535520 Fall 2024: Optimization Algorithms **(Due: 2024/12/1, 21:00)**

Homework 2: Projected GD, Frank-Wolfe, and Mirror Descent

Submission Guidelines: Your deliverables shall consist of 2 separate files – (i) A PDF file: Please compile all your write-ups and your report into one .pdf file (photos/scanned copies are acceptable; please make sure that the electronic files are of good quality and reader-friendly); (ii) A zip file: Please compress all your source code into one .zip file. Please submit your deliverables via E3.

Problem 1 (Convergence of PGD for Convex and Smooth Problems) (5+5+5+5+5=25 points)

As discussed in Lecture 8, we mentioned that for a convex and L-smooth function, the convergence rate of PGD is

$$f(x_t) - f(x^*) \leq \frac{3L\|x_0 - x^*\|^2 + \big(f(x_0) - f(x^*)\big)}{t+1}. \tag{1}$$

In this problem, let us prove this result formally in a step-by-step manner.

(a) To begin with, let us show the following lemma: For any $x, z \in C$, let $\bar{x} = \prod_C(x - \frac{1}{L}\nabla f(x))$ and $g_C(x) = L(x - \bar{x})$ (note that here we basically reuse the same notation as in our lecture slides). Then, we have

$$f(z) \geq f(\bar{x}) + g_C(x)^\top (z - x) + \frac{1}{2L}\|g_C(x)\|^2. \tag{2}$$

(Hint: Consider $f(z) - f(\bar{x}) = (f(z) - f(x)) - (f(\bar{x}) - f(x))$ and then utilize the convexity and smoothness conditions)

(b) By using the result in (a), show that the one-step improvement can be written as

$$f(x_{t+1}) - f(x_t) \leq -\frac{1}{2L}\|g_C(x_t)\|^2. \tag{3}$$

(c) Next, let us connect $\|g_C(x_t)\|$ to the objective function $f(x_t)$: Show that

$$\|g_C(x_t)\| \geq \frac{f(x_{t+1}) - f(x_t)}{\|x_t - x^*\|}. \tag{4}$$

(Hint: Find a proper way to apply the result in (a) and use Cauchy-Schwarz inequality)

(d) Define the sub-optimality gap at the t-th iteration as $\Delta_t := f(x_t) - f(x^*)$. By using the results in (a)-(c), show that $\Delta_{t+1} - \Delta_t \leq \frac{-\Delta_{t+1}^2}{2L\|x_0 - x^*\|^2}$.

(e) Finally, by using the result in (d), use an induction argument to show the convergence rate of PGD in (1).

Problem 2 (Projection Theorem) (10+10=20 points)

In this problem, let's prove the fundamental Projection Theorem, which typically involves two very useful properties as follows:

(a) Let C be a convex set. Given some vector $x \in \mathbb{R}^d$, a vector $x_C \in C$ is equal to the projection $\Pi_C(x)$ if and only if

$$(x - x_C)^\top (z - x_C) \leq 0, \quad \forall z \in C. \tag{5}$$

(Hint: You just need to prove this in both directions by leveraging the FONC-C and the basic geometric properties of projection)

(b) Given a convex set C, the projection operator $\Pi_C : \mathbb{R}^d \to C$ is non-expansive, i.e.,

$$\|x_C - z_C\| \leq \|x - z\|, \quad \forall x, z \in \mathbb{R}. \tag{6}$$

Problem 3 (Bregman Divergence) (5+10+10=25 points)

Recall from Lecture 10 that we learned the Bregman divergence, which is a key component in Mirror Descent. In this problem, you will have the oppotunity to verify a few useful properties:

(a) $\nabla_y D_\phi(y\|x) = \nabla\phi(y) - \nabla\phi(x)$.

(b) For any strictly convex functions $\phi_1 : X \to \mathbb{R}, \phi_2 : X \to \mathbb{R}$ and $\lambda \geq 0$, we have

$$D_{\phi_1+\lambda\phi_2}(y\|x) = D_{\phi_1}(y\|x) + \lambda D_{\phi_2}(y\|x) \tag{7}$$

(c) As mentioned in Lecture 10, please show that the Bregman divergence satisfies the Generalized Pythagorean Theorem, i.e.,

$$D_\phi(z\|x) \geq D_\phi(z\|\bar{x}) + D_\phi(\bar{x}\|x), \tag{8}$$

where $\bar{x}$ is the Bregman projection of x onto the feasible set C.

Problem 4 (Gurobi Optimization Solver for Frank-Wolfe) (30 points)

In this homework, you will leverage the Gurobi Optimization Solver again to solve constrained optimization problems. Specifically, let us implement Frank-Wolfe method with the help of Gurobi optimization solver, which can be used through its Python API, and reproduce the performance of Frank-Wolfe in the top-left subfigure of Figure 2 of the paper (https://arxiv.org/abs/2002.07003) on a Portfolio Management dataset (provided to you on E3). Portfolio Management can be formulated as a constrained problem can be described as

$$\min_x \quad f(x) := -\sum_{i=1}^{n} \log(a_i^\top x) \tag{9}$$

$$\text{subject to} \quad \sum_{j=1}^{p} x_j = 1, x \geq 0, \tag{10}$$

where each a_i is a p-dimensional vector. To facilitate gradient computation, you may write your code in either PyTorch or TensorFlow. If you are a beginner in learning the deep learning framework, please refer to the following tutorials:

