Synthetic Forward FX Replication

& Collateralized Discount Factors

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Synthetic Forward FX – Interest Rate Parity

Interest Rate Parity (IRP)

- IRP states that the interest rate differential between two countries equals the forward exchange rate premium or discount relative to the spot rate and ensures no arbitrage opportunities as follows[1],

$$\frac{Fwd\ FX(USD/BRL)}{Spot\ FX(USD/BRL)} = \left(\frac{1 + r^{USD}t}{1 + r^{BRL}t}\right) \quad (1)$$

- The lower interest rate currency usually trades at a forward premium relative to the higher rate currency
- Rearranging (1) gives,

$$Fwd\ FX(USD/BRL) = Spot\ FX(USD/BRL)\left(\frac{1 + r^{USD}t}{1 + r^{BRL}t}\right) \quad (2)$$

- Rewriting in terms of **Discount Factors, P(0,t)** gives,

$$Fwd\ FX(USD/BRL) = Spot\ FX(USD/BRL)\left(\frac{P(0,t)^{BRL}}{P(0,t)^{USD}}\right) \quad (3)$$

[1] For Forward FX < 1-year simple compounding is used and thereafter annual compounding.

Synthetic Forward FX – Interest Rate Parity

Synthetic Forward FX Replication – **USD/BRL Example**

Steps to replicate the value of a Forward FX where we compute the value USD 1 at a future time, t.

- **Step 1:** Borrow USD 1

The PV of the future cash flow of USD 1 is $P(0,t)^{USD}$. We simply discount USD 1.

- **Step 2:** Convert the PV from step 1 into BRL using the FX spot rate S(USD/BRL) or S.

This gives a total value in BRL of $S\,.\,P(0,t)^{USD}$

- **Step 3:** Deposit these Funds until time t

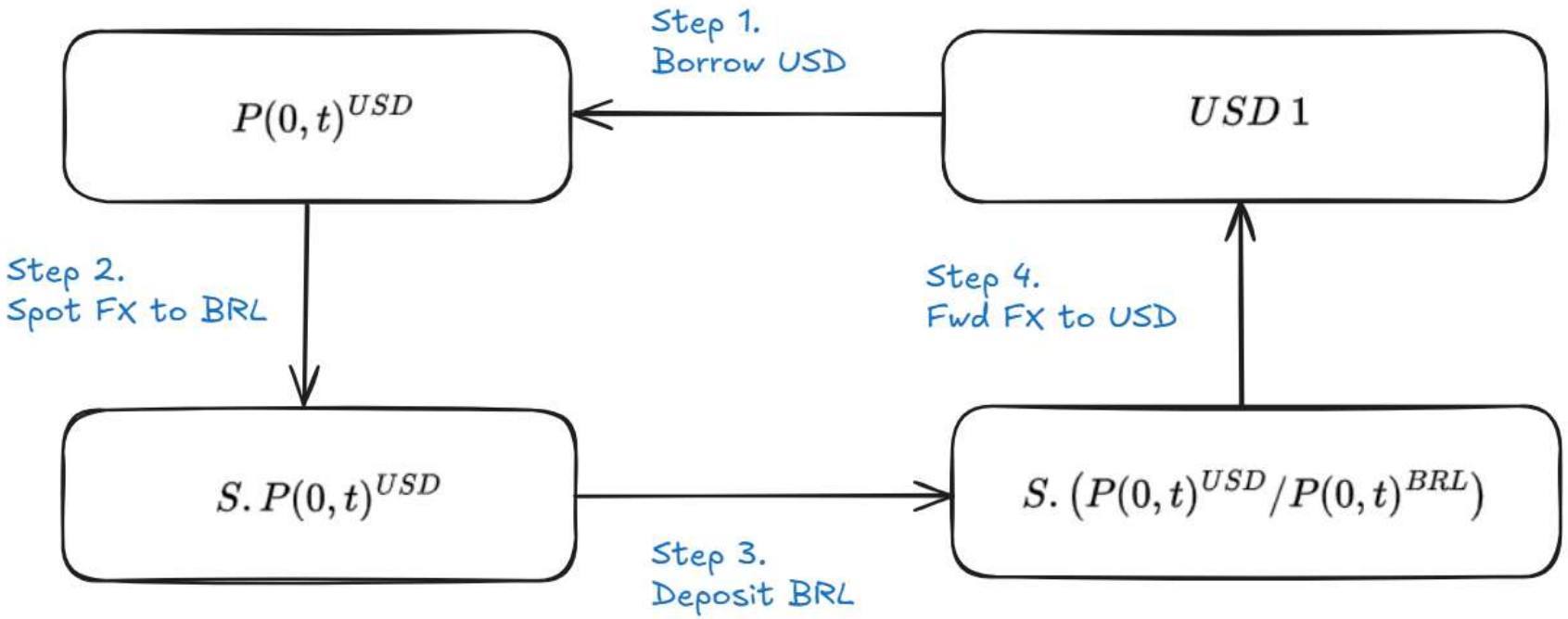
The BRL funds deposited have a growth rate of $1/P(0,t)^{BRL}$. This gives the forward FX value as,

