Homework# 1

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P86 Problem7

证明. Consider the equation

$$\begin{cases} \Delta u = 0 \text{ in } B(0,r) \\ u = g \text{ on } \partial B(0,r), \end{cases} \tag{1}$$

where $g \geq 0$ is always true. Then by Poisson's formula, we can see:

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} \mathrm{d}S(y). \tag{2}$$

The left-hand inequality means:

$$\begin{aligned} & r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq \frac{(r^2-|x|^2)}{n\alpha(n)r}\int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\mathrm{d}S(y) \\ \Leftrightarrow & n\alpha(n)r^{n-1}u(0) \leq (r+|x|)^n \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\mathrm{d}S(y) \\ \Leftrightarrow & \int_{\partial B(0,r)} g(y)\mathrm{d}S(y) \leq (r+|x|)^n \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\mathrm{d}S(y) \end{aligned} \tag{3}$$

$|y| = r$, and $g(y) \geq 0$, so:$(r+|x|)^n \geq |x-y|^n$, which means the inequality is true.

In the same way, we can see the right hand is equal to the inequality

$$(r-|x|)^n \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\mathrm{d}S(y) \leq \int_{\partial B(0,r)} g(y)\mathrm{d}S(y). \tag{4}$$

As $(r-|x|)^n \leq |x-y|^n$, the inequality is true. □

P86 Problem8

证明. (1) As $y \in \partial B(0,r)$, we can see:

$$K(x,y) \in C^\infty(B^0(0,r))\forall y \in \partial B(0,r).$$

it means that $u(x) \in C^{\infty}(B^0(0,r))$.

(2) As $K(x,y)$ is smooth related to x, we can see:

$$\begin{aligned}\Delta u &= \Delta \int_{\partial B(0,r)} K(x,y)g(y)\mathrm{d}S(y) \\ &= \int_{\partial B(0,r)} \Delta_x K(x,y)g(y)\mathrm{d}S(y) \\ &= 0.\end{aligned} \tag{5}$$

(3) Set $g \equiv 1$, we can see:

$$\int_{\partial B(0,r)} K(x,y)\mathrm{d}S(y) = 1. \tag{6}$$

Now fix $x_0 \in \partial B(0,r)$, $\epsilon > 0$, choose $\delta > 0$ s.t. $|g(y) - g(x_0)| < \epsilon$ if $|y - x_0| < \delta$. Then:

$$\begin{aligned}|u(x) - g(x_0)| &= |\int_{\partial B(0,r)} K(x,y)(g(y) - g(x_0))\mathrm{d}S(y)| \\ &\le \int_{\partial B(0,r)\cap B(x_0,\delta)} K(x,y)|g(y) - g(x_0)|\mathrm{d}S(y) \\ &+ \int_{\partial B(0,r)\setminus B(x_0,\delta)} K(x,y)|g(y) - g(x_0)|\mathrm{d}S(y) \\ &< \epsilon + 2\|g\|_{\infty} \int_{\partial B(0,r)\setminus B(x_0,\delta)} K(x,y)\mathrm{d}S(y)\end{aligned} \tag{7}$$

Set $x \to x_0$, we can see $RHS \to \epsilon$. It means that $u(x) \to g(x_0)$ if $x \to x_0$. □

P86 Problem9

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\mathrm{d}y \Rightarrow u(\lambda e_n) = \frac{2\lambda}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{(\sqrt{\lambda^2 + |y|^2})^n}\mathrm{d}y \tag{8}$$

For $u(0) = 0$, we can see:

$$\frac{u(\lambda e_n) - u(0)}{\lambda} = \frac{2}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{(\sqrt{\lambda^2 + |y|^2})^n}\mathrm{d}y \tag{9}$$

For $g(y)$ is bounded, we can see the integral

$$\frac{2}{n\alpha(n)} \int_{|y|\ge 1} \frac{g(y)}{(\sqrt{\lambda^2 + |y|^2})^n}\mathrm{d}y \tag{10}$$

is convergent. What's more:

$$\begin{aligned}&\frac{2}{n\alpha(n)}\int_{|y|\leq 1}\frac{g(y)}{(\sqrt{\lambda^2+|y|^2})^n}\mathrm{d}y\\ =&\frac{2}{n\alpha(n)}\int_{|y|\leq 1}\frac{|y|}{(\sqrt{\lambda^2+|y|^2})^n}\mathrm{d}y\\ \to&+\infty\end{aligned} \tag{11}$$

when $\lambda \to 0^+$. So we can see Du isn't bounded.

P86 Problem10

证明. (a) Just derive the Laplacian of function v, we can see $\Delta v \equiv 0$, which means v is harmonic.

(b) It suffices to show that the mean value property is true for $x \in \partial U^+$. $\forall B(x,\delta) \subset U^+$, consider the integral

$$I = \int_{\partial B(x,\delta)} v(y)\mathrm{d}S(y).$$

By the definition, we can see $v(x_1, \cdots, x_n) = -v(x_1, \cdots, -x_n)$. So we can see that $I = 0$, which means the mean value property is true for the function v in U^+. □

P87 Problem11

Let $\Omega \subset \mathbb{R}^n$ be an open subset. If $0 \notin \Omega$, we denote $x^* = \frac{x}{|x|^2}$, $\Omega^* = \{x^* | x \in \Omega\}$. For a function u defined on Ω, we define its Kelvin transformation by $K[u](x) = |x|^{2-n}u(x^*)$, $x \in \Omega^*$. Prove that u is harmonic at Ω if and only if $K[u]$ is harmonic at Ω^*.

