

全角運動量演算子 L^2 with アインシュタインの記法

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1. 概要

全角運動量演算子 L^2 をシュッと表す with アインシュタインの記法。

2. 計算

$$\begin{aligned} L^2\psi &= \frac{1}{2}g_{km}g_{ln}\left(X^k\frac{\hbar}{i}\partial^l - X^l\frac{\hbar}{i}\partial^k\right)\left(X^m\frac{\hbar}{i}\partial^n - X^n\frac{\hbar}{i}\partial^m\right)\psi \\ &= -\frac{\hbar^2}{2}g_{km}g_{ln}\left(X^k\partial^l X^m\partial^n\psi - X^k\partial^l X^n\partial^m\psi - X^l\partial^k X^m\partial^n\psi + X^l\partial^k X^n\partial^m\psi\right) \\ &= -\frac{\hbar^2}{2}\left(g_{km}g_{lp}X^k\partial^l X^m\partial^p\psi - g_{kp}g_{ln}X^k\partial^l X^n\partial^p\psi - g_{km}g_{lp}X^l\partial^k X^m\partial^p\psi + g_{kp}g_{ln}X^l\partial^k X^n\partial^p\psi\right) \\ &= -\frac{\hbar^2}{2}\left(g_{km}\delta_l^p\left(X^k\partial^l X^m - X^l\partial^k X^m\right) + \delta_k^p g_{ln}\left(X^l\partial^k X^n - X^k\partial^l X^n\right)\right)\nabla_p\psi \end{aligned}$$

ここで

$$X^k\partial^l X^m f = X^k X^m\partial^l f + X^k g^{lm} f$$

なので

$$\begin{aligned} &g_{km}\delta_l^p\left(X^k\partial^l X^m - X^l\partial^k X^m\right) + \delta_k^p g_{ln}\left(X^l\partial^k X^n - X^k\partial^l X^n\right) \\ &= g_{km}\delta_l^p\left(X^k X^m\partial^l + X^k g^{lm} - X^l X^m\partial^k - X^l g^{km}\right) + \delta_k^p g_{ln}\left(X^l X^n\partial^k + X^l g^{kn} - X^k X^n\partial^l - X^k g^{ln}\right) \end{aligned}$$

$$\begin{aligned}
 &= g_{km} \delta_l^p X^k X^m \partial^l + g_{km} \delta_l^p X^k g^{lm} - g_{km} \delta_l^p X^l X^m \partial^k - g_{km} \delta_l^p X^l g^{km} \\
 &\quad + \delta_k^p g_{ln} X^l X^n \partial^k + \delta_k^p g_{ln} X^l g^{kn} - \delta_k^p g_{ln} X^k X^n \partial^l - \delta_k^p g_{ln} X^k g^{ln} \\
 &= g_{km} \delta_q^p X^k X^m \partial^q + X^p - g_{qm} \delta_l^p X^l X^m \partial^q - 3X^p + \delta_q^p g_{ln} X^l X^n \partial^q + X^p - \delta_k^p g_{qn} X^k X^n \partial^q - 3X^p \\
 &= \left(g_{km} \delta_q^p X^k X^m - g_{qm} \delta_l^p X^l X^m + \delta_q^p g_{ln} X^l X^n - \delta_k^p g_{qn} X^k X^n \right) \partial^q - 4X^p
 \end{aligned}$$

であり

$$\begin{aligned}
 L^2 \psi &= -\frac{\hbar^2}{2} \left(\left(g_{km} \delta_q^p X^k X^m - g_{qm} \delta_l^p X^l X^m + \delta_q^p g_{ln} X^l X^n - \delta_k^p g_{qn} X^k X^n \right) \partial^q - 4X^p \right) \nabla_p \psi \\
 &= -\frac{\hbar^2}{2} \left(\delta_q^p \left(g_{km} X^k X^m + g_{ln} X^l X^n \right) - g_{qm} \delta_l^p X^l X^m - \delta_k^p g_{qn} X^k X^n \right) \nabla^q \nabla_p \psi + 2\hbar^2 X^p \nabla_p \psi \\
 &= -\hbar^2 \left(\delta_q^p \left(g_{km} X^k X^m \right) - \delta_k^p \left(g_{qm} X^k X^m \right) \right) \nabla^q \nabla_p \psi + 2\hbar^2 X^p \nabla_p \psi \\
 &= -\hbar^2 \left(\delta_q^p \left(g_{km} X^k X^m \right) - \delta_k^p \left(g_{qm} X^k X^m \right) \right) \nabla^q \nabla_p \psi + 2\hbar^2 X^p \nabla_p \psi \\
 &= -\hbar^2 \left(g_{km} X^k X^m \right) \nabla^2 \psi + \hbar^2 \left(X^p X^m \right) \nabla_m \nabla_p \psi + 2\hbar^2 X^p \nabla_p \psi
 \end{aligned}$$

したがって

$$L^2 \psi = -\hbar^2 |\vec{x}|^2 \nabla^2 \psi + \hbar^2 \vec{x} \cdot \left((\vec{x} \cdot \nabla) \nabla \psi \right) + 2\hbar^2 \vec{x} \cdot \nabla \psi$$