Matrix Realization of Spectral Bounds

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Outline

Motivation

Theorem (Stanley, 1987)

If Γ is a graph of size m with spectral radius $\rho(A)$, then

$$\rho(A) \leq \frac{-1+\sqrt{1+8m}}{2}.$$

Proof.

The i-th rowsum of $4A^2+4A$ satisfies

$$\begin{aligned} r_i(4A^2+4A) =&(4A^2+4A)_{ii} + \sum_{\partial(j,i)=1}(4A^2+4A)_{ij} + \sum_{\partial(j,i)=2}(4A^2+4A)_{ij} \\ =&4|\Gamma_1(i)| + (8|E\Gamma_1(i)| + 4|\Gamma_1(i)|) + 4|E(\Gamma_1(i),\Gamma_2(i))| \leq 8m. \end{aligned}$$

By a property we proved earlier that positive eigenvector matters, the eigenvalue $4\rho(A)^2+4\rho(A)$ of $4A^2+4A$ is at most $8m$ to have the result. $\square$

Further results

Let Γ be a graph of order n and size m with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Recall that $2m = d_1 + d_2 + \cdots + d_n$. Let A be the adjacency matrix of Γ with spectral radius $\rho(A)$. The following are extensions of

$$\rho(A) \leq \frac{-1+\sqrt{1+8m}}{2}.$$

- (Hong, 1998) $\rho(A) \leq \sqrt{2m-n+1}$.
- (Hong, Shu, Fang, 2001) $\rho(A) \leq \frac{d_n-1+\sqrt{(d_n+1)^2+4(2m-nd_n)}}{2}$.
- (Shu, Wu, 2004) $\rho(A) \leq \frac{d_\ell-1+\sqrt{(d_\ell+1)^2+4(\ell-1)(d_1-d_\ell)}}{2}$.
- (Liu, Weng, 2013) $\rho(A) \leq \frac{d_\ell-1+\sqrt{(d_\ell+1)^2+4\sum_{i=1}^{\ell-1}(d_i-d_\ell)}}{2}$.

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Matrix realizations

$$
\rho\begin{pmatrix}
0 & 1 & \cdots & & & 1 & d_1-(n-2)\\
1 & 0 & \ddots & & & \vdots & \vdots\\
\vdots & \ddots & & \ddots & & 1 & d_1-(n-2)\\
\vdots & \ddots & \ddots & & & 1 & d_\ell-(n-2)\\
 & & & & & & \vdots\\
 & & & & 0 & 1 & \\
1 & \cdots & & & 1 & 0 & d_\ell-(n-2)\\
1 & \cdots & & & & 1 & d_\ell-(n-1)
\end{pmatrix}
=\frac{d_\ell-1+\sqrt{(d_\ell+1)^2+4(\ell-1)(d_1-d_\ell)}}{2},
$$

$$
\rho\begin{pmatrix}
0 & 1 & \cdots & & 1 & 1 & d_1-(n-2)\\
1 & 0 & \ddots & & & \vdots & d_2-(n-2)\\
\vdots & \ddots & \ddots & & & & \vdots\\
 & & & & & 1 & d_\ell-(n-2)\\
 & & & & & & \vdots\\
 & & & & 0 & 1 & \\
 & & & & 1 & 0 & d_\ell-(n-2)\\
1 & \cdots & & & 1 & 1 & d_\ell-(n-1)
\end{pmatrix}
=\frac{d_\ell-1+\sqrt{(d_\ell+1)^2+4\sum_{i=1}^{\ell-1}(d_i-d_\ell)}}{2}.
$$

Main theorem

Rooted vectors

A vector $(v_1, v_2, \ldots, v_n)$ is called **rooted** if $v_i \geq v_n \geq 0$ for $1 \leq i \leq n-1$.

