# 535514: Reinforcement Learning
# Lecture 11 — Value Function Approximation and A2C Algorithm

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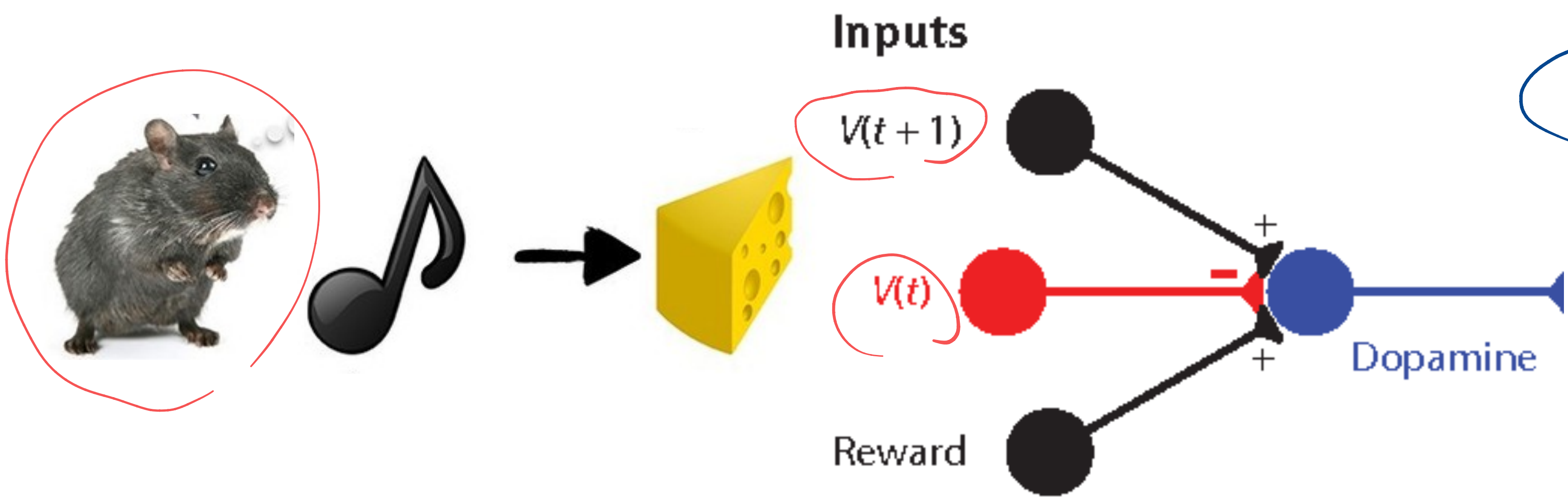
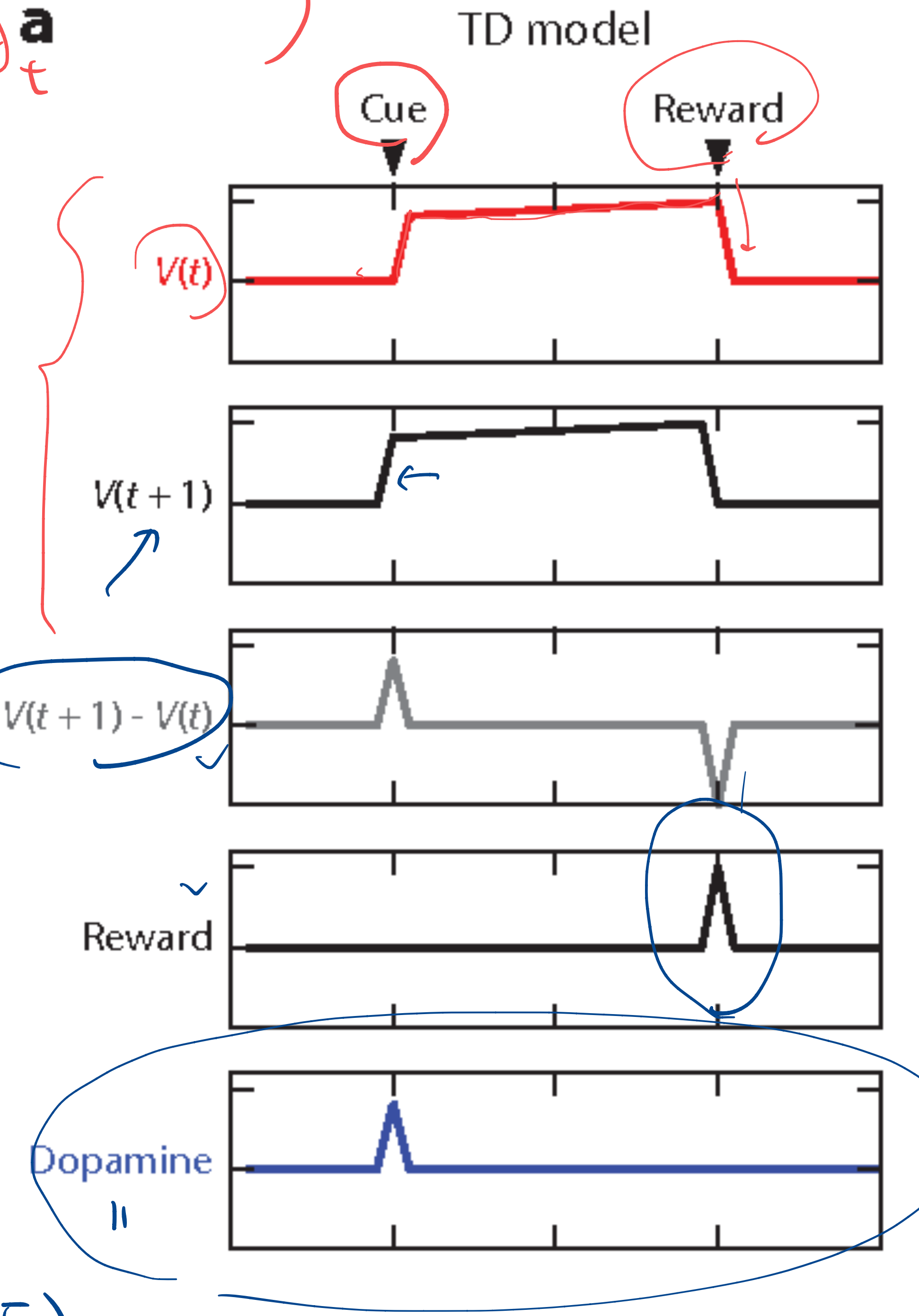
# TD Learning: From Machine Intelligence to Brain Intelligence

