A brief note for Evans PDE chapter 5-7

Author: Shuang Hu

Institute: Zhejiang University

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Chapter 1 Elliptic equation

In this chapter, we will discuss on the general elliptic PDEs and its weak form. We will exploit two essentially distinct techniques, energy methods within Sobolev spaces and maximum principle methods.

1.1 Definition

1.1.1 Elliptic equations

We will in this chapter mostly study the boundary value problem(BVP):

$$\begin{cases} Lu = f \text{ in } U; \\ u = 0 \text{ on } \partial U, \end{cases} \tag{1.1}$$

where U is an open, bounded subset of $\mathbb{R}^n$, and $u : \bar{U} \to \mathbb{R}$ is the unknown. Here f is given, and L denotes a second order differential operator have either the form:

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{1.2}$$

or else

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u. \tag{1.3}$$

The form (1.2) is in **divergence form**, while the form (1.3) is in **non-divergence form**. The requirement that $u = 0$ on $\partial\Omega$ is called **Dirichlet's boundary condition**.

Remark

- Different with Evans chapter 2, $u \in C^2(U)$ is unnecessary. So we should discuss on the weak form of equation (1.1).
- If $a^{ij} \in C^2(U)$, the form (1.2) and (1.3) are equivalent in general. But the form (1.2) is more natural for energy methods, and the form (1.3) is more appropriate for maximum principle techniques.
- Assumption: **symmetry condition**

$$a^{ij} = a^{ji}.$$

Now, give an important property of the differential operator L.

Definition 1.1

We say the partial differential operator L is (uniformly) elliptic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{1.4}$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$. ♣

Remark

- L is uniformly elliptic means the matrix $(a^{ij}(x))$ is positive definite $\forall x \in U$.
- The converse proposition of the above proposition isn't true.
- Special case: $a^{ij}(x) \equiv \delta_{ij}, b^i \equiv 0, c^i \equiv 0$. In this case, the equation (1.1) is the **Poisson equation**.

1.1.2 Weak solutions

Motivation: In general case, we can only assume that

$$a^{ij}, b^i, c \in L^\infty(U), f \in L^2(U). \tag{1.5}$$

In this case, maybe we can't find $u \in C^2(U)$ such that u satisfies (1.1), so we try to derive the **weak form** of the solution u, such that $u \in H_0^1(U)$. In practise, choose **test function** $v \in C_c^\infty(U)$, then define the linear functional

$$u^*(v) := \int_U uv\mathrm{d}x. \tag{1.6}$$

Then consider the dual form of (1.1), i.e.

$$(Lu)^*(v) = f^*(v)\ \forall v \in C_c^\infty(U). \tag{1.7}$$

It's just the **weak form** of equation (1.1). In this section, we choose the **divergence form** of operator L.

Theorem 1.1

The weak form of equation (1.1) *is*

$$\int_U \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b^i u_{x_i}v + cuv\mathrm{d}x = \int_U fv\mathrm{d}x. \tag{1.8}$$

♡

Proof It's suffices to derive the expression of $\int_U vLu\mathrm{d}x$. By Gauss-Green formula, for a vector field $F \in C^1(\mathbb{R}^n)$, we can see:

$$\int_U v\nabla\cdot F\mathrm{d}x = \int_{\partial U} vF\cdot n\mathrm{d}S(x) - \int_U F\cdot\nabla v\mathrm{d}x. \tag{1.9}$$

Choose the vector field $F = \begin{bmatrix} \sum a^{i1}(x)u_{x_i} \\ \sum a^{i2}(x)u_{x_i} \\ \vdots \\ \sum a^{in}(x)u_{x_i} \end{bmatrix}$, as $v = 0$ on $\partial\Omega$, by (1.9), we can see:

$$\begin{aligned} \int_U \sum_{i,j=1}^n \left(a^{ij}(x)u_{x_i}\right)_{x_j} v\mathrm{d}x &= \int_U v\nabla\cdot F\mathrm{d}x \\ &= -\int_U F\cdot\nabla v\mathrm{d}x \\ &= -\int_U \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j}\mathrm{d}x. \end{aligned} \tag{1.10}$$

Then:

$$\int_U vLu\mathrm{d}x = \int_U \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b^i u_{x_i}v + cuv\mathrm{d}x = \int_U vf\mathrm{d}x = f^*(v). \tag{1.11}$$

By (1.8), we can derive the following definitions.

Definition 1.2

1. *The bilinear form $B[\ ,\ ]$ associated with the divergence form elliptic operator defined by* (1.2) *is:*

$$B[u,v] := \int_U \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b^i u_{x_i}v \tag{1.12}$$

for $u, v \in H_0^1(U)$.

2. *We say that* $u \in H_0^1(U)$ *is a weak solution of the BVP* (1.1) *if*

$$B[u, v] = (f, v) \tag{1.13}$$

for all $v \in H_0^1(U)$, *where* $(\ ,\ )$ *denotes the inner product in* $L^2(U)$.

♣

More generally, let us consider the BVP

$$\begin{cases} Lu = f^0 - \sum_{i=1}^{n} f_{x_i}^i, x \in U, \\ u = 0, x \in \partial U, \end{cases} \tag{1.14}$$

where $f^i \in L^2(U)$. Then we say $u \in H_0^1(U)$ is a weak solution of problem (1.14) if

$$B[u, v] = \langle f, v \rangle \tag{1.15}$$

for all $v \in H_0^1(U)$, where $\langle \cdot, \cdot \rangle$ is the pairing of $H^{-1}(U)$ and $H_0^1(U)$.

