Spectral Excess Theorem

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Outline

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Motivation

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Graphs and Combinatorics (1999) 15 : 195–209

Graphs and Combinatorics

Spectral Characterization of Some Generalized Odd Graphs

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Abstract. Suppose G is a connected, k-regular graph such that $Spec(G) = Spec(\Gamma)$ where Γ is a distance-regular graph of diameter d with parameters $a_1 = a_2 = \cdots = a_{d-1} = 0$ and $a_d > 0$; *i.e.*, a generalized odd graph, we show that G must be distance-regular with the same intersection array as that of Γ in terms of the notion of Hoffman Polynomials. Furthermore, G is isomorphic to Γ if Γ is one of the odd polygon C_{2d+1}, the Odd graph O_{d+1}, the folded $(2d+1)$-cube, the coset graph of binary Golay code $(d = 3)$, the Hoffman-Singleton graph $(d = 2)$, the Gewirtz graph $(d = 2)$, the Higman-Sims graph $(d = 2)$, or the second subconstituent of the Higman-Sims graph $(d = 2)$.

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An unpublished result

Partially Distance-regular Graphs
and
Partially Walk-regular Graphs*

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Main Theorem

Main Theorem (Guang-Siang Lee, — (2012))

Any connected graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ must be distance-regular.

Recall notations

Notations throughout this ppt

- Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph on n vertices, with **diameter** D, **adjacency matrix** A, and **distance function** ∂.
- Assume that A has $d+1$ distinct eigenvalues $\theta_0 > \theta_1 > \ldots > \theta_d$ with corresponding multiplicities $1 = m_0$, m_1, ..., m_d.
- The **spectrum** of Γ will be denoted by the multi-set

$$\text{sp }(\Gamma) = \{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}.$$

- The parameter d is called the **spectral diameter** of Γ.
- Let $Z(x)$ denote the **minimal polynomial** of A.

Remark

$$Z(x) = \prod_{i=0}^{d} (x - \theta_i).$$

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Lemma

$D \leq d$.

Proof.

Note that

$$A^k_{xy} = \sum_{z_i \in V\Gamma} A_{xz_1} A_{z_1 z_2} \cdots A_{z_{k-1} y}$$

counts the number of x, y-walks of length k. In particular,

$$A^k_{xy} \begin{cases} \neq 0, & \text{if } k = \partial(x, y); \\ = 0, & \text{if } k > \partial(x, y). \end{cases}$$

On the contrary, suppose $d < D$, and pick vertices x, y with $\partial(x, y) = d + 1$. Then $0 = Z(A)_{xy} = A^{d+1}_{xy} \neq 0$, a contradiction. $\square$

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Orthogonal polynomials

Inner product matrix space

For two $n \times n$ symmetric matrices M, N over $\mathbb{R}$, define the **inner product**

$$\langle M, N\rangle := \frac{1}{n}\sum_{i,j}(M \odot N)_{ij} = \frac{1}{n}\sum_{i,j} M_{ij}N_{ij} = \frac{1}{n}\mathrm{tr}(MN), \tag{1}$$

and the norm

$$\|M\| = \sqrt{\langle M, M\rangle},$$

where " $\odot$ " is the entrywise or **Hadamard product** of matrices.

Inner product polynomial space

Let

$$\mathbb{R}_d[x] = \mathbb{R}[x]/Z(x)$$

denote the $(d+1)$-dimensional vector space consisting of polynomials of degrees at most d over $\mathbb{R}$. $\mathbb{R}_d[x]$ is also an algebra under the multiplication $(\mathrm{mod}\ Z)(x)$.

