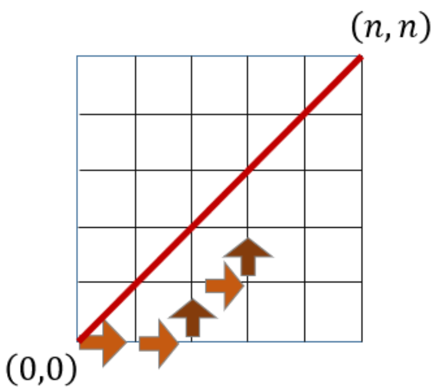
Problem 1 (20%). Let X, Y be discrete random variables. The variance of a random variable X is defined as $\text{Var}[X] := \text{E}\left[(X - \text{E}[X])^2\right]$. Prove that

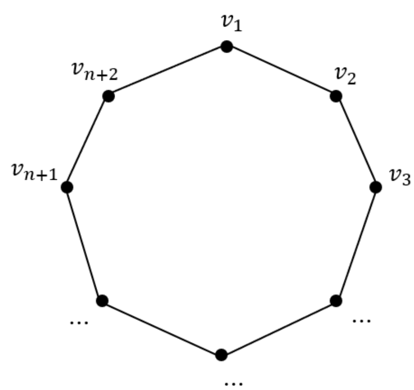
1. $\text{E}[aX + bY] = a \cdot \text{E}[X] + b \cdot \text{E}[Y]$ for any constant a, b.
2. If X and Y are independent, then $\text{E}[X \cdot Y] = \text{E}[X] \cdot \text{E}[Y]$ and
   $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$.
3. $\text{Var}[X] = \text{E}[X^2] - \text{E}[X]^2$. *Hint:* Use the fact that $\text{E}[X \cdot \text{E}[X]] = \text{E}[X]^2$.

Problem 2 (20%). Consider the slides #2. Prove that the graphs H_i defined in the proof of Theorem 3 are bicliques.

Problem 3 (20%). For any integer $n \geq 1$, consider the grid points (r, c) with $1 \leq r, c \leq n$. Let C_n be the number of possible paths from $(0, 0)$ to (n, n) that use only $\rightarrow$ and $\uparrow$ and that never cross the diagonal $r = c$. See also the Figure (a) below. For convenience, define $C_0 := 1$.



(a)



(b)

For any integer $n \geq 2$, consider the convex $(n+2)$-gon with vertices labeled with $v_1, v_2, \ldots, v_{n+2}$. Let P_n denote the number of possible ways to triangulate the polygon. It follows that $P_2 = 2$, $P_3 = 5$, etc. For convenience, also define $P_0 := 1$ and $P_1 := 1$.

1. Prove that for any $n \geq 2$, P_n satisfies the recurrence

$$P_n \;=\; \sum_{0 \,\leq\, k \,<\, n} P_k \cdot P_{n-k-1}.$$

2. Prove that for any $n \geq 2$, C_n satisfies the same recurrence

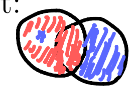
$$C_n \;=\; \sum_{0 \,\leq\, k \,<\, n} C_k \cdot C_{n-k-1}.$$

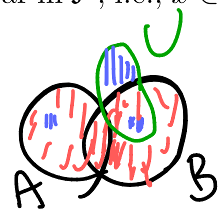
Note that this proves that P_n also equals the n^{th}-Catalan number.

Problem 4 (20%)**.** Let $\mathcal{F}$ be a family of subsets, where

$$|A| \geq 3 \text{ for any } A \in \mathcal{F} \qquad \text{and} \qquad |A \cap B| = 1 \text{ for any } A, B \in \mathcal{F},\ A \neq B.$$

Suppose that $\mathcal{F}$ is not 2-colorable. Let x, y be any elements that appear in $\mathcal{F}$, i.e., $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$ for some $A, B \in \mathcal{F}$. Prove that:





1. x belongs to at least two members of $\mathcal{F}$.

2. There exists some $C \in \mathcal{F}$ such that $\{x, y\} \subseteq C$.

Hint: Construct proper coloring to prove the properties. For (1), consider a particular A with $x \in A \in \mathcal{F}$. Color $A \setminus \{x\}$ red and the remaining blue. Show that this leads to the conclusion of (1). For (2), consider particular A, B with $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$. Color $(A \cup B) \setminus \{x, y\}$ red and the remaining blue. Prove that it leads to (2).

Problem 5 (20%)**.** Let $G = (A \cup B, E)$ be a bipartite graph, d be the minimum degree of vertices in A and D the maximum degree of vertices in B. Assume that $|A|d \geq |B|D$.

