Core Concepts

in

Asset Pricing Theory

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Contents

1 Absence of Arbitrage

1. Suppose we have two dates, S states, N securities with price p_n (vector P)

Payoff matrix: s: states, n: securities

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1s} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2s} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{ns} \end{bmatrix}$$

2. Assume all prices are martingale and $r_f = 1$

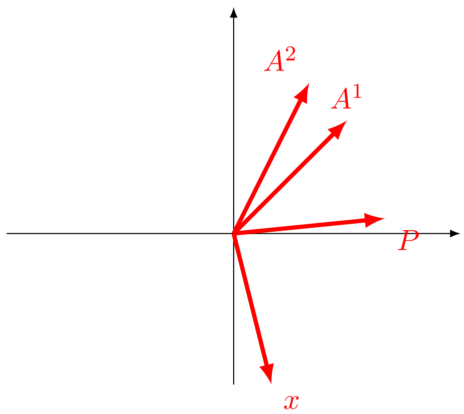
Arbitrage means:

$$\exists x_n : \sum_{i=1}^{n} x_i p_i \leq 0 \ and \ Ax \geq 0$$

$$(or \ p^{'} x < 0, \ Ax = 0)$$

Interpretation: this is effectively saying that there exists a portfolio that costs you nothing but gives you a positive return in the future. Or the portfolio will give you nothing in the future, but it gives you something now (negative cost).

Graphically:



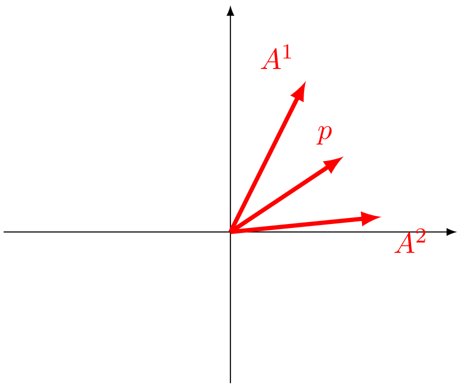
3. Theorem of Alternation:

(2) has no solution iff

$$\exists \pi_1, ..., \pi_s (\rightarrow \ vector\ \pi)$$

$$\pi > 0\ and\ p_n = \sum_s \pi_s A(s,n)$$

Graphically:



4. Normalize π so that $\sum_s \pi_s = 1$, then:

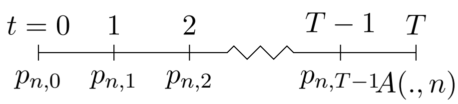
$$p_n = E[A(.,n)]$$

That is, I can write price as an expectation of payoffs.

5. This is very powerful, since prices are expectations, they have to statisfy all restrictions of expactations: Jensen's inequality, Cauchy's inequality, etc.