$1\ USD = S\,.\,P(0,t)^{USD}/P(0,t)^{BRL}\ BRL$

The final result is the same result as in equation (3)

Synthetic Forward FX – Interest Rate Parity

Synthetic Forward FX Replication - **USD/BRL Illustration**



$$Fwd\ FX(USD/BRL) = Spot\ FX(USD/BRL)\left(\frac{P(0,t)^{BRL}}{P(0,t)^{USD}}\right)$$

Synthetic Forward FX – Interest Rate Parity

Example - Computation of USD/BRL Forward FX

- Consider the USD/BRL currency pair, where the spot FX is 5.6900. If we know that the one-year USD and BRL discount factors have values of 0.9563 and 0.8717 respectively. What is the synthetic value of the one-year USD/BRL Forward FX?

- Rearranging the equation (3) we have,

$$Fwd\ FX(USD/BRL) = Spot\ FX(USD/BRL)\left(\frac{P(0,1)^{BRL}}{P(0,1)^{USD}}\right) \qquad (3)$$

- This gives the one-year discount factor $P(0,1)^{\mathrm{BRL\_}USDCSA}$ as,

$$Fwd\ FX(USD/BRL) = 5.6900 \times \left(\frac{0.9563}{0.8717}\right) = 6.2422$$

Synthetic Forward FX – Interest Rate Parity

Bloomberg FX Interest Rate Arbitrage - Bloomberg FXFA <GO>

- Implying FX forwards can identify potential arbitrage opportunities.
- Market forward FX rates usually do not match the synthetic (or implied) FX forward rates
- Similarly, we can imply domestic and foreign interest rates or yields

Imply FX Swap — Depos to FX Swap

		7) FX Swap		8) USD Yield		9) BRL Yield		FX Swap Implied		Spread	
Term	Date	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask	Bid	Ask
10 ON	02/21/25			4.6700	4.6742	13.1500	13.1500	52.9847	52.9701		
11 TN	02/24/25	38.18	38.18	4.6706	4.6748	13.1477	13.1477	5.75	5.76	-32.43	-32.42
12 SP	02/24/25	5.7059	5.7064								
13 SN	02/25/25	6.40	9.19	4.6700	4.6742	13.1464	13.1464	20.56	20.57	14.16	11.38
14 1W	03/05/25	61.98	65.49	4.6724	4.6766	13.1575	13.1575	73.32	73.38	11.34	7.89
15 1M	03/24/25	274.09	277.47	4.6782	4.6824	13.3644	13.3644	304.62	304.84	30.53	27.37
16 2M	04/24/25	632.96	638.15	4.6876	4.6918	13.7363	13.7363	703.84	704.30	70.88	66.15
17 3M	05/27/25	1024.77	1031.68	4.6976	4.7019	13.9980	13.9980	1139.20	1139.93	114.43	108.25
18 4M	06/24/25	1366.23	1424.64	4.6449	4.6492	14.1558	14.1558	1541.15	1542.11	174.92	117.47
19 5M	07/24/25	1780.17	1817.34	4.6134	4.6177	14.2869	14.2869	2032.13	2033.34	251.96	216.00
20 6M	08/25/25	2221.71	2236.21	4.5943	4.5986	14.3868	14.3868	2516.50	2517.99	294.79	281.78
21 9M	11/24/25	3536.53	3557.53	4.5462	4.5505	14.5975	14.5975	3998.20	4000.47	461.67	442.94
22 1Y	02/24/26	4822.95	4844.05	4.4990	4.5030	14.7289	14.7289	5477.68	5480.58	654.73	636.53

Note: Outright Forward FX = Spot FX + (Forward Points * Pip Size), where pip size is often 10,000

Synthetic Forward FX – FX Forward Invariance

FX Forward Invariance

- Derived from Xccy Swaps where we have collateral posted in a single currency
- FX Forward Invariance assumes forward FX is constant for any given collateral posted or CSA
- Similar to Interest Rate Parity, however discount factors are collateralized in a single common currency
- Using USD/EUR as an example the FX Forward Invariance formula looks as follows:

$$f(t)^{USD/EUR} = s^{USD/EUR}\underbrace{\left(\frac{P(0,t)^{\text{USD}\_EURCSA}}{P(0,t)^{\text{EUR}\_EURCSA}}\right)}_{\text{EUR } Collateral} = s^{USD/EUR}\underbrace{\left(\frac{P(0,t)^{\text{USD}\_USDCSA}}{P(0,t)^{\text{EUR}\_USDCSA}}\right)}_{USD\ Collateral} \quad (4)$$