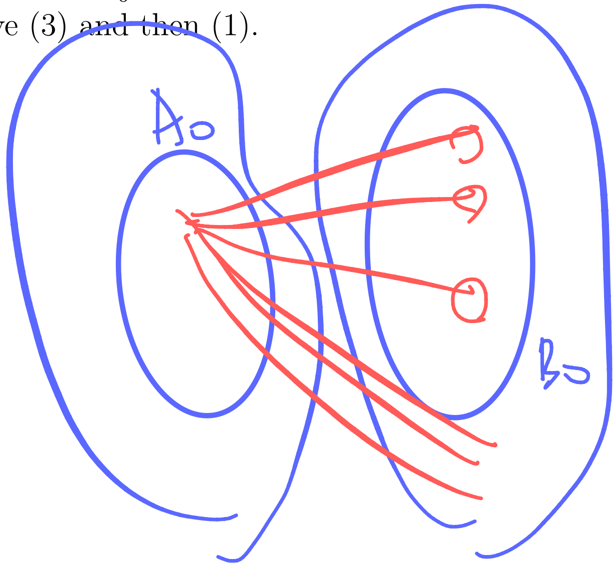
Show that, for every subset $A_0 \subseteq A$ with the density α defined as $\alpha := |A_0|/|A|$, there exists a subset $B_0 \subseteq B$ such that:

1. $|B_0| \geq \alpha \cdot |B|/2$,

2. every vertex of B_0 has at least $\alpha D/2$ neighbors in A_0, and

3. at least half of the edges leaving A_0 go to B_0.

Hint: Let B_0 consist of all vertices in B that have at least $\alpha D/2$ neighbors in A_0. First prove (3) and then (1).



廖修誼 111652017

P1 Since X and Y is discrete random variable, X and Y have finite outcome.

Let the possible outcome of X is $S_X=\{x_1,\cdots,x_n\}$ and

" Y is $S_Y=\{y_1,\cdots,y_m\}$.

1.1 $E[aX+bY]=\sum_{\substack{x\in S_X\\ y\in S_Y}}(ax+by)P(X=x,Y=y)$ (by the def. of expectation)

$=\sum_{\substack{x\in S_X\\ y\in S_Y}} axP(X=x,Y=y)+\sum_{\substack{x\in S_X\\ y\in S_Y}} byP(X=x,Y=y)$ (by the linearity of Σ)

$=\sum_{x\in S_X} axP(X=x)+\sum_{y\in S_Y} byP(Y=y)$ (since $\sum_{y\in S_Y}P(X=x,Y=y)=P(X=x)$)

$=a\sum_{x\in S} xP(X=x)+b\sum_{y\in S_Y} yP(Y=y)$ (by the linearity of Σ)

$=aE[X]+bE[Y]$. (by the def. of expectation) ▨

1.2 ① $E[X\cdot Y]=\sum_{\substack{x\in S_X\\ y\in S_Y}}(xy)P(X=x,Y=y)$ (by the def. of expectation)

$=\sum_{\substack{x\in S_X\\ y\in S_Y}}(xy)P(X=x)P(Y=y)$ (since X and Y are independent, $P(X=x,Y=y)=P(X=x)P(Y=y)$)

$=\sum_{x\in S_X} xP(X=x)\sum_{y\in S_Y} yP(Y=y)$ (by the linearity of Σ)

$=E[X]+E[Y]$ (by the def. of expectation)

② $Var(X+Y)=E[((X+Y)-(E[X+Y]))^2]$ (by the def. of variance)

$=E[((X+Y)-(E[X]+E[Y]))^2]$

(since X and Y are independent, $E[X+Y]=E[X]+E[Y]$) (P1-2)

$=E[((X-E[X])+(Y-E[Y]))^2]$

$=E[(X-E[X])^2]+2E[(X-E[X])(Y-E[Y])]+E[(Y-E[Y])^2]$

(by the linearity of expection, i.e. P1-1)

$= Var[X] + 2\{E[X]E[Y-E[Y]] - E[\underbrace{E[X]}_{E[X]}]E[Y-E[Y]]\} + Var[Y]$

(by P1-1 and P1-2)

$= Var[X] + Var[Y]$ ▨

1.3 $Var[X] = E[(X-E[X])^2]$ (by the def. of variance)

$= E[X^2 - 2XE[X] + (E[X])^2]$

$= E[X^2] - 2E[XE[X]] + E^2[X]$

$= E[X^2] - E^2[X]$ (since $E[XE[X]] = E^2[X]$)

2. Let $i \in \{1, \dots, m\}$.

Let $X_i := \{v \in K_n :$ the i^{th}-coordinate of $v = 1\}$ and

$Y_i := \{v \in K_n :$ the i^{th}-coordinate of $v = 1\}$.

