Problem 1 (20%)**.** Show that, for any positive integer n, there is a multiple of n that contains only the digits 7 or 0.

$777\ldots7 - 7\ldots7 = 7\ldots70\ldots0$ $(7\ldots7) \% n$

Hint: Consider all the numbers a_i of the form $77\ldots7$, with i sevens, for $i = 1, 2, \ldots n+1$, and the value a_i modulo n.

$\sum_{k=0}^{n} C^n_k (-1)^k 1^{n-k} = 0$ $1+(-1)$

Problem 2 (20%)**.** Prove that for any two sets I, J with $I \subseteq J$, we have

$$\sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \begin{cases} 1, & \text{if } I = J, \\ 0, & \text{if } I \neq J. \end{cases}$$

$\sum_{k=0}^{|J\setminus I|} \sum_{I \subseteq K \subseteq J} (-1)^{|K\setminus I|}$ $\sum C^{|J\setminus I|}_k (-1)^k = (1+(-1))^n$

Hint: Rewrite the summation and apply the binomial theorem (in slides # 1a).

Problem 3 (20%)**.** Let $\mathcal{F}$ be a k-uniform k-regular family, i.e., each set has k elements and each element belongs to k sets. Let $k \geq 10$. Show that there exists at least one valid 2-coloring of the elements.

Hint: Define proper events for the sets and apply the symmetric version of the local lemma.

Problem 4 (20%)**.** We proved the asymmetric version of the local lemma in lecture #4. Assume that the statement of this lemma holds. Furthermore, assume that

1. $\Pr[A_i] \leq p$ for all i, and
2. $ep(d+1) \leq 1$.

Prove that $\Pr\left[\bigcap_i \overline{A_i}\right] > 0$, i.e., use Theorem 19.2 to prove the statement of Theorem 19.1.

Hint: Let $x(A_i) = \frac{1}{d+1}$ for all $1 \leq i \leq n$. Use the inequality $\frac{1}{e} \leq \left(1 - \frac{1}{d+1}\right)^d$ obtained by the limit formula of $1/e$ and the fact that it converges from the above.

Problem 5 (20%)**.** Let X be a finite set and $A_1, A_2, \ldots, A_m$ be a partition of X into mutually disjoint blocks. Given a subset $Y \subseteq X$, consider the partition $Y = B_1 \cup B_2 \cup \cdots \cup B_m$ with the blocks B_i defined as $B_i := A_i \cap Y$. For any $1 \leq i \leq m$, we say that the block B_i is λ-*large* if

$$\frac{|B_i|}{|A_i|} \geq \lambda \cdot \frac{|Y|}{|X|}.$$

Show that, for every $\lambda > 0$, at least $(1-\lambda) \cdot |Y|$ elements of Y belong to λ-large blocks.

1. Let $n \in \mathbb{N}$.

We consider $a_i := \underbrace{7\cdots7}_{i\text{-digit number}} \% n$ for $1 \le i \le n+1$.

Since $|\{a_i : 1 \le i \le n+1\}| \le n$ and we have

$n+1$ numbers, by pigeonhole principle,

$\exists p, q \in \{1, 2, \cdots, n+1\} \ni a_p \equiv a_q \pmod n$ with $p \ne q$.

W.L.O.G., we assume that $p > q$.

Hence, $n \mid a_p - a_q$.

Thus, $n \mid \underbrace{7\cdots7}_{p\text{-digit}} - \underbrace{7\cdots7}_{q\text{-digit}} = \underbrace{7\cdots7}_{p-q\text{ digit}}\underbrace{0\cdots0}_{q\text{-digit}}$.

Hence, $\forall n \in \mathbb{N}$, $\exists$ a multiple of n that only the digits 7 or 0.