6. Also note that they are conditional expectations in a dynamic context (t=0,1,2,...)

$$p_{n,t} = E[A(.,n)|I_t]$$



7. In experiments, we can check if: $p_{n,t} = E[A(.,n)|I_t]$

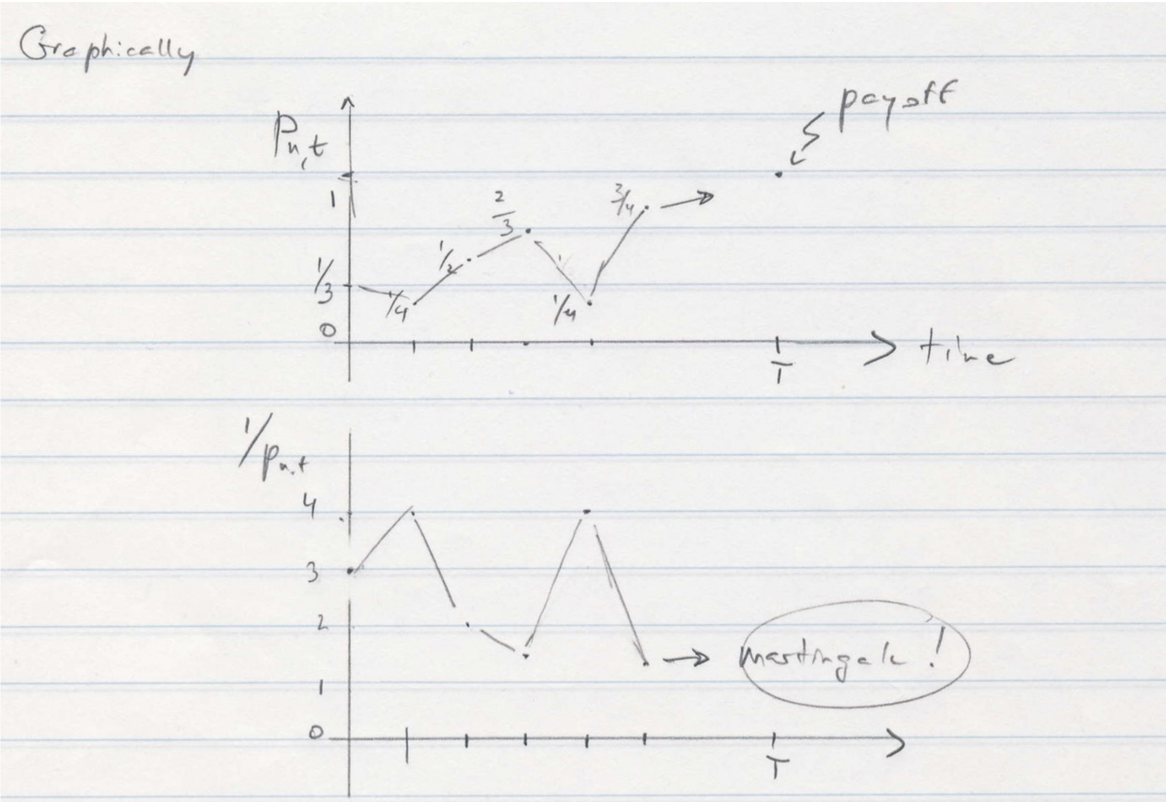
(a) implied $\pi_s \geq 0$

(b) satisfy Bayes' law in a dynamic context

8. How do we verify 7(b)?

Take $N = S$ and $A = I$ (state securities or AD security), then:

$$E[\frac{1}{p_{n,t+1}}|I_t; state = n] = \frac{1}{p_{n,t}}$$



Proof:

$$E[\frac{1}{p_{n,t+1}}|I_t; state = n]$$

$$= \int \frac{1}{p_{n,t}(signal, I_t)} L(signal|I_t, state = n) d(signal)$$

where: $L(signal|I_t, state = n)$ is the likelihood function and technically cannot be zero.

$$= \int_{signal} \frac{\sum_s L(signal|I_t, state = S) p_{s,t}}{L(signal|I_t, state = n) p_{n,t}} L(signal|I_t, state = n) d(signal)$$

$$= \frac{1}{p_{n,t}} \sum_{s} \int_{signal} L(signal|I_t, state = s) d(signal) p_{s,t}$$

$$= \frac{1}{p_{n,t}}$$

9. So it is NOT "prices should not be predictable" (Plott & Sunder, 1982) that typifies markets with absence of arbitrage, it is what we proved in 8! see (Bossaerts, 2002)

10. We generally argue: absence of arbitrage typifies a well-functioning market. However,

$$Absence\ of\ Arbitrage \Leftarrow\ Equilibrium$$

$$Absence\ of\ Arbitrage \not\Rightarrow\ Equilibrium$$

Note you don't have to be in the equilibrium to have absence of arbitrage (which is a much weaker assumption)

2 Walrasian Equilibrium

1. We define the following

x_i^j: units of asset i, agent $j \rightarrow$ vector x^j

P_i: price of asset i ("today") $\rightarrow$ vector P

R_i: payoff on asset i ("tomorrow") $\rightarrow$ vector R

e_i^j: endowment of asset i $\rightarrow$ vector e^j

u^j: utility function

2. Portfolio optimization problem determines demands:

$$\max_{x_i} \; E[u^j(R^{'}x^j)]$$

$$s.t. \; p^{'}x^j \leq p^{'}e^j$$

Note here we take price $p^{'}$ as given: "competitive analysis"

First Order Condition:

$$E[\frac{\partial u^j(R^{'}x^j)}{\partial x_i^j}] \sim P_i \quad \forall i$$

$$Solve \;\; \Rightarrow \;\; x_i^j = \hat{x}_i^j(P)$$

3. Walrasian equilibrium: $\exists P^*$ such that $\sum_j \hat{x}_i^j(P^*) = \sum_j e_i^j \qquad \forall i$

4. Existence proof: fixed-point theorem

5. Optimality:

We need complete markets (# of markets = # of states).

Question is: How many securities do you need for complete markets, assume S states and N goods?

In Walrasian Equilibrium: $S \times (N-1)$

In Arrow (1964): $S + N - 1$

Why?

In Walrasian world: we need to have all the possibilities at time $t = 0$; all trades finished at time $t = 0$ and there will be no re-trade in the future.

In Arrow world (Arrow, 1964):

Today: at time $t = 0$, we trade for state-contingent claims (like insurance contract to insure future states), therefore we do trade at $t = 0$ (S markets)

Tomorrow: at time $t = 1$, now we know one state occur, we do re-trade (e.g. trade state-contingent claims for goods, therefore we need $N - 1$ markets)

In total, $S + N - 1$

6. Remarks:

(a) Parsimony: NEED ONLY P!

Announce $P^* \Rightarrow$ Walrasian Equilibrium

We don't need: $u^{j'}$ and $e^{j'}$ for $j' \neq j$ (This is important in experiment)

We can event have: $E^{j'} \neq E^j$, which means different beliefs, and we don't need to know $E^{j'}$.

(b) With smooth u^j (All standard preferences except some ambiguity aversion!):

$$\boxed{\text{EVERYONE IS ALWAYS "MARGINAL"}}$$

The idea of a specific "marginal investor" (smart, specific preference,...) that mainstream finance has IS NOT consistent with Walrasian Equilibrium.

