# Laplace matrix

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Algebraic Graph Theory (2024 Fall)

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# Outline

# The spectrum of $L(\Gamma)$ tells the number of components

# Recall

Recall that the Laplace matrix associated with an undirected graph $\Gamma$ is $L = D - A = NN^T$, where $D$ is the degree matrix and $N$ is incidence matrix of an orientation of $\Gamma$. In particular $L\mathbf{1} = 0$ and $L$ is positive semidefinite. Hence $0$ is the smallest eigenvalue of $L$. This proves the following lemma.

## Lemma

If $\Gamma$ is an undirected graph with $c$ components, then $0$ is a Laplace eigenvalue of $\Gamma$ with multiplicity at least $c$.

□

# Example

$$\begin{pmatrix} \begin{pmatrix} 1 & -1 & \\ & 1 & -1 \\ & & \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & \\ & & \end{pmatrix} \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_1 \\ \vdots \\ s_c \\ \vdots \\ s_c \end{pmatrix} = 0.$$

# Lemma

**For any $n \times m$ matrix $N$,**

$$\{u \in \mathbb{R}^n \mid N^T u = 0\} = \{u \in \mathbb{R}^n \mid NN^T u = 0\}.$$

Proof.

$\subseteq$ is clear.

To prove $\supseteq$, suppose $NN^T u = 0$. Then

$$\|N^T u\|^2 = (N^T u)^T N^T u = u^T NN^T u = 0.$$

$\square$

# Theorem

**If $\Gamma$ is an undirected graph with $c$ components, then $0$ is a Laplace eigenvalue of $\Gamma$ with multiplicity $c$.**

## Proof.

It suffices to assume that $\Gamma$ is connected and prove that the null space of $L = NN^T$ has dimension $1$. This is equivalent to prove that the nullspace of $N^T$ has dimension $1$. Note that each row of $N^T$ has only two nonzero entries, $1$, $-1$. Hence $N^T\mathbf{1} = 0$. We prove the vectors in the nullspace of $N^T$ has the form $s\mathbf{1}$ for some $s \in \mathbb{R}$. Let $u$ be a vector in the nullspace of $N^T$, and suppose $u_i \neq u_j$ for some $i, j$. Choose such pair $i, j$ such that the distance $\partial(i, j)$ in $\Gamma$ is smallest. Note that $i, j$ have an edge $ij = e$. Hence $0 = (N^T u)_e = \pm(u_i - u_j) \neq 0$, a contradiction. Then the nullspace of $N^T$ has dimension $1$. □

# Remark

The graph $K_3 + K_2$ has $2$ components and

$$SP(K_3 + K_2) = (2, 1, -1, -1, -1).$$

Hence the multiplicity of the largest eigenvalue of a graph does not count the number of components of the graph.

# Notations

We will use the following notations:

$$\begin{aligned} \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n & \qquad \text{are eigenvalues of } A, \\ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n & \qquad \text{are eigenvalues of } L. \end{aligned}$$

## Remark

If $\Gamma$ is $k$-regular, then $L = kI - A$, and

$$\mu_1 = k - \lambda_n \geq \mu_2 = k - \lambda_{n-1} \geq \cdots \geq \mu_n = k - \lambda_1 = 0,$$

where $\mu_i$ and $\lambda_{n-i+1}$ have the same associated eigenvector.

# Laplace matrix and graph drawing

## What is $u^T L u$?

For the Laplace matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ of an edge *ab*, we have

$$\begin{aligned}
(u_a, u_b) L \begin{pmatrix} u_a \\ u_b \end{pmatrix} &= (u_a, u_b) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_a \\ u_b \end{pmatrix} \\
&= (u_a - u_b, -u_a + u_b) \begin{pmatrix} u_a \\ u_b \end{pmatrix} \\
&= u_a^2 - u_b u_a - u_a u_b + u_b^2 \\
&= (u_a - u_b)^2
\end{aligned}$$

# Lemma

**If $\Gamma$ is an undirected graph with edge set $E\Gamma$ and $u$ is a column vector, then**

$$u^T L u = \sum_{ab \in E\Gamma} (u_a - u_b)^2 \quad \text{and} \quad u^T Q u = \sum_{ab \in E\Gamma} (u_a + u_b)^2.$$

Proof.