Let $\langle \cdot, \cdot \rangle$ be the **inner product** on $\mathbb{R}_d[x]$ defined by

$$\langle p(x), q(x) \rangle := \langle p(A), q(A) \rangle = \frac{1}{n}\mathrm{tr}(p(A)q(A)) = \frac{1}{n}\sum_{i=0}^{d} m_i p(\theta_i) q(\theta_i),$$

and **norm** defined as usual by

$$\|p(x)\| = \sqrt{\langle p(x), p(x) \rangle} \qquad (p(x), q(x) \in \mathbb{R}_d[x]).$$

Remark

- $1, x, \ldots, x^d$ is a basis of $\mathbb{R}_d[x]$.
- For $p(x) \in \mathbb{R}[x]/Z(x)$,

$$\langle p(x), p(x)\rangle = \frac{1}{n}\sum_{i=0}^{d} m_i p(\theta_i)^2 = 0 \quad \Leftrightarrow \quad p(x) = 0.$$

- $$\langle xp(x), q(x)\rangle = \langle p(x), xq(x)\rangle \qquad (p(x), q(x) \in \mathbb{R}_d[x]).$$

Gram-Schmidt process

- The **projection** of $q(x)$ into $p(x)$ is defined by

$$\mathrm{Proj}_{p(x)}(q(x)) := \frac{\langle p(x), q(x)\rangle}{\|p(x)\|^2} p(x). \tag{2}$$

- Set $p'_0(x) = 1$ and

$$p'_{i+1}(x) = x^{i+1} - \sum_{k=0}^{i} \mathrm{Proj}_{p'_k(x)}(x^{i+1}) \tag{3}$$

for $0 \le i \le d-1$ recursively.
- Then $p'_0(x), p'_1(x), \ldots, p'_d(x)$ is an **orthogonal basis** of $\mathbb{R}_d[x]$ such that $p'_i(x)$ has degree i and leading coefficient 1.

Lemma

The monic polynomial $p'_i(x)$ has i distinct roots in the open interval (θ_d, θ_0). In particular, $p'_i(\theta_0) > 0$ for $0 \le i \le d$.

Proof.

- Let η_1, η_2, ..., η_h be zeros of $p'_i(x)$ in (θ_d, θ_0) for which $p'_i(x)$ takes opposite signs in left and right of η_j, where $j \le h \le i$.
- Set

$$g(x) = \prod_{j=1}^{h}(x - \eta_j) = \sum_{j \le h} c_j p'_j(x)$$

for some $c_j \in \mathbb{R}$ with $c_h = 1$, since $g(x)$ and $p'_h(x)$ are both monic.
- Then $g(x)p'_i(x) \ge 0$ or $g(x)p'_i(x) \le 0$ for all $x \in [\theta_d, \theta_0]$.
- Since $g(x)p'_i(x)$ has at most $i < d + 1 = \deg(Z(x))$ distinct zeros (those in $p'_i(x)$), there exists an eigenvalue θ_k for some $0 \le k \le d$ such that $g(\theta_k)p'_i(\theta_k) \ne 0$.

Continue the proof

- Hence

$$0 \neq \frac{1}{n}\sum_{k=0}^{d} m_j g(\theta_k) p_i'(\theta_k) = \langle g(x), p_i'(x)\rangle$$
$$= \left\langle \sum_{j\le h} c_j p_j'(x), p_i'(x) \right\rangle = \langle c_h p_h'(x), p_i'(x)\rangle,$$

  and $h = i$
- Thus $g(x) = p_i'(x)$.
- Therefore $p_i'(\theta_0) = g(\theta_0) > 0$.

$\square$

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Predistance polynomials

- Set
$$p_i(x) = \frac{p_i'(\theta_0)}{\|p_i'(x)\|^2} p_i'(x). \tag{4}$$
- Then $p_0(x), p_1(x), \ldots, p_d(x)$ is the unique system of orthogonal polynomials in $\mathbb{R}_d[x]$ satisfying
$$\deg\ p_i(x) = i$$
and
$$\|p_i(x)\|^2 = \frac{p_i'(\theta_0)^2}{\|p_i'(x)\|^2} = p_i(\theta_0) > 0.$$
for $0 \le i \le d$.
- The $p_i(x)$ is referred to as the i-th **predistance polynomial** of Γ.