For non-homogeneous elliptic PDE, i.e.

$$\begin{cases} Lu = f, x \in U, \\ u = g, x \in \partial U. \end{cases} \tag{1.16}$$

By trace theorem, $\exists w \in H^1(U)$ such that the trace of w is g. Then define $\tilde{u} := u - w$, (1.16) is equivalent to the equation:

$$\begin{cases} L\tilde{u} = \tilde{f}, x \in U, \\ \tilde{u} = 0, x \in \partial U, \end{cases} \tag{1.17}$$

where $\tilde{f} := f - Lw \in H^{-1}(U)$.

1.2 Existence of weak solutions

1.2.1 Lax-Milgram theorem

We now introduce an abstract principle from linear functional analysis.

Assume H is a real Hilbert space, with norm $\|\ \|$ and inner product $(\ ,\ )$. We let $\langle \cdot, \cdot \rangle$ denote the pairing of H with its dual space.

Theorem 1.2 (Lax-Milgram Theorem)

Assume that

$$B : H \times H \to \mathbb{R} \tag{1.18}$$

is a bilinear mapping, for which there exist constants α, β *such that*

$$|B(u, v)| \leq \alpha \|u\| \|v\| \,(u, v \in H) \tag{1.19}$$

and

$$\beta \|u\|^2 \leq B[u, u](u \in H). \tag{1.20}$$

Finally, let $f : H \to R$ *be a bounded linear functional on* H.

Then there exists a unique element $u \in H$ *such that*

$$B[u, v] = \langle f, v \rangle \tag{1.21}$$

for all $v \in H$.

Remark

1. If B is symmetry, the condition (1.53) and (1.20) means B can derive an inner product on H.
2. So, if B is symmetry, theorem 1.2 is a direct corollary of Riesz representation theorem.
3. If B isn't symmetry, Riesz representation theorem can transform $f \in H^*$ to $u_f \in H$, such that $\langle f, v\rangle = (u_f, v)$.
4. So, we should show that $\forall u_f \in H$, $\exists u \in H$ such that $B[u,v] = (u_f, v)$ for each $v \in H$.

Proof

By the remark above, we just need to show that $\forall u_f \in H$, $\exists u \in H$ such that $B[u,v] = (u_f, v)$ for each $v \in H$.

Consider an element $u \in H$. $B(u,v)$ is a bounded bilinear mapping, so the map

$$B_u(v) := B[u,v] \tag{1.22}$$

is a bounded linear functional. By Riesz representation theorem, $\exists! \tilde{u} \in H$ such that $B[u,v] = (\tilde{u}, v)\ \forall v \in H$. So we can define a map $A : H \to H$ maps u to $\tilde{u}$.

Then it's suffices to show that A is a bounded linear isomorphism.

First we should show that A is linear. $\forall v \in H$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $u_1, u_2 \in H$, we can see:

$$\begin{aligned} (A(\lambda_1 u_1 + \lambda_2 u_2), v) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \\ &= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v). \end{aligned} \tag{1.23}$$

(1.23) shows that A is a linear map.

Then we show that A is bounded. As B is bounded, we can see:

$$\begin{aligned} \|Au\|^2 &= (Au, Au) \\ &= B[u, Au] \\ &\le \alpha \|u\| \|Au\| . \end{aligned} \tag{1.24}$$

i.e. $\|Au\| \le \alpha \|u\|$. So A is bounded.

The next thing to do in the proof is to show A is an injective. By (1.20), we can see:

$$\begin{aligned} \beta \|u\|^2 &\le B[u,u] \\ &= (Au, u) \\ &\le \|Au\| \|u\| . \end{aligned} \tag{1.25}$$

i.e. $\beta \|u\| \le \|Au\|$. So $\beta \|u\| \le \|Au\| \le \alpha \|u\|$, it means that A is an injective. What's more, the range of A, marked as $R(A)$, is a closed set.

Finally, we should show that A is a surjective. If $R(A) \ne H$, since $R(A)$ is closed, there exists a nonzero element $\omega \in H$ with $\omega \in R(A)^{\perp}$. Then:

$$\beta \|\omega\|^2 \le B[\omega, \omega] = (A\omega, \omega) = 0. \tag{1.26}$$

which means that $\|\omega\| = 0$, contradict! So A is a surjective.

This completes the proof of Lax-Milgram theorem.

Lax-Milgram theorem gives an important method for us to analyze the existence of weak solution.

1.2.2 Energy estimates and First Existence theorem

Now, we return to the specific bilinear form $B[\ ,\ ]$ defined by (1.12), and try to use **Lax-Milgram theorem** to prove the first existence theorem.

