# Supplementary File for Learning Dynamic Coupling of Neural Oscillations for Motor Imagery Classification

Shoulin Huang, *Member, IEEE,* Ruifeng Zhang, Xiaohao Wen, *Member, IEEE,* and Mengchu Zhou, *Fellow, IEEE*

This supplementary file describes the optimized detail for the proposed CFC-based Tensor Decomposition (CTD) used in CFCNet. To ensure that the extractive features are discriminative, we employ the CSP technique to find the parameters of $\boldsymbol{s}$ and $\mathbf{B}$. To be more specific, the optimization objective of CTD is to maximize the following power ratios for class 1 and class 2:

$$\max_{\boldsymbol{s},\mathbf{B}} \frac{E_{\boldsymbol{\mathcal{X}}\in C_1}[\alpha(\boldsymbol{\mathcal{X}};\boldsymbol{s},\mathbf{B})]}{E_{\boldsymbol{\mathcal{X}}\in C_2}[\alpha(\boldsymbol{\mathcal{X}};\boldsymbol{s},\mathbf{B})]+E_{\boldsymbol{\mathcal{X}}\in C_1}[\alpha(\boldsymbol{\mathcal{X}};\boldsymbol{s},\mathbf{B})]} \tag{1}$$

Here, $C_y$ is the set in training data, which contains the test belonging to class $y \in \{1,2\}$; $\alpha(\boldsymbol{\mathcal{X}};\boldsymbol{s},\mathbf{B})=\boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}$ denotes the power calculated from the projection signal $\boldsymbol{z}$, i.e., $\alpha(\boldsymbol{\mathcal{X}};\boldsymbol{s},\mathbf{B})=\boldsymbol{z}^{\mathrm{T}}\boldsymbol{z}$; and $E_{\boldsymbol{\mathcal{X}}\in C_y}[\,\cdot\,]$ denotes the expected covariance matrix over $C_y$. Since it is difficult to determine $\boldsymbol{s}$ and $\mathbf{B}$ simultaneously, a standard alternating optimization strategy is adopted in the optimization process.

While $\mathbf{B}$ is fixed, $\alpha(\boldsymbol{\mathcal{X}};\boldsymbol{s},\mathbf{B})$ can be denoted by $\boldsymbol{s}^{\mathrm{T}}\mathbf{G}\mathbf{G}^{\mathrm{T}}\boldsymbol{s}$, where $\mathbf{G}=\boldsymbol{\mathcal{X}}\times_2 [\![\mathbf{B}^{\mathrm{T}}]\!] \in \mathbb{R}^{K\times L}$. Let $cov(\mathbf{A},y)$ be the function that computes the average covariance matrix for class $y$. Then it follows that $E_{\boldsymbol{\mathcal{X}}\in C_1}[\alpha(\boldsymbol{\mathcal{X}};\boldsymbol{s},\mathbf{B})]=\boldsymbol{s}^{\mathrm{T}}cov(\mathbf{G},1)\boldsymbol{s}$ and $E_{\boldsymbol{\mathcal{X}}\in C_2}[\alpha(\boldsymbol{\mathcal{X}};\boldsymbol{s},\mathbf{B})]=\boldsymbol{s}^{\mathrm{T}}cov(\mathbf{G},2)\boldsymbol{s}$. Thus, (1) can be re-expressed as a Rayleigh quotient maximization problem.

$$\max_{\boldsymbol{s}_m} J(\boldsymbol{s}_m|\mathbf{B}_m)=\frac{\boldsymbol{s}_m^{\mathrm{T}}cov(\mathbf{G}_m,d)\boldsymbol{s}_m}{\boldsymbol{s}_m^{\mathrm{T}}[cov(\mathbf{G}_m,1)+cov(\mathbf{G}_m,2)]\boldsymbol{s}_m} \tag{2}$$

Problem (2) can be solved by generalized eigenvalue decomposition:

$$cov(\mathbf{G}_m,1)\boldsymbol{s}_m=\lambda[cov(\mathbf{G}_m,1)+cov(\mathbf{G}_m,2)]\boldsymbol{s}_m \tag{3}$$

