Homework# 5

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P88 Problem16

证明. For $u_t - \Delta u = 0$, set $u_\epsilon = u - \epsilon t$, we can see:

$$\frac{\partial u_\epsilon}{\partial t} - \Delta u_\epsilon = u_t - \epsilon - \Delta u = -\epsilon < 0. \tag{1}$$

If u_ϵ attains its maximum in U_T, assume the maximum point is $(\mathbf{x}_0, t_0)$, we can see:

$$\left.\begin{aligned} \frac{\partial u_\epsilon}{\partial t}(\mathbf{x}_0, t_0) = 0 \\ \Delta(\mathbf{x}_0, t_0) \le 0 \end{aligned}\right\} \Rightarrow \frac{\partial u_\epsilon}{\partial t} - \Delta u_\epsilon \ge 0. \tag{2}$$

Contradict! So u_ϵ gets its maximal on Γ_T. It means that:

$$u_\epsilon(\mathbf{x}, t) = u - \epsilon t \le \max_{\Gamma_T} u. \tag{3}$$

Set $\epsilon \to 0$, we can see $u \le \max_{\Gamma_T} u$ is always true. □

P88 Problem17

(a)

证明. Modify the proof of theorem 3. Put v instead of u in the proof. We know that:

$$\phi'(r) = A + B = \frac{1}{r^{n+1}} \int_{E(r)} -4nv_s\psi - \frac{2n}{s} \sum_{i=1}^{n} v_{y_i} y_i \mathrm{d}y\mathrm{d}s. \tag{4}$$

Since ψ defined to be

$$\psi = -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log(r). \tag{5}$$

$\psi \geq 0$ in $E(r)$ because $\Phi(y,-s)r^n \geq 1$ in $E(r)$. Thus $4n\psi(v_s - \Delta v) \leqslant 0$, $-4n\psi v_s \geq -4n\Delta v$. Then we have inequality:

$$\begin{aligned}\phi'(r) &= \frac{1}{r^{n+1}}\int_{E(r)} -4nv_s\psi - \frac{2n}{s}\sum_{i=1}^{n} v_{y_i}y_i \mathrm{d}y\mathrm{d}s \\ &\geq \frac{1}{r^{n+1}}\int_{E(r)} -4n\Delta v\psi - \frac{2n}{s}\sum_{i=1}^{n} v_{y_i}y_i \mathrm{d}y\mathrm{d}s \\ &= 0\end{aligned} \tag{6}$$

according to ghe proof of theorem 3. So we have:

$$\phi(r) \geq \phi(\epsilon)\forall r > \epsilon > 0. \tag{7}$$

But we know:

$$\lim_{\epsilon\to 0}\phi(\epsilon) = 4v(0,0). \tag{8}$$

So we have the inequality:

$$\frac{1}{r_n}\int_{E(r)} v(y,s)\frac{|y|^2}{s^2}\mathrm{d}y\mathrm{d}s = \phi(r) \geq 4v(0,0). \tag{9}$$

WLOG, we have:

$$\frac{1}{4r^n}\int_{E(x,t;r)} v(y,s)\frac{|x-y|^2}{(t-s)^2}\mathrm{d}y\mathrm{d}s \geq v(x,t). \tag{10}$$

□

(b)

证明. Define a set

$$S := \{(\mathbf{x},t) : v(\mathbf{x},t) = \max_{\bar{U}_T} v\}. \tag{11}$$

As v is continuous, S is a relative closed set. On the other hand, choose $(\mathbf{x},t) \in S$, by (a), we can see:

$$\frac{1}{4r^n}\int_{E(\mathbf{x},t;r)} (v_{max} - v)\frac{|x-y|^2}{(t-s)^2}\mathrm{d}y\mathrm{d}s \leqslant 0. \tag{12}$$

As $\frac{|x-y|^2}{(t-s)^2} \geq 0$, (12) means that $v = v_{max}$ in E, which means that S is an open set. For U_T is a region, we can see $S = U_T$. □

(c)

$$\begin{aligned} v_t &= \phi'(u)u_t \\ v_{x_i} &= \phi'(u)u_{x_i} \\ v_{x_ix_i} &= \phi''(u)(u_{x_i})^2 + \phi'(u)u_{x_ix_i}. \end{aligned} \tag{13}$$

It means that:

$$v_t - \Delta v = -\phi''(u)\sum_{i=1}^{n}(u_{x_i})^2. \tag{14}$$

As ϕ convex, $\phi''(u) \geq 0$, which means $v_t - \Delta v \leqslant 0$.