$V'(s) \leftarrow V(s) + \alpha( \delta_t^{a} )$

## Machine Intelligence: TD Error

$$\underbrace{\delta_t = r_{t+1} + V(s_{t+1}) - V(s_t)}$$

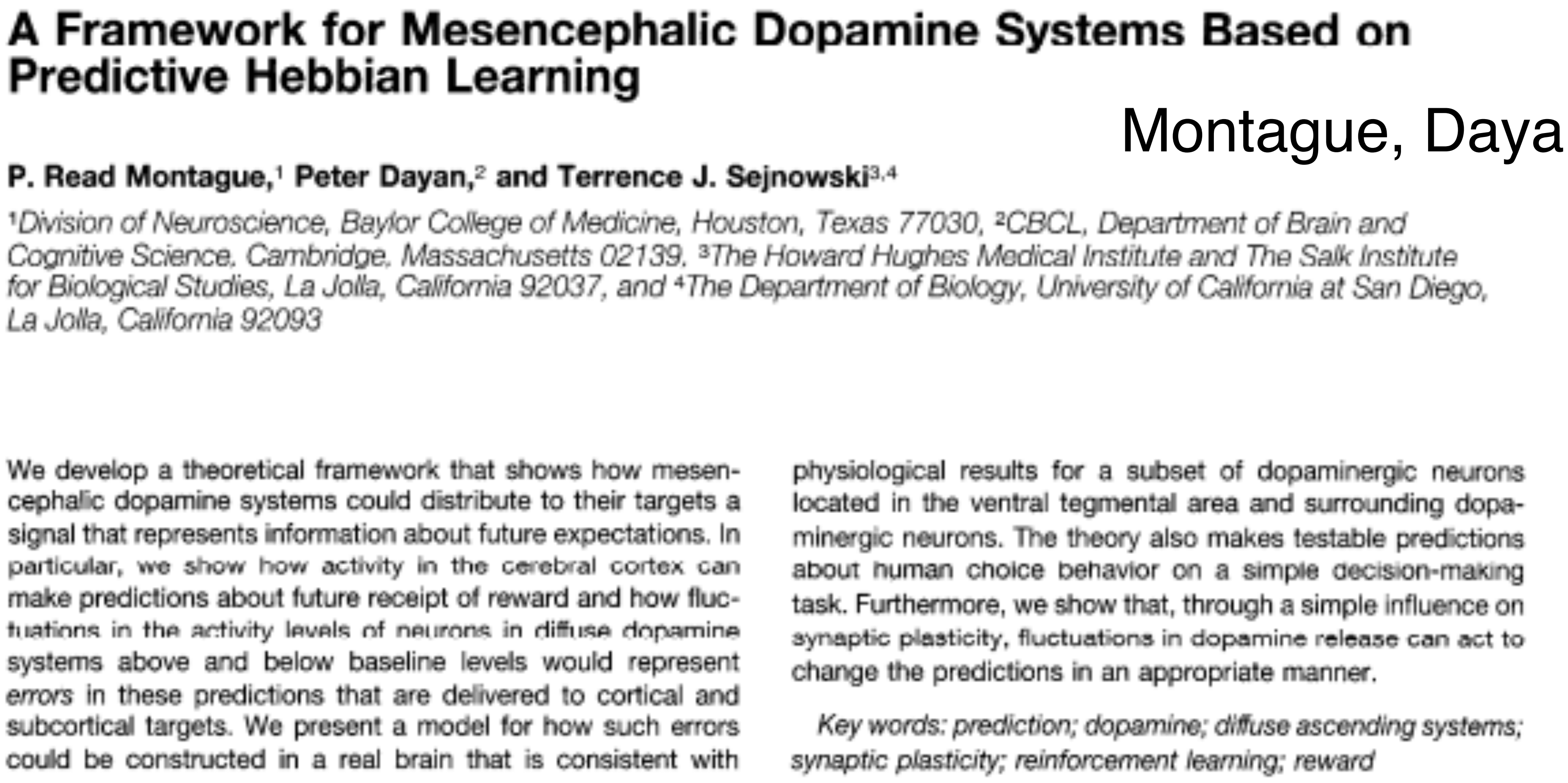




## Brain Intelligence: Reward Prediction Error (RPE)

Watabe-Uchida et al., “Neural Circuitry of Reward Prediction Error,” Annual Review of Neuroscience, 2017.