The spatial factor $\boldsymbol{s}$ is given by the eigenvector corresponding to the eigenvalue $\lambda$. The largest or smallest eigenvalues mean discriminability of the eigenvectors. We set $M_s$ as an even number $2r$. And then half of the factors, $\{\boldsymbol{s}_1,\boldsymbol{s}_2,...,\boldsymbol{s}_r\}$, are chosen as the eigenvectors corresponding to the largest $r$ eigenvalues for class 1, and the other factors, $\{\boldsymbol{s}_{r+1},\boldsymbol{s}_{r+2},...,\boldsymbol{s}_{2r}\}$ are the eigenvectors corresponding to the smallest $r$ eigenvalues for class 2.

Next, we find $\{\mathbf{B}_m,\ m=1,2,...,M\}$ assuming that $\{\boldsymbol{s}_m, m=1,2,...,M\}$ is given. Let $\boldsymbol{\beta}_m=vec(\mathbf{B}_m)\in\mathbb{R}^{KN}$, where $vec(\mathbf{B}_m)$ denotes the vectorization operation that sequentially concatenates the column vectors of $\mathbf{B}_m$ to form a new vector. Then it follows that $\alpha(\boldsymbol{\mathcal{X}};\boldsymbol{s}_m,\mathbf{B}_m)=\boldsymbol{\beta}_m^{\mathrm{T}}\mathbf{Q}_m\mathbf{Q}_m^{\mathrm{T}}\boldsymbol{\beta}_m$, where the matrix $\mathbf{Q}_m=[s_{m,1}\mathbf{X}_1^{\mathrm{T}}\ \ s_{m,2}\mathbf{X}_2^{\mathrm{T}}\ \ ...\ \ s_{m,K}\mathbf{X}_K^{\mathrm{T}}]^{\mathrm{T}}\in\mathbb{R}^{(KN)\times L}$, with $s_{m,k}$ being the $k$-th element of $\boldsymbol{s}_m$. Similarly, the optimization problem can be written as

$$\max_{\mathbf{B}_m} J(\mathbf{B}_m|\boldsymbol{s}_m)=\frac{\boldsymbol{\beta}_m^{\mathrm{T}}cov(\mathbf{Q}_m,1)\boldsymbol{\beta}_m}{\boldsymbol{\beta}_m^{\mathrm{T}}[cov(\mathbf{Q}_m,1)+cov(\mathbf{Q}_m,2)]\boldsymbol{\beta}_m} \tag{4}$$

In addition, problem (4) is solved by generalized eigenvalue decomposition. $\boldsymbol{\beta}_m$ is chosen as the eigenvector corresponding to the largest eigenvalue for $m$ within the range of 1 to $r$, and the eigenvector corresponding to the smallest eigenvalue for $m$ within the range of $r+1$ to $2r$. The $k$-th column of $\mathbf{B}_m$, $\boldsymbol{b}_{m,k}$, can be restored from $\boldsymbol{\beta}_m$ by

$$\begin{aligned}\boldsymbol{b}_{m,k}=[\beta_{m,p}\ \beta_{m,p+1}\ ...\ \beta_{m,p+N-1}]^{\mathrm{T}},&\\ k=1,2,...,K,\ m=1,2,...,2r&\end{aligned} \tag{5}$$

where $p=(k-1)N+1$, and $\beta_{m,p}$ is the $p$-th element of $\boldsymbol{\beta}_m$.

