

全角運動量演算子 L^2

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1. 概要

全角運動量演算子 L^2 をシュッと表す。

2. 計算

$$\begin{aligned}
 L^2\psi &= \left(x\frac{\hbar}{i}\partial_y - y\frac{\hbar}{i}\partial_x\right)\left(x\frac{\hbar}{i}\partial_y - y\frac{\hbar}{i}\partial_x\right)\psi + \dots \\
 &= x\frac{\hbar}{i}\partial_y x\frac{\hbar}{i}\partial_y\psi - x\frac{\hbar}{i}\partial_y y\frac{\hbar}{i}\partial_x\psi - y\frac{\hbar}{i}\partial_x x\frac{\hbar}{i}\partial_y\psi + y\frac{\hbar}{i}\partial_x y\frac{\hbar}{i}\partial_x\psi + \dots \\
 &= -\hbar^2(x\partial_y x\partial_y\psi - x\partial_y y\partial_x\psi - y\partial_x x\partial_y\psi + y\partial_x y\partial_x\psi + \dots) \\
 &= -\hbar^2(x^2\partial_y^2\psi + y^2\partial_x^2\psi - x\partial_y(y\partial_x\psi) - y\partial_x(x\partial_y\psi) + \dots)
 \end{aligned}$$

この前半 2 項 $\times 3$ は

$$\begin{aligned}
 & -\hbar^2(x^2\partial_y^2\psi + y^2\partial_x^2\psi + y^2\partial_z^2\psi + z^2\partial_y^2\psi + z^2\partial_x^2\psi + x^2\partial_z^2\psi) \\
 &= \hbar^2(x^2\partial_x^2\psi + y^2\partial_y^2\psi + z^2\partial_z^2\psi) - \hbar^2(x^2 + y^2 + z^2)(\partial_x^2\psi + \partial_y^2\psi + \partial_z^2\psi) \\
 &= \hbar^2(x^2\partial_x^2\psi + y^2\partial_y^2\psi + z^2\partial_z^2\psi) - \hbar^2|\vec{x}|^2\nabla^2\psi
 \end{aligned}$$

であり、この後半 2 項は

$$\hbar^2(x\partial_y(y\partial_x\psi) + y\partial_x(x\partial_y\psi) + \dots) = \hbar^2(x\partial_x\psi + xy\partial_y\partial_x\psi + y\partial_y\psi + yx\partial_x\partial_y\psi + \dots)$$

$$\begin{aligned}
&= \hbar^2 (x \partial_x \psi + y \partial_y \psi + 2xy \partial_y \partial_x \psi + \dots) \\
&= \hbar^2 (2x \partial_x \psi + 2y \partial_y \psi + 2z \partial_z \psi + 2xy \partial_x \partial_y \psi + 2yz \partial_y \partial_z \psi + 2zx \partial_z \partial_x \psi) \\
&= 2\hbar^2 (x \partial_x \psi + y \partial_y \psi + z \partial_z \psi) - \hbar^2 (x^2 \partial_x^2 \psi + y^2 \partial_y^2 \psi + z^2 \partial_z^2 \psi) \\
&\quad + \hbar^2 (2xy \partial_x \partial_y \psi + 2yz \partial_y \partial_z \psi + 2zx \partial_z \partial_x \psi + x^2 \partial_x^2 \psi + y^2 \partial_y^2 \psi + z^2 \partial_z^2 \psi) \\
&= 2\hbar^2 \vec{x} \cdot \nabla \psi - \hbar^2 (x^2 \partial_x^2 \psi + y^2 \partial_y^2 \psi + z^2 \partial_z^2 \psi) \\
&\quad + \hbar^2 x (x \partial_x + y \partial_y + z \partial_z) \partial_x \psi + \hbar^2 y (x \partial_x + y \partial_y + z \partial_z) \partial_y \psi + \hbar^2 z (x \partial_x + y \partial_y + z \partial_z) \partial_z \psi \\
&= 2\hbar^2 \vec{x} \cdot \nabla \psi - \hbar^2 (x^2 \partial_x^2 \psi + y^2 \partial_y^2 \psi + z^2 \partial_z^2 \psi) + \hbar^2 x (\vec{x} \cdot \nabla) \partial_x \psi + \hbar^2 y (\vec{x} \cdot \nabla) \partial_y \psi + \hbar^2 z (\vec{x} \cdot \nabla) \partial_z \psi \\
&= 2\hbar^2 \vec{x} \cdot \nabla \psi - \hbar^2 (x^2 \partial_x^2 \psi + y^2 \partial_y^2 \psi + z^2 \partial_z^2 \psi) + \hbar^2 \vec{x} \cdot ((\vec{x} \cdot \nabla) \nabla \psi)
\end{aligned}$$

したがって

$$L^2 \psi = -\hbar^2 |\vec{x}|^2 \nabla^2 \psi + 2\hbar^2 \vec{x} \cdot \nabla \psi + \hbar^2 \vec{x} \cdot ((\vec{x} \cdot \nabla) \nabla \psi)$$

3. 極座標

$$L^2 \psi = -\hbar^2 |\vec{x}|^2 \nabla^2 \psi + 2\hbar^2 \vec{x} \cdot \nabla \psi + \hbar^2 \vec{x} \cdot ((\vec{x} \cdot \nabla) \nabla \psi)$$

に

$$\vec{x} = r \vec{e}_r$$

$$r^2 \nabla^2 \psi = \partial_r \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{\sin \theta} \partial_\theta \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \partial_\varphi^2 \psi$$

$$\nabla \psi = \vec{e}_r \frac{\partial \psi}{\partial r} + \frac{1}{r} \vec{e}_\theta \frac{\partial \psi}{\partial \theta} + \frac{1}{r \sin \theta} \vec{e}_\varphi \frac{\partial \psi}{\partial \varphi}$$

$$\vec{x} \cdot \nabla = r \partial_r$$

を代入すると

$$\begin{aligned}
 L^2\psi &= -\hbar^2\left(\partial_r(r^2\partial_r\psi) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\psi) + \frac{1}{\sin^2\theta}\partial_\varphi^2\psi\right) \\
 &\quad + 2\hbar^2 r\partial_r\psi + \hbar^2\vec{x} \cdot (r\partial_r)\left(\vec{e}_r\frac{\partial\psi}{\partial r} + \frac{1}{r}\vec{e}_\theta\frac{\partial\psi}{\partial\theta} + \frac{1}{r\sin\theta}\vec{e}_\varphi\frac{\partial\psi}{\partial\varphi}\right) \\
 &= -\hbar^2\left(\partial_r(r^2\partial_r\psi) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\psi) + \frac{1}{\sin^2\theta}\partial_\varphi^2\psi\right) + 2\hbar^2 r\partial_r\psi + \hbar^2 r(r\partial_r)\frac{\partial\psi}{\partial r} \\
 &= -\hbar^2\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\psi) + \frac{1}{\sin^2\theta}\partial_\varphi^2\psi\right) - \hbar^2\partial_r(r^2\partial_r\psi) + 2\hbar^2 r\partial_r\psi + \hbar^2 r^2\frac{\partial^2\psi}{\partial r^2} \\
 &= -\hbar^2\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\psi) + \frac{1}{\sin^2\theta}\partial_\varphi^2\psi\right)
 \end{aligned}$$

となって、ラプラシアン ∇^2 の $\theta \cdot \varphi$ 絡みだけを残したやつが全角運動量演算子になることにも
 そこそこ納得がいきそうな気がしそうである。