证明. Note that

$$\frac{\partial |x|}{\partial x_i} = \frac{x_i}{|x|}$$

$$\frac{\partial x_j^*}{\partial x_i} = x_j \cdot (-2)|x|^{-3}\frac{\partial |x|}{\partial x_i} = -2x_i x_j |x|^{-4} \quad (j \neq i)$$

$$\frac{\partial x_i^*}{\partial x_i} = |x|^{-2} - 2x_i^2|x|^{-4}.$$

$$\frac{\partial K[u]}{\partial x_i} = (2-n)|x|^{1-n}\frac{\partial |x|}{\partial x_i}u(x^*) + |x|^{2-n}\sum_{j=1}^{n} u_j(x^*)\frac{\partial x_j^*}{\partial x_i}$$
$$= (2-n)|x|^{-n}x_i u(x^*) - 2|x|^{-2-n}\sum_{j=1}^{n} u_j(x^*)x_i x_j + |x|^{-n}u_i(x^*).$$

$$\frac{\partial^2 K[u]}{\partial x_i^2} = n(n-2)|x|^{-n-1}\frac{\partial |x|}{\partial x_i}x_i u(x^*) + (2-n)|x|^{-n}u(x^*) + (2-n)|x|^{-n}x_i\sum_{j=1}^{n} u_j\frac{\partial x_j^*}{\partial x_i} +$$
$$2(n+2)|x|^{-n-3}\frac{\partial |x|}{\partial x_i}\sum_{j=1}^{n} u_j x_i x_j - 2|x|^{-n-2}\sum_{j,k=1}^{n} u_{jk}\frac{\partial x_k^*}{\partial x_i}x_i x_j - 2|x|^{-n-2}\sum_{j=1}^{n} u_j x_j -$$
$$2|x|^{-n-2}u_i x_i - n|x|^{-n-1}\frac{\partial |x|}{\partial x_i}u_i + |x|^{-n}\sum_{m=1}^{n} u_{im}\frac{\partial x_m^*}{\partial x_i}$$
$$= n(n-2)|x|^{-n-2}x_i^2 u(x^*) + (2-n)|x|^{-n}u(x^*) -$$
$$2(2-n)|x|^{-n-4}x_i^2\sum_{j=1}^{n} u_j x_j + (2-n)|x|^{-n-2}u_i x_i +$$
$$2(n+2)|x|^{-n-4}x_i^2\sum_{j=1}^{n} u_j x_j + 4|x|^{-n-6}\sum_{j,k=1}^{n} u_{jk}x_i^2 x_j x_k -$$
$$2|x|^{-n-4}\sum_{j=1}^{n} u_{ij}x_i x_j - 2|x|^{-n-2}\sum_{j=1}^{n} u_j x_j -$$
$$2|x|^{-n-2}u_i x_i - n|x|^{-n-2}u_i x_i + |x|^{-n-2}u_{ii} - 2|x|^{-n-4}\sum_{m=1}^{n} u_{im}x_i x_m$$
$$= n(n-2)|x|^{-n-2}x_i^2 u(x^*) + (2-n)|x|^{-n}u(x^*) +$$
$$4n|x|^{-n-4}x_i^2\sum_{j=1}^{n} u_j x_j + 4|x|^{-n-6}\sum_{j,k=1}^{n} u_{jk}x_i^2 x_j x_k -$$
$$2|x|^{-n-4}\sum_{j=1}^{n} u_{ij}x_i x_j - 2|x|^{-n-2}\sum_{j=1}^{n} u_j x_j - 2n|x|^{-n-2}u_i x_i +$$
$$|x|^{-n-2}u_{ii} - 2|x|^{-n-4}\sum_{m=1}^{n} u_{im}x_i x_m$$

$$
\begin{aligned}
\Delta K[u] &= \sum_{i=1}^{n} \frac{\partial^2 K[u]}{\partial x_i^2} \\
&= 4|x|^{-n-6} \sum_{i,j,k=1}^{n} u_{jk} x_i^2 x_j x_k - 2|x|^{-n-4} \sum_{i,j=1}^{n} u_{ij} x_i x_j - 2|x|^{-n-4} \sum_{i,m=1}^{n} u_{im} x_i x_m + |x|^{-n-2} \Delta u \\
&= 4|x|^{-n-4} \sum_{j,k=1}^{n} u_{jk} x_j x_k - 2|x|^{-n-4} \sum_{i,j=1}^{n} u_{ij} x_i x_j - 2|x|^{-n-4} \sum_{i,m=1}^{n} u_{im} x_i x_m + |x|^{-n-2} \Delta u \\
&= |x|^{-n-2} \Delta u.
\end{aligned}
$$

Hence u is harmonic if and only if $K[u]$ is harmonic. $\square$