A brief note for Evans PDE chapter 6-7

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Chapter 1 Elliptic equation

In this chapter, we will discuss on the general elliptic PDEs and its weak form. We will exploit two essentially distinct techniques, energy methods within Sobolev spaces and maximum principle methods.

1.1 Definition

1.1.1 Elliptic equations

We will in this chapter mostly study the boundary value problem(BVP):

$$\begin{cases} Lu = f \text{ in } U; \\ u = 0 \text{ on } \partial U, \end{cases} \tag{1.1}$$

where U is an open, bounded subset of $\mathbb{R}^n$, and $u : \bar{U} \to \mathbb{R}$ is the unknown. Here f is given, and L denotes a second order differential operator have either the form:

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{1.2}$$

or else

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u. \tag{1.3}$$

The form (1.2) is in **divergence form**, while the form (1.3) is in **non-divergence form**. The requirement that $u = 0$ on $\partial\Omega$ is called **Dirichlet's boundary condition**.

Remark

- Different with Evans chapter 2, $u \in C^2(U)$ is unnecessary. So we should discuss on the weak form of equation (1.1).
- If $a^{ij} \in C^2(U)$, the form (1.2) and (1.3) are equivalent in general. But the form (1.2) is more natural for energy methods, and the form (1.3) is more appropriate for maximum principle techniques.
- Assumption: **symmetry condition**

$$a^{ij} = a^{ji}.$$

Now, give an important property of the differential operator L.

Definition 1.1

We say the partial differential operator L is (uniformly) elliptic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{1.4}$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$. ♣

Remark

- L is uniformly elliptic means the matrix $(a^{ij}(x))$ is positive definite for a.e. $x \in U$.
- The converse proposition of the above proposition isn't true.
- Special case: $a^{ij}(x) \equiv \delta_{ij}, b^i \equiv 0, c^i \equiv 0$. In this case, the equation (1.1) is the **Poisson equation**.

1.1.2 Weak solutions

Motivation: In general case, we can only assume that

$$a^{ij}, b^i, c \in L^\infty(U), f \in L^2(U). \tag{1.5}$$

In this case, maybe we can't find $u \in C^2(U)$ such that u satisfies (1.1), so we try to derive the **weak form** of the solution u, such that $u \in H_0^1(U)$. In practise, choose **test function** $v \in C_c^\infty(U)$, then define the linear functional

$$u^*(v) := \int_U uv\mathrm{d}x. \tag{1.6}$$

Then consider the dual form of (1.1), i.e.

$$(Lu)^*(v) = f^*(v)\ \forall v \in C_c^\infty(U). \tag{1.7}$$

It's just the **weak form** of equation (1.1). In this section, we choose the **divergence form** of operator L.

Theorem 1.1

The weak form of equation (1.1) *is*

$$\int_U \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b^i u_{x_i}v + cuv\mathrm{d}x = \int_U fv\mathrm{d}x. \tag{1.8}$$

♡

Proof It's suffices to derive the expression of $\int_U vLu\mathrm{d}x$. By Gauss-Green formula, for a vector field $F \in C^1(\mathbb{R}^n)$, we can see:

$$\int_U v\nabla\cdot F\mathrm{d}x = \int_{\partial U} vF\cdot n\mathrm{d}S(x) - \int_U F\cdot\nabla v\mathrm{d}x. \tag{1.9}$$

Choose the vector field $F = \begin{bmatrix} \sum a^{i1}(x)u_{x_i} \\ \sum a^{i2}(x)u_{x_i} \\ \vdots \\ \sum a^{in}(x)u_{x_i} \end{bmatrix}$, as $v = 0$ on $\partial\Omega$, by (1.9), we can see:

$$\begin{aligned} \int_U \sum_{i,j=1}^n \left(a^{ij}(x)u_{x_i}\right)_{x_j} v\mathrm{d}x &= \int_U v\nabla\cdot F\mathrm{d}x \\ &= -\int_U F\cdot\nabla v\mathrm{d}x \\ &= -\int_U \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j}\mathrm{d}x. \end{aligned} \tag{1.10}$$

Then:

$$\int_U vLu\mathrm{d}x = \int_U \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b^i u_{x_i}v + cuv\mathrm{d}x = \int_U vf\mathrm{d}x = f^*(v). \tag{1.11}$$

By (1.165), we can derive the following definitions.

Definition 1.2

1. *The bilinear form* $B[\ ,\ ]$ *associated with the divergence form elliptic operator defined by* (1.2) *is:*

$$B[u,v] := \int_U \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^n b^i u_{x_i}v \tag{1.12}$$

for $u, v \in H_0^1(U)$.

2. *We say that* $u \in H_0^1(U)$ *is a weak solution of the BVP* (1.1) *if*

$$B[u,v] = (f,v) \tag{1.13}$$

for all $v \in H_0^1(U)$*, where* $(\ ,\ )$ *denotes the inner product in* $L^2(U)$.

♣

More generally, let us consider the BVP

$$\begin{cases} Lu = f^0 - \sum_{i=1}^{n} f^i_{x_i}, x \in U, \\ u = 0, x \in \partial U, \end{cases} \tag{1.14}$$

where $f^i \in L^2(U)$. Then we say $u \in H_0^1(U)$ is a weak solution of problem (1.14) if

$$B[u,v] = \langle f, v \rangle \tag{1.15}$$

for all $v \in H_0^1(U)$, where $\langle \cdot, \cdot \rangle$ is the pairing of $H^{-1}(U)$ and $H_0^1(U)$.

For non-homogeneous elliptic PDE, i.e.

$$\begin{cases} Lu = f, x \in U, \\ u = g, x \in \partial U. \end{cases} \tag{1.16}$$

By trace theorem, $\exists w \in H^1(U)$ such that the trace of w is g. Then define $\tilde{u} := u - w$, (1.16) is equivalent to the equation:

$$\begin{cases} L\tilde{u} = \tilde{f}, x \in U, \\ \tilde{u} = 0, x \in \partial U, \end{cases} \tag{1.17}$$

where $\tilde{f} := f - Lw \in H^{-1}(U)$.

1.2 Existence of weak solutions

1.2.1 Lax-Milgram theorem

We now introduce an abstract principle from linear functional analysis.

Assume H is a real Hilbert space, with norm $\|\ \|$ and inner product $(\ ,\ )$. We let $\langle \cdot, \cdot \rangle$ denote the pairing of H with its dual space.

Theorem 1.2 (Lax-Milgram Theorem)

Assume that

$$B : H \times H \to \mathbb{R} \tag{1.18}$$

is a bilinear mapping, for which there exist constants α, β *such that*

$$|B(u,v)| \le \alpha \|u\| \|v\| \, (u, v \in H) \tag{1.19}$$

and

$$\beta \|u\|^2 \le B[u,u] (u \in H). \tag{1.20}$$

Finally, let $f : H \to R$ *be a bounded linear functional on* H.

Then there exists a unique element $u \in H$ *such that*

$$B[u,v] = \langle f, v \rangle \tag{1.21}$$

for all $v \in H$.

Remark

1. If B is symmetry, the condition (1.53) and (1.20) means B can derive an inner product on H.
2. So, if B is symmetry, theorem 1.2 is a direct corollary of Riesz representation theorem.
3. If B isn't symmetry, Riesz representation theorem can transform $f \in H^*$ to $u_f \in H$, such that $\langle f, v \rangle = (u_f, v)$.
4. So, we should show that $\forall u_f \in H$, $\exists u \in H$ such that $B[u, v] = (u_f, v)$ for each $v \in H$.

Proof

By the remark above, we just need to show that $\forall u_f \in H$, $\exists u \in H$ such that $B[u, v] = (u_f, v)$ for each $v \in H$.

Consider an element $u \in H$. $B(u, v)$ is a bounded bilinear mapping, so the map

$$B_u(v) := B[u, v] \tag{1.22}$$

is a bounded linear functional. By Riesz representation theorem, $\exists! \tilde{u} \in H$ such that $B[u, v] = (\tilde{u}, v)$ $\forall v \in H$. So we can define a map $A : H \to H$ maps u to $\tilde{u}$.

Then it's suffices to show that A is a bounded linear isomorphism.

First we should show that A is linear. $\forall v \in H$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $u_1, u_2 \in H$, we can see:

$$\begin{aligned}(A(\lambda_1 u_1 + \lambda_2 u_2), v) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \\ &= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v).\end{aligned} \tag{1.23}$$

(1.23) shows that A is a linear map.

Then we show that A is bounded. As B is bounded, we can see:

$$\begin{aligned}\|Au\|^2 &= (Au, Au) \\ &= B[u, Au] \\ &\leq \alpha \|u\| \|Au\|.\end{aligned} \tag{1.24}$$

i.e. $\|Au\| \leq \alpha \|u\|$. So A is bounded.

The next thing to do in the proof is to show A is an injective. By (1.20), we can see:

$$\begin{aligned}\beta \|u\|^2 &\leq B[u, u] \\ &= (Au, u) \\ &\leq \|Au\| \|u\|.\end{aligned} \tag{1.25}$$

i.e. $\beta \|u\| \leq \|Au\|$. So $\beta \|u\| \leq \|Au\| \leq \alpha \|u\|$, it means that A is an injective. What's more, the range of A, marked as $R(A)$, is a closed set.

Finally, we should show that A is a surjective. If $R(A) \neq H$, since $R(A)$ is closed, there exists a nonzero element $\omega \in H$ with $\omega \in R(A)^\perp$. Then:

$$\beta \|\omega\|^2 \leq B[\omega, \omega] = (A\omega, \omega) = 0. \tag{1.26}$$

which means that $\|\omega\| = 0$, contradict! So A is a surjective.

This completes the proof of Lax-Milgram theorem.

Lax-Milgram theorem gives an important method for us to analyze the existence of weak solution.

1.2.2 Energy estimates and First Existence theorem

Now, we return to the specific bilinear form $B[\ ,\ ]$ defined by (1.12), and try to use **Lax-Milgram theorem** to prove the first existence theorem.

Theorem 1.3 (Energy estimates)

There exists constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

$$|B[u,v]| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} \tag{1.27}$$

and

$$\beta \|u\|_{H_0^1(U)}^2 \leq B[u,u] + \gamma \|u\|_{L^2(U)}^2 . \tag{1.28}$$

♡

Proof First derive the inequality (1.27). According to (1.12), we can check that

$$\begin{aligned}
B[u,v] &\leq \sum_{i,j=1}^{n} \int_U \left\|a^{ij}\right\|_{L^\infty} |u_{x_i}||v_{x_j}|\mathrm{d}x + \sum_{i=1}^{n} \int_U \left\|b^i\right\|_{L^\infty} |u_{x_i}||v|\mathrm{d}x + \int_U \|c\|_{L^\infty} |u||v|\mathrm{d}x \\
&\leq \sum_{i,j=1}^{n} \int_U \left\|a^{ij}\right\|_{L^\infty} |Du||Dv|\mathrm{d}x + \sum_{i=1}^{n} \int_U \left\|b^i\right\|_{L^\infty} |Du||v|\mathrm{d}x + \int_U \|c\|_{L^\infty} |u||v|\mathrm{d}x \\
&\leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} .
\end{aligned} \tag{1.29}$$

Then, by the uniformly elliptic condition of coefficient matrix $a^{ij}(x)$, we can see:

$$\begin{aligned}
\theta \int_U |Du|^2 \mathrm{d}x &\leq \int_U \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i} u_{x_j} \mathrm{d}x \\
&= B[u,u] - \int_U \left(\sum_{i=1}^{n} b^i(x) u_{x_i} u \mathrm{d}x + cu^2 \right) \mathrm{d}x \\
&\leq B[u,u] + \sum_{i=1}^{n} \int_U \left\|b^i(x)\right\|_{L^\infty} |u||Du|\mathrm{d}x + c\int_U u^2 \mathrm{d}x
\end{aligned} \tag{1.30}$$

By Cauchy-Schwarz inequality with coefficient, we can see:

$$\int_U |u||Du|\mathrm{d}x \leq \epsilon \int_U |Du|^2 \mathrm{d}x + \frac{1}{4\epsilon} \int_U |u|^2 \mathrm{d}x. \tag{1.31}$$

Choose ϵ such that $\epsilon \sum_{i=1}^{n} \left\|b^i\right\|_{L^\infty} < \frac{\theta}{2}$, then exists constant $C > 0$ such that

$$\frac{\theta}{2} \int_U |Du|^2 \mathrm{d}x \leq B[u,u] + C \int_U u^2 \mathrm{d}x. \tag{1.32}$$

Finally, by Poincare-Friedrichs inequality, the equation (1.28) is true.

