Matrix Calculation: Homework #2

Due on Sept 26, 2022 at 3:10pm

Professor Jun Lai Monday

Shuang Hu

Problem 1

(Page 80, Problem 2.4.2)

Proof. By SVD decomposition, there exists orthogonal matrix U, V such that

$$U^t AV = \begin{bmatrix} D & O \\ O & O \end{bmatrix}, D = \text{diag}(\sigma_1, \cdots, \sigma_r). \tag{1}$$

WLOG, we set $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$, and assume that $U = [y_1, \cdots, y_m]$, $V = [x_1, \cdots, x_n]$, it means that:

$$y_i^t Ax_j = \begin{cases} \sigma_i \, (i = j, i, j \leq r) \\ 0 \, (otherwise). \end{cases} \tag{2}$$

where $\|y_i\| = \|x_i\| = 1$. As $\{y_i\}$, $\{x_j\}$ are orthonormal basis, respectively, we can write:

$$\begin{aligned} x &= \sum_{i=1}^{n} \beta_i x_i \\ y &= \sum_{i=1}^{m} \alpha_i y_i \end{aligned} \tag{3}$$

Then:

$$\begin{aligned} y^t Ax &= \sum \alpha_i \beta_i \sigma_i \\ &\leq \sigma_{\max} \sum |\alpha_i \beta_i| \\ &\leq \sigma_{\max} \sqrt{\sum \alpha_i^2 \sum \beta_i^2} \\ &= \sigma_{\max} \|x\|_2 \|y\|_2. \end{aligned} \tag{4}$$

It means that

$$\sigma_{\max} \geq \max \frac{y^t Ax}{\|x\|_2 \|y\|_2}. \tag{5}$$

On the other hand, set $x = x_1$, $y = y_1$, we can see $\frac{y^t Ax}{\|x\|_2 \|y\|_2} = \sigma_1 = \sigma_{\max}$. Q.E.D. □

Problem 2

(Page 80, Problem 2.4.6)

By corollary 2.4.7, $A = \sigma_1 u_1 v_1^t + \sigma_2 u_2 v_2^t$. In one hand, set $B = \sigma_1 u_1 v_1^t$, we can get $\|A - B\|_F = \sigma_2$.

In the other hand, assume $r(B) = 1$, then $\dim \ker B = 1$, assume that $Bx = 0$. Expand the vector x to an orthonormal basis for $\mathbb{R}^2$ as $\{x, y\}$, then:

$$\begin{aligned} \|A - B\|_F^2 &\geq \|(A - B)x\|_2^2 \\ &= \|Ax\|_2^2 \\ &= \sigma_1^2 (v_1^t x)^2 + \sigma_2^2 (v_2^t x)^2 \\ &\geq \sigma_2^2. \end{aligned} \tag{6}$$

So the nearest rank-1 matrix $B = \sigma_1 u_1 v_1^t$.

Problem 3

(Page 80, Problem 2.4.7)

Proof. By corollary 2.4.3, we can see:

$$\|A\|_F = \sqrt{\sum_{i=1}^{r} \sigma_i^2} \qquad \|A\|_2 = \sigma_i. \tag{7}$$

As $\sum_{i=1}^{r} \sigma_i^2 \leq r\sigma_1^2$, it means that $\|A\|_F \leq \sqrt{r}\sigma_1 = \sqrt{r}\|A\|_2$. □