Standard CSAs

- Discount factors with matching currency and collateral are said to have a standard or native CSA.
- We drop the CSA subscript when referencing discount factors with a standard CSA
- For example, $P(0,t)^{USD\_USDCSA} = P(0,t)^{USD}$ and similarly $P(0,t)^{EUR\_EURCSA} = P(0,t)^{EUR}$

FX Forward Invariance – Discount Factors with USD Collateral

Discount Factors with USD Collateral

- Discount factors with USD Collateral are implied from Xccy curve calibration
- Cannot use Interest Rate Parity as it does not incorporate collateral into calculations
- Knowing the Xccy Market Par Spread (S) we can solve for discount factors with USD Collateral

Example: Xccy Swap EUR/USD

- Knowing the Xccy market par spread (s) and all other terms we can solve for $P(0,t)^{EUR\_USDCSA}$

$$\underbrace{\sum N.(SOFR\ Rate).\tau.P(0,t)^{USD}}_{USD\ Trade\ Leg} = \underbrace{\sum N.(ESTR+s).\tau.P(0,t)^{EUR\_USDCSA}}_{EUR\ Trade\ Leg}$$

FX Forward Invariance – Discount Factors with USD Collateral

Bloomberg Illustration: Xccy Swap EUR/USD 5Y

- Knowing the Xccy Market Par Spread (s) we can solve for discount factors with USD Collateral

$$\underbrace{\sum N.(SOFR\ Rate).\tau.P(0,t)^{USD}}_{USD\ Trade\ Leg} = \underbrace{\sum N.(ESTR+\boldsymbol{s}).\tau.P(0,t)^{EUR\_USDCSA}}_{EUR\ Trade\ Leg}$$

FX Forward Invariance – Discount Factors with Non-USD Collateral

Discount Factors with Non-USD Collateral

- These can be derived from the FX Forward Invariance formula from (4)
- First, we must first calibrate a Xccy Curve(s) to generate discount factors with USD CSAs
- Second, compute non-USD discount factors (DFs) using (4)

$$f(t)^{USD/EUR} = s^{USD/EUR} \underbrace{\left(\frac{P(0,t)^{\text{USD}\_EURCSA}}{P(0,t)^{\text{EUR}\_EURCSA}}\right)}_{\text{EUR } Collateral} = s^{USD/EUR} \underbrace{\left(\frac{P(0,t)^{\text{USD}\_USDCSA}}{P(0,t)^{\text{EUR}\_USDCSA}}\right)}_{USD\ Collateral} \quad (4)$$

Computation from Xccy Swaps or FX Forwards

- We can imply non-USD collateral DFs from USD collateral DFs from Xccy Swaps using the RHS of (4)
- Alternatively, we can imply non-USD collateral DFs from FX rates using the LHS of (4)
- The results will be different and potential arbitrage opportunities may exist.

FX Forward Invariance – Discount Factors with Non-USD Collateral

Using Xccy Swaps for Discount Factors with Non-USD Collateral

- We use the RHS of (4) if we want to imply discount factors with Non-USD collateral from Xccy Swaps

$$f(t)^{USD/BRL} = \underbrace{s^{USD/BRL}\left(\frac{P(0,t)^{\text{USD\_}EURCSA}}{P(0,t)^{\text{EUR\_}EURCSA}}\right) = s^{USD/BRL}\left(\frac{P(0,t)^{\text{USD\_}USDCSA}}{P(0,t)^{\text{EUR\_}USDCSA}}\right)}_{RHS}$$

- Note spot FX terms cancel and using standard CSA notation we have,

$$\left(\frac{P(0,t)^{\text{USD\_}EURCSA}}{P(0,t)^{\text{EUR}}}\right) = \left(\frac{P(0,t)^{\text{USD}}}{P(0,t)^{\text{EUR\_}USDCSA}}\right)$$

- Rearranging gives,

$$P(0,t)^{\text{USD\_}EURCSA} = \left(\frac{P(0,t)^{\text{USD}}.P(0,t)^{\text{EUR}}}{\underbrace{P(0,t)^{\text{EUR\_}USDCSA}}_{From\ Xccy\ Swaps}}\right)$$

