Hamming Graphs and Tensor Products

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Tensor product of matrices

Example (Tensor product of matrices)

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \bigotimes \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} & a_{12} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \\ a_{21} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} & a_{22} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} \end{pmatrix},$$

if the rows are ordered as

$$(1,1),(1,2),(2,1),(2,2)$$

and the columns are order as

$$(1,1),(1,2),(1,3),(2,1),(2,2),(2,3).$$

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Formal definition (Tensor product of matrices)

Let $B_i = (b_{v_i u_i})$ be a matrix indexed by $V_i \times U_i$ for $1 \leq i \leq D$.

- Then the **tensor product** of $B_1, B_2, \ldots, B_D$ is the matrix

$$\bigotimes_{i=1}^{D} B_i = (b_{vu})$$

indexed by

$$(V_1 \times V_2 \times \cdots \times V_D) \times (U_1 \times U_2 \times \cdots \times U_D)$$

such that

$$b_{vu} = b_{v_1 u_1} b_{v_2 u_2} \cdots b_{v_D u_D} \qquad (v_i \in V_i, u_i \in U_i).$$

- If $B_i = B$, then we adopt the notation $B^{\otimes D} := \bigotimes_{i=1}^{D} B$.

Lemma (Multi-linear property)

If B_i are matrices for $1 \le i \le D$, C_j is a matrix with the same size of B_j, and $\alpha \in \mathbb{C}$, then

$$\begin{aligned} & B_1 \otimes \cdots \otimes B_{j-1} \otimes (B_j + \alpha C_j) \otimes B_{j+1} \otimes \cdots \otimes B_D \\ = \; & B_1 \otimes \cdots \otimes B_{j-1} \otimes B_j \otimes B_{j+1} \otimes \cdots \otimes B_D \\ & + \alpha(B_1 \otimes \cdots \otimes B_{j-1} \otimes C_j \otimes B_{j+1} \otimes \cdots \otimes B_D). \end{aligned}$$

Proof.

This is immediate from the illustration of tensor product. □

Lemma

If $v_1, v_2, \ldots v_n \in \mathbb{R}^n$ and $w_1, w_2, \ldots, w_m \in \mathbb{R}^m$ are bases then $\{v_i \otimes w_j \mid 1 \le i \le n, 1 \le j \le m\}$ is a basis of $\mathbb{R}^{nm}$.

Proof.

$$\begin{aligned}
& \sum\nolimits_{1\le i\le n, 1\le j\le m} c_{ij}(v_i \otimes w_j) = 0, \\
\Rightarrow\ & \sum\nolimits_{1\le i\le n} v_i \otimes \sum\nolimits_{1\le j\le m} c_{ij} w_j = 0 \quad \text{(multi-linear property)}, \\
\Rightarrow\ & \forall i \left(\sum\nolimits_{1\le j\le m} c_{ij} w_j = 0\right) \qquad \text{(use linear independent of } v_i), \\
\Rightarrow\ & \forall i, j\ (c_{ij} = 0).
\end{aligned}$$

Since $\{v_i \otimes w_j \mid 1 \le i \le n, 1 \le j \le m\}$ is a linearly independent set of size mn, it is a basis. $\square$

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Remark

$$
\begin{aligned}
&\{u \otimes v : \ u \in \mathbb{C}^2, v \in \mathbb{C}^3\} \\
= &\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \\ e \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ ae \\ bc \\ bd \\ be \end{pmatrix} \ : \ a, b, c, d, e \in \mathbb{C} \right\} \\
&\subsetneq \mathbb{C}^6.
\end{aligned}
$$

Tean product of vector spaces

Tensor product of vector spaces

Let V_i be vector spaces of dimension n_i over $\mathbb{C}$ of dimension n_i for $1 \leq i \leq D$.

