# Matrix Realization of Spectral Bounds

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## **Outline**

- Motivation
- 2 Main theorem
- 3 Spectral bound via a same size matrix
- 4 The case  $P = I_n$  and  $Q = I_n + \sum_{i=1}^{n-1} e_{in}$
- 6 Rooted matrices
- 6 Spectral bound via a smaller size matrix



## **Motivation**



# Theorem (Stanley, 1987)

If  $\Gamma$  is a graph of size m with spectral radius  $\rho(A)$ , then

$$\rho(A) \le \frac{-1 + \sqrt{1 + 8m}}{2}.$$

#### Proof.

The *i*-th rowsum of  $4A^2 + 4A$  satisfies

$$r_{i}(4A^{2}+4A) = (4A^{2}+4A)_{ii} + \sum_{\partial(j,i)=1} (4A^{2}+4A)_{ij} + \sum_{\partial(j,i)=2} (4A^{2}+4A)_{ij}$$
$$=4|\Gamma_{1}(i)| + (8|E\Gamma_{1}(i)| + 4|\Gamma_{1}(i)|) + 4|E(\Gamma_{1}(i),\Gamma_{2}(i))| \leq 8m.$$

By a property we proved earlier that positive eigenvector matters, the eigenvalue  $4\rho(A)^2+4\rho(A)$  of  $4A^2+4A$  is at most 8m to have the result.



### **Further results**

Let  $\Gamma$  be a graph of order n and size m with degree sequence  $d_1 \geq d_2 \geq \cdots \geq d_n$ . Recall that  $2m = d_1 + d_2 + \cdots + d_n$ . Let A be the adjacency matrix of  $\Gamma$  with spectral radius  $\rho(A)$ . The following are extensions of

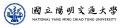
$$\rho(A) \le \frac{-1 + \sqrt{1 + 8m}}{2}.$$

- (Hong, 1998)  $\rho(A) \le \sqrt{2m-n+1}$ .
- (Hong, Shu, Fang, 2001)  $\rho(A) \leq \frac{d_n 1 + \sqrt{(d_n + 1)^2 + 4(2m nd_n)}}{2}$ .
- (Shu, Wu, 2004)  $\rho(A) \leq \frac{d_{\ell}-1+\sqrt{(d_{\ell}+1)^2+4(\ell-1)(d_1-d_{\ell})}}{2}$ .
- (Liu, Weng, 2013)  $\rho(A) \leq \frac{d_\ell 1 + \sqrt{(d_\ell + 1)^2 + 4\sum_{i=1}^{\ell-1}(d_i d_\ell)}}{2}$ .



#### Matrix realizations

$$\rho \begin{pmatrix} 0 & 1 & \cdots & 1 & d_1 - (n-2) \\ 1 & 0 & \ddots & \vdots & \ddots & \vdots \\ \vdots & \ddots & & \ddots & 1 & d_1 - (n-2) \\ \vdots & \ddots & & \ddots & 1 & d_1 - (n-2) \\ \vdots & \ddots & \ddots & & 1 & d_\ell - (n-2) \\ \vdots & \ddots & \ddots & & 1 & d_\ell - (n-2) \\ 1 & \cdots & & 1 & 0 & d_\ell - (n-2) \\ 1 & \cdots & & 1 & 1 & d_1 - (n-2) \\ 1 & 0 & \ddots & & \vdots & d_2 - (n-2) \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ & & & 1 & d_\ell - (n-2) \\ \vdots & \ddots & \ddots & & \vdots \\ & & & 0 & 1 & \vdots \\ & & & & 1 & 0 & d_\ell - (n-2) \\ 1 & \cdots & & & 1 & 1 & d_\ell - (n-1) \end{pmatrix} = \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4(\ell - 1)(d_1 - d_\ell)}}{2}.$$



## Main theorem



## **Rooted vectors**

A vector  $(v_1, v_2, \dots, v_n)$  is called **rooted** if  $v_i \ge v_n \ge 0$  for  $1 \le i \le n - 1$ .