Let $N$ denote the vertex-arc incidence matrix of an orientation $\Gamma^\sigma$. Then

$$\begin{aligned} u^T L u &= u^T N N^T u = \|N^T u\|^2 \\ &= \sum_{ab \in E\Gamma} (u_a - u_b)^2. \end{aligned}$$

The other case for $Q$ is similar. □

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# Graph drawing and energy

- Let $R$ be an $n \times m$ matrix denoting a drawing of $V\Gamma$ on $\mathbb{R}^m$, where the $i$-th row $v_i$ of $R$ denote the vertex $i$ drawn in $\mathbb{R}^m$.

$$R = (R_1 \cdots R_m) = \overbrace{\begin{pmatrix} \quad v_1 \quad \\ \vdots \\ v_n \end{pmatrix}}^{m} \qquad (n \text{ 個點放在 } m \text{ 維空間})$$

- The **energy** of a drawing $R$ is defined to be

$$\sum_{ab \in E\Gamma} \|v_a - v_b\|^2 = \sum_{ab \in E\Gamma} \sum_{i=1}^{m} (R_{ai} - R_{bi})^2 = \sum_{i=1}^{m} R_i^T L R_i = \operatorname{tr}(R^T L R).$$

- The smaller the energy of $R$ the closer in average of two adjacent vertices are drawn in $\mathbb{R}^m$.

# Assumptions on $R$ and the goal

Usually we have restriction on $R$, for example, the assumptions

$$\begin{aligned} \mathbf{1}^T R &= 0 \qquad (\text{所有點的重心在原點}), \\ R^T R &= I \qquad (\text{某種製造點差異性的方式}) \end{aligned}$$

appeared in some research papers.

**Goal:** Minimize $\mathrm{tr}(R^T L R)$ among all $n \times m$ matrices $R$ satisfying that $\mathbf{1}^T R = 0$ and $R^T R = I$.

# The case $m = 1$

**Minimize the energy trace $R^TLR$ among all $n \times 1$ matrices $R$ satisfying that $\mathbf{1}^TR = 0$ and $R^TR = 1$.**

**Solution.** Choose an orthonormal basis $\{u_1, u_2, \ldots, u_n\}$ of $\mathbb{R}^n$ such that $u_i$ is the $\mu_i$-eigenvector of $L$. Since $u_n = \frac{\mathbf{1}}{n}$, $\mathbf{1}^TR = 0$, and $R^TR = 1$, we might assume $R = \sum_{i=1}^{n-1} c_i u_i$, where $\sum_{i=1}^{n-1} c_i^2 = 1$. Then

$$\operatorname{tr}(R^TLR) = \sum_{i=1}^{n-1} c_i^2 \mu_i \geq \sum_{i=1}^{n-1} c_i^2 \mu_{n-1} = \mu_{n-1},$$

and if $R = u_{n-1}$ then $\operatorname{tr}(R^TLR) = \mu_{n-1}$. Hence $\mu_{n-1}$ is the minimum energy of $R$.

## Theorem

**The energy $\mathrm{tr}(R^TLR)$ of $R$ takes the minimum value**

$$\mu_{n-1}+\mu_{n-2}+\cdots+\mu_{n-m}$$

**among all $n\times m$ matrices $R$ satisfying $R^TR=I$ and $\mathbf{1}^TR=0$.**

Proof.

Fix $R$ satisfying the assumptions. Add the unit $\mu_n$-eigenvector $u_n=\mathbf{1}/\sqrt{n}$ of $L$ to the first column of $R$ to have an $n\times(m+1)$ matrix $R'$ satisfying $R'^TR'=I$. Since $u_n^TLu_n=0$, $\mu_n=0$ and $R'^TR'$ is $(m+1)\times(m+1)$ matrix, by Eigenvalue Interlacing Theorem,

$$\begin{aligned} & \mathrm{tr}\ (R^TLR)=\mathrm{tr}\ (R'^TLR')=\text{the sum of eigenvalues of } R'^TLR' \\ \geq\ & \mu_n+\mu_{n-1}+\mu_{n-2}+\cdots+\mu_{n-m}=\mu_{n-1}+\mu_{n-2}+\cdots+\mu_{n-m}. \end{aligned}$$