- PyTorch: https://pytorch.org/tutorials/
- Tensorflow: https://www.tensorflow.org/tutorials

For the introduction to the Python API for Gurobi Optimization Solver, please see:

- To use Gurobi, you need to install Gurobipy (https://pypi.org/project/gurobipy/)
- Examples can be found at https://www.gurobi.com/documentation/

For the deliverables, please submit the following:

- Technical report: Please summarize all your experimental results in 1 single report (and please be brief)
- All your source code

Problem 2 (a)

(a) $\Rightarrow x_C = \Pi_C(x)$

proof $\|x-z\|^2 = \|(x-x_C)+(x_C-z)\|^2$

$\geq \|x-x_C\|^2 + 2\underbrace{(x-x_C)^T(x_C-z)}_{\geq 0} + \underbrace{\|x_C-z\|^2}_{\geq 0}$

$\geq \|x-x_C\|^2 \quad \forall z \in C$ by (5).

Thus, $\|x-z\| \geq \|x-x_C\| \quad \forall z \in C$.

Hence, $x_C = \arg\min_{z \in C} \|x-z\|$

$= \Pi_C(x)$ by the definition of $\Pi_C(x)$. □

Problem 2 (a)

$x_C = \Pi_C(x) \Rightarrow$ (a)

proof $x_C = \arg\min_{z \in C} \|x-z\|$

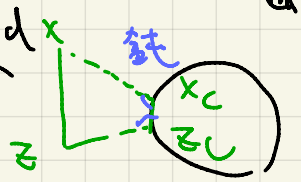
$= \arg\min_{z \in C} \frac{1}{2}\|x-z\|^2$

By FONC-C_1 since x_C is a local minimizer of $f(z) = \frac{1}{2}\|x-z\|^2$ and C is convex,

$\nabla f(x_C)^T(z-x_C) = -(x-x_C)^T(z-x_C) \geq 0 \quad \forall z \in C$.

Thus, $(x-x_C)^T(z-x_C) \leq 0 \quad \forall z \in C$. □

Problem 2. (b) Let $x, z \in \mathbb{R}^d$



$x_C = \Pi_C(x) \Rightarrow$ (b)

proof By (5), $(x-x_C)^T(z_C-x_C) \leq 0$

+) $(z-z_C)(x_C-z_C) \leq 0$.

$\Rightarrow (z-z_C+x_C-x)^T(x_C-z_C) \leq 0$

$\Rightarrow (z-x)^T(x_C-z_C) + \|x_C-z_C\|^2 \leq 0$

$\Rightarrow \|x_C-z_C\|^2 \leq (z-x)^T(x_C-z_C) \leq \|z-x\|\|x_C-z_C\|$ by 柯西 ■

If $x_C = z_C$, we are done

since $\|x_C - z_C\| = 0 \le \|x - z\|$.

Assume $x_C \neq z_C$

$\frac{(*)}{\|z_C - x_C\|}$: $\|x_C - z_C\| \le \|x - z\| \quad \forall x, z \in \mathbb{R}$. ■

Problem 3. (a)

By definition, $D_\phi(y\|x) = \phi(y) - \phi(x) - \nabla\phi^T(x)(y-x)$.

$\nabla_y D(y\|x) = \nabla\phi(y) - 0 - \nabla\phi(x) \times 1$

$= \nabla\phi(y) - \nabla\phi(x)$ ■

Problem 3. (b)

By definition, $D_\phi(y\|x) = \phi(y) - \phi(x) - \nabla\phi^T(x)(y-x)$.

Thus, $D_{\phi_1+\lambda\phi_2}(y\|x) = (\phi_1+\lambda\phi_2)(y) - (\phi_1+\lambda\phi_2)(x) - \nabla(\phi_1+\lambda\phi_2)^T(x)(y-x)$

$= \phi_1(y) - \phi_1(x) - \nabla\phi_1^T(x)(y-x)$

$+ \lambda(\phi_2(y) - \phi_2(x) - \nabla\phi_2^T(x)(y-x))$

$= D_{\phi_1}(y\|x) + \lambda D_{\phi_2}(y\|x)$ ■

Problem 3. (c)

$D_\phi(z\|x) \ge D_\phi(z\|\bar{x}) + D_\phi(\bar{x}\|x)$

$\Leftrightarrow \cancel{\phi(z)} - \cancel{\phi(x)} - \phi(x)^T(z-x) \ge \cancel{\phi(z)} - \cancel{\phi(\bar{x})} - \nabla\phi(\bar{x})^T(z-\bar{x})$

$+ \cancel{\phi(\bar{x})} - \cancel{\phi(x)} - \nabla\phi(x)^T(\bar{x}-x)$

by the definition of $D_\phi(\cdot\|\cdot)$

$\Leftrightarrow \phi(x)^T(\bar{x} - \cancel{x} + z - \cancel{x}) \ge \nabla\phi(\bar{x})^T(\bar{x}-z)$

$\Leftrightarrow (\phi(x) - \phi(\bar{x}))^T(\bar{x}-z) \ge 0$

$\Leftrightarrow \left(\nabla_x D_\phi(x\|y)\big|_{y=\bar{x}}\right)^T(\bar{x}-z) \ge 0$ by (a).

