Distance-Regular Graphs

Chih-wen Weng

Department of Applied Mathematics, National Yang Ming Chiao Tung University

Algebraic Graph Theory (2024 Fall)

Outline

Distance-Regular Graphs

Notations

We always assume $\Gamma = (V, E)$ is a connected graph with diameter D. For $x \in X$,

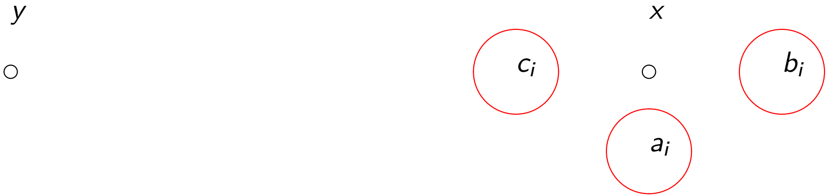
$$\Gamma_i(x) := \{y \in X \mid \partial(x, y) = i\}.$$

Distance-regular graphs

$\Gamma = (V, E)$ is **distance-regular** if and only if for $i \leq D$,

$$\begin{aligned} \mathbf{c_i} &:= |\Gamma_1(x) \cap \Gamma_{i-1}(y)|, \\ \mathbf{a_i} &:= |\Gamma_1(x) \cap \Gamma_i(y)|, \\ \mathbf{b_i} &:= |\Gamma_1(x) \cap \Gamma_{i+1}(y)| \end{aligned}$$

are **constants** subject to all vertices x, y with $\partial(x, y) = i$.

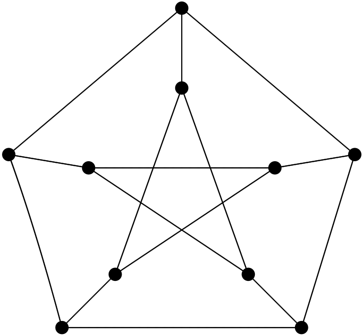


Note that $a_i + b_i + c_i = b_0$ and $k := b_0$ is the **valency** of Γ.

Example

The Petersen graph below is a distance-regular with diameter $D = 2$ and

$$b_0 = 3, c_1 = 1, a_1 = 0, b_1 = 2, c_2 = 1, a_2 = 2, b_2 = 0.$$



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Remark

If Γ is distance-regular with diameter D and fix a vertex $x \in V\Gamma$, then according to the partition

$$\Gamma_0(x) \cup \Gamma_1(x) \cup \cdots \cup D_D(x)$$

the adjacency matrix of Γ has equitable quotient matrix in the following tridiagonal form

$$\begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{D-1} & a_{D-1} & b_{D-1} \\ 0 & & & c_D & a_D \end{pmatrix}$$

Questions

From now on, we assume $\Gamma = (V, E)$ is a distance-regular graph with diameter D. For any two vertices $x, y \in X$ with distance $\partial(x, y) = k$, can you compute the **intersection numbers**

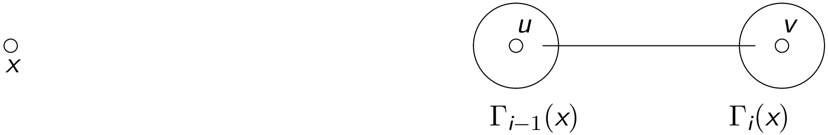
$$p_{ij}^{k}(x, y) := |\Gamma_i(x) \cap \Gamma_j(y)|?$$

Does the numbers $p_{ij}^{k}(x, y)$ depend on the choice of vertices x, y?