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Spectral excess

$$p_{\geq D}(\theta_0) := p_D(\theta_0) + p_{D+1}(\theta_0) + \cdots + p_d(\theta_0)$$

is called the **spectral excess** of Γ.

Example

Let $\Gamma = P_3$ be a path of three vertices. Then

- $D = 2$;
- $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$;
- $\mathrm{sp}(\Gamma) = \{\sqrt{2}, 0, -\sqrt{2}\}$;
- $d = 2$;
- By (3), $p'_0(x) = 1$, $p'_1(x) = x$, $p'_2(x) = x^2 - 4/3$;
- By (4), $p_0(x) = 1$, $p_1(x) = 3\sqrt{2}x/4$, $p_2(x) = 3(x^2 - 4/3)/4$;
- The spectral excess of Γ is $p_2(\theta_0) = 1/2$.

Lemma (Three-term relations)

$$xp_i(x) = c_{i+1}p_{i+1}(x) + a_ip_i(x) + b_{i-1}p_{i-1}(x) \qquad 0 \le i \le d \tag{5}$$

for some scalars c_{i+1}, a_i, $b_{i-1} \in \mathbb{R}$ with $b_{-1} = c_{d+1} := 0$.

Proof.

Since $xp_i(x)$ has degree $i+1$, write $xp_i(x) = \sum_{j=0}^{i+1} \alpha_{ij}p_j(x)$ for some $\alpha_{ij} \in \mathbb{R}$. If $j < i-1$ then

$$\begin{aligned} \alpha_{ij}\langle p_j(x), p_j(x)\rangle &= \left\langle \sum_{k=0}^{i+1} \alpha_{ik}p_k(x), p_j(x) \right\rangle = \langle xp_i(x), p_j(x)\rangle \\ &= \langle p_i(x), xp_j(x)\rangle = 0, \end{aligned}$$

so $\alpha_{ij} = 0$. This proves $xp_i(x) = \sum_{j=i-1}^{i+1} \alpha_{ij}p_j(x)$, as desired. $\square$

Hoffman polynomial

Hoffman polynomial

The polynomial

$$H(x) := n \prod_{i=1}^{d} \frac{x - \theta_i}{\theta_0 - \theta_i} = \frac{nZ(x)}{(x - \theta_0) \prod_{i=1}^{d} \theta_0 - \theta_i}$$

is called the **Hoffman polynomial** of Γ.

Perron vector

Let α be the eigenvector of A corresponding to θ_0 such that $\alpha^t\alpha = n$ and all entries of α are positive. The vector α is referred to as the **Perron vector** of A.

Remark

$\alpha = (1, 1, \ldots, 1)^t$ iff Γ is regular.

Lemma

$$H(A) = \frac{n\alpha\alpha^t}{\alpha^t\alpha} = \alpha\alpha^t.$$

Moreover, Γ is regular iff $H(A) = J$, the all 1's matrix.

Proof.

- Let u_j be the θ_j-eigenvector of A and $u_0 = \alpha$. Then

$$H(A)u_j = n\prod_{i=1}^{d} \frac{A-\theta_i}{\theta_0-\theta_i} u_j = \delta_{j0}u_0 = \frac{n\alpha\alpha^t}{\alpha^t\alpha} u_j$$

  and the first equality follows.
- The second equality follows from the assumption $\alpha^t\alpha = n$. The remaining is clear.

$\square$

The polynomial $q_i(x)$

Let

$$q_i(x) = \sum_{j=0}^{i} p_j(x)$$

be the sum of the first i predistance polynomials.

Remark

- $q_i(x)$ has degree i and $q_0(x), q_1(x), \ldots, q_d(x)$ is a basis of $\mathbb{R}_d[x]$.
- $$\|q_i(x)\|^2 = \sum_{j=0}^{i} \|p_j(x)\|^2 = \sum_{j=0}^{i} p_j(\theta_0) = q_i(\theta_0).$$

Lemma (An optimization problem)

For $p(x) \in \mathbb{R}_d[x]$ with degree at most i and $\|p(x)\| = \|q_i(x)\|$, we have $|p(\theta_0)| \leq |q_i(\theta_0)| = \|p(x)\|^2$ with equality iff $p(x) = \pm q_i(x)$.