# A Classic Paper on “Reward Prediction Error”



## A Framework for Mesencephalic Dopamine Systems Based on Predictive Hebbian Learning

P. Read Montague,[1] Peter Dayan,[2] and Terrence J. Sejnowski[3,4]

[1]Division of Neuroscience, Baylor College of Medicine, Houston, Texas 77030, [2]CBCL, Department of Brain and Cognitive Science, Cambridge, Massachusetts 02139, [3]The Howard Hughes Medical Institute and The Salk Institute for Biological Studies, La Jolla, California 92037, and [4]The Department of Biology, University of California at San Diego, La Jolla, California 92093

We develop a theoretical framework that shows how mesencephalic dopamine systems could distribute to their targets a signal that represents information about future expectations. In particular, we show how activity in the cerebral cortex can make predictions about future receipt of reward and how fluctuations in the activity levels of neurons in diffuse dopamine systems above and below baseline levels would represent *errors* in these predictions that are delivered to cortical and subcortical targets. We present a model for how such errors could be constructed in a real brain that is consistent with physiological results for a subset of dopaminergic neurons located in the ventral tegmental area and surrounding dopaminergic neurons. The theory also makes testable predictions about human choice behavior on a simple decision-making task. Furthermore, we show that, through a simple influence on synaptic plasticity, fluctuations in dopamine release can act to change the predictions in an appropriate manner.

*Key words: prediction; dopamine; diffuse ascending systems; synaptic plasticity; reinforcement learning; reward*

Montague, Dayan, and Sejnowski (1996)

Peter Dayan
@Max Planck Institute
(Known for Q-learning)

Terrence Sejnowski
@Salk Institute

Demis Hassabis
(Co-Founder of DeepMind)

# Quick Review

Given a fixed policy $\pi$, we want to estimate $V^{\pi}(s)$

1. (Incremental) Monte-Carlo update:

$$G_t \equiv r_{t+1} + \gamma \cdot r_{t+2} + \cdots$$

$$V_{k+1}(s_t) \leftarrow V_k(s_t) + \alpha(\underline{\quad G_t \quad} - V_k(s_t))$$

2. TD(0) update:

TD target

$$V_{k+1}(s_t) \leftarrow V_k(s_t) + \alpha(\underline{r_{t+1} + \gamma \cdot V_k(S_{t+1})} - V_k(s_t))$$

3. $n$-step TD update:

$$G_t^{(n)} = r_{t+1} + \gamma \cdot r_{t+2} + \cdots + \gamma^{n-1} r_{t+n} + \gamma^{n} V_k(S_{t+n})$$

$$V_{k+1}(s_t) \leftarrow V_k(s_t) + \alpha(\underline{\qquad\qquad} - V_k(s_t))$$

4. TD($\lambda$) update:

$$V_{k+1}(s_t) \leftarrow V_k(s_t) + \alpha(\underline{(1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}} - V_k(s_t))$$

$$A^{\pi}(s,a) := Q^{\pi}(s,a) - V^{\pi}(s)$$

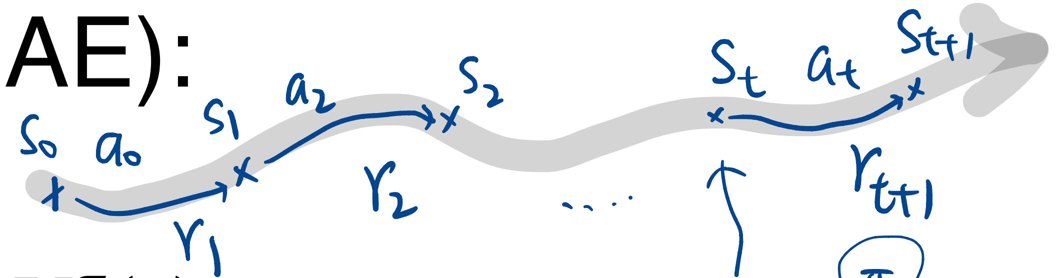
$$\hat{A}^{\pi}(s,a) = \left(r + \gamma \hat{V}^{\pi}(s')\right) - \hat{V}^{\pi}(s)$$

$s' \sim P(\cdot|s,a)$

$a \sim \pi(\cdot|s)$

# Next Question: How to Estimate $A^{\pi}(s,a)$?

# Generalized Advantage Estimator (GAE): Using TD($\lambda$) to Estimate $A^{\pi}(s,a)$



Estimate $A^{\pi}(s_t,a_t)$

Let $V(s)$ be the current estimate of true value $V^{\pi}(s)$

$$\hat{A}_t^{(1)} := \left(r_{t+1} + \gamma V(s_{t+1})\right) - V(s_t) \qquad (= \delta_t)$$

$$\hat{A}_t^{(2)} := \left(r_{t+1} + \gamma r_{t+2} + \gamma^2 V(s_{t+2})\right) - V(s_t) \qquad (= \delta_t + \gamma\delta_{t+1})$$

$$\hat{A}_t^{(3)} := r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \gamma^3 V(s_{t+3}) - V(s_t) \qquad (= \delta_t + \gamma\delta_{t+1} + \gamma^2\delta_{t+2})$$

$$\hat{A}_t^{(k)} := \left(r_{t+1} + \gamma r_{t+2} + \cdots + \gamma^k V(s_{t+k})\right) - V(s_t) \qquad \left(= \sum_{\ell=0}^{k-1} \gamma^{\ell}\delta_{t+\ell}\right)$$

- **Fact**: $\hat{A}_t^{(\infty)} = \sum_{\ell=0}^{\infty} \gamma^{\ell}\delta_{t+\ell} = G_t - V(s_t)$

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- **GAE Estimator**:

$$\hat{A}_t^{GAE(\gamma,\lambda)} = (1-\lambda)\left(\hat{A}_t^{(1)} + \lambda\hat{A}_t^{(2)} + \lambda^2\hat{A}_t^{(3)} + \cdots\right) = \sum_{\ell=0}^{\infty} (\gamma\lambda)^{\ell}\delta_{t+\ell}$$

Schulman et al., High-Dimensional Continuous Control Using Generalized Advantage Estimation, ICLR 2016