Remarks on $p^k_{ij}(x, y)$

1. $p^k_{ij}(x, y) = 0$ if $|i + j| < k$ or $|i - j| > k$.
2. $p^i_{1\,i-1} = c_i$, $p^i_{1\,i} = a_i$, and $p^i_{1\,i+1} = b_i$.
3. $$1 = c_1 \leq c_2 \leq \cdots \leq c_D,$$
   and
   $$b_0 > b_1 \geq b_2 \geq \cdots \geq b_{D-1} > b_D = 0.$$

Compute on $p^0_{ii}(x,y) = |\Gamma_i(x)|$



- In this case $x = y$. Computing the pair (u, v) of vertices $u \in \Gamma_{i-1}(x)$, $v \in \Gamma_i(x)$ with $uv \in R$ in different order, we have

$$p^0_{i-1\ i-1}(x,y)\, b_{i-1} = p^0_{i\ i}(x,y)\, c_i.$$

- Induction on i we have

$$p^0_{ii} = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i},$$

independent of the choice of x.

Distance matrices

The matrices we are concerned are square matrices with rows and columns indexed by the vertex set $V\Gamma$.

- For each i let A_i be a 01-matrix with entries

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x,y) = i; \\ 0, & \text{else.} \end{cases}$$

- A_i is called ***i*-th distance matrix**, and $A = A_1$ is also called the **adjacency matrix** of Γ.
- Note $A_0 = I$ and $A_{-1} = A_{D+1} = 0$.

Theorem

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \qquad 0 \le i \le D,$$

where $c_{D+1} := 1.$

Proof.

$$(AA_i)_{xy} = \begin{cases} b_{i-1}, & \text{if } \partial(x,y) = i-1; \\ a_i, & \text{if } \partial(x,y) = i; \\ c_{i+1}, & \text{if } \partial(x,y) = i+1. \end{cases}$$

□

Distance polynomials

In last page we show

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \qquad 0 \leq i \leq D.$$

- Consider polynomials $f_0(x) := 1,\ f_1(x) := x$ and
- $f_i(x)$ is defined recursively using

$$xf_i(x) = b_{i-1}f_{i-1}(x) + a_if_i(x) + c_{i+1}f_{i+1}(x) \qquad 2 \leq i \leq D.$$

- Note that $A_i = f_i(A)$, $f_{D+1}(A) = A_{D+1} = 0$ (using $c_{D+1} := 1$), and $f_i(x)$ has degree i.
- A has $D+1$ distinct eigenvalues, which are the roots of $f_{D+1}(x)$.
- $f_0(x), f_1(x), \ldots, f_D(x)$ are called **distance polynomials**.
- Distance polynomials are orthogonal in a kind of inner product to be mentioned later.

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Proposition

Let Γ be a distance-regular graph with diameter D. If Γ has $d+1$ distinct eigenvalues then $d = D$.

Proof.

Since the adjacency matrix $A = A(\Gamma)$ is symmetric, the minimal polynomial of $A(\Gamma)$ has degree $d+1$, where $d \geq D$ has been proved in the first Section. As $f_{D+1}(A) = 0$ in the last page, we also have $D \geq d$. Hence $D = d$. □

Algebra

An **algebra** $\mathcal{A}$ over $\mathbb{C}$ is a vector space together with a multiplication $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that

(i)

$$\begin{cases} \alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b), \\ a \cdot (b + c) = a \cdot b + a \cdot c, \\ (b + c) \cdot a = b \cdot a + b \cdot a, \end{cases} \qquad \text{(The map } \cdot \text{ is binear);}$$

(ii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (The product is associative);

(iii) $\exists 1 \in \mathcal{A},\ a \cdot 1 = 1 \cdot a = a$;

where $a, b, c \in A$ and $\alpha \in \mathbb{C}$.

If the is no confusion, we will omit the multiplication symbol $\cdot$.

Example

The set $\mathrm{Mat}_n(\mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$ is an algebra under usual addition, scalar multiplication, and multiplication of matrices.

Bose-Mesner Algebra

Let Γ be a distance-regular graph with diameter D.

- Consider the algebra $\mathcal{M} = \langle A \rangle$ generated by the adjacency matrix A of Γ.
- Then the set

$$\{I = A_0, A = A_1, A_2 = f_2(A), \ldots, A_D = f_D(A)\}$$

is a basis of $\mathcal{M}$.

