Homework# 13

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P124 Problem4

证明. If $\lambda \in \Lambda$, $\exists u \neq 0$ such that

$$\begin{cases} (L-\lambda)u = 0, x \in \Omega \\ \qquad\quad u = 0, x \in \partial\Omega \end{cases} \tag{1}$$

In Ω, $Lu = \lambda u$, it means:

$$\langle Lu, u \rangle_{L^2(\Omega)} = \langle \lambda u, u \rangle = \lambda \langle u, u \rangle . \tag{2}$$

On the other hand, for $L = L^*$, it derives the following equation:

$$\langle Lu, u \rangle = \langle u, L^* u \rangle = \langle u, Lu \rangle = \langle u, \lambda u \rangle = \bar{\lambda} \langle u, u \rangle . \tag{3}$$

As $u \neq 0$, $\langle u, u \rangle > 0$. By (3) and (2), we can see $\lambda = \bar{\lambda}$, which means that $\lambda \in \mathbb{R}$. So $\Lambda \subset \mathbb{R}$. □

P124 Problem5

For the eigen-value problem

$$\begin{cases} -\Delta u = \lambda u, x \in \Omega \\ \quad\; u = 0, x \in \partial\Omega \end{cases} \tag{4}$$

The eigen-functions $\{\omega_j\}$ forms a complete orthonormal basis on $L^2(\Omega)$. As $\langle u, \omega_1 \rangle = 0$, by Fourier extension, we can see:

$$u = \sum_{j=2}^{\infty} d_j \omega_j . \tag{5}$$

Then:

$$\frac{\langle -\Delta u, u\rangle}{\|u\|^2} = \frac{\left\langle \sum_{j=2}^{\infty} \lambda_j d_j \omega_j, \sum_{j=2}^{\infty} d_j \omega_j \right\rangle}{\sum_{j=2}^{\infty} d_j^2} = \frac{\sum_{j=2}^{\infty} \lambda_j d_j^2}{\sum_{j=2}^{\infty} d_j^2} \geq \lambda_2. \tag{6}$$

If we choose $u = \omega_2$, (6) equals to λ_2. So:

$$\lambda_2 = \inf_{u \in H_0^1(\Omega), \langle u, \omega_1\rangle = 0} \frac{\langle -\Delta u, u\rangle}{\|u\|^2}. \tag{7}$$

P135 Problem1

证明. First, we claim a lemma:

Lemma 1. *Assume $u \in C^\infty(\mathbb{R}_+^n) \cap H^{m,p}(\mathbb{R}_+^n)$, we can see:*

$$\|\nabla_h u\|_{L^p} \leq \|\frac{\partial u}{\partial x_1}\|_{L^p}. \tag{8}$$

证明. For

$$\nabla_h u = \int_0^1 \frac{\partial u}{\partial x_1}(x_1 + \lambda h, \cdot) \mathrm{d}\lambda. \tag{9}$$

We can see:

$$\int_{\mathbb{R}_+^n} |\nabla_h u|^p \leq \int_0^1 \mathrm{d}\lambda \int_{\mathbb{R}_+^n} |\frac{\partial u}{\partial x_1}(x_1 + \lambda h, \cdot)|^p \mathrm{d}x. \tag{10}$$

Then:

$$\|\nabla_h u\|_{L^p} \leq \|\frac{\partial u}{\partial x_1}\|_{L^p}. \tag{11}$$

□

Back to this theorem. It's clear to see that: $\forall u \in C^\infty(\mathbb{R}_+^n)$, $\forall$ multiple index α, the following is true:

$$\partial^\alpha \nabla_h u = \nabla_h \partial^\alpha u. \tag{12}$$

If $\alpha = (k_1, \cdots, k_n)$, we define $\beta = (k_1 + 1, \cdots, k_n)$. By (8),

$$\|\partial^\alpha \nabla_h u\|_p \leq \|\partial^\beta u\|_p. \tag{13}$$

Then:

$$\begin{aligned}\|\nabla_h u\|_{H^{m-1,p}} &= \sum_{|\alpha| \leq m-1} \|\partial^\alpha \nabla_h u\|_p \\ &\leq \sum_{|\alpha| \leq m-1} \|\partial^\beta u\|_p \\ &\leq \sum_{|\beta| \leq m} \|\partial^\beta u\|_p \\ &= \|u\|_{H^{m,p}}.\end{aligned} \tag{14}$$

□

P135 Problem2

Claim: the result is true for $-1 \leq k \leq m$.