By (1.28), if $\gamma > 0$, $B[u,v]$ isn't uniformly elliptic in general. So, $B[u,v]$ does't satisfy the hypotheses of Lax-Milgram theorem in general. So we should give some revisions on bilinear form B. Then, we derive the **first existence theorem for weak solutions.**

Theorem 1.4 (First Existence Theorem for weak solutions)

There is a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each function $f \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U)$ of the boundary-value problem

$$\begin{cases} Lu + \mu u = f, x \in U; \\ \qquad u = 0, x \in \partial\Omega. \end{cases} \tag{1.33}$$

♡

Proof Choose the parameter γ as theorem 1.3, then define the bilinear form

$$B_\mu[u,v] := B[u,v] + \mu(u,v). \tag{1.34}$$

Then the weak form of equation (1.33) is

$$B_\mu[u,v] = \langle f, v\rangle \,. \tag{1.35}$$

By (1.28), as $\mu \geq \gamma$, the bilinear form B_μ satisfies uniformly elliptic condition. So by Lax-Milgram theorem, equation (1.34) has unique weak solution.

Remark The first existence theorem for weak solutions is a milestone, but we still can't give the existence theorem of equation (1.1). We should use Fredholm alternative theorem to derive the existence of weak solution.

1.2.3 Fredholm alternative and the solvability

In this section, we show the Fredholm alternative theorem first, then employ this theorem to derive the existence theorem for weak solutions.

Theorem 1.5 (Fredholm alternative)

Let $K : H \to H$ be a compact linear operator, then:

- $\ker(I-K)$ *is finite dimensional.*
- $R(I-K)$ *is closed.*
- $R(I-K) = \ker(I-K^*)^\perp$.
- $\ker(I-K) = \{0\}$ *if and only if* $R(I-K) = H$.
- $\dim\ker(I-K) = \dim\ker(I-K^*)$.

Proof Omitted.

Then, derive the **dual problem** of equation (1.1).

Definition 1.3 (Dual problem)

1. *The operator L^*, the formal adjoint of L, is:*

$$L^*v = -\sum_{i,j=1}^{n} \left(a^{ij}v_{x_j}\right)_{x_i} - \sum_{i=1}^{n} b^i v_{x_i} + (c - \sum_{i=1}^{n} b^i_{x_i})v, \tag{1.36}$$

provided $b^i \in C^1(\bar{U})$.

2. *The adjoint bilinear form*

$$B^* : H_0^1(U) \times H_0^1(U) \to \mathbb{R} \tag{1.37}$$

is defined by

$$B^*[v,u] = B[u,v] \tag{1.38}$$

for all $u, v \in H_0^1(U)$.

3. *We say that $v \in H_0^1(U)$ is a weak solution of the adjoint problem*

$$\begin{cases} L^*v = f, x \in U; \\ \quad v = 0, x \in \partial U. \end{cases} \tag{1.39}$$

provided $B^*[v,u] = (f,u)$ for all $u \in H_0^1(U)$.

Remark

- (1.36) is called formal adjoint, for $(Lu, v) = (u, L^*v)$.

- (1.39) is the dual form of equation (1.1).

To show the solvability of problem (1.1), we derive the following theorem.

Theorem 1.6 (Second Existence Theorem for weak solutions)

1. *Precisely one of the following statements holds: either for each $f \in L^2(U)$ there exists a unique weak solution u of the boundary value problem (1.1) (marked as α), or else there exists a weak solution $u \neq 0$ of the homogeneous problem (marked as β)*
$$\begin{cases} Lu = 0, x \in U; \\ u = 0, x \in \partial U. \end{cases} \tag{1.40}$$
2. *Furthermore, should assertion (β) hold, the dimension of the subspace $N \subset H_0^1(U)$ of weak solutions of (1.40) is finite and equals the dimension of the subspace $N^* \subset H_0^1(U)$ of weak solutions of*
$$\begin{cases} L^* v = 0, x \in U; \\ v = 0, x \in \partial U. \end{cases} \tag{1.41}$$
3. *Finally, the BVP (1.1) has a weak solution if and only if*
$$(f, v) = 0 \; \forall v \in N^*. \tag{1.42}$$

♡

Remark To prove this theorem, we should try to use theorem 1.4. The main idea is to construct a compact operator K, such that the weak form of (1.1) is $(I - K)u = h$, then use theorem 1.5.

Proof First, choose γ as theorem 1.4 suggests, then for each $g \in L^2(U)$, there exists a unique $u \in H_0^1(U)$ solving:
$$B_\gamma[u, v] = \langle g, v\rangle \, \forall v \in H_0^1(U). \tag{1.43}$$
Write $u = L_\gamma^{-1} g$ if equation (1.43) holds. As the weak form of (1.1) is $B[u, v] = \langle f, v\rangle$, we can see $B_\gamma[u, v] = \langle f + \gamma u, v\rangle$, i.e. $u = L_\gamma^{-1}(f + \gamma u)$. Then choose operator $K = \gamma L_\gamma^{-1}$, $h = L_\gamma^{-1} f$, (1.43) is equivalent to
$$(I - K)u = h. \tag{1.44}$$

The next step is to show that $K : L^2(U) \to L^2(U)$ is a bounded, linear, compact operator. In fact, we only need to show K is compact. By (1.43), we can see:
$$\beta \|u\|^2_{H_0^1(U)} \leq B_\gamma[u, u] = \langle g, u\rangle \leq \|g\|_{L^2(U)} \|u\|_{L^2(U)} \leq \|g\|_{L^2(U)} \|u\|_{H_0^1(U)} \, . \tag{1.45}$$
By (1.45), there exists a constant $C > 0$ such that $\|Kg\|_{H_0^1(U)} \leq C \|g\|_{L^2(U)}$. As $H_0^1(U) \subset\subset L^2(U)$, K is a compact operator.

Then, use theorem 1.5 on equation (1.44). If $\ker(I - K) = \{0\}$, theorem 1.5 shows that $I - K$ is also a surjective, i.e. statement (α) is true. Otherwise, $\ker(I - K) \neq \{0\}$ means that $\exists u \in H_0^1(U)$ such that $(I - K)u = 0$, i.e. u satisfies equation (1.40). Then statement (β) is true. If $N \neq \{0\}$, while
$$\dim \ker(I - K) = \dim \ker(I - K^*) < \infty, \tag{1.46}$$
we can see $\dim N = \dim N^* < \infty$.

Finally, if $(I - K)u = h$ has a solution, $v \in N^*$, we can see:
$$\langle h, v\rangle = \langle (I - K)u, v\rangle = \langle u, (I - K^*)v\rangle = 0. \tag{1.47}$$
By $h = L_\gamma^{-1} f$, we can see:
$$\langle h, v\rangle = \frac{1}{\gamma} \langle Kf, v\rangle = \frac{1}{\gamma} \langle f, K^* v\rangle = \frac{1}{\gamma} \langle f, v\rangle = 0. \tag{1.48}$$

So:$\langle f, v\rangle = 0$.

1.2.4 Spectrum and third existence theorem

In this section, we will discuss on the **spectrum** of an operator L, then derive the existence of weak solutions for eigenvalue problem.

Theorem 1.7 (Third existence theorem for weak solutions)

1. *There exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the BVP*
$$\begin{cases} Lu = \lambda u + f, x \in U; \\ u = 0, x \in \partial U \end{cases} \tag{1.49}$$
has a unique weak solution for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$.
2. *If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$, the values of a nondecreasing sequence with $\lambda_k \to +\infty$.*

Remark

1. Σ is called the **spectrum** of operator L.
2. If $f = 0$, the BVP (1.49) is called eigenvalue problem, and if there exists a solution $\omega \neq 0$, λ is called an **eigenvalue** of L, and ω is a corresponding **eigenfunction**.

Proof By theorem 1.6, if $\lambda \in \Sigma$, the homogeneous eigenvalue problem
$$\begin{cases} Lu = \lambda u, x \in U, \\ u = 0, x \in \partial U \end{cases} \tag{1.50}$$
has a solution $u \neq 0$. Consider it's weak form, we can see:
$$B_\gamma[u, v] = (\lambda + \gamma)\langle u, v\rangle. \tag{1.51}$$
i.e.
$$u = L_\gamma^{-1}(\gamma + \lambda)u = \frac{\lambda + \gamma}{\gamma} K u. \tag{1.52}$$
As $u \neq 0$, u is the eigenvector of operator K, the corresponding eigenvalue is $\frac{\gamma}{\lambda+\gamma}$. By theorem 1.6, K is a compact operator, so the Spectrum set S of operator K is either finite set, or else the values of a sequence converging to zero. It means that Σ is at most countable, and if $|\Sigma| = \infty$, $\lambda_k \to \infty$.

Finally, we note the boundedness of eigenvalue problem.

Theorem 1.8 (Boundedness of the inverse)

If $\lambda \notin \Sigma$, there exists a constant C such that
$$\|u\|_{L^2(U)} \leq C \|f\|_{L^2(U)} \tag{1.53}$$
whenever $f \in L^2(U)$ and u is the unique weak solution of (1.49). The constant C depends only on λ, U and the coefficients of L.

Proof If not, there would exist sequences $\{f_k\} \subset L^2(U)$ and $\{u_k\} \subset H_0^1(U)$ such that
$$\begin{cases} Lu_k = \lambda u_k + f_k, x \in U, \\ u_k = 0, x \in \partial U, \end{cases} \tag{1.54}$$
but
$$\|u_k\|_{L^2(U)} > k \|f_k\|_{L^2(U)}. \tag{1.55}$$

Assume WLOG, $\|u_k\|_{L^2(U)} = 1$, we can see $\|f_k\| \to 0$. Then there exists a subsequence $\{u_{k_j}\}$ satisfies:

$$\begin{cases} u_{k_j} \rightharpoonup u \text{ in } H_0^1(U), \\ u_{k_j} \to u \text{ in } L^2(U). \end{cases} \tag{1.56}$$

Then u is a weak solution of (1.50). Since $\lambda \notin \Sigma$, $u \equiv 0$. But $\|u_k\|_{L^2(U)} \equiv 1$, contradict!

1.3 Interior Regularity

Now, we try to discuss on the **regularity** of weak solutions. Consider a general PDE $Lu = f$, we find the weak solution $u \in H^1(U)$. However, if we set $f \in H^m(U)$, we expect $u \in H^{m+2}(U)$, it means we derive stronger regularity about the weak solution of $Lu = f$. The point of regularity is to derive analytic estimates from the structural, algebraic assumption of ellipticity.

First, recall the definition of difference quotients, and some related properties.

Definition 1.4 (Difference quotient)

The i-th difference quotient of size h is

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h} (i = 1, \cdots, n) \tag{1.57}$$

for $x \in V$ and $h \in \mathbb{R}$, $0 < |h| < dist(V, \partial U)$. And $D^h(u) := \left(D_1^h u, \cdots, D_n^h u\right)$. ♣

The concept of difference quotients is related to weak derivatives, as the following theorem:

Theorem 1.9

1. *Suppose $1 \le p < \infty$ and $u \in W^{1,p}(U)$, for each $V \subset\subset U$,*

$$\left\|D^h u\right\|_{L^p(V)} \le C \|Du\|_{L^p(U)} \tag{1.58}$$

   for some constant C and all $0 < |h| < \frac{1}{2} dist(V, \partial U)$.

2. *Assume $1 < p < \infty$, $u \in L^p(V)$, and there exists a constant C such that*

$$\left\|D^h u\right\|_{L^p(V)} \le C \tag{1.59}$$

   for all $0 < |h| < \frac{1}{2} dist(V, \partial U)$, then $u \in W^{1,p}(V)$ with $\|Du\|_{L^p(V)} \le C$. ♡

Proof see [1] section 5.8, theorem 3.

Then, introduce two lemmas for the integrate of difference quotients.

Lemma 1.1

For a bounded open set U, and open set $W \subset\subset U$, assume $v, w \in H^1(U)$, $supp(w) \subset W$, $h < \frac{1}{2} dist(W, \partial U)$, then we can see:

$$\int_U v D_k^{-h} w \mathrm{d}x = -\int_U w D_k^h v \mathrm{d}x, \tag{1.60}$$

and

$$D_k^h(vw) = v^h D_k^h w + w D_k^h v, \tag{1.61}$$

for $v^h(x) := v(x + he_k)$. ♡

Proof For equation (1.60), as supp$w \subset W$, we can see:

$$\begin{aligned} RHS &= -\int_W w(x)\frac{v(x+he_k)-v(x)}{h}\mathrm{d}x \\ &= \frac{1}{h}\int_W w(x)v(x)\mathrm{d}x - \frac{1}{h}\int_W w(x)v(x+he_k)\mathrm{d}x. \end{aligned} \tag{1.62}$$

set $\tilde{W} := \{y : y = x + he_k, x \in W\}$, we can see:

$$\begin{aligned} LHS &= \frac{1}{h}\int_U v(x)\left(w(x) - w(x+he_k)\right)\mathrm{d}x \\ &= \frac{1}{h}\int_W v(x)w(x)\mathrm{d}x - \frac{1}{h}\int_{\tilde{W}} v(x)w(x-he_k)\mathrm{d}x. \end{aligned} \tag{1.63}$$

By integral substitution, choose $y := x - he_k$, we can see:

$$\int_{\tilde{W}} v(x)w(x-he_k)\mathrm{d}x = \int_W v(y+he_k)w(y)\mathrm{d}y. \tag{1.64}$$

So, equation (1.60) is true. Then, for equation (1.61), we can see:

$$\begin{aligned} LHS &= \frac{v(x+he_k)w(x+he_k) - v(x)w(x)}{h} \\ &= \frac{v(x+he_k)(w(x+he_k) - w(x)) + w(x)(v(x+he_k) - v(x))}{h} \\ &= v^h D_k^h w + w D_k^h v = RHS. \end{aligned} \tag{1.65}$$

Then, we give the main result for this section:

Theorem 1.10 (Interior H^2-regularity)

Assume $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$, and $f \in L^2(U)$. Suppose furthermore that $u \in H^1(U)$ is a weak solution of the elliptic PDE $Lu = f$, then:

$$u \in H^2_{loc}(U); \tag{1.66}$$

and for each open subset $V \subset\subset U$ we have the estimate

$$\|u\|_{H^2(V)} \le C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right), \tag{1.67}$$

the constant C depending only on V, U, and the coefficients of operator L.