# Algorithm: REINFORCE With GAE

Recall: (P5) REINFORCE with advantage

$$\nabla_\theta V^{\pi_\theta}(\mu) = \mathbb{E}_{\tau \sim P_\mu^{\pi_\theta}}\left[\sum_{t=0}^{\infty} \gamma^t A^{\pi_\theta}(s_t, a_t) \nabla_\theta \log \pi_\theta(a_t \mid s_t)\right]$$

- REINFORCE with GAE

Step 1: Initialize $\theta_0$ and step size $\eta$

Step 2: Sample a trajectory $\tau \sim P_\mu^{\pi_\theta}$ and make the update as

policy parameter

$$\underline{\theta_{k+1}} = \theta_k + \eta\left(\sum_{t=0}^{\infty} \gamma^t \hat{A}_t^{GAE(\gamma,\lambda)} \underline{\nabla_\theta \log \pi_\theta(a_t \mid s_t)}\right)$$

where $\hat{A}_t^{GAE(\gamma,\lambda)}$ is constructed from $V(s)$ learned by TD

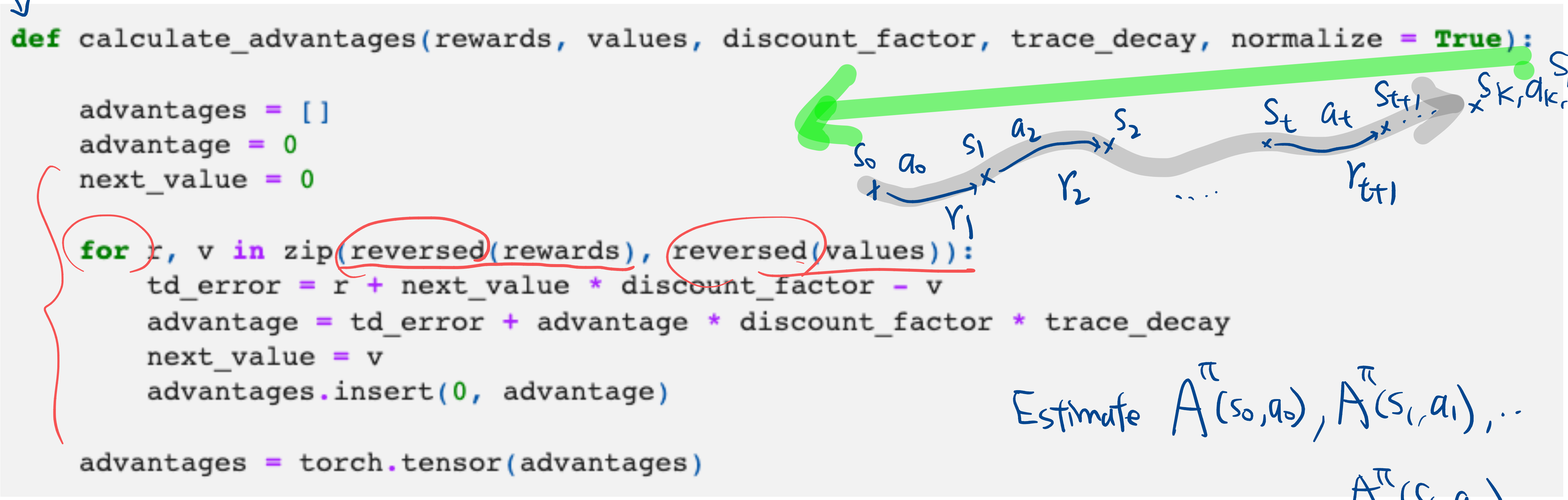
(Repeat Step 2 until termination)

# Some Discussions on GAE

1. Do we need to wait until the end of a trajectory to construct GAE?

Yes!

2. How to efficiently calculate GAE for different $t$ of the same trajectory?



```
def calculate_advantages(rewards, values, discount_factor, trace_decay, normalize = True):

    advantages = []
    advantage = 0
    next_value = 0

    for r, v in zip(reversed(rewards), reversed(values)):
        td_error = r + next_value * discount_factor - v
        advantage = td_error + advantage * discount_factor * trace_decay
        next_value = v
        advantages.insert(0, advantage)

    advantages = torch.tensor(advantages)
```

Estimate $A^{\pi}(s_0,a_0), A^{\pi}(s_1,a_1), \ldots$

$A^{\pi}(s_t,a_t), \ldots$

$$\delta_K = r_{K+1} + \gamma V(s_{K+1}) - V(s_K)$$

$$\delta_{K-1} = r_K + \gamma V(s_K) - V(s_{K-1})$$

3. Where does $V(s)$ in GAE come from?

Any model-free prediction methods!

# This Lecture:
# Let’s discuss 1 Fundamental Concept + 1 Algorithm

1. Value Function Approximation

2. Advantage Actor Critic (A2C)

References:

Richard Sutton and Andrew Barto, Reinforcement Learning: An Introduction, 2019

Mnih et al., Asynchronous Methods for Deep Reinforcement Learning, ICML 2016

Schulman et al., High-Dimensional Continuous Control Using Generalized Advantage Estimation, ICLR 2016

|  | $V^{\pi}(s)$ |
|---|---|
| $s^{(1)}$ | ← |
| $s^{(2)}$ | ← |
| ⋮ | ← |
| $s^{(100)}$ | ← |

Tabular case

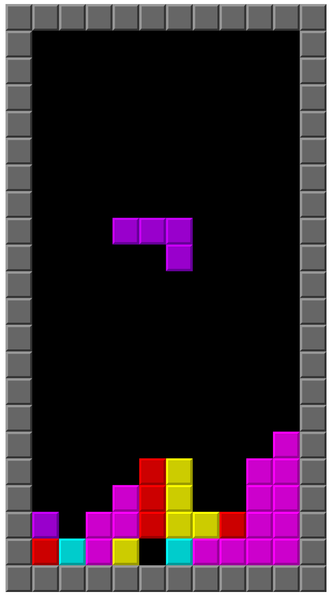
$10^{10}$

So far, we presume that $V^{\pi}(s)$ and $Q^{\pi}(s, a)$ can be predicted accurately via model-free prediction.