Proof.

Let $p(x) = \sum_{j=0}^{i} \alpha_j p_j(x)$ for some $\alpha_j \in \mathbb{R}$. As

$q_i(\theta_0) = \|q_i(x)\|^2 = \|p(x)\|^2 = \sum_{j=0}^{i} \alpha_j^2 p_j(\theta_0)$, and by Cauchy's inequality,

$$p(\theta_0)^2 = \left[\sum_{j=0}^{i} \alpha_j p_j(\theta_0)\right]^2 \leq \left[\sum_{j=0}^{i} \alpha_j^2 p_j(\theta_0)\right] \left[\sum_{j=0}^{i} p_j(\theta_0)\right] = q_i(\theta_0)^2,$$

with equality iff all α_j are equal; indeed $\alpha_j = \pm 1$. □

Lemma (The dual problem)

For $p(x) \in \mathbb{R}_d[x]$ with degree at most i and $\|p(x)\| = \|q_i(x)\|$, we have $\sum_{j=1}^{d} m_j q_i(\theta_j)^2 \leq \sum_{j=1}^{d} m_j p(\theta_j)^2$ with equality iff $p(x) = \pm q_i(x)$.

Proof.

This follows from the previous lemma and

$$\frac{1}{n}(p(\theta_0)^2+\sum_{j=1}^{d} m_j p(\theta_j)^2) = \|p(x)\|^2 = \|q_i(x)\|^2 = \frac{1}{n}(q_i(\theta_0)^2+\sum_{j=1}^{d} m_j q_i(\theta_j)^2).$$

□

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Proposition

$$H(x) = q_d(x) = p_0(x) + p_1(x) + \cdots + p_d(x). \tag{6}$$

Proof.

- Let $p(x) = c\prod_{i=1}^{d} \frac{x-\theta_i}{\theta_0-\theta_i}$ for some $c > 0$ such that $\|p(x)\| = \|q_d(x)\|$.
- By dual problem lemma, $\sum_{j=1}^{d} m_j q_d(\theta_j)^2 \leq \sum_{j=1}^{d} m_j p(\theta_j)^2 = 0$.
- Then $\sum_{j=1}^{d} m_j q_d(\theta_j)^2 = 0$ and thus $q_d(x) = \pm p(x)$.
- Hence $q_d(\theta_0) = \|q_d(x)\|^2 = (q_d(\theta_0)^2 + \sum_{j=1}^{d} m_j q_d(\theta_j)^2)/n = q_d(\theta_0)^2/n$.
- Therefore, $q_d(\theta_0) = n = c$, and $q_d(x) = n\prod_{i=1}^{d} \frac{x-\theta_i}{\theta_0-\theta_i} = H(x)$.

□

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The spectral excess theorem

Recall the distance matrix

The $n \times n$ matrix A_i with rows and columns indexed by the vertex set $V\Gamma$ such that

$$(A_i)_{uv} = \begin{cases} 1, & \text{if } \partial(u,v) = i; \\ 0, & \text{else.} \end{cases}$$

is called the i-th **distance matrix** of Γ.

Remark

$A_0 = I$ and $A_1 = A$.

Weighted distance matrices

Let α denote the Perron vector of A. Define

$$\widetilde{A}_i := A_i \odot H(A) = A_i \odot (\alpha\alpha^t).$$

The matrix $\widetilde{A}_i$ is referred as the i-th **weighted distance matrix** of Γ.

Remark

- $$(\widetilde{A}_i)_{uv} = \begin{cases} \alpha_u\alpha_v, & \text{if } \partial(u,v) = i; \\ 0, & \text{else.} \end{cases} \tag{7}$$
- $$\widetilde{A}_0 + \widetilde{A}_1 + \cdots + \widetilde{A}_D = H(A) = p_0(A) + p_1(A) + \cdots + p_d(A). \tag{8}$$

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Excess

Define

$$\delta_i := \|\widetilde{A}_i\|^2 = \frac{1}{n} \sum_{u,v \in V\Gamma, \partial(u,v)=i} \alpha_u^2 \alpha_v^2.$$

The number δ_D is referred to as the **excess** of Γ.

