Perron-Frobenius Theorem

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Outline

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Basic matrix notations and properties

Matrix Notations

1. For two matrices M, N of the same size, we write $M \leq N$ if $M_{ij} \leq N_{ij}$ for all i, j; write $M < N$ if $M_{ij} < N_{ij}$ for all i, j; write $M \lneqq N$ if $M \leq N$ and $M \neq N$.
2. M is nonnegative if $M \geq 0$.
3. $|M|$ is the matrix with entries $|M_{ij}|$.
4. An $n \times n$ matrix M is associated with a digraph Γ_M with vertex set $V\Gamma = \{1, 2, \ldots, n\}$ and arcs (i, j) whenever $M_{ij} \neq 0$.
5. A digraph Γ is **strongly connected** if for any two vertices x, y there exists a walk in Γ from x to y.
6. A matrix M is **irreducible** if Γ_M is strongly connected.

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The real value function θ

Throughout, let M be an $n \times n$ nonnegative matrix. Let $\mathbf{I} := \{x \in \mathbb{R}^n | x \geq 0, x \neq 0\}$ denote the **first orthant** in $\mathbb{R}^n$, and $\theta : \mathbf{I} \to \mathbb{R}^{>0}$ be the function satisfying

$$\theta(x) := \sup\{\eta \in \mathbb{R} \mid \eta x \leq Mx\} = \min\left\{\frac{(Mx)_i}{x_i} \mid 1 \leq i \leq n, x_i \neq 0\right\}.$$

Example

$$Mx = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 4 \end{pmatrix} \quad \Rightarrow \quad \theta(x) = \frac{1}{3}.$$

Remark

Let $x \in \mathbf{I}$. Then

- $\theta(x) \geq 0$;
- $\theta(x)x \leq Mx$;
- $\theta(cx) = \theta(x)$ for all $c > 0$.

Lemma

If $x \in \mathbf{I}$ is an eigenvector of M then $\theta(x)$ is an eigenvalue of M.

Proof.

$$\begin{aligned}\theta(x) &= \min\left\{\frac{(Mx)_i}{x_i} \mid 1 \le i \le n, x_i \neq 0\right\}\\ &= \min\left\{\frac{(\lambda x)_i}{x_i} \mid 1 \le i \le n, x_i \neq 0\right\}\\ &= \lambda.\end{aligned}$$

$\square$

Thus we might guess that $\theta_0 := \sup_{x\in\mathbf{I}} \theta(x)$ is an eigenvalue of M if θ_0 is well-defined.

Remark

Notice that $\theta(x)$ is not continuous on $\mathbf{I}$, as the following example shows.

Example

$$\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-t \\ t \end{pmatrix} = \begin{pmatrix} 2+t \\ t \end{pmatrix}$$

$$\Rightarrow \quad \lim_{t \to 0^+} \theta \begin{pmatrix} 1-t \\ t \end{pmatrix} = \lim_{t \to 0^+} \min \left(\frac{2+t}{1-t}, \frac{t}{t} \right) = 1$$

$$\theta \left(\lim_{t \to 0^+} \begin{pmatrix} 1-t \\ t \end{pmatrix} \right) = \theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \min \left(\frac{2}{1} \right) = 2.$$

However $\theta(x)$ is continuous on $\mathbf{I}^+ := \{x \in \mathbb{R}_n \mid x > 0\}$.

Question

If $\theta_0 := \sup_{x\in\mathbf{I}} \theta(x)$ is an eigenvalue of M, what is the θ_0-eigenvector?

The vector x_0

Let

$$S = \{x \in \mathbf{I} \mid \|x\| := x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}$$

be the unit sphere in $\mathbb{R}^n$ and guess that the vector

$$x_0 = \lim_{t\to\infty} \frac{M^t x}{\|M^t x\|} \qquad \text{for} \quad x \in I \cap S$$

will be the eigenvector of M.

Example

The above limit might not exist.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \limsup_{n\to\infty} \frac{M^n x}{\|M^n x\|} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \liminf_{n\to\infty} \frac{M^n x}{\|M^n x\|} \qquad \left(x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

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Remark

Recall that M is an $n \times n$ nonnegative matrix.