♡

Before the proof of this theorem, we should give some remarks.

Remark

1. In this theorem, we don't require $u \in H^1_0(U)$.
2. Since $u \in H^2_{loc}(U)$, we have

$$Lu = f \text{ a.e. in U}. \tag{1.68}$$

The idea : First, construct a truncated function $\zeta \in C^\infty(U)$ such that $V \subset\subset W \subset\subset U$, $\zeta|_V \equiv 1$, $\zeta|_{U\setminus W} \equiv 0$. Then, choose a test function

$$v(x) = -D_k^{-h}\left(\zeta^2 D_k^h u(x)\right) \in H^1_0(U). \tag{1.69}$$

In fact, if $u \in C^2(U)$, v is a difference quotient approximation for D^2u. Finally, use the relation $B[u,v] = (f,v)$ to approximate $\left\|D_k^h Du\right\|_{L^2(V)}$, and use the second part of theorem 1.9.

Proof By the definition of weak solution, $B[u,v] = (f,v)$. Then we can see:

$$\sum\int_U a^{ij}u_{x_i}v_{x_j}\mathrm{d}x = \int_U (f - \sum b_i u_{x_i} - cu)v\mathrm{d}x. \tag{1.70}$$

mark

$$A = \sum \int_U a^{ij} u_{x_i} v_{x_j} \mathrm{d}x, \tag{1.71}$$

and choose v as (1.69), according to lemma 1.1, we can derive:

$$\begin{aligned} A &= -\int_U \sum a^{ij} u_{x_i} \left(D_k^{-h}\left(\zeta^2 D_k^h u(x)\right)\right)_{x_j} \mathrm{d}x \\ &= \sum \int_U D_k^h \left(a^{ij} u_{x_i}\right) \left(\zeta^2 D_k^h u(x)\right)_{x_j} \mathrm{d}x \\ &= \sum \int_U \left(a^{ij,h} D_k^h u_{x_i} + u_{x_i} D_k^h (a^{ij})\right) \left(2\zeta\zeta_{x_j} D_k^h u(x) + \zeta^2 \left(D_k^h u(x)\right)_{x_j}\right) \mathrm{d}x \\ &= \sum \int_U a^{ij,h} \zeta^2 D_k^h u_{x_i} D_k^h u_{x_j} \mathrm{d}x \\ &+ \sum \int_U \left(2\zeta\zeta_{x_j} u_{x_i} D_k^h (a^{ij}) D_k^h u(x) + 2\zeta\zeta_{x_j} a^{ij,h} D_k^h u_{x_i} D_k^h u(x) + \zeta^2 u_{x_i} D_k^h (a^{ij}) \left(D_k^h u(x)\right)_{x_j}\right) \mathrm{d}x \\ &:= A_1 + A_2. \end{aligned} \tag{1.72}$$

By the uniform elliptic property, there exists $\theta > 0$ such that

$$A_1 \geq \theta \int_U \zeta^2 |D_k^h Du|^2 \mathrm{d}x. \tag{1.73}$$

Then we try to approx $|A_2|$. As $a^{ij} \in C^1(U)$, $\zeta \in C^\infty(U)$, we can see:

$$\begin{aligned} |A_2| &\leq \sum \int_U \left(C_1 \zeta |u_{x_i}||u(x)| + C_2 \zeta |D_k^h u_{x_i}||D_k^h u(x)| + C_3 \zeta |u_{x_i}| \left(D_k^h u(x)\right)_{x_j}\right) \mathrm{d}x \\ &\leq C \int_U \left(\zeta |Du||u| + \zeta |D_k^h Du||D_k^h u| + \zeta |Du||D_k^h Du|\right) \mathrm{d}x \\ &\leq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 \mathrm{d}x + \tilde{C} \int_U |Du|^2 + |D_k^h u|^2 \mathrm{d}x \\ &\leq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 \mathrm{d}x + M \int_U |Du|^2 \mathrm{d}x. \end{aligned} \tag{1.74}$$

In equation (1.74), $C_1, C_2, C_3, C, \tilde{C}, M$ are all constants. The first step follows from $a^{ij}, \zeta \in C^1(U)$, the second and the third steps from Cauchy-Schwarz inequality, and the last from theorem 1.9.

Combining (1.73) and (1.74), we can see

$$A \geq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 \mathrm{d}x - M \int_U |Du|^2 \mathrm{d}x. \tag{1.75}$$

The next step is to approximate the right-hand integral

$$B := \int_U (f - \sum b_i u_{x_i} - cu) v \mathrm{d}x. \tag{1.76}$$

First, as $b_i \in C^1(U)$, we can see

$$|B| \leq C \int_U (|f| + |Du| + |u|)|v| \mathrm{d}x. \tag{1.77}$$

Then, consider

$$\begin{aligned}\int_U v^2 \mathrm{d}x &= \int_U |D_k^{-h}(\zeta^2 D_k^h u)|^2 \mathrm{d}x \\ &\leq C \int_U D|\zeta^2 D_k^h u|^2 \mathrm{d}x \\ &\leq C \int_U \zeta^2 |D_k^h Du|^2 \mathrm{d}x + C \int_W |D_k^h u|^2 \mathrm{d}x \\ &\leq C \int_U \left(|Du|^2 + \zeta^2 |D_k^h Du|^2\right) \mathrm{d}x.\end{aligned} \tag{1.78}$$

The second step follows from theorem 1.9, the third step follows from the Leibniz formula, and the final step follows from theorem 1.9 as well.

Finally, by (1.77) , (1.78) and Cauchy-Schwarz inequality, there exists constant C such that

$$|B| \leq \frac{\theta}{4} \int_U \zeta^2 |D_k^h Du|^2 \mathrm{d}x + C \int_U \left(|f|^2 + |u|^2 + |Du|^2\right) \mathrm{d}x. \tag{1.79}$$

By the estimation (1.167) and (1.79), it's clear that $\forall h > 0$, $\exists$ constant $C > 0$, such that

$$\int_V |D_k^h Du|^2 \mathrm{d}x \leq C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}). \tag{1.80}$$

It means that $Du \in H^1(V)$, and

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right). \tag{1.81}$$

Finally, choose auxiliary function $\xi \in C^\infty(U)$, supp$\xi \subset U$ and $\xi \equiv 1$ on W, set $v = \xi^2 u$, according to equation (1.70), we can see:

$$\int_U \xi^2 |Du|^2 \mathrm{d}x \leq C \int_U (f^2 + u^2) \mathrm{d}x. \tag{1.82}$$

Then:

$$\|u\|_{H^1(W)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right). \tag{1.83}$$

Remark Theorem 1.10 shows the condition for the higher regularity of weak solutions.

Then, we introduce the higher interior regularity.

Theorem 1.11 (Higher interior regularity)

Let m be a nonnegative integer, and assume $a^{ij}, b^i, c \in C^{m+1}(U)$, and $f \in H^m(U)$. Suppose $u \in H^1(U)$ is a weak solution of the elliptic PDE

$$Lu = f, \tag{1.84}$$

then $u \in H^{m+2}_{loc}(U)$ and for each $V \subset\subset U$ we have the estimate

$$\|u\|_{H^{m+2}(V)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}). \tag{1.85}$$

♡

Proof We prove this theorem by induction. If $m = 0$, theorem 1.11 is equivalent to theorem 1.10, so the induction basis is true. Assume theorem 1.11 is true for $m = k$, i.e. if $a^{ij}, b^i, c \in C^{k+1}(U)$, $f \in H^k(U)$, the weak solution $u \in H^1(U)$ satisfies

$$\|u\|_{H^{k+2}(W)} \leq C(\|f\|_{H^k(U)} + \|u\|_{L^2(U)}). \tag{1.86}$$

for each $W \subset\subset U$ and approximate constant C, and $u \in H^{k+2}_{loc}(U)$. Then let α be any multiindex with $|\alpha| = m + 1$ and choose any auxiliary function $\tilde{v} \in C_c^\infty(W)$. Choose the test function $v = (-1)^{|\alpha|} D^\alpha \tilde{v}$, insert it into the weak form $B[u,v]=(f,v)$, perform some integrations by parts, we discover:

$$\begin{aligned}B[u,v] &= \sum \int_U a^{ij} u_{x_i} (-1)^{|\alpha|} (D^\alpha \tilde{v})_{x_j} \mathrm{d}x + \sum \int_U b^i u_{x_i} (-1)^{|\alpha|} D^\alpha \tilde{v} \mathrm{d}x + \int_U cu(-1)^{|\alpha|} D^\alpha \tilde{v} \mathrm{d}x \\ &= \sum \int_U D^\alpha (a^{ij} u_{x_i}) \tilde{v}_{x_j} \mathrm{d}x + \sum \int_U D^\alpha (b^i u_{x_i}) \tilde{v} \mathrm{d}x + \int_U D^\alpha (cu) \tilde{v} \mathrm{d}x \\ &= B[D^\alpha u, \tilde{v}] - \sum_{i,j} \sum_{\beta \le \alpha} \int_U \binom{\alpha}{\beta} (D^{\alpha-\beta} a^{ij} D^\beta u_{x_i})_{x_j} \tilde{v} \mathrm{d}x + \sum_i \sum_{\beta \le \alpha} \int_U D^{\alpha-\beta} b^i D^\beta u_{x_i} \tilde{v} \mathrm{d}x \\ &\quad + \sum_{\beta \le \alpha} \int_U D^{\alpha-\beta} c D^\beta u \tilde{v} \mathrm{d}x. \end{aligned} \tag{1.87}$$

So if we write $\tilde{u} := D^\alpha u$, we can see:

$$B[\tilde{u}, \tilde{v}] = (\tilde{f}, \tilde{v}). \tag{1.88}$$

Where

$$\tilde{f} := D^\alpha f - \sum_{\beta < \alpha} \binom{\alpha}{\beta} \left[-\sum (D^{\alpha-\beta} a^{ij} D^\beta u_{x_i})_{x_j} + \sum D^{\alpha-\beta} b^i D^\beta u_{x_i} + D^{\alpha-\beta} c D^\beta u \right]. \tag{1.89}$$

We can see that $\tilde{u}$ is the weak solution of equation $L\tilde{u} = \tilde{f}$. Since $a^{ij}, b^i, c \in C^{k+2}(U)$, $f \in H^{k+1}(U)$, we can see $\tilde{f} \in L^2(W)$, and

$$\left\| \tilde{f} \right\|_{L^2(W)} \le C \left(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)} \right). \tag{1.90}$$

In light of theorem 1.10, we see $\tilde{u} \in H^2(V)$ with

$$\|\tilde{u}\|_{H^2(V)} \le \ C(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)}). \tag{1.91}$$

As the inequality holds for all multiindex α satisfies $|\alpha| = m+1$, the proof is completed.

1.4 Boundary regularity

Now, we extend to give the regularity of weak solution for homogeneous Dirichlet boundary value problem. In fact, when it comes to Dirichlet BVPs, the result comes to be stronger.

Theorem 1.12 (Boundary H^2-regularity)

Assume $a^{ij} \in C^1(\bar{U}), b^i, c \in L^\infty(U)$ and $f \in L^2(U)$. Suppose that $u \in H_0^1(U)$ is a weak solution of the elliptic BVP (1.1). Assume finally ∂U is C^2, then $u \in H^2(U)$ and we have the estimate

$$\|u\|_{H^2(U)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}). \tag{1.92}$$

The constant C depending only on U and the coefficients of L.

♡

Remark

- Main difference from theorem 1.10: now $u \in H^2(U)$ rather than $u \in H^2_{loc}(U)$.
- If equation (1.1) has unique solution, we can see $\|u\|_{H^2(U)} \le C \|f\|_{L^2(U)}$ from theorem 1.12. This result shows the **well-posed** of elliptic equation.
- In this theorem, we assume $u \equiv 0$ along ∂U in the trace case.

