Introduction to Analysis Homework 7

October, 24, 2024

1. (a) Let $\mathbf{x} \in \mathbb{R}^k$. Show that for any $r > 0$, $B_r(\mathbf{x})$ is arcwise connected.

   (b) Let $S \subset \mathbb{R}^k$. Show that if S is open and connected, then S is arcwise connected.

2. Let $S \subset \mathbb{R}^k$. Show that if S is connected and $S \subset T \subset \overline{S}$, then T is connected.

3. Let M be a metric space. Show that M is disconnected if and only if $\exists E \subsetneq M$ and $E \neq \emptyset$ such that E is both open and closed.

4. (a) Show that the image of a Cauchy sequence under a uniformly continuous function is a Cauchy sequence.

   (b) Let $f(x) = \sin(\frac{1}{x})$ for $x > 0$. Prove or disprove that f is uniformly continuous.