Lemma

$$\langle \widetilde{A}_i, \widetilde{A}_j \rangle = 0 \qquad \text{if } j \neq i, \tag{9}$$

$$\langle \widetilde{A}_i, p_j(A) \rangle = 0 \qquad \text{if } j < i. \tag{10}$$

Proof.

- (9) is immediate from the definition of Weighted distance matrices.
- (10) is immediate from the definition of inner product of matrices in (1), since $(\widetilde{A}_i)_{uv} \neq 0$ occurs only when $\partial(u, v) = i$, but in this situation a u, v-walk can not have length t for any $t \leq j < i$, so $(p_j(A))_{uv} = 0$.

□

Lemma

Let $p_{\geq D}(x) = \sum_{i=D}^{d} p_i(x)$. Then the projection of $\widetilde{A}_D$ into $p_{\geq D}(A)$ is

$$\mathbf{Proj}_{p_{\geq D}(A)} \widetilde{A}_D = \frac{\delta_D}{p_{\geq D}(\theta_0)} \, p_{\geq D}(A).$$

Proof.

$$\begin{aligned}
\mathrm{Proj}_{p_{\geq D}(A)} \widetilde{A}_D &= \frac{\langle \widetilde{A}_D, \sum_{i=0}^{d} p_i(A) \rangle}{\| p_{\geq D}(A) \|^2} \, p_{\geq D}(A) \qquad \text{(by (2), (10))}, \\
&= \frac{\langle \widetilde{A}_D, H(A) \rangle}{p_{\geq D}(\theta_0)} \, p_{\geq D}(A) = \frac{\langle \widetilde{A}_D, \sum_{i=0}^{D} \widetilde{A}_i \rangle}{p_{\geq D}(\theta_0)} \, p_{\geq D}(A) \qquad \text{(by (6), (8))} \\
&= \frac{\delta_D}{p_{\geq D}(\theta_0)} \, p_{\geq D}(A) \qquad \text{(by (9))}.
\end{aligned}$$

□

Theorem (Spectral Excess Eheorem)

Let Γ be a connected graph with diameter D. Then $\delta_D \le p_{\ge D}(\theta_0)$. Moreover, $\delta_D = p_{\ge D}(\theta_0)$ if and only if $\widetilde{A}_D = p_{\ge D}(A)$.

Proof.

By previous lemma,

$$0 \le \|\widetilde{A}_D\|^2 - \|\mathrm{Proj}_{p_{\ge D}(A)}\widetilde{A}_D\|^2 = \delta_D - \frac{\delta_D^2}{p_{\ge D}(\theta_0)}.$$

The equality is attained iff $\widetilde{A}_D = \mathrm{Proj}_{p_{\ge D}(A)}\widetilde{A}_D = p_{\ge D}(A)$. $\square$

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Example

Revisiting the case that $\Gamma = P_3$ is a path of three vertices,

- $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$;
- $\mathrm{sp}(\Gamma) = \{\sqrt{2}, 0, -\sqrt{2}\}$, $d = D = 2$, and $p_{\geq D}(\theta_0) = p_2(\theta_0) = 1/2$ is the spectral excess;
- The Perron vector $\alpha = (\sqrt{3}/2, \sqrt{6}/2, \sqrt{3}/2)^t$;
- $$\widetilde{A}_D = \begin{pmatrix} 0 & 0 & 3/4 \\ 0 & 0 & 0 \\ 3/4 & 0 & 0 \end{pmatrix}.$$
- $\delta_D = 3/8 \leq 1/2 = p_{\geq D}(\theta_0)$.

The excess is at most the spectral excess.