The idea of the proof: First, consider the special case that U is a half-ball, then $\forall x_0 \in \partial U$, choose a homologeous map from $B(x_0, \epsilon) \cap U$ to the half ball. Finally, use the compactness of ∂U.

Proof First, consider the case $U = B(0,1) \cap \mathbb{R}^n_+$. The target is to show that $\exists V \subset\subset U$ with $\partial V \cap \partial U \ne \emptyset$

such that

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}). \tag{1.93}$$

To derive the H^2 norm on subset $V := B(0, \frac{1}{2}) \cap \mathbb{R}^n_+$, we first derive the truncated function $\zeta \in C^\infty(U)$ satisfies:

$$\begin{cases} \zeta \equiv 1, x \in B(0, \dfrac{1}{2}) \\ \zeta \equiv 0, x \notin B(0,1), \\ 0 \leq \zeta \leq 1, \end{cases} \tag{1.94}$$

such that ζ vanishes near the curved part of ∂U. Since u is a weak solution of equation (1.1), we have $B[u,v] = (f,v)$ for all $v \in H_0^1(V)$. Consequently, we can also write the auxiliary equation (1.43). Then we should derive the form of $v \in H_0^1(U)$.

Consider $1 \leq k \leq n-1$, set $v = -D_k^{-h}(\zeta^2 D_k^h u)$, first we claim that $v \in H_0^1(U)$. In fact:

$$v(x) = \frac{1}{h^2}\left[\zeta^2(x - he^k)u(x) + \zeta^2(x)u(x) - \zeta^2(x)u(x+he^k) - \zeta^2(x-he^k)u(x-he^k)\right]. \tag{1.95}$$

On the face $\{x_n = 0\}$, by the definition of weak solution, $u(x) = 0$. And near the curved part of ∂U, by the definition of ζ, $\zeta(x) = 0$. So on ∂U, we can see $v(x) \equiv 0$, i.e. $v(x) \in H_0^1(U)$.

Then, by equation (1.43) and the same method as theorem 1.10, we can see that for $1 \leq k \leq n-1$,

$$\int_V |D_k^h Du|^2 \mathrm{d}x \leq C\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right). \tag{1.96}$$

i.e.

$$\sum_{\substack{k+l<2n \\ k=l=1}}^{n} \int_V |u^2_{x_k x_l}| \mathrm{d}x \leq C\left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right). \tag{1.97}$$

But we can't estimate $\|u_{x_n x_n}\|_{L^2(V)}$ by this method, for $x - he^n$ may lie outside the region U. Now we rewrite the equation (1.1) by nondivergence form, i.e.

$$-\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} \tilde{b}^i u_{x_i} + cu = f. \tag{1.98}$$

As $a^{ij}(x)$ satisfies **uniformly elliptic condition**, there exists $\theta > 0$ such that $a^{nn}(x) \geq \theta\ \forall x \in U$. So:

$$|u_{x_n x_n}| \leq C\left(\sum_{\substack{i,j=1 \\ i+j<2n}}^{n} |u_{x_i x_j}| + |Du| + |u| + |f|\right). \tag{1.99}$$

By (1.97), (1.99) and (1.83), we conclude $u \in H^2(V)$ and

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right). \tag{1.100}$$

Then, we drop the assumption that U is a half-ball. In general case, we choose any point $x^0 \in \partial U$ and note that since ∂U is C^2, we may assume that

$$U \cap B(x^0, r) = \{x \in B(x^0, r) | x_n > \gamma(x_1, \cdots, x_{n-1})\} \tag{1.101}$$

for some $r < 0$ and some C^2 function γ. Consider the "flatten out" map as follows:

$$\Phi^i(x) = \begin{cases} x_i, i = 1, \cdots, n-1; \\ x_n - \gamma(x_1, \cdots, x_{n-1}), i = n. \end{cases} \tag{1.102}$$

Then write $y = \Phi(x)$, $x = \Psi(y)$ as the linear map from $U \cap B(x^0, r)$ to $\tilde{U}$. We can see $\tilde{U}$ has a piece of boundary as $y_n = 0$. Finally, we define $u'(y) := u(\Psi(y))$, it's clear that $u' \in H^1(\tilde{U})$ and $u' = 0$ on $\partial\tilde{U} \cap \{y_n = 0\}$.

Now, we transform equation (1.1) to the equation related to u' on $\tilde{U}$.

Write the PDE related to u' as

$$L'u' = f', \tag{1.103}$$

with the function $f'(y) := f(\Psi(y))$, and the operator

$$L'u' := -\sum_{k,l=1}^{n}(a'^{kl}u'_{y_k})_{y_l} + \sum_{k=1}^{n} b'^{k}u'_{y_k} + c'u'. \tag{1.104}$$

where

$$a'^{kl}(y) = \sum_{r,s=1}^{n} a^{rs}(\Psi(y))\Phi^k_{x_r}(\Psi(y))\Phi^l_{x_s}(\Psi(y)), \tag{1.105}$$

$$b'^{k}(y) = \sum_{r=1}^{n} b^r(\Psi(y))\Phi^k_{x_r}(\Psi(y)), \tag{1.106}$$

and

$$c'(y) = c(\Psi(y)), \tag{1.107}$$

for $y \in U'$, $k, l = 1, \cdots, n$. In fact, the operator is derived using the chain rule of derivatives.

If $v' \in H^1_0(U')$ and $B'[\cdot,\cdot]$ denotes the bilinear form associated with the operator L', we have:

$$B'[u', v'] = \int_{\tilde{U}} \sum a'^{kl}u'_{y_k}v'_{y_l} + \sum b'^{k}u'_{y_k}v' + c'u'v'\mathrm{d}y. \tag{1.108}$$

Set the function $v'(x) = v(\Psi(x))$, and we define $u'(x) = u(\Psi(x))$ before, by (1.108), we can see:

$$B'[u', v'] = \sum \int_{\tilde{U}} a'^{kl}u_{x_i}\Psi^i_{y_k}v_{x_j}\Psi^j_{y_l}\mathrm{d}y + \sum \int_{\tilde{U}} b'^{k}u_{x_i}\Psi^i_{y_k}v\mathrm{d}y + \int_{\tilde{U}} c'uv\mathrm{d}y. \tag{1.109}$$

In the following equation, u, v means $u(\Psi(y))$, $v(\Psi(y))$ respectively. By equation (1.105), we can see:

$$\sum_{k,l} a'^{kl}\Psi^i_{y_k}\Psi^j_{y_l} = \sum_{r,s}\sum_{k,l} a^{rs}\Phi^k_{x_r}\Phi^l_{x_s}\Psi^i_{y_k}\Psi^j_{y_l} = a^{ij} \tag{1.110}$$

since $D\Psi = (D\Phi)^{-1}$. Similarly,

$$\sum_{k=1}^{n} b'^{k}\Psi^i_{y_k} = b^i. \tag{1.111}$$

So:

$$B'[u', v'] = B[u, v] = \langle f, v\rangle = \langle f', v'\rangle, \tag{1.112}$$

i.e. the auxiliary PDE has the same solution as original PDE.

Then we check the operator L' is uniformly elliptic in U'. If $y \in U'$ and $\xi \in \mathbb{R}^n$, we note that

$$\sum_{k,l=1}^{n} a'^{kl}(y)\xi_k\xi_l = \sum_{k,l}\sum_{r,s} a^{rs}(\Psi(y))\Phi^k_{x_r}\Phi^l_{x_s}\xi_k\xi_l = \sum_{r,s} a^{rs}(\Psi(y))\eta_r\eta_s \geq \theta|\eta|^2. \tag{1.113}$$

where $\eta = \xi D\Phi$. So L' is uniformly elliptic on $\tilde{U}$. It's clear that $a'^{ij}, b'^{i}, c' \in C^1(\tilde{U})$.

Finally, by equation (1.100) and (1.112), we can see that

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right) \tag{1.114}$$

for an open neighborhood V of x^0.

Since ∂U is compact, we can cover ∂U with finitely many sets $V_1, \cdots, V_N$ as above. Sum them up, then complete the proof.

Then, derive the higher boundary regularity

Theorem 1.13 (Higher boundary regularity)

Let m be a nonnegative integer, and assume $a^{ij}, b^i, c \in C^{m+1}(\bar{U})$, and $f \in H^m(U)$, suppose that $u \in H_0^1(U)$ is a weak solution of the BVP

$$\begin{cases} Lu = f, x \in U; \\ \quad u = 0, x \in \partial U. \end{cases} \tag{1.115}$$

Assume finally $\partial U \in C^2$, then $u \in H^{m+2}(U)$ and we have the estimate

$$\|u\|_{H^{m+2}(U)} \leq C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)}). \tag{1.116}$$

Proof Skip. See [1] Page 343-345.

In this section, the basic tool of integration by parts has eventually taken us from weak solution $u \in H_0^1(U)$ to smooth, classical solutions.

1.5 Maximum Principles

In this section, we concentrate on the properties of elliptic operator L, and we assume that $u \in C^2(U)$. As $u \in C^2(U)$, we can see if $u(x)$ attains its maximum over U, then $Du(x_0) = 0, D^2u(x_0) \leq 0$. Now, we consider elliptic operators L having the nondivergence form:

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu, \tag{1.117}$$

where the coefficients a^{ij}, b^i, c are continuous and the uniform ellipticity condition 1.1 holds. Without loss of generality, we assume the symmetry condition $a^{ij} = a^{ji}$.

1.5.1 Weak maximum principle

First, assume $U \subset \mathbb{R}^n$ is open and bounded, we will show that if $Lu \leq 0$ in U and $c \equiv 0$, u must attain its maximum on ∂U.

Theorem 1.14 (Weak maximum principle)

Assume $u \in C^2(U) \cap C^1(\bar{U})$ and $c \equiv 0$ in U, if $Lu \leq 0$ in U, then

$$\max_{\bar{U}} u = \max_{\partial U} u. \tag{1.118}$$

Proof First, consider a stronger case, i.e. $Lu < 0$. Assume there exists a point $x_0 \in U$ such that $u(x_0) = \max_U u(x)$. By the basic vector calculus, we can see:

$$Du(x_0) = \mathbf{0}, \tag{1.119}$$

and

$$D^2u(x_0) \leq 0. \tag{1.120}$$

Mark that $D^2u(x_0) \leq 0$ means that the Hesse matrix $D^2u(x_0)$ is semi negative definite. Since $A(x_0)$ is symmetric and positive definite, there exists an orthogonal matrix O such that

$$OA(x_0)O^T = D, \tag{1.121}$$

while D is a diagonal matrix. Now write $y = x_0 + O(x - x_0)$, i.e. $x - x_0 = O^T(y - x_0)$. Mark $O = (o_{ij})$, by the chain rule of derivatives:

$$u_{x_i} = \sum_{k=1}^{n} u_{y_k} o_{ki}, u_{x_i x_j} = \sum_{k,l=1}^{n} u_{y_k y_l} o_{ki} o_{lj}. \tag{1.122}$$

Then, at the point x_0,

$$\sum_{i,j} a^{ij} u_{x_i} u_{x_j} = \sum_{k,l} \sum_{i,j} a^{ij} u_{y_k y_l} o_{ki} o_{lj} = \sum_{i} d^i u_{y_i y_i} \tag{1.123}$$

As $D^2 u(x_0) \leq 0$, we can see $u_{y_i y_i}(x_0) \leq 0$, and for $A(x_0)$ is positive definite, $d^i \geq 0$, i.e.

$$\sum_{i,j} a^{ij} u_{x_i} u_{x_j} \leq 0. \tag{1.124}$$

On the other hand, at the point x_0:

$$\sum_{i=1}^{n} b^i u_{x_i}(x_0) = \sum_{i=1}^{n} \sum_{k=1}^{n} u_{y_k} o_{ki} b^i = 0. \tag{1.125}$$

It means that $Lu(x_0) \leq 0$, contradict! So if $Lu < 0$ in U, the maximum value of u in $\bar{U}$ appears on ∂U.

In general case, i.e. $Lu \leq 0$, we derive the auxiliary function $u^\epsilon(x) = u(x) + \epsilon e^{\lambda x_1}$, then:

$$\begin{aligned} Lu^\epsilon &= Lu + \epsilon L(e^{\lambda x_1}) \\ &\leq \epsilon e^{\lambda x_1}(-\lambda^2 a^{11} + \lambda b^1) \\ &\leq \epsilon e^{\lambda x_1}(-\lambda^2 \theta + \lambda \|b\|_{L^\infty}). \end{aligned} \tag{1.126}$$

if we choose sufficiently large λ, we can see $Lu^\epsilon < 0$ for any $\epsilon > 0$. By the proof above, u^ϵ gets its maximum on ∂U. Let $\epsilon \to 0$, we can see u gets its maximum on ∂U.

