Introduction to Analysis Homework 11

December, 05, 2024

1. Find all (local) extremum values of

$$f(x,y,z,w) = 3x + y + w,$$

subjects to $3x^2 + y + 4z^3 = 1$ and $-x^3 + 3z^4 + w = 0$
(Hint: There is not any extremum in the case.)

2. (a) Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x,y) = y^2 - x$. Show that there does not exist a function ϕ defined on a neighborhood W of 0 such that $F(x, \phi(x)) = 0$ for all $x \in W$.

   (b) Can you apply the implicit function theorem here?

3. Write Taylor's formula of $f(x,y) = cos(x + 2y)$ around $\mathbf{x_0} = (0,0)$ to the second order; that is, for $r \geq 3$, it can be replaced by the residual term.

4. (Thm 26.7)Let $S \subset \mathbb{R}^k$ be open and $f : S \to \mathbb{R}^m$ with $f = (f_1, f_2, \ldots, f_m)$. Then "the partial derivatives $\partial_j f_i$ all exist and are continuous on S" is equivalent to "f is continuously differentiable on S."

5. Let $G : \mathbb{R}^2 \to \mathbb{R}$. Prove or disprove that if G is differentiable at every point of $\mathbb{R}^2$, then the partial derivatives $\frac{\partial G}{\partial x_1}$ and $\frac{\partial G}{\partial x_2}$ are bounded near $(0,0)$.

6. Let

$$f(x) = \begin{cases} x + 2x^2 \sin \dfrac{1}{x}, & if\ x \neq 0 \\ 0, & if\ x = 0. \end{cases}$$

   (a) Find $f'(0)$.

   (b) Is f locally invertible near 0? If not, why can the inverse function theorem not be applied here?