Eigenvalue Interlacing Theorem

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Outline

Quick computation of eigenvalues and eigenvectors

Examples

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ -6 \end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix} 3 & 0 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -6 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix} 3 & 0 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Problem

Determine the eigenvalues of

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Solution.

- The matrix has nullity 2, so 0 is an eigenvalue with multiplicity 2.
- The remaining two eigenvalues are 3 and 4 by using the following matrix partition to simplify the computation of eigenvalues.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ \hline 1 & 2 & 3 & 4 \end{array}\right), \begin{pmatrix} 3 & 0 \\ 6 & 4 \end{pmatrix}.$$

Quotient matrix

Let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_t\}$ be a partition of $[n] := \{1, 2, \ldots, n\}$, and $M = (m_{ij})$ be an $n \times n$ matrix.

- The **quotient matrix** of M with respect to Π is a $t \times t$ matrix $N = (n_{ab})$ such that
$$n_{ab} = \frac{1}{|\pi_a|} \sum_{(i,j) \in \pi_a \times \pi_b} m_{ij},$$
i.e., n_{ab} is the average rowsum of the sumbatrix $M_{\alpha_a \times \alpha_b}$.
- The above quotient matrix $N = (n_{ab})$ is **equitable** if for all $a, b \in [t]$,
$$n_{ab} = \sum_{s \in \pi_b} m_{ks} \qquad (\forall k \in \pi_a).$$

Example

Find the the quotient matrix of the matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ \hline 1 & 2 & 3 & 4 \end{array}\right)$$

with respect to the partition $\Pi = \{\{1,2,3\},\{4\}\}$ **of** $[4]$. **Is this an equitable quotient matrix?**

Solution. (i)

$$\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$$

(ii) No. $\square$

Example

Find the quotient matrix of the matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ \hline 1 & 2 & 3 & 4 \end{array}\right)$$

with respect to the partition $\Pi = \{\{1,2,3\},\{4\}\}$ **of** $[4]$. **Is this an equitable quotient matrix?**

Solution. (i)

$$\begin{pmatrix} 3 & 0 \\ 6 & 4 \end{pmatrix}.$$

(ii) Yes. □

Matrix description of an equitable quotient matrix

Incident matrix of a partition

Let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_t\}$ be a partition of $[n]$. Let $U = (u_{ij})$ be an $n \times t$ matrix with

$$u_{ib} = \begin{cases} 1, & \text{if } i \in \pi_b \text{ ;} \\ 0, & \text{else,} \end{cases}$$

Then the matrix U is called the **incident matrix** of Π.

Example

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & 0 & 2 \\ \hline 1 & 2 & 3 & 4 \end{array}\right) \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 2 \\ 3 & 2 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}.$$

$$\Pi = \{\{1,2,3\},\{4\}\}.$$

The above equation shows $MU = UN$ as formal stated in the next lemma. The 2×2 matrix N is an equitable quotient matrix of the 4×4 matrix M. The 4×2 matrix U is the incident matrix of the partition Π.

Lemma

Let M be an $n \times n$ nonnegative matrix and Π a partition of $[n]$ with an $n \times t$ incident matrix $U = (u_{ib})$. If $\Pi(M)$ is an equitable quotient matrix of M with respect to Π, then $MU = U\Pi(M)$.

Proof.

This is immediate from the definition of equitable quotient matrix and matrix multiplication. □

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Theorem

Let M be an $n \times n$ nonnegative matrix and Π a partition of $[n]$ into t blocks. If $t \times t$ matrix $\Pi(M)$ is an equitable quotient matrix of M with respect to Π of $[n]$, then the eigenvalues of $\Pi(M)$ are also the eigenvalues of M.

Proof.

Let λ be an eigenvalue of $\Pi(M)$ with associated eigenvector w, i.e., $\Pi(M)w = \lambda w$. Then in the notation of previous lemma,

$$MUw = U\Pi(M)w = \lambda Uw.$$

Hence λ is an eigenvalue of M with associated eigenvector Uw. $\square$

Corollary

Let M be an $n \times n$ nonnegative matrix and Π a partition of $[n]$. If the quotient matrix $\Pi(M)$ of M with respect to Π is equitable, then $\rho(M) = \rho(\Pi(M))$, where ρ is the spectral radius.

Proof.