1. If x is an eigenvector of M then x is also an eigenvector of $(I+M)^k$ for any $k \in \mathbb{N}$.
2. If M is irreducible then $(I+M)^{n-1} > 0$.
3. If M is irreducible then

$$x \lneq y \quad \Rightarrow \quad (I+M)^{n-1}x < (I+M)^{n-1}y \qquad (x, y \in \mathbb{R}^n).$$

Lemma

Let M be a nonnegative $x \in \mathbf{I}$ then $\theta(x) \leq \theta((I+M)^{n-1}x)$. Moreover, if M is irreducible then $\theta(x) = \theta((I+M)^{n-1}x)$ if and only if x is an eigenvector of M.

Proof.

Recall that $\theta(x)x \leq Mx$. By $(I+M)^{n-1} \geq 0$, we have

$$\theta(x)[(I+M)^{n-1}x] = (I+M)^{n-1}\theta(x)x \leq (I+M)^{n-1}Mx = M[(I+M)^{n-1}x].$$

By the definition of $\theta((I+M)^{n-1}x)$, we have $\theta(x) \leq \theta((I+M)^{n-1}x)$. If M is irreducible then $(I+M)^{n-1} > 0$, so the following (i)-(iii) are equivalent.

(i) $\theta(x)x = Mx$;

(ii) $(I+M)^{n-1}\theta(x)x = (I+M)^{n-1}Mx$;

(iii) $\theta(x) = \theta((I+M)^{n-1}x)$.

□

Remark

1. The set $\mathbf{I} \cap S$ is closed and bounded, and so is the set $(I + M)^{n-1}(\mathbf{I} \cap S) \subseteq \mathbf{I}^{+}$.
2. If M is an $n \times n$ nonnegative irreducible matrix, then θ is continuous on the closed and bounded set $D = (I + M)^{n-1}(\mathbf{I} \cap S)$.
3. $\sup_{x \in D} \theta(x)$ exists.

Perron-Frobenius Theorem

Perron–Frobenius Theorem I

Let M be an $n \times n$ nonnegative irreducible matrix and define

$$\theta_0 := \sup_{x \in \mathbf{I}} \theta(x) \in \mathbb{R} \cup \{\infty\}.$$

Then $\theta_0 = \theta(x_0)$ for some $x_0 \in \mathbf{I}^+$. Moreover x_0 is θ_0-eigenvector of M.

Proof.

Recall that θ is continuous on the set $D := (I + M)^{n-1}(\mathbf{I} \cap S) \subseteq \mathbf{I}^+$. By the last lemma in the previous section,

$$\sup_{x \in \mathbf{I} \cap S} \theta(x) = \sup_{x \in \mathbf{I}} \theta(x) \geq \sup_{x \in D} \theta(x) = \sup_{c \in \mathbf{I} \cap S} \theta((I + M)^{n-1} c) \geq \sup_{c \in I \cap S} \theta(c).$$

Since D is closed and bounded, there exists $x_0 \in D \subseteq \mathbf{I}^+$ and $\theta(x_0) = \sup_{x \in I} \theta(x) = \theta_0$. As $\theta((I + M)^{n-1} x_0) = \theta(x_0)$, x_0 is a θ_0-eigenvector of M. □

Spectral radius

For an $n \times n$ matrix M, the number

$$\rho(M) = \max\{|\theta| \mid \theta \in \mathbb{C}, Mx = \lambda x \text{ for some } 0 \neq x \in \mathbb{C}^n\}$$

is called the **spectral radius** of M.

Perron–Frobenius Theorem II

Let M be a nonnegative irreducible matrix. Then $\theta_0 = \rho(M)$. Moreover x_0 is the unique θ_0-eigenvector of M up to a scalar.

Proof.

$$\begin{aligned} & Mx = \theta x \quad (x \text{ is an } \theta\text{-eigenvector}) \\ \Rightarrow \quad & |\theta||x| = |\theta x| = |Mx| \le M|x| \\ \Rightarrow \quad & |\theta| \le \theta(|x|) \\ \Rightarrow \quad & |\theta| \le \theta_0, \quad (|\theta| = \theta_0 \Rightarrow M|x| = \theta_0|x|). \end{aligned}$$

If $M|x| = \theta_0|x|$ and $|x|_i = 0$ then $\sum_{k=1}^{n} M_{ik}|x|_k = 0$, so $|x|_k = 0$ if $M_{ik} > 0$. By the strong connectivity of Γ_M, $|x|_i \neq 0$ for all i, so $|x| > 0$. If $x \neq cx_0$, we might choose an i such that $x - cx_0 \neq 0$ is an θ_0-eigenvector of M and $(x - cx_0)_i = 0$, a contradiction. □