Theorem 1.3 (Energy estimates)

There exists constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

$$|B[u,v]| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} \tag{1.27}$$

and

$$\beta \|u\|^2_{H_0^1(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}. \tag{1.28}$$

♡

Proof First derive the inequality (1.27). According to (1.12), we can check that

$$\begin{aligned}
B[u,v] &\leq \sum_{i,j=1}^{n} \int_U \left\|a^{ij}\right\|_{L^\infty} |u_{x_i}||v_{x_j}| \mathrm{d}x + \sum_{i=1}^{n} \int_U \left\|b^i\right\|_{L^\infty} |u_{x_i}||v| \mathrm{d}x + \int_U \|c\|_{L^\infty} |u||v| \mathrm{d}x \\
&\leq \sum_{i,j=1}^{n} \int_U \left\|a^{ij}\right\|_{L^\infty} |Du||Dv| \mathrm{d}x + \sum_{i=1}^{n} \int_U \left\|b^i\right\|_{L^\infty} |Du||v| \mathrm{d}x + \int_U \|c\|_{L^\infty} |u||v| \mathrm{d}x \\
&\leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}.
\end{aligned} \tag{1.29}$$

Then, by the uniformly elliptic condition of coefficient matrix $a^{ij}(x)$, we can see:

$$\begin{aligned}
\theta \int_U |Du|^2 \mathrm{d}x &\leq \int_U \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i} u_{x_j} \mathrm{d}x \\
&= B[u,u] - \int_U \left(\sum_{i=1}^{n} b^i(x) u_{x_i} u \mathrm{d}x + cu^2 \right) \mathrm{d}x \\
&\leq B[u,u] + \sum_{i=1}^{n} \int_U \left\|b^i(x)\right\|_{L^\infty} |u||Du| \mathrm{d}x + c \int_U u^2 \mathrm{d}x
\end{aligned} \tag{1.30}$$

By Cauchy-Schwarz inequality with coefficient, we can see:

$$\int_U |u||Du| \mathrm{d}x \leq \epsilon \int_U |Du|^2 \mathrm{d}x + \frac{1}{4\epsilon} \int_U |u|^2 \mathrm{d}x. \tag{1.31}$$

Choose ϵ such that $\epsilon \sum_{i=1}^{n} \left\|b^i\right\|_{L^\infty} < \frac{\theta}{2}$, then exists constant $C > 0$ such that

$$\frac{\theta}{2} \int_U |Du|^2 \mathrm{d}x \leq B[u,u] + C \int_U u^2 \mathrm{d}x. \tag{1.32}$$

Finally, by Poincare-Friedrichs inequality, the equation (1.28) is true.

By (1.28), if $\gamma > 0$, $B[u,v]$ isn't uniformly elliptic in general. So, $B[u,v]$ does't satisfy the hypotheses of Lax-Milgram theorem in general. So we should give some revisions on bilinear form B. Then, we derive the **first existence theorem for weak solutions.**

Theorem 1.4 (First Existence Theorem for weak solutions)

There is a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each function $f \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U)$ of the boundary-value problem

$$\begin{cases} Lu + \mu u = f, x \in U; \\ \qquad u = 0, x \in \partial\Omega. \end{cases} \tag{1.33}$$

♡

Proof Choose the parameter γ as theorem 1.3, then define the bilinear form

$$B_\mu[u,v] := B[u,v] + \mu(u,v). \tag{1.34}$$

Then the weak form of equation (1.33) is

$$B_\mu[u,v] = \langle f, v\rangle. \tag{1.35}$$

By (1.28), as $\mu \geq \gamma$, the bilinear form B_μ satisfies uniformly elliptic condition. So by Lax-Milgram theorem, equation (1.34) has unique weak solution.

Remark The first existence theorem for weak solutions is a milestone, but we still can't give the existence theorem of equation (1.1). We should use Fredholm alternative theorem to derive the existence of weak solution.

1.2.3 Fredholm alternative and the solvability

In this section, we show the Fredholm alternative theorem first, then employ this theorem to derive the existence theorem for weak solutions.

Theorem 1.5 (Fredholm alternative)

Let $K : H \to H$ be a compact linear operator, then:

- $\ker(I-K)$ *is finite dimensional.*
- $R(I-K)$ *is closed.*
- $R(I-K) = \ker(I-K^*)^\perp$.
- $\ker(I-K) = \{0\}$ *if and only if* $R(I-K) = H$.
- $\dim\ker(I-K) = \dim\ker(I-K^*)$.

♡

Proof Omitted.

Then, derive the **dual problem** of equation (1.1).

Definition 1.3 (Dual problem)

1. *The operator L^*, the formal adjoint of L, is:*

$$L^*v = -\sum_{i,j=1}^n \left(a^{ij}v_{x_j}\right)_{x_i} - \sum_{i=1}^n b^i v_{x_i} + \Big(c - \sum_{i=1}^n b^i_{x_i}\Big)v, \tag{1.36}$$

provided $b^i \in C^1(\bar{U})$.

2. *The adjoint bilinear form*

$$B^* : H_0^1(U) \times H_0^1(U) \to \mathbb{R} \tag{1.37}$$

is defined by

$$B^*[v,u] = B[u,v] \tag{1.38}$$

for all $u, v \in H_0^1(U)$.

3. *We say that $v \in H_0^1(U)$ is a weak solution of the adjoint problem*

$$\begin{cases} L^*v = f, x \in U; \\ \quad v = 0, x \in \partial U. \end{cases} \tag{1.39}$$

provided $B^*[v,u] = (f,u)$ for all $u \in H_0^1(U)$.

♣

Remark

- (1.36) is called formal adjoint, for $(Lu, v) = (u, L^*v)$.

- (1.39) is the dual form of equation (1.1).

To show the solvability of problem (1.1), we derive the following theorem.