Remark

If Γ is regular with diameter $D = 2$, then the equality in Spectral Excess Theorem holds. Indeed

$$\widetilde{A}_2 = A_2 = J - I - A = H(A) - I - A = p_{\geq 2}(A).$$

$$D = d$$

Lemma

$\widetilde{A}_0 = p_0(A) = I$ iff Γ is regular.

Proof.

- Since $p_0(x) = 1$, $p_0(A) = I$ is always true. .
- The Perron vector of Γ is $\alpha^t = (1, 1, \dots, 1)$ iff Γ is regular.
- The 0-th weighted distance matrix $\widetilde{A}_0$ is a diagonal matrix.
- From (7), $(\widetilde{A}_0)_{uu} = \alpha_u^2$ for $u \in V\Gamma$.

$\square$

The above simple lemma plays a key role in proving the regularity of a graph.

Theorem (Characterization Theorem of DRG)

If $D = d$, then $\widetilde{A}_D = p_D(A)$ iff $\widetilde{A}_i = p_i(A)$ for $0 \le i \le D-1$. Moreover, if $\widetilde{A}_D = p_D(A)$ then Γ is distance-regular.

Proof. ($\Leftarrow$) Delete $\widetilde{A}_i = p_i(A)$ for $0 \le i \le D-1$ in both sides of

$$\widetilde{A}_0 + \widetilde{A}_1 + \cdots + \widetilde{A}_D = H(A) = p_0(A) + p_1(A) + \cdots + p_D(A). \tag{11}$$

($\Rightarrow$) We use (backward) induction on $0 \le i \le D$. The base case is the assumption that $\widetilde{A}_D = p_D(A)$. Suppose now that $p_k(A) = \widetilde{A}_k$ for $D \ge k \ge i$. It remains to show that $p_{i-1}(A) = \widetilde{A}_{i-1}$. Then deleting these common terms from both sides of (11), we have

$$\widetilde{A}_0 + \widetilde{A}_1 + \cdots + \widetilde{A}_{i-1} = p_0(A) + p_1(A) + \cdots + p_{i-1}(A). \tag{12}$$

In particular, for $\partial(u, v) \ge i-1$, $(p_{i-1}(A))_{uv} = (\widetilde{A}_{i-1})_{uv}$ by (12).

Continue the proof

Applying the three-term recurrence in (5),

$$\begin{aligned} A\widetilde{A}_i =& c_{i+1}p_{i+1}(A) + a_i p_i(A) + b_{i-1}p_{i-1}(A) \\ =& c_{i+1}\widetilde{A}_{i+1} + a_i\widetilde{A}_i + b_{i-1}p_{i-1}(A). \end{aligned} \tag{13}$$

Hence for $\partial(u,v) < i-1$, $(A\widetilde{A}_i)_{uv} = \sum_{w\in\Gamma(u)}(\widetilde{A}_i)_{wv} = 0$, where the last equality follows since $\partial(w,v) \leq 1 + \partial(u,v) < i$. Then $(p_{i-1}(A))_{uv} = 0$ by (13) and since $b_{i-1} \neq 0$. This proves the necessity.

Suppose $\widetilde{A}_D = p_D(A)$. Then $\widetilde{A}_0 = p_0(A) = I$, and Γ is regular by previous lemma. Thus $A_i = \widetilde{A}_i$ for all $0 \leq i \leq D$. Hence (13) becomes

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{u-1}A_{i-1},$$

which is the defining equation of a distance-regular graph. $\square$

Corollary

A regular graph with $d = 2$ is distance-regular of diameter 2.

Proof.

Since $0 \leq D \leq d = 2$, and K_n is the only graph which has $0 \leq D = d \leq 1$, we have $D = 2$. Since Γ is regular with diameter $D = 2$, we have $\widetilde{A}_2 = p_{\geq 2}(A)$. By the above characterization theorem, Γ is DRG. □

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Graphs with odd-girth $2d + 1$

The odd girth of a graph

The **odd-girth** of a graph Γ is the length of a shortest odd cycle in Γ.

