Matrices associated to a graph

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2024 Fall

Outline

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Adjacency, Laplace, signless Laplace matrices

Let Γ be a graph with multiple edges and without loops. Let $V\Gamma$ and $E\Gamma$ denote the vertex set and edge set respectively of Γ. All the matrices here have rows and columns indexed by $V\Gamma$.

- The diagonal matrix D with diagonals $D_{xx} = d_\Gamma(x)$ is called the **degree matrix** of Γ, where $d_\Gamma(x)$ is the degree of $x \in V\Gamma$.
- The matrix A with xy-entry $A_{xy} = m_{xy} \quad (x, y \in V(\Gamma))$ is called the **adjacency matrix** of A, where m_{xy} is the number of edges incident on vertices x and y.
- The matrix $L := D - A$ is called the **Laplace matrix** of Γ.
- The matrix $Q := D + A$ is called the **signless Laplace matrix** of Γ.

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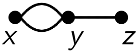
Example



$$A = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, L = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}, Q = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

There might have different ways to present these matrices according to different orders of the vertices, but usually the properties (e.g. eigenvalues, characteristic polynomials) we will investigate on these matrices are invariant of the order.

Other matrices associated with a graph

$$
\begin{aligned}
\mathcal{D} &= \{\partial(x,y)\}_{x,y\in V\Gamma} && (\textbf{distance matrix})\\
\widetilde{L} &= I - D^{-1/2}AD^{-1/2} && (\textbf{normalized Laplacian})\\
\Omega &= \{\omega_{xy}\}_{x,y\in V\Gamma} && (\textbf{effective resistance matrix})\\
A_\alpha &= \alpha D + (1-\alpha)A && (A_\alpha\textbf{- matrix}) \quad (\alpha\in[0,1])\\
&\vdots
\end{aligned}
$$

(Less involved in this lecture)

Remark

$$Q = 2A_{\frac{1}{2}}$$

V. Nikiforov, Merging the A-and Q-spectral theories, *Appl. Anal. Discrete Math.* 11 (2017) 81-107.

An important property for adjacent matrix

Proposition

A^i_{xy} is the number of walks of length i from x to y.

Proof.

$$A^i_{xy} = \sum_{x_1,x_2,\ldots,x_{i-1}\in V\Gamma} A_{xx_1}A_{x_1x_2}\cdots A_{x_{i-2}x_{i-1}}A_{x_{i-1}y} = \sum_{\substack{x_1,x_2,\ldots,x_{i-1}\in V\Gamma\\ x\sim x_1\sim x_2\sim\cdots\sim x_{i-1}\sim y}} 1.$$

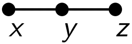
□

Signless Laplace matrix and incident matrix

Incident matrix

The **incidence matrix** M of Γ is a matrix with entries in $V\Gamma \times E\Gamma$ such that

$$M_{xe} = \begin{cases} 1, & x \in e; \\ 0, & x \notin e. \end{cases}$$



$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Lemma

If Γ is a graph with signless Laplace matrix Q and incident matrix M then

$$Q = MM^T.$$

Proof.

$$(MM^T)_{xy} = \sum_e M_{xe}(M^T)_{ey} = \begin{cases} d_x, & \text{if } x = y; \\ A_{xy}, & \text{if } x \neq y \end{cases} = Q_{xy}.$$

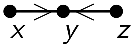
□

Laplace matrix and graph orientation

Graph orientation

An **orientation** σ of a graph Γ is an assignment of each edge of Γ a direction to form a digraph Γ^σ. Let N denote the (vertex-arc) **incidence matrix** of Γ^σ, i.e. N is a matrix with rows indexed by vertices and columns indexed by arcs such that

$$N_{xe} = \begin{cases} 1, & x \text{ is the tail of } e; \\ -1 & x \text{ is the head of } e; \\ 0, & \text{else.} \end{cases}$$



$$N = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad NN^T = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Lemma

If Γ is a graph with Laplace matrix L, then $L = NN^T$ for any incidence matrix N of Γ^σ, independent of the choice of orientation σ.

Proof.

