Stochastic Calculus

Black Scholes and Itô Process

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Contents

1 Itô Process

0. Useful in finance for a variety of reasons

(1) Price changes = drift + unpredictable part [1] [2]

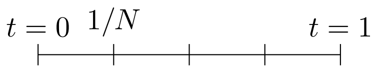
(2) Linearity property [3]

(3) Under the probability measure that represents pricing, price processes are still Itô, merely with drift replaced [4]

For (3), we use a simple example to illustrate. Imagine the payoffs are Gaussian, under the state price probability measure, will these payoffs still be Gaussian?
The answer is no. State-price probabilities are marginal utilities multiply by probabilities. If the aggregate payoff is low, the payoff for that particular asset is higher. Therefore, the distribution is skewed because there are more marginal utilities when payoffs are low, and the state prices are higher (and hence the state price probabilities are higher).

Therefore, if you start with Itô process, under state-price probabilities, you will actually end up with Itô process. [5]

1. In integral form.



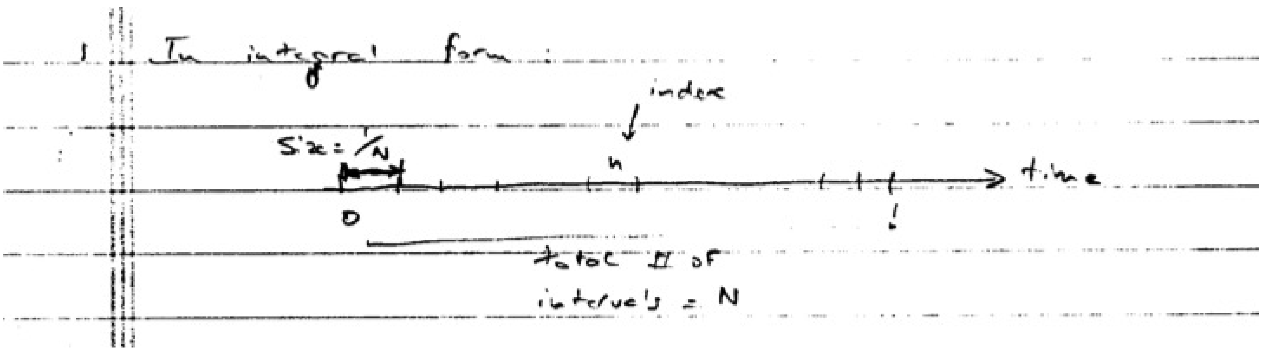
[1] If you have portfolio of securities, then the evolution of the value/prices of the portfolio weighted sum of drifts + weighted sum of unpredictable term

[2] Note that this does not apply in the real world in general

[3] Everything is locally linear, that is, everything in a short period of time is Gaussian

[4] Note that the link between physical probability and the state price probability in the pricing world is only at when there is 0 probabilities (equivalent martingale measures)

[5] This is a result of Girsanov theorem



$$s(t) = s(0) + \int_0^t \mu(s,k)dk + \int_0^t \sigma(s,k)dW_k$$

Notes:

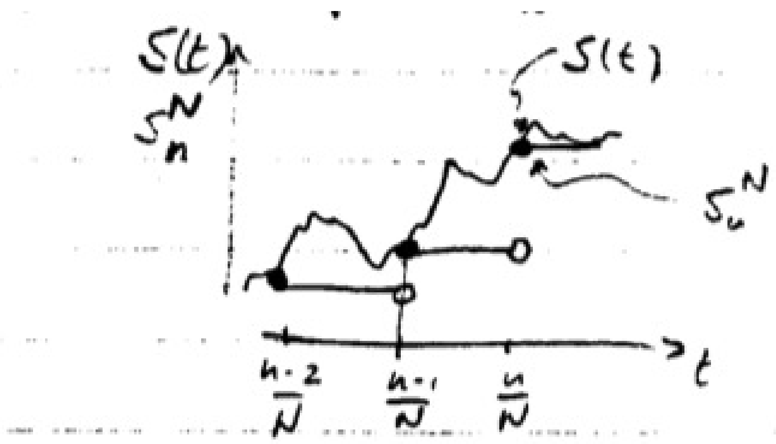
(1). time from 0 to 1 in N intervals (equal length).

