A brief note for Evans PDE chapter 5-7

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Contents

Chapter 1 Elliptic equation

In this chapter, we will discuss on the general elliptic PDEs and its weak form. We will exploit two essentially distinct techniques, energy methods within Sobolev spaces and maximum principle methods.

1.1 Definition

1.1.1 Elliptic equations

We will in this chapter mostly study the boundary value problem(BVP):

$$\begin{cases} Lu = f \text{ in } U; \\ \quad u = 0 \text{ on } \partial U, \end{cases} \tag{1.1}$$

where U is an open, bounded subset of $\mathbb{R}^n$, and $u : \bar{U} \to \mathbb{R}$ is the unknown. Here f is given, and L denotes a second order differential operator have either the form:

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{1.2}$$

or else

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u. \tag{1.3}$$

The form (1.2) is in **divergence form**, while the form (1.3) is in **non-divergence form**. The requirement that $u = 0$ on $\partial\Omega$ is called **Dirichlet's boundary condition**.

Remark

- Different with Evans chapter 2, $u \in C^2(U)$ is unnecessary. So we should discuss on the weak form of equation (1.1).
- If $a^{ij} \in C^2(U)$, the form (1.2) and (1.3) are equivalent in general. But the form (1.2) is more natural for energy methods, and the form (1.3) is more appropriate for maximum principle techniques.
- Assumption: **symmetry condition**
$$a^{ij} = a^{ji}.$$

Now, give an important property of the differential operator L.

Definition 1.1

We say the partial differential operator L is (uniformly) elliptic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \tag{1.4}$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$. ♣

Remark

- L is uniformly elliptic means the matrix $(a^{ij}(x))$ is positive definite $\forall x \in U$.
- The converse proposition of the above proposition isn't true.
- Special case: $a^{ij}(x) \equiv \delta_{ij}, b^i \equiv 0, c^i \equiv 0$. In this case, the equation (1.1) is the **Poisson equation**.

1.1.2 Weak solutions

Motivation: In general case, we can only assume that

$$a^{ij}, b^i, c \in L^\infty(U), f \in L^2(U). \tag{1.5}$$

In this case, maybe we can't find $u \in C^2(U)$ such that u satisfies (1.1), so we try to derive the **weak form** of the solution u, such that $u \in H_0^1(U)$. In practise, choose **test function** $v \in C_c^\infty(U)$, then define the linear functional

$$u^*(v) := \int_U uv\mathrm{d}x. \tag{1.6}$$

Then consider the dual form of (1.1), i.e.

$$(Lu)^*(v) = f^*(v)\ \forall v \in C_c^\infty(U). \tag{1.7}$$

It's just the **weak form** of equation (1.1). In this section, we choose the **divergence form** of operator L.

Theorem 1.1

The weak form of equation (1.1) *is*

$$\int_U \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv\mathrm{d}x = \int_U fv\mathrm{d}x. \tag{1.8}$$

♡

Proof It's suffices to derive the expression of $\int_U vLu\mathrm{d}x$. By Gauss-Green formula, for a vector field $F \in C^1(\mathbb{R}^n)$, we can see:

$$\int_U v\nabla \cdot F\mathrm{d}x = \int_{\partial U} vF\cdot n\mathrm{d}S(x) - \int_U F\cdot\nabla v\mathrm{d}x. \tag{1.9}$$

Choose the vector field $F = \begin{bmatrix} \sum a^{i1}(x)u_{x_i} \\ \sum a^{i2}(x)u_{x_i} \\ \vdots \\ \sum a^{in}(x)u_{x_i} \end{bmatrix}$, as $v = 0$ on $\partial\Omega$, by (1.9), we can see:

$$\begin{aligned} \int_U \sum_{i,j=1}^n \left(a^{ij}(x)u_{x_i}\right)_{x_j} v\mathrm{d}x &= \int_U v\nabla\cdot F\mathrm{d}x \\ &= -\int_U F\cdot\nabla v\mathrm{d}x \\ &= -\int_U \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j}\mathrm{d}x. \end{aligned} \tag{1.10}$$

Then:

$$\int_U vLu\mathrm{d}x = \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv\mathrm{d}x = \int_U vf\mathrm{d}x = f^*(v). \tag{1.11}$$

By (1.8), we can derive the following definitions.

Definition 1.2

1. *The bilinear form $B[\ ,\ ]$ associated with the divergence form elliptic operator defined by* (1.2) *is:*

$$B[u,v] := \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v \tag{1.12}$$

for $u, v \in H_0^1(U)$.

2. *We say that* $u \in H_0^1(U)$ *is a weak solution of the BVP* (1.1) *if*

$$B[u, v] = (f, v) \tag{1.13}$$

for all $v \in H_0^1(U)$, *where* $(\ ,\ )$ *denotes the inner product in* $L^2(U)$.