Theorem 1.6 (Second Existence Theorem for weak solutions)

1. *Precisely one of the following statements holds: either for each $f \in L^2(U)$ there exists a unique weak solution u of the boundary value problem (1.1) (marked as α), or else there exists a weak solution $u \neq 0$ of the homogeneous problem (marked as β)*
$$\begin{cases} Lu = 0, x \in U; \\ u = 0, x \in \partial U. \end{cases} \tag{1.40}$$
2. *Furthermore, should assertion (β) hold, the dimension of the subspace $N \subset H_0^1(U)$ of weak solutions of (1.40) is finite and equals the dimension of the subspace $N^* \subset H_0^1(U)$ of weak solutions of*
$$\begin{cases} L^*v = 0, x \in U; \\ v = 0, x \in \partial U. \end{cases} \tag{1.41}$$
3. *Finally, the BVP (1.1) has a weak solution if and only if*
$$(f, v) = 0 \; \forall v \in N^*. \tag{1.42}$$

♡

Remark To prove this theorem, we should try to use theorem 1.4. The main idea is to construct a compact operator K, such that the weak form of (1.1) is $(I - K)u = h$, then use theorem 1.5.

Proof First, choose γ as theorem 1.4 suggests, then for each $g \in L^2(U)$, there exists a unique $u \in H_0^1(U)$ solving:
$$B_\gamma[u, v] = \langle g, v \rangle \; \forall v \in H_0^1(U). \tag{1.43}$$
Write $u = L_\gamma^{-1} g$ if equation (1.43) holds. As the weak form of (1.1) is $B[u, v] = \langle f, v \rangle$, we can see $B_\gamma[u, v] = \langle f + \gamma u, v \rangle$, i.e. $u = L_\gamma^{-1}(f + \gamma u)$. Then choose operator $K = \gamma L_\gamma^{-1}$, $h = L_\gamma^{-1} f$, (1.43) is equivalent to
$$(I - K)u = h. \tag{1.44}$$

The next step is to show that $K : L^2(U) \to L^2(U)$ is a bounded, linear, compact operator. In fact, we only need to show K is compact. By (1.43), we can see:
$$\beta \|u\|_{H_0^1(U)}^2 \le B_\gamma[u, u] = \langle g, u \rangle \le \|g\|_{L^2(U)} \|u\|_{L^2(U)} \le \|g\|_{L^2(U)} \|u\|_{H_0^1(U)} . \tag{1.45}$$
By (1.45), there exists a constant $C > 0$ such that $\|Kg\|_{H_0^1(U)} \le C \|g\|_{L^2(U)}$. As $H_0^1(U) \subset\subset L^2(U)$, K is a compact operator.

Then, use theorem 1.5 on equation (1.44). If $\ker(I - K) = \{0\}$, theorem 1.5 shows that $I - K$ is also a surjective, i.e. statement (α) is true. Otherwise, $\ker(I - K) \neq \{0\}$ means that $\exists u \in H_0^1(U)$ such that $(I - K)u = 0$, i.e. u satisfies equation (1.40). Then statement (β) is true. If $N \neq \{0\}$, while
$$\dim \ker(I - K) = \dim \ker(I - K^*) < \infty, \tag{1.46}$$
we can see $\dim N = \dim N^* < \infty$.

Finally, if $(I - K)u = h$ has a solution, $v \in N^*$, we can see:
$$\langle h, v \rangle = \langle (I - K)u, v \rangle = \langle u, (I - K^*)v \rangle = 0. \tag{1.47}$$
By $h = L_\gamma^{-1} f$, we can see:
$$\langle h, v \rangle = \frac{1}{\gamma} \langle Kf, v \rangle = \frac{1}{\gamma} \langle f, K^* v \rangle = \frac{1}{\gamma} \langle f, v \rangle = 0. \tag{1.48}$$

So:$\langle f, v\rangle = 0$.

1.2.4 Spectrum and third existence theorem

In this section, we will discuss on the **spectrum** of an operator L, then derive the existence of weak solutions for eigenvalue problem.

Theorem 1.7 (Third existence theorem for weak solutions)

1. *There exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the BVP*
$$\begin{cases} Lu = \lambda u + f, x \in U; \\ u = 0, x \in \partial U \end{cases} \tag{1.49}$$
has a unique weak solution for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$.
2. *If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$, the values of a nondecreasing sequence with $\lambda_k \to +\infty$.*

♡

Remark

1. Σ is called the **spectrum** of operator L.
2. If $f = 0$, the BVP (1.49) is called eigenvalue problem, and if there exists a solution $\omega \neq 0$, λ is called an **eigenvalue** of L, and ω is a corresponding **eigenfunction**.

Proof By theorem 1.6, if $\lambda \in \Sigma$, the homogeneous eigenvalue problem
$$\begin{cases} Lu = \lambda u, x \in U, \\ u = 0, x \in \partial U \end{cases} \tag{1.50}$$
has a solution $u \neq 0$. Consider it's weak form, we can see:
$$B_\gamma[u, v] = (\lambda + \gamma)\langle u, v\rangle . \tag{1.51}$$
i.e.
$$u = L_\gamma^{-1}(\gamma + \lambda)u = \frac{\lambda + \gamma}{\gamma} K u. \tag{1.52}$$
As $u \neq 0$, u is the eigenvector of operator K, the corresponding eigenvalue is $\frac{\gamma}{\lambda+\gamma}$. By theorem 1.6, K is a compact operator, so the Spectrum set S of operator K is either finite set, or else the values of a sequence converging to zero. It means that Σ is at most countable, and if $|\Sigma| = \infty$, $\lambda_k \to \infty$.

