Homework# 4

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P87 Problem12

(a)

$$\left.\begin{aligned}\frac{\partial u_\lambda}{\partial t} &= \lambda^2\frac{\partial u}{\partial t}(\lambda x, \lambda^2 t)\\ \frac{\partial^2 u_\lambda}{\partial x_i^2} &= \lambda^2\frac{\partial^2 u}{\partial x_i^2}(\lambda x, \lambda^2 t)\end{aligned}\right\} \Rightarrow \frac{\partial u_\lambda}{\partial t} - \Delta u_\lambda = \lambda^2(\frac{\partial u}{\partial t} - \Delta u) = 0. \tag{1}$$

It means that u_λ solves the heat equation.

(b)

$$\frac{\mathrm{d}u_\lambda}{\mathrm{d}\lambda} = x \cdot Du(\lambda x, \lambda^2 t) + 2\lambda t u_t(x,t) \xrightarrow{\lambda\to 1} v(x,t). \tag{2}$$

As $\forall\lambda$, u_λ satisfies the heat equation, we can see $\frac{u_{\lambda+\Delta\lambda}-u_\lambda}{\Delta\lambda}$ satisfies heat equation for all $\Delta\lambda$. It means that v satisfies the heat equation.

P87 Problem13

(a) We can see:

$$\begin{aligned}\frac{\partial u}{\partial t} &= v'(z)\frac{\partial(\frac{x}{\sqrt{t}})}{\partial t} = -\frac{1}{2}xt^{-\frac{3}{2}}v'(z)\\ \frac{\partial u}{\partial x} &= v'(z)\frac{1}{\sqrt{t}}\\ \frac{\partial^2 u}{\partial x^2} &= v''(z)\frac{1}{t}\end{aligned} \tag{3}$$

So:

$$\begin{aligned}&u_{xx} = u_t\\ \Leftrightarrow& -\frac{1}{2}v'(z)xt^{-\frac{3}{2}} = v''(z)t^{-1}\\ \Leftrightarrow& v''(z)t^{-1} + \frac{1}{2}v'(z)xt^{-\frac{3}{2}} = 0\\ \Leftrightarrow& v''(z) + \frac{z}{2}v'(z) = 0\end{aligned} \tag{4}$$

To get the general solution for 4, we can see:

$$\begin{aligned} v'' + \frac{z}{2}v' = 0 &\Rightarrow \frac{\mathrm{d}v'}{\mathrm{d}z} + \frac{z}{2}v' = 0 \\ &\Rightarrow \frac{\mathrm{d}v'}{v'} + \frac{z\mathrm{d}z}{2} = 0 \\ &\Rightarrow \mathrm{d}\log(v') + \mathrm{d}\frac{z^2}{4} = 0 \\ &\Rightarrow \frac{z^2}{4} + \log(v') \equiv C \\ &\Rightarrow v' = e^{C - \frac{z^2}{4}} \\ &\Rightarrow v(z) = c\int_0^z e^{-\frac{s^2}{4}}\mathrm{d}s + d. \end{aligned} \tag{5}$$

(b) As (a) suggests, for $t > 0$, we can see:

$$u(x,t) = v(z) = c\int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{s^2}{4}}\mathrm{d}s + d = c\int_0^x e^{-\frac{w^2}{4t}}\frac{1}{\sqrt{t}}\mathrm{d}w + d. \tag{6}$$

So:

$$\frac{\partial u}{\partial x} = \frac{c}{\sqrt{t}}e^{-\frac{x^2}{4t}}. \tag{7}$$

Just set $c = \frac{1}{\sqrt{4\pi}}$, we get the foundamental solution Φ.

Then I should explain the reason of this phinomenon. By the definition $u(x,t) = v(\frac{x}{\sqrt{t}})$, we can see:

$$u(x,0) = \begin{cases} \lim\limits_{z\to+\infty} v(z) = \sqrt{\pi}c + d, x > 0 \\ v(0) = d, x = 0 \\ \lim\limits_{z\to-\infty} v(z) = d - \sqrt{\pi}c, x < 0 \end{cases} \tag{8}$$

So:$\frac{\partial u(x,0)}{\partial x} = \delta_0(x)$. It shows that this solution is the foundamental solution.

P87 Problem14

Define $v(x,t) = u(x,t)e^{ct}$, then:

$$\begin{cases} v_t - \Delta v = e^{ct}f \text{ in } \mathbb{R}^n \times (0,\infty) \\ v = g \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{9}$$

So:

$$v(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\mathrm{d}y + \int_0^t\int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)e^{cs}\mathrm{d}y\mathrm{d}s. \tag{10}$$

And $u(x,t) = e^{-ct}v(x,t)$.

P87 Problem15

Set $v(x,t) = u(x,t) - g(t)$, then we can see:

$$\begin{cases} v_t - v_{xx} = -g'(t) \text{ in } \mathbb{R}_+ \times (0,\infty) \\ v(x,0) = 0 \\ v(0,t) = 0 \end{cases} \tag{11}$$

Then extend v to $x < 0$: just set $v(x,t) = -v(-x,t)$ on $x < 0$, as the equation satisfies the compatible condition, this extension is okay. We can see v satisfies:

$$\begin{cases} v_t - v_{xx} = f(x,t) \\ v(x,0) = 0 \end{cases} \tag{12}$$

where $f(x,t) = -g'(t)$ if $x \geq 0$, and $f(x,t) = g'(t)$ if $x < 0$. Solve this equation, we can see

$$u(x,t) = v(x,t) + g(t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{\frac{3}{2}}} e^{-\frac{x^2}{4(t-s)}} g(s)\mathrm{d}s. \tag{13}$$

P89 Problem22

$$\begin{cases} u_t + u_x = d(v-u) \\ v_t - v_x = d(u-v) \end{cases} \Rightarrow u_{tt} + u_{tx} = d(v_t - u_t), u_{tx} + u_{xx} = d(v_x - u_x)$$
$$\Rightarrow u_{tt} - u_{xx} = d(v_t - u_t - v_x + u_x) \tag{14}$$

What's more, $v_t - v_x = -(u_t + u_x)$, so $u_{tt} - u_{xx} = -2du_t$. In the same way, $v_{tt} + 2dv_t - v_{xx} = 0$.

P90 Problem24

(a)

$$\begin{aligned} \frac{\mathrm{d}k+p}{\mathrm{d}t} &= \int_{-\infty}^{+\infty} (u_t u_{tt} + u_x u_{xt})\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} (u_t u_{xx} + u_x u_{xt})\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} \frac{\partial u_t u_x}{\partial x}\mathrm{d}x \\ &= 0. \end{aligned} \tag{15}$$

The final equation is true for g and h both have compact support.

(b) By D.Almbert's formula, we can see:

$$u(x,t) = \frac{g(x+t)+g(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\mathrm{d}y. \tag{16}$$

It means that:

$$\begin{aligned} u_x &= \frac{g'(x+t)+g'(x-t)}{2} + \frac{1}{2}(h(x+t)-h(x-t)). \\ u_t &= \frac{g'(x+t)-g'(x-t)}{2} + \frac{1}{2}(h(x+t)+h(x-t)). \end{aligned} \tag{17}$$

As g, h both have compact support, we can choose large t such that $u_x^2 = u_t^2 \forall x$, which means $k(t) = p(t)$.