(c) Walrasian Equilibrium is the core concept behind most mainstream (static) asset pricing theory: CAPM, (equilibrium) APT

And complete-markets (but NOT dynamically complete markets) equilibrium: Arrow-Debrew

3 Radner Equilibrium 1

Question: can you have fewer securities than sates?

Yes but you need dynamically complete market.

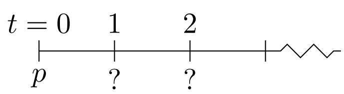
Dynamically complete markets are markets where you can actually generate anything you want, even if there are fewer securities than states, by re-trading in between. (BSM: you can replicate the payoff of an option by buying and selling two securities, as long as you do it as often as enough, but this requires that you can re-trade at the price you expect, you have to have the correct expectations of prices)

BSM is wrong in the sense that in the real world, people do not have the correct expectations of prices (rational expectations)

This is the core equilibrium theorem for incomplete markets.

Core: Rational Expectation Equilibrium, or Perfect foresight equilibrium, meaning that we know the mapping from states to prices. (or we have the correct foresight of prices given states)

1. Core equilibrium concept for multi-period, incomplete markets (including dynamically complete markets)



For $t = 1, 2$, since these markets have not open yet, people cannot get any information about prices.

In complete market (Arrow-Debrew), there is no re-trade, and we can predict the price from prices at $t = 0$, they have to wait until the market open tomorrow open to get prices.

2. Basically, agents need to posit

$\Leftrightarrow$ Mapping from all possible states ω at $t = 1$ to price line.

They need to make conjecture of prices given states.

(e.g. They need to make conjecture of prices of ice-cream when sunshine and prices of ice-cream when rain.)

3. Radner Equilibrium:

1). Markets clear like Walrasian Equilibrium

2). Everyone uses the correct mapping from states $\rightarrow$ price to determine demands. Demands depend on return on securities R_i, which are partly dividends (exogenous, state-independent) and partly future prices (state-dependent):

$$R_i(\omega) = D_i(\omega) + P_i(\omega)$$

4. Remarks:

(a) Now we need more than prices for markets to be at equilibrium.

We need to prove equilibrium prices P^* and mapping for future prices $\omega \rightarrow P(\omega)$

(b) Equilibrium may not be Pareto optimal (need dynamically complete markets-more later)

Note: why do so many people claim that competitive financial markets are "good?"

This is WRONG, we need complete market or dynamically complete market and rational expectations (perfect foresight)

(c) Remember (a) is problematic even in the experiments

5. Simplest possible example (Myth):

The drive home point is that Radner DOES NOT assume that beliefs E^i (i = individual) are the same, which is usually an empirical auxiliary assumption (Xiong, Scheinkman)

(a) Two types of producers $i = 1, 2$

One good with demand $Q = ap + u^1$ $(a < 0)$.

u is unknown, so it is a random variable $(r.v.)$ [2] that lives on (Ω, F) [3]

Beliefs of i: probability p_i on (Ω, F) (this won't be a common knowledge)

Assume (common knowledge) that: $E^i(u) = E^i[E^j(u)]$ $(= \lambda_i)$ (Known as the "Average Opinion Rule") [4]

(b) Production set before price determined:[5]

$$q_i^* = \frac{E^i(p) - \beta}{\alpha}$$

Cost:

$$Cost: \quad \frac{1}{2}\alpha q_i^2 + \beta_{q_i} + \gamma$$

Walrasian part of equilibrium (tomorrow):

$$Demand = Supply$$

$$Q = q_1^* + q_2^*$$

$$Q = \frac{E^1(p) - \beta}{\alpha} + \frac{E^2(p) - \beta}{\alpha}$$

(d) Radner part of equilibrium: E^i is based on knowing true mapping from states $(u, \lambda^1, \lambda^2)$ [6] to price line.

(e) Posit (A linear mapping):

That prices in equilibrium depends on $(u, \lambda^1, \lambda^2)$

$$P(\lambda^1, \lambda^2, u) = \delta_0 + \delta_1\lambda^1 + \delta_2\lambda^2 + \delta_3 u$$

Somehow, someone tell them δ_0, δ_1, δ_2 values

[1] u is shock

[2] a way to treat uncertainty

[3] Probability space where Ω is the outcome space and F is the events.

[4] What i thinks about u is, on average, what i thinks j thinks about u. (In equilibrium) I expect to think that you expect to think the same way as a I do.

[5] q_i^* is the optimal quantity.

[6] λ^1 is what first individual thinks about u

Since individual $i = 1$ knows λ^1 and expect $\lambda^2 = \lambda^1$ as a result of "Average Opinion Rule" (see above, note $i = 1$ does not actually know what λ^2 is, but he/she expects $\lambda^2 = \lambda^1$):

$$E^1(p|P(\lambda^1, \lambda^2, u)) = \delta_0 + (\delta_1 + \delta_2 + \delta_3)\lambda^1$$

$$E^2(p|P(\lambda^1, \lambda^2, u)) = \delta_0 + (\delta_1 + \delta_2 + \delta_3)\lambda^2$$

Then use equilibrium condition (see part (c)) to solve for:

$$\delta_0 = \frac{1}{\alpha a}(\delta_0 - \beta)$$

$$\delta_1 = \frac{1}{\alpha a}(\delta_1 + \delta_2 + \delta_3)$$
$$\delta_2 = \frac{1}{\alpha a}(\delta_1 + \delta_2 + \delta_3)$$
$$\delta_3 = -\frac{1}{a}$$

Note that the above results show that it is feasible for us to think that linear pricing is okay.

