Introduction to Analysis Homework 5

October, 10, 2024

1. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Prove the following statements.
   (a) If $\{x_n\}_{n=1}^{\infty}$ converges, then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.
   (b) If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then $\{x_n\}_{n=1}^{\infty}$ is bounded.
   (c) If a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ converges to x, then $\{x_n\}_{n=1}^{\infty}$ converges to x.
2. (a) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ with $|x_n - x_{n+1}| \leq \frac{1}{2^n}$, for all $n \in \mathbb{N}$. Prove that $\{x_n\}_{n=1}^{\infty}$ converges.
   (b) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ with $|x_n - x_{n+1}| \to 0$ as $n \to \infty$. Does $\{x_n\}_{n=1}^{\infty}$ still converge? If not, please provide a counterexample.
3. Let $A = \{x_n\}_{n=1}^{\infty}$ be a bounded sequence of distinct points in $\mathbb{R}$. Show that the set of accumulation points of A is nonempty.
4. Prove that a closed subset of a compact set is compact.
5. Give an example of a set A which is closed and bounded under a defined metric space but not compact.