



Problem 1 (20%). How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 = 21$ with

- 1. $x_i \geq 0$.
- 2. $x_i > 0$.
- 3. $0 \leq x_i \leq 12$.



Problem 2 (20%). Prove the following identities **using path-walking argument**.



- For any $n, r \in \mathbb{Z}^{\geq 0}$,

$$\sum_{0 \leq k \leq r} \binom{n+k}{k} = \binom{n+r+1}{r}.$$



- For any $m, n, r \in \mathbb{Z}^{\geq 0}$ with $0 \leq r \leq m+n$,

$$\sum_{0 \leq k \leq r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}.$$



Problem 3 (20%). Let \mathcal{F} be a set family on the ground set X and $d(x)$ be the degree of any $x \in X$, i.e., the number of sets in \mathcal{F} that contains x . Use the double counting principle to prove the following two identities.

$$\sum_{x \in Y} d(x) = \sum_{A \in \mathcal{F}} |Y \cap A| \text{ for any } Y \subseteq X.$$

$$\sum_{x \in X} d(x)^2 = \sum_{A \in \mathcal{F}} \sum_{x \in A} d(x) = \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} |A \cap B|.$$



Problem 4 (20%). Let H be a 2α -dense 0-1 matrix. Prove that at least an $\alpha/(1 - \alpha)$ fraction of its rows must be α -dense.



Problem 5 (20%). Let \mathcal{F} be a family of subsets defined on an n -element ground set X . Suppose that \mathcal{F} satisfies the following two properties:



- 1. $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$.
- 2. For any $A \subsetneq X$, $A \notin \mathcal{F}$, there always exists $B \in \mathcal{F}$ such that $A \cap B = \emptyset$.

Prove that

$$2^{n-1} - 1 \leq |\mathcal{F}| \leq 2^{n-1}.$$



Hint: Consider any set $A \subseteq X$ and its complement \bar{A} . Apply the conditions given above and prove the two inequalities " \leq " and " \geq " separately.

1-1

Let us map this problem to an equivalent problem.

Suppose there are $N=21$ the same blue balls and $M=4$ the same red balls.

Q: How many ways to permute them?

We can assign $X_1 = \#$ of blue balls which are to the left of the leftmost red ball,

$X_2 = \#$ of blue balls which are between the leftmost red ball and the second leftmost red ball,

$X_3 = \#$ of blue balls which are between the second leftmost, red ball and the third leftmost red ball, and

$X_4 = \#$ of blue balls which are to the right of the rightmost red ball.

Since such mapping function is 1-1 and onto, these two problems have the same answer.

Hence, # of integer solution (X_1, X_2, X_3, X_4)

$$\Rightarrow \sum_{i=1}^4 X_i = 21 \quad X_i \geq 0$$

$$= \frac{24!}{3! 2!} = \frac{24 \times 23 \times 22}{6} = 2024$$

1.2 Notice that the condition " $x_i > 0 \wedge x_i \in \mathbb{Z}$ " is equivalent to " $x_i \geq 1 \wedge x_i \in \mathbb{Z}$ ".

Moreover, # of integer solution (x_1, x_2, x_3, x_4)

$$\Rightarrow \textcircled{1} \sum_{i=1}^4 x_i = 21 \quad \textcircled{2} \quad x_i \geq 1 \quad \Rightarrow \text{equal to}$$

of integer solution (y_1, y_2, y_3, y_4)

$$\Rightarrow \textcircled{1} \sum_{i=1}^4 y_i = 17 \quad \textcircled{2} \quad y_i \geq 0$$

by constructing a 1-1 onto mapping

$$\begin{aligned} f(y_1, y_2, y_3, y_4) &= (y_1 + 1, y_2 + 1, y_3 + 1, y_4 + 1) \\ &=: (x_1, x_2, x_3, x_4). \end{aligned}$$

Hence, we only need to calculate

of integer solution (y_1, y_2, y_3, y_4)

$$\Rightarrow \textcircled{1} \sum_{i=1}^4 y_i = 17 \quad \textcircled{2} \quad y_i \geq 0$$

$$= \binom{17 + (4-1)}{4-1} = \binom{20}{3} = \frac{20 \times 19 \times 18}{6} = 1140 \quad \textcircled{3}$$

1.3 Define $E_j = \{(x_1, x_2, x_3, x_4) : \sum_{i=1}^4 x_i = 21 \text{ and } x_i \geq 13 \wedge x_j \geq 0$
 $\text{for } j=1, 2, 3, 4\}$ for $i=1, 2, 3, 4$.

$$|E_1 \cup E_2 \cup E_3 \cup E_4|$$

$$\begin{aligned} &= |E_1| + |E_2| + |E_3| + |E_4| + \sum_{|I|=2, I \subseteq \{1, 2, 3, 4\}} (-1)^{2+1} |A_I| + \sum_{|I|=3, I \subseteq \{1, 2, 3, 4\}} (-1)^{3+1} |A_I| \\ &\quad + \sum_{|I|=4, I \subseteq \{1, 2, 3, 4\}} (-1)^{4+1} |A_I| \quad \text{by Prop 1.14} \end{aligned}$$

$$= 4|E_1| \text{ since } |A_I|=0 \quad \forall I \subseteq \{1, 2, 3, 4\} \text{ with}$$

size ≥ 2 and the symmetric of E_j .