Remark

- Maximum principle shows an important property for elliptic operator L.
- Since $-u$ is a subsolution whenever u is a supersolution, if $Lu \geq 0$ in U, we can see $\min_{\bar{U}} u = \min_{\partial U} u$.
- Write $u^+ = \max(u, 0)$, if $c \geq 0$ and $Lu \leq 0$ in U, we can see $\max_{\bar{U}} u \leq \max_{\partial U} u^+$. Just consider the operator $K = L - cI$ to prove it.

1.5.2 Strong maximum principle

In this section, we establish the stronger result known as the strong maximum principle. In Theorem 1.14, we prove that u attains its maximum on the boundary ∂U. In fact, if there exists $x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} u(x)$, then $u(x)$ must be constant, which is referred to as the **strong maximum principle**. First, we should introduce the **Hopf's lemma**, which derives a subtle analysis of the outer normal derivative $\frac{\partial u}{\partial \nu}$.

Lemma 1.2 (Hopf's Lemma)

Assume $u \in C^2(U) \cap C^1(\bar{U})$ and $c \equiv 0$ in U, suppose further $Lu \leq 0$ in U and there exists $x^0 \in \partial U$ such that

$$u(x^0) > u(x) \tag{1.127}$$

for all $x \in U$. Assume finally that U satisfies the interior ball condition at x^0, that is, there exists an open ball $B \subset U$ with $x^0 \in \partial B$.

- *Then, $\frac{\partial u}{\partial \nu}(x^0) > 0$ where ν is the outer unit normal to B at x^0.*
- *If $c \geq 0$ in U, the same conclusion holds provided $u(x^0) \geq 0$.*

Remark

- The inequality $\frac{\partial u}{\partial \nu}(x^0) \geq 0$ is trivial, so the strict inequality is necessary.
- If the boundary ∂U is C^2, the interior ball condition is satisfied.
- The idea of this proof is to construct an auxiliary function v that satisfies the following conditions:
    - $v|_{\partial B} = 0$,
    - $Lv \leq 0$,
    - $\frac{\partial v}{\partial \nu}(x^0) < 0$.

Proof Consider the case $c \geq 0$. Without loss of generality, assume the center of the ball B is the origin point, and we write $B = B(0, r)$. Set the auxiliary function

$$v(x) = e^{-\lambda |x|^2} - e^{-\lambda r^2}, \tag{1.128}$$

first verify the three properties above. On ∂B, $|x| = r$ means that $\forall x \in \partial B$, $v(x) = 0$. Then derive the expression of Lv:

$$\begin{aligned} Lv &= -\sum a^{ij} v_{x_i x_j} + \sum b^i v_{x_i} + cv \\ &= -\sum a^{ij}(x) 4\lambda^2 e^{-\lambda |x|^2} x_i x_j + 2\lambda e^{-\lambda |x|^2} \sum a^{ii}(x) - 2\lambda \sum_{i=1}^{n} b^i e^{-\lambda |x|^2} + c(e^{-\lambda |x|^2} - e^{-\lambda r^2}) \\ &\leq e^{-\lambda |x|^2} \left(-4\theta \lambda^2 |x|^2 + 2\lambda \mathrm{tr}(A) + 2\lambda \left\| b \right\|_{L^\infty} |x| + c \right). \end{aligned} \tag{1.129}$$

Choose λ sufficiently large, we can see $Lv \leq 0$ for $x \in B(0, r) \setminus B(0, \frac{r}{2})$. Mark $u^\epsilon(x) := u(x) + \epsilon v(x) - u(x^0)$, we can see $Lu^\epsilon(x) \leq -cu(x^0) \leq 0$.

By the equation $u(x^0) > u(x)$, we can choose sufficiently small $\epsilon > 0$ such that

$$u^\epsilon(x) \leq u^\epsilon(x^0) = 0 \tag{1.130}$$

on $\partial B(0, \frac{r}{2})$. As $v(x) = 0$ on $\partial B(0, r)$, $u^\epsilon(x) \leq 0$ for $x \in \partial B(0, r)$. By theorem 1.14, $u^\epsilon(x) \leq 0$ for all $x \in B(0, r) \setminus B(0, \frac{r}{2})$. And $u^\epsilon(x^0) = 0$, so we can see:

$$\frac{\partial u^\epsilon}{\partial \nu}(x^0) \geq 0. \tag{1.131}$$

Consequently:

$$\frac{\partial u}{\partial \nu}(x^0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x^0) = -\frac{\epsilon}{r} Dv(x^0) \cdot x^0 = 2\lambda \epsilon r e^{-\lambda r^2} > 0. \tag{1.132}$$

Remark

- If $c \equiv 0$, we don't require $u(x^0) \geq 0$, because $Lu^\epsilon \leq 0$ is true for $u(x^0) < 0$.
- If we write $u^\epsilon(x) = u(x) + \epsilon v(x)$, we can see that $Lu^\epsilon \leq 0$ as well, and $\max_{\partial B(0, \frac{r}{2}) \cup \partial B(0, r)} u^\epsilon(x) = u(x^0)$. By remark 3 of theorem 1.14, if $u(x^0) \geq 0$, we can see $u^\epsilon(x) \leq u(x^0)$ as well; but if $u(x^0) < 0$, we can only derive $u^\epsilon(x) \leq 0$. So, for $c \geq 0$, the condition $u(x^0) \geq 0$ is necessary.

Hopf's lemma is the primary technical tool in the next proof:

Theorem 1.15 (Strong maximum principle)

Assume $u \in C^2(U) \cap C^1(\bar{U})$ and $c \equiv 0$ in U, suppose also U is connected, open and bounded. If $Lu \leq 0$ in U and u attains its maximum over $\bar{U}$ at an interior point, then u is constant within U. ♡

Remark Theorem 1.15 is strictly stronger than theorem 1.14. If L satisfies the strong maximum principle, then the maximum of the function u must be attained on the boundary ∂U. However, it should be noted that $\max_U u = \max_{\bar{U}} u$ does not imply that if u attains its maximum value in U, then u is constant.

Proof Write $M = \max_{\bar{U}} u$ and $C := \{x \in U | u(x) = M\}$, assume that $C \neq \emptyset$. Then if $u \not\equiv M$, set

$$V := \{x \in U | u(x) < M\}, \tag{1.133}$$

then choose $y \in V$ such that $\text{dist}(y, C) < \text{dist}(y, \partial U)$, and let B denote the largest ball with center y whose interior lies in V, then there exists $x^0 \in C \cap \partial B$. By Hopf's lemma, $\frac{\partial u}{\partial \nu}(x^0) > 0$. But on the other hand, $x^0 \in C$ means $Du(x^0) = \mathbf{0}$, contradict!

Remark If $c \geq 0$, $Lu \leq 0$ in U, and u attains its **non-negative** maximum over $\bar{U}$ at an interior point, then u is constant within U.

1.6 Harnack's Inequality

Harnack's inequality states that the values of a nonnegative solution are comparable, i.e. if $u \geq 0$ and u is the solution of elliptic equation $Lu = 0$, the supremum of function u can be controled by its infimum.

Theorem 1.16 (Harnack's inequality)

Assume $u \geq 0$ is a C^2 solution of $Lu = 0$ in U, and suppose $V \subset\subset U$ is connected. Then there exists a constant C such that

$$\sup_V u \leq C \inf_V u. \tag{1.134}$$

The constant C depends only on V and the coefficients of L.

The idea of this proof: Without loss of generality, we assume that $u > 0$ in U. Set $v(x) := \log u(x)$, choose $x \in V$, $y \in V$ and $x \neq y$, as V is connected, there is a path comes through the point x and y, marked as l. As l is compact, there exists finite number of balls $B(x_i, r_i)$ such that $l \subset \cup_{i=1}^N B(x_i, r_i)$. If $\|Dv\|_{L^\infty(V)} < \infty$, on the ball $B(x_i, r_i)$, choose $x, y \in B(x_i, r_i) \cap l$, we can see that

$$|v(x) - v(y)| \leq N|Dv|r \leq CNr, \tag{1.135}$$

while r means the length of the arc from x to y. As the cover is finite, mark the arc length of l is s_{xy}, we can see:

$$\left|\frac{u(x)}{u(y)}\right| \leq e^{|v(x)-v(y)|} \leq e^{CNs_{xy}} \leq C(V). \tag{1.136}$$

So, it suffices to show that $\|Dv\|_{L^\infty(V)} < \infty$.

Proof We just consider a simple case, i.e. $a^{ij} \in C^\infty(U)$ and $b^i \equiv 0, c \equiv 0$. As u is the solution, we can see $\sum a^{ij} u_{x_i x_j} = 0$, i.e.

$$\sum a^{ij} v_{x_i x_j} + a^{ij} v_{x_i} v_{x_j} = 0. \tag{1.137}$$

Then derive the expression for:

$$a^{ij} D_i D_j |Dv|^2 = \sum_{k=1}^n 2a^{ij}(D_i D_k v D_j D_k v + D_k v D_i D_j D_k v). \tag{1.138}$$

By (1.137), we can see:

$$\sum_{i,j=1}^n \left(D_k a^{ij} v_{x_i x_j} + a^{ij} v_{x_i x_j x_k} + D_k a^{ij} D_i v D_j v + 2a^{ij} D_k D_i v D_j v\right) = 0. \tag{1.139}$$

Combine (1.139) and (1.138), we can see:

$$\begin{aligned} \sum a^{ij} D_i D_j |Dv|^2 &= \sum \left(2a^{ij} D_i D_k v D_j D_k v - 2D_k v D_k a^{ij} \left(D_i D_j v + D_i v D_j v\right) - 4D_k v a^{ij} D_k D_i v D_j v\right) \\ &= \sum \left(2a^{ij} D_i D_k v D_j D_k v - 2D_k v D_k a^{ij} \left(D_i D_j v + D_i v D_j v\right) - 2a^{ij} D_i |Dv|^2 D_j v\right). \end{aligned} \tag{1.140}$$

To be continued...

1.7 Eigenvalues and Eigenfunctions

In this section, we consider the boundary-value problem:

$$\begin{cases} Lw = \lambda w, x \in U; \\ w = 0, x \in \partial U, \end{cases} \tag{1.141}$$

where U is open and bounded, and we call λ an **eigenvalue** of L with a nontrivial **eigenfunction** w. Now, we will discuss the properties of operator L, its eigenvalues λ and the corresponding eigenfunctions w.

1.7.1 Eigenvalues of symmetric elliptic operators

For simplicity, we consider now an elliptic operator having the divergence form:

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j}, \tag{1.142}$$

where $a^{ij} \in C^\infty(\bar{U})$. We suppose the coefficients a^{ij} satisfy the symmetry condition $a^{ij} = a^{ji}$, then the operator L is thus formally **symmetric**, and in particular the associated bilinear form $B[\ ,\ ]$ satisfies $B[u,v] = B[v,u]$. Assume also U is connected.

Theorem 1.17 (Eigenvalues of symmetric elliptic operators)

For the operator L as (1.142):

- *Each eigenvalue of L is real.*
- *If we repeat each eigenvalue according to its (finite) multiplicity, we have:*

$$\Sigma = \{\lambda_k\}_{k=1}^{\infty} \tag{1.143}$$

  where

$$0 < \lambda_1 \le \lambda_2 \le \cdots \tag{1.144}$$

  and $\lim_{k\to\infty} \lambda_k = \infty$.
- *Finally, there exists an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(U)$, where $w_k \in H_0^1(U)$ is an eigenfunction corresponding to λ_k:*

$$\begin{cases} Lw_k = \lambda_k w_k, x \in U \\ w_k = 0, x \in \partial U. \end{cases} \tag{1.145}$$

♡

Remark Owing to the regularity theorem, $w_k \in C^\infty(U)$, and furthermore $\omega_k \in C^\infty(\bar{U})$ if ∂U is smooth.

Proof First, by the form of operator L, we can see the corresponding bilinear form $B[u,v]$ of L is **bounded** and **uniformly elliptic**. By the discussion about theorem 1.6, the operator L is a linear, bounded bijective from $L^2(U)$ to $L^2(U)$, and its inverse operator $S := L^{-1}$ is a compact operator.

Then, claim that S is symmetric. First, if $Sf = u$, it means that

$$\begin{cases} Lu = f, x \in U; \\ u = 0, x \in \partial U, \end{cases} \tag{1.146}$$

and if $Sg = v$, it means that

$$\begin{cases} Lv = g, x \in U; \\ v = 0, x \in \partial U, \end{cases} \tag{1.147}$$

then

$$(Sf, g) = (u, g) = (u, Lv) = (Lv, u) = B[v, u], \tag{1.148}$$

and

$$(f, Sg) = (f, v) = (Lu, v) = B[u, v], \tag{1.149}$$

as $B[u, v] = B[v, u]$, i.e. $(Sf, g) = (f, Sg)$, so S is symmetric.

