What does the spectrum of a graph tell?

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Algebraic Graph Theory (2024 Fall)

Outline

Regularity

Theorem

Let Γ be a simple graph with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then the following are equivalent.

(i) **Γ is regular with valency λ_1.**

(ii) $\sum_{i=1}^{n} \lambda_i^2 = n\lambda_1$.

Proof.

Note that $A^2_{xx} = \deg(x)$ for $x \in V\Gamma$.

(i) $\Rightarrow$ (ii) This follows from $\displaystyle\sum_{i=1}^{n} \lambda_i^2 = \mathrm{tr} A^2 = \sum_{x \in V\Gamma} A^2_{xx} = n\lambda_1$.

(ii) $\Rightarrow$ (i) The average degree $\displaystyle\overline{k} = \frac{\mathrm{tr} A^2}{n} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^2 = \lambda_1$. Hence Γ is regular with valency λ_1. $\square$

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Bipartite graphs

Bipartite matrix

- A square matrix M is **bipartite** if after permuting the rows and columns M has the following block form.

$$M = \begin{pmatrix} \mathbf{0} & M_{21} \\ M_{12} & \mathbf{0} \end{pmatrix},$$

  where the $\mathbf{0}$'s are zero submatrices of suitable sizes.
- A graph is **bipartite** if its adjacency matrix is bipartite.

Lemma

Let M be a bipartite matrix and $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of M iff $-\lambda$ is an eigenvalue of M. Moreover λ and $-\lambda$ has the same geometry multiplicity.

Proof.

Observe in block form product

$$\begin{pmatrix} \mathbf{0} & M_{21} \\ M_{12} & \mathbf{0} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

iff

$$\begin{pmatrix} \mathbf{0} & M_{21} \\ M_{12} & \mathbf{0} \end{pmatrix} \begin{pmatrix} -x \\ y \end{pmatrix} = -\lambda \begin{pmatrix} -x \\ y \end{pmatrix}.$$

$\square$

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Lemma

Let M be a bipartite matrix. Then M^2 is reducible.

Proof.

This follows from in block form product

$$M^2 = \begin{pmatrix} \mathbf{0} & M_{21} \\ M_{12} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & M_{21} \\ M_{12} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} M_{21}M_{12} & 0 \\ 0 & M_{12}M_{21} \end{pmatrix}.$$

□

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Proposition

If a matrix M is irreducible, M^2 is reducible and $M \geq 0$, then M is bipartite.

This is Homework 2.

Theorem

Let Γ be a connected undirected graph with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$. Then Γ is bipartite if and only if $\lambda_n = -\lambda_1$.

Proof.

($\Rightarrow$) This follows from the symmetric property of spectrum.
($\Leftarrow$) Suppose $\lambda_n = -\lambda_1$. Then λ_1^2 is a maximal eigenvalue of A^2 with multiplicity at least 2. By Perron Frobenius Theorem, A^2 is reducible. Hence A is bipartite by previous proposition. □

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Remark

- If Γ is bipartite, then $SP(L(\Gamma)) = SP(Q(\Gamma))$.
- $$SP(Q(K_{1,3}) = SP\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = (4,1,1,0).$$
- $$SP(Q(K_1+K_3)) = SP\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} = (4,1,1,0).$$
- $$SP(L(K_1+K_3)) = SP\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix} = (3,3,0,0).$$

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Exercise

Let Γ be a graph. Then Γ is bipartite if and only if

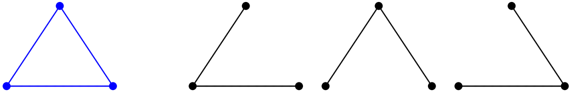
$$SP(L(\Gamma)) = SP(Q(\Gamma)).$$

Hint. Check the possible diagonal matrix E with unit lengths of diagonal entries and $Q = ELE^{-1}$.

Matrix-Tree Theorem

Tree

1. A **cycle** in a graph is a closed walk without repeated vertices except that the first vertex and the last vertex are the same.
2. A **tree** is a connected graph without cycles.
3. A **spanning tree** of an undirected graph Γ is a subgraph of Γ which is a tree and contains all vertices of Γ.

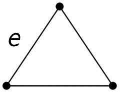


K_3 has three spanning trees.