Main Theorem (Cheng, –)

Let $M = (m_{ab})$ be an $\ell \times \ell$ matrix whose first $\ell - 1$ columns and row-sum vector are all rooted. If $C = (c_{ij})$ is an $n \times n$ nonnegative matrix and there exists a partition $\Pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of $[n] := \{1, 2, \ldots, n\}$ such that

$$\max_{i \in \pi_a} \sum_{j \in \pi_b} c_{ij} \leq m_{ab} \quad \text{and} \quad \max_{i \in \pi_a} \sum_{j=1}^{n} c_{ij} \leq \sum_{c=1}^{\ell} m_{ac}$$

for $1 \leq a \leq \ell$ and $1 \leq b \leq \ell - 1$, then $\rho(C) \leq \rho_r(M)$.

Before we prove this theorem, we will see some example first.

Example

If $\ell = n$, $\Pi = \{\{1\}, \{2\}, \ldots, \{n\}\}$, and $|c_{ij}| \leq m_{ij}$, then $\rho(C) \leq \rho(M)$ by an application of Perron-Frobenius Theorem. In this case, we don't need the rooted assumption of M.

Example

If $\ell = 1$, $C \geq 0$, $\Pi = \{[n]\}$, **and** $m_{11} = \max\limits_{i \in [n]} \sum_{j=1}^{n} c_{ij}$, **then** $\rho(C) \leq \rho(M)$.

Example

If $\ell = n = 2$, $\Pi = \{\{1\},\{2\}\}$, and

$$C = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 5 & 2 \\ 4 & -1 \end{pmatrix},$$

then

$$3 + 2\sqrt{2} = \rho(C) \leq \rho(M) = 2 + \sqrt{17}.$$

Remark

The above matrix M is rooted, since $5 \geq 4 \geq$ and $5 + 2 \geq 4 + (-1) \geq 0$. The last column of M is unrelated.

Example

With $n = 7$, $\ell = 3$, $\Pi = \{\{1,2,3\},\{4,5\},\{6,7\}\}$, and

$$C = \left(\begin{array}{ccc|cc|cc} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{array}\right), \quad M = \begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix},$$

where the row-sum vectors of C and M are

$$(24, 23, 22|20, 19|13, 12), \quad (24, 20, 13),$$

respectively, we have $\rho(C) \le \rho_r(M) \approx 18.6936$.

Spectral bound via a same size matrix

Lemma

Let C, C', P and Q be $n \times n$ matrices. Assume that (i) $PCQ \leq PC'Q$; (ii) $C'Qu = \lambda' Qu$ for some column vector $0 \neq u \geq 0$ and $\lambda' \in \mathbb{R}$; (iii) $v^T PC = \lambda v^T P$ for some row vector $0 \neq v^T \geq 0$ and $\lambda \in \mathbb{R}$; and (iv) $v^T PQu > 0$. Then $\lambda \leq \lambda'$.

Proof.

$$
\begin{aligned}
& PCQ \leq PC'Q \\
\Rightarrow \quad & PCQu \leq PC'Qu = \lambda' PQu \\
\Rightarrow \quad & \lambda v^T PQu = v^T PCQu \leq v^T PC'Qu = \lambda' v^T PQu \\
\Rightarrow \quad & \lambda \leq \lambda'.
\end{aligned}
$$

□

Remark

If $P = Q = I$, C' is irreducible, $0 \leq C \leq C'$, $u > 0$ is λ'-eigenvector of C' and $v^T \geq 0$ is left λ-eigenvector of C, then the assumption (iv) $v^T u > 0$ in the previous lemma holds and the conclusion $\lambda \leq \lambda'$ holds. Indeed this is an application of Perron-Frobenius Theorem.

Lemma

Let C, C', P and Q be $n \times n$ matrices. Assume that (i) $PCQ \geq PC'Q$; (ii) $C'Qu = \lambda' Qu$ for some column vector $0 \neq u \geq 0$ and $\lambda' \in \mathbb{R}$; (iii) $v^T PC = \lambda v^T P$ for some row vector $0 \neq v^T \geq 0$ and $\lambda \in \mathbb{R}$; and (iv) $v^T PQu > 0$. Then $\lambda \geq \lambda'$.