Since $\Pi(M)$ is nonnegative, $\rho(\Pi(M))$ is an eigenvalue of $\Pi(M)$. By last theorem, $\rho(\Pi(M))$ is an eigenvalue of M. Hence $\rho(M) \geq \rho(\Pi(M))$ Let U denote the incident matrix of Π and u be a nonnegative left $\rho(M)$-eigenvector of M. Then $uU \neq 0$ and

$$\rho(M)uU = uMU = uU\Pi(M).$$

Hence uU is a $\rho(M)$-eigenvector of $\Pi(M)$. This proves $\rho(M) \leq \rho(\Pi(M))$. □

Example

The complete bipartite graph K_{pq} has the adjacency matrix of the form

$$\begin{pmatrix} 0 & J_{pq} \\ J_{qp} & 0 \end{pmatrix}.$$

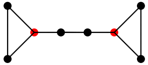
Find all the eigenvalues of the above matrix.

Solution. According to the bipartite, the quotient matrix is equitable, equals

$$\begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}$$

and has eigenvalues $\pm\sqrt{pq}$, which are the nonzero eigenvalues of K_{pq}. Since the above matrix has nullity 2, the other 2 eigenvalues are 0's. $\square$

Example (Mckay's graph)



With respect to the (black, red)-vertex partition, the adjacency matrix A of the above graph has equitable quotient matrix N as

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix},$$

where N has two eigenvalues $\frac{1\pm\sqrt{13}}{2}$.

Matrix description of a quotient matrix

Example

$$\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ \hline 1 & 2 & 3 & 4 \end{array}\right) \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}.$$

This is $D^{-1}U^TMU = \Pi(M)$ in the next page.

Lemma

Let M be an $n \times n$ matrix, $\Pi = \{\pi_1, \pi_2, \ldots, \pi_t\}$ be a partition of $[n]$ with incident matrix $U = (u_{ib})$, and $D = \mathrm{diag}(|\pi_1|, |\pi_2|, \ldots, |\pi_t|)$. If $\Pi(M)$ is a quotient matrix of M with respect to Π, then

$$\Pi(M) = D^{-1} U^T M U.$$

Proof.

This is immediate from the definition of quotient matrix and matrix multiplication. $\square$

Remark

$$
\begin{aligned}
& \Pi(M) = D^{-1} U^T M U \\
\Rightarrow \quad & \sqrt{D}\Pi(M)\sqrt{D}^{-1} = \left(U\sqrt{D}^{-1}\right)^T M U \sqrt{D}^{-1} \\
\Rightarrow \quad & \sqrt{D}\Pi(M)\sqrt{D}^{-1} = S^T M S \quad (S := U\sqrt{D}^{-1}) \\
& S^T S = I \quad (S \text{ is a real orthonormal matrix.})
\end{aligned}
$$

Example

$$S = U\sqrt{D}^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \end{pmatrix},$$

one can check directly that $S^T S = I$.

Another example of $S^TS = I$

For

$$S := \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right).$$

we have $S^TS = I$.

Remark

If M is a 4×4 matrix then S^TMS is the 2×2 submatrix of M in the first two rows and columns.

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One more example of $S^T S = I$

If M is an $n \times n$ symmetric matrix and we take the $n \times t$ matrix $S = \left(u_1 u_2 \cdots u_t \right)$, where u_i are orthonormal eigenvectors of M corresponding to λ_i for $1 \leq i \leq t$, then

$$S^T S = I.$$

Remark

$$S^T M S = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_t \end{pmatrix}.$$

Eigenvalue Interlacing Theorem

Questions

- **Can you say that the eigenvalues of $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$ are a part of the eigenvalues of**

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}?$$

Solution. No! □

- **If there a relation between the eigenvalues of M and its quotient matrix N?**

Solution. Yes, when M is **symmetric**. This is the eigenvalue interlacing theorem. □

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The two most important theorems in AGT

- **Perron-Frobenius Theorem** for spectral radius needs the discussing matrices to be **nonnegative**.
- **Cauchy's Interlace Theorem** for eigenvalues needs the discussing matrices to be **symmetric**.

Interlacing

For $t < n$, the sequence $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_t$ is said to **interlace** the sequence $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$ whenever

$$\theta_i \geq \eta_i \geq \theta_{n-t+i}$$

for $1 \leq i \leq t$.

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Examples

- The sequence $6, 4, 3, 1$ interlaces the sequence $6, 5, 4, 3, 2, 1$.
- The sequence $6, 5, 2, 1$ interlaces the sequence $6, 5, 4, 3, 2, 1$.
- Any subsequence of a sequence interlaces the sequence.