# Main Theorem (Cheng, -)

Let  $M=(m_{ab})$  be an  $\ell \times \ell$  matrix whose first  $\ell-1$  columns and row-sum vector are all rooted. If  $C=(c_{ij})$  is an  $n \times n$  nonnegative matrix and there exists a partition  $\Pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$  of  $[n]:=\{1,2,\ldots,n\}$  such that

$$\max_{i \in \pi_a} \sum_{j \in \pi_b} c_{ij} \leq m_{ab} \quad \text{and} \quad \max_{i \in \pi_a} \sum_{j=1}^n c_{ij} \leq \sum_{c=1}^\ell m_{ac}$$

for  $1 \le a \le \ell$  and  $1 \le b \le \ell - 1$ , then  $\rho(C) \le \rho_r(M)$ .

Before we prove this theorem, we will see some example first.

## **Example**

If  $\ell=n$ ,  $\Pi=\{\{1\},\{2\},\ldots,\{n\}\}$ , and  $|c_{ij}|\leq m_{ij}$ , then  $\rho(C)\leq \rho(M)$  by an application of Perron-Frobenius Theorem. In this case, we don't need the rooted assumption of M.



If 
$$\ell = 1$$
,  $C \ge 0$ ,  $\Pi = \{[n]\}$ , and  $m_{11} = \max_{i \in [n]} \sum_{j=1}^{n} c_{ij}$ , then  $\rho(C) \le \rho(M)$ .



If 
$$\ell = n = 2$$
,  $\Pi = \{\{1\}, \{2\}\}$ , and

$$C = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 5 & 2 \\ 4 & -1 \end{pmatrix},$$

then

$$3 + 2\sqrt{2} = \rho(\mathit{C}) \le \rho(\mathit{M}) = 2 + \sqrt{17}.$$

#### Remark

The above matrix M is rooted, since  $5 \ge 4 \ge$  and  $5 + 2 \ge 4 + (-1) \ge 0$ . The last column of M is unrelated.



With n=7,  $\ell=3$ ,  $\Pi=\{\{1,2,3\},\{4,5\},\{6,7\}\}$ , and

$$C = \begin{pmatrix} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{pmatrix}, \quad M = \begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix},$$

where the row-sum vectors of C and M are

$$(24, 23, 22|20, 19|13, 12), (24, 20, 13),$$

respectively, we have  $\rho(\mathit{C}) \leq \rho_{\mathit{r}}(\mathit{M}) \approx 18.6936$ .



# Spectral bound via a same size matrix



#### Lemma

Let C, C, P and Q be  $n \times n$  matrices. Assume that (i)  $PCQ \leq PCQ$ ; (ii)  $C'Qu = \lambda'Qu$  for some column vector  $0 \neq u \geq 0$  and  $\lambda' \in \mathbb{R}$ ; (iii)  $v^TPC = \lambda v^TP$  for some row vector  $0 \neq v^T \geq 0$  and  $\lambda \in \mathbb{R}$ ; and (iv)  $v^TPQu > 0$ . Then  $\lambda \leq \lambda'$ .

#### Proof.

$$PCQ \leq PC'Q$$

$$\Rightarrow PCQu \leq PC'Qu = \lambda'PQu$$

$$\Rightarrow \lambda v^T PQu = v^T PCQu \leq v^T PC'Qu = \lambda' v^T PQu$$

$$\Rightarrow \lambda \leq \lambda'.$$



#### Remark

If P=Q=I, C' is irreducible,  $0\leq C\leq C'$ , u>0 is  $\lambda'$ -eigenvector of C' and  $v^T\geq 0$  is left  $\lambda$ -eigenvector of C, then the assumption (iv)  $v^Tu>0$  in the previous lemma holds and the conclusion  $\lambda\leq \lambda'$  holds. Indeed this is an application of Perron-Frobenius Theorem.