(f) In Radner equilibrium, agents know and use these values of δ_0, δ_1, δ_2, δ_3.

6. Where do agents get knowledge at δ_0, δ_1, δ_2, δ_3 from? Repetition of environment is not enough (Jordan, etc.)

That is, is it possible for people to learn (or to update) overtime to become rational [7]? If you repeat the environment, will you get this equilibrium, will the market ultimately converge to equilibrium? Jordan proves that this may never happen, that is, the economy will never converge.

> However, this is equilibrium concept of modern dynamic asset pricing theory
>
> Merton, Lucas,......

Simple example:

At $t = 0$, you expect that ice-cream is gonna be cheap when the sunshine. At $t = 1$, the sunshines (fact) and the ice-cream is expensive, so you say "okay, I made a mistake but I want

[7] Note the definition of rational: perfect foresight of prices given states, NOT finance rational

more ice-cream", you now want to correct yourself. Since now you know that ice-cream is more expensive than what you expect, you are now going to make sure that I will have more money when sunshine. Then you will buy a contingent claim that gives me a lot of money when the sunshines. What is going to happen the next time sunshines? We do not know yet, but it is not going to be the same as the first time because now everyone else will change plans accordingly. You learn about the wrong thing overtime.

Note here: you NEED the assumption of "Average Opinion Rule".

7. Note: if agent 1 in (5) incorrectly assume that $E^1[E^2(u)] \neq \lambda^1$ then there will be an inconsistency between:

a. announced δ_0, δ_1, δ_2, δ_3

b. coefficients that she herself could derive from structural knowledge of economy and her beliefs. (Bossaerts, 2002).

I.E. Once you drop Average Opinion Rule, the equilibrium collapse.

What do we do then?

Note however, Radner shows the rational expectation equilibrium generically where individual read information from prices. (Another type of Equilibrium).

The other "rational expectation equilibrium", an equilibrium where you read information from prices. (Radner, 1979)

In this particular setting, I am a professor, I know nothing about anything in the world. I go to the market and I trade against that I know are better informed, Goldman Sachs. What I do is, I am careful because I know when I see a higher price, it may be because of two reasons. First one is there is little supply, second reason is there are people who knows inside information. So I should adjust my portfolio based on that. What Radner shows is that prices, generically, fully reveal all information.

4 Radner Equilibrium 2

Generate optimal allocations [a] in an incomplete market

[a] Or complete market allocations

1. This is a type of equilibrium used in asset pricing models with heterogeneous information (Admati, 1985; Biais, Bossaerts, & Spatt, 2010; Grossman & Stiglitz, 1980). Under this model, asymmetric information exists (Noise) and does matter. If you observe price increase, there are two possible reasons, first is there is more demand, and second there is asymmetric information (Someone knows something that you don't know). The proportion of this stock in your portfolio goes up (value), so you should adjust the weight accordingly. Biais et al. (2010) shows that this can generate more return than a simple buy and hold index strategy.

2. Also referred to as "Rational Expectation Equilibrium" or "Noisy Rational Expectation Equilibrium" (to avoid (generic) full revelation of information).

3. It is the theoretical foundation of the "Efficient Market Hypothesis". Note here we have very strong assumption here, everyone knows the mapping from prices to states, i.e. the information that insiders have, everyone knows what they are discussing relates to the stock price (and how they are related).

4. Without noise, agents know SO MUCH (Much more that is revealed in standard real-world markets) that equilibrium is generally fully revealing (markets are generically strong-form efficient (if we use Fama's words)).

5. Simplest possible example:

(a) Forward contract, pays $x - p$ to long side when $p =$ forward rate/price, x is unknown.

(b) Hedgers take exogenous position S, unknown to speculators.

(b) Speculators [8]:

Type H: receive signal H

[8] H is going to estimate N, and N is going to estimate H

Type N: receive signal N

$$x = H + N$$

(d) S, N, H, random variables, live on (Ω, F) [9]

True probability: p^*

Beliefs: p^i $i = H, N$, $\lambda^i = E^i(S)$ (p^i, p^* such that all variables are Gaussian $N(0,1)$) [10]

Average opinion rule: [11]

$$E^N(A^H) = \lambda^N$$

$$E^H(A^N) = \lambda^H$$

(e) Speculators maximize:

$$\max_{D^i} \quad E^i[D^i(x-p)] - \frac{1}{2}(D^i)^2$$

where: $\frac{1}{2}(D^i)^2$ is the quadratic position cost.

