Partially Distance-regular Graphs and Partially Walk-regular Graphs*

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Abstract

We study partially distance-regular graphs and partially walk-regular graphs as generalizations of distance-regular graphs and walk-regular graphs respectively. We conclude that the partially distance-regular graphs can be viewed as some extremal graphs of partially walk-regular graphs. In the special case that the graph is assumed to be regular with four distinct eigenvalues, a well known class of walk-regular graphs, we show that there exists a rational function f in the expression of the order and the four eigenvalues of the graph such that $k_2(x)$, the number of vertices with distance 2 to a vertex x, satisfies $k_2(x) \geq f$; furthermore we show the equality holds for each vertex x if and only if the graph is distance-regular with diameter 3.

Keywords: Partially distance-regular graphs; partially walk-regular graphs, eigenvalues.

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1 Introduction

Partially distance-regular graphs and partially walk-regular graphs (formal definition in Section 2 and Section 3) are generalizations of distance-regular graphs and walk-regular graphs respectively. They were introduced by M. A. Fiol and E. Garriga when they studied the range of the spectrum of a graph [6].

We study these two classes of graphs and find their properties by following the classic way in the study of distance-regular graphs and walk-regular graphs respectively. We study the link between these two classes of graphs, and conclude that the partially distance-regular graphs can be viewed as some extremal graphs of partially walk-regular graphs. See Theroem 3.3 for detailed description.

We apply Theorem 3.3 to the case when the graph Γ is assumed to be regular and has exactly four distinct eigenvalues. It is well known that Γ is walk-regular. See Lemma 5.1 for the generalization of this result. We show that there exists a rational function f in the expression of the order and the four eigenvalues of Γ such that $k_2(x)$, the number of vertices with distance 2 to a vertex x, satisfies $k_2(x) \geq f$. Furthermore we show the equality holds for each vertex x of Γ if and only if Γ is distance-regular with diameter 3. See Theorem 6.1 for detail. Theorem 6.1 answers a conjecture in [4].

It is well known that the Godsil switching, a way of switching edges of a graph (described in Section 4), will destroy the distance-regularity of a graph while preserving its spectrum. We show in Corollary 4.3 that the walk-regularity of a graph is preserved by Godsil switching provided the switching exists.

2 Partially Distance-regular Graphs

Throughout the paper, let $\Gamma = (X, R)$ denote a simple connected graph with diameter d, order n, and length distance function ∂. For an integer i and $x \in X$, set

$$\Gamma_i(x) := \{y \mid y \in X,\ \partial(y, x) = i\}.$$

Γ is *k-regular* or *regular* for short if $|\Gamma_1(x)| = k$ for all $x \in X$. The i-th *distance matrix* A_i is an $n \times n$ matrix with rows and columns indexed by X such that

$$(A_i)_{xy} := \begin{cases} 1, & \text{if } \partial(x,y) = i; \\ 0, & \text{else,} \end{cases} \qquad (x, y \in X).$$

Hence $A_i = 0$ for $i < 0$ or $i > d$. $A = A_1$ is called the *adjacency matrix* of Γ.

Definition 2.1. We say that Γ is *t-partially distance-regular* whenever for each integer $0 \le i \le t$, there exists a polynomial $v_i(\lambda) \in \mathbb{R}[\lambda]$ of degree i such that $A_i = v_i(A)$. Γ is *distance-regular* if Γ is d-partially distance-regular.

Hence any graph is 1-partially distance-regular from the above definition.

Definition 2.2. Fix integers $0 \le i, j, h \le d$. We say P^h_{ij} is *well-defined* in Γ whenever for any two vertices $x, y \in X$ with $\partial(x,y) = h$, the number

$$p^h_{ij}(x,y) := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of the choice of x, y. In this case we write p^h_{ij} for $p^h_{ij}(x,y)$, and call p^h_{ij} the *intersection number* of Γ. For convenience we will write c_h, a_h, b_h, and k_h for the symbols $p^h_{1\ h-1}$, $p^h_{1\ h}$, $p^h_{1\ h+1}$ and p^0_{hh} respectively.

Note that $k_1(x)$ is the valency of $x \in X$. Observe that for $h \ge 1$, $a_h(x,y) + b_h(x,y) + c_h(x,y) = k_1(x)$ for $x, y \in X$ with $\partial(x,y) = h$. The following proposition is similar to a well-known property in the study of distance-regular graphs [1, Proposition 1.1].