Proof.

$$\begin{aligned} & PCQ \geq PC'Q \\ \Rightarrow \quad & PCQu \geq PC'Qu = \lambda' PQu \\ \Rightarrow \quad & \lambda v^T PQu = v^T PCQu \geq v^T PC'Qu = \lambda' v^T PQu \\ \Rightarrow \quad & \lambda \geq \lambda'. \end{aligned}$$

□

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The case $P = I_n$ and $Q = I_n + \sum_{i=1}^{n-1} e_{in}$

The matrix $Q = I_n + \sum_{i=1}^{n-1} e_{in}$

We will apply the previous two lemmas by letting $P = I_n$ and

$$Q = I_n + \sum_{i=1}^{n-1} e_{in} = \begin{pmatrix} I_{n-1} & J_{(n-1)\times 1} \\ O_{1\times(n-1)} & 1 \end{pmatrix}.$$

Hence

$$C'Q = \left(\begin{array}{c|c} & r'_1 \\ & r'_2 \\ C'[n|n-1] & \vdots \\ & r'_n \end{array} \right),$$

where $C'[n|n-1]$ is the submatrix of C on the first $n-1$ columns and where $(r'_1, r'_2, \ldots, r'_n)$ is the row-sum vector of C'.

Lemma

Let $Q = I_n + \sum_{i=1}^{n-1} e_{in}$ and u be a column vector. Then Qu is rooted if and only if u is nonnegative.

Proof.

This follows from the definition of rooted vector and using

$$Qu = \begin{pmatrix} u_1 + u_n & u_2 + u_n & \cdots & u_{n-1} + u_n & u_n \end{pmatrix}.$$

$\square$

Theorem

Let C and C' be $n \times n$ matrices. Assume that
(i) $C[n|n-1] \leq C'[n|n-1]$ **and** $(r_1, r_2, \ldots, r_n) \leq (r'_1, r'_2, \ldots, r'_n)$**;**
(ii) $C'w = \lambda' w$ **for some column rooted vector** $0 \neq w \geq 0$ **and** $\lambda' \in \mathbb{R}$**;**
(iii) $v^T C = \lambda v^T$ **for some row vector** $0 \neq v^T \geq 0$ **and** $\lambda \in \mathbb{R}$**; and**
(iv) $v^T w > 0$.
Then $\lambda \leq \lambda'$**.**

Proof.

Applying the lemma in the previous section with $P = I_n$, $Q = I_n + \sum_{i=1}^{n-1} e_{in}$ $w = Qu$, and noticing that u is nonnegative if and only if w is rooted, we immediately have the theorem. □

Theorem

Let C and C' be $n \times n$ matrices. Assume that
(i) $C[n|n-1] \geq C'[n|n-1]$ **and** $(r_1, r_2, \ldots, r_n) \geq (r'_1, r'_2, \ldots, r'_n)$**;**
(ii) $C'w = \lambda' w$ **for some column rooted vector** $0 \neq w \geq 0$ **and** $\lambda' \in \mathbb{R}$**;**
(iii) $v^T C = \lambda v^T$ **for some row vector** $0 \neq v^T \geq 0$ **and** $\lambda \in \mathbb{R}$**; and**
(iv) $v^T w > 0$.
Then $\lambda \geq \lambda'$**.**

Proof.