#### Lemma

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$$PCQ \ge PC'Q$$

$$\Rightarrow PCQu \ge PC'Qu = \lambda'PQu$$

$$\Rightarrow \lambda v^T PQu = v^T PCQu \ge v^T PC'Qu = \lambda' v^T PQu$$

$$\Rightarrow \lambda \ge \lambda'.$$



The case 
$$P = I_n$$
 and  $Q = I_n + \sum_{i=1}^{n-1} e_{in}$ 



# The matrix $Q = I_n + \sum_{i=1}^{n-1} e_{in}$

We will apply the previous two lemmas by letting  $P = I_n$  and

$$Q = I_n + \sum_{i=1}^{n-1} e_{in} = \begin{pmatrix} I_{n-1} & J_{(n-1)\times 1} \\ O_{1\times (n-1)} & 1 \end{pmatrix}.$$

Hence

$$C'Q = \left( \begin{array}{c|c} C'[n|n-1] & r'_1 \\ c'_2 & \vdots \\ c'_n \end{array} \right),$$

where C'[n|n-1] is the submatrix of C on the first n-1 columns and where  $(\mathbf{r}_1',\mathbf{r}_2',\ldots,\mathbf{r}_n')$  is the row-sum vector of C'.



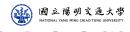
#### Lemma

Let  $Q = I_n + \sum_{i=1}^{n-1} e_{in}$  and u be a column vector. Then Qu is rooted if and only if u is nonnegative.

#### Proof.

This follows from the definition of rooted vector and using

$$Qu = \begin{pmatrix} u_1 + u_n & u_2 + u_n & \cdots & u_{n-1} + u_n & u_n \end{pmatrix}.$$



#### **Theorem**

Let C and C' be  $n \times n$  matrices. Assume that

- (i)  $C[n|n-1] \le C'[n|n-1]$  and  $(r_1, r_2, \dots, r_n) \le (r'_1, r'_2, \dots, r'_n)$ ;
- (ii)  $C'w = \lambda'w$  for some column rooted vector  $0 \neq w \geq 0$  and  $\lambda' \in \mathbb{R}$ ;
- (iii)  $v_{\underline{\ }}^T C = \lambda v^T$  for some row vector  $0 \neq v^T \geq 0$  and  $\lambda \in \mathbb{R}$ ; and
- (iv)  $v^T w > 0$ .

Then  $\lambda \leq \lambda'$ .

#### Proof.

Applying the lemma in the previous section with  $P = I_n$ ,

 $Q = I_n + \sum_{i=1}^{n-1} e_{in} \ w = Qu$ , and noticing that u is nonnegative if and only if w is rooted, we immediately have the theorem.



#### **Theorem**

Let C and C' be  $n \times n$  matrices. Assume that

- (i)  $C[n|n-1] \ge C'[n|n-1]$  and  $(r_1, r_2, \dots, r_n) \ge (r'_1, r'_2, \dots, r'_n)$ ;
- (ii)  $C'w = \lambda'w$  for some column rooted vector  $0 \neq w \geq 0$  and  $\lambda' \in \mathbb{R}$ ;
- (iii)  $v_T^T C = \lambda v^T$  for some row vector  $0 \neq v^T \geq 0$  and  $\lambda \in \mathbb{R}$ ; and
- (iv)  $v^T w > 0$ .

Then  $\lambda \geq \lambda'$ .

#### Proof.

Applying the lower bound lemma in the previous section with  $P = I_n$ ,  $Q = I_n + \sum_{i=1}^{n-1} e_{in} \ w = Qu$ , and noticing that u is nonnegative if and only if w is rooted, we immediately have the theorem.

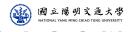


Consider the following three matrices

$$C'_{\ell} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad C'_{r} = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

with  $C_{\ell}[3|2] \leq C[3|2] \leq C_{r}[3|2]$ , and the same row-sum vector (5,3,3). Note that  $C_{\ell}$  has a rooted 3-eigenvector  $w_{\ell} = (1,0,0)^{T}$ , and  $C_{r}$  has a rooted 4-eigenvector  $r_{r} = (2,1,1)^{T}$ . Since C is irreducible, it has a positive left  $\rho(C)$ -eigenvector  $(v_{1},v_{2},v_{3})$ . Hence

$$3 \le \rho(C) \le 4$$
.



