Green Function And Discretized Laplacian Operator

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2023.3.4

Statement

I would like to acknowledge the help from Huiteng Li.

1 Introduction

Definition 1.1. For a fixed $\bar{x} \in [0,1]$, the **Green's function** $G(x;\bar{x})$ is the function of x that solves the BVP

$$\begin{cases} u''(x) = \delta(x-\bar{x}); \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $\delta(x-\bar{x})$ is the Dirac delta function.

Discretize equation (1), then find the inverse of matrix

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \tag{2}$$

by Green's function. In (2), for $n \times n$ matrix A, $h = \frac{1}{n+1}$. A is **discretized Laplacian operator**.

2 Discretize the dirac delta function

Dirac delta function isn't a usual continuous function, so we should find a new method to discretize it. In fact, when we discretize a function, we exactly find a **grid function** to approximate it. For delta function $\delta(x-\bar{x})$, it satisfies:

- $\forall x \neq \bar{x}$, $\delta(x-\bar{x}) = 0$.
- $\int_{-\infty}^{+\infty} \delta(x-\bar{x})\mathrm{d}x = 1$.

So, assume $\bar{x} = x_j$ is on the discretized grid set $X := \{x_j\}_{j=1}^{n}$, the discretized grid function $\delta_g : X \to \mathbb{R}$ satisfies:

- $\delta_g(x_i) = \delta_{ij}$ while δ_{ij} is the Kronecker symbol.
- $\|\delta_g\|_1 = 1$, $\|\cdot\|$ means the **q-norm** of grid function.

Then:

$$\delta_g(x_i) = \begin{cases} 0, i \neq j; \\ \dfrac{1}{h}, i = j. \end{cases} \tag{3}$$

So, for $\bar{x} = x_j$, the discretized linear system for equation (1) is:

$$-AU = \frac{1}{h}e_j. \tag{4}$$

When $h \to 0$, $\delta_g \to \delta$ and $A \to -\Delta$, so we can use (4) to discretize equation (1).

3 Inverse of discretized Laplacian operator

If we admit (4) has no truncated error, we can use green's function to derive the inverse of matrix A. For $B = A^{-1}$, mark $B = [b_1, \cdots, b_n]$, we can see:

$$Ab_j = e_j. \tag{5}$$

If U_j satisfies (4), $b_j = -hU_j$.

By the definition of green's function, when $\bar{x} = x_j$, the solution of equation (1) is $G(x; x_j)$. Its restriction operator on X is $G(x_i; x_j)$. So, if truncated error $\tau = 0$, we can see:

$$U_j = \begin{bmatrix} G(x_1; x_j) \\ G(x_2; x_j) \\ \vdots \\ G(x_n; x_j) \end{bmatrix}. \tag{6}$$

So, if $B = A^{-1}$, we can see $b_{ij} = -hG(x_i; x_j)$. We can verify that B is indeed the inverse of A.

4 Discussion

How to show (4) has no truncated error?