Fix an orientation of Γ. Note that

$$(NN^T)_{xy} = \sum_e N_{xe}(N^T)_{ey} = \begin{cases} -A_{xy}, & x \neq y; \\ d_\Gamma(x), & x = y; \end{cases} = L_{xy}.$$

□

Positive semidefinite matrices

Positive semidefinite matrices

Recall from linear algebra a symmetry matrix P is **positive semidefinite** if $u^T P u \geq 0$ for all column vectors u.

Remark

Since

$$u^T |L| u = u^T M M^T u = u^T M (u^T M)^T \geq 0$$

and

$$u^T L u = u^T N N^T u = (u^T N)(u^T N)^T \geq 0,$$

both Q and L are positive semidefinite.

Theorem

If M is a real symmetric matrix, then the following are equivalent.

(i) **M is positive semidefinite.**

(ii) **All eigenvalues of $M[\alpha|\alpha]$ are nonnegative for all $\alpha \subseteq [n]$.**

(iii) **All eigenvalues of M are nonnegative.**

(iv) **$M = PP^T$ for some matrix P.**

Proof.

(i)⇒(ii) Let x be an eigenvector of $M[\alpha|\alpha]$ associated with an eigenvalue λ. Then $\lambda x^T x = x^T M[\alpha|\alpha]x = y^T My \geq 0$, where y is the vector from x by filling 0's.

(ii)⇒(iii) Take $\alpha = [n]$.

(iii)⇒(iv) Since M is symmetric, $M = UDU^T$ for some diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where λ_i are eigenvalues of M and $\lambda_i \geq 0$ by (iii). Take $P = U\mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_n})$ to have $M = PP^T$.

(iv)⇒(i) By (iv) $x^T Mx = x^T PP^T x = (x^T P)(x^T P)^T \geq 0$. □

Summary

1. The diagonal entries of an adjacency matrix are all zero.
2. The sum of entries in each row of a Laplace matrix is 0.
3. The eigenvalues of a Laplace matrix are nonnegative with one being zero.
4. The diagonal entry in a row of a signless Laplace matrix is the sum of the remaining entries in the row.
5. The adjacency matrix can be defined in a directed graph by $A_{xy} = 1$ iff xy is an arc.

The adjacency matrix of a directed triangle is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

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The spectrum of a graph

Recall from linear algebra

1. Let M be an $n \times n$ matrix over $\mathbb{R}$. $\lambda \in \mathbb{C}$ is an **eigenvalue** of M if there exists a nonzero column vector u such that $Mu = \lambda u$. The column vector u is called an **eigenvector** of M associated with λ.
2. Eigenvalues are the zeros of the characteristic polynomial $p_M(t) := \det(tI - M)$ of M and vice versa.
3. If M is symmetric then M has n orthogonal eigenvectors over $\mathbb{R}$. Hence $p_M(t) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n)$, where $\lambda_i \in \mathbb{R}$ are eigenvalues of M.
4. If M is symmetric with k **distinct** eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$ over $\mathbb{R}$, then the minimal polynomial of M is $(t - \mu_1)(t - \mu_2) \cdots (t - \mu_k)$.

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Example

Find the minimal polynomials of

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, M_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Solution. M_1 has minimal polynomial $(t-1)(t-2)$, M_2 has minimal polynomial $(t-1)^2(t-2)$, and M_3 has minimal polynomial $(t-1)^2(t-2)^3$. □

Spectrum of a graph

The eigenvalues (resp. Laplace eigenvalues) of a graph are the eigenvalues of its adjacency matrix (resp. Laplace matrix). The sequence of eigenvalues of a graph Γ is called the **spectrum** of Γ. Usually we assume the graph is undirected to ensure the existence of n eigenvalues with n corresponding orthogonal eigenvectors, where n is the number of vertices.

Example

Let $\Gamma = P_3$ be a path with three vertices. Find the spectrum and Laplace spectrum of Γ.