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Remark

The above two theorems and their proofs implies that

$$x_0 = (I + M)^{n-1} \lim_{t\to\infty} \frac{((I+M)^{n-1})^t x}{\|((I+M)^{n-1})^t x\|}$$

for any $x \in I$, not as what we guessed earlier as

$$x_0 = \lim_{t\to\infty} \frac{M^t x}{\|M^t x\|} \qquad \text{for} \quad x \in I \cap S.$$

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Applications of Perron–Frobenius Theorem

Corollary

Let M be an $n \times n$ nonnegative irreducible matrix with spectral radius θ_0. Suppose that $\theta \in \mathbb{C}$ is an eigenvalue of M whose associated eigenvector $x \in \mathbf{I}$. Then $\theta = \theta_0$.

Proof.

Apply the above proof to left eigenvector $y_0 > 0$ associated with θ_0, i.e. we have $y_0^T M = \theta_0 y_0^T$. Then

$$\theta_0 y_0^T x = y_0^T M x = y_0^T \theta x$$

and $y_0^T x > 0$, so $\theta = \theta_0$. □

(左右開攻!)

Corollary

Let M be an $n \times n$ nonnegative irreducible matrix with spectral radius θ_0. Then the following (i)-(ii) hold.

(i) **If $Mx \leq px$ for some $x \in I$ and $p > 0$ then $p \geq \theta_0$, and equality holds iff $x > 0$ is an eigenvector of θ_0; and**

(ii) **If $Mx \geq px$ for some $x \in I$ and $p > 0$ then $p \leq \theta_0$, and equality holds iff $x > 0$ is an eigenvector of θ_0.**

Proof.

As before, let $y_0 > 0$ be a left θ_0-eigenvector. Suppose that $Mx \leq px$ for some $p > 0$ and $0 \neq x \geq 0$. Then

$$\theta_0 y_0^T x = y_0^T Mx \leq y_0^T px.$$

Since $y_0^T x > 0$, we have $p \geq \theta_0$. Note that $p = \theta_0$ iff $Mx = px$. The other statement is similar. □

Corollary

Let N be an $n \times n$ complex matrix such that $|N| \leq M$ for some nonnegative irreducible matrix M with spectral radius θ_0. Then $\rho(N) \leq \theta_0$. Moreover $\rho(N) = \theta_0$ if and only if $|N| = M$ and there is a diagonal matrix E with diagonal entries of absolute value 1 and a constant c of absolute value 1, such that $N = cEME^{-1}$.

Proof.

Suppose s is an θ-eigenvector of N, where $|\theta| = \rho(N)$. Then $M|s| \geq |N||s| \geq |Ns| = |\theta| \cdot |s|$. By previous theorem, $\theta_0 \geq |\theta|$. One direction for the second statement is clear. Suppose $\theta_0 = |\theta|$. Then $|s| > 0$ is an $|\theta|$-eigenvector of M, and $M|s| = |N||s| = |Ns|$, which implies $M = |N|$, and for each i, there exists an $e_i \in \mathbb{C}$ with $|e_i| = 1$ and $N_{ij}s_j = e_i|N_{ij}s_j|$ for all j. Let $E = \mathrm{diag}(e_1, \ldots, e_n)$. Then $\theta s = Ns = E|\theta||s|$, so $c := |s_j|e_j/s_j = \theta/|\theta|$ is independent of j, and

$$N_{ij} = \frac{e_i|N_{ij}s_j|}{s_j} = \frac{e_i|N_{ij}||s_j|}{s_j} = \frac{|s_j|e_j}{s_j} \cdot \frac{e_iM_{ij}}{e_j} = c(EME^{-1})_{ij}. \qquad \square$$

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Question

Why the spectral radii of the following nonnegative matrices are not strictly increasing?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \lneqq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} \lneqq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix} \lneqq \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}$$

Solution. The four matrices all have spectral radius 3. They are not irreducible. □

The algebraic multiplicity

The determinant of a matrix

1. A permutation of a set α is a bijection on α. Let S_α denote the set of permutations of α. If $\alpha = \{1, 2, \ldots, n\}$, write S_n for S_α.
2. A permutation can be expressed uniquely in a form containing disjoint directed cycles, e.g. $(1, 2, 3, 5)(4, 6)$ to denote the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 5 & 6 & 1 & 4 & 7 \end{pmatrix}.$$

3. The sign $\operatorname{sgn}(\sigma)$ of a permutation σ is -1 if σ has odd number of even cycles (cycles in graph language, including single edge as a cycle of length 2), and is 1 if σ has even number of even cycles, e.g. $\operatorname{sgn}((1, 2, 3, 5)(4, 6)) = 1$.
4. For an $n \times n$ matrix M, the **determinant** $\det(M)$ of M is defined to be

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) M_{1\sigma(1)} M_{2\sigma(2)} \cdots M_{n\sigma(n)}.$$

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Characteristic polynomial of a matrix

Let M be an $n \times n$ matrix.