$$\text{Similar to Problem 1.2, } |E_1| = \binom{21-13+(4-1)}{4-1} = \binom{11}{3} = \frac{11 \times 10 \times 9}{6} = 165$$

$$\begin{aligned}\text{Hence, } |E_1 \cup E_2 \cup E_3 \cup E_4| &= 4|E_1| \\ &= 4 \times 165 \\ &= 660 \quad -\star\end{aligned}$$

We only need to calculate

of integer solution (x_1, x_2, x_3, x_4)

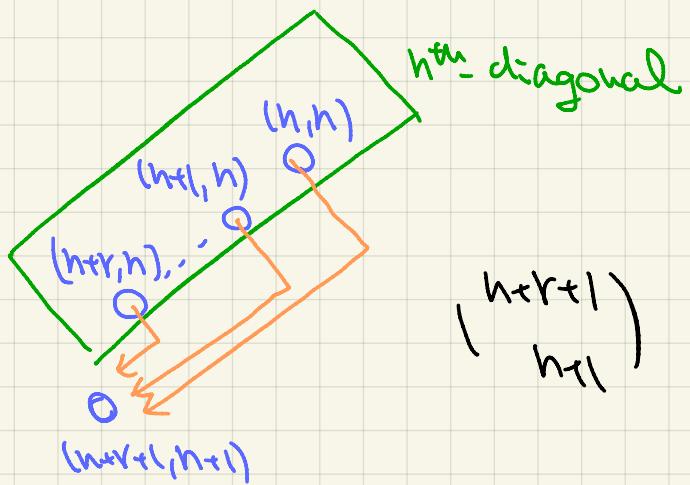
$$\Rightarrow \sum_{i=1}^4 x_i = 21 \quad (x_i \geq 1) \text{ minus to } (E_1 \cup E_2 \cup E_3 \cup E_4)$$

$$= \binom{24}{3} - 660 \text{ by Problem 1.1 and } \star$$

$$= 2024 - 660$$

$$= 1364 \quad \text{④}$$

2.1



Consider where R is the last 'R' when we walk to $(h+r+1, h+r+1)$.

There are $\sum_{0 \leq k \leq r} \binom{n+k}{n} = \binom{n+r+1}{n+r+1}$ such paths.

By double counting principle, $\sum_{0 \leq k \leq r} \binom{n+k}{n} = \binom{n+r+1}{n+r+1}$. ⊗

By symmetry of walk, $\binom{n+k}{n} = \binom{n+k}{k}$ and $\binom{n+r+1}{n+r+1} = \binom{n+r+1}{r}$. ⊗

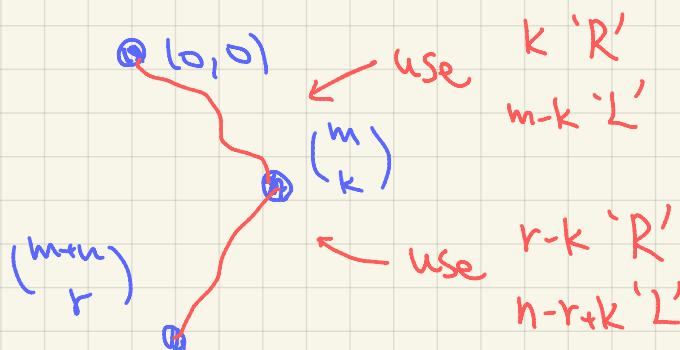
Hence, $\sum_{0 \leq k \leq r} \binom{n+k}{k} = \sum_{0 \leq k \leq r} \binom{n+k}{n}$ by ⊗

$$= \binom{n+r+1}{n+r+1} \quad \text{by } \text{⊗}$$

$$= \binom{n+r+1}{r} \quad \text{by } \text{⊗}$$

Thus, $\sum_{0 \leq k \leq r} \binom{n+k}{k} = \binom{n+r+1}{r}$. ⊗

2.2



Consider we reach with row using $k 'R'$ first, and finally, we reach $(m+n, r)$.

There are $\sum_{0 \leq k \leq r} \binom{m}{k} \binom{n}{r-k}$ path.

By double counting principle, $\sum_{0 \leq k \leq r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$. ⊗

$$3.1 \left(\sum_{x \in Y} d(x) = \sum_{A \in F} |Y \cap A| \quad \forall Y \subseteq X \right)$$

Let $Y \subseteq X$ be a set.

Consider $|X| \times |F|$ incidence cube $M = (m_{x,A})$,

$$\text{where } m_{x,A} = \begin{cases} 1 & \text{if } x \in Y \wedge x \in A \\ 0 & \text{else.} \end{cases}$$

Then $d(x)$ for each $x \in Y$ is the # of 1s in x -th row and

$|Y \cap A|$ for each $A \in F$ is the # of 1s in A -th column.

