Matrix Calculation: Homework #4

Due on Oct 17, 2022 at 3:10pm

Professor Jun Lai Monday

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Problem 1

(P136 Problem1)
By the definition of matrix production, we can see:

$$\begin{bmatrix} a_1^t \\ a_2^t \\ \vdots \\ a_n^t \end{bmatrix} = \begin{bmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n1} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} u_1^t \\ u_2^t \\ \vdots \\ u_n^t \end{bmatrix} \tag{1}$$

As $l_{ii} = 1$, we can see that

$$u_i^t = a_i^t - \sum_{j=1}^{i-1} l_{ij} u_j^t. \tag{2}$$

To prove $\|U\|_\infty \leq 2^{n-1}\|A\|_\infty$, I use induction on the row order $k \leq n$. First, I claim that

$$\|u_k^t\|_1 \leq 2^{k-1}\|A\|_\infty. \tag{3}$$

For $k = 1$, we can see $\|u_1^t\|_1 = \|a_1^t\|_1 \leq \|A\|_\infty$.
Assume the result is true for $k \leq k_0$, for $k = k_0 + 1$, we can see:

$$\|u_{k_0+1}\|_1 \leq \|a_{k_0+1}\|_1 + \sum_{j=1}^{k_0} |l_{ij}|\|u_j\|_1 \leq \|A\|_\infty + \sum_{j=1}^{k_0} 2^{j-1}\|A\|_\infty = 2^{k_0}\|A\|_\infty \tag{4}$$

This equation shows that (3) is correct. And (3) suggests that $\|U\|_\infty \leq 2^{n-1}\|A\|_\infty$, as $\|U\|_\infty = \max \|u_i\|_1$.

Problem 2

(P158 Problem1)

Assume the matrix A is singular, it suggests that $\exists x \neq 0$ such that $Ax = 0$. Write $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, assume $|x_i| = \max_j\{|x_j|\}$, consider the ith row of matrix A, we can see:

$$\sum a_{ij} x_j = 0. \tag{5}$$

(5) suggests that

$$a_{ii} = -\sum_{j \neq i} a_{ij} \frac{x_j}{x_i}. \tag{6}$$

Then:

$$|a_{ii}| \leq \sum_{j \neq i} |a_{ij}|. \tag{7}$$

Contradict to the strict diagonal advantage. So matrix A must be nonsingular.

Problem 3

(P158 Problem2)
A is row diagonally dominant means that A^t is column diagonally dominant, then use theorem 4.1.2, we can see that $\|(A^t)^{-1}\|_1 \leq \frac{1}{\delta}$. It's clear that $A_1^t = \|A\|_\infty$, so $\|A^{-1}\|_\infty \leq \frac{1}{\delta}$.