$\Leftrightarrow \left(\nabla_y D_\phi(y\|x)\big|_{y=\bar{x}}\right)^T(z-\bar{x}) \ge 0$ by (a)

Define $f(y) = D_\phi(y \| x)$.

We only need to proof $\nabla f(\bar{x})^T (z - \bar{x}) \geq 0$.

Since $\bar{x} = \arg\min_{y \in C} D_\phi(y \| x)$, and $f(y)$ is convex by Property 3 of Bregman Divergence, by Lec1 Cb, $\nabla f^T(\bar{x})(z - \bar{x}) \geq 0 \ \forall z \in C$.

Hence, $D_\phi(z \| x) \geq D_\phi(z \| \bar{x}) + D_\phi(\bar{x} \| x)$. ∎

Problem 1(a)

$$f(z) - f(x) = (f(z) - f(\bar{x})) - (f(x) - f(\bar{x}))$$

$$\geq \underbrace{\nabla f^T(\bar{x})(z - \bar{x})}_{f \text{ is convex}} - \underbrace{(\nabla f^T(x)(\bar{x} - x) + \frac{L}{2}\|\bar{x} - x\|^2)}_{L\text{-Smooth}}$$

$$= \nabla f^T(x)(z - \bar{x}) - \frac{L}{2}\|\bar{x} - x\|^2$$

$$\geq g_C^T(x)(z - \bar{x}) - \frac{L}{2}\|\bar{x} - x\|^2 \quad \text{by Claim: } \nabla f^T(x)(z - \bar{x}) \geq g_C^T(x)(z - \bar{x})$$

$$= g_C^T(x)(z - x + x - \bar{x}) - \frac{L}{2}\|\bar{x} - x\|^2$$

$$= g_C^T(x)(z - x) + L\|x - \bar{x}\|^2 - \frac{L}{2}\|\bar{x} - x\|^2$$

Since $g_L(x) = L(x - \bar{x})$

$$= g_C^T(x)(z - x) + \frac{1}{2L}\|g_L(x)\|^2 \quad \text{by } g_L(x) = L(x - \bar{x})$$ ∎

Claim: $\nabla f^T(x)(z - \bar{x}) \geq g_C^T(x)(z - \bar{x})$

By Projection Optimality Condition,

$$(x - \frac{1}{L}\nabla f(x) - \bar{x})^T (z - \bar{x}) \leq 0 \ \forall z \in C$$

$$\Rightarrow (x - \bar{x} - \frac{1}{L}\nabla f(x))^T (z - \bar{x}) \leq 0$$

$$\Rightarrow (\frac{1}{L} g_L(x) - \frac{1}{L}\nabla f(x))^T (z - \bar{x}) \leq 0$$

Since $g_L(x) = L(x - \bar{x})$

$$\Rightarrow \nabla f^T(x)(z - \bar{x}) \geq g_L^T(x)(z - \bar{x}).$$ ∎

Projection Optimality Condition: $(x-\frac{1}{L}\nabla f(x)-\bar{x})^T(z-\bar{x}) \le 0 \quad \forall z \in C$

It is a direct result by Projection Theorem. ■

Problem1. (b)

Let $z = x_{t+1}$ into (1).

Thus, $f(x)-f(x)+g_C^T(x)(x-x) \le \frac{-1}{2L}\|g_C(x)\|^2$

Since $x_{t+1}=\bar{x}$ and let $x=x_t$, $f(x_{t+1})-f(x_t) \le \frac{-1}{2L}\|g_C(x_t)\|^2$. ■

Problem1. (c)

$$\Delta_{t+1}-\Delta_t = (f(x_{t+1})-f(x^*))-(f(x_t)-f(x^*))$$
$$= f(x_{t+1})-f(x_t)$$
$$\le -\frac{1}{2L}\|g_C(x_t)\|^2 \quad \text{by (3)}-\text{②}$$

Since $g_C^T(x)(x-z) \ge f(x)-f(z)+\frac{1}{2L}\|g_C(x)\|^2$,

($x \to x_t$, $z \to x^*$, $x \to x_{t+1}$, $z \to x^*$, $x \to x_t$)

$$g_C^T(x_t)(x_t-x^*) \ge f(x_{t+1})-f(x^*)+\frac{1}{2L}\|g_C(x_t)\|^2 \ge f(x_{t+1})-f(x^*)$$

Thus, by 柯西, $\|g_C^T(x)\|\|x_t-x^*\| \ge g_C^T(x_t)(x_t-x^*)$
$$\ge f(x_{t+1})-f(x^*).$$

Hence, $\|g_C^T(x)\| \ge \frac{f(x_{t+1})-f(x^*)}{\|x_t-x^*\|}$.

