

# Matrix Calculation: Homework #9

Due on Nov 28, 2022 at 3:10pm

*Professor Jun Lai Monday*

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## Problem 1

(P355 Problem 7.1.3)

Assume that  $Q = [q_1 \ \cdots \ q_n]$ , by the definition of Schur decomposition, we can see that

$$A [q_1 \ \cdots \ q_n] = [q_1 \ \cdots \ q_n] T. \quad (1)$$

while

$$T := \begin{bmatrix} \lambda_1 & t_{12} & \cdots & t_{1n} \\ & \lambda_2 & \cdots & t_{2n} \\ & & \ddots & \vdots \\ & & & \lambda_n \end{bmatrix}. \quad (2)$$

As matrix  $A$  has distinct eigenvalues, we can see:  $\lambda_i \neq \lambda_j, \forall i \neq j$ . Then we suffice to show that:  $\forall m \leq n$ ,  $\text{span}\{q_1, \dots, q_m\}$  is a B-invariant subspace.

Then, we prove this result by induction. First, for  $m = 1$ , we can see that  $Aq_1 = \lambda_1 q_1$ , and  $AB = BA$  means  $ABq_1 = BAq_1 = \lambda_1 Bq_1$ , which means that  $Bq_1$  belongs to the eigen-subspace of  $A$  related to eigenvalue  $\lambda_1$ . So  $Bq_1 = \mu_1 q_1$ . Assume  $\text{span}\{q_1, \dots, q_t\}$  is a B-invariant subspace with  $t \leq m$ , consider the subspace  $V_{m+1} = \text{span}\{q_1, \dots, q_{m+1}\}$ . By (??), we can see:

$$Aq_{m+1} = \sum_{i=1}^m t_{i,m+1} q_i + \lambda_{m+1} q_{m+1}. \quad (3)$$

As  $AB = BA$ , we can see:

$$ABq_{m+1} = BAq_{m+1} = B\left(\sum_{i=1}^m t_{i,m+1} q_i + \lambda_{m+1} q_{m+1}\right). \quad (4)$$

It means:

$$(A - \lambda_{m+1} I)Bq_{m+1} = B\left(\sum_{i=1}^m t_{i,m+1} q_i\right). \quad (5)$$

If  $Bq_{m+1} \notin \text{span}\{q_1, \dots, q_{m+1}\}$ , i.e.  $Bq_{m+1} = \sum_{i=1}^n \alpha_i q_i$  while  $\exists i > m+1, \alpha_i \neq 0$ , we can see:

$$(A - \lambda_{m+1} I)Bq_{m+1} = v + \sum_{i=m+2}^n (\lambda_i - \lambda_{m+1}) \alpha_i q_i. \quad (6)$$

while  $v \in V_{m+1}$ . Meanwhile:

$$B\left(\sum_{i=1}^m t_{i,m+1} q_i\right) \in V_m. \quad (7)$$

Contradict! So  $\text{span}\{q_1, \dots, q_{m+1}\}$  is an invariant subspace as well.

By induction,  $Q^H BQ$  is also an upper-triangular matrix.

## Problem 2

(P355 Problem 7.1.5)

By Schur decomposition, exists  $Q \in \mathbb{C}^{n \times n}$  such that:

$$Q^H A Q = T \quad (8)$$

as  $T$  upper-triangular. If  $T$  has distinct eigenvalues, we just set  $B = A$ ,  $B$  is diagonalizable actually. If there exists  $i, j$  such that  $\lambda_i = \lambda_j$ , we set diagonal matrix

$$D = \text{diag}\{d_1, \dots, d_n\}. \quad (9)$$

$d_i$  satisfies the following conditions:

- $|d_i| \leq \frac{\epsilon}{n}$ .
- $\lambda_i + d_i \neq \lambda_j + d_j$ .

As  $n < \infty$ , such matrix  $D$  exists. We set

$$B = Q(T + D)Q^H, \quad (10)$$

then:

$$\|A - B\|_2 \leq \|D\|_2 \leq \epsilon. \quad (11)$$

As the eigenvalues of  $B$  are distinct,  $B$  is diagonalizable. So  $B$  is what we want to see.

### Problem 3

(Page 374, Problem 7.3.7)

(a) As  $\|\cdot\|_2$  is a norm, we can derive  $\|\cdot\|_X$  is a norm directly.

Then, as  $\|\cdot\|_2$  is a matrix norm, we can see:

$$\|AB\|_X = \|X^{-1}AXX^{-1}BX\|_2 \leq \|X^{-1}AX\|_2 \|X^{-1}BX\|_2 = \|A\|_X \|B\|_X. \quad (12)$$

(b) Assume the Schur decomposition of matrix  $A$  is:

$$Q^H A Q = D + N. \quad (13)$$

Assume  $D_a = \text{diag}\{1, a, \dots, a^{n-1}\}$ , set  $X = QD_a$ , we can see that:

$$X^{-1}AX = D_a^{-1}(D + N)D_a = D + D_a^{-1}ND_a. \quad (14)$$

We can set  $a$  such that

$$\|D_a^{-1}ND_a\| \leq \epsilon. \quad (15)$$

Then:

$$\|X^{-1}AX\|_2 \leq \|D + D_a^{-1}ND_a\|_2 \leq \rho(D) + \epsilon = \rho(A) + \epsilon. \quad (16)$$

For given matrix  $X$ ,  $\|\cdot\|_2$  is equivalent with  $\|\cdot\|_X$ . So:

$$\|A^k\|_2 \leq M\|A^k\|_X \leq M(\rho(A) + \epsilon)^k. \quad (17)$$