So

$$D^{i*} = E^i[x - p|information]$$

(f) Information:

1). signal H or N plus:

2). mapping states (H, N, λ^H, λ^N [12], S) to prices (Invest prices for information)

(g) Radner equilibrium:

Posit [13]:

$$P() = b(H+N) + c(\lambda^H + \lambda^N) + dS$$

$P()$ solves equilibrium: $S = D^H + D^N$

where demands are from:

$$E^N[x - p|N, P, P()]$$

[9] Probability space, S is the supply of the hedges, or the net stuff that is not generated by the speculators.

[10] H, N could have different beliefs, so here we allow to have disagreement.

[11] H thinks N thinks about the supplies, on average

[12] If H is optimistic about N, this will affect prices

[13] Here we use the same coefficient b, c because of the same ex ante beliefs for both H, N

$$E^H[x-p|H,P,P()]$$

For example[14]:

$$E^N[x-p|H,P,P()]$$

$$= H + E^H[N|H,P,P()] - p \quad E^H \textit{ is projection}$$

$$= H - \frac{b}{b^2+c^2+d^2}(bH + c\lambda^H + cE^H[\lambda^N|H,P()] + d\lambda^H)$$

$$+\frac{b}{b^2+c^2+d^2}P - P$$

$$= H - \frac{b}{b^2+c^2+d^2}(b+1+(2c+d)\lambda^H)$$

$$+\frac{b-(b^2+c^2+d^2)}{b^2+c^2+d^2}P$$

Use equilibrium condition to derive b,c,d

6. As with Radner equilibrium 1, if N incorrectly assumes that [15]

$$E^N(\lambda^H) \neq \lambda^N$$

Then there is a conflict between:

1). "Announced" b,c,d in equilibrium

2). what she herself can derive using structural knowledge of economy.

[14]Note in (d) we assume all Gaussian, so the regression is the same as linear projection. $\frac{b}{b^2+c^2+d^2}$ is just the slope coefficient, *covariance* over *variance*

[15]This is similar to Radner Equilibrium 1, if H,N has different beliefs over the other on average. then the equilibrium collapse.

5 Dynamic Completeness

1. Start with complete market. Gives Pareto Optimal allocations.

Question: can you implement this(complete market allocation) in an incomplete market when information "diffuses" overtime? or is it always possible to generate same allocations of complete market in the incomplete market?

Yes but you need to construct a Radner Equilibrium. (Duffie & Huang, 1986; Kreps, 1981)[16]

2. Example of complete market

(a) Two agents i, three states (probability: $\pi_s = \frac{1}{3}$)

(b) (Note here we have the perfect rank correlation (Pareto Optimality))

States	Endowment e_{is}	Optimal w_{is} [a]	Trade x_{is} (state security) Equilibrium imposed
$S = 1$	$(e_{i=1} = 1,\ e_{i=2} = 2)$[b]	(1.2, 1.8)	(+0.2, -0.2)
2	(1, 3)	(1.5, 2.5)	(+0.5, -0.5)
3	(4, 1)	(2, 3)	(-2, +2)

[a] Here we have perfect rank correlation
[b] Agent $i = 1, 2$

(c) Define marginal utilities u_s for $i = 1$ and v_s for $i = 2$, Lagrange multiplier λ for $i = 1$, μ for $i = 2$.

First Order Condition (FOC) for portfolio problem:

$$\frac{1}{3}u_s = \lambda p_s$$

$$\frac{1}{3}\sum u_s = \lambda$$

$$\frac{1}{3}v_s = \mu p_s$$

$$\frac{1}{3}\sum v_s = \mu$$

Budget Constraints:

$$p_1 0.2 + p_2 0.5 - p_3 2 = 0$$

[16] First raised by Kreps, Duffie & Huang proved this.

(Note Utility only defined up to positive linear transformation) Set $\lambda = \mu = 1$ and $p_2 = \frac{1}{3}$ ($p_3 = 1 - p_1 - p_2$)

So: Budget Constraint is:

$$p_1(2.2) = 2 - \frac{2.5}{3}$$

Find u_s, v_s, p_1.

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.2 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \\ p_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{3} \\ 2 - \frac{2.5}{3} \\ 1 \\ 0 \\ \frac{1}{3} \end{bmatrix}$$

Solution:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \\ p_1 \end{bmatrix} = \begin{bmatrix} 1.5909 \\ 1 \\ 0.4091 \\ 1.5909 \\ 1 \\ 0.4901 \\ 0.5303 \end{bmatrix}$$

$$p_3 = 0.1364$$

Notice:

graph

Complete market $\Rightarrow \exists$ representative agent supporting the prices.

3. **Now assume that markets are incomplete.** However, information about state S becomes available early, at which point re-trade is allowed.

(a) Asset Y, price b_t ("bond") and Asset Z, price s_t "stock")

(b)

			Y(s)	Z(s)
ϕ [a]	a	s=1	1	0
		s=2	1	1
	b	s=3	1	2

[a] Half way through we will be told whether state 3 will occur or not

(c) Imagine the following prices for Y,Z

$$b_0 = b_{1a} = b_{1b} = 1$$

b is price of bond, subscript 1 is time, a, b are intermediate states

$$S_{1b} = 2$$

From the complete market case $\dfrac{p_1}{p_1 + p_2} = 0.6410$

$$S_{1a} = 0\frac{p_1}{p_1 + p_2} + 1\frac{p_1}{p_1 + p_2} = 0.3860$$

$$S_0 = 0p_1 + 1p_2 + 2p_3$$

$$= \frac{1}{3} + 2(0.1364) = 0.6061$$

(d) Question:

Can our agents implement a trading strategy in two securities that:

1) generate net trades from complete markets

2) costs NOTHING (since they don't have any endowment at $t = 0$.