The above lower bound is obtained by $R=[u_{n-1}u_{n-2}\cdots u_{n-m}]$. □

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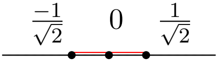
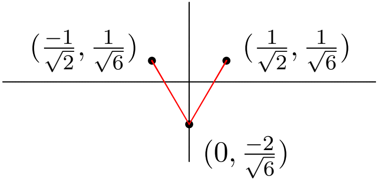
# Application

Let $\Gamma = P_3$ be a path with three vertices. Then the Laplace spectrum of $\Gamma$ is $(0, 1, 3)$ with corresponding sequence of unit eigenvectors

$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}.$$




應用上取 $m = 2$ 及 $L$ 的最小兩個非 $0$ 特徵值 (可重複) 所對應的兩向量, 對每一點以第一向量該點的位置當 $x$ 軸座標, 第二向量該點的位置當 $y$ 軸座標, 即可在平面上畫出圖.

# The complement of a graph

# The complement of a graph

**Let $\Gamma$ be a simple graph of order $n$. The complement $\overline{\Gamma}$ of $\Gamma$ is the graph with vertex set $V\overline{\Gamma} = V\Gamma$ and edges $xy \in E\overline{\Gamma}$ iff $xy \notin E\Gamma$ for distinct $x, y \in V\Gamma$. Note that**

$$L(\Gamma) + L(\overline{\Gamma}) = nI - J.$$

## Proposition

**$\mu_i(\overline{\Gamma}) + \mu_{n-i}(\Gamma) = n$ for $1 \le i \le n-1$. In particular $\mu_1(\Gamma) \le n$ with equality iff $\overline{\Gamma}$ is disconnected.**

### Proof.

Fix $1 \le i \le n-1$. Since a $\mu_{n-i}(\Gamma)$-eigenvector $u$ of $L(\Gamma)$ in $\mathbf{1}^\perp$ is also an $n$-eigenvector of $nI - J$,

$$L(\overline{\Gamma})u = (nI - J)u - L(\Gamma)u = (n - \mu_{n-i}(\Gamma))u.$$

This proves the first statement. Take $i = n-1$ and use $\mu_{n-1}(\overline{\Gamma}) \ge 0$ with equality iff $\overline{\Gamma}$ is disconnected to have the second statement. □

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# Example

**Determine the spectrum $SP(L(K_{pq}))$ of complete bipartite graph $K_{pq}$.**

**Solution.** Note that $K_{pq} = \overline{K_p \cup K_q}$ and

$$SP(L(K_p \cup K_q)) = (q^{q-1}, p^{p-1}, 0, 0).$$

Hence

$$SP(L(K_{pq})) = (p+q, q^{p-1}, p^{q-1}, 0).$$

# Algebraic connectivity

# Algebraic connectivity

Let $\Gamma$ be a graph with at least two vertices. The second (from the least) Laplace eigenvalue $\mu_{n-1}(\Gamma)$ is called the **algebraic connectivity** of $\Gamma$. Note that $\mu_{n-1}(\Gamma) \geq 0$ with equality if and only if $\Gamma$ is disconnected.

# Proposition

**Let $\Gamma$ and $\Delta$ be two edge-disjoint graphs on the same vertex set.**
**Then** $\mu_{n-1}(\Gamma) \le \mu_{n-1}(\Gamma) + \mu_{n-1}(\Delta) \le \mu_{n-1}(\Gamma \cup \Delta) \le \mu_{n-1}(\Gamma) + 2|E\Delta|$.

**Proof.** Let $z$ a $\mu_{n-1}(\Gamma \cup \Delta)$-eigenvector of $\Gamma \cup \Delta$. Then

$$\begin{aligned}
\mu_{n-1}(\Gamma \cup \Delta)\|z\|^2 &= z^T L(\Gamma \cup \Delta) z \\
&= \sum_{ab \in E(\Gamma \cup \Delta)} (z_a - z_b)^2 \\
&= \sum_{ab \in E\Gamma} (z_a - z_b)^2 + \sum_{ab \in E\Delta} (z_a - z_b)^2 \\
&= z^T L(\Gamma) z + z^T L(\Delta) z \\
&\ge (\mu_{n-1}(\Gamma) + \mu_{n-1}(\Delta)) z^T z
\end{aligned}$$

by Rayleigh's principle. Hence $\mu_{n-1}(\Gamma \cup \Delta) \ge \mu_{n-1}(\Gamma) + \mu_{n-1}(\Delta)$.

