Homework# 12

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P111 Problem2

证明. Now the equation is:

$$\begin{cases} Lu = f, x \in \Omega \\ \dfrac{\partial u}{\partial \nu} + \sigma u = 0, x \in \partial\Omega \end{cases} \tag{1}$$

By $P110$ equation $(1.13), (1.14), (1.15)$, we can see: if $u \in H^2(\Omega)$, for $Lu = f$, $\forall v \in H^1(\Omega)$, the following equation is true:

$$(-f, v)_{L^2(\Omega)} = a(u, v) - \left(\frac{\partial u}{\partial \nu}, v\right)_{L^2(\partial\Omega)}. \tag{2}$$

By condition:

$$a(u, v) + (\sigma u, v)_{L^2(\partial\Omega)} = -(f, v), \forall v \in H^1(\Omega) \tag{3}$$

(2) and (3) shows that:

$$\left(\frac{\partial u}{\partial \nu} + \sigma u, v\right)_{L^2(\partial\Omega)} = 0 \forall v \in H^1(\Omega). \tag{4}$$

Assume T is the trace map from $H^1(\Omega)$ to $L^2(\partial\Omega)$, by (4), it means that:

$$\left(\left(\frac{\partial u}{\partial \nu} + \sigma u\right)|_{\partial\Omega}, Tv\right)_{L^2(\partial\Omega)} = 0 \forall v \in H^1(\Omega). \tag{5}$$

(5) shows that $\forall \varphi \in L^2(\partial\Omega)$,

$$\left(\left(\frac{\partial u}{\partial \nu} + \sigma u\right)|_{\partial\Omega}, \varphi\right)_{L^2(\partial\Omega)} = 0. \tag{6}$$

It means that $\frac{\partial u}{\partial \nu} + \sigma u = 0$ on $\partial\Omega$ in the meaning of trace. □

P111 Problem3

By (1.18), the generalized solution $u \in H^1(\Omega)$ of this equation satisfies:

$$\int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \sigma \int_{\partial\Omega} uv\mathrm{d}S = -\int_\Omega fv\mathrm{d}x, \forall v \in H^1(\Omega). \tag{7}$$

We set the functional

$$J[u] = \frac{1}{2}\sigma \int_{\partial\Omega} u^2 \mathrm{d}S + \int_\Omega (\frac{1}{2}|Du|^2 + uf)\mathrm{d}x. \tag{8}$$

Then $\forall v \in H^1(\Omega)$, we can see:

$$\frac{J[u+\epsilon v] - J[u]}{\epsilon} \to \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x + \sigma \int_{\partial\Omega} uv\mathrm{d}S + \int_\Omega fv\mathrm{d}x \tag{9}$$

when $\epsilon \to 0$. From (7) and (9), we can see (8) is the functional for the 3rd BVP equation.

P111 Problem4

The functional related to a homogenious Dirichlet problem for Laplacian equation is

$$J[u] = \int_\Omega (\frac{1}{2}|Du|^2)\mathrm{d}x. \tag{10}$$

Now we should consider the target function $u - f \in H^1(B)$, by $(1.9), (1.10), (1.11)$, it is suffices to find a generalized solution for the Laplacian equation

$$\begin{cases} \Delta u = 0 (x \in \Omega) \\ u = f (x \in \partial\Omega) \end{cases} \tag{11}$$

By Poisson's formula, the solution is:

$$u(x) = \frac{1 - |x|^2}{2\pi} \int_{\partial B(0,1)} \frac{f(y)}{|x-y|^2} \mathrm{d}S(y) \tag{12}$$

And, by $\Delta u = 0$ in Ω, use Gauss-Green formula, we can see:

$$\int_B |\nabla u|^2 \mathrm{d}x = \int_B \nabla \cdot (u \nabla u)\mathrm{d}x = \int_{\partial B} u \frac{\partial u}{\partial n} \mathrm{d}S(x). \tag{13}$$

By (12) and (13), the result is π.

P124 Problem2

证明. Extend the boundary condition $g \in H^{\frac{1}{2}}(\partial\Omega)$ to the region Ω, by the extension theorem, we can extend it to $G \in H^1(\Omega)$, and $G|_{\partial\Omega} = g$. Set $w = u - G$, we can see:

$$\begin{cases} Lw = f - LG, x \in \Omega \\ w = 0, x \in \partial\Omega \end{cases} \tag{14}$$

Then use theorem 2.2, $\exists\lambda$ such that:

$$\|-Lw+\lambda w\|_{-1} \geq C\|w\|_1. \tag{15}$$

By (15),(14) and triangular inequality, we can see:

$$\|f-LG\|_{-1}+|\lambda|\|u-G\|_{-1} \geq C\|u-G\|_1 \geq C\|u\|_1 - C\|G\|_1. \tag{16}$$

After the extension, $\|G\|_1 \leq C_2\|g\|_{\frac{1}{2}}$, $\|G\|_{-1} \leq C_3\|g\|_{\frac{1}{2}}$by (16), we can see:

$$C\|u\|_1 \leq CC_2\|g\|_{\frac{1}{2}} + \|f\|_{-1} + \|LG\|_{-1} + \|\lambda u\|_{-1} + \|\lambda C_3 g\|_{\frac{1}{2}}. \tag{17}$$

Finally, as L is eliptic, $\|LG\|_{-1} \leq C_4\|G\|_{-1}$. Above all, we can see: $\exists C>0$ such that:

$$\|u\|_{H^1(\Omega)} \leq C(\|f\|_{H^{-1}(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|u\|_{H^{-1}(\Omega)}). \tag{18}$$

$\square$

P124 Problem3

Assume the result is false, i.e. for $\lambda \notin \Lambda$, $\exists$ function sequence $\{u_n\}$ such that:

- $\|u_n\|_1 = 1$.
- $\|-Lu_n+\lambda u_n\|_{-1} < \frac{1}{n}\|u_n\|_1$.

$u_n \in C_c^\infty(\Omega)$, $C_c^\infty(\Omega)$ is a complete space, it means that $\exists u_{n_i} \to u_0$ in $C_c^\infty(\Omega)$. Then on the second condition, we can see:

$$\|-Lu_{n_i}+\lambda u_{n_i}\|_{-1} < \frac{1}{n_i}\|u_{n_i}\|_1. \tag{19}$$

Set $i\to\infty$, $RHS \to 0$, which means $-Lu_0+\lambda u_0 = 0$. Then u_0 is the solution of equation

$$\begin{cases} (L-\lambda)u = 0, x\in\Omega \\ \qquad\quad u = 0, x \in \partial\Omega. \end{cases} \tag{20}$$

Contradict! So $\exists C$ such that $\|-Lu+\lambda u\|_{-1} \geq C\|u\|_1$ is always true for $u \in C_c^\infty(\Omega)$.