(e) Yes, focus on $i = 1$

e_1 : At $t = 1$, state a :, need quantities $y^{i=1}_{time=1,\ state=a}, z^1_{1a}$

So that:

$$y^1_{1a}1 + z^1_{1a}0 = 0.2$$

$$y^1_{1a}1 + z^1_{1a}1 = 0.5$$

Hence: $y^1_{1a} - 0.2$; $z^1_{1a} = 0.3$

COST?

$$y^1_{1a} b_{1a} + z^1_{1a} s_{1a} = 0.2 * 1 + 0.3(0.386) = 0.3158$$

e_2 : At $t = 1$, state b :,

$$y^1_{1b}1 + z^1_{1b}2 = -2$$

Take $y^1_{1b} = -2$; $z^1_{1b} = 0$

COST? $= -2 * 1 = -2$

e_3 : At $t = 0$, need to invest to generate Income at $t = 1$, namely:

0.3158 at state a.

-2 at state b.

$$y^1_0 1 + z^1_0(0.3860) = 0.3158$$

$$y^1_0 1 + z^1_0(2) = -2$$

So $y^1_0 = 0.87$, $z^1_0 = -1.435$

COST? (0.87) 1 + (-1.435)(0.6061)=0

4. There exists a Radner equilibrium [17] but this "implementation" of complete-market (optimal) allocations in incomplete market requires knowledge of complete-market state prices! (Walrasian Equilibrium)

or, equivalently, the continuation prices at $t = 1$ $(s_{1a}, s_{1b}, b_{1a}, b_{1b})$ consistent with complete markets case! (Radner 1 equilibrium)

Easy: Walrasian Equilibrium

Difficult: Radner Equilibrium 1

5. Conclusion: the standard dynamically complete optimal equilibrium in finance imposes very stringent condition on the markets!

How do agents know the true future state price mapping?

[17] There could be other Radner equilibriums

6 Equilibrium (Stability)

1. Back to Walrasian equilibrium. Is is equilibrium a foregone conclusion?

IT IS NOT!

2. Perspective: in physics , "law of entropy" justifies focusing on equilibrium (situation with minimal potential energy)

3. Walras' own defence: "Walrasian tatonnement"

(Auctioneers) change prices in direction of excess demand.

$\triangle p > 0$ if demand $>$ supply

$\triangle p < 0$ if demand $<$ supply

4. Formally, assume adjustment mechanism at time t is:

$$\frac{dp}{dt}(p^*)\ E(p)$$

where $E(p)$ is excess demand. Note that here p is a vector, we have multiple goods/assets.

5. Scarf:

(a) Three (types of) agents: $i = 1, 2, 3$

Three goods with prices: p_j $j = 1, 2, 3$, and quantities: x_j^i

Preferences:

For $i = 1$

$$u_1(x_1^1, x_2^1, x_3^1) = min(x_1^1, x_2^1)$$

$$(Endowment)\ \ e^1 = (e_1^1, e_2^1, e_3^1) = (1, 0, 0)$$

Excess demands:

$$E_1^1 = \frac{p_1}{p_1 + p_2} - 1$$

$$E_2^1 = \frac{p_1}{p_1 + p_2} - 1$$

$$E_3 = 0$$

For $I = 2$

$$u_2(x_1^2, x_2^2, x_3^2) = min(x_1^2, x_2^2)$$

$$(Endowment) \quad e^2 = (e_1^2, e_2^2, e_3^2) = (0, 1, 0)$$

...

For $I = 3$

$$u_3(x_1^3, x_2^3, x_3^3) = min(x_1^3, x_2^3)$$

$$(Endowment) \quad e^3 = (e_1^3, e_2^3, e_3^3) = (0, 1, 0)$$

...

(b) Total Excess Demands:

$$E_1(p) = -\frac{p_2}{p_1 + p_2} + \frac{p_3}{p_1 + p_3}$$

$$E_2(p) = -\frac{p_3}{p_2 + p_3} + \frac{p_1}{p_1 + p_3}$$

$$E_3(p) = -\frac{p_1}{p_1 + p_3} + \frac{p_2}{p_2 + p_3}$$

Walrasian Equilibrium ($E(p^*) = 0$): $p_1^* = p_2^* = p_3^*$ (c) Dynamics:

Normally adjust two markets; 3rd clears automatically.

Here: adjust all 3; change all 3 prices.

$$p^{'} = E(p)$$

So: $p_1^2 + p_2^2 + p_3^2 = c$ because $\frac{d}{dt}(p_1^2 + p_2^2 + p_3^2) = \sum p_j p_j^{'} = \sum p_j E_j(p) = 0$ by Walrasian's law.