(i) The **characteristic polynomial** of M is

$$p_M(t) = \det(tI - M).$$

(ii) The **algebraic multiplicity** of an eigenvalue θ of M is the multiplicity of θ as a root in $p_M(t)$.

Lemma

Let M be an $n \times n$ matrix. Then

$$p_M(t) = \sum_{k=0}^{n} \sum_{\substack{\alpha \subseteq [n] \\ |\alpha| = k}} \det(-M[\alpha]) t^{n-k},$$

where $[n] = \{1, 2, \dots, n\}$ and $M[\alpha]$ is the principal submatrix of M restricted to rows and columns in α.

Proof.

This follows from the following expansion

$$\det(tI - M) = \sum_{k=0}^{n} t^{n-k} \sum_{\substack{\alpha \subseteq [n] \\ |\alpha| = k}} \sum_{\sigma \in S_\alpha} \operatorname{sgn}(\sigma) \Big(\prod_{i \in \alpha} -M_{i\sigma(i)} \Big).$$

$\square$

Theorem

Let M be an $n \times n$ matrix and $M(i) := M[[n] - \{i\}]$. Then

$$\frac{d}{dt}p_M(t) = \sum_{i=1}^{n} p_{M(i)}(t).$$

Proof.

$$\frac{d}{dt}p_M(t) = \sum_{k=0}^{n} \sum_{\alpha \subseteq [n],\ |\alpha|=k} (n-k)\det(-M[\alpha])t^{n-k-1}$$

$$= \sum_{k=0}^{n} \sum_{\alpha \subseteq [n],\ |\alpha|=k} \sum_{i \notin \alpha} \det(-M[\alpha])t^{n-k-1} = \sum_{i=1}^{n} p_{M(i)}(t).$$

□

Corollary

Let M be an $n \times n$ nonnegative irreducible matrix with spectral radius θ_0. Then θ_0 has algebraic multiplicity 1, i.e. θ_0 is a simple root of the characteristic polynomial $p_M(t)$ of M.

Proof.

We have shown that

$$\frac{dp_M(t)}{dt} = \sum_{i=1}^{n} p_{M(i|i)}(t).$$

By filling a column and a row of zeros, we can view as $M(i|i) \leq M$. Note that $M(i|i) \neq M$ since M has no zero row. Hence θ_0 is strictly larger than the absolute value of any eigenvalue of $M(i|i)$. This implies $p_{M(i|i)}(\theta_0) > 0$ for all i. Since $p'_M(\theta_0) \neq 0$, θ_0 is a simple root of $p_M(t)$. □

Positive eigenvector matters

The i-th rowsum $r_i(A)$

For an $n \times n$ matrix A, the value

$$r_i(A) := \sum_{j=1}^{n} A_{ij}$$

is called the i-th **rowsum** of A.

Corollary

If $M \geq 0$ has spectral radius θ_0 then

$$\min_{1\leq i\leq n} r_i(M) \leq \theta_0 \leq \max_{1\leq i\leq n} r_i(M).$$

Moreover, if M is irreducible then each equality holds if and only if $r_i(A) = r_j(A)$ for all i, j.

Proof.

With $x = (1, 1, \ldots, 1)^T$ we have

$$Ax \leq \left(\max_{1\leq i\leq n} r_i(A)\right) x.$$

Hence $\theta_0 \leq \max_{1\leq i\leq n} r_i(A)$. The equality holds iff x is a θ_0-eigenvector of M, which is equivalent to $r_i(A) = r_j(A)$ for all i, j. By a dual proof, we have $\min_{1\leq i\leq n} r_i(M) \leq \theta_0$, and the same equality conditions. □

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Theorem

If M is symmetric with a λ-eigenvector $x \geq 0$, then

$$\min_{1\leq i\leq n} r_i(M) \leq \lambda \leq \max_{1\leq i\leq n} r_i(M).$$

Moreover, if $x > 0$ then each equality holds if and only if $r_i(A) = r_j(A)$ for all i, j.

Proof.