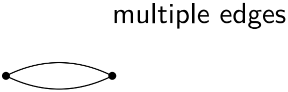
Deletion and contraction

Let Γ be an undirected graphs with possible multi-edges and without loops. Let e be an edge in Γ. Then $\Gamma \setminus e$ is the graph obtained by deleting the edge e from Γ, and Γ/e is the graph obtained from Γ by identifying the two ends of e to a vertex, deleting e and all multi-edges of e to remain loopless.

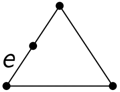
Γ:



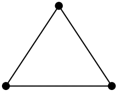
Γ/e:



Γ:



Γ/e:



Number of spanning trees

Let $\tau(\Gamma)$ denote the number of spanning trees on Γ. Then

$$\tau(\Gamma) = \tau(\Gamma - e) + \tau(\Gamma / e)$$

for any edge e in G.

Proof.

$\tau(\Gamma - e)$ counts the number of spanning trees on G that do not contain e, and $\tau(\Gamma / e)$ counts the number of spanning trees that contain e. □

Multi-linear property of determinant

From linear algebra

$$\det\begin{pmatrix} u+v & B \end{pmatrix} = \det\begin{pmatrix} u & B \end{pmatrix} + \det\begin{pmatrix} v & B \end{pmatrix},$$

where u, v are column vectors, and B has size $n \times (n-1)$.

Example

$$\det\begin{pmatrix} 8 & 7 \\ 6 & 5 \end{pmatrix} = \det\begin{pmatrix} 5 & 7 \\ 6 & 5 \end{pmatrix} + \det\begin{pmatrix} 3 & 7 \\ 0 & 5 \end{pmatrix} = -2.$$

Matrix notation

Let M be an $n \times n$ matrix and $i \in [n]$. Let $M(i|i)$ be the principal submatrix of M obtained by deleting the row and column indexed by i.

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Lemma

Let e be an edge in G with one end i. Then

$$\det\left(L(\Gamma)(i|i)\right) = \det\left(L(\Gamma - e)(i|i)\right) + \det\left(L(\Gamma/e)(i|i)\right).$$

Proof.

Suppose $e = 12$, $i = 1$, and $d_1, d_2, \dots$ are the degrees. Then

$$\begin{aligned}
\det\left(L(\Gamma)(1|1)\right) &= \det\begin{pmatrix} d_2 & * & \\ * & d_3 & \\ & & \end{pmatrix} \\
&= \det\begin{pmatrix} d_2 - 1 & * & \\ * & d_3 & \\ & & \end{pmatrix} + \det\begin{pmatrix} 1 & * & \\ 0 & d_3 & \\ & & \end{pmatrix} \\
&= \det(L(\Gamma - e)(1|1)) + \det(L(\Gamma/e)(1|1))
\end{aligned}$$

□

Theorem (Matrix-Tree Theorem)

$$\tau(\Gamma) = \det\left(L(\Gamma)(u|u)\right)$$

for any $u \in VG$.

Proof.

Since $\tau(G)$ and $\det L(G)(u|u)$ satisfy the same pattern of recurrence relation, we only need to check the initial conditions, $c \geq 2$ copies of K_1 and K_2. This follows from

$$\tau(cK_1) = 0 = \det\left(L(cK_1)(u|u)\right),$$
$$\tau(K_2) = 1 = \det\left(L(K_2)(u|u)\right)$$

for any vertex u. □

Theorem (Cayley, 1889)

$$\tau(K_n) = n^{n-2}.$$

Proof.

Note that $L(K_n)(1|1) = nI_{n-1} - J_{n-1}$, where I_{n-1} and J_{n-1} are $(n-1) \times (n-1)$ identity matrix and all ones matrix respectively. Note that J_{n-1} has eigenvalues $n-1$ and 0 with multiplicities 1 and $n-2$ respectively. Hence $L(K_n)(1|1)$ has eigenvalues 1 and n with multiplicities 1 and $n-2$ respectively. This

$$\tau(K_n) = \det(L(K_n)(1|1)) = 1 \cdot n \cdot n \cdots n = n^{n-2}.$$

$\square$

A. Cayley, A theorem on trees, Quart. J. Pure Appl. Math. 23 (1889) 376–378.