Matrix Calculation: Homework #9

Due on Nov 21, 2022 at 3:10pm

Professor Jun Lai Monday

Shuang Hu

Problem 1

(Page 296,Problem 5.5.1)
By the definition,

$$A = \begin{bmatrix} T & S \\ O & O \end{bmatrix}, X = \begin{bmatrix} T^{-1} & O \\ O & O \end{bmatrix}. \tag{1}$$

We can see that

$$AX = \begin{bmatrix} I & O \\ O & O \end{bmatrix} = (AX)^T, \tag{2}$$

and

$$AXA = \begin{bmatrix} I & O \\ O & O \end{bmatrix} A = A. \tag{3}$$

So X is a (1,3) pseudoinverse of A.
By the definition, if $x_B = Xb$, we can see that

$$x_B = \Pi \begin{bmatrix} R_{11}^{-1} c \\ O \end{bmatrix} = \Pi \begin{bmatrix} R_{11}^{-1} & O \\ O & O \end{bmatrix} Q^T b. \tag{4}$$

So

$$X = \Pi \begin{bmatrix} R_{11}^{-1} & O \\ O & O \end{bmatrix} Q^T. \tag{5}$$

Then:

$$AX = A\Pi \begin{bmatrix} R_{11}^{-1} & O \\ O & O \end{bmatrix} Q^T = Q \begin{bmatrix} R_{11} & R_{12} \\ O & O \end{bmatrix} \begin{bmatrix} R_{11}^{-1} & O \\ O & O \end{bmatrix} Q^T = \begin{bmatrix} I & O \\ O & O \end{bmatrix} = (AX)^T. \tag{6}$$

and

$$AXA = A. \tag{7}$$

It means that X is the (1,3) pseudoinverse of A.

Problem 2

By SVD, we can see that

$$A = U\Sigma V^T, A^+ = V\Sigma^+ U^T. \tag{8}$$

Then:

$$B(\lambda) = (A^T A + \lambda I)^{-1} A^T = V(\Sigma^T \Sigma + \lambda I)^{-1} V^T A^T. = V(\Sigma^T \Sigma + \lambda I)^{-1} \Sigma U^T. \tag{9}$$

So

$$B(\lambda) - A^+ = V \left[(\Sigma^T \Sigma + \lambda I)^{-1} \Sigma - \Sigma^+ \right] U^T. \tag{10}$$

It means that

$$\|B(\lambda) - A^+\|_2 = \|(\Sigma^T \Sigma + \lambda I)^{-1} \Sigma - \Sigma^+\|_2. \tag{11}$$

Consider the matrix righthand, on the i-th position, we can see

$$a_{ii} = \frac{\sigma_i}{\sigma_i^2 + \lambda} - \frac{1}{\sigma_i} = \frac{-\lambda}{\sigma_i(\sigma_i^2 + \lambda)}, i \leq r. a_{ii} = 0, i > r. a_{ij} = 0, i \neq j. \tag{12}$$

So

$$\|B(\lambda) - A^+\|_2 = \frac{\lambda}{\sigma_r(\sigma_r^2 + \lambda)}. \tag{13}$$

Problem 3

(Page 301, P5.6.2)

$$Ax = b \Rightarrow x = \lambda_1 e_1 + \frac{1}{2}\lambda_2 e_2 + \frac{1}{3}\lambda_3 e_3. \tag{14}$$

while $\lambda_1 + \lambda_2 + \lambda_3 = 1$. To minimize $\|x\|$, just set $\lambda_3 = 1$ and $\lambda_1 = \lambda_2 = 0$, then $x = \frac{1}{3}e_3$ is the minimal norm solution.