(2). To properly prove the Itô process without N interval chops, one needs measure theory.

(3). $\int_0^t \mu(s,k)dk$ is the drift function, it is standard Lebesgue Integral, which can be viewed as the limit:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{[tN]} \mu(S_n^N, \frac{n}{N})$$

where S_n^N = "cadlag" jump process generated by combining $S(\frac{n-1}{N})$ and $S(\frac{n}{N})$: [6]



Note that here we have $\mu(S(t))$.

The function in the limit is like an average, so in the limit as $N \to \infty$, I expect that summation to exist because of the law of large numbers (LLN).

[6] Note that the limit may not exist if the function has too much variability, this function is an example so it has to be the Lebesgue Integral

2. $\int_0^t \sigma(S,k)dW_k$ is a stochastic integral of the Itô type [7]. Here we are integrating over a function of time W_t.

W_t is a Wiener process on $[0,1]$ (standard Brownian Motion).

$$dW_t = \varepsilon\sqrt{dt}, \Delta W_t = \varepsilon\sqrt{\Delta r}$$

$$\Delta W_t \sim N(0, \Delta t)$$

The integral can be viewed as the limit of [8]

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{n=1}^{[tN]} \sigma(S_n^N, \frac{n}{N})\varepsilon_n^N$$

Where ε_n^N are shocks with mean $E(\varepsilon_n^N) = 0$ and variance $E(\varepsilon_n^{N^2}) = 1$ and $W_t = \frac{\varepsilon_n^N}{\sqrt{N}}$, so the Brownian motion is a process that$\sim Gaussian(0, \sigma^2 T)$

Conclusion: An Itô process is, at any moment of time, accumulation of drifts and the stochastic terms.

The Lebesgue integral is well understood. It is also known that the limit of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{[tN]} \mu(S_n^N, \frac{n}{N})$$

would be no different if we were to evaluate μ at other points in the interval $[\frac{n}{N}, \frac{n+1}{N}]$ than the beginning.

However, the latter in the equation is **NOT** the case for the Itô process. [9] [10]

To see that Itô integral makes sense, i.e., that its existence can be shown, we take the following special case.

[7] Note that there are other stochastic integrals such as backward Itô integral

[8] Central Limit Theorem (CLT)

[9] When Itô proves the Itô integral make sense, he used functional central limit theorems.

[10] We set up the process by looking at the beginning of the interval and see what is the value of S and then keep it the same. This is purposely done because if we wait until the middle of the interval, we might get different result. The stochastic integral/the summation will still exist, but it is going to be different. So why do Merton choose the Itô integral? Because one makes decision at the beginning, not in the middle and one cannot know what it will be like in the middle of the interval.

Assume $\sigma(.) = \sigma$, a constant

Then by the Central Limit Theorem (CLT):

$$\lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{n=1}^{[tN]} \sigma \varepsilon_n^N \Leftrightarrow \int_0^t \sigma dW_t = \sigma W(t) \sim N(0, \sigma^2 t)^{11}$$

$$\varepsilon_n^N \sim N(0,1)$$

The fact that the limit above is **WEAK** (convergence in distribution) will be bothersome for finance, though.

Itô processes are continuous since they are the sum of drift and unpredictable part, they necessarily are not differentiable (as a function of time).

2. Shorthand notation:

$$dS = \mu(S,t)dt + \sigma(S,t)dW_t$$

$$\mu(S,t): \text{ drifts}$$

$$\sigma(S,t): \text{ stochastic terms}$$

[11] This is the Brownian motion

2 Itô's Lemma

1. Take $f : \mathbb{R}^2 \to \mathbb{R}$ to be a twice continously differentiable function of S and t.

What is the process of f?

