Matrix Calculation: Homework #5

Due on Oct 24, 2022 at 3:10pm

Professor Jun Lai Monday

Shuang Hu

Problem 1

(P173 Problem4.2.3)
(a) By definition,

$$\|A^{-1}\|_2 = \sup_{x\in\mathbb{R}^n} \frac{\|A^{-1}x\|}{\|x\|} = \sup_{y\in\mathbb{R}^n} \frac{\|y\|}{\|Ay\|} \tag{1}$$

so:

$$\begin{aligned} \|A^{-1}\|_2^{-1} &= \inf_{\|y\|=1} \|Ay\| \\ &= \inf_{\|y\|=1} \|Ay\|\|y\| \\ &\geq \inf_{\|y\|=1} y^tAy = \inf_{\|y\|=1} y^tTy = \|T^{-1}\|_2^{-1} \end{aligned} \tag{2}$$

The final equation is derived by the positive definity of matrix A. So it's clear that $\|A^{-1}\|_2 \leq \|T^{-1}\|_2$.
By definition, $T^{-1} - A^{-1} = T^{-1}(A-T)A^{-1} = T^{-1}SA^{-1}$, we can see:

$$\begin{aligned} &x^t(T^{-1} - A^{-1})x \\ =&x^tT^{-1}SA^{-1}x \\ =&(A^{-1}x + (T^{-1} - A^{-1})x)^tSA^{-1}x \\ =&(A^{-1}x + T^{-1}SA^{-1}x)^tSA^{-1}x \\ =&(A^{-1}x)^tSA^{-1}x + (SA^{-1}x)^t(T^{-1})^tSA^{-1}x \\ \geq&0 \end{aligned} \tag{3}$$

and the final inequality is derived by the anti-symmetrical of S, and positive-definity of T^{-1}.
(b)By the definition of matrix norm,

$$\|D^{-1}\|_2 = \|M^tA^{-1}L\|_2 \leq \|A^{-1}\|_2 \leq \|T^{-1}\|_2. \tag{4}$$

So $\frac{1}{d_k} \leq \|T^{-1}\|_2 \ \forall k \Rightarrow d_k \geq \frac{1}{\|T^{-1}\|_2}$.

Problem 2

Example:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{5}$$

then:

$$\begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + y^2 \geq 0. \tag{6}$$

For real vector, the value equals zero if and only if $x = y = 0$. For complex vector, we can see:

$$\begin{bmatrix} i & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = 2 - 2i \notin \mathbb{R}. \tag{7}$$

so A isn't positive definite as $\mathbb{C}^{2\times 2}$.

Problem 3

Using theorem 4.2.3, to show A positive definite, it suffices to show $T = \frac{A+A^t}{2}$ has positive eigenvalues. As both A and A^t are strictly diagonally dominant, we can see that T is strictly diagonally dominant. The diagonal elements of T are positive, i.e. T is positive definite, so the eigenvalues of T are all positive.So A positive definite.