Notice that $(Sf, f) = (u, f) = (u, Lu) = B[u, u] \geq 0$, we can see S is a positive operator. Then: we can see S is **compact, symmetric, linear and positive**. By the property of compact and positive operator, we can see the eigenvalues of S are all real, $\eta_k \to 0$, and the eigenvectors of S, marked as $\{w_k\}$, form an orthonormal basis of $L^2(U)$. As $S = L^{-1}$, w_k is also the eigenfunction of L with eigenvalue $\frac{1}{\eta_k}$. So the eigenvalues of L satisfies $\lim_{k\to\infty} \lambda_k = \infty$, and the eigenfunctions $\{w_k\}$ form an orthonormal basis for $L^2(U)$.

Remark Specially, if $L = -\Delta$, and the region $U \subset \mathbb{R}^n$ is bounded and open, **Weyl's Law** asserts

$$\lim_{k\to\infty} \frac{\lambda_k^{\frac{n}{2}}}{k} = \frac{(2\pi)^n}{|U|\alpha(n)}. \tag{1.150}$$

Now, we scrutinize more carefully the **first eigenvalue** of L.

Definition 1.5

We call $\lambda_1 > 0$ the principal eigenvalue of L.

Theorem 1.18 (Variational principle for the principal eigenvalue)

- *We have*

$$\lambda_1 = \min\{B[u, u] | u \in H_0^1(U), \|u\|_{L^2} = 1\}. \tag{1.151}$$

- *Furthermore, the above minimum is attained for a function w_1, positive within U, which solves*

$$\begin{cases} Lw_1 = \lambda_1 w_1, x \in U; \\ w_1 = 0, x \in \partial U. \end{cases} \tag{1.152}$$

- *Finally, if $u \in H_0^1(U)$ is any weak solution of*

$$\begin{cases} Lu = \lambda_1 u, x \in U; \\ u = 0, x \in \partial U, \end{cases} \tag{1.153}$$

then u is a multiple of w_1.

Remark Theorem 1.18 suggests that the principal eigenvalue λ_1 is simple.

Proof First, define $B[\ ,\ ]$ as an inner product in $H_0^1(U)$, find the **orthonormal basis** of the Hilbert space $H_0^1(U)$. By theorem 1.17, the eigenfunctions $\{w_k\}$ form an orthonormal basis on $L^2(U)$, $\|w_k\|_{L^2(U)} = 1$, and

$$\begin{aligned} B[w_k, w_k] &= \lambda_k \|w_k\|_{L^2(U)}^2 = \lambda_k, \\ B[w_k, w_l] &= 0 (k \neq l), \end{aligned} \tag{1.154}$$

so the set $\left\{\frac{w_k}{\lambda_k^{\frac{1}{2}}}\right\}_{k=1}^{\infty}$ forms an orthonormal subset of $H_0^1(U)$. Then we show this orthonormal subset is complete. Assume $u \in H_0^1(U)$ satisfies

$$B[u, w_k] = 0, k = 1, 2, \cdots, \tag{1.155}$$

as $\{w_k\}$ is an orthonormal basis in $L^2(U)$, by Fourier expansion:

$$u = \sum_{k=1}^{\infty} (u, w_k) w_k := \sum_{k=1}^{\infty} d_k w_k. \tag{1.156}$$

Then $B[u, w_k] = 0$ means that $\lambda_k d_k = 0$, i.e. $d_k \equiv 0$, so $u \equiv 0$. Now the set $\left\{ \frac{w_k}{\lambda_k^{\frac{1}{2}}} \right\}$ forms an orthonormal basis for $H_0^1(U)$ equipped with inner product $B[\ ,\ ]$.

Now, recall the expression $B[u, u]$. If $\|u\|_{L^2(U)} = 1$, write the Fourier expansion **on space** $H_0^1(U)$ as $u = \sum_{k=1}^{\infty} d_k w_k$, with $\sum_{k=1}^{\infty} d_k^2 = 1$, we can see:

$$B[u, u] = B[\sum d_k w_k, \sum d_k w_k] = \sum d_k^2 \lambda_k \geq \lambda_1, \tag{1.157}$$

and the equality holds for $u = w_1$. We complete the proof of (1.151).

Then, if u satisfies (1.152), i.e. $Lu = \lambda_1 u$, and $\|u\|_{L^2(U)} = 1$, we can see $B[u, u] = (Lu, u) = \lambda_1 \|u\|_{L^2(U)}^2 = \lambda_1$. On the other hand, if $B[u, u] = 1$ and $\|u\|_{L^2(U)} = 1$, consider the Fourier series of u as $u = \sum d_k u_k$ with $\sum d_k^2 = 1$, we have

$$\sum d_k^2 \lambda_1 = \lambda_1 = B[u, u] = \sum d_k^2 \lambda_k. \tag{1.158}$$

i.e. for $k \geq 2$, $d_k = (u, w_k) = 0$. By **Fredholm alternative**, the eigenvalue λ_1 has finite multiplicity, it follows that

$$u = \sum_{k=1}^{m} (u, w_k) w_k \tag{1.159}$$

for $m < \infty$ and $Lw_k = \lambda_1 w_k$. Therefore:

$$Lu = \sum_{k=1}^{m} (u, w_k) Lw_k = \lambda_1 u. \tag{1.160}$$

Finally, we will show that the multiplicity of eigenvalue λ_1 is 1. To show this result, we should derive the following lemma:

Lemma 1.3

If $u \in H_0^1(U)$ is a weak solution of (1.152), $u \not\equiv 0$, then either $u > 0$ in U or $u < 0$ in U. ♡

Proof Without loss of generality, assume $\|u\|_{L^2(U)} = 1$, mark $\alpha = \int_U (u^+)^2 \mathrm{d}x$, $\beta = \int_U (u^-)^2 \mathrm{d}x$, we can see $\alpha + \beta = 1$. Then:

$$\lambda_1 = B[u, u] = B[u^+, u^+] + B[u^-, u^-] \geq \lambda_1(\alpha + \beta) = \lambda_1. \tag{1.161}$$

The equality holds, so $B[u^+, u^+] = \lambda_1 \|u^+\|_{L^2(U)}^2$, and $B[u^-, u^-] = \lambda_1 \|u^-\|_{L^2(U)}^2$. It means that u^+ and u^- are both the weak solution of $Lu = \lambda_1 u$. What's more, $Lu^+ \geq 0$, by strong maximum principle, $u^+ > 0$ in U or $u^+ \equiv 0$ in U. Similar arguments apply to u^-. So either $u > 0$ in U or $u < 0$ in U.

Back to the proof of main theorem. If u and $\tilde{u}$ are both nontrivial solution of equation (1.152), by the above lemma, $\int_U u \mathrm{d}x \neq 0$ and $\int_U \tilde{u} \mathrm{d}x \neq 0$. So $\exists k \in \mathbb{R}$ such that $\int_U (u - k\tilde{u}) \mathrm{d}x = 0$. On the other hand, $u - k\tilde{u}$ still satisfies equation (1.152) and $\int_U (u - k\tilde{u}) \mathrm{d}x = 0$, by the above lemma, we can see $u - k\tilde{u} \equiv 0$. So $u = k\tilde{u}$ in U. Hence the eigenvalue λ_1 is simple.

1.8 Eigenvalues of nonsymmetric elliptic operators

Now consider a uniformly elliptic operator L in the nondivergence form:

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu. \tag{1.162}$$

Suppose that $a^{ij}, b^i, c \in C^\infty(\bar{U})$ and U is open, bounded and connected, assume $a^{ij} = a^{ji}$ and $c \geq 0$ in U. In this condition, the operator L is no longer symmetry, i.e. L will in general have complex eigenvalues and eigenfunctions. However, there are still some properties about the principal eigenvalue of L.

Theorem 1.19 (Principal eigenvalue for nonsymmetric elliptic operators)

- *There exists a real eigenvalue λ_1 for the operator L, taken with zero boundary conditions, such that if $\lambda \in \mathbb{C}$ is any other eigenvalue, we have $\Re(\lambda) \geq \lambda_1$.*
- *There exists a corresponding eigenfunction w_1, which is positive within U.*
- *The eigenvalue λ_1 is simple.*

♡

Proof skip.

1.9 Exercises

Exercise 1.1 $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of

$$-\Delta u + c(u) = f, \tag{1.163}$$

where $f \in L^2(\mathbb{R}^n)$, $c : \mathbb{R} \to \mathbb{R}$ smooth, $c(0) = 0$, $c'(0) \geq 0$, and $c(u) \in L^2(\mathbb{R}^n)$. Derive the estimate:

$$\left\|D^2 u\right\|_{L^2(\mathbb{R}^n)} \leq C \left\|f\right\|_{L^2(\mathbb{R}^n)}. \tag{1.164}$$

Solution *Assume $\operatorname{supp} u = \Omega$ is a convex compact subset of $\mathbb{R}^n$, by the definition of weak solution, $\forall v \in H_0^1(\Omega)$, we have:*

$$\int_\Omega -\Delta u v + c(u) v \mathrm{d}x = \int_\Omega f v \mathrm{d}x. \tag{1.165}$$

By Gauss-Green formula and $v = 0$ on $\partial\Omega$, we can see:

$$\int_\Omega \nabla v \cdot \nabla u \mathrm{d}x = \int_\Omega (f - c(u)) v \mathrm{d}x. \tag{1.166}$$

Mark $A := \int_\Omega \nabla v \cdot \nabla u \mathrm{d}x$, $B := \int_\Omega (f - c(u)) v \mathrm{d}x$, and choose $v = -D_k^{-h}\left(D_k^h u\right)$ $(1 \leq k \leq n)$, we can see:

$$\begin{aligned}
A &= \sum_{i=1}^n \int_\Omega v_{x_i} u_{x_i} \mathrm{d}x \\
&= -\sum_{i=1}^n \int_\Omega \left(D_k^{-h}\left(D_k^h u\right)\right)_{x_i} u_{x_i} \mathrm{d}x \\
&= \sum_{i=1}^n \int_\Omega \left(D_k^h u\right)_{x_i} \left(D_k^h u\right)_{x_i} \mathrm{d}x \\
&= \left\|D_k^h D u\right\|_{L^2}^2.
\end{aligned} \tag{1.167}$$

Then mark $B_1 := \int_\Omega f v \mathrm{d}x$, $B_2 := -\int_\Omega c(u) v \mathrm{d}x$, we can see $B = B_1 + B_2$. By Cauchy-Schwarz inequality:

$$B_1 \leq \|f\|_{L^2} \|v\|_{L^2} \leq C \|f\|_{L^2} \left\|D_k^h D u\right\|_{L^2}. \tag{1.168}$$

And:

$$\begin{aligned}
B_2 = -\int_\Omega c(u) v \mathrm{d}x &= -\int_\Omega D_k^h(c(u)) D_k^h u \mathrm{d}x \\
&= -\int_\Omega \frac{u(x + he_k) - u(x)}{h} \frac{c(u(x + he_k)) - c(u(x))}{h} \mathrm{d}x \\
&= -\int_\Omega \left(\frac{u(x + he_k) - u(x)}{h}\right)^2 c'(\xi(x)) \mathrm{d}x \\
&\leq 0,
\end{aligned} \tag{1.169}$$

while the third equality holds from the mean value theorem on convex region, and the final inequality holds from *the condition $c' \geq 0$.*

So: $\forall 1 \leq k \leq n$, we can see:

$$\left\|D_k^h Du\right\|_{L^2} \leq C_k \|f\|_{L^2}. \tag{1.170}$$

Set $h \to 0$, we can derive the result

$$\left\|D^2 u\right\|_{L^2} \leq C \|f\|_{L^2}. \tag{1.171}$$

✍ **Exercise 1.2** Assume u is a smooth solution of $Lu = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j} = f$ in U, $u = 0$ on ∂U, where f is bounded. Fix $x^0 \in \partial U$. A barrier at x^0 is a C^2 function w such that

$$Lw \geq 1 (x \in U), w(x^0) = 0, w \geq 0 (x \in \partial U). \tag{1.172}$$

Show that if w is a barrier at x^0, there exists a constant C such that:

$$|Du(x^0)| \leq C \left|\frac{\partial w}{\partial \nu}(x^0)\right|. \tag{1.173}$$

Solution *First, set $\varphi_1 := u + w \|f\|_{L^\infty}$, we can see:*

$$L\varphi_1 = f + \|f\|_\infty Lw \geq 0, \tag{1.174}$$

and $\varphi_1|_{\partial\Omega} \geq 0$. By weak maximum principle, $\min_{\bar{U}} \varphi_1 = 0$.

Then, set $\varphi_2 := u - w \|f\|_{L^\infty}$, we can see:

$$L\varphi_2 = f - \|f\|_\infty Lw \leq 0, \tag{1.175}$$

and $\varphi_2|_{\partial\Omega} \leq 0$. By weak maximum principle, $\max_{\bar{U}} \varphi_2 = 0$. By Hopf's lemma, $\frac{\partial \varphi_1}{\partial \nu}(x_0) \leq 0$, $\frac{\partial \varphi_2}{\partial \nu}(x_0) \geq 0$.

