\left(1 - \frac{1}{n/(1+k)}\right).$$

$$\geq \frac{n^2}{2}\left(1 - \frac{1}{\lceil n/(1+k) \rceil - 1}\right)$$

since $x \geq \lceil x \rceil - 1$.

By the previous discussion, we have $\alpha(G) \geq \frac{n}{k+1}$.

Problem 4.1

Let $k \in \mathbb{R}^+$.

Define a random variable $X = \begin{cases} 0, & \text{with probability } 1-\frac{1}{k}, \\ 1, & \text{with probability } \frac{1}{k}. \end{cases}$

Then $E[X] = 0\cdot\left(1-\frac{1}{k}\right) + 1\cdot\frac{1}{k}$

$= \frac{1}{k}$. $\circledast$

And $\Pr[X \geq kE[X]] = \Pr[X \geq 1]$ by $\circledast$

$= \frac{1}{k}$ by the def. of X.

Problem 4.2

Let $t \in \mathbb{R}^+$.

Define a random variable $Y = \begin{cases} -t & \text{with probability } \frac{1}{2t^2}, \\ 0 & \text{with probability } 1-\frac{1}{t^2}, \\ t & \text{with probability } \frac{1}{2t^2}. \end{cases}$

Then $E[Y] = (-t)\times\frac{1}{2t^2} + 0\cdot\left(1-\frac{1}{t^2}\right) + t\cdot\left(\frac{1}{2t^2}\right) = 0$, $\circledast$

$Var[Y] = E[(Y-E[Y])^2]$

$= E[Y^2]$ (Since $\circledast$, $E[Y]=0$)

$= (-t)^2\times\frac{1}{2t^2} + 0^2\cdot\left(1-\frac{1}{t^2}\right) + t^2\cdot\left(\frac{1}{2t^2}\right) = 0$

$= 1$.

Moreover, $\Pr[|Y-E[Y]| \geq t] = \Pr[|Y| \geq t]$

$= 2\cdot\frac{1}{2t^2} = \frac{1}{t^2}$

$= \frac{Var[Y]}{t^2}$.

Hence, Chebyshev's inequality is tight.

Problem 5.1

Let X_i be a random variable $= \begin{cases} 0, \text{ with probability } 1/2, \\ 1, \text{ with probability } 1/2 \end{cases}$

for $1 \le i \le n$.

Suppose Y_i is obtained by (X_i XOR X_j with $i<j$)

Then $P(Y_i=0) = P((X_i, X_j)=(0,0) \vee (X_i, X_j)=(1,1))$

$= P((X_i, X_j)=(0,0)) + P((X_i, X_j)=(1,1))$

$= P(X_i=0)P(X_j=0) + P(X_i=1)P(X_j=1)$

(since coin is fair)

$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = 1/2$

Since $Y_i \in \{0,1\}$, $P(Y_i=1) = 1 - P(Y_i=0) = 1/2$. $\blacksquare$

Problem 5.2.1

Suppose $Y_i = X_a$ XOR X_b and $Y_j = X_c$ XOR X_d

with $c<d$.

Case 1: $a \neq c$ and $b \neq d$

$E[Y_i \cdot Y_j] = 1 \cdot (P(Y_i=1 \text{ and } Y_j=1) + 0 \cdot [1 - P(Y_i=1 \text{ and } Y_j=1)]$

$= 1 \cdot 1 \cdot \frac{1}{2} \times \frac{1}{2} + 0$ (by Pronble 5.1 and Y_i, Y_j are independent)

$= \frac{1}{4}$

$= \frac{1}{2} \cdot \frac{1}{2} = E[Y_i] \cdot E[Y_j]$ (by Pronble 5.1)

Case $a=c$ and $b=d$

$E[Y_i \cdot Y_j] = E[Y_i^2]$

$= 1 \cdot P[Y_i=1] + 0 \cdot (1 - P(Y_i=1))$

$= \frac{1}{2}$ (by Pronble 5.1)

$\neq \frac{1}{4} = E[Y_i^2] = E[Y_i \cdot Y_j]$.

Case 3 $a=c$ and $b\neq d$

$$E[Y_i \cdot Y_j] = 1\cdot(P(Y_i=1 \text{ and } Y_j=1) + 0\cdot[1-P(Y_i=1 \text{ and } Y_j=1)]$$
$$= P((X_a, X_b, X_d) = (1,0,1) \text{ or } (X_a, X_b, X_d) = (0,1,0))$$
$$= P((X_a, X_b, X_d) = (1,0,1)) + P((X_a, X_b, X_d) = (0,1,0))$$
$$= \left(\frac{1}{2}\right)^3 \times 2 = \frac{1}{4}.$$
$$= \frac{1}{2}\cdot\frac{1}{2} = E[Y_i]\cdot E[Y_j] \text{ (by Prouble 5.1)}$$

Case 4 $a\neq c$ and $b=d$

Similarly to Case 3, we have $E[Y_i \cdot Y_j] = E[Y_i]E[Y_j]$

Conclusion

$E[Y_i \cdot Y_j] = E[Y_i]E[Y_j]$ for all $1 \le i,j \le n$ with $i \neq j$.

Problem 5.2.2

Since $E[Y_i] = \frac{1}{2}$ for all $1\le i \le n$, $E[Y] = E[\sum_{1\le i\le m} Y_i] = \frac{m}{2}$.

$$E[Y^2] = E\left[\sum_{i=1}^{m}\sum_{j=1}^{m} Y_i Y_j\right] = E\left[\sum_{\substack{i\neq j \\ 1\le i,j\le m}} Y_i Y_j\right] + E\left[\sum_{\substack{i=j \\ 1\le i,j\le m}} Y_i Y_j\right]$$
$$= \binom{m}{2} \times 2! \cdot \frac{1}{4} + m\cdot\frac{1}{2}$$
$$= \frac{m(m-1)}{4} + \frac{m}{2}$$

Hence, $\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2$
$$= \frac{m^2-m}{4} + \frac{m}{2} - \frac{m^2}{4}$$
$$= \frac{m}{4}. \quad \blacksquare$$

Problem 5.3.

By Chebyshev's inequality, $\Pr(|Y-E[Y]| \ge t) \le \frac{\mathrm{Var}[Y]}{t^2}$ $\forall t>0$.

Hence, $\Pr(|Y-E[Y]| \ge n) \le \frac{\mathrm{Var}[Y]}{n^2} = \frac{m}{4n^2} = \frac{n(n-1)}{8n^2} = \frac{n-1}{8n} \le \frac{n}{8n} = \frac{1}{8}. \quad \blacksquare$