Homework# 8

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P41 Problem1

(2)

引理 1. $F(e^{-|x|}) = \frac{2}{1+\xi^2}$.

证明.

$$\begin{aligned} F(e^{-|x|}) &= \int_{\mathbb{R}} e^{-|x|} e^{i\xi x} \mathrm{d}x \\ &= \int_0^{+\infty} e^{x(i\xi-1)} \mathrm{d}x + \int_{-\infty}^0 e^{x(i\xi+1)} \mathrm{d}x \\ &= \frac{2}{1+\xi^2}. \end{aligned} \tag{1}$$

□

So:

$$F(e^{-a|x|}) = \frac{2a}{a^2+\xi^2}. \tag{2}$$

Then: by inversion formula, we can see:

$$e^{-a|x|} = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2a}{a^2+\xi^2} e^{i\xi x} \mathrm{d}\xi. \tag{3}$$

set $x \leftarrow -x$, we can see:

$$F(\frac{1}{a^2+x^2}) = \frac{\pi}{a} e^{-a|\xi|}. \tag{4}$$

(4) As:

$$F(1) = 2\pi\delta, F(x) = -DF(1). \tag{5}$$

We can see:

$$F(2x^2+x+1) = \delta - D\delta + 2D^2\delta. \tag{6}$$

(6)

$$\log|x| = D(P.V.(\frac{1}{x})). \tag{7}$$

So:

$$F(\log|x|) = \xi F(P.V.(\frac{1}{x})) = -i\pi\xi\mathrm{sgn}\xi \tag{8}$$

P41 Problem3

证明. (1) $\Leftrightarrow$ (2): according to the property of Fourier transformation, we can see:

$$F(D^\alpha f) = \xi^\alpha \hat{f}(\xi). \tag{9}$$

By Theorem 3.2(Parseval equality), we can see:

$$\int_{\mathbb{R}^n} |D^\alpha f|^2 \mathrm{d}x = (2\pi)^{-n} \int_{\mathbb{R}^n} |F[D^\alpha f]|^2 \mathrm{d}x. \tag{10}$$

According to (**??**) and (**??**), we cna see (1) $\Leftrightarrow$ (2).

For (3) $\Rightarrow$ (4), set

$$h(\xi) = \begin{cases} 2, |\xi| \le 1 \\ 2|\xi|^2, |\xi| > 1. \end{cases} \tag{11}$$

we can see $(1+|\xi|^2)^{\frac{m}{2}} \le (h(\xi))^{\frac{m}{2}}$.

$\forall P(\xi)$, $P(\xi)\hat{f}(\xi) \in L^2(\mathbb{R}^n)$, so:

$$\begin{aligned} \|(1+\xi^2)^{\frac{m}{2}} \hat{f}(\xi)\|_{L^2} &\le \|\hat{f}(\xi)\|_{L^2} + \||\xi|^m \hat{f}(\xi)\|_{L^2} \\ &< +\infty. \end{aligned} \tag{12}$$

It means (3) $\Rightarrow$ (4).

Then we need to show that (4) $\Rightarrow$ (2). By mean-value inequality, for $|\alpha| \le m$, we can see:

$$|\xi^\alpha| \le (1+|\xi|^2)^{\frac{m}{2}}. \tag{13}$$

So

$$\|\xi^\alpha \hat{f}(\xi)\|_{L^2} \le \|(1+|\xi|^2)^{\frac{m}{2}} \hat{f}(\xi)\|_{L^2}. \tag{14}$$

It means that (4) $\Rightarrow$ (2). Above all, (1), (2), (3), (4) are all equivalent. □

P41 Problem4

证明. As $S \in \mathscr{S}'(\mathbb{R}^n)$, $T \in \mathscr{E}'(\mathbb{R}^n)$, we can see $T * S \in \mathscr{S}'$.

Let (α_j) be a sequence in $C_c^\infty(\mathbb{R}^n)$ converging to δ in $\mathscr{E}'(\mathbb{R}^n)$ as $j \to \infty$, we can see the sequence of functions

$$\phi_j = \alpha_j * T \in C_c^\infty \tag{15}$$

converges to T in ϵ'. Then $\phi_j * S \to T * S$ in $\mathscr{S}'$. Hence, taking Fourier transforms:

$$F[T * S] = \lim_{j\to\infty} F[\phi_j * S] = \lim_{j\to\infty} F[\phi_j]F[S]. \tag{16}$$

On the other hand, since $\phi_j \to T$ in $\mathscr{E}'$, it also converges in $\mathscr{S}'$. Hence, by Fourier transform $F[\phi_j] \to F[T]$ in $\mathscr{S}'$. So it means that

$$\lim_{j\to\infty} F[\phi_j]F[S] = F[T]F[S]. \tag{17}$$

Finally, we can see that

$$F[T * S] = F[T]F[S]. \tag{18}$$

□

P42 Problem7

(3)

$$\begin{aligned} F(xe^{-\pi y^2}) &= F_x(x)F_y(e^{-\pi y^2}) \\ &= -2\pi\delta'(\xi)e^{-\frac{\eta^2}{4\pi}} \end{aligned} \tag{19}$$

.

(4)

$$\begin{aligned} F(\delta'(x)e^{-\frac{y^2}{2}}) &= F_x(\delta'(x))F_y(e^{-\frac{y^2}{2}}) \\ &= \sqrt{2\pi}\xi e^{-\frac{|\eta|^2}{2}}. \end{aligned} \tag{20}$$

P42 Problem8

题都没看懂...

P42 Problem9

证明. By definition, $Au(x) = \int e^{i\langle x,\xi\rangle} a(x,\xi)\hat{u}(\xi)\mathrm{d}\xi$. Fourier transformation is a linear map from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^n)$, so $\hat{u}(\xi) \in \mathscr{S}(\mathbb{R}^n)$. Now we consider the expression $\partial^p Au(x)$.

We can see:

$$\int |\partial^p(e^{i\langle x,\xi\rangle} a(x,\xi)\hat{u}(\xi))|\mathrm{d}\xi \leq \int |e^{i\langle x,\xi\rangle}|\xi|^p\hat{u}(\xi)|\mathrm{d}\xi < \infty. \tag{21}$$

So, by dominent convergent theorem(DCT), we can swap the order of integral and derivative. Now, consider

$$Bu(x) = \int e^{i\langle x,\xi\rangle}|\xi|^p\hat{u}(\xi)\mathrm{d}\xi. \tag{22}$$

As $\hat{u} \in \mathscr{S}$, this integral must be convergent, and $\exists$ constant C such that $Bu(x) \leq C^p u(x)$.
$u \in \mathscr{S} \Rightarrow Au(x) \in \mathscr{S}$. $\square$