Remark

$$\sum_{j=0}^{d} m_j \theta_j^i = \mathrm{tr}(A^i) = \sum_{x \in V\Gamma} A^i_{xx} \geq 0,$$

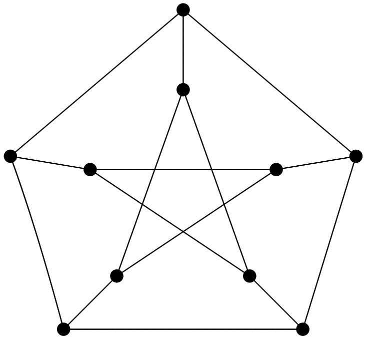
and the following (i)-(iii) are equivalent.

(i) $\mathrm{tr}(A^i) = 0$;

(ii) there is no closed walk of length i;

(iii) there is no cycle of length j and $j \leq i$ with the same parity.

The Petersen graph has odd girth 5



$$\Pi = (\{x\}, \Gamma_1(x), \Gamma_2(x))$$

$$\Pi(A) = \begin{pmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ 0 & c_2 & a_2 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$(\theta_0^{m_0}, \theta_1^{m_1}, \theta_2^{m_2}) = (3^1, 1^5, (-2)^4).$$

Question

Providing a graph Γ with eigenvalues $3, 1, -2$ and respective multiplicities $1, 5, 4$, can you recover the graph Γ to be Petersen graph.

Remark

From the spectrum $sp(\Gamma) = \{\theta_1^{m_1}, \theta_2^{m_2}, \theta_3^{m_3}\} = \{3^1, 1^5, (-2)^4\}$, we know $D \leq d = 2$ and

$$\mathrm{tr}(A^i) = \sum_{j=0}^{d} m_j \theta_j^i \begin{cases} = 0, & \text{if } i < 2d+1 \text{ is odd;} \\ \neq 0, & \text{if } i = 2d+1 \end{cases}$$

to conclude that Γ has odd-girth $2d+1$, and use this to further conclude that $D = d = 2$.

Goal

From now on, let Γ be a graph with an additional assumption that Γ has **odd-girth** $2d+1$. This will implies $D = d$.

Our goal is to show that Γ must be distance-regular.

The leading coefficient c of $H(x)$

Let $c = n / \prod_{i=1}^{d}(\theta_0 - \theta_i)$ be the leading coefficient of the Hoffman polynomial

$$H(x) = n \prod_{i=1}^{d} \frac{x - \theta_i}{\theta_0 - \theta_i} = cx^d + \cdots .$$

Then for two vertices $u, v \in V\Gamma$ with $\partial(u, v) = d$,

$$(A^d)_{uv} = H(A)_{uv}/c. \tag{14}$$

Lemma

$$(A^{d+1})_{uv} = \left(\sum_{i=0}^{d} \theta_i\right) H(A)_{uv}/c \qquad (\partial(u,v) = d). \tag{15}$$

Proof.

From

$$Z(x) = \prod_{i=0}^{d}(x - \theta_i) = x^{d+1} - x^d \sum_{i=0}^{d} \theta_i + \cdots ,$$

for two vertices $u, v \in V\Gamma$ with $\partial(u,v) = d$,

$$(A^{d+1})_{uv} = Z(A)_{uv} + \left(\sum_{i=0}^{d} \theta_i\right)(A^d)_{uv} = \left(\sum_{i=0}^{d} \theta_i\right) H(A)_{uv}/c.$$

□

Lemma

$$\delta_d = c^2 \mathbf{tr}(A^{2d+1}) / \left(n \sum_{i=0}^{d} \theta_i \right) > 0.$$

In particular, $D = d$.

