14

$x_i \geq 13$ $\quad$ $\underbrace{\quad}_{}^{8}$

Problem 1 (20%). How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 = 21$ with

1. $x_i \geq 0$. $\quad$ 2. $x_i > 0$. $\quad$ 3. $0 \leq x_i \leq 12$.

Problem 2 (20%). Prove the following identities **using path-walking argument**.

0000|0 0|0 ... 0

1. For any $n, r \in \mathbb{Z}^{\geq 0}$,

$$\sum_{0 \leq k \leq r} \binom{n+k}{k} = \binom{n+r+1}{r}.$$

2. For any $m, n, r \in \mathbb{Z}^{\geq 0}$ with $0 \leq r \leq m+n$,

$$\sum_{0 \leq k \leq r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

Problem 3 (20%). Let $\mathcal{F}$ be a set family on the ground set X and $d(x)$ be the degree of any $x \in X$, i.e., the number of sets in $\mathcal{F}$ that contains x. Use the double counting principle to prove the following two identities.

$$\sum_{x \in Y} d(x) = \sum_{A \in \mathcal{F}} |Y \cap A| \text{ for any } Y \subseteq X.$$

$2 - 2\alpha > \alpha$

$3\alpha < 2, \ \alpha < \frac{2}{3}$

$$\sum_{x \in X} d(x)^2 = \sum_{A \in \mathcal{F}} \sum_{x \in A} d(x) = \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} |A \cap B|.$$

$2\alpha^2$

$(1-2\alpha)\cdot\alpha$

$\frac{\alpha}{1-\alpha} < 2$ $\quad$ $\frac{1-\alpha^2}{1-\alpha}$

most

Problem 4 (20%). Let H be a 2α-dense 0-1 matrix. Prove that at least an $\alpha/(1-\alpha)$ fraction of its rows must be α-dense.

$|H| \leq \frac{\alpha}{1-\alpha}\cdot m\cdot(\cancel{2\alpha\cdot n}) + (1 - \frac{\alpha}{1-\alpha})m(\alpha\cdot n) < 1-\alpha$

怎麼想到的?

Problem 5 (20%). Let $\mathcal{F}$ be a family of subsets defined on an n-element ground set X. Suppose that $\mathcal{F}$ satisfies the following two properties:

$A = \{2,4,5\}$. $\quad$ $B \in \bar{A}$, $B \in \mathcal{F}$

1. $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$.

$A \subset X$, $A \notin \mathcal{F}$

2. For any $A \subsetneq X$, $A \notin \mathcal{F}$, there always exists $B \in \mathcal{F}$ such that $A \cap B = \emptyset$.

$\bar{A} \in \mathcal{F}$

Prove that $\quad$ ϕ. X, $\notin$

$$2^{n-1} - 1 \leq |\mathcal{F}| \leq 2^{n-1}.$$

(A or $\bar{A}$ in $\mathcal{F}$) $\quad$ $A \in \mathcal{F}$ $\to \bar{A} \notin \mathcal{F}$

12 $\quad$ 12345 $\quad$ F

Hint: Consider any set $A \subseteq X$ and its complement $\overline{A}$. Apply the conditions given above and prove the two inequalities "$\leq$" and "$\geq$" separately.

1 → 1

if $A \in \mathcal{F}$. $\quad$ if $A \in \mathcal{F} \to \bar{A} \notin \mathcal{F}$

2 → 2 $\quad$ if $\bar{A} \in \mathcal{F}$ too,

3 →

$A, \bar{A}$ not in $\mathcal{F}$

HW1

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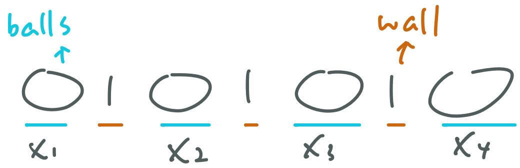
Problem 1

(1.)

$x_1 + x_2 + x_3 + x_4 = n$ can interpret as permutation of balls and walls



Consider there are 21 same balls and 3 same walls permutating.

then the # of solutions $= \frac{24!}{21! \cdot 3!} = 2024$ #

(2.)

distribute each x_i one ball to ensure $x_i > 0$,

so there are 17 balls remaining

Consider there are 17 same balls and 3 same walls permutating,

then the # of solutions $: \frac{20!}{17! \cdot 3!} = 1140$

#

(3.)

Consider the opposite question $x_i \geq 13$ where $x_1 + x_2 + x_3 + x_4 = 21$,

let's distribute 13 balls to a x_i, so their are 8 balls remaining.

$\rightarrow$ # of permutations $= \frac{11!}{8! \cdot 3!} \times 4 = 528$

$\therefore$ # of $0 \leq x_i \leq 13$ $= (\# \text{ of } x_i \geq 0) - (\# \text{ of } x_i \geq 13)$

$= 1364$ #

Problem 2

1.

Consider the set of all possible upward and rightward paths in $\mathbb{N}^2$ space

① There are $\binom{n+r+1}{r}$ such paths from $(0,0)$ to $(n+1, r)$

② Decompose path $(0,0) \rightarrow (n+1, r)$ to two part $(0,0) \rightarrow (n,k) \wedge (n,k) \rightarrow (n+1, r)$ $\forall\, 0 \leq k \leq r$

$\because$ $(n,k) \rightarrow (n+1, r)$ has only one path with 1 'R's and $r-k$ 'U's

$\therefore$ there are $\sum_{k=0}^{r} \binom{n+k}{n} \cdot 1$ paths

By ① ② + Doubing counting principle, $\sum_{k=0}^{r}\binom{n+k}{k} = \binom{n+r+1}{r}$ #

2.

Consider the set of all possible upward and rightward paths in $\mathbb{N}^2$ space

① There are $\binom{m+n}{r}$ from $(0,0)$ to $(m+n-r, r)$

② Decompose path $(0,0) \to (n+1, r)$ to two part $(0,0) \to (m-k, k)$ $\wedge$ $(m-k, k) \to (m+n-r, r)$ $\forall\, 0 \le k \le r$

$(0,0) \to (n+1, r)$ has $\binom{m}{k}$ paths

$(m-k, k) \to (m+n-r, r)$ has $\binom{n}{r-k}$ paths

$\because$ $\forall\, k \in \{1, 2, \dots, m\}$, path from $(0,0)$ to $(m-k, k)$ are mutually exclusive

$\therefore$ total paths $= \sum_{k=0}^{r}\binom{m}{k}\cdot\binom{n}{r-k}$

By ① ② + Doubing counting principle, $\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{k}\cdot\binom{n}{r-k}$

#

Problem 3

1.

Consider the $|X| \times |F|$ incidence matrix $M = (m_{x,A})$

where

$m_{x,A} = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$ $\wedge$ $\begin{cases} d(x) \text{ is the number of 1's in the } x\text{-th row} \\ A \text{ is the set of 1's in } A\text{-th col} \end{cases}$

For any $Y \in X$,

① let iterate each row whose x in Y, plusing the sum of each row

total $m_{x,A} = \sum_{x \in Y} d(x)$

② let iterate each column, plusing the sum of each row whose x in Y

Total $m_{x,A} = \sum_{A \in F} |Y \cap A|$

By ①, ② and double counting principle, $\sum_{x \in Y} d(x) = \sum_{A \in F} |Y \cap A|$

#

1.