Proposition 2.3. *Fix an integer $1 \le t \le d$. Then the following are equivalent.*

(i) Γ is t-partially distance-regular.

(ii) p^k_{ij} are well-defined for $0 \le i+j, k \le t$.

(iii) c_i, a_{i-1}, b_{i-2} are well-defined for $1 \le i \le t$.

Proof. ((i)$\Longrightarrow$(ii)) Fix two integers i, j with $0 \le i+j \le t$. By the assumption (i) we know $\{A_0, A_1, \ldots, A_t\}$ is a basis of the vector space $\{\ f(A) \mid f(\lambda) \in \mathbb{R}[\lambda]$ has degree at most $t\}$ over $\mathbb{R}$, and A_iA_j is in the vector space. Hence

$$A_iA_j = \sum_{k=0}^{t} c^k_{ij} A_k \tag{2.1}$$

for some constants $c_{ij}^k \in \mathbb{R}$. For any two vertices $x, y \in X$ with $\partial(x,y) = k \leq t$, comparing the xy entry on both sides of (2.1), we find that $p_{ij}^k(x,y) = c_{ij}^k$ is independent of x, y.
((ii)$\Longrightarrow$(iii)) For $1 \leq i \leq t$, $c_i = p_{1\ i-1}^i$, $a_{i-1} = p_{1\ i-1}^{i-1}$, $b_{i-2} = p_{1\ i-1}^{i-2}$ are well-defined, since the sum is $i \leq t$ in each of the subscripts of intersection numbers.
((iii)$\Longrightarrow$(i)) Note that

$$AA_{i-1} = b_{i-2}A_{i-2} + a_{i-1}A_{i-1} + c_iA_i$$

by comparing entries on both sides, or equivalently

$$A_i = c_i^{-1}((A - a_{i-1}I)A_{i-1} - b_{i-2}A_{i-2})$$

for $1 \leq i \leq t-1$. Hence $A_i = v_i(A)$ where $v_i(\lambda) \in \mathbb{R}[\lambda]$ has degree i and is defined recursively by $v_0(\lambda) = 1$, $v_1(\lambda) = \lambda$, and

$$v_i(\lambda) = c_i^{-1}((\lambda - a_{i-1})v_{i-1}(\lambda) - b_{i-2}v_{i-2}(\lambda))$$

for $2 \leq i \leq t$. □

It is not clear at this moment that $k_t = p_{tt}^0$ is well-defined from Proposition 2.3(ii). The following lemma explains this.

Lemma 2.4. *Suppose Γ is t-partially distance-regular, where $t \geq 2$. Then b_{t-1} and k_i are well-defined for $0 \leq i \leq t$.*

Proof. We apply Proposition 2.3. Note b_0 is well-defined since $t \geq 2$. Since $b_{t-1} = b_0 - a_{t-1} - c_{t-1}$, we find b_{t-1} is well-defined. As in [2, Chapter 5], we have $k_i = b_0b_1 \cdots b_{i-1}/(c_1c_2 \cdots c_i)$, and hence k_i is well-defined for $0 \leq i \leq t$. □

3 Partially Walk-regular Graphs

We now give the definition of the second class of graphs in the title of the paper.

Definition 3.1. We say that Γ is *t-partially walk-regular* whenever for each integer $1 \leq i \leq t$, $(A^i)_{xx}$ is a constant depending on i, but not on $x \in X$. Γ is *walk-regular* if Γ is t-partially walk-regular for any integer t.

Hence in a t-partially walk-regular graph, the number of closed walks of length i from a vertex x to itself is a constant, depending on $i \leq t$ not on $x \in X$. In particular, a 2-partially walk-regular graph is regular with valency $(A^2)_{xx}$ for any $x \in X$.

Lemma 3.2. *Suppose Γ is t-partially distance-regular, where $t \geq 2$. Then Γ is $2t$-partially walk-regular.*

Proof. Fix a positive integer $u \leq t$. Suppose

$$A^{u-1} = \sum_{i=0}^{u-1} t_i A_i, \quad A^u = \sum_{i=0}^{u} s_i A_i$$

for some $t_i, s_i \in \mathbb{R}$. Note that k_i is well-defined for $0 \leq i \leq t$ by Lemma 2.4. Then

$$\begin{aligned} (A^{2u-1})_{xx} &= \sum_{y \in X} (A^{u-1})_{xy} (A^u)_{yx} \\ &= \sum_{y \in X} (\sum_{i=0}^{u-1} t_i (A_i)_{xy})(\sum_{i=0}^{u} s_i (A_i)_{xy}) \\ &= \sum_{i=0}^{u-1} k_i t_i s_i, \end{aligned}$$

and

$$\begin{aligned} (A^{2u})_{xx} &= \sum_{y \in X} (\sum_{i=0}^{u} s_i (A_i)_{xy})^2 \\ &= \sum_{i=0}^{u} k_i s_i^2 \end{aligned}$$

are independent of the choice of $x \in X$. $\square$

The converse of the above lemma is false. $\overline{C_6}$, the complement of a cycle of length 6, is a graph of diameter 2, which is walk-regular, but not distance-regular.