The answer is that it is Itô, and:

$$f(s,t) = f(s,0) + \int_0^t (f_s\mu + f_t + \frac{1}{2}f_{ss}\sigma^2)dt + \int_0^t f_s\sigma dW_t$$

Where:

$$f_s = \frac{\partial f}{\partial s}, \quad f_t = \frac{\partial f}{\partial t}, \quad f_{ss} = \frac{\partial^2 f}{\partial s^2}$$

Notes:

(1). The equality holds almost surely - path by path. This is important for finance: it is not an approximation (see proof below). That is, if you give a path for an underlying process that is constructed for a particular dW_t, a particular brownian motion, then $f(s,t)$ uses the same dW_t

(2). In shorthand:

$$df = (f_s\mu + f_t + \frac{1}{2}f_{ss}\sigma^2)dt + f_s\sigma dW_t$$

The latter highlights the difference with the chain rule of calculus:

$$\text{Chain Rule } df = f_s ds + f_t dt$$

$$\begin{aligned} \text{Itô:}^{12} \quad df &= (f_s\mu + f_t + \frac{1}{2}f_{ss}\sigma^2)dt + f_s\sigma dW_t \\ &= f_s\mu dt + f_s\sigma dW_t + f_t dt + \frac{1}{2}f_{ss}\sigma^2 dt \\ &= f_s ds + f_t dt + \frac{1}{2}f_{ss}\sigma^2 dt \end{aligned}$$

(3) Itô's lemma can be extended to the case where s is a vector Itô process.

2. Ourc "proof" will use shorthand notations, we will come across tons of the following forms:

$$\alpha dt^{3/2}, \quad \alpha dW_t dt, \quad \alpha dW_t^2$$

[12] for higher order terms see proof below

where α usually is some function of s and t, but we will just take it to be constant here.

(1). $\alpha dt^{3/2}$ is shorthand notation for the limit [13]

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \frac{1}{N} \sum_{n=1}^{[tN]} \alpha$$

Of course, $\frac{1}{\sqrt{N}} \frac{1}{N} \sum_{n=1}^{[tN]} \alpha \to 0$, so we will state that $\alpha dt^{3/2} = 0$, and, $\alpha dt^{\beta} = 0$ for $\beta \geq \frac{3}{2}$

(2). $\alpha dW_t dt$ is really shorthand for the limit:

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \frac{1}{N} \sum_{n=1}^{[tN]} \alpha \varepsilon_n \tag{1}$$

Again, equation (1) $\to 0$, so $dW_t dt = 0$

(3). αdW_t^2 is shorthand for the limit [14]:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{[tN]} \varepsilon_n^2 \tag{2}$$

By Law of Large Numbers (LLN), equation (2) $\to t$, so we have:

$$\alpha dW_t^2 = \alpha dt$$

Note here the stochastic term is gone. Think the sum and LLN!

3. "Proof". Apply a Taylor Series Expansion:

$$df = f_t dt + f_s ds + \frac{1}{2} f_{ss} ds^2 + \text{higher order terms}$$

For higher order terms dt^{β}, since $\beta \geq \frac{3}{2}$, higher orders terms $= 0$.

We have:

$$ds = \mu dt + \sigma dW_t$$

$$ds^2 = \mu^2 dt + \sigma^2 dW_t^2$$

$$df = f_t dt + f_s \mu dt + f_s \sigma dW_t + \frac{1}{2} f_{ss} \mu^2 dt^2 + \frac{1}{2} f_{ss} \sigma^2 dW_t^2$$

[13] dt is $\frac{1}{N}$
[14] dW_t is $\frac{\varepsilon}{N}$

Again for dt^2, $2 > \frac{3}{2}$ so $\frac{1}{2}f_{ss}\mu^2 dt^2 = 0$

$$df = f_t dt + f_s \mu dt + f_s \sigma dW_t + \frac{1}{2} f_{ss} \sigma^2 dt$$

3 The Self-Financing Constraint

Self-Financing Constraint: you can get pay from the old portfolio, realize all the assets, and pay for the new portfolio, in a way that you don't have add money but also there is no money left over.

(1). consider a portfolio of n units of an asset, with price S (the "stock") and m units of another asset, with price C (the "option"). m and n may change over time, as a function of s, c or/and time t. S and C follow Itô process, so m and n are Itô process as well.