Finally, we note the boundedness of eigenvalue problem.

Theorem 1.8 (Boundedness of the inverse)

If $\lambda \notin \Sigma$, there exists a constant C such that
$$\|u\|_{L^2(U)} \leq C\,\|f\|_{L^2(U)} \tag{1.53}$$
whenever $f \in L^2(U)$ and u is the unique weak solution of (1.49). The constant C depends only on λ, U and the coefficients of L.

♡

Proof If not, there would exist sequences $\{f_k\} \subset L^2(U)$ and $\{u_k\} \subset H_0^1(U)$ such that
$$\begin{cases} Lu_k = \lambda u_k + f_k, x \in U, \\ u_k = 0, x \in \partial U, \end{cases} \tag{1.54}$$
but
$$\|u_k\|_{L^2(U)} > k\,\|f_k\|_{L^2(U)} . \tag{1.55}$$

Assume WLOG, $\|u_k\|_{L^2(U)} = 1$, we can see $\|f_k\| \to 0$. Then there exists a subsequence $\{u_{k_j}\}$ satisfies:

$$\begin{cases} u_{k_j} \rightharpoonup u \text{ in } H_0^1(U), \\ u_{k_j} \to u \text{ in } L^2(U). \end{cases} \tag{1.56}$$

Then u is a weak solution of (1.50). Since $\lambda \notin \Sigma$, $u \equiv 0$. But $\|u_k\|_{L^2(U)} \equiv 1$, contradict!

1.3 Interior Regularity

Now, we try to discuss on the **regularity** of weak solutions. Consider a general PDE $Lu = f$, we find the weak solution $u \in H^1(U)$. However, if we set $f \in H^m(U)$, we expect $u \in H^{m+2}(U)$, it means we derive stronger regularity about the weak solution of $Lu = f$. The point of regularity is to derive analytic estimates from the structural, algebraic assumption of ellipticity.

First, recall the definition of difference quotients, and some related properties.

Definition 1.4 (Difference quotient)

The i-th difference quotient of size h is

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h} (i = 1, \cdots, n) \tag{1.57}$$

for $x \in V$ and $h \in \mathbb{R}$, $0 < |h| < dist(V, \partial U)$. And $D^h(u) := \left(D_1^h u, \cdots, D_n^h u\right)$. ♣

The concept of difference quotients is related to weak derivatives, as the following theorem:

Theorem 1.9

1. *Suppose $1 \le p < \infty$ and $u \in W^{1,p}(U)$, for each $V \subset\subset U$,*

$$\left\| D^h u \right\|_{L^p(V)} \le C \|Du\|_{L^p(U)} \tag{1.58}$$

for some constant C and all $0 < |h| < \frac{1}{2} dist(V, \partial U)$.

2. *Assume $1 < p < \infty$, $u \in L^p(V)$, and there exists a constant C such that*

$$\left\| D^h u \right\|_{L^p(V)} \le C \tag{1.59}$$

for all $0 < |h| < \frac{1}{2} dist(V, \partial U)$, then $u \in W^{1,p}(V)$ with $\|Du\|_{L^p(V)} \le C$. ♡

Proof see [1] section 5.8, theorem 3.

Then, introduce two lemmas for the integrate of difference quotients.

Lemma 1.1

For a bounded open set U, and open set $W \subset\subset U$, assume $v, w \in H^1(U)$, $supp(w) \subset W$, $h < \frac{1}{2} dist(W, \partial U)$, then we can see:

$$\int_U v D_k^{-h} w \mathrm{d}x = -\int_U w D_k^h v \mathrm{d}x, \tag{1.60}$$

and

$$D_k^h(vw) = v^h D_k^h w + w D_k^h v, \tag{1.61}$$

for $v^h(x) := v(x + he_k)$. ♡

Proof For equation (1.60), as supp$w \subset W$, we can see:

$$\begin{aligned} RHS &= -\int_W w(x)\frac{v(x+he_k)-v(x)}{h}\mathrm{d}x \\ &= \frac{1}{h}\int_W w(x)v(x)\mathrm{d}x - \frac{1}{h}\int_W w(x)v(x+he_k)\mathrm{d}x. \end{aligned} \tag{1.62}$$

set $\tilde{W} := \{y : y = x + he_k, x \in W\}$, we can see:

$$\begin{aligned} LHS &= \frac{1}{h}\int_U v(x)\left(w(x) - w(x+he_k)\right)\mathrm{d}x \\ &= \frac{1}{h}\int_W v(x)w(x)\mathrm{d}x - \frac{1}{h}\int_{\tilde{W}} v(x)w(x-he_k)\mathrm{d}x. \end{aligned} \tag{1.63}$$

By integral substitution, choose $y := x - he_k$, we can see:

$$\int_{\tilde{W}} v(x)w(x-he_k)\mathrm{d}x = \int_W v(y+he_k)w(y)\mathrm{d}y. \tag{1.64}$$

So, equation (1.60) is true. Then, for equation (1.61), we can see:

$$\begin{aligned} LHS &= \frac{v(x+he_k)w(x+he_k) - v(x)w(x)}{h} \\ &= \frac{v(x+he_k)(w(x+he_k) - w(x)) + w(x)(v(x+he_k) - v(x))}{h} \\ &= v^h D_k^h w + w D_k^h v = RHS. \end{aligned} \tag{1.65}$$

Then, we give the main result for this section:

Theorem 1.10 (Interior H^2-regularity)

Assume $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$, and $f \in L^2(U)$. Suppose furthermore that $u \in H^1(U)$ is a weak solution of the elliptic PDE $Lu = f$, then:

$$u \in H^2_{loc}(U); \tag{1.66}$$

and for each open subset $V \subset\subset U$ we have the estimate

$$\|u\|_{H^2(V)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right), \tag{1.67}$$

the constant C depending only on V, U, and the coefficients of operator L. ♡

Before the proof of this theorem, we should give some remarks.