By the double counting principle, they must be the same.

$$\text{Hence, } \sum_{x \in Y} d(x) = \sum_{A \in F} |Y \cap A|.$$

Since $Y \subseteq X$ is arbitrary chosen, $\sum_{x \in Y} d(x) = \sum_{A \in F} |Y \cap A| \quad \forall Y \subseteq X$.

$$3.2 \left(\sum_{x \in X} d(x)^2 = \sum_{A \in F} \sum_{x \in A} d(x) = \sum_{A \in F} \sum_{B \in F} |A \cap B| \right)$$

Consider $|X| \times |F| \times |F|$ incidence matrix $M = (m_{x,A,B})$,

$$\text{where } m_{x,A,B} = \begin{cases} 1 & \text{if } x \in A \wedge x \in B \\ 0 & \text{else.} \end{cases}$$

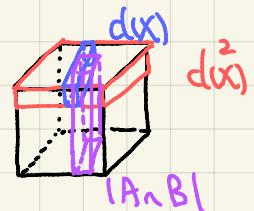
Then $d(x)$ for each $x \in X$ is the # of 1s in x -th YZ -plane,

$d(x)$ for each $A \in F, x \in A$ is the # of 1s in

(x, A, \cdot) straight line, and

$|A \cap B|$ for each $A \in F, B \in F$ is the # of 1s in

(\cdot, A, B) straight line.



Hence, $\sum_{x \in X} d(x)^2, \sum_{A \in F} \sum_{x \in A} d(x), \sum_{A \in F} \sum_{B \in F} |A \cap B|$ is the # of 1s in the cube.

By the double counting principle, they must be the same.

$$\text{Hence, } \sum_{x \in X} d(x)^2 = \sum_{A \in F} \sum_{x \in A} d(x) = \sum_{A \in F} \sum_{B \in F} |A \cap B|.$$

4. By definition of dense, $2d < 1$, i.e. $d < 0.5$.

If "at least an $d/(1-d)$ fraction of its row must be d -dense" is false, it means

that at most or equal an $d/(1-d)$ fraction of its row are d -dense.

Hence, # of 1s in $H = \#$ of 1s in d -dense row

$$+ \# \text{ of 1s in non-}d\text{-dense row}$$

$$< \left(\frac{d}{1-d} m \right) n + \left(1 - \frac{d}{1-d} m \right) d n$$

by hypothesis

at most $\frac{d}{1-d} m$ rows
are all 1s

the other rows
are not d -dense
(strictly larger
than 2nd term
in LHS)

$$= mn \left(\frac{d}{1-d} + \frac{d-2d^2}{1-d} \right)$$

$$= \frac{2d-2d^2}{1-d} mn = 2d mn$$

Moreover, since H is $2d$ -dense, # of 1s in $H \geq 2d mn$ ~~*~~

Hence, $2d mn > \# \text{ of 1s in } H \geq 2d mn$. *

Thus, our hypothesis is false.

Therefore, at least an $d/(1-d)$ fraction of its row must be d -dense. \blacksquare

5.1 ($|F| \leq 2^{n-1}$)

If not, $|F| > 2^n$.

Let $F = \{f_1, \dots, f_m\}$ with distinct elements and $M > 2^{n-1}$. \otimes

Consider $S = F \cup \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m\}$.

Note that F and $\{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m\}$ must overlap.

(Otherwise, $|S| = |F| + m = 2m > 2^n$ by $\oplus \star$)

Since $f_i \neq f_j$ and $\bar{f}_i \neq \bar{f}_j$ for all $i \neq j, \exists p, q \in \{1, \dots, m\} \Rightarrow f_i = \bar{f}_j$.

And $f_i \cap f_j = \bar{f}_j \cap \bar{f}_i = \emptyset$. (\star to Condition!)

Hence, $|F| \leq 2^{n-1}$.

5.2 ($2^{n-1} - 1 \leq |F|$)

If not, $2^{n-1} - 1 > |F|$.

Let $F = \{f_1, \dots, f_m\}$ with distinct elements and $M < 2^{n-1} - 1$. \otimes

Consider $S = F \cup \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m\} \cup \emptyset \cup \{X\}$.

Note that $|S| \leq |F| + m + 1 + 1 = 2m + 2 < 2^n$ by \oplus .

Hence, $\exists A \subseteq X, A \neq \emptyset \Rightarrow A \notin S$.

Since $F \subseteq S$ and $A \notin S, A \notin F$.

Claim $\bar{A} \notin F$

If not, $\bar{A} = A \in \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_m\} \subseteq S$. \star

Claim $\#B \in F \Rightarrow A \cap B = \emptyset$.

If not, $\exists B \in F \Rightarrow A \cap B \neq \emptyset$.

Thus, $B = \bar{A}$ or \emptyset .

By claim 1, $B = \emptyset$.

Thus, $\exists B = B = \emptyset \in F \Rightarrow B \cap B = \emptyset$. (\star to Condition!)

Hence, $2^{n-1} - 1 \leq |F|$

By 5.1 and 5.2, $2^{n-1} - 1 \leq |F| \leq 2^{n-1}$. \square