Consider the $|X| \times |F| \times |F|$ rank-3 incidence tensor

where

$$m_{x,A,B} = \begin{cases} 1, & \text{if } x \in A \wedge x \in B \\ 0, & \text{otherwise} \end{cases}$$

+

- $d(x)$ is the sum of 1 in the area $m_{x,A,B}$ for all $B \in F$
- $|A \cap B|$ is the sum of 1 in the area $m_{x,A,B}$ for all $x \in X$

① let's iterate each layer $x \in X$,

$\because m_{x,A,B} = 1 \Leftrightarrow m_{x,B,A} = 1 \quad \forall A, B \in F$

By definition of $d(x)$,

$\therefore$ Total number of 1 in each layer $= d(x)^2$

$\Rightarrow$ Total $m_{x,A,B} = \sum_{x \in X} d(x)^2$

② let's iterate each $A \in F$, then iterate each $x \in A$,

By definition of $d(x)$,

$\Rightarrow$ Total $m_{x,A,B} = \sum_{A \in F} \sum_{x \in A} d(x)$

③ let's iterate each $A \in F$, then iterate each $B \in F$,

By definition of $|A \cap B|$

$\Rightarrow$ Total $m_{x,A,B} = \sum_{A \in F} \sum_{B \in F} |A \cap B|$

By ①②③ and Double counting principle, $\sum_{x \in X} d(x)^2 = \sum_{A \in F} \sum_{x \in X} d(x) = \sum_{A \in F} \sum_{B \in F} |A \cap B|$

#

Problem 4

if at most $\frac{\alpha}{1-\alpha}$ fraction of its row are α-dense,

$$\begin{aligned} \#\text{ of 1 in } H &\le (\#\text{ of 1 in } \alpha\text{-dense row}) + (\#\text{ of 1 in non-}\alpha\text{-dense row}) \\ &< \left(\frac{\alpha}{1-\alpha} \cdot m\right) \cdot n + \left(1 - \frac{\alpha}{1-\alpha}\right) \cdot m \cdot (\alpha \cdot n) \\ &= \frac{2\alpha - 2\alpha^2}{1-\alpha} = 2\alpha \end{aligned}$$

$\rightarrow H$ is not 2α-dense $(*)$

$\therefore$ there are at least $\frac{\alpha}{1-\alpha}$ fraction of its row are α-dense #

Problem 5

$(\leq)$

Claim 1 : A or $\bar{A}$ in F $\forall A \subset X \wedge A \neq \phi \wedge A \neq X$

Suppose A and $\bar{A}$ not in F,

By property (2)

there exist $B_1 \subset \bar{A}$, $B_2 \subset A$ that $B_1, B_2 \in F$

but $B_1 \cap B_2 = \phi$ (※. contradict to property (1))

$\therefore$ A or $\bar{A}$ in F ▨

There are $2^n - 2$ total subsets A excluding X, ϕ

By claim 1, $|F|$ has at least $\frac{2^n - 2}{2} = 2^{n-1} - 1$

$(\geq)$

Claim 2 : if $A \in F \rightarrow \bar{A} \notin F$

if $A \in F$,

suppose $\bar{A} \in F$ too,

then $A \cap \bar{A} = \phi$ (※. contradict to property (1))

$\therefore$ $\bar{A} \notin F$ ▨

There are 2^n total subsets of A

By claim 2, $|F|$ has at most $\frac{2^n}{2} = 2^{n-1}$

By $(\geq)$, $(\leq)$, $2^{n-1} - 1 \leq |F| \leq 2^{n-1}$

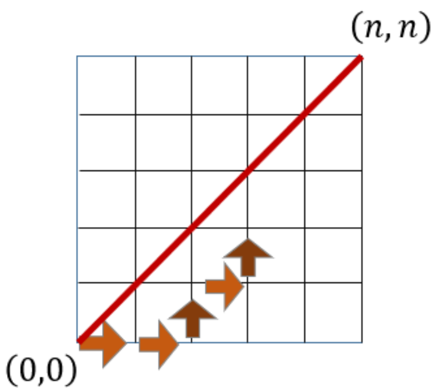
#

Problem 1 (20%). Let X, Y be discrete random variables. The variance of a random variable X is defined as $\mathrm{Var}[X] := \mathrm{E}\left[(X - \mathrm{E}[X])^2\right]$. Prove that

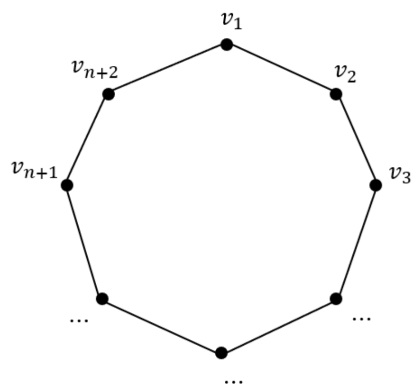
1. $\mathrm{E}[aX + bY] = a \cdot \mathrm{E}[X] + b \cdot \mathrm{E}[Y]$ for any constant a, b.
2. If X and Y are independent, then $\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y]$ and $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.
3. $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$. *Hint:* Use the fact that $\mathrm{E}[X \cdot \mathrm{E}[X]] = \mathrm{E}[X]^2$.

Problem 2 (20%). Consider the slides #2. Prove that the graphs H_i defined in the proof of Theorem 3 are bicliques.

Problem 3 (20%). For any integer $n \geq 1$, consider the grid points (r, c) with $1 \leq r, c \leq n$. Let C_n be the number of possible paths from $(0, 0)$ to (n, n) that use only $\rightarrow$ and $\uparrow$ and that never cross the diagonal $r = c$. See also the Figure (a) below. For convenience, define $C_0 := 1$.



(a)



(b)

For any integer $n \geq 2$, consider the convex $(n+2)$-gon with vertices labeled with $v_1, v_2, \ldots, v_{n+2}$. Let P_n denote the number of possible ways to triangulate the polygon. It follows that $P_2 = 2$, $P_3 = 5$, etc. For convenience, also define $P_0 := 1$ and $P_1 := 1$.

1. Prove that for any $n \geq 2$, P_n satisfies the recurrence

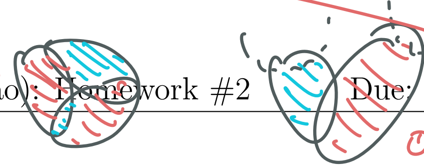
$$P_n = \sum_{0 \leq k < n} P_k \cdot P_{n-k-1}.$$

2. Prove that for any $n \geq 2$, C_n satisfies the same recurrence

$$C_n = \sum_{0 \leq k < n} C_k \cdot C_{n-k-1}.$$

Note that this proves that P_n also equals the n^{th}-Catalan number.

如果有一個x，x只包含於A
這個A可以讓所有人滿2-colorable?



① 反證
② 期望值：distribution
機率法

Problem 4 (20%). Let $\mathcal{F}$ be a family of subsets, where

$$|A| \geq 3 \text{ for any } A \in \mathcal{F} \quad \text{and} \quad |A \cap B| = 1 \text{ for any } A, B \in \mathcal{F},\ A \neq B.$$

∃ monocromatic set

Suppose that $\mathcal{F}$ is not 2-colorable. Let x, y be any elements that appear in $\mathcal{F}$, i.e., $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$ for some $A, B \in \mathcal{F}$. Prove that:

1. x belongs to at least two members of $\mathcal{F}$. (只有1個 member 的機率?) 期望值
2. There exists some $C \in \mathcal{F}$ such that $\{x, y\} \subseteq C$.

Hint: Construct proper coloring to prove the properties. For (1), consider a particular A with $x \in A \in \mathcal{F}$. Color $A \setminus \{x\}$ red and the remaining blue. Show that this leads to the conclusion of (1). For (2), consider particular A, B with $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$. Color $(A \cup B) \setminus \{x, y\}$ red and the remaining blue. Prove that it leads to (2).

Problem 5 (20%). Let $G = (A \cup B, E)$ be a bipartite graph, d be the minimum degree of vertices in A and D the maximum degree of vertices in B. Assume that $|A|d \geq |B|D$.