# Continue the proof

Similarly let $z'$ denote an eigenvector corresponding to the Laplace eigenvalue $\mu_{n-1}(\Gamma)$. Then

$$
\begin{aligned}
\mu_{n-1}(\Gamma\cup\Delta)\|z'\|^2 &\le z'^T L(\Gamma\cup\Delta)z' \\
&= \sum_{ab\in E(\Gamma\cup\Delta)} (z'_a - z'_b)^2 \\
&= \mu_{n-1}(\Gamma)\|z'\|^2 + \sum_{ab\in E\Delta} (z'_a - z'_b)^2 \\
&\le \mu_{n-1}(\Gamma)\|z'\|^2 + \sum_{ab\in E\Delta} 2({z'_a}^2 + {z'_b}^2) \\
&\le \mu_2(\Gamma)\|z'\|^2 + \sum_{ab\in E\Delta} 2\|z'\|^2 \\
&= (\mu_{n-1}(\Gamma) + 2|E\Delta|)\|z'\|^2.
\end{aligned}
$$

$\square$

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# Edge interlacing property for Laplace matrix

# Lemma

**Let $N$ be an $n \times m$ matrix. Then there exists a one-one correspondence between the nonzero eigenvalues of $NN^T$ and $N^TN$.**

Proof.

Suppose $\mu$ is a nonzero eigenvalue of $NN^T$ with corresponding eigenvector $u$. Then $NN^Tu = \mu u \neq 0$. In particular $N^Tu \neq 0$. Since $N^TNN^Tu = \mu N^Tu$, $N^Tu$ is an eigenvector of $N^TN$ corresponding to the eigenvalue $\mu$. Suppose $\mu$ has multiplicity $m$ as an eigenvalue of $NN^T$. Let $u_1, u_2, \ldots, u_m$ be the corresponding orthogonal eigenvectors. If $c_1N^Tu_1 + \cdots + c_mN^Tu_m = 0$ then $0 = N(c_1N^Tu_1 + \cdots + c_mN^Tu_m) = \mu(c_1u_1 + \cdots + c_mu_m)$, and hence $c_1 = c_2 = \cdots = c_m = 0$. This proves that the multiplicity of $\mu$ in $NN^T$ is no larger than that in $N^TN$. Similarly for the other side, so the two multiplicities are the same. $\square$

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# Proposition (Edge Interlacing)

**Let $\Gamma \setminus e$ denote the graph obtained from $\Gamma$ by deleting the edge $e$. Then $\mu_{i+1}(\Gamma) \leq \mu_i(\Gamma \setminus e) \leq \mu_i(\Gamma) \quad *$[1].**

## Proof.

Let $N$ denote the vertex-arc incidence matrix of an orientation $\Gamma^\sigma$ and recall that $L(\Gamma) = NN^T$. Note that the vertex-arc incidence matrix $N'$ of the corresponding orientation $(\Gamma \setminus e)^\sigma$ is obtained from $N$ be deleting a column. Hence $N'^T N'$ is a principal submatrix of $N^T N$. By interlacing property, the $i$th largest eigenvalues of $N^T N$ is no less than that of $N'^T N'$. By previous lemma, we have $\mu_{i+1}(\Gamma) \leq \mu_i(\Gamma \setminus e) \leq \mu_i(\Gamma)$. □

[1] Same for signless Laplace matrix

## Remark

If $\Gamma' = \Gamma - x$ for some vertex of $\Gamma$, then

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda'_{n-1} \geq \lambda_n \quad \text{(vertex interlacing).}$$