What's more, $u|_{\partial\Omega} = 0$, i.e. $\nabla u \parallel \nu$. So there exists $C > 0$ such that

$$|Du(x_0)| \leq C \left|\frac{\partial w}{\partial \nu}(x_0)\right|. \tag{1.176}$$

✍ **Exercise 1.3** Assume U is connected. Use (a) energy methods and (b) the maximum principle to show that the only smooth solutions of the Neumann boundary-value problem

$$\begin{cases} -\Delta u = 0, x \in U; \\ \dfrac{\partial u}{\partial \nu} = 0, x \in \partial U \end{cases} \tag{1.177}$$

are $u \equiv C$, for some constant C.

Solution *(a)Energy method: as $\Delta u = 0$ and $\frac{\partial u}{\partial \nu}\big|_{\partial U} \equiv 0$, we can see:*

$$0 = -\int_U u \Delta u \mathrm{d}x = \int_U |\nabla u|^2 \mathrm{d}x - \int_U u \frac{\partial u}{\partial \nu} \mathrm{d}S = \int_U |\nabla u|^2 \mathrm{d}x. \tag{1.178}$$

It means that $\nabla u \equiv 0$ a.e. And u is smooth, so $u \equiv C$.

(b)The maximum principle: First, we derive the WMP. Set $\varphi := u + \epsilon|x|^2$, we can see $-\Delta\varphi = -\Delta u - \Delta(\epsilon|x|^2) = -2n\epsilon < 0$, and $\frac{\partial \varphi}{\partial \nu} = \epsilon \frac{\partial |x|^2}{\partial \nu} > 0$, then $\max_{\bar{U}} \varphi = \max_{\partial U} \varphi$, i.e.:

$$u(x) + \epsilon|x|^2 \leq \max_{x \in \partial U} u(x) + \epsilon d^2, \tag{1.179}$$

while d is the diagram of region U. Choose $\epsilon \to 0$, we get the weak maximum principle (WMP) of u. Then derive the strong maximum principle. Set:

$$\begin{aligned} V &:= \{x \in U | u(x) < M\}, \\ C &:= \{x \in U | u(x) = M\}, \end{aligned} \tag{1.180}$$

choose $x \in V$, construct a ball $B(x)$ such that $B(x) \cap C = \{x^0\}$ or $B(x) \cap \partial U = \{x^0\}$. By Hopf's lemma, $\frac{\partial u}{\partial \nu}(x^0) > 0$. If $x^0 \in \partial C$, contradict to $u(x) \equiv M$; if $x^0 \in \partial U$, contradict to the boundary condition. So $u \equiv C$.

Exercise 1.4 Assume $u \in H^1(U)$ is a bounded weak solution of equation

$$-\sum_{i,j} \left(a^{ij} u_{x_i}\right)_{x_j} = 0, x \in U. \tag{1.181}$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex and smooth, and set $w = \phi(u)$. Show w is a weak subsolution, i.e. $B[w, v] \leq 0$ for all $v \in H_0^1(U), v \geq 0$.

Solution *By the definition of weak solution, we can see:*

$$\begin{aligned} B[\phi(u), v] &= \sum_{i,j} \int_V a^{ij} \phi'(u) u_{x_i} v_{x_j} \mathrm{d}x \\ &= \sum_{i,j} \int_V a^{ij} u_{x_i} \frac{\partial(\phi'(u)v)}{\partial x_j} \mathrm{d}x - \sum_{i,j} \int_V a^{ij} u_{x_i} v \phi''(u) u_{x_j} \mathrm{d}x \\ &= -\sum_{i,j} \int_V a^{ij} u_{x_i} u_{x_j} v \phi''(u) \mathrm{d}x \\ &\leq -\theta \int_V v\phi''(u) |\nabla u|^2 \mathrm{d}x \leq 0. \end{aligned} \tag{1.182}$$

While the second equation follows from integration by parts, the third equation follows from the fact that u is a bounded weak solution, and the final step comes from the uniform elliptic condition.

Exercise 1.5 We say that the uniformly elliptic operator

$$Lu = -\sum a^{ij} u_{x_i x_j} + \sum b^i u_{x_i} + cu \tag{1.183}$$

satisfies the weak maximum principle if for all $u \in C^2(U) \cap C^1(\bar{U})$,

$$\begin{cases} Lu \leq 0, x \in U \\ \quad u \leq 0, x \in \partial U \end{cases} \tag{1.184}$$

implies that $u \leq 0$ in U. Suppose that there exists a function $v \in C^2(U) \cap C^1(\bar{U})$ such that $Lv \geq 0$ in U and $v > 0$ on $\bar{U}$. Show that L satisfies the weak maximum principle.

Solution *Set $w = \frac{u}{v}$, derive $a^{ij} w_{ij}$ as:*

$$a^{ij} w_{ij} = a^{ij} \left[\frac{v u_{ij} - u v_{ij}}{v^2} + \frac{2u v_i v_j - 2v u_j v_i}{v^3} + \frac{u_j v_i - u_i v_j}{v^2} \right]. \tag{1.185}$$

Sum them up, we can see:

$$-\sum a^{ij} w_{ij} = \sum \frac{u a^{ij} v_{ij}}{v^2} - \sum \frac{v a^{ij} u_{ij}}{v^2} + \frac{2}{v^3} \sum \left(a^{ij} v u_j v_i - a^{ij} u v_i v_j\right). \tag{1.186}$$

On the region $\{u > 0\}$, by the condition $-\sum a^{ij} v_{ij} + \sum b^i v_i + cv > 0$, $-\sum a^{ij} u_{ij} + \sum b^i u_i + cu \leq 0$, then estimate (1.186), we have:

$$\begin{aligned} -\sum a^{ij} w_{ij} &\leq \frac{u}{v^2} \left(\sum b^i v_i + cv\right) - \frac{1}{v} \left(\sum b^i u_i + cu\right) + \frac{2}{v^3} \sum a^{ij} v_i (v u_j - u v_j) \\ &= \sum \frac{b^j}{v^2} (u v_j - v u_j) - \frac{2}{v^3} \sum a^{ij} v_i (u v_j - v u_j) \\ &= \sum \left(\frac{b^j}{v^2} - \frac{2}{v^3} \sum a^{ij} v_i \right) (u v_j - v u_j) \\ &= \sum \left(\frac{2}{v} \sum a^{ij} v_i - b^j \right) w_j. \end{aligned} \tag{1.187}$$

So if we define:

$$Mw := -\sum a^{ij} w_{ij} + \sum \left(b^j - \frac{2}{v} \sum a^{ij} v_i \right) w_j, \tag{1.188}$$

we can see $Mw \leq 0$ on the region $\{u > 0\}$. L is uniformly elliptic means that M is uniformly elliptic, so the *operator M satisfies weak maximum principle. If $\{u > 0\} \neq \emptyset$, mark $\Omega := \{x|u(x) > 0\}$, $Mw \leq 0$ means $0 < \sup_{\bar{\Omega}} w = \sup_{\partial\Omega} w = 0$, contradict!*

So $\Omega = \emptyset$, i.e. L satisfies weak maximum principle.

Exercise 1.6 Let $Lu = -\sum_{i,j=1}^{n} \left(a^{ij}u_{x_i}\right)_{x_j}$, where (a^{ij}) is symmetric. Assume the operator L with zero boundary conditions has eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$.Show:

$$\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{u \in S^{\perp}, \|u\|_{L^2}=1} B[u,u]. \tag{1.189}$$

Here Σ_{k-1} means the collection of $(k-1)$-dimensional subspaces of $H_0^1(U)$.

Solution *By theorem 1.17, $\exists$ an orthogonal basis $\{u_i\}$ for Hilbert space $H_0^1(U)$, such that*

$$\begin{aligned} Lu_i &= \lambda_i u_i, \\ B[u_i, u_j] &= \lambda_i \delta_{ij}, \\ \langle u_i, u_j \rangle_{L^2(U)} &= \delta_{ij}. \end{aligned} \tag{1.190}$$

Assume $\|u\|_{L^2} = 1$, write $u = \sum r_i u_i$, we can see $\sum r_i^2 = 1$. Then:

$$B[u,u] = B\left[\sum r_i u_i, \sum r_i u_i\right] = \sum \lambda_i r_i^2. \tag{1.191}$$

Set the subspace $S_0 := sp\{u_1, \cdots, u_{k-1}\}$, $S_1 := sp\{u_1, \cdots, u_k\}$, then $S^{\perp} \cap S_1 \neq \emptyset$. Choose $\tilde{u} \in S^{\perp} \cap S_1$, we can see

$$B[\tilde{u}, \tilde{u}] = \sum_{i=1}^{k} \lambda_i r_i^2 \leq \lambda_k. \tag{1.192}$$

So the right hand of equation (1.189) is less or equal to λ_k. On the other hand, if we choose $\Sigma_{k-1} = S_0$, as $B[u_k, u_k] = \lambda_k$, the "=" can be satisfied. Now, we complete the proof.

Exercise 1.7 Let λ_1 be the principal eigenvalue of the uniformly elliptic, nonsymmetric operator

$$Lu = -\sum a^{ij}u_{x_i x_j} + \sum b^i u_{x_i} + cu, \tag{1.193}$$

taken with zero boundary conditions. Prove the max-min representation formula

$$\lambda_1 = \sup_u \inf_x \frac{Lu(x)}{u(x)}, \tag{1.194}$$

the "sup" taken over functions $u \in C^{\infty}(\bar{U})$ with $u > 0$ in U, $u = 0$ on ∂U, and the "inf" taken over points $x \in U$.

Solution *Set $X := \{u \in C^{\infty}(\bar{U}) : u > 0 (x \in U), u = 0 (x \in \partial U)\}$. Assume $w_1 \in H^1(U)$, $w_1 > 0$ is the eigenfunction related to λ_1, choose $\{u_n\} \subset X$ such that $u_n \to w_1$ in $H^1(U)$, then*

$$\sup_u \inf_x \frac{Lu(x)}{u(x)} \geq \inf_x \frac{Lu_n(x)}{u_n(x)}. \tag{1.195}$$

Set $n \to \infty$, we can see that

$$\sup_u \inf_x \frac{Lu(x)}{u(x)} \geq \lambda_1. \tag{1.196}$$

Then it suffices to show $\sup_u \inf_x \frac{Lu(x)}{u(x)} \leq \lambda_1$, i.e. $\inf_{x\in U}(Lu - \lambda_1 u) \leq 0$. As $\lambda_1 \in \mathbb{R}$, by the definition of dual operator, λ_1 is also the principal eigenvalue of operator L^*, i.e. $\exists w_1^* > 0$ such that $L^* w_1^* = \lambda_1 w_1^*$. Then:

$$\langle Lu - \lambda_1 u, w_1^* \rangle = \langle u, L^* w_1^* - \lambda_1 w_1^* \rangle = \langle u, 0 \rangle = 0. \tag{1.197}$$

As $w_1^* > 0$, if $Lu - \lambda_1 u > 0$, then $\langle Lu - \lambda_1 u, w_1^* \rangle > 0$, contradict! So $\inf_{x\in U}(Lu - \lambda_1 u) \leq 0$.

Exercise 1.8 (Eigenvalues and domain variations) Consider a family of smooth, bounded domains $U(\tau) \subset \mathbb{R}^n$ that depend smoothly upon the param eter $\tau \in \mathbb{R}$. As τ changes, each points on $\partial U(\tau)$ moves with velocity $\mathbf{v}$.