## **Rooted matrices**



#### Rooted matrices

- An  $n \times n$  real matrix  $C' = (c'_{ij})$  is called **rooted** if there is a constant d such that the first n-1 columns and the row-sum vector of  $C' + dI_n$  are all rooted.
- If C' has no real eigenvalue, set  $\rho_r(C') = \infty$ ; otherwise let  $\rho_r(C')$  denote the largest real eigenvalue of C'.



The following three matrices

$$C'_{\ell} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad C'_{r} = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

are all rooted.



#### Remark

•

$$Q = I_{n} + \sum_{i=1}^{n-1} e_{in} = \begin{pmatrix} I_{n-1} & J_{(n-1)\times 1} \\ O_{1\times(n-1)} & 1 \end{pmatrix}$$

$$\Rightarrow Q^{-1} = I_{n} - \sum_{i=1}^{n-1} e_{in} = \begin{pmatrix} I_{n-1} & -J_{(n-1)\times 1} \\ O_{1\times(n-1)} & 1 \end{pmatrix}$$

$$Q^{-1}C'Q = Q^{-1} \begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1} & -1 & t'_{1} \\ c'_{21} & c'_{22} & \cdots & c'_{2} & -1 & t'_{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c'_{n-1} & 1 & c'_{n-1} & 2 & \cdots & c'_{n-1} & -1 & t'_{n-1} \\ c'_{n1} & c'_{n2} & \cdots & c'_{nn-1} & t'_{n} \end{pmatrix}.$$

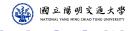
•  $Q^{-1}(C' + dI_n)Q \ge 0$  for some  $d \in \mathbb{R}$  if and only if C' is rooted.



If C is an  $n \times n$  rooted matrix, then  $\rho_r(C') < \infty$ , and C has a rooted eigenvector  $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)^T$  for  $\rho_r(C)$ . Moreover, if the row vector  $(c'_{n1}, c'_{n2}, \dots, c'_{nn-1})$  is positive then  $\sqrt{c'_{n1}}$  is positive.

#### Proof.

Let  $Q = I_n + \sum_{i=1}^{n-1} e_{in}$ . Since  $Q^{-1}(C' + dI_n)Q \ge 0$  for some  $d \in \mathbb{R}$ , there is a  $\eta$ -eigenvector  $u \geq 0$ , where  $\eta = \rho(C' + dI_n) \in \mathbb{R}$ . Since  $\rho(C' + dI_n) = \rho_r(C') + d$ , V = Qu is a rooted  $\rho_r(C')$ -eigenvector. This proves the first statement. Suppose  $(c'_{n1}, c'_{n2}, \dots, c'_{nn-1}) > 0$ . If  $u_n > 0$ then  $\sqrt{\ }$  is clear to be positive. Suppose  $u_n=0$ . Then  $\sqrt{\ }_n=0$  and  $\sum_{i=1}^{n-1} c'_{ni} v'_i = \sum_{i=1}^n c'_{ni} v'_i = (C'v')_n = \rho_r(C') v'_n = 0$ . Hence  $v'_i = 0$  for  $1 \le i \le n-1$ . Thus v is a zero vector, a contradiction.



#### Theorem

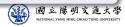
Let C and C be  $n \times n$  matrices such that C > 0, C is rooted,  $0 \le C[n|n-1] \le C'[n|n-1]$  and  $(r_1, r_2, \dots, r_n) \le (r'_1, r'_2, \dots, r'_n)$ . Then  $\rho(C) < \rho_r(C')$ .