To facilitate fast convergence, we initialize $\{\mathbf{B}_m, m=1,2,...,M\}$ with effective initial values. Let us define a coupling factor $\boldsymbol{w}\in\mathbb{R}^{KN}$ as

$$\boldsymbol{w}=\boldsymbol{s}\odot vec(\mathbf{B})=[s_1\boldsymbol{b}_1^{\mathrm{T}}\ \ s_2\boldsymbol{b}_2^{\mathrm{T}}\ \ ...\ \ s_K\boldsymbol{b}_K^{\mathrm{T}}]^{\mathrm{T}} \tag{6}$$

where $\odot$ denotes the KhatriRao product of two vectors. It is easy to find $\alpha(\boldsymbol{\mathcal{X}};\boldsymbol{s},\mathbf{B})=\boldsymbol{w}^{\mathrm{T}}\mathbf{R}\mathbf{R}^{\mathrm{T}}\boldsymbol{w}$, where $\mathbf{R}=[\mathbf{X}_1;\ \mathbf{X}_2;...;\ \mathbf{X}_K]\in\mathbb{R}^{(KN)\times L}$. Thus, (1) can be re-expressed as

$$\max_{\boldsymbol{w}} J(\boldsymbol{w})=\frac{\boldsymbol{w}^{\mathrm{T}}cov(\mathbf{R},1)\boldsymbol{w}}{\boldsymbol{w}^{\mathrm{T}}[cov(\mathbf{R},1)+cov(\mathbf{R},2)]\boldsymbol{w}} \tag{7}$$

**Algorithm 1** Optimization details for CTD

**Input:** The training set $\{\mathbf{X}_{i\text{-th}} \in \mathbb{R}^{K\times L}, i=1,2,...,I\}$ with their labels $\mathbf{y} \in \mathbb{R}^I$, the maximum number of iterations $T$ and a small value $\varepsilon$ for testing convergence, and the number of spatial-spectral factors $2r$.
**Output:** $\{(\boldsymbol{s}_m, \mathbf{B}_m), m=1,2,...,M\}$ (note that $M=2r$).
calculate $\boldsymbol{\mathcal{X}}_i \in \mathbb{R}^{K\times N\times L}$ for $1 \le i \le I$;
calculate average covariance: $cov(\mathbf{R},1)$ and $cov(\mathbf{R},2)$;
calculate $\{\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_{2r}\}$ according to (7);
initialize $\{\mathbf{B}_m, m=1,2,...,2r\}$ according to (8);
**for** $1 \le t \le T$ **do**
calculate $cov(\mathbf{G}_m,1)$ and $cov(\mathbf{G}_m,2)$;
obtain $\{\boldsymbol{s}_m^{(t)}, m=1,2,...,2r\}$ by solving (2);
calculate $cov(\mathbf{Q}_m,1)$ and $cov(\mathbf{Q}_m,2)$;
obtain $\{\boldsymbol{b}_{m,k}^{(t)}, k=1,2,...,K,\ m=1,2,...,2r\}$ by solving (4) and (5);
calculate $\{\mathbf{B}_m^{(t)}, m=1,2,...,2r\}$ according to (8);
if $t=T$ or $J(\boldsymbol{s}_m^{(t)}|\mathbf{B}_m^{(t-1)}) - J(\mathbf{B}_m^{(t)}|\boldsymbol{s}_m^{(t)}) \le \varepsilon$ for all $m$
break;
**end for**
return $\{(\boldsymbol{s}_m^{(t)}, \mathbf{B}_m^{(t)}), m=1,2,...,M\}$

Problem (7) also can be solved by generalized eigenvalue decomposition: $\gamma[cov(\mathbf{R},1)+cov(\mathbf{R},2)]\boldsymbol{w}$. Half of the factors, $\{\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_r\}$, are chosen as eigenvectors corresponding to the largest $r$ eigenvalues for class 1, and the others are eigenvectors corresponding to the smallest $r$ eigenvalues for class 2. The initial values of $\mathbf{B}_m$ can be estimated from $\boldsymbol{w}_m$, as follows:

$$\begin{aligned} \boldsymbol{b}_{m,k} &= [w_{m,p} \ \ w_{m,p+1} \ ... \ w_{m,p+N-1}]^{\mathrm{T}}/h_p, \\ & k=1,2,...,K,\ m=1,2,...,2r \end{aligned} \tag{8}$$