*But, is this always true?*

# Tabular Methods for Large RL Problems?

- So far: Value functions ($V(s)$ or $Q(s,a)$) are represented by a lookup table
- Such tabular methods may be impractical for large RL problems
  - ✓ Too many states (or actions) to store in memory
  - ✓ It is slow to learn $V(s)$ or $Q(s,a)$ for each state separately
- Examples:

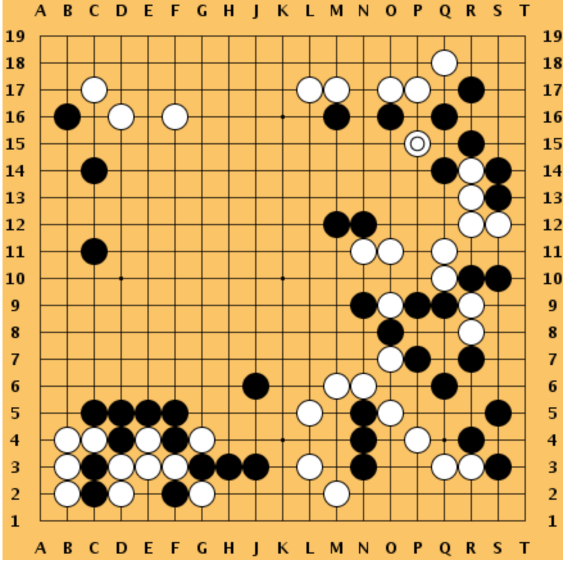


Tetris: ~$2^{200}$ states

19

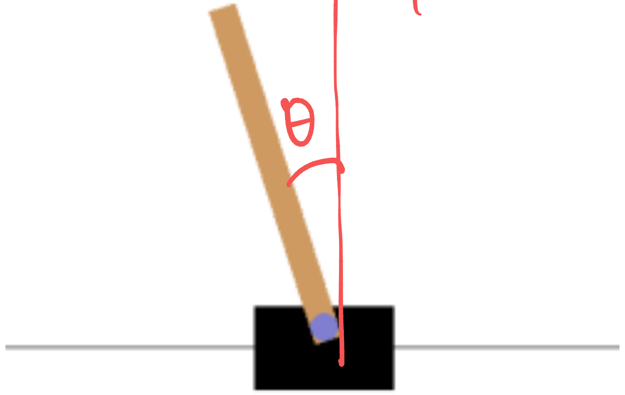
361

$3^{361}$



Go: ~$10^{170}$ states

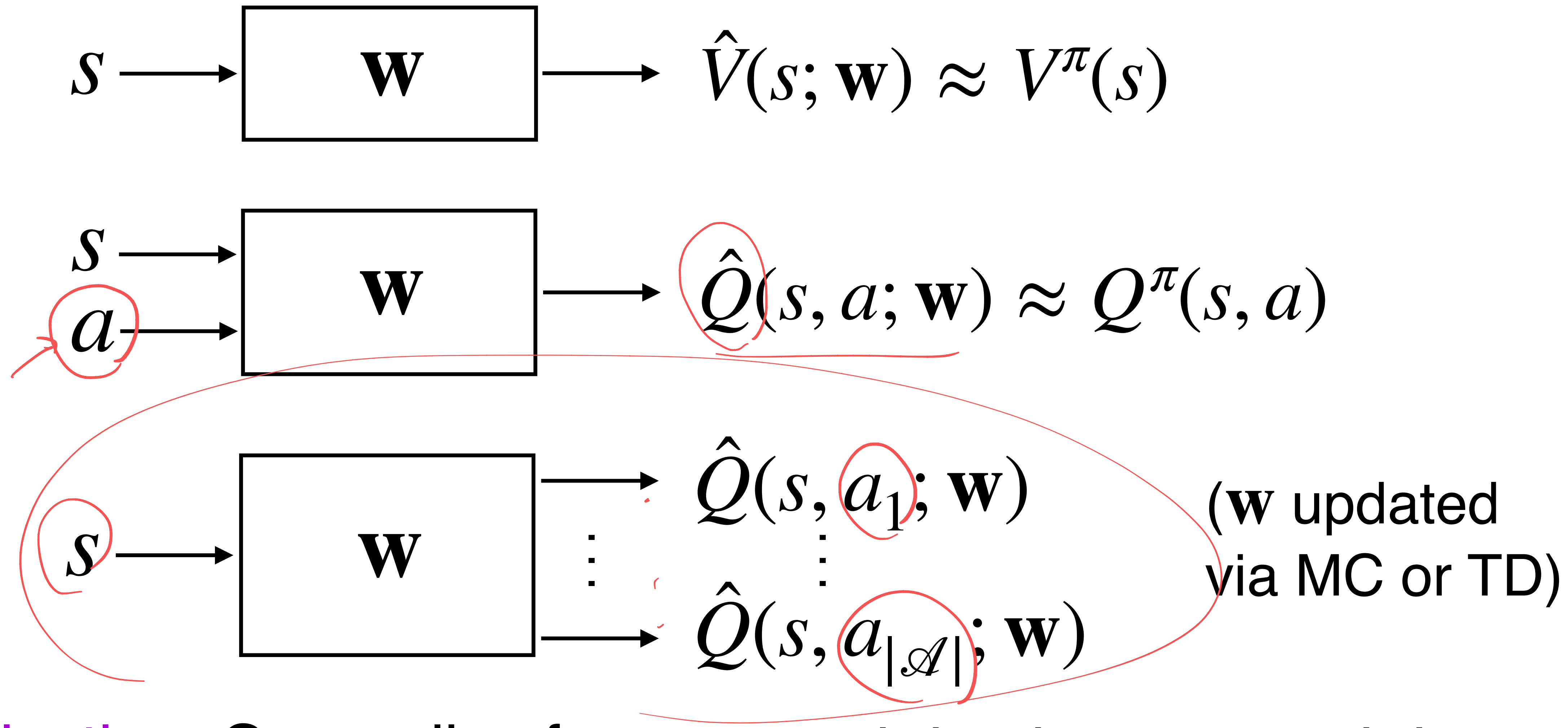
State representation $= (x, v, \theta, \omega)$