Remark

1. In this theorem, we don't require $u \in H^1_0(U)$.
2. Since $u \in H^2_{loc}(U)$, we have

$$Lu = f \text{ a.e. in U.} \tag{1.68}$$

The idea : First, construct a truncated function $\zeta \in C^\infty(U)$ such that $V \subset\subset W \subset\subset U$, $\zeta|_V \equiv 1$, $\zeta|_{U\setminus W} \equiv 0$. Then, choose a test function

$$v(x) = -D_k^{-h}\left(\zeta^2 D_k^h u(x)\right) \in H^1_0(U). \tag{1.69}$$

In fact, if $u \in C^2(U)$, v is a difference quotient approximation for D^2u. Finally, use the relation $B[u,v] = (f,v)$ to approximate $\left\|D_k^h Du\right\|_{L^2(V)}$, and use the second part of theorem 1.9.

Proof By the definition of weak solution, $B[u,v] = (f,v)$. Then we can see:

$$\sum\int_U a^{ij}u_{x_i}v_{x_j}\mathrm{d}x = \int_U (f - \sum b_i u_{x_i} - cu)v\mathrm{d}x. \tag{1.70}$$

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$$A=\sum\int_U a^{ij}u_{x_i}v_{x_j}\mathrm{d}x, \tag{1.71}$$

and choose v as (1.69), according to lemma 1.1, we can derive:

$$\begin{aligned}
A &= -\int_U\sum a^{ij}u_{x_i}\left(D_k^{-h}\left(\zeta^2 D_k^h u(x)\right)\right)_{x_j}\mathrm{d}x \\
&=\sum\int_U D_k^h\left(a^{ij}u_{x_i}\right)\left(\zeta^2 D_k^h u(x)\right)_{x_j}\mathrm{d}x \\
&=\sum\int_U\left(a^{ij,h}D_k^h u_{x_i}+u_{x_i}D_k^h(a^{ij})\right)\left(2\zeta\zeta_{x_j}D_k^h u(x)+\zeta^2\left(D_k^h u(x)\right)_{x_j}\right)\mathrm{d}x \\
&=\sum\int_U a^{ij,h}\zeta^2 D_k^h u_{x_i}D_k^h u_{x_j}\mathrm{d}x \\
&+\sum\int_U\left(2\zeta\zeta_{x_j}u_{x_i}D_k^h(a^{ij})D_k^h u(x)+2\zeta\zeta_{x_j}a^{ij,h}D_k^h u_{x_i}D_k^h u(x)+\zeta^2 u_{x_i}D_k^h(a^{ij})\left(D_k^h u(x)\right)_{x_j}\right)\mathrm{d}x \\
&:= A_1+A_2.
\end{aligned} \tag{1.72}$$

By the uniform elliptic property, there exists $\theta>0$ such that

$$A_1\geq\theta\int_U\zeta^2|D_k^h Du|^2\mathrm{d}x. \tag{1.73}$$

Then we try to approx $|A_2|$. As $a^{ij}\in C^1(U)$, $\zeta\in C^\infty(U)$, we can see:

$$\begin{aligned}
|A_2| &\leq\sum\int_U\left(C_1\zeta|u_{x_i}||u(x)|+C_2\zeta|D_k^h u_{x_i}||D_k^h u(x)|+C_3\zeta|u_{x_i}|\left(D_k^h u(x)\right)_{x_j}\right)\mathrm{d}x \\
&\leq C\int_U\left(\zeta|Du||u|+\zeta|D_k^h Du||D_k^h u|+\zeta|Du||D_k^h Du|\right)\mathrm{d}x \\
&\leq\frac{\theta}{2}\int_U\zeta^2|D_k^h Du|^2\mathrm{d}x+\tilde{C}\int_U|Du|^2+|D_k^h u|^2\mathrm{d}x \\
&\leq\frac{\theta}{2}\int_U\zeta^2|D_k^h Du|^2\mathrm{d}x+M\int_U|Du|^2\mathrm{d}x.
\end{aligned} \tag{1.74}$$

In equation (1.74), $C_1, C_2, C_3, C, \tilde{C}, M$ are all constants. The first step follows from $a^{ij},\zeta\in C^1(U)$, the second and the third steps from Cauchy-Schwarz inequality, and the last from theorem 1.9.