**Laplace matrix does not have vertex interlacing property.** Let $x$ be a vertex of degree $2$ in $P_3$. Then $P_3 - x = 2K_1$, and

$$SP(L(2K_1)) = (0,0) \text{ is not interlacing } SP(L(P_3)) = (3,1,0).$$

## Remark

If $\Gamma' = \Gamma - e$ for some edge of $\Gamma$, then

$$\mu_1 \geq \mu_1' \geq \mu_2 \geq \mu_2' \geq \cdots \geq \mu_{n-1} \geq \mu_{n-1}' \geq \mu_n = 0 \geq 0 = \mu_n'.$$

(edge interlacing)

**Adjacency matrix does not have edge interlacing property.** Let $e$ be an edge in $P_3$. Then $P_3 - e = K_1 + K_2$ and $SP(A(K_1 + K_2)) = (1, 0, -1)$ is not interlacing $SP(A(P_3)) = (\sqrt{2}, 0, -\sqrt{2})$. (兩者項數一樣，只差最後一項).

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# Question

**If $\Gamma' = \Gamma - e$ for some edge $e$ of $\Gamma$, can we have**

$$\lambda_1 \geq \lambda'_1 \geq \lambda_2 \geq \lambda'_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda'_{n-1} \geq \lambda_n?$$

**Solution.** No. Let $e$ be an edge in $K_3$. Then $P_3 = K_3 - e$ and $SP(A(P_3)) = (\sqrt{2}, 0, -\sqrt{2})$. The sequence $(\sqrt{2}, 0)$ of first two values is not interlacing $SP(A(K_3)) = (2, -1, -1)$. $\square$

**The adjacency matrix does not have edge interlacing property in any sense.**

# Corollary

**Let $\Delta$ be a subgraph of $\Gamma$. Then $\mu_i(\Delta) \leq \mu_i(\Gamma)$.**

Proof.

Use previous proposition to delete edges until all the edges are in $\Delta$. Then delete isolated vertices. Delete an isolated vertex only decreases one zero eigenvalue. $\square$

# Example

**Find $SP(L(K_{1,n-1}))$, where $K_{1,n-1}$ is a star with Laplace matrix**

$$L(K_{1,n-1}) = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & 1 & & 0 \\ \vdots & & \ddots & \\ -1 & 0 & & 1 \end{pmatrix}.$$

**Solution.** Since $L(K_{1,n-1}) = M + I$, where

$$M = \begin{pmatrix} n-2 & -1 & \cdots & -1 \\ -1 & 0 & & 0 \\ \vdots & & \ddots & \\ -1 & 0 & & 0 \end{pmatrix}$$

has nullity $n-2$, $(-1)$-eigenvector $\mathbf{1}$ and $(n-1)$-eigenvector $(1-n, 1, \ldots, 1)$. Hence $SP(L(K_{1,n-1})) = (n, 1, \ldots, 1, 0)$.

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## Corollary

**Let $\Gamma$ be an simple undirected graph on $n$ vertices with at least an edge. Then the largest Laplace eigenvalue satisfying $\mu_1 \geq 1 + k_{\max}$.**

Proof.

Since the star $K_{1,k_{\max}}$ has largest Laplace eigenvalue $1 + k_{\max}$ and is a subgraph of $\Gamma$, the corollary follows from the interlacing property. □

# Line graph and signless Laplace matrix

# Line graph

The line graph of a graph $\Gamma$ is the graph $\ell(\Gamma)$ with vertex set $E\Gamma$ and $ee' \in V\ell(\Gamma)$ if $e$ and $e'$ are incident with a common vertex in $\Gamma$. The graph $\ell(\Gamma)$ is called the **line graph** of $\Gamma$.

## Remark

$$A(\ell(\Gamma)) = \begin{cases} 0, & \text{if } e = e'; \\ 1, & \text{if } e, e' \text{ have a uinque common end point;} \\ 0, & \text{else.} \end{cases}$$

# Theorem

**Let $\Gamma$ be a graph with signless Laplace matrix $Q$. The the following (i)-(ii) hold.**

1. $A(\ell(\Gamma))$ **has least eigenvalue at least** $-2$.
2. $\lambda_i(|L|) = \begin{cases} \lambda_i(A(\ell(\Gamma))) + 2, & \textbf{if } \lambda_i(|L|) \neq 0; \\ 0, & \textbf{else.} \end{cases}$

Proof.

