Algebraic Graph Theory HW

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Abstract

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A collection $\{S_1, S_2, \ldots, S_t\}$ of subsets of S is called a *partition* if $S_i \neq \emptyset$, $S_i \cap S_j = \emptyset$, and $S_1 \cup S_2 \cup \cdots \cup S_t = S$, where $1 \leq i \neq j \leq t$. A *digraph* $\overrightarrow{\Gamma}$ is a pair (V, E) such that V is a finite set and E is a subset of $V \times V$. An element in E is called a *directed edge* of an *arc*. Sometimes, we use $E\overrightarrow{\Gamma}$ and $V\overrightarrow{\Gamma}$ to denote E and V respectively for referring their digraph $\overrightarrow{\Gamma}$. For $x, y \in V$, a walk in $\overrightarrow{\Gamma}$ from x to y is a sequence $x = x_0, x_1, \ldots, x_t = y$ such that $x_{i-1}x_i \in E$ for all $1 \leq i \leq t$.

1. Let $\overrightarrow{\Gamma}$ be a digraph with vertex set $V\overrightarrow{\Gamma}$ and directed edge set $E\overrightarrow{\Gamma}$. Prove the following two statements.

   (a) If there exists a partition $\{X, Y\}$ of $V\overrightarrow{\Gamma}$ such that $xy \notin E\overrightarrow{\Gamma}$ for every pair $(x, y) \in X \times Y$, then $\overrightarrow{\Gamma}$ is not strongly connected.

   (b) If $\overrightarrow{\Gamma}$ is not strongly connected then there exists a partition $\{X, Y\}$ of $V\overrightarrow{\Gamma}$ such that $xy \notin E\overrightarrow{\Gamma}$ for every pair $(x, y) \in X \times Y$.

Proof. (a) We prove this by the contradiction method. Suppose $\overrightarrow{\Gamma}$ is strongly connected. Take $x \in X$ and $y \in Y$. Since $\overrightarrow{\Gamma}$ is strongly connected, a walk $x = x_0 - x_1 - x_2 - \cdots - x_k = y$ exists. Let x_p be the first node that belongs to Y in the walk. Clearly, $p \neq 0$ and such p always exists. (because at least we can take $p = k$) Hence, $x_{p-1}x_p \in E\overrightarrow{\Gamma}$, where $x_{p-1} \in X$ and $x_p \in Y$. It contradicts our assumption. Therefore, $\overrightarrow{\Gamma}$ is not strongly connected. □

Proof. (b) We provide an algorithm to create a partition $\{X, Y\}$.

Algorithm 1 Find a partition $\{X, Y\}$.

Input: A directed graph $\overrightarrow{\Gamma}$.
Output: A partition $\{X, Y\}$ of $V\overrightarrow{\Gamma}$ satisfies the condition that $xy \notin E\overrightarrow{\Gamma}$ for every pair $(x, y) \in X \times Y$.
1. Find strong connected component of $\overrightarrow{\Gamma}$, called $C_1, C_2, \ldots, C_k$.
2. Compress the graph $\overrightarrow{\Gamma}$ to $\overrightarrow{G}$ with $V\overrightarrow{G} = \{v_1, v_2, \ldots, v_k\}$.
And $E\overrightarrow{G} = \{(v_i, v_j) \ : \ \text{there exists } pq \in E\overrightarrow{G} \text{ with } p \in C_i \text{ and } q \in C_j\}$.
3. Use the topology sort algorithm to sort $\overrightarrow{G}$.
Then we get an order $v_{p_1}, v_{p_2}, \ldots, v_{p_k}$ with $\cancel{\mathrm{indeg}(v_{p_1}) = 0}$ $\mathrm{outdeg}(v_{p_1}) = 0$.
4. Return $X = \{x \ : \ x \in C_{p_1}\}$ and $Y = V\overrightarrow{\Gamma} \setminus X$.

Note that by the definition of the strongly connected component which is a maximal subgraph where every pair of vertices is mutually reachable, $\overrightarrow{G}$ is a directed acyclic graph (DAG). Therefore, we can apply the topology sort algorithm. Meanwhile, since $\mathrm{indeg}(v_{p_1}) = 0$, there is no any edge $xy \in E\overrightarrow{G}$. Hence, there is no any edge $xy \in E\overrightarrow{\Gamma}$ with $x \in X$ and $y \in Y$. □

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Let M be a nonnegative irreducible square matrix corresponding the graph G such that M^2 is reducible. Let G^2 be the graph corresponding the adjacency matrix M^2.

Clearly, G is strongly connected, and G^2 is not strongly connected by the definition of irreducibility.

Lemma 1. *Let a, b be vertices in G. If an even length walk exists from a to b in G, then a walk exists from a to b in G^2.*

Proof. By assumption, there exists an even length walk from a to b is $a = x_0 - x_1 - \cdots - x_n = b$, where n is an even integer. Notice that $x_k - x_{k+2}$ is an edge in G^2 for $k = 0, 1, \cdots n-2$. Thus, a walk $a = x_0 - x_2 - \cdots - x_n = b$ exists. $\square$

Lemma 2. *If C is an odd cycle in G, then C is strongly connected in G^2, i.e., for any $u, v \in C$, there are paths from u to v and from v to u in G^2.*

Proof. Let $C = x_0 - x_1 - \cdots - x_n - x_0$, where n is an even integer. By Lemma 1, there exists a walk $x_0 - x_2 - \cdots - x_n - x_1 - x_3 - \cdots - x_{n-1} - x_0$ in G^2. Thus, each of the two points in C is strongly connected. Hence, C is strongly connected in G^2. $\square$

Lemma 3. *Let a be any vertex in G but not C. Then there exists a vertex b in C such that there exists a walk from a to b in G^2.*

Proof. Let a be a vertex in G but not C. Since G is strongly connected, a vertex b in G exists such that a walk $a = x_0 - x_1 - \cdots - x_n = b$ exists. If n is an even integer, by Lemma 1, we are done. If not, a vertex y exists connecting b in C. Thus, $a = x_0 - x_1 - \cdots - x_n = b - y$ is an even length walk from a to y. Thus, a walk exists from a to y in G^2. By Lemma 2, a walk exists from y to b in G^2. Hence, a walk exists from a to b in G^2. $\square$

Lemma 4. *Let a be any vertex in G but not C. Then there exists a vertex b in C such that there exists a walk from b to a in G^2.*

Proof. Similar to the above proof. Let a be a vertex in G but not C. Since G is strongly connected, a vertex b in G exists such that a walk $b = x_0 - x_1 - \cdots - x_n = a$ exists. If n is an even integer, by Lemma 1, we are done. If not, a vertex y exists connecting b in C. Thus, $y - b = x_0 - x_1 - \cdots - x_n = a$ is an even length walk from y to a. Thus, a walk exists from y to a in G^2. By Lemma 2, a walk exists from b to y in G^2. Hence, a walk exists from b to a in G^2. $\square$

We return to prove this homework. The proof is done by using the method of contradiction.

Theorem 2.0.1. *Prove that G is bipartite.*

Proof. Suppose G is not bipartite. By the property of bipartite graphs, an odd cycle C exists in G. By Lemma 2, C is strongly connected in G^2. Combining Lemma 3 and Lemma 4, we conclude that G^2 is strongly connected. It contradicts our assumption that G^2 is not strongly connected; thus, G is bipartite. □