We might assume $\sum_{i=1}^n x_i = 1$. Using $Mx = \lambda x$ and $A_{ij} = A_{ji}$, we have

$$\lambda = \lambda \sum_{i=1}^n x_i = \sum_{i=1}^n (Mx)_i = \sum_{i=1}^n \sum_{j=1}^n M_{ij}x_j = \sum_{i=1}^n \sum_{j=1}^n M_{ji}x_j = \sum_{j=1}^n r_j(M)x_j.$$

The proof is finished, since λ is a convex combination of $r_i(M)$ for i in $\{1, 2, \ldots, n\}$. $\square$

Remark

- The theorem in the last page is first appeared in the following paper.

  M. N. Ellingham and X. Zha, The spectral radius of graphs on surfaces, *Journal of Combinatorial Theory Ser.*, B 78 (1) (2000) 45-56.

- We will show in the next chapter that if M is a symmetric matrix with spectral radius θ_0 then

$$\theta_0 \geq \frac{1}{n} \sum_{i=1}^{n} r_i(M).$$

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Applications to graph theory

Planar graphs

A **planar graph** is a graph which has a drawing on a plane without edge crossing. It is well known a planar graph of order $n \geq 3$ has at most $3n - 6$ edges.

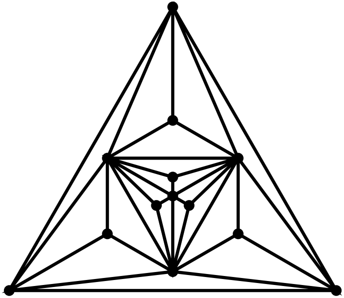

Figure: A planar graph of order 13 and size 33.

Outerplanar graphs

An **outerplanar graph** is a planar graph that has a drawing on the plane such that all vertices are appeared in the boundary of an unbounded face. It is well known a planar graph of order $n \geq 2$ has at most $2n-3$ edges.

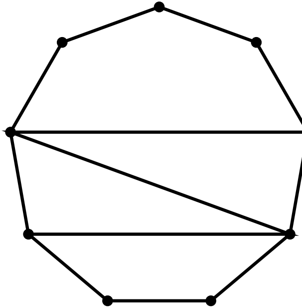

Figure: An outerplanar graph of order 9.

Notations

Let Γ be a graph, $i \in V\Gamma$ and $A, B \subseteq V\Gamma$ with $A \cap B = \emptyset$.

- The set $\Gamma(i) = \{j \mid ji \in E\Gamma\}$ is called the **neighbor set** of i in G.
- The set $\Gamma[i] = \{j \mid ji \in E\Gamma\} \cup \{i\}$ is called the **closed neighbor set** of i in G.
- The set $EA = \{ij \in E\Gamma \mid i, j \in A\}$ is called the **edge set induced on** A.
- The set $E(A, B) = \{ij \in E\Gamma \mid i \in A, j \in B\}$ is called the **set of edges between** A and B.

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Lemma

If Γ is a planar graph of order $n \geq 3$ with adjacency matrix A, then

$$r_i(A^2) \leq 3n + 2r_i(A) - 9.$$

Proof.

We lease the case $|\Gamma(i)| = 1$ as an exercise, and suppose $|\Gamma(i)| \geq 2$. Then $|E\Gamma(i)| \leq 2r_i(A) - 3$, and

$$\begin{aligned} r_i(A^2) = \sum\nolimits_{j,k} A_{ik}A_{kj} &= r_i(A) + 2|E\Gamma(i)| + |E(\Gamma(i), \overline{\Gamma[i]})| \\ &\leq r_i(A) + 2|E\Gamma(i)| + (3n - 6 - r_i(A) - |E\Gamma(i)|) \\ &\leq 3n - 6 + 2r_i(A) - 3 = 3n + 2r_i(A) - 9. \end{aligned}$$

□

Theorem

If Γ is a planar graph of order $n \geq 3$ with adjacency matrix A, then the spectral radius $\rho(A)$ of A satisfies

$$\rho(A) \leq 1 + \sqrt{3n - 8}.$$

Proof.

Let x be a nonnegative $\rho(A)$-eigenvector of A. Then x is also an λ-eigenvector of the matrix $A^2 - 2A$, where $\lambda = \rho(A)^2 - 2\rho(A)$. By a theorem in the section that positive eigenvector matters,

$$\rho(A)^2 - 2\rho(A) \leq \max_i r_i(A^2 - 2A) \leq 3n - 9,$$

where the last inequality is by the previous lemma. Hence the theorem follows. □