#### Proof.

We assume first  $(c_{n1}, c_{n2}, \dots, c_{nn-1}) > 0$ . By the previous lemma, C' has a positive rooted  $\rho_r(C')$ -eigenvector V. By Perron-Frobenius Theorem, C has a nonnegative  $\rho(C)$ -eigenvector v. Since  $v^T v' > 0$ , we have  $\rho(C) < \rho_r(C')$ by the upper bound theorem in the previous section.

In general let  $\epsilon > 0$  and  $C_{\epsilon} := C' + \epsilon J_n$ . By the argument above we have  $\rho(C) < \rho_r(C_c)$ . Hence

$$\rho(C) \leq \lim_{\epsilon \to 0^+} \rho_r(C'_{\epsilon}) = \rho_r(C').$$



#### Theorem

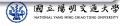
Let C and C be  $n \times n$  matrices such that C > 0, C is rooted,  $0 \le C'[n|n-1] \le C[n|n-1]$  and  $(r'_1, r'_2, \dots, r'_n) \le (r_1, r_2, \dots, r_n)$ . Then  $\rho(C) \leq \rho_r(C')$ .

#### Proof.

We assume first  $(c_{n1}, c_{n2}, \dots, c_{nn-1}) > 0$ . By the previous lemma, C' has a positive rooted  $\rho_r(C')$ -eigenvector V. By Perron-Frobenius Theorem, C has a nonnegative  $\rho(C)$ -eigenvector v. Since  $v^T v' > 0$ , we have  $\rho(C) > \rho_r(C')$ by the lower bound theorem in the previous section.

In general let  $\epsilon > 0$  and  $C_{\epsilon} := C' + \epsilon J_n$ . By the argument above we have  $\rho(C) > \rho_r(C_c)$ . Hence

$$\rho(C) \ge \lim_{\epsilon \to 0^+} \rho_r(C'_{\epsilon}) = \rho_r(C').$$



#### Remark

The condition 'C' is a rooted matrix' can be replaced by 'C' has a rooted  $\rho_r(C')$ -eigenvector in the above two theorems with almost the same proofs.



# Spectral bound via a smaller size matrix



#### **Motivation**

For

$$C = \begin{pmatrix} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{pmatrix}, C' = \begin{pmatrix} 2 & 2 & 3 & 3 & 3 & 12 & -1 \\ 4 & 2 & 1 & 4 & 2 & 6 & 5 \\ 2 & 3 & 2 & 4 & 2 & 8 & 3 \\ 4 & 5 & 3 & 1 & 1 & 3 & 3 \\ 5 & 6 & 1 & 1 & 1 & 3 & 3 \\ 1 & 2 & 1 & 2 & 2 & 6 & -1 \\ 2 & 2 & 0 & 2 & 2 & 1 & 4 \end{pmatrix}$$

with  $C[9|6] \le C'[7|6]$  and corresponding row-sum vectors,

$$(24, 23, 22, 20, 19, 13, 12) \le (24, 24, 24, 20, 20, 13, 13),$$

we still can't apply the upper bound theorem in the previous section to conclude  $\rho(C) \leq \rho_r(C')$ . This is because C' is not a rooted matrix, since  $c'_{61} = 0 < 1 = c'_{71}$ .



#### Observation

For

$$C' = \begin{pmatrix} 2 & 2 & 3 & 3 & 3 & 12 & -1 \\ 4 & 2 & 1 & 4 & 2 & 6 & 5 \\ 2 & 3 & 2 & 4 & 2 & 8 & 3 \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & 3 \\ 5 & 6 & 1 & 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & -1 \\ 2 & 2 & 0 & 2 & 2 & 1 & 4 \end{pmatrix}, M = \begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix},$$

the rooted matrix M is an equitable quotient matrix of C with respect to the partition  $\Pi = \{\{1,2,3\},\{4,5\},\{6,7\}\}$  of [7].

• To have  $\rho_r(C') = \rho(M)$ , the  $\rho_r(C')$ -eigenvector of C' needs to be extended from the  $\rho(M)$ -eigenvector of M.