Prices remain on the surface of sphere, equilibrium is a point on this surface.

(d) Do dynamics converge?

Notice: $p_1 p_2 p_3 = d$ (constant) as well.

proof:

$$\frac{d}{dt}(p_1 p_2 p_3) = p_1^{'} p_2 p_3 + p_1 p_2^{'} p_3 + p_1 p_2 p_3^{'} = 0$$

Note: demands (check!)

Then, starting from, say: $p_1^2 + p_2^2 + p_3^2 = 3$

1) Equilibrium on this sphere is $p_1 = p_2 = p_3 = 1$

2) If $p_1 p_2 p_3 \neq 1$ initially, dynamics can never reach equilibrium. So, $p^{'} = E(p)$ is bad dynamics!

6. How bad?

Sonnenschein-Mantel-Debreu theorem:

E(p) can be anything $\Rightarrow p^{'}$ can do anything.

So, very bad.

7. Do we care?

Economists/Finance people seem not to care. They continue to insist on $p^{'} = E(p)$

BUT!

(A). $p^{'} = E(p)$ may NOT be a good description of adjustment in actual markets.

(B). $E(p)$ may be well behave (eg. CAPM)

How do we know? from experiments.

8. Final remark

The problem raised in (5) is caused by the multi-good/security ($\# > 2$) nature of economy (Relevance of General Equilibrium!)

7 Contract Equilibrium

1. Contract (between principal and agents)

Standard analysis: "participation constraint" is kept constant/ But is really endogenous- if others offer contacts!

Let's see how this works if:

1) Adverse Selection

2) Principle compete to offer contracts

2. What does competition mean?

1). Game theories: # of principles is finite, lots of common knowledge, specific rules (extensive form).

2). "Institution XXXXXX": (Rothschild & Stiglitz, 1976)

Expected profits are zero on a contract by contract basis

3. Rothschild-Stiglitz example of insurance/bank loans/mortgages.(Rothschild & Stiglitz, 1976)

Agents: insurees sustain loss L with probability $\pi \in \{\pi_H, \pi_L\}$

$$\pi_H > \pi_L$$

$$\lambda \quad 1 - \lambda$$

Where $1 - \lambda$ is the fraction of potential insurees.

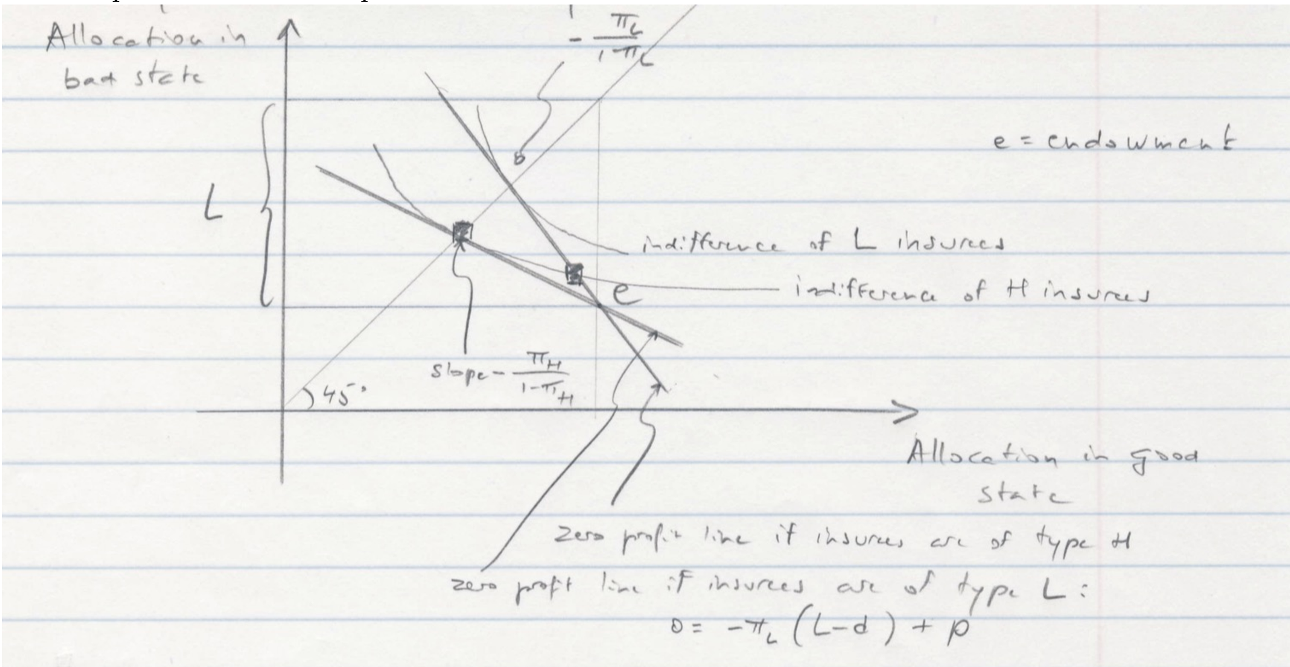
Principles: insurers offer contracts with

$$\text{Premium } p$$

$$\text{Deductible } d$$

(Note: in bank loans/mortgages, $p =$ credit spread, $d =$ downpayments

4. RS Equilibrium in state space:



5. Notice:

1) Separating equilibrium

2) Bad guys are fully insured (→ "subprime mortgages")