Let $Q = NN^T$, where $N$ is the vertex-edge incident matrix of $\Gamma$. Observe that

$$(N^T N)_{ee'} = (2I + A(\ell(\Gamma)))_{ee'}.$$

The theorem follows since $Q = NN^T$ and $N^T N = 2I + A(\ell(\Gamma))$ have the same nonzero eigenvalues and the same corresponding multiplicities. □

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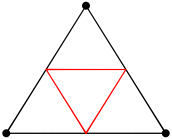
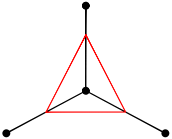
# Example

$\ell(K_3) = K_3$

$K_3$ 

| | eigenvalues |
|---|---|
| $A(\ell(K_3))$: | $2, -1, -1$ |
| $\lvert L\rvert(K_3)$: | $4, 1, 1$ |
| $L(K_3)$: | $3, 3, 0$ |

$\ell(K_{1,3}) = K_3$

$K_{1,3}$ 

| | eigenvalues |
|---|---|
| $A(\ell(K_{1,3}))$: | $2, -1, -1$ |
| $\lvert L\rvert(K_{1,3})$: | $4, 1, 1, 0$ |
| $L(K_{1,3})$: | $4, 1, 1, 0$ |

# Remark

If $\Gamma$ is bipartite then $L$ and $|L|$ have the form

$$L = \begin{pmatrix} D & -M \\ -M^T & D' \end{pmatrix}, \qquad |L| = \begin{pmatrix} D & M \\ M^T & D' \end{pmatrix}.$$

Hence

$$\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} L \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = Q.$$

Thus $Q$ and $L$ have the same set of eigenvalues.

# Theorem

**Let $\Gamma$ be a graph with signless Laplace matrix $Q$. Then the nullity of $Q$ is the number of bipartite component in $\Gamma$.**

Proof.

As the notation $|L| = NN^T$ in previous page, note that $|L|u = 0$ iff $N^T u = 0$, and this implies $u_x = -u_y$ for $xy \in E\Gamma$, so the support of $u$ does not contain odd cycles. Hence if $u$ in the kernel of $L$, then $u_x = 0$ for all $x$ in a non-bipartite component of $\Gamma$. Restricted to a bipartite component, $u$ is uniquely determined up to a scalar. $\square$

# Summary

Let $\lambda_1(\ell(\Gamma)) \geq \ldots \geq \lambda_m(\ell(\Gamma))$ denote the eigenvalues of the line graph of $\Gamma$ and $\rho_1 \geq \ldots \geq \rho_r$ denote the positive signless Laplace eigenvalues of $\Gamma$, where $m = |E\Gamma|$. Then

1. $n - r$ is the number of bipartite components in $\Gamma$;
2. $\lambda_i((\ell(\Gamma))) = \rho_i - 2$ for $1 \leq i \leq r$;
3. $\lambda_i((\ell(\Gamma))) = -2$ for $r < i \leq m$.

The graphs with the least eigenvalue at least $-2$ are classified by Cameron, Goethals, Seidel and Shult (1976).

# Proposition

**$\mu_1 \leq \rho_1$. Moreover, if $\Gamma$ is connected, then the equality holds if and only if $\Gamma$ is bipartite.**

**Proof.**

Exercise. $\square$

Let $d_x := |\Gamma_1(x)|$ be the degree of the vertex $x$ in $\Gamma$. A bipartite graph is **biregular** if all vertices in the same part of the bipartite have the same valency.

# Proposition

**Let $\Gamma$ be a graph with largest signless Laplace eigenvalue $\rho_1$. Then**

$$\rho_1 \leq \max_{xy \in E\Gamma}(d_x + d_y).$$

**Moreover, if $\Gamma$ is connected then the above equality holds iff $\Gamma$ is regular or bipartite biregular.**

Proof.

Observe that

$$\rho_1 = \lambda_1(\ell(\Gamma)) + 2 \leq d_{\max}(\ell(\Gamma)) + 2 = \max_{xy \in E\Gamma}(d_x + d_y - 2) + 2.$$

The equality holds iff the line graph of $\Gamma$ is regular. This is equivalent to $\Gamma$ regular or bipartite biregular (exercise). □

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