Cartpole: continuous state space

# Value Function Approximation (VFA)

- To scale up the model-free methods, function approximation is commonly used to learn value functions
- Idea: Approximate a value function by a parametric function



$s \rightarrow \boxed{\mathbf{w}} \rightarrow \hat{V}(s;\mathbf{w}) \approx V^{\pi}(s)$

$s, a \rightarrow \boxed{\mathbf{w}} \rightarrow \hat{Q}(s,a;\mathbf{w}) \approx Q^{\pi}(s,a)$

$s \rightarrow \boxed{\mathbf{w}} \rightarrow \hat{Q}(s,a_1;\mathbf{w}), \ \vdots \ , \hat{Q}(s,a_{|\mathscr{A}|};\mathbf{w})$

($\mathbf{w}$ updated via MC or TD)

- Motivation: Generalize from seen states to unseen states

# 2 Commonly-Used Function Approximators

$$s \longrightarrow \boxed{\;w\;} \longrightarrow \hat{V}(s;w) \approx V^{\pi}(s)$$

1. Linear combinations of features

- Construct a feature vector $\phi(s)$, for each $s$

2. Neural networks

- Assume linear functions

$$\hat{V}(s;w) = \phi(s)^{T} w$$

(like linear regression)

- Both are differentiable function approximators (Why?)

# How to Quantify the Accuracy of VFA?

- For each state $s$, the squared error between $V^{\pi}(s)$ and $\hat{V}(s;\mathbf{w})$:

$$\left(V^{\pi}(s) - \hat{V}(s;\mathbf{w})\right)^2$$

- To jointly consider all states, we use mean squared error (MSE):

$$F(\mathbf{w}) := \sum_{s\in\mathcal{S}} \rho(s)\left(V^{\pi}(s) - \hat{V}(s;\mathbf{w})\right)^2$$

  - $\rho(s)$ are the weights for mixing the MSE of the states
  - If $\rho(s)$ is a probability distribution, the objective function becomes:

$$F(\mathbf{w}) := \mathbb{E}_{s\sim\rho(s)}\left[\left(V^{\pi}(s) - \hat{V}(s;\mathbf{w})\right)^2\right]$$

# Some Natural Choices of $\rho(s)$

- For continuing environments:

1. choose $\rho(s) \equiv d^{\pi}_{\mu}(s) := \mathbb{E}_{s_0 \sim \mu}\left[(1-\gamma)\sum_{t=0}^{\infty} \gamma^t P(s_t = s \mid s_0, \pi)\right]$

2. choose $\rho(s) \equiv$ undiscounted stationary state distribution

- For episodic environments:

1. choose $\rho(s) \equiv d^{\pi}_{\mu}(s) := \dfrac{\mathbb{E}_{s_0 \sim \mu}\left[\sum_{t=0}^{T} \gamma^t P(s_t = s \mid s_0, \pi)\right]}{\text{normalization constant}}$

- Remark: $\rho(s)$ usually corresponds to the sampling strategy (will be discussed momentarily)

# Value Function Approximation via GD

- Goal: find $\mathbf{w}$ that minimizes MSE between $V^{\pi}(s)$ and $\hat{V}(s; \mathbf{w})$

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \underbrace{\mathbb{E}_{s \sim \rho(s)}\left[\left(V^{\pi}(s) - \hat{V}(s; \mathbf{w})\right)^2\right]}_{=:F(\mathbf{w})}$$

- Suppose: We are given an oracle for querying $V^{\pi}(s)$
- Iterative GD update:

$$\begin{aligned} \mathbf{w}_{k+1} &= \mathbf{w}_k - \alpha_k \nabla_{\mathbf{w}} F(\mathbf{w}) \\ &= \mathbf{w}_k - \alpha_k \mathbb{E}_{s \sim \rho(s)}\left[\left(V^{\pi}(s) - \hat{V}(s; \mathbf{w})\right) \nabla_{\mathbf{w}} \hat{V}(s; \mathbf{w})\right] \end{aligned}$$

- GD finds a local minimum under proper step sizes $\alpha_k$ (Why?)

# Value Function Approximation via SGD

- Iterative GD update:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \alpha_k \mathbb{E}_{s \sim \rho(s)} \left[ \left( V^\pi(s) - \hat{V}(s; \mathbf{w}_k) \right) \nabla_{\mathbf{w}} \hat{V}(s; \mathbf{w}_k) \right]$$

- Iterative SGD update by the sampled state $S \sim \rho$:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \alpha_k \left[ \left( V^\pi(S) - \hat{V}(S; \mathbf{w}_k) \right) \nabla_{\mathbf{w}} \hat{V}(S; \mathbf{w}_k) \right]$$

- The SGD update converges to a local minimum under stochastic approximation conditions of $\alpha_k$ (even for NN)

In practice, we don't have an oracle for $V^{\pi}(s)$.

What shall we do?