FX Forward Invariance – Discount Factors with Non-USD Collateral

Using FX Forwards for Discount Factors with Non-USD Collateral

- We use the LHS of (4) if we want to imply discount factors with Non-USD collateral from FX Forwards

$$\underbrace{f(t)^{USD/EUR} = s^{USD/EUR}\left(\frac{P(0,t)^{\text{USD\_}EURCSA}}{P(0,t)^{\text{EUR\_}EURCSA}}\right)}_{LHS} = s^{USD/EUR}\left(\frac{P(0,t)^{\text{USD\_}USDCSA}}{P(0,t)^{\text{EUR\_}USDCSA}}\right)$$

- This is similar to Interest Rate Parity (IRP), except that discount factor terms need to be collateralized, in this case with EUR CSA collateral as follows,

$$f(t)^{USD/EUR} = s^{USD/EUR}\left(\frac{P(0,t)^{\text{USD\_}EURCSA}}{P(0,t)^{\text{EUR\_}EURCSA}}\right)$$

- Rearranging and noting that EUR_EURCSA is a standard CSA we have,

$$P(0,t)^{\text{USD\_}EURCSA} = P(0,t)^{\text{EUR\_}EURCSA}\left(\frac{f(t)^{USD/BRL}}{s^{USD/BRL}}\right)$$

FX Forward Invariance – Yield Curve Dependencies

Yield Curve Dependencies

- Discount Factors with Collateral have yield curve dependencies
- We must build yield curves in the correct order, outlined below

Yield Curve Calibration Order for Collateralized Discount Factors

- Firstly, calibrate regular **Swap Curves** for Discount Factors with a **Standard CSA**
- Secondly, calibrate **Xccy Curves** for Discount Factors with a **USD CSA**
- Thirdly, use the **FX Forward Invariance Formula** for Discount Factors with a **Non-USD CSA**

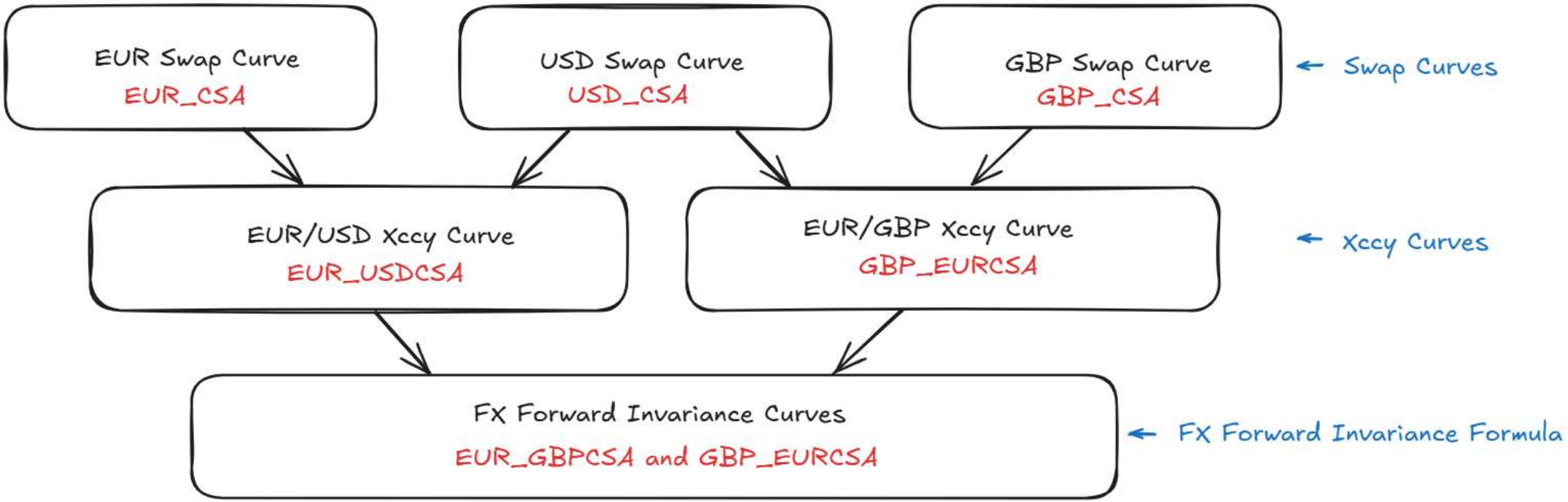
Interest Rate Forwards and Discount Factors

- Forward Rates **do not** require collateral adjustments
- Only Discount Factors are adjusted for collateral

FX Forward Invariance – Yield Curve Dependencies

Illustration: Yield Curve Dependencies

- We illustrate the yield curve dependencies below collateralized discount factor calculations.
- Firstly, compute Standard CSAs, then USD CSAs then Non-USD CSAs as shown below.



Have questions or want further info?

Contact

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