Lemma (Rayleigh's principle)

Let M be a real symmetric matrix of order n with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and respective orthonormal eigenvectors $u_1, u_2, \ldots, u_n$. Then

1. **$\frac{u^TMu}{u^Tu} \geq \lambda_i$ for any $u \in \mathrm{Span}(u_1, u_2, \ldots, u_i)$, and equality holds iff u is a λ_i-eigenvector;**
2. **$\frac{u^TMu}{u^Tu} \leq \lambda_{i+1}$ for any $u \in \mathrm{Span}(u_1, u_2, \ldots, u_i)^\perp$, and equality holds iff u is a λ_{i+1}-eigenvector.**

Proof.

1. Write $u = c_1 u_1 + \ldots + c_i u_i$ for some $c_j \in \mathbb{R}$. Then

$$\frac{u^TMu}{u^Tu} = \frac{c_1^2\lambda_1 + c_2^2\lambda_2 + \cdots + c_i^2\lambda_i}{c_1^2 + c_2^2 + \cdots + c_i^2} \geq \lambda_i.$$

2. Use a dual argument (exercise).

□

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Recall (Dimension Theorem)

For vector spaces $U, V \subseteq \mathbb{R}^n$,

$$\dim(U) + \dim(V) = \dim(U + V) + \dim(U \cap V).$$

Assumptions

From now on, we assume that

- M is an $n \times n$ symmetric matrix, S is an $n \times t$ matrix $(t < n)$ such that $S^T S = I$ and $N = S^T M S$;
- $u_1, u_2, \ldots, u_n$ are orthonormal eigenvectors of M corresponding to eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ respectively, and
- $v_1, v_2, \ldots, v_t$ are orthonormal eigenvectors of N corresponding to eigenvalues $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_t$ respectively.

Eigenvalue Interlacing Theorem

The eigenvalues of N interlace those of M, i.e.

$$\lambda_i \geq \eta_i \geq \lambda_{n-t+i} \qquad (1 \leq i \leq t).$$

Proof.

Note that $\dim(\mathrm{Span}(v_1, v_2, \ldots, v_i) \cap \mathrm{Span}(S^T u_1, S^T u_2, \ldots, S^T u_{i-1})^{\perp})$ is at least $i + (t - i + 1) - t = 1$. Hence there exists a nonzero vector $s_i \in \mathrm{Span}(v_1, v_2, \ldots, v_i) \cap \mathrm{Span}(S^T u_1, S^T u_2, \ldots, S^T u_{i-1})^{\perp}$. Note that $(Ss_i)^T u_j = s_i^T S^T u_j = 0$ for $1 \leq j \leq i-1$, hence $Ss_i \in \mathrm{Span}(u_1, u_2, \ldots, u_{i-1})^{\perp}$ and by Rayleigh's principle,

$$\lambda_i \geq \frac{(Ss_i)^T M (Ss_i)}{(Ss_i)^T (Ss_i)} = \frac{s_i^T N s_i}{s_i^T s_i} \geq \eta_i.$$

By applying the above inequality to $-M$ and $-N$ we get $\lambda_{n-t+i} \leq \eta_i$. $\square$

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Corollary

1. **If N is a principal submatrix of a symmetric matrix M, then the eigenvalues of N interlace the eigenvalues of M. In particular if Γ is an undirected graph then the eigenvalues of an induced subgraph of Γ interlace the eigenvalues of Γ.**

2. **Let M be a symmetric matrix and N' a quotient matrix of M. Then the eigenvalues of N' interlace the eigenvalues of M.**

Proof.

This is immediate from the eigenvalue interlacing theorem and the setting in the beginning. □

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Corollary

Let λ_1 be the largest eigenvalue and $\overline{k}$ be the average degree of an undirected graph G. Then $\lambda_1 \geq \overline{k}$, and equality holds iff the graph is regular.

Proof.

The first statement follows from that the 1×1 matrix $(\overline{k})$ is a quotient matrix of A. If G is k-regular then $k = \overline{k}$ and $\lambda_1 = k$ by Perron-Frobenius Theorem. Let $J = (1, 1, \ldots, 1)^T$ and write $J/\sqrt{n} = c_1 u_1 + \cdots + c_n u_n$ as a linear combination of orthonormal eigenvectors $u_1, u_2, \ldots, u_n$ of A, where $c_1^2 + c_2^2 + \cdots + c_n^2 = 1$. Then

$$\overline{k} = (J^T/\sqrt{n})A(J/\sqrt{n}) = c_1^2\lambda_1 + c_2^2\lambda_2 + \cdots + c_n^2\lambda_n \leq \lambda_1.$$

Moreover, if $\lambda_1 = \overline{k}$ then $c_1 = 1$ and $\frac{J}{\sqrt{n}} = u_1$, so G is λ_1-regular. This proves the second statement. □

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Remark

From the above Corollary, if G is connected then we can now conclude that

$$k_{min} \leq \overline{k} \leq \lambda_1 \leq k_{\max}$$

and any one of the equality holds if and only if the graph is regular.