For each τ, we consider eigenvalues $\lambda = \lambda(\tau)$ and corresponding eigenfunctions $w = w(x,\tau)$:

$$\begin{cases} -\Delta w = \lambda w, x \in U(\tau), \\ \quad w = 0, x \in \partial U(\tau), \end{cases} \tag{1.198}$$

normalized so that $\|w\|_{L^2(U(\tau))} = 1$. Suppose that λ and w are smooth functions of τ and x, prove **Hadamard's variational formula**

$$\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau} = -\int_{\partial U(\tau)} \left|\frac{\partial w}{\partial \nu}\right|^2 \mathbf{v}\cdot\nu \mathrm{d}S. \tag{1.199}$$

Solution *By (1.198), we can see:*

$$\int_{U(\tau)} -w\Delta w \mathrm{d}x = \int_{U(\tau)} \lambda(\tau) w^2 \mathrm{d}x. \tag{1.200}$$

By the boundary conditions and Gauss-Green formula:

$$f(\tau) = \int_{U(\tau)} \left(|\nabla w|^2 - \lambda(\tau) w^2\right) \mathrm{d}x \equiv 0. \tag{1.201}$$

*For $\|w\|_{L^2} = 1$, it means that $\lambda(\tau) = \int_{U(\tau)} |\nabla w|^2 \mathrm{d}x$. Then, by **Reynold's transport theorem**:*

$$\begin{aligned} \frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau} &= \int_{\partial U(\tau)} |\nabla w|^2 \mathbf{v}\cdot\nu \mathrm{d}S + \int_{U(\tau)} \frac{\partial |\nabla w|^2}{\partial \tau}\mathrm{d}x \\ &= \int_{\partial U(\tau)} \left|\frac{\partial w}{\partial \nu}\right|^2 \mathbf{v}\cdot\nu \mathrm{d}S + 2\int_{U(\tau)} \nabla w \cdot \nabla w_t \mathrm{d}x. \end{aligned} \tag{1.202}$$

On the other hand:

$$\begin{aligned} &2\int_{U(\tau)} \nabla w \cdot \nabla w_\tau \mathrm{d}x \\ =&2\int_{U(\tau)} \nabla\cdot(w\nabla w_\tau)\mathrm{d}x - 2\int_{U(\tau)} w\Delta w_\tau \mathrm{d}x \\ =&-2\int_{U(\tau)} w\frac{\partial \Delta w}{\partial \tau}\mathrm{d}x \\ =&2\int_{U(\tau)} w\frac{\partial \lambda(\tau) w}{\partial \tau}\mathrm{d}x \\ =&2\lambda'(\tau)\int_{U(\tau)} w^2\mathrm{d}x + 2\int_{U(\tau)} \lambda(\tau) w\frac{\partial w}{\partial \tau}\mathrm{d}x \\ =&2\lambda'(\tau) + 2\int_{U(\tau)} \lambda w \frac{\partial w}{\partial \tau}\mathrm{d}x = 2\lambda'(\tau). \end{aligned} \tag{1.203}$$

So:

$$\frac{\mathrm{d}\lambda(\tau)}{\mathrm{d}\tau} = -\int_{\partial U(\tau)} \lambda(\tau) w^2 \mathrm{d}x. \tag{1.204}$$

Chapter 2 Linear Evolution Equations

In this chapter, we will study the linear PDEs that involves time. We call such PDE **evolution equations**. Now, we study the **general second-order parabolic and hyperbolic equations** first. In this chapter, **the Fourier transform** and **the semigroup techniques** provide alternative approaches.

2.1 The space $H^{-1}(U)$

In this short section, we will introduce the definition and some basic properties of the Sobolev space $H^{-1}(U)$.

Definition 2.1

We denote by $H^{-1}(U)$ the dual space to $H_0^1(U)$. ♣

In other words, f belongs to $H^{-1}(U)$ provided f is a bounded linear functional on $H_0^1(U)$. In fact, $H_0^1(U)$ isn't reflexive, and we have:

$$H_0^1(U) \subset L^2(U) \subset H^{-1}(U). \tag{2.1}$$

We write $\langle \cdot, \cdot \rangle$ to denote the pairing between $H^{-1}(U)$ and $H_0^1(U)$.

Definition 2.2

If $f \in H^{-1}(U)$, we define the norm

$$\|f\|_{H^{-1}(U)} := \sup \left\{ \langle f, u \rangle \,|u \in H_0^1(U), \|u\|_{H_0^1(U)} \leq 1 \right\}. \tag{2.2}$$

♣

Then, we will introduce the characterizations of $H^{-1}(U)$.

Theorem 2.1 (characterization of H^{-1})

- *Assume $f \in H^{-1}(U)$, there exist functions $f^0, f^1, \cdots, f^n$ in $L^2(U)$ such that*

$$\langle f, v \rangle = \int_U f^0 v + \sum_{i=1}^n f^i v_{x_i} \mathrm{d}x. \tag{2.3}$$

- *Furthermore,*

$$\|f\|_{H^{-1}(U)} = \inf \left\{ \left(\int_U \sum_{i=0}^n |f^i|^2 \mathrm{d}x \right)^{\frac{1}{2}} \right\}. \tag{2.4}$$

- *In particular, we have*

$$(v, u)_{L^2(U)} = \langle v, u \rangle \tag{2.5}$$

for all $u \in H_0^1(U)$, $v \in L^2(U) \subset H^{-1}(U)$. ♡

2.2 Second-order Parabolic Equations

Second-order parabolic PDEs are natural generalizations of the heat equations. We will study the existence and uniqueness of appropriately defined weak solutions, their smoothness and other properties.

2.2.1 Definitions

2.2.1.1 Parabolic equations

In this chapter, we assume U to be an open bounded subset of $\mathbb{R}^n$ and $U_T := U \times (0, T]$ for some fixed time $T > 0$.

Consider the initial/boundary-value problem:

$$\begin{cases} u_t + Lu = f, x \in U_T; \\ \qquad u = 0, x \in \partial U \times [0, T]; \\ \qquad u = g, x \in U \times \{t = 0\}. \end{cases} \tag{2.6}$$

where $f : U_T \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are given and $u : \bar{U_T} \to \mathbb{R}$ is the unknown. The operator L is the elliptic operator related to time parameter t as divergence form (1.2) or non-divergence form (1.3), with the uniformly parabolic condition.

Definition 2.3 (Uniformly parabolic)

We say that the partial differential operator $\frac{\partial}{\partial t} + L$ is uniformly parabolic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x, t)\xi_i\xi_j \geq \theta|\xi|^2 \tag{2.7}$$

for all $(x, t) \in U_T$, $\xi \in \mathbb{R}^n$. ♣

In physical terms, the second-order term $\sum a^{ij}u_iu_j$ represents **diffusion**, the first-order term $\sum b^iu_i$ represents **transport**, and the zeroth-order term cu represents **creation**.

2.2.1.2 Weak solutions

Now, we consider the operator L has the divergence form , and:

$$\begin{aligned} &a^{ij}, b^i, c \in L^\infty(U_T), \\ &f \in L^2(U_T), \\ &g \in L^2(U). \end{aligned} \tag{2.8}$$

We will always suppose $a^{ij} = a^{ji}$.

First, we temporarily suppose that $u(x, t)$ is a smooth function on U_T. Define a series of function $\mathbf{u} : [0, T] \to H_0^1(U)$ as

$$[\mathbf{u}(t)](x) = u(x, t). \tag{2.9}$$

and a series $\mathbf{f} : [0, T] \to L^2(U)$ as:

$$[\mathbf{f}(t)](x) := f(x, t). \tag{2.10}$$

Now, choose a fixed function $v \in H_0^1(U)$, multiply the PDE $\frac{\partial u}{\partial t} + Lu = f$ by v and integrate by parts, define the bilinear form

$$B[u, v; t] := \int_U \left(\sum_{i,j=1}^{n} a^{ij}(\cdot, t)u_iv_j + \sum_{i=1}^{n} b^i(\cdot, t)u_iv + cuv \right) \mathrm{d}x, \tag{2.11}$$

we can see that

$$(\mathbf{u}', v) + B[\mathbf{u}, v; t] = (\mathbf{f}, v) \tag{2.12}$$

for each $0 \leq t \leq T$. The pairing $(\cdot, \cdot)$ denoting the inner product in $L^2(U)$.

Then, by (2.6), we can see:

$$u_t = g^0 + \sum_{j=1}^{n} g^j_{x_j}, x \in U_T \tag{2.13}$$

for $g^0 := f - \sum_{i=1}^n b^i u_{x_i} - cu$ and $g^j := \sum_{i=1}^n a^{ij} u_{x_i}$. By the definition of $\|\cdot\|_{H^{-1}(U)}$, we can see:

$$\|u_t\|_{H^{-1}(U)} \leq \left(\sum_{j=0}^{n} \|g^j\|^2_{L^2(U)} \right)^{\frac{1}{2}} \leq C \left(\|u\|_{H_0^1(U)} + \|f\|_{L^2(U)} \right). \tag{2.14}$$

So, it's reasonable to demand $u_t \in H^{-1}(U)$. In the case where u smooth on U_T, we observe that $u_t(\tau) \in L^2(U)$. Consequently, the term $(\mathbf{u}', v)$ in equation (2.12) can be generally expressed as $\langle \mathbf{u}', v \rangle$ on $H_0^1(U)$. From this, we can derive the definition of a weak solution.

Definition 2.4 (Weak Solution)

We say a function

$$\mathbf{u} \in L^2(0, T; H_0^1(U)), \mathbf{u}' \in L^2(0, T; H^{-1}(U)), \tag{2.15}$$

*is a **weak solution** of the parabolic initial/boundary value problem (2.6) provided*

$$\langle \mathbf{u}', v \rangle + B[\mathbf{u}, v; t] = (\mathbf{f}, v) \tag{2.16}$$

for each $v \in H_0^1(U)$ and a.e. time $0 \leq t \leq T$ and $\mathbf{u}(0) = g$. ♣

Remark The space $L^2(0, T; H_0^1(U))$ means that:

- $\forall t \in [0, T]$, $\mathbf{u}(t) \in H_0^1(U)$.
- Mark $f(t) := \|\mathbf{u}(t)\|_{H_0^1(U)}$, then $f(t) \in L^2(0, T)$.

2.2.2 Existence and Uniqueness

In this section, we will discuss the **existence** and **uniqueness** of weak solutions. In Chapter 1, we explored the well-posedness of elliptic equations through the analysis of compact operators. Here, our focus shifts to evolution equations. As the parameter t varies, we select finite-dimensional subspaces V_i to approximate the functional space $H_0^1(U)$. We then seek the approximated solution on V_i and take the limit, employing the method known as **Galerkin approximations**.

2.2.2.1 Galerkin approximations

For the parabolic problem (2.6), we construct the solutions of certain finite-dimensional approximations first. More precisely, by theorem 1.17, if we choose the set of eigenfunctions $\{w_k(x)\}$ for the operator $L = -\Delta$, then $\{w_k\}_{k=1}^{\infty}$ forms an orthogonal basis of $H_0^1(U)$, and $\{w_k\}_{k=1}^{\infty}$ forms an orthonormal basis of $L^2(U)$.

Now, fix $m \in \mathbb{N}^+$, we will look for a function $\mathbf{u}_m : [0, T] \to H_0^1(U)$ with the form:

$$\mathbf{u}_m(t) := \sum_{k=1}^{m} d_m^k(t) w_k, \tag{2.17}$$

by the definition 2.4, the finite dimensional approximation $d_m^k(t)$ satisfies:

- $$d_m^k(0) = (g, w_k). \tag{2.18}$$

- $$(\mathbf{u}'_m, w_k) + B[\mathbf{u}_m, w_k; t] = (\mathbf{f}, w_k). \tag{2.19}$$

In fact, equation (2.18) is obtained from the boundary value $\mathbf{u}(0) = g$. On the subspace V_k, we can observe that $(\mathbf{u}_m(0), w_k) = d_m^k(0) = (g, w_k)$. Equation (2.19) is derived from (2.16) by replacing v with w_k. Thus, the function $\mathbf{u}_m(t)$ is the **projection** of the solution $\mathbf{u}$ onto the subspace span$\{w_1, \cdots, w_m\}$.

Theorem 2.2 (Construction of approximate solutions)

For each integer $m = 1, 2, \cdots$ there exists a unique function $\mathbf{u}_m$ of the form (2.17) satisfying (2.18) and (2.19). ♡

Proof By the definition of $\mathbf{u}_m$, we can see:

$$(\mathbf{u}'_m, w_k) = {d_m^k}'(t) \tag{2.20}$$

and

$$B[\mathbf{u}_m, w_k; t] = \sum_{l=1}^{m} B[w_l, w_k; t] d_m^l(t). \tag{2.21}$$

Now we get an ODE system:

$${d_m^k}'(t) = \sum_{l=1}^{m} B[w_l, w_k; t] d_m^l(t), \tag{2.22}$$

with the initial values

$$d_m^k(0) = (g, w_k). \tag{2.23}$$

By the standard existence theory for ODEs, the equation (2.22) equipped with initial value (2.23) has unique solution $d_m^k(t)$. Then we can see the solution $\mathbf{u}_m$ exists and unique.

2.2.2.2 Energy estimates

In this section, we need some uniform estimates for the L^2 norm of $\mathbf{u}_m$. It's just the **energy estimates** for parabolic equation.

Theorem 2.3 (Energy estimates)

There exists a constant C, depending only on U, T and the coefficients of L, such that:

$$\max_{0 \le t \le T} \|\mathbf{u}_m(t)\|_{L^2(U)} + \|\mathbf{u}_m\|_{L^2(0,T;H_0^1(U))} + \|\mathbf{u}'_m\|_{L^2(0,T;H^{-1}(U))} \le C \left(\|\mathbf{f}\|_{L^2(0,T;L^2(U))} + \|g\|_{L^2(U)} \right). \tag{2.24}$$

♡

Bibliography

[1] Lawrence C Evans. *Partial differential equations*. Vol. 19. American Mathematical Society, 2022.