Applying the lower bound lemma in the previous section with $P = I_n$, $Q = I_n + \sum_{i=1}^{n-1} e_{in}$ $w = Qu$, and noticing that u is nonnegative if and only if w is rooted, we immediately have the theorem. $\square$

Example

Consider the following three matrices

$$C'_\ell = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad C'_r = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

with $C'_\ell[3|2] \leq C[3|2] \leq C'_r[3|2]$, and the same row-sum vector $(5, 3, 3)$. Note that C'_ℓ has a rooted 3-eigenvector $w_\ell = (1, 0, 0)^T$, and C'_r has a rooted 4-eigenvector $r_r = (2, 1, 1)^T$. Since C is irreducible, it has a positive left $\rho(C)$-eigenvector (v_1, v_2, v_3). Hence

$$3 \leq \rho(C) \leq 4.$$

Rooted matrices

Rooted matrices

- An $n \times n$ real matrix $C' = (c'_{ij})$ is called **rooted** if there is a constant d such that the first $n-1$ columns and the row-sum vector of $C' + dI_n$ are all rooted.
- If C' has no real eigenvalue, set $\rho_r(C') = \infty$; otherwise let $\rho_r(C')$ denote the largest real eigenvalue of C'.

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Example

The following three matrices

$$C'_\ell = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad C'_r = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix}$$

are all rooted.

Remark

- $$Q = I_n + \sum_{i=1}^{n-1} e_{in} = \begin{pmatrix} I_{n-1} & J_{(n-1)\times 1} \\ O_{1\times(n-1)} & 1 \end{pmatrix}$$
$$\Rightarrow \qquad Q^{-1} = I_n - \sum_{i=1}^{n-1} e_{in} = \begin{pmatrix} I_{n-1} & -J_{(n-1)\times 1} \\ O_{1\times(n-1)} & 1 \end{pmatrix}$$
$$Q^{-1} C' Q = Q^{-1} \begin{pmatrix} c'_{11} & c'_{12} & \cdots & c'_{1\ n-1} & r'_1 \\ c'_{21} & c'_{22} & \cdots & c'_{2\ n-1} & r'_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c'_{n-1\ 1} & c'_{n-1\ 2} & \cdots & c'_{n-1\ n-1} & r'_{n-1} \\ c'_{n1} & c'_{n2} & \cdots & c'_{nn-1} & r'_n \end{pmatrix}.$$
- $Q^{-1}(C' + dI_n)Q \geq 0$ for some $d \in \mathbb{R}$ if and only if C' is rooted.

Lemma

If C' is an $n \times n$ rooted matrix, then $\rho_r(C') < \infty$, and C' has a rooted eigenvector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ for $\rho_r(C')$. Moreover, if the row vector $(c'_{n1}, c'_{n2}, \ldots, c'_{nn-1})$ is positive then v' is positive.

Proof.

Let $Q = I_n + \sum_{i=1}^{n-1} e_{in}$. Since $Q^{-1}(C' + dI_n)Q \geq 0$ for some $d \in \mathbb{R}$, there is a η-eigenvector $u \geq 0$, where $\eta = \rho(C' + dI_n) \in \mathbb{R}$. Since $\rho(C' + dI_n) = \rho_r(C') + d$, $v' = Qu$ is a rooted $\rho_r(C')$-eigenvector. This proves the first statement. Suppose $(c'_{n1}, c'_{n2}, \ldots, c'_{nn-1}) > 0$. If $u_n > 0$ then v' is clear to be positive. Suppose $u_n = 0$. Then $v'_n = 0$ and $\sum_{j=1}^{n-1} c'_{nj} v'_j = \sum_{j=1}^{n} c'_{nj} v'_j = (C'v')_n = \rho_r(C') v'_n = 0$. Hence $v'_j = 0$ for $1 \leq j \leq n-1$. Thus v is a zero vector, a contradiction. □

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Theorem

Let C and C' be $n \times n$ matrices such that $C \geq 0$, C' is rooted, $0 \leq C[n|n-1] \leq C'[n|n-1]$ and $(r_1, r_2, \ldots, r_n) \leq (r'_1, r'_2, \ldots, r'_n)$. Then $\rho(C) \leq \rho_r(C')$.

Proof.

We assume first $(c'_{n1}, c'_{n2}, \ldots, c'_{nn-1}) > 0$. By the previous lemma, C' has a positive rooted $\rho_r(C')$-eigenvector v'. By Perron-Frobenius Theorem, C has a nonnegative $\rho(C)$-eigenvector v. Since $v^T v' > 0$, we have $\rho(C) \leq \rho_r(C')$ by the upper bound theorem in the previous section.