Claim1: H_i is a bipartite graph

By the construction of H_i, $\nexists x, y \ni (x,y) \in X_i$ or $(x,y) \in Y_i$,

since the i^{th}-coordinate is the same.

Hence, the Claim1 holds. ▨

Claim2: $\forall u \in X_i, \forall v \in Y_i, \exists e = (u,v) \in E(H_i)$

Let $u \in X_i$ and $v \in Y_i$.

Since $u \in X_i$ and $v \in Y_i$, since the i^{th}-coordinate of u and v

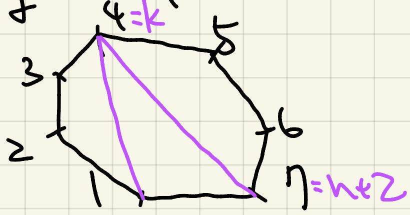
are differ, $\exists (u,v) \in E(H_i)$.

Since $u \in X_i$ and $v \in Y_i$ are arbitrary, the Claim2 holds. ▨

By Claim1 and Claim2, since i is arbitrary,

H_i is a biclique for all $i = 1, 2, \cdots, m$. ▨

3.1. Note that if we fixed the side $(1, n+2)$, and let $k \in \{2, 3, \cdots, n+1\}$ be the partition point.
There are a convex k-gon on the LHS and convex $n+2-k+1 = (n+3-k)$-gon on the LHS.
A toy example is below:



Moreover, there are P_{k-2} way to triangulate a convex $(k-2)$-gon and P_{n+1-k} way to triangulate a convex $(k-2)$-gon.

Hence, $P_n = \sum_{k=2}^{n-1} P_{k-2} P_{n+1-k} = \sum_{k=0}^{n-1} P_k P_{n-k-1}$.

3.2. Let $k \in \mathbb{N} \ni$ the coordinate of 1^{st} at the diagonal is (k, k).

Clearly, when 1^{st} step, we can only move "$\rightarrow$",
when $(2k-1)^{th}$ step, we can only move "$\uparrow$",
and # of "$\rightarrow$" $\geq$ # of "$\uparrow$" during 2^{nd} step to $(2k-2)^{th}$ step.

Therefore, there are h_{k-1} way from $(0,0)$ to (k,k).

Similarly, when $(2k+1)^{th}$ step, we can only move "$\rightarrow$",
when $(2n-1)^{th}$ step, we can only move "$\uparrow$",
and # of "$\rightarrow$" $\geq$ # of "$\uparrow$" during $(2k+1)^{th}$ step to $(2n-2)^{th}$ step.

Therefore, there are h_{n-k} way from (k,k) to (n,n).

Hence, $C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k} = \sum_{k=0}^{n-1} C_k C_{n-k-1}$

4.1 Suppose "statement 1" is false.

(We want to show that $\mathcal{F}$ is 2-colorable.)

Since "statement 1" is false, $\exists!\, A \in \mathcal{F} \ni x \in A$.

Thus, $\forall B \in \mathcal{F} \setminus \{A\},\ x \notin B$. — $\circledast$

Since $|A \cap B| = 1$ and $\circledast$, $\exists y \in$ ground set $N \ni y \neq x$ and $\{y\} = A \cap B$. — $\circledast\circledast$

Claim $\mathcal{F}$ is 2-colorable.

Consider the coloring of the element in ground set N:

Color $A \setminus \{x\}$ red and the remaining blue.

(Case 1: A is not monochromatic)

By apply this coloring, $g(x) = B$, where $x \in A$

Since $|A| \geq 3$, $\exists z \in A \setminus \{x\}$ with $g(z) = R$.

Hence, A is not monochromatic.

(Case 2: $\forall B \in \mathcal{F} \setminus \{A\}$, B is not monochromatic)

If $\mathcal{F} \setminus \{A\} = \emptyset$, the statement holds clearly.

Assume $\mathcal{F} \setminus \{A\} \neq \emptyset$.

Let $B \in \mathcal{F} \setminus \{A\}$.

By $\circledast\circledast$, $\exists y \in N \ni y \neq x$ and $\{y\} = A \cap B$.

By apply this coloring, $g(y) = R$ since $y \in A \setminus \{x\}$.

Since $|B| \geq 3$, $\exists z \in B \setminus \{y\}$ with $g(z) = B$.