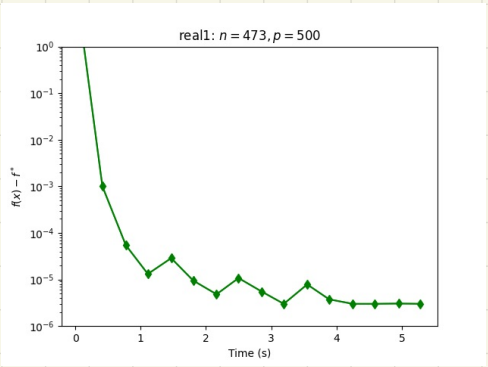
Problem1. (d)

By (4), $\|g_C^T(x)\| \ge \frac{\|f(x_{t+1})-f(x^*)\|}{\|x_t-x^*\|}$. since $\|x_t-x^*\|>0$ and $f(x_{t+1}) \ge f(x^*)$. -③

By ②, $\Delta_{t+1}-\Delta_t \le -\frac{1}{2L}\|g_C(x_t)\|^2$
$$\le \frac{-1}{2L}\frac{\|f(x_{t+1})-f(x^*)\|^2}{\|x_t-x^*\|^2} \quad \text{by ③} \times (-\tfrac{1}{2L})$$
$$= \frac{-\Delta_{t+1}^2}{2L\|x_0-x^*\|^2} \quad \text{by the definition of } \Delta_{t+1}$$

Problem 4.

Result



Problem 1. (e)

We use the following Lemma proven in Reference P.6

Lemma $\sum_{i=0}^{k} (f(x_{i+1}) - f(x^*)) \le \frac{L}{2}\|x_0 - x^*\|^2$

And $f(x_t) \ge f(x_{t+1})$, since by (2),

put $z = x = x_t$, we have $f(x_t) \ge f(x_{t+1}) + \frac{1}{2L}\|g_t(x)\|^2$.

Thus, $f(x_t) - f(x^*) \ge f(x_{t+1}) - f(x^*)$. — (1)

By Lemma $\sum_{i=0}^{t} (f(x_{i+1}) - f(x^*)) + f(x_1) - f(x^*) \le \frac{L}{2}\|x_0 - x^*\|^2 + (f(x_1) - f(x^*))$. — (2)

By (1) and (2), $(t+2)(f(x_{t+1}) - f(x^*)) \le \frac{L}{2}\|x_0 - x^*\|^2 + (f(x_1) - f(x^*))$.

Thus, $f(x_{t+1}) - f(x^*) \le \frac{3L\|x_0 - x^*\|^2 + (f(x_1) - f(x^*))}{t+2}$.

Hence, $f(x_t) - f(x^*) \le \frac{3L\|x_0 - x^*\|^2 + (f(x_1) - f(x^*))}{t+1}$. (4)

Lecture 15: Projected Gradient Descent

Yudong Chen

Consider the problem

$$\min_{x\in\mathcal{X}} f(x), \tag{P}$$

where f is continuously differentiable and $\mathcal{X} \subseteq \mathrm{dom}(f) \subseteq \mathbb{R}^n$ is a closed, convex, nonempty set.

In this lecture, we further assume f is L-smooth (w.r.t. $\|\cdot\|_2$).

1 Projected gradient descent and gradient mapping

Recall the first-order condition for L-smoothness:

$$\forall x, y: \quad f(y) \le f(x) + \langle \nabla f(x), y - x\rangle + \frac{L}{2}\|y-x\|_2^2. \tag{1}$$

For unconstrained problem, recall that each iteration of gradient descent (GD) minimizes the RHS above:

$$\begin{aligned} \textbf{(GD)} \qquad x_{k+1} &= \operatorname*{argmin}_{y\in\mathbb{R}^d} \left\{ f(x_k) + \langle \nabla f(x_k), y - x_k\rangle + \frac{L}{2}\|y-x_k\|_2^2 \right\} \\ &= x_k - \frac{1}{L}\nabla f(x_k). \end{aligned}$$

Projected Gradient Descent (PGD) For constrained problem, we consider PGD, which minimizes the RHS of (1) *over the feasible set* $\mathcal{X}$*:*

$$\begin{aligned} \textbf{(PGD)} \qquad x_{k+1} &= \operatorname*{argmin}_{y\in\mathcal{X}} \left\{ f(x_k) + \underbrace{\langle \nabla f(x_k), y - x_k\rangle + \frac{L}{2}\|y-x_k\|_2^2}_{\text{complete this square}} \right\} \\ &= \operatorname*{argmin}_{y\in\mathcal{X}} \left\{ \frac{L}{2}\left\| y - x_k + \frac{1}{L}\nabla f(x_k)\right\|_2^2 \right\} \\ &= P_{\mathcal{X}}\left(x_k - \frac{1}{L}\nabla f(x_k)\right). \end{aligned}$$

As in GD, we can also use some other stepsize $\frac{1}{\eta}$ with $\eta \ge L$:

$$x_{k+1} = P_{\mathcal{X}}\left(x_k - \frac{1}{\eta}\nabla f(x_k)\right).$$

It will be useful later to recall that Euclidean projection is characterized by the minimum principle

$$\forall y \in \mathcal{X}: \quad \langle P_{\mathcal{X}}(x) - x, y - P_{\mathcal{X}}(x)\rangle \ge 0. \tag{2}$$

1.1 Gradient mapping

Many results for GD can be generalized to PGD, where the role of the gradient is replaced by the gradient mapping defined below.

Definition 1 (Gradient Mapping)**.** Suppose $\mathcal{X} \subseteq \mathbb{R}^d$ is closed, convex and nonempty, and f is differentiable. Given $\eta > 0$, the *gradient mapping* $G_\eta : \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$G_\eta(x) = \eta \left(x - P_{\mathcal{X}} \left(x - \frac{1}{\eta} \nabla f(x) \right) \right) \qquad \text{for } x \in \mathbb{R}^d.$$

Using the above definition, we can write PGD in a form that resembles GD:

$$x_{k+1} = x_k - \frac{1}{\eta} G_\eta(x_k).$$

The fixed points of PGD are those that satisfy $G_\eta(x) = 0$.