Show that, for every subset $A_0 \subseteq A$ with the density α defined as $\alpha := |A_0|/|A|$, there exists a subset $B_0 \subseteq B$ such that:

1. $|B_0| \geq \alpha \cdot |B|/2$,
2. every vertex of B_0 has at least $\alpha D/2$ neighbors in A_0, and
3. at least half of the edges leaving A_0 go to B_0.

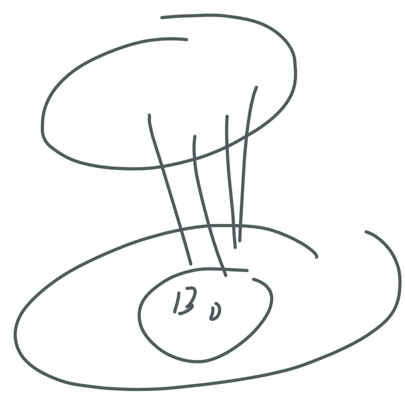
反面想，如果 edges 沒有走到 B_0 呢？

Hint: Let B_0 consist of all vertices in B that have at least $\alpha D/2$ neighbors in A_0. First prove (3) and then (1).

$\alpha \cdot D/2$

$\sum_{i=1}^{|A_0|} d_a \geq \frac{1}{2} \cdot \sum_{i=1}^{|A_0|} d \geq |B| \cdot \frac{\alpha \cdot D}{2}$

$\geq |B - B_0| \cdot \frac{\alpha D}{2}$



$|B_0| \cdot D \geq \frac{1}{2} \cdot d_a \geq \frac{1}{2} \cdot |A_0| \cdot d$

$\geq \frac{1}{2} \cdot \alpha \cdot |B| \cdot D$

HW 2

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Problem 1

(1)

① $E[aX] = \sum_{x=-\infty}^{\infty} x \cdot P(a \cdot X = x) = \sum_{x=-\infty}^{\infty} x \cdot P(X = \frac{x}{a})$

let $x = a \cdot k \rightarrow E[aX] = \sum_{k=-\infty}^{\infty} (a \cdot k) \cdot P(X = k) = a \cdot E[X]$

② $E[X+Y] = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} (x+y) \cdot P(X=x, Y=y)$

$= \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} x \cdot P(X=x, Y=y) + \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} y \cdot P(X=x, Y=y)$

[Given $\sum_{y=-\infty}^{\infty} P(X=x, Y=y) = P(X=x)$]

$= \sum_{x=-\infty}^{\infty} x \cdot P(X=x) + \sum_{y=-\infty}^{\infty} y \cdot P(Y=y) = E[X] + E[Y]$

$\therefore E[aX + bY] \underset{\uparrow \text{ by ①}}{=} E[aX] + E[bY] \underset{\uparrow \text{ by ②}}{=} a \cdot E[X] + b \cdot E[Y]$

(2)

$E[X \cdot Y] = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} x \cdot y \cdot P_{XY}(x,y)$

X, Y is independent $\rightarrow = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} x \cdot y \cdot P_X(x) \cdot P_Y(y)$

$= \left(\sum_{x=-\infty}^{\infty} x \cdot P_X(x)\right) \cdot \left(\sum_{y=-\infty}^{\infty} y \cdot P_Y(y)\right) = E[X] \cdot E[Y]$

(3)

$Var(X) = E[(X - E[X])^2]$

$= E[X^2 - 2 \cdot X \cdot E[X] + (E[X])^2]$

$= E[X^2] - E[2 \cdot X \cdot E[X]] + E[(E[X])^2]$ (linearity) (by (1))

$= E[X^2] - 2 \cdot E[X]^2 + E[(E[X])^2]$ (by the fact)

$= E[X^2] - (E[X])^2$ ($E[X]^2$ is constant)

Problem 2

① $\because$ vertices of K_n is coordinate $\{0,1\}^m$ and

$(u,v) \notin E(H_i)$ if i th - coordinate are the same

$\therefore H_i : A_i \cup B_i$ is a bipartite

with $A_i = \{ v \mid$ the i th - coordinate of $v = 0 \}$

$B_i = \{ v \mid$ the i th - coordinate of $v = 1 \}$

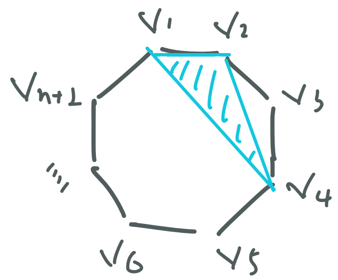
② $\because (u,v) \in E(H_i)$ if i th - coordinate of u, v are different

$\therefore \forall u \in A_i, v \in B_i, (u,v) \in E(H_2)$

By ①, ②, H_i is a $K_{|A_i|, |B_i|}$ biclique #

Problem 3

(1)



fix an edge $\overline{v_1 v_2}$, consider a triangle

whose point is v_1, v_2, v_{k+3}, $\forall k \in \{0, \dots, n-1\}$

① $v_{k+3} \neq v_3$ and $v_{k+3} \neq v_{n+2}$,

there are two polygons remain.

One is $\langle v_2, v_3, \dots, v_{k+3} \rangle$, a $(k+2)$ - gon

, one is $\langle v_1, v_{k+3}, \dots, v_{n+2} \rangle$, a $(n-k+1)$ - gon

$\because$ Triangulization of 2 polygons are independent

$\therefore \# = \sum_{k=1}^{n-2} P_k \cdot P_{n-k-1}$

② there is one polygon $\langle v_1, v_3, v_4, \dots, v_{n+2} \rangle$ or

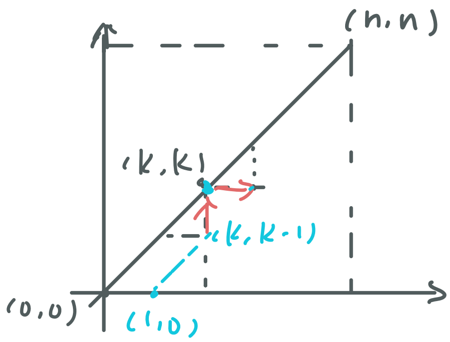
$\langle v_2, v_3, \dots, v_{n+1} \rangle$ remain, a $(n+1)$ - gon

$\# = 1 \cdot P_{n-1} = P_0 \cdot P_{n-1} + P_{n-1} \cdot P_0$

By ①②, $\# = \sum_{k=0}^{n-1} P_k \cdot P_{n-k-1}$

#

(2)



Consider the path that touch diagonal first on (k,k). $k \neq 0$ $k \in \{1,2,\cdots,n\}$

path from $(1,0)$ to $(k,k-1)$ is a subproblem

that path cannot crossing $y = k-1$ -> # : C_{k-1}

path from (k,k) to (n,n) is a subproblem

that path cannot crossing $y = k$ -> # : C_{n-k}

$\therefore$ Total # $= \sum_{k=1}^{n} C_{k-1} \cdot C_{n-k} = \sum_{m=0}^{n-1} C_m \cdot C_{n-m-1}$ #

Problem 4

(1)

Suppose $\exists\, y \in A \in F$ that y belongs to only one member,

let vertex in $U \setminus A$ be blue where U : {all vertices}

let $A \setminus \{y\}$ be red

let y be blue

① For $B \in F \setminus \{A\}$,

$\because$ $|A \cap B| = 1$ $\Leftrightarrow$ $\exists\, z \in A \cap B$ that z is red

$\therefore$ $B / \{z\}$ are blue $\wedge$ z is red where $|B / \{z\}| \geq 3 - 1 = 2$