2. $$\sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \sum_{m=0}^{|J \setminus I|} \sum_{\substack{I \subseteq K \subseteq J \\ |K| = |I| + m}} (-1)^{|K \setminus I|}$$

$$= \sum_{m=0}^{|J \setminus I|} \sum_{\substack{I \subseteq K \subseteq J \\ |K| = |I| + m}} (-1)^m$$

$$= \sum_{m=0}^{|J \setminus I|} C_m^{|J \setminus I|} (-1)^m$$

$$= (1-1)^{|J \setminus I|} = \begin{cases} 0, & \text{if } I \ne J, \\ 1, & \text{if } I = J. \end{cases}$$

3. Define A_i = the set i is monochromatic for all set $i \in F$.

Consider a uniform random coloring of the element using 2 colors.

Since each set has k elements, $P(A_i) = 2 \cdot \frac{1}{2^k}$. Define $2 \cdot \frac{1}{2^k} = p$.
(F is k-uniformly)

Since each element belongs to k sets and each set has k elements, the maximum degree of dependency graph $\le \underbrace{k}_{\text{element}}\underbrace{(k-1)}_{\text{other set in } F}$

Moreover, $4pd \le 4k(k-1)\frac{2}{2^k} = \frac{k(k-1)}{2^{k-3}}$.

We claim that $k(k-1) \le 2^{k-3}$ for all $k \in \{10, 11, \cdots\}$.

If the claim holds, by symmetric version of the local lemma, we have

$$P(\cap \overline{A_i}) = P(F \text{ is 2-colorable}) > 0.$$

Hence, there exists at least one valid 2-coloring of the elements. $\square$

Claim: $k(k-1) \le 2^{k-3}$ for all $k \in \{10, 11, \cdots\}$.

Base case: $k = 10$

Since, $k(k-1) = 10 \cdot 9 \le 128 = 2^{k-3}$, the claim holds for $k = 10$.

Suppose the claim holds for $k = n$.

When $k = n+1$,

$$2^{n+1-3} = 2 \cdot 2^{n-3} \ge 2 \cdot (k(k-1)) \text{ by induction}$$
$$\ge k(k+1) \quad (\text{since } 2k^2 - 2k \ge k^2 + k \Leftrightarrow k^2 \ge 3k \Leftrightarrow k \ge 3)$$

Hence, the claim holds for $k \in \{10, 11, \cdots\}$. $\blacksquare$

4. Let $G(V,E)$ be a dependency graph of events $A_1, \cdots, A_n$.
Since we want to use Asymmetric LLL,
we need to show there exists real number $x_1, x_2, \cdots, x_n$ with $0 \le x_i < 1$
such that for all i, $\Pr[A_i] \le x_i \cdot \prod_{j:(i,j)\in E} (1-x_j)$ first, then we have

$\Pr(\bigcap_{i=1}^{n} \bar{A_i}) \ge \prod_{1\le i\le n} (1-x_i)$.

Let $x_i = \frac{1}{d+1}$ for $1 \le i \le n$.

Then, $p \le \frac{1}{e(d+1)}$ (by Assume 2)

$\le \frac{1}{d+1} \cdot (1-\frac{1}{d+1})^d$ (since $\frac{1}{e} \le (1-\frac{1}{d+1})^d$)

$\le x_i \cdot \prod_{(i,j)\in E} (1-x_j)$. for all $1 \le i \le n$

(since the maximum degree of dependency graph $= d$)

Clearly, $0 \le x_i < 1$ for all $1 \le i \le n$.

Hence, by Asymmetric LLL, $\Pr(\bigcap_{i=1}^{n} \bar{A_i}) \ge \prod_{1\le i\le n} (1-x_i) > 0$. ∎

5. W.L.O.H. Suppose $B_1, B_2, \cdots, B_k$ are not λ-large.

Hence, $\sum_{i=1}^{k} |B_i| < \sum_{i=1}^{k} \lambda \frac{|Y|}{|X|} |A_i|$ (since B_i is not λ-large for $1 \le i \le k$)

$\le \lambda \frac{|Y|}{|X|} \sum_{i=1}^{m} |A_i| = \lambda \frac{|Y|}{|X|} |X|$ (since $A_1, \cdots, A_m$ is a mutually disjoint partition of X.)

$= \lambda |Y|$

Thus, at most $\lambda|Y|$ elements of Y belong to non λ-large blocks.

Therefore, at least $(1-\lambda)|Y|$ elements of Y belong to λ-large blocks.