where $p=(k-1)N+1$; $w_{m,p}$ is the $p$-th element of $\boldsymbol{w}_m$; and $h_p = \sqrt{(w_{m,p})^2 + (w_{m,p+1})^2 + ... + (w_{m,p+N-1})^2}$. In this way, $\boldsymbol{b}_{m,k}$ is assumed to be unit $l_2$-norm. The pseudo-code of the CTD finding $\{(\boldsymbol{s}_m, \mathbf{B}_m), m=1,2,...,M\}$ is given in **Algorithm 1**. Similarly to CSP, CTD can easily be extended to multi-class applications, by adopting strategies, such as Pair-Wise and One-Versus-Rest.

The proposed CTD method finds spatial-spectral factors alternatively by solving a single Rayleigh quotient maximization objective function, where we never decrease the value of cost function between two successive iterations. The convergence of the alternating optimization process is proved in $Theorem$ 1.

$Theorem$ 1: The alternating optimization process for CTD guarantees monotonic convergence:

$$\begin{aligned} \sigma &\le J(\boldsymbol{s}_m^{(1)}|\mathbf{B}_m^{(0)}) \le J(\mathbf{B}_m^{(1)}|\boldsymbol{s}_m^{(1)}) \le J(\boldsymbol{s}_m^{(2)}|\mathbf{B}_m^{(1)}) \le ... \\ &\le J(\boldsymbol{s}_m^{(t)}|\mathbf{B}_m^{(t-1)}) \le J(\mathbf{B}_m^{(t)}|\boldsymbol{s}_m^{(t)}) < 1 \end{aligned} \tag{9}$$

where $\sigma$ is a limiting value in the $\mathbb{R}^+$ space ($0<\sigma<1$), and $J(\mathbf{B}_m^{(t)}|\boldsymbol{s}_m^{(t)})$ is calculated with the given $\boldsymbol{s}_m^{(t)}$ in the $t$-th ($t \ge 1$) iteration. $\mathbf{B}_m^{(0)}$ is initialized according to (8).

$Proof$: Let $\sigma_m = J(\boldsymbol{w}_m)$ for $m=1,2,...,2r$, and it is easy to know that $0<\sigma_m<1$. Since $\boldsymbol{s}_m^{(1)}$ is calculated by arg maximizing $J(\boldsymbol{s}_m|\mathbf{B}_m)$ with the given $\mathbf{B}_m^{(0)}$, we have the following inequality: $\sigma \le J(\boldsymbol{s}_m^{(1)}|\mathbf{B}_m^{(0)})$, where $\sigma = \min(\sigma_1, \sigma_2, ..., \sigma_{2r})$.

According to (4), we have $J(\mathbf{B}_m^{(t)}|\boldsymbol{s}_m^{(t)}) < 1$. In addition, because $\boldsymbol{s}_m^{(t)}$ maximizes $J(\boldsymbol{s}_m^{(t)}|\mathbf{B}_m^{(t-1)})$ with fixed $\mathbf{B}_m^{(t-1)}$ and then $\mathbf{B}_m^{(t)}$ maximizes $J(\mathbf{B}_m^{(t)}|\boldsymbol{s}_m^{(t)})$ with calculated $\boldsymbol{s}_m^{(t)}$, the relationship $J(\mathbf{B}_m^{(t-1)}|\boldsymbol{s}_m^{(t-1)}) \le J(\boldsymbol{s}_m^{(t)}|\mathbf{B}_m^{(t-1)}) \le J(\mathbf{B}_m^{(t)}|\boldsymbol{s}_m^{(t)})$ is given. Hence, we finally obtain the following relationship: $\sigma \le J(\boldsymbol{s}_m^{(1)}|\mathbf{B}_m^{(0)}) \le J(\mathbf{B}_m^{(1)}|\boldsymbol{s}_m^{(1)}) \le ... \le J(\boldsymbol{s}_m^{(t)}|\mathbf{B}_m^{(t-1)}) \le J(\mathbf{B}_m^{(t)}|\boldsymbol{s}_m^{(t)}) < 1$.