Homework 1 of Algebra Graph Theory

9/12/2024

Due on 9/26/2024 in class

請同學使用 A4 紙張，以打字或黑色或藍色筆書寫作業。請用英文作答，並注意下列事項：1. 清楚標示及區別逗點及句點；2. 句首字母要大寫；3. 不以數學符號當句首；4. 句子間要有連接詞、不用數學符號取代連接詞；5. 詳述推論原因不跳躍思考；6. 清楚標示出現的變數是存在還是所有，不用代號 ∀ 及 ∃ ；7. 文法正確、不省略動詞、注意單複數。

評分標準：邏輯正確 (60%)、敘述詳盡及文法正確 (30%)、字體端正格式整齊 (10%)。

A collection $\{S_1, S_2, \ldots, S_t\}$ of subsets of S is called a *partition* if $S_i \neq \emptyset$, $S_i \cap S_j = \emptyset$, and $S_1 \cup S_2 \cup \cdots \cup S_t = S$, where $1 \leq i \neq j \leq t$. A *digraph* $\overrightarrow{\Gamma}$ is a pair (V, E) such that V is a finite set and E is a subset of $V \times V$. An element in E is called a *directed edge* of an *arc*. Sometimes, we use $E\overrightarrow{\Gamma}$ and $V\overrightarrow{\Gamma}$ to denote E and V respectively for referring their digraph $\overrightarrow{\Gamma}$. For $x, y \in V$, a walk in $\overrightarrow{\Gamma}$ from x to y is a sequence $x = x_0, x_1, \ldots, x_t = y$ such that $x_{i-1}x_i \in E$ for all $1 \leq i \leq t$.

1. Let $\overrightarrow{\Gamma}$ be a digraph with vertex set $V\overrightarrow{\Gamma}$ and directed edge set $E\overrightarrow{\Gamma}$. Prove the following two statements.

   (a) If there exists a partition $\{X, Y\}$ of $V\overrightarrow{\Gamma}$ such that $xy \notin E\overrightarrow{\Gamma}$ for every pair $(x, y) \in X \times Y$, then $\overrightarrow{\Gamma}$ is not strongly connected.

   Proof. Since X and Y are nonempty, we can pick $x \in X$ and $y \in Y$. We show no directed walk in $\overrightarrow{\Gamma}$ from x to y. On the contrary, assume there exists a directed walk $x = x_0,\ x_1,\ \ldots,\ x_t = y$ from x to y. Let x_i be the first vertex in Y along this walk. Such i exists since $x_t \in Y$ and $i \geq 1$ since $x_0 \in X$. Then $x_{i-1} \in X$, $x_i \in Y$ and $x_{i-1}x_i \in E\overrightarrow{\Gamma}$, a contradiction to the assumption. □

   (b) If $\overrightarrow{\Gamma}$ is not strongly connected then there exists a partition $\{X, Y\}$ of $V\overrightarrow{\Gamma}$ such that $xy \notin E\overrightarrow{\Gamma}$ for every pair $(x, y) \in X \times Y$.

   Proof. Since $\overrightarrow{\Gamma}$ is not strongly connected, there exist $x \in X$ and $y \in Y$ with no directed walk in $\overrightarrow{\Gamma}$ from x to y. Let

   $$X = \{z \in V\overrightarrow{\Gamma} \mid \text{There is a directed walk from } x \text{ to } z\},$$

   and $Y = V\overrightarrow{\Gamma} - X$. The collection $\{X, Y\}$ is clear a partition of $V\overrightarrow{\Gamma}$. It remains to show $x'y' \notin E\overrightarrow{\Gamma}$ for every pair $(x', y') \in X \times Y$. On the contrary, assume $x'y' \in E\overrightarrow{\Gamma}$ for some pair $(x', y') \in X \times Y$. Since $x' \in X$, there is a directed walk $x = x_0, x_1, \ldots, x_t = x'$ in $\overrightarrow{\Gamma}$. Then $x = x_0, x_1, \ldots, x_t, y'$ is a directed walk from x to y. Hence $y \in X \cap Y = \emptyset$, a contradiction. □