Remark 1. When $\mathcal{X} = \mathbb{R}^d$, $G_\eta(x) = \nabla f(x)$. Hence the gradient mapping generalizes the gradient.

For constrained problems, gradient mapping acts as a "proxy" for the gradient and has properties similar to the gradient.

- If $G_\eta(x) = 0$, then x is a stationary point, meaning that $-\nabla f(x) \in N_{\mathcal{X}}(x)$. If $\left\| G_\eta(x) \right\|_2 \leq \epsilon$, we get a near-stationary point.
- A Descent Lemma holds for PGD: if we use $\eta \geq L$, then $f(x_{k+1}) - f(x_k) \leq -\frac{1}{2\eta} \left\| G_\eta(x_k) \right\|_2^2$.

We elaborate below.

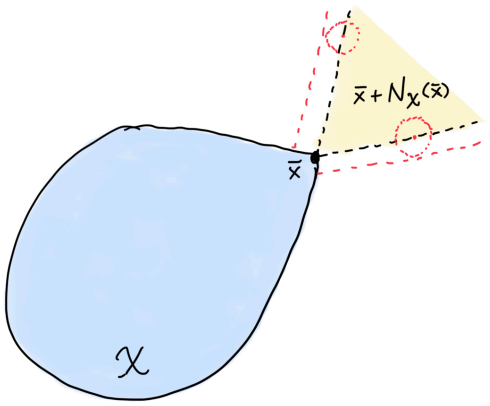
1.2 Gradient mapping and stationarity

Let $\mathcal{B}_2(z, r) := \left\{ x \in \mathbb{R}^d : \|x - z\|_2 \leq r \right\}$ denotes the Euclidean ball of radius r centered at z. For two sets $S_1, S_2 \subset \mathbb{R}^d$, let $S_1 + S_2 = \{x + y : x \in S_1, y \in S_2\}$ denote their Minkowski sum.

The first lemma says if $\left\| G_\eta(x) \right\|_2$ is small, then x almost satisfies the first-order optimality condition and can be considered a near-stationary point.

Lemma 1 (Gradient mapping as a surrogate for stationarity)**.** *Consider (P), where f is L-smooth, and $\mathcal{X}$ is closed, convex and nonempty. Denote $\bar{x} = P_{\mathcal{X}} \left(x - \frac{1}{\eta} \nabla f(x) \right)$, so that $G_\eta(x) = \eta(x - \bar{x})$. If $\left\| G_\eta(x) \right\|_2 \leq \epsilon$ for some $\epsilon \geq 0$, then:*

$$\begin{aligned} & -\nabla f(\bar{x}) \in N_{\mathcal{X}}(\bar{x}) + \mathcal{B}_2 \left(0, \epsilon \left(\frac{L}{\eta} + 1 \right) \right) \\ \iff & \forall u \in \mathcal{X} : \langle -\nabla f(\bar{x}), u - \bar{x} \rangle \leq \epsilon \left(\frac{L}{\eta} + 1 \right) \|u - \bar{x}\|_2 \\ \implies & \forall u \in \mathcal{X} \cap \mathcal{B}_2(\bar{x}, 1) : \langle -\nabla f(\bar{x}), u - \bar{x} \rangle \leq \epsilon \left(\frac{L}{\eta} + 1 \right). \end{aligned}$$



Proof. Suppose that $\left\|G_\eta(x)\right\|_2 \leq \epsilon$. By definition:

$$\bar{x} = P_{\mathcal{X}}\left(x - \frac{1}{\eta}\nabla f(x)\right) = \underset{y\in\mathcal{X}}{\operatorname{argmin}} \left\{\frac{1}{2}\left\|y - \left(x - \frac{1}{\eta}\nabla f(x)\right)\right\|_2^2\right\}.$$

Hence $\bar{x}$ satisfies the optimality condition of the minimization problem above:

$$-\left(\bar{x} - x + \frac{1}{\eta}\nabla f(x)\right) \in N_{\mathcal{X}}(\bar{x}).$$

Adding and subtracting $-\frac{1}{\eta}\nabla f(\bar{x})$:

$$-\frac{1}{\eta}\nabla f(\bar{x}) - \underbrace{\left(\bar{x} - x + \frac{1}{\eta}\nabla f(x) - \frac{1}{\eta}\nabla f(\bar{x})\right)}_{\rho} \in N_{\mathcal{X}}(\bar{x}).$$

Note that

$$\begin{aligned}
\|\rho\|_2 &= \Big\| \underbrace{\bar{x} - x}_{-\frac{1}{\eta}G_\eta(x)} + \frac{1}{\eta}\left(\nabla f(x) - \nabla f(\bar{x})\right) \Big\|_2 \\
&\leq \frac{1}{\eta}\left\|G_\eta(x)\right\|_2 + \frac{1}{\eta}\underbrace{\|\nabla f(x) - \nabla f(\bar{x})\|_2}_{\leq L\|x-\bar{x}\|_2 = \frac{L}{\eta}\left\|G_\eta(x)\right\|_2} \\
&\leq \frac{1}{\eta}\left(1 + \frac{L}{\eta}\right)\left\|G_\eta(x)\right\|_2 \\
&\leq \frac{\epsilon}{\eta}\left(1 + \frac{L}{\eta}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&-\frac{1}{\eta}\nabla f(\bar{x}) \in N_{\mathcal{X}}(\bar{x}) + \rho \\
\iff &-\nabla f(\bar{x}) \in N_{\mathcal{X}}(\bar{x}) + \eta\rho \\
\implies &-\nabla f(\bar{x}) \in N_{\mathcal{X}}(\bar{x}) + \mathcal{B}_2\left(0, \epsilon\left(1 + \frac{L}{\eta}\right)\right).
\end{aligned}$$

□

The next lemma shows that x^* is a stationary point of (P) if and only if $G_\eta(x^*) = 0$.