# Monte-Carlo Value Function Approximation

- Recall: Iterative SGD update by the sampled state $S \sim \rho$:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \alpha_k \left[ \left( V^\pi(S) - \hat{V}(S; \mathbf{w}_k) \right) \nabla_{\mathbf{w}} \hat{V}(S; \mathbf{w}_k) \right]$$

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- Idea: Use MC, i.e. sample return $G_t$ to estimate $V^\pi(S)$

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \alpha_k \left[ \left( G_t - \hat{V}(S; \mathbf{w}_k) \right) \nabla_{\mathbf{w}} \hat{V}(S; \mathbf{w}_k) \right]$$

$G_t$ is an unbiased estimate of $V^\pi(S)$

- Equivalent to *supervised learning* with (noisy) training data as

$$(s_0, G_0), (s_1, G_1), \cdots, (s_t, G_t) \cdots$$

# Monte-Carlo Value Function Approximation (Cont.)

- First-Visit MC Value Function Approximation:

```
Step 1: Initialize $\mathbf{w} = 0$ and $k = 1$
Step 2: Sample $\tau_k = (s_0, a_0, r_1, \cdots, s_{L_k-1}, a_{L_k-1}, r_{L_k}) \sim P_\mu^{\pi_\theta}$
            For each step of the current episode $t = 0,1,\cdots L_k - 1$
                If First visit to state $s$ in episode $k$
                    $G_t(s) = \sum_{i=t+1}^{L_k} r_i$
                    $\Delta\mathbf{w}_k \leftarrow \Delta\mathbf{w}_k + \alpha_k\left[\left(G_t(s) - \hat{V}(S;\mathbf{w}_k)\right)\nabla_{\mathbf{w}}\hat{V}(S;\mathbf{w}_k)\right]$
        $\mathbf{w}_{k+1} \leftarrow \mathbf{w}_k + \Delta\mathbf{w}_k,\ k \leftarrow k+1$
```

- Convergence to a local minimum can be achieved with proper step sizes $\alpha_k$, for general function approximators (Why?)

# Temporal-Difference Value Function Approximation

- Recall: Iterative SGD update by the sampled state $S \sim \rho$:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \alpha_k \Big[ \big( V^\pi(S) - \hat{V}(S; \mathbf{w}_k) \big) \nabla_{\mathbf{w}} \hat{V}(S; \mathbf{w}_k) \Big]$$

---

- Idea: Use bootstrapping (e.g.TD(0)) to estimate $V^\pi(S)$

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \alpha_k \Big[ \big( r + \gamma \hat{V}(S'; \mathbf{w}_k) - \hat{V}(S; \mathbf{w}_k) \big) \nabla_{\mathbf{w}} \hat{V}(S; \mathbf{w}_k) \Big]$$

$r + \gamma \hat{V}(S'; \mathbf{w}_k)$ is a biased estimate of $V^\pi(S)$

- Equivalent to *supervised learning* with (noisy) training data as

$$(s_0, r_1 + \gamma \hat{V}(s_1; \mathbf{w})), \cdots, (s_t, r_{t+1} + \gamma \hat{V}(s_{t+1}; \mathbf{w}))\cdots$$

- This can be easily extended to the more general TD($\lambda$)

# TD Value Function Approximation (Cont.)

- TD(0) Value Function Approximation:

```
Step 1: Initialize $\mathbf{w} = 0$ and $k = 1$
Step 2: Sample $\tau_k = (s_0, a_0, r_1, \cdots, s_{L_k-1}, a_{L_k-1}, r_{L_k}) \sim P_\mu^{\pi_\theta}$
        For each step of the current episode $t = 0, 1, \cdots L_k - 1$
            $\mathbf{w} \leftarrow \mathbf{w} + \alpha \left[ \left( r_{t+1} + \gamma \hat{V}(s_{t+1}; \mathbf{w}) - \hat{V}(s_t; \mathbf{w}) \right) \nabla_{\mathbf{w}} \hat{V}(s_t; \mathbf{w}) \right]$
```

## Put Everything Together:

## PG + TD + VFA = Advantage Actor-Critic (A2C)

# Advantage Actor-Critic (A2C) via TD

- An example of actor-critic algorithm:
  - Critic: estimate $V^{\pi_\theta}$ by TD(0) bootstrapping
  - Actor: updates policy parameters $\theta$ by policy gradient
- Advantage Actor-Critic (A2C):

```
Step 1: Initialize $\theta_0$ and step size $\alpha$
Step 2: Sample a trajectory $\tau = (s_0, a_0, r_1, \cdots) \sim P_\mu^{\pi_\theta}$
            For each step of the current trajectory $t = 0,1,2,\cdots$
                $\Delta\theta_k \leftarrow \Delta\theta_k + \alpha\gamma^t\left(r_t + \gamma\hat{V}(s_{t+1}) - \hat{V}(s_t)\right)\nabla_\theta \log \pi_\theta(a_t | s_t)$
                Update value function $\hat{V}(s_t)$ by TD(0)
        $\theta_{k+1} \leftarrow \theta_k + \Delta\theta_k$
```

# A2C With Value Function Approximation

- $\hat{V}_w(s)$ is learned by value function approximation (e.g. by using a neural network or linear combinations of features)

- Advantage Actor-Critic (A2C) with Value Function Approximation:

```
Step 1: Initialize $\theta_0$, $w_0$ and step sizes $\alpha_\theta, \alpha_w$
Step 2: Sample a trajectory $\tau = (s_0, a_0, r_1, \cdots) \sim P_\mu^{\pi_\theta}$
        For each step of the current trajectory $t = 0,1,2,\cdots$
            $\Delta\theta_k \leftarrow \Delta\theta_k + \alpha_\theta \gamma^t \left(r_t + \gamma \hat{V}_{w_k}(s_{t+1}) - \hat{V}_{w_k}(s_t)\right) \nabla_\theta \log \pi_\theta(a_t | s_t)$
            $\Delta w_k \leftarrow \Delta w_k + \alpha_w \left(r_t + \gamma \hat{V}_{w_k}(s_{t+1}) - \hat{V}_{w_k}(s_t)\right) \nabla_w \hat{V}_w(s_t)|_{w=w_k}$
        $\theta_{k+1} \leftarrow \theta_k + \Delta\theta_k, w_{k+1} \leftarrow w_k + \Delta w_k$
```