Combining (1.73) and (1.74), we can see

$$A\geq\frac{\theta}{2}\int_U\zeta^2|D_k^h Du|^2\mathrm{d}x-M\int_U|Du|^2\mathrm{d}x. \tag{1.75}$$

The next step is to approximate the right-hand integral

$$B:=\int_U(f-\sum b_i u_{x_i}-cu)v\mathrm{d}x. \tag{1.76}$$

First, as $b_i\in C^1(U)$, we can see

$$|B|\leq C\int_U(|f|+|Du|+|u|)|v|\mathrm{d}x. \tag{1.77}$$

Then, consider

$$\begin{aligned}\int_U v^2 \mathrm{d}x &= \int_U |D_k^{-h}(\zeta^2 D_k^h u)|^2 \mathrm{d}x \\ &\leq C \int_U D|\zeta^2 D_k^h u|^2 \mathrm{d}x \\ &\leq C \int_U \zeta^2 |D_k^h Du|^2 \mathrm{d}x + C \int_W |D_k^h u|^2 \mathrm{d}x \\ &\leq C \int_U \left(|Du|^2 + \zeta^2 |D_k^h Du|^2\right) \mathrm{d}x.\end{aligned} \tag{1.78}$$

The second step follows from theorem 1.9, the third step follows from the Leibniz formula, and the final step follows from theorem 1.9 as well.

Finally, by (1.77) , (1.78) and Cauchy-Schwarz inequality, there exists constant C such that

$$|B| \leq \frac{\theta}{4} \int_U \zeta^2 |D_k^h Du|^2 \mathrm{d}x + C \int_U \left(|f|^2 + |u|^2 + |Du|^2\right) \mathrm{d}x. \tag{1.79}$$

By the estimation (1.75) and (1.79), it's clear that $\forall h > 0$, $\exists$ constant $C > 0$, such that

$$\int_V |D_k^h Du|^2 \mathrm{d}x \leq C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}). \tag{1.80}$$

It means that $Du \in H^1(V)$, and

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right). \tag{1.81}$$

Finally, choose auxiliary function $\xi \in C^\infty(U)$, supp$\xi \subset U$ and $\xi \equiv 1$ on W, set $v = \xi^2 u$, according to equation (1.70), we can see:

$$\int_U \xi^2 |Du|^2 \mathrm{d}x \leq C \int_U (f^2 + u^2) \mathrm{d}x. \tag{1.82}$$

Then:

$$\|u\|_{H^1(W)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right). \tag{1.83}$$

Remark Theorem 1.10 shows the condition for the higher regularity of weak solutions.

Then, we introduce the higher interior regularity.

Theorem 1.11 (Higher interior regularity)

Let m be a nonnegative integer, and assume $a^{ij}, b^i, c \in C^{m+1}(U)$, and $f \in H^m(U)$. Suppose $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f, \tag{1.84}$$

then $u \in H^{m+2}_{loc}(U)$ and for each $V \subset\subset U$ we have the estimate

$$\|u\|_{H^{m+2}(V)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}). \tag{1.85}$$

Proof We prove this theorem by induction. If $m = 0$, theorem 1.11 is equivalent to theorem 1.10, so the induction basis is true. Assume theorem 1.11 is true for $m = k$, i.e. if $a^{ij}, b^i, c \in C^{k+1}(U)$, $f \in H^k(U)$, the weak solution $u \in H^1(U)$ satisfies

$$\|u\|_{H^{k+2}(W)} \leq C(\|f\|_{H^k(U)} + \|u\|_{L^2(U)}). \tag{1.86}$$

for each $W \subset\subset U$ and approximate constant C, and $u \in H^{k+2}_{loc}(U)$. Then let α be any multiindex with $|\alpha| = m + 1$ and choose any auxiliary function $\tilde{v} \in C_c^\infty(W)$. Choose the test function $v = (-1)^{|\alpha|} D^\alpha \tilde{v}$, insert it into the weak form $B[u,v]=(f,v)$, perform some integrations by parts, we discover:

$$\begin{aligned} B[u,v] &= \sum \int_U a^{ij} u_{x_i} (-1)^{|\alpha|} (D^\alpha \tilde{v})_{x_j} \mathrm{d}x + \sum \int_U b^i u_{x_i} (-1)^{|\alpha|} D^\alpha \tilde{v} \mathrm{d}x + \int_U cu(-1)^{|\alpha|} D^\alpha \tilde{v} \mathrm{d}x \\ &= \sum \int_U D^\alpha (a^{ij} u_{x_i}) \tilde{v}_{x_j} \mathrm{d}x + \sum \int_U D^\alpha (b^i u_{x_i}) \tilde{v} \mathrm{d}x + \int_U D^\alpha (cu) \tilde{v} \mathrm{d}x \\ &= B[D^\alpha u, \tilde{v}] - \sum_{i,j} \sum_{\beta \le \alpha} \int_U \binom{\alpha}{\beta} (D^{\alpha-\beta} a^{ij} D^\beta u_{x_i})_{x_j} \tilde{v} \mathrm{d}x + \sum_i \sum_{\beta \le \alpha} \int_U D^{\alpha-\beta} b^i D^\beta u_{x_i} \tilde{v} \mathrm{d}x \\ &+ \sum_{\beta \le \alpha} \int_U D^{\alpha-\beta} c D^\beta u \tilde{v} \mathrm{d}x. \end{aligned} \tag{1.87}$$

