Homework# 14

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P150 Problem2

证明. u_t times Lu, then integral it on Q_t, we can see:

$$\int_{Q_t} u_t Lu \mathrm{d}x\mathrm{d}t = I_1(t) + I_2(t). \tag{1}$$

While

$$I_1(t) = -\int_{Q_t} \left(\sum_i b_i \frac{\partial u}{\partial x_i} + cu\right) u_t \mathrm{d}x\mathrm{d}t, \tag{2}$$

$$I_2(t) = \int_{Q_t} \left(\frac{\partial^2 u}{\partial t^2} - \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)\right) u_t \mathrm{d}x\mathrm{d}t. \tag{3}$$

Then: set the vector function

$$F(x) = \left(\sum a_{1j}\frac{\partial u}{\partial x_j}, \cdots, \sum a_{nj}\frac{\partial u}{\partial x_j}\right) \tag{4}$$

By (a_{ij}) elliptic, we can see:

$$\begin{aligned}
&-\int_0^t \int_\Omega \nabla \cdot F \mathrm{d}x\mathrm{d}t \\
=&\int_0^t \int_\Omega \sum_{i,j} \frac{\partial^2 u}{\partial x_i t} a_{ij} \frac{\partial u}{\partial x_j} \mathrm{d}x\mathrm{d}t - \int_0^t \int_{\partial\Omega} u_t F \cdot n \mathrm{d}S\mathrm{d}t. \\
\leq &\int_0^t \int_\Omega \sum_{i,j} \frac{\partial^2 u}{\partial x_i t} a_{ij} \frac{\partial u}{\partial x_j} \mathrm{d}x\mathrm{d}t + \sigma\alpha \int_0^t \int_{\partial\Omega} u u_t \mathrm{d}S\mathrm{d}t.
\end{aligned} \tag{5}$$

If we denote $A = (a_{ij})$, the second inequality is derived from:

$$(A\begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n}\end{bmatrix}) \cdot \mathbf{n} = \begin{bmatrix} \frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u}{\partial x_n}\end{bmatrix} A\mathbf{n} \geq \alpha \frac{\partial u}{\partial \mathbf{n}} = -\sigma\alpha\mathbf{n}. \tag{6}$$

Then by P145 Theorem 1.1, we can see that:

$$\begin{aligned} I_2(t) \leq & \frac{1}{2}\int_0^t \int_\Omega \frac{\partial}{\partial t}\left(\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i,j} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)\mathrm{d}x\mathrm{d}t \\ & - \frac{1}{2}\int_0^t \int_\Omega \sum_{i,j}\frac{\partial a_{ij}}{\partial t}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\mathrm{d}x\mathrm{d}t + \sigma\alpha\int_0^t \int_{\partial\Omega} uu_t \mathrm{d}S\mathrm{d}t \end{aligned} \tag{7}$$

On the other hand:

$$\int_0^t \int_{\partial\Omega} uu_t \mathrm{d}S\mathrm{d}t = \frac{1}{2}(\int_{\partial\Omega} u^2(\mathbf{x},t)\mathrm{d}S - \int_{\partial\Omega} u^2(\mathbf{x},0)\mathrm{d}S) \tag{8}$$

Define the energy norm

$$E(t) = \int_\Omega (u^2 + u_t^2 + \sum u_{x_i}^2)\mathrm{d}x + \int_{\partial\Omega} u^2 \mathrm{d}S. \tag{9}$$

We can see:

$$\int_0^t \int_\Omega u_t L u \mathrm{d}x\mathrm{d}t = \frac{1}{2}\left[\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i,j} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \int_{\partial\Omega} u^2 \mathrm{d}S\right]_{t=0}^{t=t} + \tilde{I}_1(t). \tag{10}$$