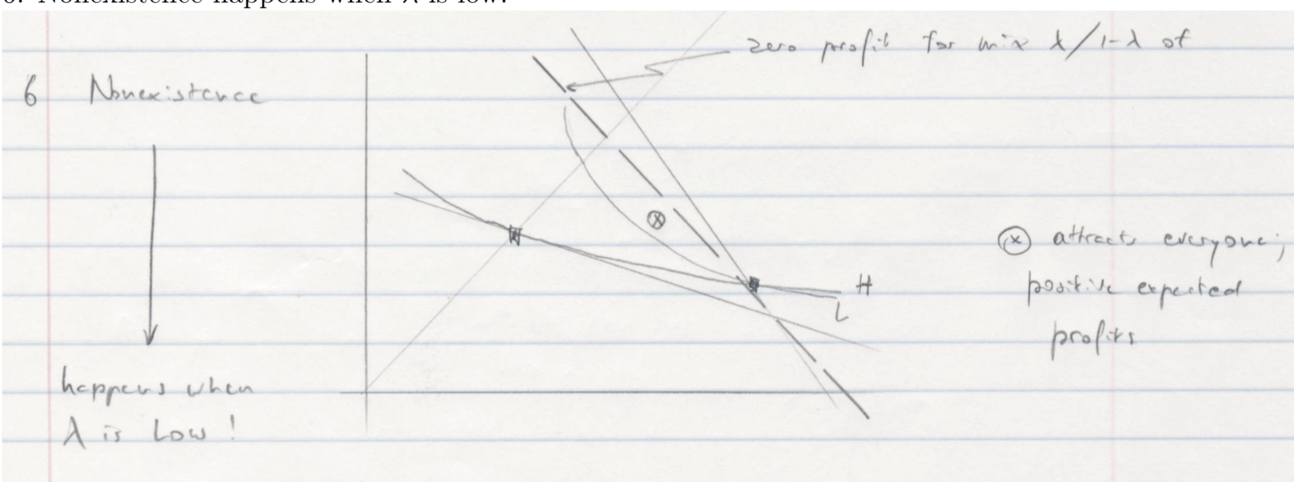
3) No need to check quality (because of separation)

4) Can never have pooling equilibrium

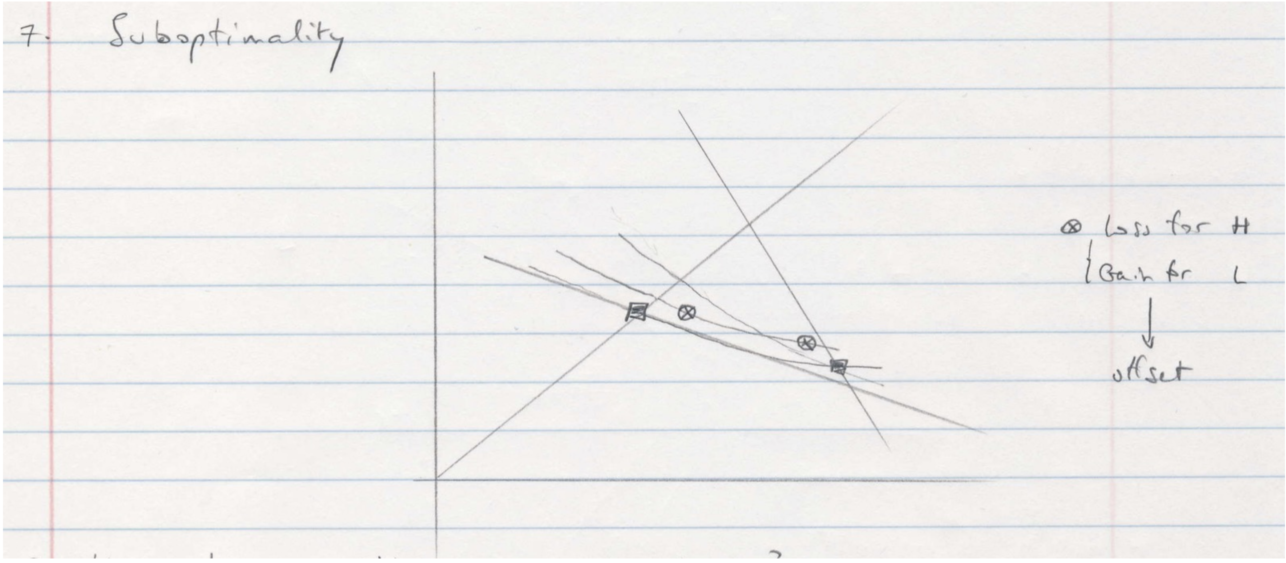
5) Equilibrium MAY NOT EXIST

6) Equilibrium MAY NOT be optimal!

6. Nonexistence happens when λ is low!



7. Suboptimality



8. How does equilibrium come about?

What institution to use?

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