$\Rightarrow$ B is not monocromatic

② For A,

$\because$ $A \setminus \{y\}$ are red $\wedge$ y is blue where $|A \setminus \{y\}| \geq 2$

$\therefore$ A is not monocromatic

By ①②, F is 2-colorable $(*)$

$\therefore$ y belongs to at least two members $\forall\, y \in A \in F$

(2)

Suppose $\nexists\, C \in F$ such that $\{x, y\} \in C$ where $x \in A \wedge y \in B$,

let $U/(A \cup B)$ be blue where $U : \{\text{all vertices}\}$

let $(A \cup B)/\{x, y\}$ be red

let x, y be blue

$\forall\, C \in (F/A)/B$, $|C \cap A| = 1 \Leftrightarrow \exists\, z \in A \cap C$

① For C, if $z \neq x$

$\rightarrow$ $C/\{A \cup B\}$ are blue $\wedge$ z is red where $|C/\{A \cap B\}| = |C| - |\{z\}| \geq 2$

$\rightarrow$ C is not monocromatic

② For C, if $z = x$, take $u \in B \cap C$ for $|C \cap B| = 1$

Suppose $u = y$, then $\exists\, C \in F$ s.t. $\{x, y\} \in C$ (*)

$\rightarrow$ there must $\exists\, u \in B$ that $u \neq y$

$\rightarrow$ $C/\{A \cup B\}$ are blue $\wedge$ u is red where $|C/\{A \cap B\}| = |C| - |\{z, u\}| \geq 1$

$\rightarrow$ C is not monocromatic

③ For A, B,

$\because$ $A/\{x\}$ are red $\wedge$ x is blue,

$B/\{y\}$ are red $\wedge$ y is blue

$\therefore$ A, B are not monocromatic

By ①②③, F is 2-colorable (*)

$\therefore$ $\exists\, C \in F$ such that $\{x, y\} \in C$ #

Problem 5

let B_0 consist of all vertices in B with at least $\alpha \cdot D/2$ neighbors in A_0

(2)

by definition of B_0, property (2) satisfy

(3)

$\because$ $A_0 \subset A$ is a independent set

$\therefore$ edges leaving A_0 $= \sum_{a \in A_0} d_a$ where d_a is the degree of vertice a

Since $\frac{1}{2} \cdot \sum_{a \in A_0} d_a \geq \frac{1}{2} \cdot \sum_{a \in A_0} d \geq \frac{1}{2} \cdot |A_0| \cdot d \geq \frac{1}{2} \cdot \alpha \cdot |A| \cdot d$

$\geq \frac{1}{2} \cdot \alpha \cdot |B| \cdot D \geq \sum_{b \in B/B_0} \frac{\alpha \cdot D}{2} \geq$ edges leaving A_0 go to B/B_0

$\Rightarrow$ at most half of the edges leaving A_0 go to B/B_0

$\Rightarrow$ at least half of the edges leaving A_0 go to B_0

(1)

define $\tilde{d_b}$ be the number of neighbors in A_0 for vertice b

By (3), $|B_0| \cdot D = \sum_{b \in B_0} \tilde{d_b} \geq \frac{1}{2} \cdot \sum_{a \in A_0} d_a \geq \frac{1}{2} \cdot \alpha \cdot |B| \cdot D$

$\Rightarrow |B_0| \geq \frac{1}{2} \cdot \alpha \cdot |B|$

prove B_0 exist :

Suppose $\nexists\, b \in B$ having at least $\alpha \cdot D/2$ neighbors in A_0,

$\Rightarrow$ edges leaving A_0 go to B : $\sum_{a \in A_0} d_a \leq |B| \cdot \alpha \cdot D/2$

$\leq |A| \cdot d \cdot \alpha/2 \leq |A_0| \cdot d/2$

$\Rightarrow d_a \leq d/2$ (*)

$\therefore \exists\, b \in B$ having at least $\alpha \cdot D/2$ neighbors in A_0 $\Rightarrow$ B_0 exist #

Problem 1 (20%). Consider the following two problems regarding Markov's and Chebyshev's inequalities.

- For any positive integer k, describe a non-negative random variable X such that

$$\Pr[\, X \geq k \cdot \mathrm{E}[X]\,] = \frac{1}{k}.$$

  Note that, this shows that Markov's inequality is as tight as it could possibly be.

- Can you provide an example that shows that Chebyshev's inequality is tight? If not, explain why not.

$3 - K + \log_2 k < 0.$

Problem 2 (20%). Prove that for any two sets I, J with $I \subseteq J$, we have

$$\sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k \qquad \sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \begin{cases} 1, & \text{if } I = J, \\ 0, & \text{if } I \neq J. \end{cases} \qquad 8 \cdot 2^{-k} \cdot 2^{\log_2 K}$$

$$= (1+x)^n \qquad\qquad = 2^3 \cdot 2^{-K + \log_2 K}$$

Hint: Rewrite the summation and apply the binomial theorem (in slides # 1a).

A is monocromatic $\leq 2 \cdot 2^{-K}$ $\quad \bar{A} \cap \bar{A} \cap \bar{A} \cdots \quad 2^{\log K}$

Problem 3 (20%). Let $\mathcal{F}$ be a k-uniform k-regular family, i.e., each set has k elements and each element belongs to k sets. Let $k \geq 10$. Show that there exists at least one valid 2-coloring of the elements.

Hint: Define proper events for the sets and apply the symmetric version of the local lemma.

$4 \cdot (2 \cdot 2^{-k}) \cdot K = 4 \cdot 2^{-k} \cdot K \leq 1$

Problem 4 (20%). We proved the asymmetric version of the local lemma in lecture #4. Assume that the statement of this lemma holds. Furthermore, assume that

1. $\Pr[A_i] \leq p$ for all i, and
2. $ep(d+1) \leq 1$.

$p \leq \frac{1}{e(d+1)}$

goal: $p \leq x_{A_i} \cdot \prod_{(i,j) \in E} (1 - x_j)$

$\Rightarrow \frac{1}{e(d+1)} \leq x_{A_i} \cdot \prod_{(i,j) \in E} (1 - x_j)$

Prove that $\Pr\left[\bigcap_i \overline{A_i}\right] > 0$, i.e., use Theorem 19.2 to prove the statement of Theorem 19.1.

Hint: Let $x(A_i) = \frac{1}{d+1}$ for all $1 \leq i \leq n$. Use the inequality $\frac{1}{e} \leq \left(1 - \frac{1}{d+1}\right)^d$ obtained by the limit formula of $1/e$ and the fact that it converges from the above.

$\left(1 - \frac{1}{d+1}\right)^d$

Problem 5 (20%). Suppose that we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these random bits in any order. Let Y_i be the exclusive-or (XOR) of the i^{th} pair of bits, and let $Y := \sum_{1 \leq i \leq m} Y_i$. 只考慮 pair

$E[(Y - E[Y])^2]$

- Show that $Y_i = 0$ and $Y_i = 1$ with probability $1/2$ each.
- Show that $\mathrm{E}[\, Y_i \cdot Y_j \,] = \mathrm{E}[Y_i] \cdot \mathrm{E}[Y_j]$ for any $1 \leq i, j \leq m$ and derive $\mathrm{Var}[Y]$. $\;= E[Y^2] - (E[Y])^2$
- Use Chebyshev's inequality to derive a bound on $\Pr[\, |Y - \mathrm{E}[Y]| \geq n \,]$.

