Introduction to Analysis Homework 9

November, 14, 2024

1. (a) Let $\alpha \in (0,1]$. Use the mean value theorem to show that $(1+x)^\alpha \leq 1+\alpha x$ for all $x \in [-1,\infty)$.

   (b) Use (a) to prove that the sequence $\{(1+\frac{1}{n})^n\}_{n\in\mathbb{N}}$ is increasing. Also show that $L = \lim_{n\to\infty}(1+\frac{1}{n})^n$ exists and satisfying $L \in (2,3]$.

2. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function, where $a < b$. Show that if $s \in \mathbb{R}$ with $f'(a) < s < f'(b)$ or $f'(a) > s > f'(b)$, then there exists $c \in (a,b)$ such that $f'(c) = s$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} 1, \ if \ x \geq 0 \\ 0, \ if \ x < 0 \end{cases}$$
Prove or disprove that there exists a differentiable function $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = f(x), \ \forall x \in \mathbb{R}$.

4. Prove or disprove the following statement.
"Let $f, g : [0,\infty) \to \mathbb{R}$ be differentiable on $(0,\infty)$ satisfying that $g'(x) \neq 0$ for all $x \in (0,\infty)$. If $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = \infty$, then $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)}$."

5. Let $S \subset \mathbb{R}^k$ be open, $a \in S$, and $f : S \to \mathbb{R}$. Show that if the partial derivatives $\partial_i f(x)$ all exist and are bounded for all $x \in S$, then f is continuous on S.