Matrix Calculation: Homework #9

Due on Nov 28, 2022 at 3:10pm

Professor Jun Lai Monday

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Problem 1

(P355 Problem 7.1.3)
Assume that $Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$, by the definition of Schur decomposition, we can see that

$$A \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} T. \tag{1}$$

while

$$T := \begin{bmatrix} \lambda_1 & t_{12} & \cdots & t_{1n} \\ & \lambda_2 & \cdots & t_{2n} \\ & & \ddots & \vdots \\ & & & \lambda_n \end{bmatrix}. \tag{2}$$

As matrix A has distinct eigenvalues, we can see: $\lambda_i \neq \lambda_j, \forall i \neq j$. Then we suffice to show that: $\forall m \leq n$, $span\{q_1, \cdots, q_m\}$ is a B-invariant subspace.
Then, we prove this result by induction. First, for $m = 1$, we can see that $Aq_1 = \lambda_1 q_1$, and $AB = BA$ means $ABq_1 = BAq_1 = \lambda_1 Bq_1$, which means that Bq_1 belongs to the eigen-subspace of A related to eigenvalue λ_1. So $Bq_1 = \mu_1 q_1$. Assume $span\{q_1, \cdots, q_t\}$ is a B-invariant subspace with $t \leq m$, consider the subspace $V_{m+1} = span\{q_1, \cdots, q_{m+1}\}$. By (1), we can see:

$$Aq_{m+1} = \sum_{i=1}^{m} t_{i,m+1} q_i + \lambda_{m+1} q_{m+1}. \tag{3}$$

As $AB = BA$, we can see:

$$ABq_{m+1} = BAq_{m+1} = B(\sum_{i=1}^{m} t_{i,m+1} q_i + \lambda_{m+1} q_{m+1}). \tag{4}$$

It means:

$$(A - \lambda_{m+1} I) Bq_{m+1} = B(\sum_{i=1}^{m} t_{i,m+1} q_i). \tag{5}$$

If $Bq_{m+1} \notin span\{q_1, \cdots, q_{m+1}\}$, i.e. $Bq_{m+1} = \sum_{i=1}^{n} \alpha_i q_i$ while $\exists i > m+1$, $\alpha_i \neq 0$, we can see:

$$(A - \lambda_{m+1} I) Bq_{m+1} = v + \sum_{i=m+2}^{n} (\lambda_i - \lambda_{m+1}) \alpha_i q_i. \tag{6}$$

while $v \in V_{m+1}$. Meanwhile:

$$B(\sum_{i=1}^{m} t_{i,m+1} q_i) \in V_m. \tag{7}$$

Contradict! So $span\{q_1, \cdots, q_{m+1}\}$ is an invariant subspace as well.
By induction, $Q^H BQ$ is also an upper-triangular matrix.

Problem 2

(P355 Problem 7.1.5)
By Schur decomposition, exists $Q \in \mathbb{C}^{n \times n}$ such that:

$$Q^H AQ = T \tag{8}$$

as T upper-triangular. If T has distinct eigenvalues, we just set $B = A$, B is diagonalizable actually. If there exists i, j such that $\lambda_i = \lambda_j$, we set diagonal matrix

$$D = \text{diag}\{d_1, \cdots, d_n\}. \tag{9}$$

d_i satisfies the following conditions:

- $|d_i| \leq \frac{\epsilon}{n}$.
- $\lambda_i + d_i \neq \lambda_j + d_j$.

As $n < \infty$, such matrix D exists. We set

$$B = Q(T + D)Q^H, \tag{10}$$

then:

$$\|A - B\|_2 \leq \|D\|_2 \leq \epsilon. \tag{11}$$

As the eigenvalues of B are distinct, B is diagonalizable. So B is what we want to see.

Problem 3

(Page 374, Problem 7.3.7)
(a) As $\|\cdot\|_2$ is a norm, we can derive $\|\cdot\|_X$ is a norm directly.
Then, as $\|\cdot\|_2$ is a matrix norm, we can see:

$$\|AB\|_X = \|X^{-1}AXX^{-1}BX\|_2 \leq \|X^{-1}AX\|_2\|X^{-1}BX\|_2 = \|A\|_X\|B\|_X. \tag{12}$$

(b) Assume the Schur decomposition of matrix A is:

$$Q^H AQ = D + N. \tag{13}$$

Assume $D_a = \text{diag}\{1, a, \cdots, a^{n-1}\}$, set $X = QD_a$, we can see that:

$$X^{-1}AX = D_a^{-1}(D + N)D_a = D + D_a^{-1}ND_a. \tag{14}$$

We can set a such that

$$\|D_a^{-1}ND_a\| \leq \epsilon. \tag{15}$$

Then:

$$\|X^{-1}AX\|_2 \leq \|D + D_a^{-1}ND_a\|_2 \leq \rho(D) + \epsilon = \rho(A) + \epsilon. \tag{16}$$

For given matrix X, $\|\cdot\|_2$ is equivalent with $\|\cdot\|_X$. So:

$$\|A^k\|_2 \leq M\|A^k\|_X \leq M(\rho(A) + \epsilon)^k. \tag{17}$$