Lemma 2 (Wright-Recht Prop 7.8). *Consider (P), where f is L-smooth, and $\mathcal{X}$ is closed, convex and nonempty. Then, $x^* \in \mathcal{X}$ satisfies the first-order condition $-\nabla f(x^*) \in N_{\mathcal{X}}(x^*)$ if and only if $x^* = P_{\mathcal{X}}\left(x^* - \frac{1}{\eta}\nabla f(x^*)\right)$ (equivalently, $G_\eta(x^*) = 0$).*

Proof. **"if" part:** Suppose $G_\eta(x^*) = 0$. Applying Lemma 1 with $\epsilon = 0$ and noting that $\bar{x} = x^*$, we get $-\nabla f(x^*) \in N_{\mathcal{X}}(x^*)$.

- Explicit proof: $G_\eta(x^*) = 0$ means

$$x^* = P_{\mathcal{X}}\left(x^* - \frac{1}{\eta}\nabla f(x^*)\right) = \operatorname*{argmin}_{y\in\mathcal{X}} \left\{\frac{1}{2}\left\|y - \left(x^* - \frac{1}{\eta}\nabla f(x^*)\right)\right\|_2^2\right\}.$$

  By first-order optimality condition applied to the above minimization problem, we have

$$N_{\mathcal{X}}(x^*) \ni -\nabla\left[\frac{1}{2}\left\|y - \left(x^* - \frac{1}{\eta}\nabla f(x^*)\right)\right\|_2^2\right]\Bigg|_{y=x^*} = -\frac{1}{\eta}\nabla f(x^*),$$

  which is equivalent to $N_{\mathcal{X}}(x^*) \ni -\frac{1}{\eta}\nabla f(x^*)$.

"only if" part: Suppose $-\nabla f(x^*) \in N_{\mathcal{X}}(x^*)$. By definition of $N_{\mathcal{X}}(x^*)$, we have

$$\begin{aligned}\forall y \in \mathcal{X}: \qquad 0 &\geq \frac{1}{\eta}\left\langle -\nabla f(x^*), y - x^*\right\rangle \\ &= \left\langle x^* - \frac{1}{\eta}\nabla f(x^*) - x^*, y - x^*\right\rangle.\end{aligned}$$

By the minimum principle (2) with $x = x^* - \frac{1}{\eta}\nabla f(x^*)$, the above inequality implies

$$x^* = P_{\mathcal{X}}(x) = P_{\mathcal{X}}\left(x^* - \frac{1}{\eta}\nabla f(x^*)\right).$$

□

1.3 Sufficient descent property/descent lemma

The gradient mapping also inherits the descent lemma.

Lemma 3 (Thm 2.2.13 in Nes'18). *Consider (P), where f is an L-smooth function. If $\eta \geq L$ and $\bar{x} = x - \frac{1}{\eta}G_\eta(x)$, then:*

$$f(\bar{x}) \leq f(x) - \frac{1}{2\eta}\left\|G_\eta(x)\right\|_2^2.$$

Proof. From the first-order condition for L-smoothness (Lecture 4, Lemma 1),

$$\begin{aligned}f(\bar{x}) &\leq f(x) + \left\langle \nabla f(x), \bar{x} - x\right\rangle + \frac{\eta}{2}\left\|\bar{x} - x\right\|_2^2 \\ &= f(x) - \frac{1}{\eta}\left\langle \nabla f(x), G_\eta(x)\right\rangle + \frac{1}{2\eta}\left\|G_\eta(x)\right\|_2^2 && \bar{x} - x = -\frac{1}{\eta}G_\eta(x) \\ &= f(x) - \frac{1}{2\eta}\left\|G_\eta(x)\right\|_2^2 + \frac{1}{\eta}\left\langle G_\eta(x) - \nabla f(x), G_\eta(x)\right\rangle. && \text{add/subtract } \frac{1}{\eta}\left\langle G_\eta(x), G_\eta(x)\right\rangle = \frac{1}{\eta}\left\|G_\eta(x)\right\|_2^2\end{aligned}$$