In general let $\epsilon > 0$ and $C'_\epsilon := C' + \epsilon J_n$. By the argument above we have $\rho(C) \leq \rho_r(C'_\epsilon)$. Hence

$$\rho(C) \leq \lim_{\epsilon \to 0^+} \rho_r(C'_\epsilon) = \rho_r(C').$$

□

Theorem

Let C and C' be $n \times n$ matrices such that $C \geq 0$, C' is rooted, $0 \leq C'[n|n-1] \leq C[n|n-1]$ and $(r'_1, r'_2, \ldots, r'_n) \leq (r_1, r_2, \ldots, r_n)$. Then $\rho(C) \leq \rho_r(C')$.

Proof.

We assume first $(c'_{n1}, c'_{n2}, \ldots, c'_{nn-1}) > 0$. By the previous lemma, C' has a positive rooted $\rho_r(C')$-eigenvector v'. By Perron-Frobenius Theorem, C has a nonnegative $\rho(C)$-eigenvector v. Since $v^T v' > 0$, we have $\rho(C) \geq \rho_r(C')$ by the lower bound theorem in the previous section.

In general let $\epsilon > 0$ and $C'_\epsilon := C' + \epsilon J_n$. By the argument above we have $\rho(C) \geq \rho_r(C'_\epsilon)$. Hence

$$\rho(C) \geq \lim_{\epsilon \to 0^+} \rho_r(C'_\epsilon) = \rho_r(C').$$

□

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Remark

The condition 'C' is a rooted matrix' can be replaced by 'C' has a rooted $\rho_r(C')$-eigenvector in the above two theorems with almost the same proofs.

Spectral bound via a smaller size matrix

Motivation

For

$$C = \left(\begin{array}{cccccc|c} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{array}\right), C' = \left(\begin{array}{cccccc|c} 2 & 2 & 3 & 3 & 3 & 12 & -1 \\ 4 & 2 & 1 & 4 & 2 & 6 & 5 \\ 2 & 3 & 2 & 4 & 2 & 8 & 3 \\ 4 & 5 & 3 & 1 & 1 & 3 & 3 \\ 5 & 6 & 1 & 1 & 1 & 3 & 3 \\ 1 & 2 & 1 & 2 & 2 & 6 & -1 \\ 2 & 2 & 0 & 2 & 2 & 1 & 4 \end{array}\right)$$

with $C[9|6] \leq C'[7|6]$ and corresponding row-sum vectors,

$$(24, 23, 22, 20, 19, 13, 12) \leq (24, 24, 24, 20, 20, 13, 13),$$

we still can't apply the upper bound theorem in the previous section to conclude $\rho(C) \leq \rho_r(C')$. This is because C' is not a rooted matrix, since $c'_{61} = 0 < 1 = c'_{71}$.

Observation

- For
$$C' = \left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & -1 \\ 4 & 2 & 1 & 4 & 2 & 6 & 5 \\ 2 & 3 & 2 & 4 & 2 & 8 & 3 \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & 3 \\ 5 & 6 & 1 & 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & -1 \\ 2 & 2 & 0 & 2 & 2 & 1 & 4 \end{array}\right), M = \begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix},$$
the rooted matrix M is an equitable quotient matrix of C' with respect to the partition $\Pi = \{\{1,2,3\},\{4,5\},\{6,7\}\}$ of $[7]$.
- To have $\rho_r(C') = \rho(M)$, the $\rho_r(C')$-eigenvector of C' needs to be extended from the $\rho(M)$-eigenvector of M.

Lemma

If C' is an $n \times n$ matrix, $\Pi = \{\pi_1, \ldots, \pi_\ell\}$ is an equitable partition of C' with $n \in \pi_\ell$ and $\Pi(C')$ is a rooted matrix, then C' has a rooted eigenvector Su for $\rho_r(\Pi(C'))$, where S is the incident matrix of Π and u is a rooted eigenvector of $\Pi(C')$ for $\rho_r(C')$.