Mnih et al., “Asynchronous Methods for Deep Reinforcement Learning”, ICML 2016

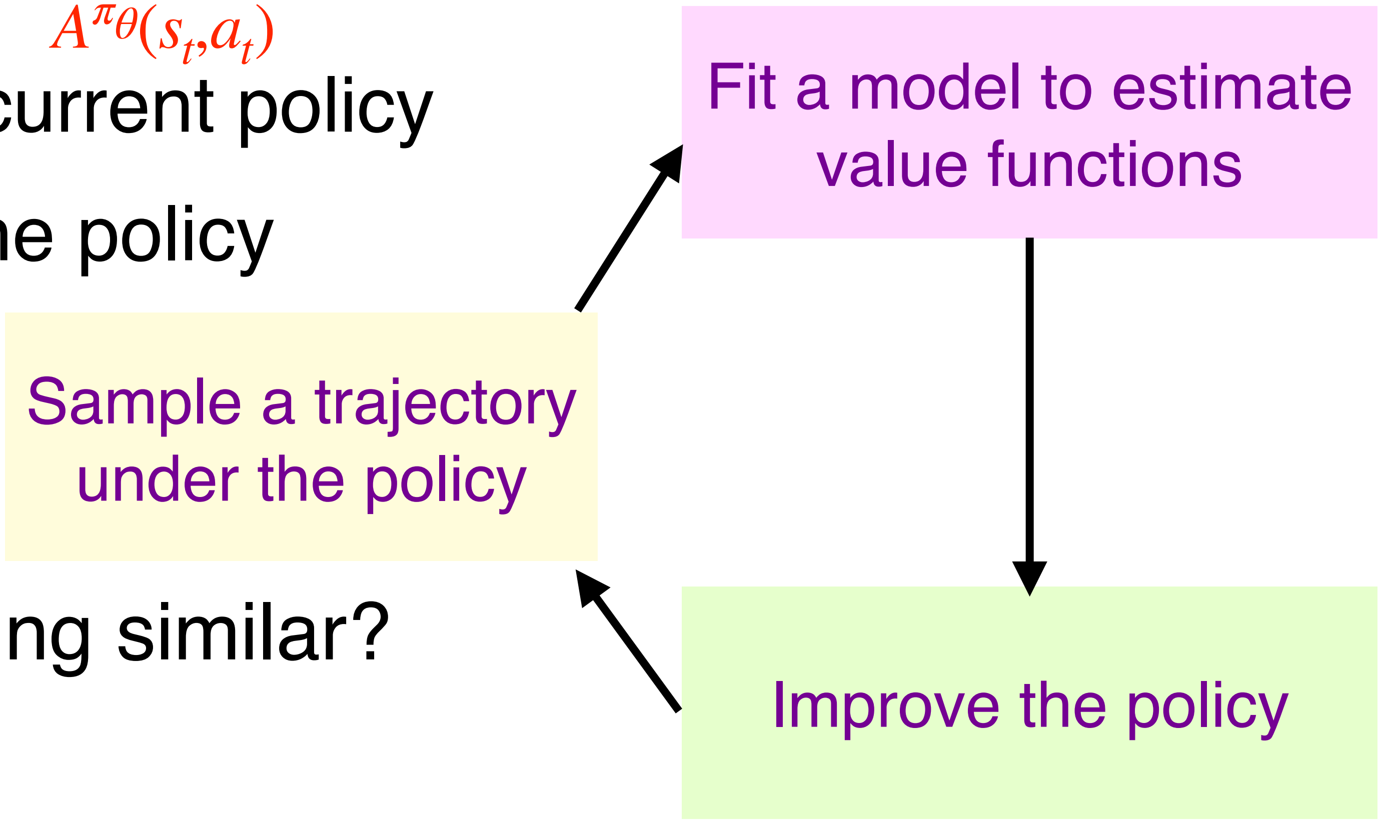
# Another Interpretation of A2C

- In A2C, the policy parameters $\theta$ are updated as:

$$\Delta\theta_k \leftarrow \Delta\theta_k + \alpha_\theta \gamma^t \underbrace{\Big(\overbrace{r_t + \gamma \hat{V}_{w_k}(s_{t+1})}^{\hat{Q}^{\pi_\theta}(s_t,a_t)} - \hat{V}_{w_k}(s_t)\Big)}_{\hat{A}^{\pi_\theta}(s_t,a_t)} \nabla_\theta \log \pi_\theta(a_t \mid s_t)$$

Step 1: Estimate $A^{\pi_\theta}(s_t, a_t)$ for the current policy

Step 2: Use $\hat{A}^{\pi_\theta}(s_t, a_t)$ to improve the policy


Fit a model to estimate value functions
Improve the policy
Sample a trajectory under the policy

- Question: Have we seen anything similar?

  *Policy iteration!*

Step 1: Policy evaluation for the current policy (i.e. find $Q^\pi(s, a)$)

Step 2: Policy improvement based on Bellman optimality equations

(Slide Credit: Sergey Levine)

# A2C $\approx$ Policy Iteration, But With Slight Difference

*Policy iteration:*

Step 1: Policy evaluation for the current policy (i.e. find $Q^{\pi}(s, a)$)

Step 2: Policy improvement based on Bellman optimality equations

*A2C:*

Step 1: Estimate $A^{\pi_\theta}(s_t, a_t)$ for the current policy

Step 2: Use $\hat{A}^{\pi_\theta}(s_t, a_t)$ to improve the policy

- Question: Is policy guaranteed to be improved in Step 2 of A2C?