- The set
$$\{v_1 \otimes v_2 \otimes \cdots \otimes v_D \ : \ v_i \in V_i\} \subsetneq \mathbb{C}^{\prod_{i=1}^{D} n_i}$$
is **not** necessary to be a subspace.
- The the **tensor product** of V_i is the vector space defined by
$$V_1 \otimes V_2 \otimes \cdots \otimes V_D := \mathrm{Span}\{v_1 \otimes v_2 \otimes \cdots \otimes v_D \ : \ v_i \in V_i\}.$$
- Comparing the dimensions by previous Lemma,
$$V_1 \otimes V_2 \otimes \cdots \otimes V_D = \mathbb{C}^{n_1 n_2 \cdots n_D}.$$
- If $V_i = V$ for all $1 \leq i \leq D$, we use the notation $V^{\otimes D} := \bigotimes_{i=1}^{D} V_i$.
- Note that $\dim(V^{\otimes D}) = (\dim(V))^D$.

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Lemma

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD),$$

where A, B, C, D are matrices of suitable sizes that make AC and BD well-defined.

Proof.

Let $x = (i, j)$, $y = (k, s)$ be fixed and $z = (u, v)$ be variable. Then

$$\begin{aligned}[(A \otimes B)(C \otimes D)]_{x,y} &= \sum_z (A \otimes B)_{xz}(C \otimes D)_{zy} = \sum_{u,v} A_{iu}B_{jv}C_{uk}D_{vs} \\ &= (AC)_{ik}(BD)_{js} = [(AC) \otimes (BD)]_{x,y}.\end{aligned}$$

$\square$

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Corollary

If square matrix A has λ-eigenvector u and B has η-eigenvector v, then $A \otimes B$ has $\lambda\eta$-eigenvector $u \otimes v$.

Proof.

By the previous Lemma,

$$(A \otimes B)(u \otimes v) = Au \otimes Bv = \lambda u \otimes \eta v = \lambda\eta(u \otimes v).$$

$\square$

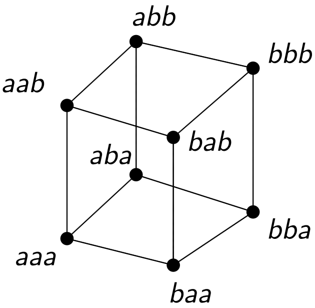
Hamming graphs $H(D, q)$

Hamming graphs $H(D, q)$

- Let X be a finite set of q elements, $V = X^D$, and

$$E = \{uv \mid u, v \in V \text{ differ in exact one cordinate}\}.$$

- $\Gamma = (V, E)$ is called the **Hamming graph** $H(D, q)$.
- Note that $H(1, q) = K_q$, $H(2, 2)$ is a square and $H(3, 2)$ is a cube.



$X = \{a, b\}$, $D = 3$,

$H(3, 2)$ is the cube.

Remarks

- For vertices x and y in $H(D,q)$, $\partial(x,y) = i$ if and only if x and y differ in exactly i coordinates.
- Hence $H(D,q)$ is $D(q-1)$-regular with order q^D and diameter D.

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Adjacency matrix $A(D, q)$ of $H(D, q)$

Adjacency matrix of $H(D, q)$

The adjacency matrix $A = A(D, q) = (a_{uv})$ of $H(D, q)$ satisfies

$$a_{uv} = \begin{cases} 1, & u, v \text{ differ in exact one cordinate;} \\ 0, & \text{otherwise} \end{cases} \qquad \left(u, v \in X^D\right).$$

Lemma

Let $A(D,q)$ denote the adjacency matrix of Hamming graph $H(D,q)$. Then

$$
\begin{aligned}
A(1,q) &= J - I \qquad (q \times q \textbf{ matrices}),\\
A(D,q) &= \sum_{i=1}^{D} \underbrace{I \otimes \cdots \otimes I}_{i-1 \textbf{ times}} \otimes A(1,q) \otimes \underbrace{I \otimes \cdots \otimes I}_{D-i \textbf{ times}} \qquad (q^D \times q^D \textbf{ matrices})\\
&= A(D-1,q) \otimes I + I^{\otimes D-1} \otimes A(1,q).
\end{aligned}
$$

Proof.