Proof. For vertices $u, v \in V\Gamma$ with $\partial(u,v) < d$, we have $(A^d)_{uv} = 0$ or $(A^{d+1})_{vu} = 0$ since no odd cycle has length less than $2d+1$. By (1), (8), (14), (15),

$$\begin{aligned}
& n \left(\sum_{i=0}^{d} \theta_i \right) \delta_d = \left(\sum_{i=0}^{d} \theta_i \right) \sum_{u,v \in V\Gamma} [(\widetilde{A}_d)_{uv}]^2 \\
= & \left(\sum_{i=0}^{d} \theta_i \right) \sum_{u \in V\Gamma} \sum_{v \in \Gamma_d(u)} [H(A)_{uv}]^2 = c^2 \sum_{u \in V\Gamma} \sum_{v \in V\Gamma} (A^d)_{uv} (A^{d+1})_{uv} \\
= & c^2 \mathrm{tr}(A^{2d+1}) \neq 0.
\end{aligned}$$

Hence

$$\sum_{i=0}^{d} \theta_i \neq 0,$$

and

$$\delta_d = c^2 \mathrm{tr}(A^{2d+1})/(n \sum_{i=0}^{d} \theta_i) > 0.$$

Note that $\delta_d > 0$ implies $D \geq d$. Hence $D = d$. $\square$

Lemma

Referring the notations of three-term recurrence in (5),

(i) $a_{j-1} = 0$ **for** $1 \le j \le d$**;**

(ii) $p_j(x)$ **is an even or odd polynomial depending on whether** j **is even or odd for** $0 \le j \le d$**.**

Moreover,

$$p_{\ge D}(\theta_0) = c^2\mathbf{tr}(A^{2d+1}) / \left(n \sum_{i=0}^{d} \theta_i \right) = \delta_D.$$

Proof. Note that $d = D$ by previous lemma. Clearly, $p_0(x) = 1$ is even.

- We prove (i)-(ii) by induction on $j \ge 1$.
- By (4),

$$p_1(x) = \frac{n\theta_0 x}{\sum_{i=0}^{d} m_i \theta_i^2}$$

  is odd.
- Setting $i = 0$ in (5), $a_0 = 0$. Hence we have (i)-(ii) in the case $j = 1$.

Continue the proof

- By (5),

$$a_k p_k(\theta_0) = \langle a_k p_k(x), p_k(x)\rangle = \langle x p_k(x), p_k(x)\rangle = \mathrm{tr}(A p_k^2(A))/n \quad (16)$$

  for $0 \le k \le d$.
- Now suppose (i)-(ii) for $j = k < d$.
- Since $x p_k^2(x)$ is an odd polynomial of degree $2k+1 < 2d+1$, the last term in (16) is zero.
- Hence $a_k = 0$ and (i) holds for $j = k+1$. From (i) and setting $i = k$ in (5), the polynomial $p_{k+1}(x)$ satisfies (ii). This proves (i)-(ii) in any j.
- Equation (16) with $k = d$ is

$$a_d p_d(\theta_0) = \mathrm{tr}(A p_d^2(A))/n = c^2 \mathrm{tr}(A^{2d+1})/n. \quad (17)$$

Continue the proof

- To prove
$$p_d(\theta_0) = c^2 \mathrm{tr}(A^{2d+1}) / \left(n \sum_{i=0}^{d} \theta_i \right),$$
it suffices to show $a_d = \sum_{i=0}^{d} \theta_i$.
- For two vertices u and v at distance d, we have
$$a_d H(A)_{uv} = a_d p_d(A)_{uv} = (A p_d(A))_{uv} = c(A^{d+1})_{uv} = \left(\sum_{i=0}^{d} \theta_i \right) H(A)_{uv},$$
where the third equality follows because $x p_d(x)$ has no term of degree d by (5), (6), (15).
- Dividing both sides by $H(A)_{uv}$, we have $a_d = \sum_{i=0}^{d} \theta_i$.

$\square$

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Theorem (Odd Girth Theorem)

Any connected graph with $d+1$ distinct eigenvalues and odd-girth $2d+1$ must be distance-regular.

Proof.

By previous two lemmas, we have $d = D$ and $p_{\geq D}(\theta_0) = \delta_D$. By Spectral Excess Theorem, we have $\widetilde{A}_D = p_{\geq D}(A)$. Hence Γ is distance-regular by the characterization Theorem. □