Schur's inequality and Courant-Weyl inequalities

$\lambda_i(M)$

In this section, for a symmetric $n \times n$ matrix M, let

$$\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$$

denote the sequence of eigenvalues of M in decreasing order.

Theorem (Schur's inequality, 1923)

Let M be a real symmetric matrix with diagonal elements $d_1 \geq d_2 \geq \cdots \geq d_n$. Then for each $1 \leq t \leq n$,

$$\sum_{i=1}^{t} d_i \leq \sum_{i=1}^{t} \lambda_i(M).$$

Proof.

Let N be the principal submatrix of M obtained by deleting the rows and columns containing $d_{t+1}, \ldots, d_n$. By Eigenvalue Interlacing Theorem

$$\sum_{i=1}^{t} d_i = \mathrm{tr}(N) = \sum_{i=1}^{t} \lambda_i(N) \leq \sum_{i=1}^{t} \lambda_i(M).$$

$\square$

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Theorem (Courant-Weyl inequalities)

Let M and N be real symmetric matrices of order n, and let $1 \le i, j \le n$. If $i+j-1 \le n$ then $\lambda_{i+j-1}(M+N) \le \lambda_i(M) + \lambda_j(N)$.

Proof.

Let U (resp. V) be the space spanned by the eigenvectors corresponding to the first $i-1$ eigenvalues of M (resp. the first $j-1$ eigenvalues of N). Then $(U+V)^\perp$ has dimension at least $n-(i+j-2)$. Let W be the space spanned by the eigenvectors corresponding to eigenvalues of $M+N$ which are strictly larger than $\lambda_i(M)+\lambda_j(N)$. Then for $w \in W$, $w^T(M+N)w > (\lambda_i(M)+\lambda_j(N))w^Tw$. But for a vector $u \in (U+V)^\perp$, $u^T(M+N)u \le (\lambda_i(M)+\lambda_j(N))u^Tu$ by Rayleigh's principle. Hence $W \cap (U+V)^\perp = 0$. Thus the dimension of W is at most $n-(n-i-j+2) = i+j+2$. Thus the dimension of W is at most $n - \dim((U+V)^\perp) = n-(n-i-j+2) = i+j-2$. Hence $\lambda_{i+j-1}(M+N) \le \lambda_i(M)+\lambda_j(N)$. □

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The Courant-Weyl inequalities

Let M and N be real symmetric matrices of order n, and let $1 \le i, j \le n$.

(i) **If $i + j - n \ge 1$ then $\lambda_i(M) + \lambda_j(N) \le \lambda_{i+j-n}(M + N)$.**

(ii) **If all eigenvalues of N are nonnegative then $\lambda_i(M + N) \ge \lambda_i(M)$.**

Proof.

(i) Recall that if $i + j - 1 \le n$ then $\lambda_{i+j-1}(M + N) \le \lambda_i(M) + \lambda_j(N)$. Apply this with (i, j, M, N) to $(n - i + 1, n - j + 1, -M, -N)$, and use $\lambda_i(-M) = -\lambda_{n-i+1}(M)$. We have

$$\begin{aligned} \lambda_i(M) + \lambda_j(N) &= -(\lambda_{n-i+1}(-M) + \lambda_{n-j+1}(-N)) \\ &\le -\lambda_{2n-i-j+1}(-M - N) \\ &= \lambda_{-n+i+j}(M + N). \end{aligned}$$

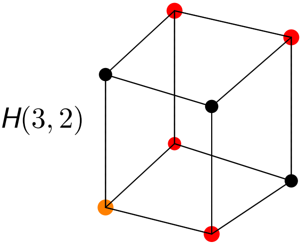
(ii) Apply the case $j = n$ of (i) and use $\lambda_n(N) \ge 0$. □

The Sensitivity Conjecture

Conjecture of Chung, Furedi, Graham, and Seymour

For every $n \geq 1$, let S be an arbitrary $(2^{n-1}+1)$-vertex induced subgraph of the n-dimensional cube $H(n,2)$ with maximum degree $\Delta(S)$. Then

$$\Delta(S) \geq \sqrt{n}.$$



$\Rightarrow$

$$|S| = 2^{3-1}+1 = 5$$

$$\Delta(S) \geq \sqrt{3}$$

The Sensitivity Conjecture

If the conjecture of Chung, Furedi, Graham, and Seymour holds, then an outstanding foundational problem in theoretical computer science, called the Sensitivity Conjecture of Nisan and Szegedy, is solved.