The first equation follows from $H(1,q) = K_q$. The second equation holds since both matrices coincide at each position uv for $u, v \in X^D$. The third equation follows from the second. □

The spectrum of $H(D, q)$

Spectrum and algebraic Diameter

- Sometimes, we use $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ to denote the eigenvalues of a graph Γ of order n.
- Sometimes, we use $\theta_0 = \lambda_1, \theta_1, \ldots, \theta_d$ to denote the **distinct** eigenvalues of Γ.
- The multiset $SP(\Gamma) = \{m_0 \cdot \theta_0, m_1 \cdot \theta_1, \ldots, m_d \cdot \theta_d\}$ is called the **spectrum** of Γ, where m_i is the multiplicity of θ_i.
- Note that $d \geq D$, since G has at least $D + 1$ distinct eigenvalues, where D is the diameter of Γ.
- The number of d is called the **algebraic diameter** of Γ.

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Proposition

The Hamming graph $H(D, q)$ has $D+1$ distinct eigenvalues

$$\theta_i(D, q) = D(q-1) - qi$$

with multiplicities

$$m_i(D, q) = \binom{D}{i}(q-1)^i$$

for $0 \le i \le D$.

Proof. We prove by induction on D. For $D = 1$, $H(1, q) = K_q$ has eigenvalues $\theta(1, q)_0 = q - 1$ and $\theta(1, q)_1 = -1$ with multiplicities $m(1, q)_0 = 1$ and $m(1, q)_1 = q - 1$ respectively. Let u_0 be $\theta(1, q)_0$-eigenvector and $u_1, u_1, \ldots, u_{q-1}$ be the independent $\theta(1, q)_1$-eigenvector of $A(1)$. Assume the statement is true for $D - 1$. Let w_i be an $\theta_i(D-1, q)$-eigenvector of $A(D-1, q)$ for $0 \le i \le D-1$.

Continue the proof

Then

$$\begin{aligned}
A(D,q)(w_i \otimes u_0) =& (A(D-1,q) \otimes I_q + I_q^{\otimes D-1} \otimes A(1,q))(w_i \otimes u_0) \\
=& ((D-1)(q-1) - qi + q - 1)(w_i \otimes u_0) \\
=& (D(q-1) - qi)(w_i \otimes u_0) \quad (0 \le i \le D-1), \\
A(D)(w_{i-1} \otimes u_1) =& (A(D-1) \otimes I_q + I_q^{\otimes D-1} \otimes A(1))(w_{i-1} \otimes u_1) \\
=& ((D-1)(q-1) - q(i-1) - 1)(w_{i-1} \otimes u_1) \\
=& (D(q-1) - qi)(w_{i-1} \otimes u_1) \quad (1 \le i \le D).
\end{aligned}$$

This prove $\theta_i(D) = D(q-1) - qi$ is an eigenvalue of $A(D,q)$ with multiplicity $m(D,q)_i \ge m(D-1,q)_i + m(D-1,q)_{i-1}(q-1) = \binom{D}{i}(q-1)^i$ by induction. Since $\sum_{i=0}^{D} \binom{D}{i}(q-1)^i = (1+q-1)^D = q^D$, we have indeed $m(D)_i = \binom{D}{i}(q-1)^i$. $\square$

Corollary

The Hamming graph $H(n,2)$ has $n+1$ distinct eigenvalues $\theta_i = n-2i$ with multiplicities $m_i = \binom{n}{i}$ for $0 \le i \le n$. $\square$

Remark

The spectrum

$$
\begin{aligned}
&SP(H(n,2)) \\
=&\left\{\begin{pmatrix} n \\ 0 \end{pmatrix} \cdot n, \begin{pmatrix} n \\ 1 \end{pmatrix} \cdot (n-2), \ldots, \begin{pmatrix} n \\ n-1 \end{pmatrix} \cdot (2-n), \begin{pmatrix} n \\ n \end{pmatrix} \cdot (-n)\right\}
\end{aligned}
$$

of $H(n,2)$ is symmetric to 0. The adjacency matrix $A(n,2)$ has a zero eigenvalue if and only if n is even.