Hence, B is not monochromatic.

Thus, $\forall B \in \mathcal{F} \setminus \{A\}$, B is not monochromatic.

By Case 1 and Case 2, $\mathcal{F}$ is 2-colorable. $\divideontimes$ ($\mathcal{F}$ is not 2-colorable)

Thus, "statement 1" is true. ■

4.2 Suppose "statement 2" is false. Let $A, B \in \mathcal{F} \ni x \in A$ and $y \in B$.

(We want to show that $\mathcal{F}$ is 2-colorable)

Claim $\mathcal{F}$ is 2-colorable.

Consider the coloring of the element in ground set N:

Color $(A \cup B) \setminus \{x, y\}$ red and the remaining blue.

(Case1: A is not monochromatic)

By apply this coloring, $g(x) = B$, where $x \in A$

Since $|A| \geq 3$, $\exists z \in A \setminus \{x\}$ with $g(z) = R$.

Hence, A is not monochromatic.

(Case2: $B \in \mathcal{F}$, B is not monochromatic)

If $B = A$ or $\mathcal{F} = \{A\}$, the statement holds clearly.

Assume $B \neq A$.

Similar to case1, B is not monochromatic.

(Case3: $\forall C \in \mathcal{F} \setminus \{A, B\}$, C is not monochromatic)

If $\mathcal{F} \setminus \{A \cup B\} = \emptyset$, the statement holds clearly.

Assume $\mathcal{F} \setminus \{A \cup B\} \neq \emptyset$.

Suppose $C \in \mathcal{F} \setminus \{A \cup B\}$ is monochromatic.

By assumption since $C \neq A$, $\exists z_A \in N \ni \{z_A\} = A \cap C$.
$C \neq B$, $\exists z_B \in N \ni \{z_B\} = B \cap C$, where $z_A \neq z_B$.

($z_A = x$)

If not, $g(z_A) = R$ since $z_A \in (A \cup B) \setminus \{x, y\}$.

Since $|C| \geq 3$, $|C \cap B| = 1$, and $|C \cap A| = 1$, $\exists z \in N \setminus (A \cup B)$
$\ni z \in C$.

Thus, $g(z) = B$ $\divideontimes$ (C is monochromatic.)

Hence, $z_A = x$.

$(z_B = y)$

Similar to $z_A = x$, we have $z_B = y$.

Hence, $z_A \in C$ and $z_B \in C$. $\ast$ ("statement 2" is false)

Thus, $\forall C \in F \setminus \{A, B\}$, C is not monochromatic.

Thus, F is 2-colorable. $\ast$ (F is not 2-colorable)

Hence, "statement 2" is true. ∎

P5 Let $A_0 \subseteq A$ be a subset of A

and define $\alpha = \frac{|A_0|}{|A|}$.

Let B_0 consist of vertices in B that have at least $\alpha D/2$ neighbors in A_0

(Condition 2)

By the construction, B_0 satisfies the condition 2. ∎

(Condition 3)

It is sufficient to proof "at most half of the edges leaving A_0 go to $B \setminus B_0$.

We denote $d(a)$ = the degree of vertex a $\forall a \in (A \cup B)$.

Then, $\frac{1}{2} \sum_{a \in A_0} d(a) \geq \frac{1}{2} \sum_{a \in A_0} d$ since $d \leq d(x)$ $\forall x \in A$

$$
\begin{aligned}
&= \frac{1}{2} |A_0| d \\
&= \frac{1}{2} \alpha |A| d \quad \text{since } \alpha = \frac{|A_0|}{|A|} \\
&\geq \frac{1}{2} \alpha |B| D \quad \text{since } |A| d \geq |B| D \ (\ast) \\
&\geq \frac{1}{2} \alpha (|B| - |B_0|) D = \sum_{b \in |B| \setminus |B_0|} \frac{\alpha D}{2} \quad \text{since } |B_0| \geq 0 \\
&\geq \sum_{b \in |B| \setminus |B_0|} d(b) \quad \text{since the construct of } B_0
\end{aligned}
$$

Hence, at most half of the edges leaving A_0 go to $B \setminus B_0$. ∎

(Condition 1)

$$|B_0| D \geq \sum_{b \in B_0} d(b) \quad \text{since } D \geq d(x) \ \forall x \in B$$

$$\geq \frac{1}{2} \sum_{a \in A_0} d(a) \quad \text{by Condition 3}$$

$$\geq \frac{1}{2} \alpha |B| D \quad \text{by } \circledast$$

Hence, $|B_0| \geq \alpha |B|/2$. $\blacksquare$