Matrix Calculation: Homework #3

Due on Oct 10, 2022 at 3:10pm

Professor Jun Lai Monday

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Problem 1

(Page 92, Problem 2.6.5)

Proof. Set $u := e_i$, $v := \epsilon e_j$, we can see $A + \epsilon a_{ij} E_{ij} = A + uv^T$. Set $x = A^{-1}b$, $\tilde{x} = (A + \epsilon a_{ij})^{-1}b$, and $C := A^{-1}$, we can see:

$$\tilde{x} = (A + uv^t)^{-1}b = (C - \frac{Cuv^tC}{1 + v^tCu})b. \tag{1}$$

Then:

$$x - \tilde{x} = \frac{Cuv^tCb}{1 + v^tCu} = \frac{Cuv^tx}{1 + \epsilon c_{ji}} = \frac{\epsilon x_j C e_i}{1 + \epsilon c_{ji}} = \frac{\epsilon x_j}{1 + \epsilon c_{ji}} \begin{bmatrix} c_{1i} \\ c_{2i} \\ \vdots \\ c_{ni} \end{bmatrix}. \tag{2}$$

We can see:

$$\frac{\tilde{x}_k - x_k}{\epsilon} = -\frac{x_j}{1 + \epsilon c_{ji}} c_{ki} \to -x_j c_{ki}. \tag{3}$$

□

Problem 2

(Page 120, Problem 3.2.2)

Proof. For matrix $A(\epsilon) = (a_{ij}(\epsilon))_{i,j=1}^n$, as the problem suggests, $a_{ij}(\epsilon)$ must be C^1 function. Then, for the principal matrices $A_k(\epsilon) = A(1:k, 1:k)$, $\det(A_k(\epsilon))$ must be C^1 as well. And $\det(A_k(0)) \neq 0$, so we can choose $\epsilon_k \in (0, w_k)$ such that $\det(A_k(\epsilon_k)) \neq 0$, where $w_k > 0$. Then set $w = \min_{k=1}^N w_k$, we can see $w > 0$ and $\forall \epsilon \in (0, w)$, $\det(A_k(\epsilon)) \neq 0$. This means that $A(\epsilon)$ has an LU factorization.
Then, using induction to prove the property of the LU factorization. For $n = 1$, we can see $L = (1)$ and $U = (a_{11}(\epsilon))$, L and U are both C^1. Assume this factorization exists for $n = k$, consider $n = k + 1$, in this condition,

$$A(\epsilon) = \begin{bmatrix} A_k(\epsilon) & \beta(\epsilon) \\ \alpha^t(\epsilon) & \varphi(\epsilon) \end{bmatrix}. \tag{4}$$

By induction, $A_k(\epsilon) = L_k(\epsilon)U_k(\epsilon)$ where L_k, U_k both non-regular.
Then, set

$$L(\epsilon) = \begin{bmatrix} L_k(\epsilon) & 0 \\ U_k^{-1}(\epsilon)\alpha^t(\epsilon) & 1 \end{bmatrix}, U(\epsilon) = \begin{bmatrix} U_k(\epsilon) & L_k(\epsilon)^{-1}\beta(\epsilon) \\ 0 & \varphi(\epsilon) - \alpha(\epsilon)(U_k^{-1}(\epsilon))^t L_k^{-1}(\epsilon)\beta(\epsilon) \end{bmatrix}. \tag{5}$$

We can see $L(\epsilon)U(\epsilon) = A(\epsilon)$, and they are both C^1. □