♣

More generally, let us consider the BVP

$$\begin{cases} Lu = f^0 - \sum_{i=1}^{n} f^i_{x_i}, x \in U, \\ u = 0, x \in \partial U, \end{cases} \tag{1.14}$$

where $f^i \in L^2(U)$. Then we say $u \in H_0^1(U)$ is a weak solution of problem (1.14) if

$$B[u, v] = \langle f, v \rangle \tag{1.15}$$

for all $v \in H_0^1(U)$, where $\langle \cdot, \cdot \rangle$ is the pairing of $H^{-1}(U)$ and $H_0^1(U)$.

For non-homogeneous elliptic PDE, i.e.

$$\begin{cases} Lu = f, x \in U, \\ u = g, x \in \partial U. \end{cases} \tag{1.16}$$

By trace theorem, $\exists w \in H^1(U)$ such that the trace of w is g. Then define $\tilde{u} := u - w$, (1.16) is equivalent to the equation:

$$\begin{cases} L\tilde{u} = \tilde{f}, x \in U, \\ \tilde{u} = 0, x \in \partial U, \end{cases} \tag{1.17}$$

where $\tilde{f} := f - Lw \in H^{-1}(U)$.

1.2 Existence of weak solutions

1.2.1 Lax-Milgram theorem

We now introduce an abstract principle from linear functional analysis.

Assume H is a real Hilbert space, with norm $\|\ \|$ and inner product $(\ ,\ )$. We let $\langle \cdot, \cdot \rangle$ denote the pairing of H with its dual space.

Theorem 1.2 (Lax-Milgram Theorem)

Assume that

$$B : H \times H \to \mathbb{R} \tag{1.18}$$

is a bilinear mapping, for which there exist constants α, β *such that*

$$|B(u, v)| \leq \alpha \|u\| \|v\| \,(u, v \in H) \tag{1.19}$$

and

$$\beta \|u\|^2 \leq B[u, u](u \in H). \tag{1.20}$$

Finally, let $f : H \to R$ *be a bounded linear functional on* H.

Then there exists a unique element $u \in H$ *such that*

$$B[u, v] = \langle f, v \rangle \tag{1.21}$$

for all $v \in H$.

Remark

1. If B is symmetry, the condition (1.19) and (1.20) means B can derive an inner product on H.
2. So, if B is symmetry, theorem 1.2 is a direct corollary of Riesz representation theorem.
3. If B isn't symmetry, Riesz representation theorem can transform $f \in H^*$ to $u_f \in H$, such that $\langle f, v \rangle = (u_f, v)$.
4. So, we should show that $\forall u_f \in H$, $\exists u \in H$ such that $B[u,v] = (u_f, v)$ for each $v \in H$.

Proof

By the remark above, we just need to show that $\forall u_f \in H$, $\exists u \in H$ such that $B[u,v] = (u_f, v)$ for each $v \in H$.

Consider an element $u \in H$. $B(u,v)$ is a bounded bilinear mapping, so the map

$$B_u(v) := B[u,v] \tag{1.22}$$

is a bounded linear functional. By Riesz representation theorem, $\exists! \tilde{u} \in H$ such that $B[u,v] = (\tilde{u}, v)$ $\forall v \in H$. So we can define a map $A : H \to H$ maps u to $\tilde{u}$.

Then it's suffices to show that A is a bounded linear isomorphism.

First we should show that A is linear. $\forall v \in H$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $u_1, u_2 \in H$, we can see:

$$\begin{aligned}(A(\lambda_1 u_1 + \lambda_2 u_2), v) &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \\ &= \lambda_1 B[u_1, v] + \lambda_2 B[u_2, v] \\ &= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v).\end{aligned} \tag{1.23}$$

(1.23) shows that A is a linear map.

Then we show that A is bounded. As B is bounded, we can see:

$$\begin{aligned}\|Au\|^2 &= (Au, Au) \\ &= B[u, Au] \\ &\leq \alpha \|u\| \|Au\| .\end{aligned} \tag{1.24}$$

i.e. $\|Au\| \leq \alpha \|u\|$. So A is bounded.

The next thing to do in the proof is to show A is an injective. By (1.20), we can see:

$$\begin{aligned}\beta \|u\|^2 &\leq B[u,u] \\ &= (Au, u) \\ &\leq \|Au\| \|u\| .\end{aligned} \tag{1.25}$$

i.e. $\beta \|u\| \leq \|Au\|$. So $\beta \|u\| \leq \|Au\| \leq \alpha \|u\|$, it means that A is an injective. What's more, the range of A, marked as $R(A)$, is a closed set.

Finally, we should show that A is a surjective. If $R(A) \neq H$, since $R(A)$ is closed, there exists a nonzero element $\omega \in H$ with $\omega \in R(A)^{\perp}$. Then:

$$\beta \|\omega\|^2 \leq B[\omega, \omega] = (A\omega, \omega) = 0. \tag{1.26}$$

which means that $\|\omega\| = 0$, contradict! So A is a surjective.

This completes the proof of Lax-Milgram theorem.