Solution.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \pm 2 \\ \sqrt{2} \end{pmatrix} = \pm\sqrt{2} \begin{pmatrix} \sqrt{2} \\ \pm 2 \\ \sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0, \quad \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \qquad SP(A) = (\sqrt{2}, 0, -\sqrt{2}) \quad \text{and} \quad SP(L) = (3, 1, 0). \qquad \square$$

Example

Determine all eigenvalues of

$$J_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Solution.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since J has rank 1, it has three eigenvalues being zero. □

Example

Let $\Gamma = K_n$ be the complete graph with n vertices, i.e. each pair of vertices has an edge. Find the spectrum, Laplace spectrum and signless Laplace spectrum of K_n.

Solution. From $A = J_n - I_n$, $L = (n-1)I - A$, $Q = (n-1)I + A$,

$$\begin{aligned} SP(A) =& (n, 0, 0, \ldots, 0) - (1, 1, \ldots, 1) = (n-1, -1, -1, \ldots, -1) \\ SP(L) =& (n-1, n-1, \ldots, n-1) - (n-1, -1, -1, \ldots, -1) \\ =& (0, n, n, \ldots, n) \\ SP(Q) =& (n-1, n-1, \ldots, n-1) + (n-1, -1, -1, \ldots, -1) \\ =& (2n-2, n-2, n-2, \ldots, n-2). \end{aligned}$$

□

Remark

$$SP(A - B) \neq SP(A) - SP(B).$$

Example

Determine all eigenvalues of

$$A(K_{2,3}) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Solution.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ x \\ y \\ y \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ x \\ y \\ y \\ y \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\lambda = \pm\sqrt{6}, \qquad (x, y) = (\sqrt{6}, \pm 2).$$

The remaining three eigenvalues are zero, since $A(K_{2,3})$ has rank 2. □

Example

Find two eigenvalues of

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

Solution.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ x \\ y \\ y \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ x \\ y \\ y \\ y \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\lambda = \pm\sqrt{6}, \qquad (x, y) = (\sqrt{6}, \pm 2).$$

$\square$

$\begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$ is called the **front divisor** or **row equitable quotient** of M.

Example

Find two eigenvalues of

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution.

$$\begin{pmatrix} x \\ x \\ y \\ y \\ y \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{pmatrix} = \lambda \begin{pmatrix} x \\ x \\ y \\ y \\ y \end{pmatrix}^T \quad \Leftrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}^T.$$

$$\lambda = \pm\sqrt{6}, \qquad (x, y) = (\sqrt{6}, \pm 2).)$$

□

$\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$ is called the **rear divisor** or **column equitable quotient** of M.

Remark

"Most introductory linear algebra courses impart the belief that the way to compute the eigenvalues of a matrix is to find the zeros of its characteristic polynomial. For matrices with order greater than two, this is false. Generally, the best way to obtain eigenvalues is to find eigenvectors."

(Algebraic Graph Theory by Chris Godsil and Gordon Royle, page 171.)

Diameter of a graph

Diameter of a graph

Let Γ be an undirected graph.

- The **distance** $\partial(x, y)$ of two vertices x, y is the smallest integer t such that there exists a walk $x = x_0, x_1, \ldots, x_t = y$ of length t.
- Γ is **connected** if $\partial(x, y)$ exists for all vertices x, y.
- The **diameter** of a connected graph Γ is the maximum value $\partial(x, y)$ among all pairs (x, y) of vertices.

Proposition.

Let Γ be a connected graph with diameter d. Then Γ has at least $d+1$ distinct eigenvalues (respectively, Laplace eigenvalues and signless Laplace eigenvalues).

Proof.

Let M be A, L, or Q. Suppose M has k distinct eigenvalues. Let $f(x) = x^k + \cdots$ denote the minimal polynomials of M. Suppose $k \leq d$, and pick two vertices x, y with $\partial(x, y) = k$. Recall that $A^k_{xy} \neq 0$ is the number of walks from x to y. Then

$$0 = f(M)_{xy} = M^k_{xy} = \begin{cases} A^k_{xy}, & \text{if } M = A \text{ or } M = |L|; \\ (-1)^k A^k_{xy}, & \text{if } M = L \end{cases} \neq 0,$$

a contradiction. Hence $k \geq d+1$. □

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