$f(x) = \frac{1}{x^d}$ $\qquad \int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = 1$

HW 3

Problem 1

- Consider the random variable $X = \begin{cases} 0 & . \text{ with } Pr(x) = 1 - \frac{1}{K} \\ 1 & . \text{ with } Pr(x) = \frac{1}{K} \end{cases}$

$\rightarrow E[X] = 1 \cdot \frac{1}{K}$

$\rightarrow Pr[X \geq K \cdot \frac{1}{K}] = \frac{1}{K}$

- Consider the random variable $Y = \begin{cases} -K & . \text{ with } Pr(x) = \frac{1}{2K^2} \\ 0 & . \text{ with } Pr(x) = 1 - \frac{2}{K^2} \\ K & . \text{ with } Pr(x) = \frac{1}{2K^2} \end{cases}$

$\rightarrow E[Y] = (-K) \cdot \frac{1}{2K^2} + K \cdot \frac{1}{2K^2} = 0$

$\rightarrow Var[Y] = E[(Y - E[Y])^2] = 2K^2 \cdot \frac{1}{2K^2} = 1$

$\rightarrow Pr[|Y - 0| \geq K] = \frac{1}{K^2} = \frac{Var[Y]}{K^2} = \frac{1}{K^2}$

$\because$ the Chebyshev's inequality hold when $t = K$ and distribution $= Y$

$\therefore$ the Chebyshev's inequality is tight

Problem 2

$\because I \subseteq J \quad \therefore$ let $n = |J| - |I|$

Define a $2^n \times n$ matrix M with $\begin{cases} \text{row } K & : \text{subset between } I \text{ and } J \\ \text{col } m & : \text{the size of subset} \end{cases}$

$\rightarrow \begin{cases} M_{ij} = 1 & \text{when } |K_i| = m_i \\ M_{ij} = 0 & \text{else} \end{cases} \rightarrow$ the sum of col $m_i = C^n_m$

By double counting theorem and binomial theorem,

$$\Rightarrow \sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \sum_{k=0}^{n} C_k^n \cdot (-1)^k = (1-1)^n = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

Namely,

$$\sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \begin{cases} 0, & I \neq J \\ 1, & I = J \end{cases}$$

#

Problem 3

let A_i = the probability that A is monocromatic

$\because$ each set has K elements

$\therefore$ $P(A_i) = p \leq 2 \cdot 2^{-k}$

$\because$ F is K-uniform K-regular family

$\therefore$ there are $K(K-1)$ sets intersecting with A_i $\forall A_i \in F$

$\rightarrow$ d = dependent number $= k(k-1)$

$\Rightarrow 4 \cdot p \cdot d = 4 \cdot (2 \cdot 2^{-k}) \cdot K(k-1)$

$\Rightarrow$ when $K \geq 10$, $4pd \leq \frac{8 \cdot 90}{1024} \leq 1$

By Lovasz local lemma, $P(F \text{ is } 2\text{-colorable}) = P(\bigcap_{i=1}^{|F|} \bar{A_i}) > 0$ #

Problem 4

let $x(A_i) = \frac{1}{d+1}$ $\forall 1 \leq i \leq n$

$p \leq \frac{1}{d+1} \cdot \frac{1}{e} \leq \frac{1}{d+1} \cdot \left(1 - \frac{1}{d+1}\right)^d$ (by inequality)

$\leq x(A_i) \cdot (1 - x(A_i))^d \underset{\substack{\uparrow \\ \text{the maximum degree} = d}}{\leq} x(A_i) \cdot \prod_{(i,j) \in E} (1 - x(A_j))$ $\forall 1 \leq i \leq n$

Continue from problem 4,

By Thm 19.2, $Pr(\bigcap_i \bar{A_i}) \geq \prod_{1 \leq i \leq n} (1-x_i) > 0$ #

$\because 0 < x(A_i) = \frac{1}{d+1} < 1$

Problem 5

- When $Y_i = 0 \Leftrightarrow$ # of 1 is even ; When $Y_i = 1 \Leftrightarrow$ # of 1 is odd

  $P(Y_i = 0) = P(\text{`11'} \cup \text{`00'}) = \frac{1}{2}$

  $P(Y_i = 1) = P(\text{`10'} \cup \text{`01'}) = \frac{1}{2}$

- $E[Y_i \cdot Y_j] = 1 \cdot 1 \cdot P((Y_i = 1) \cap (Y_j = 1)) + 0 \cdot (1 - P((Y_i = 1) \cap (Y_j = 1)))$

  $= 1 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

  $E[Y_i] \cdot E[Y_j] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ $\therefore E[Y_i \cdot Y_j] = E[Y_i] \cdot E[Y_j]$ if $i \neq j$

  Note, if $i = j$,

  $E[Y_i^2] = 1 \cdot P(Y_i = 1) + 0 \cdot P(Y_i = 0) = \frac{1}{2} \neq E[Y_i] \cdot E[Y_i] = \frac{1}{4}$

- $E[Y] = E[\sum_{i=1}^{m} Y_i] = \sum_{i=1}^{m} E[Y_i] = \frac{m}{2}$ (linearity)

  $E[Y^2] = E[\sum_{i=1}^{m} \sum_{j=1}^{m} Y_i \cdot Y_j] = \sum_{\substack{i=j \\ 1 \leq i,j \leq n}} E[Y_i \cdot Y_j] + \sum_{\substack{i \neq j \\ 1 \leq i,j \leq n}} E[Y_i \cdot Y_j]$

  $= m \cdot \frac{1}{2} + (m^2 - m) \cdot \frac{1}{4}$

  $Var[Y] = E[Y^2] - (E[Y])^2 = \frac{1}{4} m$

- $Pr[|Y - E[Y]| \geq n] \leq \frac{Var[Y]}{n^2} = \frac{\frac{1}{4} m}{n^2} = \frac{\frac{1}{4} \cdot \binom{n}{2}}{n^2} = \frac{1}{8} \cdot \frac{n-1}{n} \leq \frac{1}{8}$

Problem 1 (20%)**.** Show that, for any positive integer n, there is a multiple of n that contains only the digits 7 or 0.

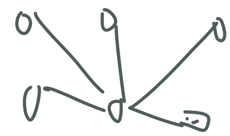
Hint: Consider all the numbers a_i of the form $77\ldots7$, with i sevens, for $i = 1, 2, \ldots n+1$, and the value a_i modulo n.

(2, 3) (3, 4)

Problem 2 (20%)**.** Prove that every set of $n+1$ distinct integers chosen from $\{1, 2, \ldots, 2n\}$ contains a pair of consecutive numbers and a pair whose sum is $2n+1$.

For each n, exhibit two sets of size n to show that the above results are the best possible, i.e., sets of size $n+1$ are necessary.

Hint: Use pigeonholes $(2i, 2i-1)$ and $(i, 2n-i+1)$ for $1 \le i \le n$.



$\chi(G)$

Problem 3 (20%)**.** Let $G = (V, E)$ be a graph. Denote by $\chi(G)$ the minimum number of colors needed to color the vertices in V such that, no adjacent vertices are colored the same.

Prove that, $\chi(G) \le \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices.

Hint: Order the vertices $v_1, v_2, \ldots, v_n$ and use greedy coloring. Show that it is possible to color the graph using $\Delta(G) + 1$ colors.

