Matrix Calculation: Homework #11

Due on Dec 12, 2022 at 3:10pm

Professor Jun Lai Monday

Shuang Hu

Problem 1

(P383 Problem7.4.4)
Denote Hessenberg matrix

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ & \ddots & \ddots & \vdots \\ & & h_{n-1,n} & h_{nn} \end{bmatrix} \tag{1}$$

and the diagonal matrix

$$D = \mathrm{diag}\{d_1, d_2, \cdots, d_n\}. \tag{2}$$

As the subdiagonal elements of $D^{-1}HD$ are all 1, we can see that:

$$\frac{d_{i+1}}{d_i} = h_{i+1,i}, i \in [1, n-1]. \tag{3}$$

It derives the value of matrix D. Its condition number:

$$\kappa_2(D) = \frac{\max \prod_i |h_{i,i-1}|}{\min \prod_i |h_{i,i-1}|}. \tag{4}$$

Problem 2

(P383 Problem7.4.5)
Assume the eigenvector related to λ is $X_1 + iX_2$, we can see:

$$(W + iY)(X_1 + iX_2) = \lambda(X_1 + iX_2). \tag{5}$$

i.e.

$$(WX_1 - YX_2) + i(YX_1 + WX_2) = \lambda X_1 + i\lambda X_2. \tag{6}$$

Then:

$$\begin{cases} WX_1 - YX_2 = \lambda X_1, \\ YX_1 + WX_2 = \lambda X_2. \end{cases} \tag{7}$$

i.e.

$$\begin{bmatrix} W & -Y \\ Y & W \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \lambda \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{8}$$

So λ is an eigenvalue of B, with the eigenvector $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$.

Problem 3

(Page 392, Problem7.5.5)
For $j = 2$, consider the induction basis:

$$\begin{aligned} U_1U_2R_2R_1 &= U_1(H_2 - \mu_2 I)R_1 \\ &= U_1H_2R_1 - \mu_2(H - \mu_1 I) \\ &= U_1(R_1U_1 + \mu_1 I)R_1 - \mu_2(H - \mu_1 I) \\ &= (H - \mu_1 I)^2 + \mu_1(H - \mu_1 I) - \mu_2(H - \mu_1 I) \\ &= H^2 - (\mu_1 + \mu_2)H + \mu_1\mu_2 I \\ &= (H - \mu_1 I)(H - \mu_2 I). \end{aligned} \tag{9}$$

Problem 3 continued on next page...

Assume the result holds for $j = n$, i.e.

$$U_1 \cdots U_n R_n \cdots R_1 = (H - \mu_1 I) \cdots (H - \mu_n I). \tag{10}$$

Then:

$$U_1 \cdots U_{n+1} R_{n+1} \cdots R_1 = U_1 \cdots U_n H_{n+1} R_n \cdots R_1 - \mu_{n+1} (H - \mu_1 I) \cdots (H - \mu_n I). \tag{11}$$

As:

$$H_{j+1} R_j = (R_j U_j + \mu_j I) R_j = R_j (\mu_j I + U_j R_j) = R_j H_j. \tag{12}$$

It means that:

$$U_1 \cdots U_n H_{n+1} R_n \cdots R_1 = U_1 \cdots U_n R_n \cdots R_1 H. \tag{13}$$

So the result holds for $j = n + 1$.