It remains to show that $\langle G_\eta(x) - \nabla f(x), G_\eta(x)\rangle \le 0$. Plugging in the definition of $G_\eta(x)$, we have

$$\begin{aligned}
&\langle G_\eta(x) - \nabla f(x), G_\eta(x)\rangle \\
=&\left\langle \eta\left[x - P_{\mathcal{X}}\left(x - \frac{1}{\eta}\nabla f(x)\right)\right] - \nabla f(x), \eta\left[x - P_{\mathcal{X}}\left(x - \frac{1}{\eta}\nabla f(x)\right)\right]\right\rangle \\
=&\eta^2\left\langle \underbrace{x - \frac{1}{\eta}\nabla f(x)}_{y} - P_{\mathcal{X}}\left(x - \frac{1}{\eta}\nabla f(x)\right), x - P_{\mathcal{X}}\left(x - \frac{1}{\eta}\nabla f(x)\right)\right\rangle \\
=&\eta^2\langle y - P_{\mathcal{X}}(y), x - P_{\mathcal{X}}(y)\rangle \\
\le&0
\end{aligned}$$

by the minimum principle (2). □

2 Convergence guarantees for projected gradient descent

Consider the PGD update

$$x_{k+1} = P_{\mathcal{X}}\left(x_k - \frac{1}{L}\nabla f(x_k)\right) = x_k - \frac{1}{L}G_L(x_k),$$

where we fix the stepsize to be $\frac{1}{L}$, with L being the smoothness parameter of f.

The convergence guarantees of PGD parallel those of GD.

2.1 Nonconvex case

Suppose f is L-smooth.

By the Descent Lemma 3:

$$f(x_{k+1}) - f(x_k) \le -\frac{1}{2L}\|G_L(x_k)\|_2^2.$$

Summing up over k and noting that the LHS telescopes:

$$f(x_{k+1}) - f(x_0) \le -\frac{1}{2L}\sum_{i=0}^{k}\|G_L(x_i)\|_2^2.$$

If $\bar{f} := \inf_{x\in\mathcal{X}} f(x) > -\infty$, then

$$\frac{1}{2L}\sum_{i=0}^{k}\|G_L(x_k)\|_2^2 \le f(x_0) - \bar{f}.$$

Hence

$$\min_{0\le i\le k}\|G_L(x_i)\|_2 \le \sqrt{\frac{2L\left(f(x_0) - \bar{f}\right)}{k+1}}.$$

Equivalently, after at most $k = \frac{8L\left(f(x_0)-\bar{f}\right)}{\epsilon^2}$ iterations of PGD, we have

$$\begin{aligned}
&\min_{0\le i\le k}\|G_L(x_i)\|_2 \le \frac{\epsilon}{2} \\
\implies &\exists i \in \{1,\dots,k+1\} : -\nabla f(x_i) \in N_{\mathcal{X}}(x_i) + \mathcal{B}_2(0,\epsilon)
\end{aligned}$$

where the last line follows from Lemma 1.

2.2 Convex case

Suppose f is L-smooth and convex, with a global minimizer x^*.

1) From HW 4: $\|G_L(x_k)\|_2 \leq \|G_L(x_{k-1})\|_2, \forall k$. (In HW3 we proved a similar monotonicity property for the gradient.) The result above thus implies

$$\|G_L(x_k)\|_2 \leq \sqrt{\frac{2L\left(f(x_0) - \bar{f}\right)}{k+1}}.$$

2) From Descent Lemma 3:

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2L}\|G_L(x_k)\|_2^2 \leq f(x_k),$$

so the function value is non-increasing in k.

3) Convexity gives the lower bound

$$f(x^*) \geq f(x_k) + \langle \nabla f(x_k), x^* - x_k \rangle,$$

whence

$$\begin{aligned} f(x_{k+1}) - f(x^*) &\leq f(x_{k+1}) - f(x_k) - \langle \nabla f(x_k), x^* - x_k \rangle \\ &= f(x_{k+1}) - f(x_k) - \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \langle \nabla f(x_k), x_{k+1} - x^* \rangle. \end{aligned} \tag{3}$$

(In the analysis of GD, we then use $\nabla f(x_k) = L(x_k - x_{k+1})$ and the 3-point identity). Recall that

$$x_{k+1} = \operatorname*{argmin}_{y \in \mathcal{X}} \left\{ \langle \nabla f(x_k), y - x_k \rangle + \frac{L}{2}\|y - x_k\|_2^2 \right\}.$$

The first-order optimality condition gives

$$\forall y \in \mathcal{X} : \langle \nabla f(x_k) + L(x_{k+1} - x_k), y - x_{k+1} \rangle \geq 0.$$

Taking $y = x^*$ gives

$$\begin{aligned} \langle \nabla f(x_k), x_{k+1} - x^* \rangle &\leq L\langle x_{k+1} - x_k, x^* - x_{k+1} \rangle \\ &= \frac{L}{2}\left(\|x_k - x^*\|_2^2 - \|x_{k+1} - x^*\|_2^2 - \|x_{k+1} - x_k\|_2^2\right). \qquad \text{3-point identity} \end{aligned}$$

Plugging into (3), we get

$$\begin{aligned} f(x_{k+1}) - f(x^*) &\leq \underbrace{f(x_{k+1}) - f(x_k) - \langle \nabla f(x_k), x_{k+1} - x_k \rangle - \frac{L}{2}\|x_{k+1} - x_k\|_2^2}_{\leq 0 \text{ by } L\text{-smoothness}} + \frac{L}{2}\|x_k - x^*\|_2^2 - \frac{L}{2}\|x_{k+1} - x^*\|_2^2 \\ &\leq \frac{L}{2}\|x_k - x^*\|_2^2 - \frac{L}{2}\|x_{k+1} - x^*\|_2^2. \end{aligned}$$

