Homework 4

1. Show that for any real number x there exists a natural number n such that $n > x$.

2. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be bounded sequences in $\mathbb{R}$ with $x_n, y_n \geq 0$ for all $n \in \mathbb{N}$.

(i) Show that

$$\limsup_{n\to\infty}(x_n y_y) \leq (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n)$$

(ii) If one of the sequences converge, and the product is not of the form $0 \times \infty$ or $\infty \times 0$, show that

$$\limsup_{n\to\infty}(x_n y_y) = (\limsup_{n\to\infty} x_n)(\limsup_{n\to\infty} y_n)$$

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$, and let $A \in \mathbb{R}$ and $B \in \mathbb{R} \bigcup \{\pm\infty\}$. Show that the followings hold.

3. (i)The limit superior $\limsup_{n\to\infty} x_n = A$

$\Leftrightarrow$ (ii)the sequence $\{sup\{x_n : n \geq m\}\}_{m=1}^{\infty}$ is bounded below with the limit A

$\Leftrightarrow$ (iii)for any $\epsilon > 0$, there are infinitely many n's such that $A - \epsilon < x_n$, but only finitely many n's such that $A + \epsilon < x_n$.

(Just need to prove (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(i))

4. The limit superior $\limsup_{n\to\infty} x_n$ is the limit of a subsequence of $\{x_n\}_{n=1}^{\infty}$

5. If B is the limit of the sequence $\{x_n\}_{n=1}^{\infty} \Leftrightarrow \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = B$