Theorem 3.3. *Suppose Γ is regular and t-partially distance-regular. Then for $x \in X$, we have $|\Gamma_{t+1}(x)| \geq f$, where f is a function of intersection* *numbers and* $(A^{2t+1})_{xx}, (A^{2t+2})_{xx}$*. Furthermore suppose* Γ *is* $(2t+2)$*-partially walk-regular. Then the above equality holds for each* $x \in X$ *if and only if* Γ *is* $(t+1)$*-partially distance-regular.*

Proof. If $\Gamma_{t+1}(x) = \emptyset$, we choose $f = 0$ and the first part of the theorem holds clearly. We assume $\Gamma_{t+1}(x) \neq \emptyset$. By the assumption we can write $A^t = \sum_{i=0}^{t} s_i A_i$ for some $s_i \in \mathbb{R}$ with $s_t \neq 0$. Also c_i, b_{i-1}, a_{i-1} and k_i are well-defined in Γ for $1 \leq i \leq t$ by Proposition 2.3 and Lemma 2.4. Then

$$
\begin{aligned}
(A^{2t+1})_{xx} &= \sum_{y \in X} (\sum_{z \in X} \sum_{i=0}^{t} s_i (A_i)_{xz} A_{zy}) (\sum_{i=0}^{t} s_i (A_i)_{yx}) \\
&= \sum_{i=0}^{t-1} k_i (c_i s_{i-1} + a_i s_i + b_i s_{i+1}) s_i \\
&\qquad + (k_t c_t s_{t-1} + \sum_{y \in \Gamma_t(x)} a_t(y,x) s_t) s_t \qquad (3.1)
\end{aligned}
$$

by summing y according to its distance i to x. From (3.1) we find

$$\sum_{y \in \Gamma_t(x)} a_t(y,x)$$

can be determined from the well-defined intersection numbers of Γ and an additional constant $(A^{2t+1})_{xx}$. Similarly,

$$
\begin{aligned}
(A^{2t+2})_{xx} &= \sum_{y \in X} ((A^{t+1})_{xy})^2 \\
&= \sum_{y \in X} (\sum_{z \in X} \sum_{i=0}^{t} s_i (A_i)_{xz} (A)_{zy})^2 \\
&= \sum_{i=0}^{t-1} k_i (c_i s_{i-1} + a_i s_i + b_i s_{i+1})^2 \qquad (3.2) \\
&\qquad + \sum_{y \in \Gamma_t(x)} (c_t s_{t-1} + a_t(y,x) s_t)^2 \qquad (3.3) \\
&\qquad + \sum_{y \in \Gamma_{t+1}(x)} (c_{t+1}(y,x) s_t)^2. \qquad (3.4)
\end{aligned}
$$

By applying Cauchy's inequality in (3.3),

$$\begin{aligned}
&\sum_{y\in\Gamma_t(x)} (c_t s_{t-1} + a_t(y,x)s_t)^2 \\
\geq\ & \frac{1}{k_t}\Big(\sum_{y\in\Gamma_t(x)} (c_t s_{t-1} + a_t(y,x)s_t)\Big)^2. \qquad (3.5)
\end{aligned}$$

and equality holds in (3.5) iff $a_t(y,x)$ is independent of the choice of $y \in \Gamma_t(x)$. Similarly for (3.4) we have

$$\begin{aligned}
&\sum_{y\in\Gamma_{t+1}(x)} (c_{t+1}(y,x)s_t)^2 \\
\geq\ & \frac{1}{k_{t+1}(x)}\Big(\sum_{y\in\Gamma_{t+1}(x)} c_{t+1}(y,x)s_t\Big)^2 \\
=\ & \frac{1}{k_{t+1}(x)}\Big(\sum_{y\in\Gamma_t(x)} (b_0 - c_t - a_t(y,x))s_t\Big)^2, \qquad (3.6)
\end{aligned}$$