(2). The value of this portfolio, V, is:

$$V = mC + nS$$

(3). By Itô's lemma (Multivariate version):

$$dV = mdC + ndS + Cdm + Sdn + dCdm + dSdn$$

(4). We are going to impose a self-financing constraint: each time the portfolio is rebalanced, inflows and outflows of cash should cancel. This will imply:

$$Cdm + Sdn + dCdm + dSdn = 0$$

Hence,

$$dV = mdC + ndS$$

(5). It is best to see how this obtains in discrete time: rebalance a portfolio (established at time t) at $t + \Delta t$

$$V_t = m_t C_t + n_t S_t$$

$$V_{t+\Delta t} = m_t C_{t+\Delta t} + n_t S_{t+\Delta_t}$$

for the portfolio to be self-financed, it must be that:

$$V_{t+\Delta t} = m_{t+\Delta t} C_{t+\Delta t} + n_{t+\Delta t} S_{t+\Delta_t}$$

i.e.

$$m_t C_{t+\Delta t} + n_t S_{t+\Delta_t} = m_{t+\Delta t} C_{t+\Delta t} + n_{t+\Delta t} S_{t+\Delta_t}$$

or

$$(m_{t+\Delta t} - m_t) C_{t+\Delta t} + (n_{t+\Delta t} - n_t) S_{t+\Delta t} = 0$$

Rewrite this:

$$= (m_{t+\Delta t} - m_t)(C_{t+\Delta t} - C_t) + (n_{t+\Delta t} - n_t)(S_{t+\Delta t} - S_t) + (m_{t+\Delta t} - m_t) C_t + (n_{t+\Delta t} - n_t) S_t = 0$$

Let $\Delta t \to 0$ (in limit term/continuous term):

$$dmdC + dndS + mdC + ndS = 0$$

This is the self-financing constraint in part (4) above.

Note that self-financing constraint is linear in m and n, this is because Black-Scholes depends on a self-financed portfolio of the stock and bank account/bond that mimics the payoff of an option, or another way to say this, a portfolio of stocks and options that is risk-free, but it needs to be self-financing.

Therefore, two constraints:

(A) First is has to be risk-free, hopefully it is a linear constraint

(B) Second is self-financing constraint.

For two unknowns, two constraints, if the system is linear, then we have solutions. If not linear, then it is hard to proof that there exists a solution pair.

4 The Black Scholes Model

(1). Assume:

$$dS = \mu S dt + \sigma S dW_t \text{ (Stock)}$$

$$C = C(S,t), \text{ twice differentiable (Call option)}$$

Value of a riskfree bank account increases at a constant rate: **rdt**

(2). Note: if $T = Maturity$, $K = Stike\ Price$, then we will have the following boundary conditions [15]:

$$C(S,T) = \max(0, S-K)$$

$$C(0,t) = 0$$

$$\lim_{S\to 0} C_s(S,t) = 1 \text{ where } C_s = \frac{\partial C}{\partial S}$$

Note why is $C(0,t) = 0$?, that is, when stock price moves towards zero, the call price move towards to zero as well?

Because of the boundary conditions in economics. The original assumption here is that the stock price follows a Geometric Brownian Motion, hence once the price hits zero, then it will not move any more, $ds = 0$. The assumption implies that there is zero probabilities that the price will hit zero because when the stock price move towards zero, the variance is also zero.

Now what will the call price behave when stock price is high? Be careful here, the call price does not converge to stock price when stock price is high, we can only say that the stock price is always greater than the call price. However, it is the growth rate/derivative that converges to one. That is,

$$\lim_{S\to 0} C_s(S,t) = 1 \text{ where } C_s = \frac{\partial C}{\partial S}$$

(3). We are going to prove that solving for C will give us the following equation:

$$\frac{1}{2}\sigma^2 S^2 C_{ss} + rSC_s + C_t - rC = 0$$

[15] Note that if we use a put options, the boundary condition values are different from that of using call options

subject to boundary conditions in (2).

(4). Proof

Consider a portfolio of n stocks and m options such that:

(a) self-financed

(b) risk-free

If constructing such a portfolio is possible, then its value V must increase like that of a riskfree bank account. [16]

$$dV = rVdt \tag{3}$$

Then by Itô's lemma:

$$dC = \mu SC_s dt + \sigma SC_s dW_t + \frac{1}{2}\sigma^2 S^2 C_{ss} dt + C_t dt$$

Hence, if we choose m to make the portfolio self-financed,

$$dV = mdC + ndS = m\mu SC_s dt + m\sigma SC_s dW_t + m\frac{1}{2}\sigma^2 S^2 C_{ss} + n\mu Sdt + n\sigma SdW_t \tag{4}$$

To make increments of V riskfree, choose n such that:

$$m\sigma SC_S dW_t = -n\sigma SdW_t \tag{5}$$

So:

$$n = -mC_s$$

C_s is called "hedging delta".