So if we write $\tilde{u} := D^\alpha u$, we can see:

$$B[\tilde{u}, \tilde{v}] = (\tilde{f}, \tilde{v}). \tag{1.88}$$

Where

$$\tilde{f} := D^\alpha f - \sum_{\beta < \alpha} \binom{\alpha}{\beta} \left[-\sum (D^{\alpha-\beta} a^{ij} D^\beta u_{x_i})_{x_j} + \sum D^{\alpha-\beta} b^i D^\beta u_{x_i} + D^{\alpha-\beta} c D^\beta u \right]. \tag{1.89}$$

We can see that $\tilde{u}$ is the weak solution of equation $L\tilde{u} = \tilde{f}$. Since $a^{ij}, b^i, c \in C^{k+2}(U)$, $f \in H^{k+1}(U)$, we can see $\tilde{f} \in L^2(W)$, and

$$\left\| \tilde{f} \right\|_{L^2(W)} \le C \left(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)} \right). \tag{1.90}$$

In light of theorem 1.10, we see $\tilde{u} \in H^2(V)$ with

$$\|\tilde{u}\|_{H^2(V)} \le \ C(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)}). \tag{1.91}$$

As the inequality holds for all multiindex α satisfies $|\alpha| = m+1$, the proof is completed.

1.4 Boundary regularity

Now, we extend to give the regularity of weak solution for homogeneous Dirichlet boundary value problem. In fact, when it comes to Dirichlet BVPs, the result comes to be stronger.

Theorem 1.12 (Boundary H^2-regularity)

Assume $a^{ij} \in C^1(\bar{U}), b^i, c \in L^\infty(U)$ and $f \in L^2(U)$. Suppose that $u \in H_0^1(U)$ is a weak solution of the elliptic BVP (1.1). Assume finally ∂U is C^2, then $u \in H^2(U)$ and we have the estimate

$$\|u\|_{H^2(U)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}). \tag{1.92}$$

The constant C depending only on U and the coefficients of L.

♡

Remark

- Main difference from theorem 1.10: now $u \in H^2(U)$ rather than $u \in H^2_{loc}(U)$.
- If equation (1.1) has unique solution, we can see $\|u\|_{H^2(U)} \le C \|f\|_{L^2(U)}$ from theorem 1.12. This result shows the **well-posed** of elliptic equation.
- In this theorem, we assume $u \equiv 0$ along ∂U in the trace case.

The idea of the proof: First, consider the special case that U is a half-ball, then $\forall x_0 \in \partial U$, choose a homologeous map from $B(x_0, \epsilon) \cap U$ to the half ball. Finally, use the compactness of ∂U.

Proof First, consider the case $U = B(0,1) \cap \mathbb{R}^n_+$. The target is to show that $\exists V \subset\subset U$ with $\partial V \cap \partial U \neq \emptyset$

such that
$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}). \tag{1.93}$$
To derive the H^2 norm on subset $V := B(0, \frac{1}{2}) \cap \mathbb{R}^n_+$, we first derive the truncated function $\zeta \in C^\infty(U)$ satisfies:
$$\begin{cases} \zeta \equiv 1, x \in B(0, \dfrac{1}{2}) \\ \zeta \equiv 0, x \notin B(0,1), \\ 0 \le \zeta \le 1, \end{cases} \tag{1.94}$$
such that ζ vanishes near the curved part of ∂U. Since u is a weak solution of equation (1.1), we have $B[u,v] = (f,v)$ for all $v \in H^1_0(V)$. Consequently, we can also write the auxiliary equation (1.43). Then we should derive the form of $v \in H^1_0(U)$.

Consider $1 \le k \le n-1$, set $v = -D_k^{-h}(\zeta^2 D_k^h u)$, first we claim that $v \in H^1_0(U)$. In fact:
$$v(x) = \frac{1}{h^2}\left[\zeta^2(x-he^k)u(x) + \zeta^2(x)u(x) - \zeta^2(x)u(x+he^k) - \zeta^2(x-he^k)u(x-he^k)\right]. \tag{1.95}$$
On the face $\{x_n = 0\}$, by the definition of weak solution, $u(x) = 0$. And near the curved part of ∂U, by the definition of ζ, $\zeta(x) = 0$. So on ∂U, we can see $v(x) \equiv 0$, i.e. $v(x) \in H^1_0(U)$.

Then, by equation (1.43) and the same method as theorem 1.10, we can see that for $1 \le k \le n-1$,
$$\int_V |D_k^h Du|^2 \mathrm{d}x \le C\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right). \tag{1.96}$$
i.e.
$$\sum_{\substack{k+l<2n \\ k=l=1}}^{n} \int_V |u^2_{x_k x_l}| \mathrm{d}x \le C\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right). \tag{1.97}$$
But we can't estimate $\|u_{x_n x_n}\|_{L^2(V)}$ by this method, for $x - he^n$ may lie outside the region U. Now we rewrite the equation (1.1) by nondivergence form, i.e.
$$-\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i u_{x_i} + cu = f. \tag{1.98}$$
As $a^{ij}(x)$ satisfies **uniformly elliptic condition**, there exists $\theta > 0$ such that $a^{nn}(x) \ge \theta \ \forall x \in U$. So:
$$|u_{x_n x_n}| \le C\left(\sum_{\substack{i,j=1 \\ i+j<2n}}^{n} |u_{x_i x_j}| + |Du| + |u| + |f|\right). \tag{1.99}$$
By (1.97), (1.99) and (1.83), we conclude $u \in H^2(V)$ and
$$\|u\|_{H^2(V)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right). \tag{1.100}$$

Bibliography

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