We then follow the same steps as in the analysis of GD, summing up and telescoping the above inequality:

$$\sum_{i=0}^{k} (f(x_{i+1}) - f(x^*)) \leq \frac{L}{2}\|x_0 - x^*\|_2^2 - \frac{L}{2}\|x_{k+1} - x^*\|_2^2 \leq \frac{L}{2}\|x_0 - x^*\|_2^2.$$

But LHS $\geq (k+1)(f(x_{k+1}) - f(x^*))$ due to monotonicity $f(x_{k+1}) \leq f(x_k) \leq \cdots \leq f(x_0)$. It follows that

$$f(x_{k+1}) - f(x^*) \leq \frac{L\|x_0 - x^*\|_2^2}{2(k+1)}.$$

2.3 Strongly convex case

Suppose f is m-strongly convex and L-smooth, with a unique global minimizer x^*.

Since x^* satisfies the first-order optimality condition, we have $P_{\mathcal{X}}\left(x^* - \frac{1}{L}\nabla f(x^*)\right) = x^*$ (Lemma 2). By nonexpansiveness of $P_{\mathcal{X}}$, we have

$$\begin{aligned}\|x_{k+1} - x^*\|_2^2 &= \left\|P_{\mathcal{X}}\left(x_k - \frac{1}{L}\nabla f(x_k)\right) - P_{\mathcal{X}}\left(x^* - \frac{1}{L}\nabla f(x^*)\right)\right\|_2^2 \\ &\le \left\|\left(x_k - \frac{1}{L}\nabla f(x_k)\right) - \left(x^* - \frac{1}{L}\nabla f(x^*)\right)\right\|_2^2 \\ &= \|x_k - x^*\|_2^2 + \frac{1}{L^2}\|\nabla f(x_k) - \nabla f(x^*)\|_2^2 - \frac{2}{L}\langle x_k - x^*, \nabla f(x_k) - \nabla f(x^*)\rangle\,.\end{aligned}$$

But

$$\|\nabla f(x_k) - \nabla f(x^*)\|_2^2 \le L\langle \nabla f(x_k) - \nabla f(x^*), x_k - x^*\rangle$$

by HW2 Q1, hence

$$\|x_{k+1} - x^*\|_2^2 \le \|x_k - x^*\|_2^2 - \frac{1}{L}\langle x_k - x^*, \nabla f(x_k) - \nabla f(x^*)\rangle\,. \tag{4}$$

By strong convexity of f:

$$\begin{aligned}f(x_k) &\ge f(x^*) + \langle \nabla f(x^*), x_k - x^*\rangle + \frac{m}{2}\|x_k - x^*\|_2^2\,, \\ f(x^*) &\ge f(x_k) + \langle \nabla f(x_k), x^* - x_k\rangle + \frac{m}{2}\|x_k - x^*\|_2^2\,.\end{aligned}$$

Adding up the two inequalities gives

$$\langle \nabla f(x_k) - \nabla f(x^*), x_k - x^*\rangle \ge m\|x_k - x^*\|_2^2\,.$$

(this is called the *strong monotonicity* property of the gradient.) Plugging into (4), we obtain

$$\begin{aligned}&\|x_{k+1} - x^*\|_2^2 \le \left(1 - \frac{m}{L}\right)\|x_k - x^*\|_2^2 \\ \implies &\|x_{k+1} - x^*\|_2^2 \le \left(1 - \frac{m}{L}\right)^{k+1}\|x_0 - x^*\|_2^2\,.\end{aligned}$$

Exercise 1. Generalize the above results to PGD with a general stepsize $\frac{1}{\eta}$, where $\eta \ge L$.

3 Extensions

3.1 Acceleration (optional)

Nesterov's acceleration scheme can be extended to PGD:

$$\begin{aligned}y_k &= x_k + \beta_k(x_k - x_{k-1})\,, && \text{momentum step} \\ x_{k+1} &= P_{\mathcal{X}}(y_k - \alpha_k\nabla f(y_k))\,. && \text{projected gradient step}\end{aligned}$$

This is a special case of the *accelerated proximal gradient method* (a.k.a. fast iterative shrinkage-thresholding algorithm, FISTA), which applies to problems of the form

$$\min_{x\in\mathbb{R}^d} f(x) + g(x), \tag{5}$$

where $f : \mathbb{R}^d \to \mathbb{R}$ is convex and smooth, and $g : \mathbb{R}^d \to \bar{\mathbb{R}}$ is convex and lower semicontinuous with a computable proximal operator. Equation (5) is called a *composite problem*. As discussed in Lecture 1–2, the constrained problem (P) corresponds to a special case of the composite problem (5) with $g(x) = I_{\mathcal{X}}(x)$ being the indicator function of $\mathcal{X}$.

For details see the chapter from Beck's book.

3.2 Other search direction?

Recall that for unconstrained problems, we may use some other search direction p_k instead of the negative gradient direction and still guarantee descent in function value (Lecture 7–8).

For constrained problem, can we use some other direction $p_k \neq -\nabla f(x_k)$ in the update $x_{k+1} = P_{\mathcal{X}}\left(x_k + \frac{1}{\eta} p_k\right)$? In general, doing so does *not* guarantee the descent property $f(x_{k+1}) < f(x_k)$, even when p_k satisfies $\langle p_k, -\nabla f(x_k)\rangle > 0$. See below for an illustration.