Matrix Calculation: Homework #1

Due on Sept 15, 2022 at 3:10pm

Professor Jun Lai Monday

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Problem 1

(Page 67, Problem 2.1.1)

Proof. By the normalized standard form, as $r(A) = r$, we can write :

$$PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} \tag{1}$$

Then:

$$\begin{aligned} A &= P^{-1} \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} Q^{-1} \\ &= \left(P^{-1} \begin{bmatrix} I_r \\ O \end{bmatrix} \right) \left(\begin{bmatrix} I_r, O \end{bmatrix} Q^{-1} \right) \end{aligned} \tag{2}$$

Set $X = P^{-1} \begin{bmatrix} I_r \\ O \end{bmatrix}$, $Y^t = \begin{bmatrix} I_r, O \end{bmatrix} Q^{-1}$, we can see $r(X) = r(Y) = r$, which satisfies the decomposition. □

Problem 2

(Page 67, Problem 2.1.2)
(a).

Proof.

$$\begin{aligned} LHS &= \lim_{t \to 0} \frac{A(\alpha + t)B(\alpha + t) - A(\alpha)B(\alpha)}{t} \\ &= \lim_{t \to 0} \frac{[A(\alpha + t) - A(\alpha)]B(\alpha + t) + A(\alpha)[B(\alpha + t) - B(\alpha)]}{t} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}\alpha} A(\alpha) \right] B(\alpha) + A(\alpha) \left[\frac{\mathrm{d}}{\mathrm{d}\alpha} B(\alpha) \right] \end{aligned} \tag{3}$$

□

(b)

Proof. As $I = A(\alpha)(A(\alpha)^{-1})$, $\frac{\mathrm{d}I}{\mathrm{d}\alpha} = 0$ and **(a)** suggests, we can see:

$$\frac{\mathrm{d}I}{\mathrm{d}\alpha} = 0 = \frac{\mathrm{d}}{\mathrm{d}\alpha} \left[(A(\alpha))^{-1} \right] A(\alpha) + (A(\alpha))^{-1} \frac{\mathrm{d}}{\mathrm{d}\alpha} A(\alpha). \tag{4}$$

Then:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \left[A(\alpha)^{-1} \right] = -A(\alpha)^{-1} \left[\frac{\mathrm{d}}{\mathrm{d}\alpha} A(\alpha) \right] A(\alpha)^{-1}. \tag{5}$$

□

Problem 3

(Page 70, Problem 2.2.1)

Problem 3 continued on next page...

Proof. In one hand:

$$\|\mathbf{x}\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \geq (\|\mathbf{x}\|_\infty^p)^{\frac{1}{p}} = \|\mathbf{x}\|_\infty. \tag{6}$$

In the other hand:

$$\|\mathbf{x}\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \leq (n\|\mathbf{x}\|_\infty^p)^{\frac{1}{p}} \to \|\mathbf{x}\|_\infty. \tag{7}$$

Set $p \to \infty$, we can see: $\|\mathbf{x}\|_p \to \|\mathbf{x}\|_\infty$. □

Problem 4

Proof. On one hand:

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\| \Rightarrow \|x\| - \|y\| \leq \|x - y\|. \tag{8}$$

On the other hand:

$$\|y\| = \|y - x + x\| \leq \|x\| + \|y - x\| = \|x\| + \|x - y\| \Rightarrow \|y\| - \|x\| \leq \|x - y\|. \tag{9}$$

Then:

$$|\|x\| - \|y\|| \leq \|x - y\|. \tag{10}$$

□