Matrix Calculation: Homework #8

Due on Nov 14, 2022 at 3:10pm

Professor Jun Lai Monday

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Problem 1

Set the matrix

$$Q := [\beta_1, \cdots, \beta_n]. \tag{1}$$

we can see that

$$Q^T = \begin{bmatrix} \beta_1^T \\ \vdots \\ \beta_n^T \end{bmatrix} \tag{2}$$

Assume the matrix R can be represented as

$$R = \begin{bmatrix} l_1^T \\ \vdots \\ l_n^T \end{bmatrix} \tag{3}$$

while $l_n = re_n$. By the equation $Q^T A - DQ^T = R$, we can see:

$$(A^T - d_i I)\beta_i = l_i^T. \tag{4}$$

First we consider the vector β_n. By equation $(A^t - d_n I)\beta_n = l_n^t = re_n$, we set

$$\beta_n = \frac{(A^t - d_n I)^{-1} e_n}{\|(A^t - d_n I)^{-1} e_n\|}. \tag{5}$$

β_n is the induction basis, and by β_n we can write the vector l_n.
Then, assume $\beta_n, \cdots, \beta_m$ are determined, for β_{m-1}, set linear space $V = \operatorname{span}\{e_{m-1}, e_m, \cdots, e_n\}$, we can see that $\beta_{m-1} \in (A^T - d_{m-1}I)^{-1}V$. Finally, by Gram-Schmidt orthogonal process, we can derive β_{m-1} with aid of $\beta_m, \cdots, \beta_n$.

Problem 2

Solution

First, we can see that

$$\hat{r} - r = A(x - \hat{x}). \tag{6}$$

Then, as the following equations are both true:

$$\begin{aligned} &\left.\begin{aligned} (A^T A + F)x &= A^T b + Fx, \\ (A^T A + F)\hat{x} &= A^T b. \end{aligned}\right\} \Rightarrow (A^T A + F)(x - \hat{x}) = Fx \\ &\qquad\qquad\qquad\qquad\quad \Rightarrow x - \hat{x} = (A^T A + F)^{-1} Fx \\ &\qquad\qquad\qquad\qquad\quad \Rightarrow \hat{r} - r = A(A^T A + F)^{-1} Fx. \end{aligned} \tag{7}$$

Then, assume the SVD of A is $A = P\Sigma Q^T$, then we can see that:

$$\begin{aligned} \|A(A^T A + F)^{-1}\|_2 &= \|\Sigma Q^T (A^T A + F)^{-1}\|_2 \\ &= \|\Sigma Q^T (Q\Sigma^2 Q^T + F)^{-1}\|_2 \\ &= \|\Sigma \left[(Q\Sigma^2 Q^T + F)Q\right]^{-1}\|_2 \\ &= \|\Sigma (Q\Sigma^2 + FQ)^{-1}\|_2 \\ &\le 2\|A^{-1}\|_2. \end{aligned} \tag{8}$$

So:

$$\|\hat{r} - r\|_2 \le 2\|A^{-1}\|_2 \|F\|_2 \|x\|_2 \le 2\kappa_2(A) \frac{\|F\|_2}{\|A\|_2} \|x\|_2. \tag{9}$$