and equality holds iff $c_{t+1}(y,x)$ is independent of the choice $y \in \Gamma_{t+1}(x)$. Set T_1, T_2, $(k_{t+1}(x))^{-1}T_3$ to be the expressions in (3.2), (3.5) and (3.6) respectively, and note that T_1, T_2, T_3 can be computed from the intersection numbers of Γ and the additional constant $(A^{2t+1})_{xx}$. Now we have

$$(A^{2t+2})_{xx} \geq T_1 + T_2 + (k_{t+1}(x))^{-1}T_3.$$

Note that $T_3 > 0$. Then $(A^{2t+2})_{xx} - T_1 - T_2 > 0$. So we can rewrite the above inequality as

$$k_{t+1}(x) \geq \frac{T_3}{(A^{2t+2})_{xx} - T_1 - T_2}. \qquad (3.7)$$

The first part of the theorem is obtained by setting f to be the right hand side of (3.7).

Suppose Γ is $(2t+2)$-partially walk-regular. Then the f is a constant, not depending on $x \in X$. We consider two cases according to $f = 0$ or not. Note that $f = |\Gamma_{t+1}(x)| = 0$ for all $x \in X$ iff $d \leq t$, and this is equivalent to that Γ is distance-regular. Suppose $|\Gamma_{t+1}(x)| \neq 0$ for some $x \in X$. Then the equality hold in (3.7) for each $x \in X$ iff c_{t+1}, a_t, $b_t = b_0 - c_t - a_t$ are well-defined, and this is equivalent to that Γ is $(t+1)$-partially distance-regular. $\square$

Remark 3.4. The inequality in Theorem 3.3 essentially comes from Cauchy's inequality. A similar argument also appears in [5].

4 Godsil Switching

We shall prove the walk-regularity are preserved by two operations on graphs in this section.

The *complement* $\overline{\Gamma}$ of $\Gamma = (X, R)$ is a graph with vertex set X and adjacency matrix $\overline{A} = J - I - A$, where A is the adjacency matrix of Γ, I is the identity matrix and J is the all 1's matrix.

Lemma 4.1. *Suppose $\Gamma = (X, R)$ is t-partially walk-regular. Then the complement $\overline{\Gamma}$ of Γ is t-partially walk regular.*

Proof. This is clear if $t \leq 1$ since every graph is 1-partially walk-regular. Assume $t \geq 2$. In particular Γ is k-regular for some nonegative integer k. Since $JA = AJ = kJ$ and $JJ = nJ$, we find $\overline{A}^{\,i} = (J - I - A)^i$ is a linear combination of J, I, A; in particular $\overline{A}^{\,i}$ has identical diagonal entries for $0 \leq i \leq t$. □

Suppose $\pi = (C_1, C_2, \ldots, C_k, C_{k+1})$ is a partition of the vertex set X such that the following (i)-(ii) hold.

(i) For $1 \leq i, j \leq k$, there exists a constant t_{ij} such that

$$|\Gamma_1(x) \cap C_j| = t_{ij} \tag{4.1}$$

for all $x \in C_i$.

(ii) For $x \in C_{k+1}$ and $1 \leq i \leq k$,

$$|\Gamma_1(x) \cap C_i| = 0, |C_i|/2, \text{ or } |C_i|.$$

Suppose the above partition π exists in $\Gamma = (X, R)$. Let $\Gamma^{(\pi)}$ denote the graph with same vertex set X and the same edges of Γ except that for each $x \in C_{k+1}$ and each $1 \leq i \leq k$ with $|\Gamma_1(x) \cap C_i| = |C_i|/2$, the edges between x and C_i are deleted and all the edges from x to vertices in $C_i - \Gamma_1(x)$ are added. We say the graph $\Gamma^{(\pi)}$ is obtained from Γ by the *Godsil switching with respect to* π. To describe the adjacency matrix $A^{(\pi)}$ of $\Gamma^{(\pi)}$ as shown in [7], we need the following setting. For positive integers m, t, let J_m (resp. j_m) denote the $m \times m$ (resp. $m \times 1$) all 1's matrix, I_m denote the $m \times m$ identity matrix and $Q_m = (2/m)J_m - I_m$. The the following (a)-(d) are easily verified.

(a) $Q_m^2 = I_m$,

(b) If X is an $m \times t$ matrix with a constant row sum and a constant column sum, then $Q_m X Q_t = X$,

(c) If X is an $m \times 1 (1 \times m)$ matrix with column sum (row sum) $m/2$, then $Q_m X = j_m - X$ $((j_m)^T - X = XQ_m)$,

(d) $Q_m j_m = j_m$.