(d)

$$\begin{aligned} v_t &= 2u_t u_{tt} + \sum_{i=1}^{n} 2u_i u_{it} \\ v_i &= 2\sum_{j=1}^{n} u_{ij}u_j + 2u_t u_{it} \\ v_{ii} &= 2\sum_{j=1}^{n} u_{iij}u_j + 2\sum_{j=1}^{n}(u_{ij})^2 + 2(u_{it})^2 + 2u_t u_{iit} \end{aligned} \tag{15}$$

It means:

$$v_t - \Delta v = 2u_t(u_{tt} - \Delta u_t) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} u_j(u_{jt} - u_{iij}) - 2\sum_{j=1}^{n}(u_{ij})^2 \leqslant 0. \tag{16}$$

So v is a subsolution.

Modern PDE

In the following problems, we set the notation:

$$\alpha(x) = \begin{cases} e^{\frac{1}{|x|^2-1}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \tag{17}$$

and $\alpha_\epsilon(x) = \frac{1}{\epsilon^n}\alpha(\frac{x}{\epsilon})$.

定义 1. *The convolution of function f and g which is defined on Ω is:*

$$f * g(x) = \int_\Omega f(y)g(x-y)\mathrm{d}y = \int_\Omega f(x-y)g(y)\mathrm{d}y \tag{18}$$

P9 Problem1

Get $u \in C^0(\mathbb{R}^n)$, set $u_\epsilon = u * \alpha_\epsilon$, by theorem 1.1, we can see:

$$\lim_{\epsilon\to 0} u_\epsilon = u \text{ in } C^0(\mathbb{R}^n). \tag{19}$$

We just choose the sequence $f_n = u_{\frac{1}{n}}$, u and $\alpha_{\frac{1}{n}}$ both have compact support, so $f_n \in C_c^\infty(\mathbb{R}^n)$, and by (19), we can see

$$f_n \rightrightarrows u. \tag{20}$$

which means $C_c^\infty(\mathbb{R}^n)$ is dense in $C^0(\mathbb{R}^n)$.

If $u \in L^p(\mathbb{R}^n)$, by Riesz theorem, $\forall \frac{1}{n} > 0$, $\exists v_n$ s.t. $\|u - v_n\|_p < \frac{1}{n}$. By theorem 1.1, the following equation is true:

$$\lim_{\epsilon \to 0} v_{n\epsilon} = v_n(L^p(\mathbb{R}^n)). \tag{21}$$

Set $v_{nm} = v_n * \alpha_{\frac{1}{m}}$, we can see

$$\lim_{m \to \infty, n \to \infty} v_{nm} = u. \tag{22}$$

So C_c^∞ is dense in $L^p(\mathbb{R}^n)$.

P9 Problem2

$\forall$ compact set K and multiple index β, consider:

$$\|\partial^\beta(J_\epsilon u - u)\| \leqslant \int_{\|y\| \leqslant \epsilon} \|\partial^\beta u(x-y) - \partial^\beta u(x)\| \alpha_\epsilon(y) \mathrm{d}y. \tag{23}$$

As $u \in C^\infty(\mathbb{R}^n)$, we can see $\partial^\beta u \in C^\infty(\mathbb{R}^n)$, which means $\partial^\alpha u$ is uniform continuous in K. It means:

$$\forall \epsilon > 0, \exists \delta, \forall |y| < \delta, |u(x-y) - u(x)| \leqslant \epsilon \Rightarrow |J_\delta u - u| \leqslant \epsilon. \tag{24}$$

So $J_\epsilon u \to u(C^\infty(\mathbb{R}^n))$.

