Homework# 13

Shuang Hu(26)

2022 年 5 月 21 日

P124 Problem4

证明. If $\lambda \in \Lambda$, $\exists u \neq 0$ such that

$$\begin{cases} (L-\lambda)u = 0, x \in \Omega \\ u = 0, x \in \partial\Omega \end{cases} \tag{1}$$

In Ω, $Lu = \lambda u$, it means:

$$\langle Lu, u \rangle_{L^2(\Omega)} = \langle \lambda u, u \rangle = \lambda \langle u, u \rangle . \tag{2}$$

On the other hand, for $L = L^*$, it derives the following equation:

$$\langle Lu, u \rangle = \langle u, L^* u \rangle = \langle u, Lu \rangle = \langle u, \lambda u \rangle = \bar{\lambda} \langle u, u \rangle . \tag{3}$$

As $u \neq 0$, $\langle u, u \rangle > 0$. By (3) and (2), we can see $\lambda = \bar{\lambda}$, which means that $\lambda \in \mathbb{R}$. So $\Lambda \subset \mathbb{R}$. □

P124 Problem5

For the eigen-value problem

$$\begin{cases} -\Delta u = \lambda u, x \in \Omega \\ u = 0, x \in \partial\Omega \end{cases} \tag{4}$$

The eigen-functions $\{\omega_j\}$ forms a complete orthonormal basis on $L^2(\Omega)$. As $\langle u, \omega_1 \rangle = 0$, by Fourier extension, we can see:

$$u = \sum_{j=2}^{\infty} d_j \omega_j . \tag{5}$$

Then:

$$\frac{\langle -\Delta u, u\rangle}{\|u\|^2} = \frac{\left\langle \sum_{j=2}^{\infty} \lambda_j d_j \omega_j, \sum_{j=2}^{\infty} d_j \omega_j \right\rangle}{\sum_{j=2}^{\infty} d_j^2} = \frac{\sum_{j=2}^{\infty} \lambda_j d_j^2}{\sum_{j=2}^{\infty} d_j^2} \geq \lambda_2. \tag{6}$$

If we choose $u = \omega_2$, (6) equals to λ_2. So:

$$\lambda_2 = \inf_{u \in H_0^1(\Omega), \langle u, \omega_1\rangle = 0} \frac{\langle -\Delta u, u\rangle}{\|u\|^2}. \tag{7}$$