$K < n$ $\quad \le (1 - \frac{1}{k}) \cdot \frac{n^2}{2}$

Problem 4 (20%)**.** Let $\alpha(G)$ be the *independence number* of a graph G, i.e., the maximum size of any independent set of G. Prove the following dual version of Turán's theorem:

If G is a graph with n vertices and $nk/2$ edges, where $k \ge 1$, then we have

$c(K+1)$

$$\alpha(G) \ge n/(k+1).$$

$= \frac{n}{2} \cdot (1 - \frac{1}{y})$

$t'(n) = \binom{n}{2} - \frac{n \cdot k}{2} = \frac{n(n-1)}{2} - \frac{nk}{2} = \frac{n^2 - n - nk}{2} = \frac{n(n-k-1)}{2} \quad > (1 - \frac{1}{k}) \cdot \frac{n}{2}$

Problem 5 (20%)**.** Let X be a finite set and $A_1, A_2, \ldots, A_m$ be a partition of X into mutually disjoint blocks. Given a subset $Y \subseteq X$, consider the partition $Y = B_1 \cup B_2 \cup \cdots \cup B_m$ with the blocks B_i defined as $B_i := A_i \cap Y$. For any $1 \le i \le m$, we say that the block B_i is *λ-large* if

(132

$$\frac{|B_i|}{|A_i|} \ge \lambda \cdot \frac{|Y|}{|X|}.$$

Show that, for every $\lambda > 0$, at least $(1 - \lambda) \cdot |Y|$ elements of Y belong to λ-large blocks.

HW4

Problem 1

Consider all the numbers a_i of the form $\underbrace{77\ldots7}_{i}$ with i seven.
$i \in \{1, 2, \cdots, n+1\}$

$\because$ there are $n+1$ a_i $\wedge$ there are n remainders if we mod n

By Pigeonhole principle

$\therefore$ There exist two number a_i, a_j having the same remainder

$\rightarrow$ let $a_i = m \cdot n + r$

$a_j = k \cdot n + r$. $(m > k)$

$\rightarrow$ $a_i - a_j = (m-k) \cdot n = \underbrace{7\ldots7}_{i-j}\underbrace{0\ldots0}_{j}$

$a_i - a_j$

$\therefore$ $(m-k) \cdot n$ is a multiple of n containing only 7 and 0

Problem 2

① if set $S = \{a_1, a_2, \cdots, a_n\}$ of size n,

we could take $a_i = 2 \cdot i$. $\forall i \in \{1, 2, \ldots, n\}$.

$\rightarrow$ it doesn't contain a pair of consecutive numbers

② if size $S = \{a_1, a_2, \cdots, a_n, a_{n+1}\}$ of size $n+1$,

Consider boxes $(2i, 2i-1)$. $i = 1, 2, \cdots, n$

let's put $n+1$ a in S into n boxes where each box has capacity 2

By Pigeonhole principle,

there are 2 a in the same box

$\rightarrow$ it contains a pair of consecutive numbers

$\therefore$ By ①, ②, S with size $n+1$ are the best possible

Problem 3

take v is the vertex with $\deg(G) = \Delta(G)$

prove By induction,

$n = 1$ is true for sure

let $G' = G - v$

Suppose $\chi(G') \le \Delta(G') + 1$,

Then, consider v,

$\because$ v has at most $\Delta(G)$ number in its neighborhood

$\therefore$ we can choose the missing color in its neighbor

s.t. $\chi(G) \le \Delta(G) + 1$

$\therefore$ $\chi(G) \le \Delta(G) + 1$ $\blacksquare$

Problem 4

let $E' = \{ e \mid e \notin E(G) \}$

$|E'| = \binom{n}{2} - \frac{n \cdot k}{2} = \frac{n^2 - n - n \cdot k}{2} = \frac{n}{2} \cdot (n - 1 - k)$

When $n - k - 1 = 0$,

$\rightarrow |E'| = 0. \rightarrow G = K_n$

$\Rightarrow \alpha(G) = \alpha(K_n) = 1 \ge \frac{k+1}{k+1} = 1$ $\blacksquare$

When $n - k - 1 > 0$

$\because (n-k-1)^2 > (n-k-1)^2 - (k-1)^2$

$\rightarrow (n-k-1)^2 > n \cdot (n - 2k - 2)$

$$\Rightarrow n\cdot(n-K-1)^2 > n^2\cdot(n-2K-2)$$

$$\Rightarrow \frac{n}{2}\cdot(n-K-1) > \frac{n^2}{2}\cdot\frac{n-2K-2}{n-K-1} = \left(1-\frac{1}{\frac{n-K-1}{K+1}}\right)\cdot\frac{n^2}{2}$$

By Turan's theorem,

$\because |E'| > \left(1-\frac{1}{\frac{n-K-1}{K+1}}\right)\cdot\frac{n^2}{2}$ $\therefore$ we have $\frac{n-K-1}{K+1}+1 = \frac{n}{K+1}$ clique in G'

then in G, the $\frac{n}{K+1}$ vertices will form a independent set

$\therefore \alpha(G) \geq \frac{n}{K+1}$

#

Problem 5

let $B_1, B_2, \cdots, B_K$ belong to non-λ-large block

$$\sum_{i=1}^{K}|B_i| < \sum_{i=1}^{K}\lambda\cdot\frac{|Y|}{|X|}\cdot|A_i| \leq \lambda\cdot\frac{|Y|}{|X|}\cdot\left(\underline{\sum_{i=1}^{n}|A_i|}\right)$$

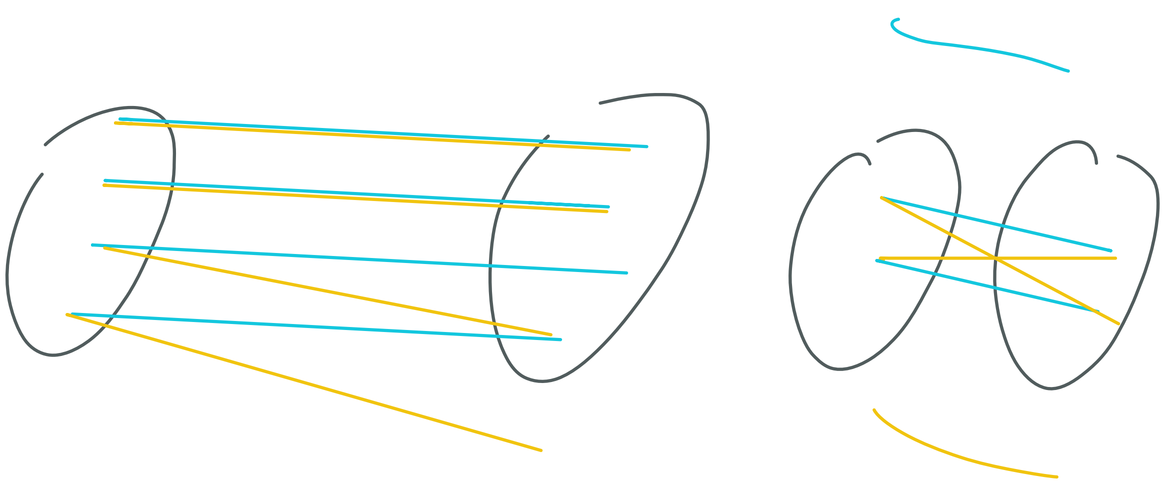
each A_i is disjoint blocks

$$\leq \lambda\cdot\frac{|Y|}{|X|}\cdot\left(\underline{\left|\bigcup_{i=1}^{n}A_i\right|}\right) = \lambda\cdot\frac{|Y|}{\cancel{|X|}}\cdot\cancel{|X|} = \lambda\cdot|Y|$$

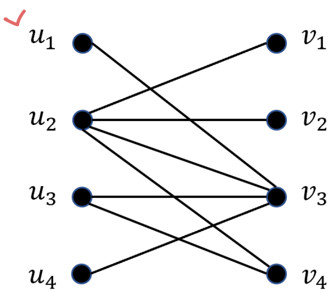
$\Rightarrow$ at most $\lambda\cdot|Y|$ elements belongs to non-λ-large block

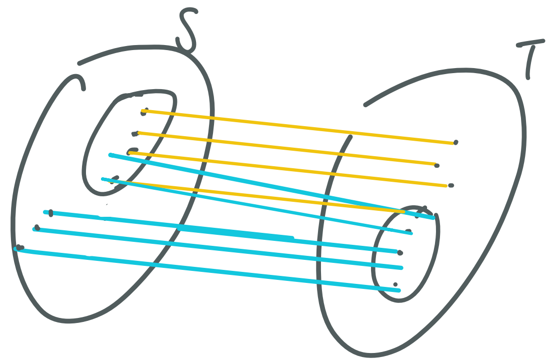
$\Rightarrow$ at least $(1-\lambda)\cdot|Y|$ elements belongs to non-λ-large block

#



Problem 1 (20%)**.** Consider the following graph. Identify a maximum-size matching and a minimum-size vertex cover for it.