Combining equation (3), (4) and (5):

$$rVdt = rmCdt - rmC_s Sdt$$

$$= m\mu SC_s dt + m\frac{1}{2}\sigma^2 S^2 C_{ss} dt - mC_s \mu Sdt + mC_t dt$$

$$rCdt - rC_s Sdt = \mu SC_s dt + \frac{1}{2}\sigma^2 S^2 C_{ss} dt - C_s \mu Sdt + C_t dt$$

[16] The typical risk free arbitrage free, or no free lunch proof

We have:

$$\mu S C_s dt = 0$$

$$C_s \mu dt = 0$$

Hence, re-arrange the equation, we obtain:

$$\frac{1}{2}\sigma^2 S^2 C_{ss} + rSC_s + C_t = rC$$

In summary, the steps are:

1. Construct a portfolio that combines n stock and m option (Call)

2. m to figure out the self-financing constraint, n to make sure that the portfolio is risk-free

3. Apply Itô's lemma and then impose constraints the portfolio should accumulate value that a bank account does (at risk-free rate)

4. Generate the Black-Scholes equation

(5) Notes:

(A) The proof is based on an arbitrage-free argument, both the bank account and the portfolio of stock and option are risk-free. It is crucial that the latter is indeed riskfree. For that to be true, Itô's lemma must NOT be an approximation, but a pathwise equality.

(B) μ, the drift term, does not enter the BS partial differential equation. That is, the option price does not depends on μ. This is specific to the case where S is an Itô process. The results does not obtain in general.

(C) The weakness of this theory is that the limiting Itô processes are only approximately actual processes in distribution (from the Central Limit Theorem (CLT) arguments in the discussion of the Itô integral). This means that they approximate a whole family of discrete-time price processes with vastly different sample paths. For example, the binomial model, where one can perfectly replicate the payoff on a risk-free bank account with only the stock and an option, as in the limiting BS world.

Contrast this with the trinomial model, which also has BS as limit, that is, the model also converges to Black-Scholes model, but on each step, there is implied no arbitrage because where it is **impossible** to replicate the bank account with only the stock and an option (three possible outcomes and yet you only have two securities).

This is very **important**. In our world, the no arbitrage principle only applies when we assume a binomial process, in the limits, yes we will obtain Black-Scholes model because all kinds of processes (binomial, trinomial, etc.) will converge to Brownian motions. However, we cannot even prove that binomial is the right process when we have discrete time.

(D) It is interesting that we can use σ, the volatility of the stock to price an option. Remember from the general theory, the link between the state-price probability and physical probabilities is at zero probability. So why is the volatility important? That turns out to be very specific to Itô process. The underlying process (the process of the stock) is an Itô process. An Itô process is a process that is continuous but not differentiable (continuous but unpredictable). Finance academics insist on non-predictable but continuous processes (hence, can only be non-differentiable processes) so the processes must be locally-Gaussian processes (i.e. the changes are Gaussian). The reason is that you can carry information (volatility in this case) from the physical world to the pricing world (state-price probability), i.e. you can price options using information from the real world is because you are in Itô world. In a Itô world, under the no-arbitrage principle, when you change the measure, and you make things martingale, you can only work with other Itô processes. (put it into another way, if your process is Itô, the equivalent martingale measure must be Itô as well)

(E) If one assumes continuous non-differentiable processes, we are not allowed to price using poisson process because that is a non-continuous almost always differentiable process. Under absence of arbitrage principle, there exists an equivalent martingale measure with which you can price everything, that is, there are probabilities such that all returns on all assets that are equal to the risk-free rate, equivalent meaning only at zero probability events will carry over. In this world of Itô process, one has to stay in this world to be able to carry information from the physical world to pricing world (in this case, volatility). If one wants to use another process, it has to satisfy the Girsanov theorem.

Black-Scholes equations **CANNOT** be applied under processes other than Itô process.

Black-Scholes model uses probability-pricing world where non-continuous discreteness and differentiability have zero probability happen.

Please do not use Black-Scholes if you believe that there are jumps out there.

The Black-Scholes-Merton differential equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial s^2} = r_f$$