- $\mathcal{M}$ is called the **Bose-Mesner Algebra** of Γ.

Theorem

$$p_{ij}^k(x, y) = p_{ij}^k,$$

independent the choice x, y.

Proof.

Note that $A_i A_j \in \mathcal{M}$. Hence

$$A_i A_j = \sum_{k=0}^{D} p_{ij}^k A_k$$

for some constant $p_{ij}^k \in \mathbb{R}$. For $x, y \in X$ with $\partial(x, y) = k$ consider the xy entry of both sides of the above equality, we find $p_{ij}^k(x, y) = p_{ij}^k$, independent the choice x, y. $\square$

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Strongly regular graphs

A distance-regular graph of diameter 2 is called a **strongly regular graph**.

Strongly regular graphs are studied earlier than the study of distance-regular graphs. They have relations to designs, finite geometries, graph eigenvalues, etc.

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Distance-transitive graphs

Let $\Gamma = (V, E)$ be a graph.

- A map $\phi : V \to V$ is an **automorphism** of Γ if ϕ is bijection and

$$uv \in E \quad \Leftrightarrow \quad \phi(u)\phi(v) \in E.$$

- Γ is **vertex-transitive (VT)** if for any two vertices u, v there exists an automorphism ϕ of Γ such that $\phi(u) = (v)$.
- Γ is **distance-transitive (DT)** if for any two pairs $(u, v), (x, y)$ of vertices with $\partial(u, v) = \partial(x, y)$, there exists an automorphism ϕ of Γ such that $x = \phi(u)$ and $y = \phi(v)$.
- Note that a distance transitive graph is distance-regular.

Recall the Hamming graphs $H(D, q)$

- Let X be a finite field of q elements, $V = X^D$, and

$$E = \{uv \mid u, v \in V \text{ differ in exact one cordinate}\}.$$

- $\Gamma = (V, E)$ is called the **Hamming graph** $H(D, q)$.
- Note that $H(1, q) = K_q$, $H(2, 2)$ is a square and $H(3, 2)$ is a cube.
- For vertices x and y in $H(D, q)$, $\partial(x, y) = i$ if and only if x and y differ in exactly i coordinates.
- Hence $H(D, q)$ is $D(q-1)$-regular with order q^D and diameter D.

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Theorem

The Hamming graph $H(D,q)$ is distance-regular with intersection numbers

$$a_i = i(q-2), \quad b_i = (D-i)(q-1), \quad c_i = i.$$

Proof.

For two vertices x and y in $H(D,q)$ at distance $\partial(x,y) = i$, x and y differ at exactly i coordinates. Hence for $z \in \Gamma_i(x) \cap \Gamma_1(y)$, the unique coordinate that y and z differ is in one of the above i coordinates and at this coordinate z has one of the $q-2$ values which also differs to that of both x and y. This proves $a_i = i(q-2)$. The proof of c_i is similar. Use $a_i + b_i + c_i = b_0 = D(q-1)$ to find b_i. □

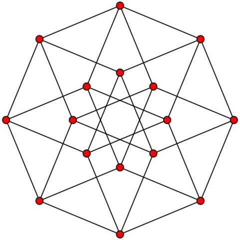
Coding Theory

In $H(D,2)$ for each positive integer c find $C \subseteq X$ with $|C| = c$ to maximize

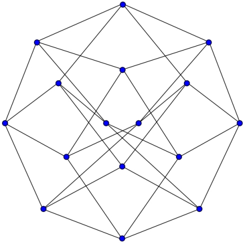
$$\mathrm{Min}\{\partial(x,y) \mid x,y \in C, x \neq y\}.$$

P. Delsarte studied coding theory in any distance-regular graphs in his 1973 PH. D. thesis "An Algebraic Approach to the Association Schemes of Coding Theory".

Two cospectral graphs



The 4-cube.