We may assume that the vertices of Γ are ordered so that A can be written as

$$\begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1k+1} \\ B_{21} & B_{22} & \cdots & B_{2k+1} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k+1\ 1} & B_{k+1\ 2} & \cdots & B_{k+1k+1} \end{pmatrix},$$

where B_{ii} is the adjacency matrix of the graph induced by C_i. Note that B_{ij} has a constant row sum and a constant column sum for each pair $1 \leq i, j \leq k$. The partition $\pi = (C_1, C_2, \ldots, C_k, C_{k+1})$ of X is *equitable* if (4.1) holds for $1 \leq i, j \leq k+1$; in this case B_{ij} has a constant row sum and a constant column sum for each pair $1 \leq i, j \leq k+1$. Let Q be the block diagonal matrix with $k+1$ blocks, where the i-th diagonal block is Q_{m_i} if $i \leq k$ and the $(k+1)$-th block is the identity matrix $I_{m_{k+1}}$, with $m_i = |C_i|$. From (a)-(d) above and the constriction, we have $Q^2 = I$ and

$$A^{(\pi)} = QAQ = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1k} & Q_{m_1} B_{1k+1} \\ B_{21} & B_{22} & \cdots & B_{2k} & Q_{m_2} B_{2k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{k1} & B_{k2} & & B_{kk} & Q_{m_k} B_{kk+1} \\ B_{k+1\ 1} Q_{m_1} & B_{k+1\ 2} Q_{m_2} & \cdots & B_{k+1k} Q_{m_k} & B_{k+1k+1} \end{pmatrix}.$$

Suppose A^s is written in the block matrix form as

$$A^s = \begin{pmatrix} B_{11}^{(s)} & B_{12}^{(s)} & \cdots & B_{1k+1}^{(s)} \\ B_{21}^{(s)} & B_{22}^{(s)} & \cdots & B_{2k+1}^{(s)} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k+1\ 1}^{(s)} & B_{k+1\ 2}^{(s)} & \cdots & B_{k+1k+1}^{(s)} \end{pmatrix} \tag{4.2}$$

for any nonnegative integer s, where $B_{ij}^{(1)} = B_{ij}$ for $1 \leq i, j \leq k+1$.

Proposition 4.2. *Let $\Gamma = (X, R)$ be a regular graph, and let π be an equitable partition of X satisfying (ii) above. Fix a nonnegative integer s and suppose A^s is as in (4.2). Then*

$$(A^{(\pi)})^s = \begin{pmatrix} B_{11}^{(s)} & B_{12}^{(s)} & \cdots & B_{1k}^{(s)} & Q_{m_1}B_{1k+1}^{(s)} \\ B_{21}^{(s)} & B_{22}^{(s)} & \cdots & B_{2k}^{(s)} & Q_{m_2}B_{2k+1}^{(s)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{k1}^{(s)} & B_{k2}^{(s)} & & B_{kk}^{(s)} & Q_{m_k}B_{kk+1}^{(s)} \\ B_{k+1\ 1}^{(s)}Q_{m_1} & B_{k+1\ 2}^{(s)}Q_{m_2} & \cdots & B_{k+1k}^{(s)}Q_{m_k} & B_{k+1k+1}^{(s)} \end{pmatrix}.$$

Proof. The B_{ij} described above has a constant row sum and a constant column sum for each pair $1 \le i, j \le k+1$, since π is equitable. This implies that

$$B_{ij}^{(s)} = \sum_{1 \le p_1, p_2, \ldots, p_{s-1} \le k+1} B_{ip_1} B_{p_1 p_2} \cdots B_{p_{s-2}p_{s-1}} B_{p_{s-1}j}$$

has a constant row sum and a constant column sum. Applying the above (b) to $(A^{(\pi)})^s = QA^sQ$ and simplifying, we have the proposition. □

Corollary 4.3. *Let $\Gamma = (X, R)$ denote a t-partially walk-regular graph, and let π be an equitable partition of X satisfying (ii) above. Then $\Gamma^{(\pi)}$ is t-partially walk-regular.*

Proof. The corollary follows immediately from Proposition 4.2 since A^s and $(A^{(\pi)})^s$ have the same diagonal blocks for any nonnegative integer s. □

Remark 4.4. ([8]) The *Gosset graph* $\Gamma = (X, R)$ is the unique distance-regular graph on 56 vertices of diameter 3 with $b_0 = 27$, $b_1 = 10$, $b_2 = 1$, $c_2 = 10$, and $c_3 = 27$. There exists an equitable partition π of X that satisfies (ii) above. The graph $\Gamma^{(\pi)}$ obtained from Γ by Godsil switching with respect to π is not distance-regular.