If $u \in C_c^\infty(\mathbb{R}^n)$, assume K is the compact support set of u, then the support set of u_ϵ must be a subset of

$$K_\epsilon = \{x : \exists y \in K s.t. |x - y| \leqslant \epsilon\}. \tag{25}$$

Then $\forall \epsilon \leqslant 1$, $K_\epsilon \subset K_1$. In the same way, in K_1, $\sup_{x \in K} |\partial^\alpha(u_\epsilon - u)| \to 0$.

P10 Problem3

Mark the consistent compact support set as K, all the regular points in K is

$$\{q_1, q_2, \cdots, q_n, \cdots\}. \tag{26}$$

Then $\forall$ i, $\{\phi_m(q_i)\}$ is a Cauchy sequence in $\mathbb{R}$, so we can set $\phi(q_i) = \lim_{m \to \infty} \phi_m(q_i)$. Moreover, as the definition of Cauchy sequence, the convergence of $\{\phi_m(q_i)\}$ is consistent, i.e. $\forall \epsilon > 0$, $\exists N$, $\forall m > N$ we have $|\phi_m(q_i) - \phi(q_i)| < \epsilon$ $\forall$ i.

For irregular point $\mathbf{x}$ in K, there exists a sequence of regular points such that $\lim_{j\to\infty} q_{m_j} = \mathbf{x}$. Consider the sequence $a_j = \phi(q_{m_j})$, claim a_j is a Cauchy sequence.

For $j_1 \neq j_2$, $\forall n \in \mathbb{N}$, we have:

$$|\phi(q_{m_{j_1}})-\phi(q_{m_{j_2}})| \leqslant |\phi(q_{m_{j_1}})-\phi_n(q_{m_{j_1}})|+|\phi_n(q_{m_{j_1}})-\phi_n(q_{m_{j_2}})|+|\phi(q_{m_{j_2}})-\phi_n(q_{m_{j_2}})|. \tag{27}$$

As $\phi_n(q_i)$ uniformly converge, and ϕ_n is continuous, we can see $\{\phi(q_{m_i})\}$ is a Cauchy sequence, which means that $\phi(x) := \lim_{j\to\infty} \phi(q_{m_j})$ is well-defined.

By such definition, we can see $\phi(x)$ is continuous.

Then, we show that $\phi_n \rightrightarrows \phi$ in K. As ϕ continuous in K, $\forall \epsilon > 0$, $\exists$ $\delta_1(x)$, $\forall |x - x_0| < \delta_1$, $|\phi(x) - \phi(x_0)| < \epsilon$.

As $\phi_n(q_i)$ is uniformly converge, $\exists N$, $\forall n > N$, $|\phi(q_i) - \phi_n(q_i)| < \epsilon$.

As $\phi_n(x)$ is continuous, $\exists \delta_2(n, x)$ s.t. $\forall |x - x_1| < \delta_2$, $|\phi_n(x) - \phi_n(x_1)| < \epsilon$.

Regular number is dense in $\mathbb{R}$. So $\forall x \in K$, $\exists q_i$ s.t. $|x - q_i| < \delta_1, |x - q_i| < \delta_2$.

Summery the above cases, $\forall n > N$, we can see:

$$\begin{aligned} |\phi(x) - \phi_n(x)| &\leq |\phi(x) - \phi(q_i)| + |\phi(q_i) - \phi_n(q_i)| + |\phi_n(q_i) - \phi_n(x)| \\ &\leq 3\epsilon. \end{aligned} \tag{28}$$

Which means that $\phi_n \rightrightarrows \phi$ in K. As $\phi_n \in C_c^\infty(\mathbb{R}^n)$, we can see $\phi_m \to \phi(C_c^\infty(\mathbb{R}^n))$.