Proof.

Since u is rooted and $n \in \pi_\ell$, Su is rooted. Note that

$$C'Su = S\Pi(C')u = \rho_r(C')Su.$$

□

Main Theorem

Let $M = (m_{ab})$ be an $\ell \times \ell$ matrix whose first $\ell - 1$ columns and row-sum vector are all rooted. If $C = (c_{ij})$ is an $n \times n$ nonnegative matrix and there exists a partition $\Pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of $[n] := \{1, 2, \ldots, n\}$ such that

$$\max_{i \in \pi_a} \sum_{j \in \pi_b} c_{ij} \le m_{ab} \quad \text{and} \quad \max_{i \in \pi_a} \sum_{j=1}^{n} c_{ij} \le \sum_{c=1}^{\ell} m_{ac}$$

for $1 \le a \le \ell$ and $1 \le b \le \ell - 1$, then $\rho(C) \le \rho_r(M)$.

Proof. Rearranging the indices of C if necessary, we might assume $n \in \pi_\ell$. We first consider the case that $(m_{\ell 1}, m_{\ell 2}, \ldots, m_{\ell \ell - 1}) > 0$. We construct an $n \times n$ matrix C' such that $C[n|n-1] \le C'[n|n-1]$, $(r_1, r_2, \ldots, r_n) \le (r'_1, r'_2, \ldots, r'_n)$, and $\Pi(C') = M$ is an equitable partition of C'.

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Continue the proof

Since M has a positive rooted $\rho_r(M)$-eigenvector, by the previous lemma, C' has a positive rooted $\rho_r(M)$-eigenvector. By Perron-Frobenius Theorem, C has a nonnegative $\rho(C)$-eigenvector v. Since $v^T v' > 0$, we have $\rho(C) \le \rho_r(M) \le \rho_r(C')$ by the upper bound theorem.

In general let $\epsilon > 0$ and $M_\epsilon := M + \epsilon J_\ell$. By the argument above we have $\rho(C) \le \rho_r(M_\epsilon)$. Hence

$$\rho(C) \le \lim_{\epsilon \to 0^+} \rho_r(M_\epsilon) = \rho_r(M).$$

Main Theorem

Let $M = (m_{ab})$ be an $\ell \times \ell$ matrix whose first $\ell - 1$ columns and row-sum vector are all rooted. If $C = (c_{ij})$ is an $n \times n$ nonnegative matrix and there exists a partition $\Pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of $[n] := \{1, 2, \ldots, n\}$ such that

$$\max_{i \in \pi_a} \sum_{j \in \pi_b} c_{ij} \geq m_{ab} \quad \textbf{and} \quad \max_{i \in \pi_a} \sum_{j=1}^{n} c_{ij} \geq \sum_{c=1}^{\ell} m_{ac}$$

for $1 \leq a \leq \ell$ and $1 \leq b \leq \ell - 1$, then $\rho(C) \geq \rho_r(M)$.

Proof.

This is by a dual proof. □

Example

For

$$
C = \left(\begin{array}{ccc|cc|cc}
2 & 1 & 3 & 3 & 3 & 12 & 0 \\
4 & 2 & 1 & 4 & 2 & 6 & 4 \\
2 & 3 & 1 & 4 & 1 & 8 & 3 \\
\hline
3 & 5 & 3 & 1 & 1 & 3 & 4 \\
5 & 6 & 1 & 1 & 0 & 3 & 3 \\
\hline
0 & 2 & 1 & 2 & 2 & 6 & 0 \\
2 & 2 & 0 & 2 & 1 & 1 & 4
\end{array}\right), M = \begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix},
$$

with corresponding row-sum vectors

$$
(24, 23, 22|20, 19|13, 12), \quad (24, 20, 13),
$$

we have $\rho(C) \le \rho_r(M) \approx 18.6936.$