5 Graphs with s Distinct Eigenvalues

In this section we assume $\Gamma = (X, R)$ is a simple connected k-regular graph with diameter d, s distinct eigenvalues

$$k > \lambda_1 > \lambda_2 > \ldots > \lambda_{s-1},$$

and order n. It is well-known that $s \geq d+1$ [7, Lenna 5.2], and

$$J = \frac{n}{q(k)} q(A), \tag{5.1}$$

where $q(\lambda) := \prod_{i=1}^{s-1} (\lambda - \lambda_i) \in \mathbb{R}[\lambda]$ [3, Corollary 3.3].

Lemma 5.1. *Suppose that Γ is $(s-2)$-partially walk-regular, where $s \geq 4$. Then Γ is walk-regular. In particular, for any nonnegative integers t, $(A^t)_{xx}$ is determined by n and the eigenvalues of Γ for all $x \in X$.*

Proof. For $s \geq 4$, Γ is regular. Note that the $q(\lambda)$ of Γ has degree $s-1$ and

$$k^t J = \frac{n}{q(k)} q(A) A^t$$

for all nonnegative integers t, where k is the valency of Γ. Hence $(A^{s-1+t})_{xx}$ is a function of $k = (A^2)_{xx}, (A^3)_{xx}, \ldots, (A^{s-2})_{xx}$ for all nonnegative integers t. □

Lemma 5.2. *Suppose that Γ is $(d-1)$-partially distance-regular and has $d+1$ distinct eigenvalues, where $d \geq 3$. Then Γ is distance-regular.*

Proof. Γ is regular since $d \geq 3$. The $q(\lambda)$ of Γ has degree d since Γ has $d+1$ eigenvalues. Hence by referring to (5.1), for any two vertices $x, y \in X$ with $\partial(x, y) = d$,

$$(A^d)_{xy} = \frac{q(k)}{n}$$

is independent of the choice of x, y, and note that

$$(A^d)_{xy} = c_d(y, x) c_{d-1} c_{d-2} \cdots c_1.$$

Hence c_d is well-defined. Similarly, for any two vertices $x, y \in X$ with $\partial(x, y) = d-1$,

$$(A^d)_{xy} = \frac{q(k)}{n} - c_{d-1} c_{d-2} \cdots c_1 \times \text{the coefficient of } \lambda^{d-1} \text{ in } q(\lambda)$$

is independent of the choice of x, y, and note that

$$(A^d)_{xy} = (a_{d-1}(y, x) + a_{d-2} + \ldots + a_1)(c_{d-1} c_{d-2} \cdots c_1).$$

Hence a_{d-1} is well-defined. Since Γ is regular with diameter d, we find c_d, a_{d-1}, b_{d-2} are well-defined. □

6 Regular Graphs with Four Eigenvalues

We apply the previous results to the connected regular graphs with four distinct eigenvalues in this section.

Theorem 6.1. *Let* $\Gamma = (X, R)$ *denote a connected* k*-regular graph with* n *vertices and four distinct eigenvalues* $k > \lambda_1 > \lambda_2 > \lambda_3$*. Then* Γ *is walk-regular with diameter* 2 *or* 3*. Moreover there exists a rational function* $f(n, k, \lambda_1, \lambda_2, \lambda_3)$ *in the variables* n, k, λ_1, λ_2, λ_3 *such that for any* $x \in X$

$$k_2(x) \geq f(n, k, \lambda_1, \lambda_2, \lambda_3). \tag{6.1}$$

Furthermore the equality holds for each $x \in X$ *if and only if* Γ *is distance-regular with diameter* 3*.*

Proof. Γ is walk-regular by Lemma 5.1 and clear to have diameter 2 or 3. It is well known that if Γ has diameter 2 then it is not distance-regular[7, Lemma 4.1]. Now the theorem follows from Theorem 3.3 with the case $t = 1$. □

Remark 6.2. The second part of Theorem 6.1 is essentially a result of E. R. van Dam and W. H. Haemers [4] with slightly different variables in the expression of f [4]. The inequality (6.1) is also obtained there with other additional assumptions. They conjectured these additional assumptions can be removed. Theorem 6.1 fulfills their conjecture.

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