Problem 2 (20%)**.** Let G be a bipartite graph with partite sets A and B, and M, M' be two matchings. Suppose that, M matches the vertices in $S \subseteq A$ and M' matches the vertices in $T \subseteq B$. Prove that there is a matching that matches all the vertices in $S \cup T$.

Hint: Consider $M \cup M'$.

Problem 3 (20%)**.** Let G be a bipartite graph with partite sets X and Y. Prove that G has a matching of size t if and only if for all $A \subseteq X$,

$$|N(A)| \;\geq\; |A| + t - |X| \;=\; \underline{t - |X - A|}.$$

Hint: Add $|X| - t$ new vertices to Y and connect these vertices to every vertex in X.

Problem 4 (20%)**.** Let G be a bipartite graph with partite sets X and Y. Define

$$\delta(G) := \max_{A \subseteq X} \left(|A| - |N(A)|\right),$$

i.e., $\delta(G)$ measures the worst violation of the Hall's matching condition. Note that, $\delta(G) \geq 0$ since $A = \emptyset$ is considered as a subset of X. Use the statement in Problem 3 to prove that, G has a maximum matching of size $|X| - \delta(G)$.

Problem 5 (20%)**.** Let G be a bipartite graph with partite sets X and Y. Assume the same notation $\delta(G)$ as Problem 4. Show that, the largest independent set of G has size $|Y| + \delta(G)$.

$\forall s^* \in S$, $M(s) = t^* \in T$, (s^*, t^*) is matching in M

, (t^*, s^*) is also a matching in M

HW5

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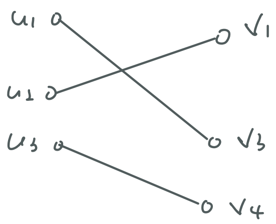
Problem 1

maximum matching : 3. minimum-size vertex cover : 3



vertex cover : $\{u_2, u_3, v_3\}$

Problem 2

$m(v) = u$ where $(u, v) \in M$

define $U_S = \{ s \mid s \in S \wedge m(s) \in T \}$, $U_T = \{ t \mid t \in T \wedge m(s) \in S \}$

There are three kinds of components in P that are even cycle, even path, odd path where $(s^*, t^*) \in P$, $\forall s^* \in U_S, \forall t^* \in U_T \}$

$M_1 = \{ (s, m(s)) \mid (s, m(s)) \in M, \forall s \in S \setminus U_S \}$

$M_2 = \{ (t, m(t)) \mid (t, m(t)) \in M', \forall t \in T \setminus U_t \}$

$M_3 = \{ (s^*, t^*) \mid (s^*, t^*) \in M \cap M' \wedge \forall s^* \in U_S, \forall t^* \in U_T \}$

$M_4 = \{ (\bigcup_i P_i) \cap M \mid (s^*, t^*) \in P_i$ where P_i is even path or even cycle that M, M' alternate, $\forall s^* \in U_S, \forall t^* \in U_T \}$

$M_5 = \{ (\bigcup_i P_i) \cap M \mid (s^*, t^*) \in P_i$ where P_i is odd path that M, M' alternate, $|M \cap P| > |M' \cap P|$, $\forall s^* \in U_S, \forall t^* \in U_T \}$

$M_6 = \{ (\bigcup_i P_i) \cap M' \mid (s^*, t^*) \in P_i$ where P_i is odd path that M, M' alternate, $|M' \cap P| > |M \cap P|$, $\forall s^* \in U_S, \forall t^* \in U_T \}$

$\therefore \bigcup_{i=1}^{6} M_i$ is a matching in $S \cup T$ #

Problem 3

($\Rightarrow$)

define $B = \{x \mid x \in A \wedge x \text{ is a endpoint in matching}\}$

By Hall's condition, $|N(B)| \geq |B|$

Since $B \subset A$, $|N(A)| \geq |N(B)|$

Since $A/B \subset \{x \mid x \in X \wedge x \text{ is not in matching}\}$,

$\rightarrow |A/B| \leq |X| - t$

$\rightarrow |N(A)| \geq |N(B)| \geq |B| = |A| - |A \setminus B|$

$\geq |A| + t - |X|$

($\Leftarrow$)

Add new vertices Z to Y and connect these vertices in Z to every vertex in X, where $|Z| = |X| - t$

define new neighbors $N(A)^* = \{N(A) \cup Z\}$

Since $|N(A)| \geq |A| + t - |X| \rightarrow |N(A)| + |X| - t = |N(A)^*| \geq |A|$

$\because$ ① By Hall's condition, there is a matching for A

② at most $|X| - t$ matching connect to Z

$\therefore$ at least t matching connect to Y

By ($\Rightarrow$)($\Leftarrow$), Q.E.D #

Problem 4

By Problem 3, $t \le |X| - (|A| - |N(A)|)$ where t is matching size $\forall A \subseteq X$

choose matching size $= |X| - \delta(G)$

if there is a larger matching t',

$t' > |X| - \delta(G) \to \delta(G) = \max_{A \subseteq X} \{|A| - |N(A)|\} > |X| - t'$

but by problem 3, $|A| - |N(A)| \le |X| - t$ $(*)$

$\therefore$ maximum matching size $= |X| - \delta(G)$ #

Problem 5

By König - Egevary, maximum matching $\overset{\alpha_1(G)}{}$ $=$ minimum vertex cover $\overset{\sigma(G)}{}$ in barpartite graph

By lemma 1, $\to \alpha(x)$ = independence number

$= |Y| + \delta(G)$ #

lemma 1. $\sigma(G) + \alpha(G) = |G|$

<p.f.>

① Let S be a vertex cover s.t. $|S| = \sigma(G)$.

Since $V(G)/S$ is an independent set, $|V(G) \setminus S| \le \alpha(G)$

$\Rightarrow |G| - \sigma(G) \le \alpha(G) \Rightarrow |G| \le \sigma(G) + \alpha(G)$

② Let S' be a independence set s.t. $|S'| = \alpha(G)$.

Since $V(G)/S'$ is an vertex cover of G, $|V(G) \setminus S'| \ge \sigma(G)$

$\Rightarrow |G| \ge \alpha(G) + \sigma(G)$

By ①, ②, $\sigma(G) + \alpha(G) = |G|$ ∎

Problem 1 (20%)**.** Prove that, for any vector $v \in \mathbb{R}^n$,

$$\frac{|v|_1}{\sqrt{n}} \;\le\; \|v\|_2 \;\le\; |v|_1,$$

where $|v|_1 := \sum_i |v_i|$ is the L_1-norm and $\|v\|_2 := \left(\sum_i v_i^2\right)^{1/2}$ is the L_2-norm of v.

Hint: Use the Cauchy-Schwarz inequality, i.e., $|u \cdot v| \le \|u\|_2 \|v\|_2$ for any $u, v \in \mathbb{R}^n$.

Problem 2 (20%)**.** Let A be a square symmetric matrix and λ be an eigenvalue of A. Prove that, for any $k \in \mathbb{N}$, λ^k is an eigenvalue of A^k.