#### Lemma

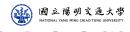
If C' is an  $n \times n$  matrix,  $\Pi = \{\pi_1, \dots, \pi_\ell\}$  is an equitable partition of C' with  $n \in \pi_\ell$  and  $\Pi(C')$  is a rooted matrix, then C' has a rooted eigenvector Su for  $\rho_r(\Pi(C'))$ , where S is the incident matrix of  $\Pi$  and u is a rooted eigenvector of  $\Pi(C')$  for  $\rho_r(C')$ .

#### Proof.

Since u is rooted and  $n \in \pi_{\ell}$ , Su is rooted. Note that

$$C'Su = S\Pi(C')u = \rho_r(C')Su.$$





#### Main Theorem

Let  $M=(m_{ab})$  be an  $\ell \times \ell$  matrix whose first  $\ell-1$  columns and row-sum vector are all rooted. If  $C=(c_{ij})$  is an  $n \times n$  nonnegative matrix and there exists a partition  $\Pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$  of  $[n]:=\{1,2,\ldots,n\}$  such that

$$\max_{i \in \pi_a} \sum_{j \in \pi_b} c_{ij} \le m_{ab} \quad \text{and} \quad \max_{i \in \pi_a} \sum_{j=1}^n c_{ij} \le \sum_{c=1}^\ell m_{ac}$$

for  $1 \le a \le \ell$  and  $1 \le b \le \ell - 1$ , then  $\rho(C) \le \rho_r(M)$ .

**Proof.** Rearranging the indices of C if necessary, we might assume  $n \in \pi_{\ell}$ . We first consider the case that  $(m_{\ell 1}, m_{\ell 2}, \ldots, m_{\ell \ell - 1}) > 0$ . We construct an  $n \times n$  matrix C such that  $C[n|n-1] \leq C'[n|n-1]$ ,  $(r_1, r_2, \ldots, r_n) \leq (r'_1, r'_2, \ldots, r'_n)$ , and  $\Pi(C') = M$  is an equitable partition of C'.



# Continue the proof

Since M has a positive rooted  $\rho_r(M)$ -eigenvector, by the previous lemma, C' has a positive rooted  $\rho_r(M)$ -eigenvector. By Perron-Frobenius Theorem, C has a nonnegative  $\rho(C)$ -eigenvector v. Since  $v^Tv'>0$ , we have  $\rho(C) \leq \rho_r(M) \leq \rho_r(C')$  by the upper bound theorem.

In general let  $\epsilon>0$  and  $M_\epsilon:=M+\epsilon J_\ell$ . By the argument above we have  $\rho(\mathcal{C})\leq \rho_r(M_\epsilon)$ . Hence

$$\rho(\mathit{C}) \leq \lim_{\epsilon \to 0^+} \rho_{\mathit{r}}(\mathit{M}_{\epsilon}) = \rho_{\mathit{r}}(\mathit{M}).$$



#### Main Theorem

Let  $M=(m_{ab})$  be an  $\ell \times \ell$  matrix whose first  $\ell-1$  columns and row-sum vector are all rooted. If  $C=(c_{ij})$  is an  $n \times n$  nonnegative matrix and there exists a partition  $\Pi=(\pi_1,\pi_2,\ldots,\pi_\ell)$  of  $[n]:=\{1,2,\ldots,n\}$  such that

$$\max_{i \in \pi_a} \sum_{j \in \pi_b} c_{ij} \geq m_{ab} \quad \text{and} \quad \max_{i \in \pi_a} \sum_{j=1}^n c_{ij} \geq \sum_{c=1}^\ell m_{ac}$$

for  $1 \le a \le \ell$  and  $1 \le b \le \ell - 1$ , then  $\rho(C) \ge \rho_r(M)$ .

Proof.

This is by a dual proof.



For

$$C = \begin{pmatrix} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{pmatrix}, M = \begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix},$$

with corresponding row-sum vectors

$$(24, 23, 22|20, 19|13, 12), (24, 20, 13),$$

we have  $\rho(C) \leq \rho_r(M) \approx 18.6936$ .