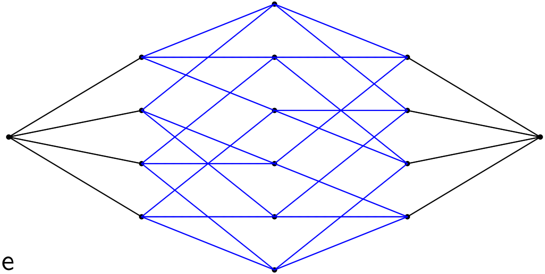


The Hoffman graph.

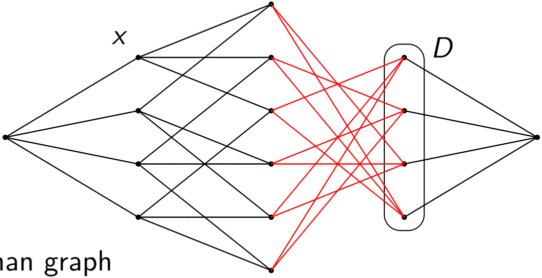
Copy from http://en.wikipedia.org/wiki/Hoffman_graph

In the Hoffman graph, every vertex has distance at most 3 to the southwestern vertex. So Hoffman graph is not distance-transtive.

Another drawing of Hoffman graph



The 4-cube



The Hoffman graph

Example (Johnson graphs $J(n, D)$, $2D \leq n$)

- Set $[n] = \{1, 2, \ldots, n\}$,

$$V = \binom{[n]}{D}, \quad E = \{uv \mid u, v \in V, \ |u \cap v| = D - 1\}.$$

- Then $\Gamma = (V, E)$ is called the **Johnson graph** $J(n, D)$.
- $J(n, D)$ is a distance-regular graph with diameter D (Exercise).

Design theory

- A t-(n, D, λ) **design** is a pair $(P, \mathcal{B})$, where $P = [n]$ and $\mathcal{B}$ is a family of D-subsets of P such that any t elements in P are contained in exactly λ D-subsets in $\mathcal{B}$.
- Note that with $P = [n]$ and $\mathcal{B} = \binom{[n]}{D}$, we have a t-$(n, D, \binom{n-t}{D-t})$ design.
- Any t-(n, D, λ) $(P, \mathcal{B})$ for $2D \leq n$ can be viewed as a subgraph of $J(n, D)$ induced on $\mathcal{B}$ with some nice properties to be specified (a design on $J(n, D)$).

Question

What is the related design on arbitrary distance-regular graph. (P. Delsarte's thesis)

Grassmann graph $J_q(n, D)$, $2D \leq n$

- Let F_q be a finite field of q elements and $V = \binom{F_q^n}{D}$ be the set of D-dimensional subspaces of F_q^n, and

$$E = \{uv \mid u, v \in V,\ \dim(u \cap v) = D - 1\}.$$

- Then $\Gamma = (V, E)$ is called the **Grassmann graph** $J_q(n, D)$, or the **q-analog of Johnson graph**.
- $J_q(n, D)$ is a distance-regular graph with diameter D (Exercise).

A class of non-VT DRG

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Inventiones mathematicae

A new family of distance-regular graphs with unbounded diameter

E.R. van Dam[1,⋆], **J.H. Koolen**[2,⋆⋆]

[1] Tilburg University, Dept. Econometrics and O.R., PO Box 90153, 5000 LE Tilburg, The Netherlands (e-mail: `Edwin.vanDam@uvt.nl`)
[2] POSTECH, Dept. Mathematics, Pohang 790-784, South Korea (e-mail: `koolen@postech.ac.kr`)

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Abstract. We construct distance-regular graphs with the same – classical – parameters as the Grassmann graphs on the e-dimensional subspaces of a $(2e + 1)$-dimensional space over an arbitrary finite field. This provides the first known family of non-vertex-transitive distance-regular graphs with unbounded diameter.