The Sensitivity Conjecture: The sensitivity and degree of a boolean function are polynomially related.

Theorem (Hao Huang, 2019)

The conjecture of Chung, Furedi, Graham, and Seymour holds.

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Induced subgraphs of hypercubes and a proof of the Sensitivity Conjecture

By Hao Huang

Abstract

In this paper, we show that every $(2^{n-1}+1)$-vertex induced subgraph of the n-dimensional cube graph has maximum degree at least $\sqrt{n}$. This is the best possible result, and it improves a logarithmic lower bound shown by Chung, Füredi, Graham and Seymour in 1988. As a direct consequence, we prove that the sensitivity and degree of a boolean function are polynomially related, solving an outstanding foundational problem in theoretical computer science, the Sensitivity Conjecture of Nisan and Szegedy.

1. **Introduction**

Let Q^n be the n-dimensional hypercube graph, whose vertex set consists of vectors in $\{0,1\}^n$. Two vectors are adjacent if they differ in exactly one coordinate. For an undirected graph G, we use the standard graph-theoretic notation $\Delta(G)$ for its maximum degree, and we use $\lambda_1(G)$ for the largest eigenvalue of its adjacency matrix. In 1988, Chung, Füredi, Graham, and Seymour [3] proved that if H is an induced subgraph of more than 2^{n-1} vertices of Q^n, then the maximum degree of H is at least $(1/2-o(1))\log_2 n$. Moreover, they constructed a $(2^{n-1}+1)$-vertex induced subgraph whose maximum degree is $\lceil\sqrt{n}\,\rceil$.

In this short paper, we prove the following result, establishing a sharp lower bound that matches their construction. Note that the 2^{n-1} even vertices of Q^n induce an empty subgraph. This theorem shows that any subgraph with just one more vertex would have its maximum degree suddenly jump to $\sqrt{n}$.

Theorem 1.1. *For every integer $n \ge 1$, let H be an arbitrary $(2^{n-1}+1)$-vertex induced subgraph of Q^n. Then*

$$\Delta(H) \ge \sqrt{n}.$$

Moreover this inequality is tight when n is a perfect square.

Keywords: Sensitivity Conjecture, boolean function, hypercube, eigenvalue interlacing
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Hao Huang (mathematician)

From Wikipedia, the free encyclopedia

Hao Huang is a mathematician known for solving the sensitivity conjecture.[1][2] Huang is currently an associate professor in the mathematics department at National University of Singapore.[3]

Huang was an assistant professor from 2015 to 2021 in the Department of Mathematics at Emory University. He obtained his Ph.D in mathematics from UCLA in 2012 advised by Benny Sudakov.[4] His postdoctoral research was done at the Institute for Advanced Study in Princeton and DIMACS at Rutgers University in 2012-2014, followed by a year at the Institute for Mathematics and its Applications at University of Minnesota.

In July 2019, Huang announced a breakthrough, which gave a proof of the sensitivity conjecture.[5] At that point the conjecture had been open for nearly 30 years, having been posed by Noam Nisan and Mario Szegedy in 1992.[6]

Theoretical computer scientist Scott Aaronson said of Huang's ingenious two-page proof, "I find it hard to imagine that even God knows how to prove the Sensitivity Conjecture in any simpler way than this."[7]

Huang received an NSF Career Award in 2019[8] and a Sloan Research Fellowship in 2020.[9]

References [edit]

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Proof

Define $2^n \times 2^n$ symmetric matrix A_n recursively as follows,

$$A_1 = A(H(1,2)) = A(K_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_n = \begin{pmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{pmatrix}.$$

Indeed $|A_n|$ is the adjacency matrix of $H(n,2)$. One can prove recursively

$$A_n^2 = \begin{pmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + I \end{pmatrix} = nI.$$

Hence the eigenvalues of A_n are $\pm\sqrt{n}$. Moreover, since $\mathrm{tr}(A_n) = 0$, the two eigenvalues $\pm\sqrt{n}$ have the same multiplicity 2^{n-1}. By Eigenvalue Interlacing Theorem,

$$\Delta(S) \geq \lambda_1(|A_n[S]|) \geq \lambda_{(2^{n-1}+1)-2^{n-1}}(A_n[S]) \geq \lambda_{2^n-2^{n-1}}(A_n) = \sqrt{n}.$$

□

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