Problem 3 (20%)**.** Let G be an n-vertex d-regular bipartite graph and A be the normalized adjacency matrix of G. Prove that, there exists a vector $v \in \mathbb{R}^n$ such that

$$Av = -v.$$

$1 \ge \alpha_1 \ge \cdots \ge \alpha_n \ge -1$

Generalize the construction to non-regular bipartite graphs, i.e., for any bipartite graph G' with column-normalized adjacency matrix A', prove that A' has an eigenvalue -1.

Note: A' is also called the *random-walk* matrix of G'.

Problem 4 (20%)**.** Let $G = (V, E)$ be a d-regular graph and P be a random walk of length t in G. Prove that, for any edge $e \in E$ and any $1 \le i \le t$,

$$\Pr\left[\ e \text{ is the } i^{th}\text{-edge of } P\ \right] \;=\; \frac{1}{|E|}.$$

Hint: Prove by induction on i.

$$\begin{bmatrix} \frac{1}{\sqrt{d(0)}} & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{1}{\sqrt{d(n)}} \end{bmatrix} A \cdot \begin{bmatrix} \frac{1}{d(0)} & \cdots & 0 \\ 0 & \cdots & \frac{1}{d(n)} \end{bmatrix} \cdot \begin{bmatrix} \vdots \end{bmatrix}$$

Problem 5 (20%)**.** Let $G = (V, E)$ be an (n, d, λ)-expander and $S \subseteq V$ be a vertex subset. Prove that,

$$\Pr_{(u,v)\in E}\left[\, u, v \in S \,\right] \;\le\; \frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right),$$

i.e., for any $(u, v) \in E$, the probability that both u, v are in S is bounded by $\frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right)$.

Hint: Use the fact that $|E(S,S)| = (d|S| - |E(S,T)|)/2$. Apply the crossing lemma.

$\left(\frac{1}{2} - \frac{1-\lambda}{2}\right)$ $\quad |d| \cdot |S|)$

$$\begin{bmatrix} \frac{1}{\sqrt{d(0)}} & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{1}{\sqrt{d(n)}} \end{bmatrix} A \cdot \begin{bmatrix} \frac{1}{\sqrt{d(0)}} \\ \vdots \\ \frac{1}{\sqrt{d(n)}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{d(0)}} & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & & \frac{1}{\sqrt{d(n)}} \end{bmatrix}$$

$- \left(\frac{1-\lambda}{2}\right) \cdot d \cdot |S|$

$= \left(\frac{1+\lambda}{2}\right) \cdot d \cdot |S|$

HW6

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Problem 1

① By Cauchy-Schwarz inequality,

$$\left(\sum_i |v_i|^2\right)\cdot\left(\sum_i 1^2\right) \geq \left(\sum_i |v_i|\right)^2 \rightarrow \frac{1}{\sqrt{n}}\cdot\sum_i |v_i| \leq \left(\sum_i |v_i|^2\right)^{\frac{1}{2}} = \|v\|_2$$

② $\because \left(\sum_i v_i^2\right) \leq \left(\sum_i |v_i|\right)^2 = \sum_i v_i^2 + 2\cdot\sum_{i,j}^{i\neq j} v_i\cdot v_j$

$\therefore \|v\|_2 = \left(\sum_i v_i^2\right)^{\frac{1}{2}} \leq |v|_1 = \sum_i |v_i|$

By ①②,

$$\therefore \frac{|v|_1}{\sqrt{n}} = \frac{1}{\sqrt{n}}\cdot\sum_{i=1}^{n}|v_i| \leq \|v\|_2 \leq |v|_1$$

#

Problem 2

let v be the eigenvector of eigenvalue λ

When $k=1$, $A\cdot v = \lambda\cdot v$ by def

Suppose $k=n$, $A^n\cdot v = \lambda^n\cdot v$

Then $k=n+1$, $A\cdot(A^n\cdot v) = A\cdot(\lambda^n\cdot v) = \lambda^n\cdot\lambda\cdot v = \lambda^{n+1}\cdot v$

By induction, $A^k\cdot v = \lambda^k\cdot v \quad \forall k\in\mathbb{N}$

$\therefore$ By def of eigenvalue, λ^k is eigenvalue of A^k

#

Problem 3

let $v_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{n} \\ \vdots \\ \frac{1}{n} \end{bmatrix}$. $\rightarrow A' \cdot v_1 = 1 \cdot v_1$

$\because A'$ is bipartite graph. $A' = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{cases} B \cdot y = x \\ B^T \cdot x = y \end{cases}$

let $v_2 = \begin{bmatrix} x \\ -y \end{bmatrix}$

$\rightarrow A' \cdot v_2 = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} -By \\ B^T x \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$

$= (-1) \cdot \begin{bmatrix} x \\ -y \end{bmatrix} = (-1) \cdot v_2$

$\therefore A'$ has eigenvalue -1 #

Problem 4

let the starting distribution = uniform distribution = u

Claim: $\Pr[$ start at v_k at the i-th round $] = \frac{1}{|V|}$

When $i = 1$, $P \cdot u = \begin{bmatrix} \sum_{z=1}^{d} \frac{1}{d} \cdot \frac{1}{|V|} \\ \vdots \\ \end{bmatrix} = \begin{bmatrix} \frac{1}{|V|} \\ \vdots \\ \end{bmatrix}$

Suppose $i = n$, $P^n \cdot u = \begin{bmatrix} \frac{1}{|V|} \\ \vdots \\ \end{bmatrix}$

Then, $i = n+1$, $P^{n+1} \cdot u = \begin{bmatrix} \sum_{z=1}^{d} \frac{1}{d} \cdot \frac{1}{|V|} \\ \vdots \\ \end{bmatrix} = \begin{bmatrix} \frac{1}{|V|} \\ \vdots \\ \end{bmatrix}$ ∴ Claim is true

By claim, at i th round, the probability of reaching each vertex is the same

Hence, the probability of each edge choosing is $\frac{2}{|V|\cdot d}$.

$$\text{the probability} = \frac{\frac{2}{|V|\cdot d}}{\sum_{i=1}^{|E|} \frac{2}{|V|\cdot d}} = \frac{1}{|E|} \quad \#$$

Problem 5

① If $|S| \le n/2$, let $S' = S$

The expander crossing lemma says that $|E(S', T)| \ge \frac{1-\lambda}{2} \cdot d \cdot |S'|$

$$\Rightarrow |E(S', S')| = (d \cdot |S'| - |E(S, T)|)/2$$
$$\le \frac{d \cdot |S'|}{2} - \frac{(1-\lambda)\cdot d \cdot |S'|}{2}$$
$$= \frac{\lambda}{2} \cdot d \cdot |S'|$$

$$\therefore \Pr_{(u,v)\in E}[u,v \in S'] \le \frac{\frac{\lambda}{2} \cdot d \cdot |S'|}{\frac{d \cdot n}{2}} = \lambda \cdot \frac{|S'|}{n}$$
$$\le \left(\frac{|S'|}{n} + \lambda\right) \cdot \frac{|S'|}{n}$$

② If $|S| \ge n/2$, let $S' = T$,

Similar to above, $\Pr_{(u,v)\in E}[u,v \in S'] \le \left(\frac{|S'|}{n} + \lambda\right) \cdot \frac{|S'|}{n}$

$$\le \left(\frac{|S|}{n} + \lambda\right) \cdot \frac{|S|}{n}$$

By ①②, $\Pr_{(u,v)\in E}[u,v \in S'] \le \left(\frac{|S|}{n} + \lambda\right) \cdot \frac{|S|}{n} \quad \forall S \subseteq V$

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