Where:

$$|\tilde{I}_1(t)| \leq C\int_0^t E(\tau)\mathrm{d}\tau. \tag{11}$$

Then, by the same tragedy with the proof of Theorem 1.1, we can derive the enermy inequality as:

$$E(t) \leq C(E(0) + \int_{Q_t} f^2 \mathrm{d}x\mathrm{d}t). \tag{12}$$

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P150 Problem3

As $I(t) \in C^2(\mathbb{R})$, first we induce the following derivatives:

$$(e^{\lambda t}I(t))' = e^{\lambda t}(I'(t) + \lambda I(t)). \tag{13}$$

$$(e^{\lambda t}I(t))'' = e^{\lambda t}(\lambda^2 I(t) + 2\lambda I'(t) + I''(t)). \tag{14}$$

Then we can see:

$$(e^{\lambda t}I(t))'' + k(e^{\lambda t}I(t))' = e^{\lambda t}(I''(t) + (2\lambda + k)I'(t) + (\lambda^2 + \lambda k)I(t)). \tag{15}$$

If we set:

$$\begin{cases} 2\lambda + k = -C_1 \\ \lambda^2 + \lambda k = -C_2 \end{cases} \tag{16}$$

by the condition, we can derive that

$$(e^{\lambda t}I(t))'' + k(e^{\lambda t}I(t))' \le e^{\lambda t}M. \tag{17}$$

Then set $h(t) = e^{kt}(e^{\lambda t}I(t))'$, we can derive that $h'(t) \le e^{(\lambda+k)t}M$.

As $\lim_{t\to-\infty} h(t) = 0$, we can see that $h(t) \le \frac{e^{(\lambda+k)t}}{\lambda+k}M$, so:

$$(e^{\lambda t}I(t))' \le \frac{e^{\lambda t}}{\lambda+k}M \tag{18}$$

Then:

$$e^{\lambda t}I(t) \le \frac{e^{\lambda t}}{\lambda(\lambda+k)}M \Rightarrow I(t) \le \frac{M}{\lambda(\lambda+k)}. \tag{19}$$

By (16), $I(t) \le -\frac{M}{C_2}$. If $C_2 \ge 0$, there is no such $I(t)$.

P155 Problem1

证明. First, consider the evaluation of $\|\partial_t u\|_{r-1}^2$, by (2.1), we can see:

$$\|\partial_t u(h)\|_{r-1}^2 \le C_{r-1}\left(\|u_t(0,\cdot)\|_{r-1}^2 + \|u_{tt}(0,\cdot)\|_{r-2}^2 + \int_0^h \|\partial_t f(t,\cdot)\|_{r-2}^2 \mathrm{d}t\right) \tag{20}$$

On the other hand, $Lu = u_{tt} - \tilde{L}u$, while $\tilde{L}$ is an elliptic operator, so:

$$\|u_{tt}(0,\cdot)\|_{r-2}^2 \le \|f + \tilde{L}u\|_{r-2}^2 \le C\|f(0,\cdot)\|_{r-2}^2. \tag{21}$$

While C is a constant. So: by (20)

$$\|\partial_t u(h)\|_{r-1}^2 \le C_{r-1}(\|\phi_1\|_{r-1}^2 + \int_0^h \|\partial_t f(t,\cdot)\|_{r-1}^2 \mathrm{d}t + \|f(0,\cdot)\|_{r-2}^2). \tag{22}$$

If $j > 1$, we can do this evaluate in the same way. Then, choose $j \in [0, r]$ and sum them up, we can see:

$$\sum_{j=0}^{r} \|\partial_t^j u(h)\|_{r-j}^2 \le C_r\left(\|\phi_0\|_r^2 + \|\phi\|_1^2 + \int_0^h \sum_{j=0}^{r-1} \|\partial_t^j f(t,\cdot)\|_{r-j-1}^2 \mathrm{d}t + \sum_{j=0}^{r-2} \|\partial_t^j f(0,\cdot)\|_{r-j-2}^2\right). \tag{23}$$

□

P155 Problem2

The inequality: Mark $\|u(h)\|_r$ as the $H^r - norm$ on region Ω when $t = h$, then:

$$\|u(h)\|_r^2 \le C_r\left(\|\varphi_0\|_r^2 + \|\varphi_1(x)\|_{r-1}^2 + \int_0^h \|Lu(\cdot,t)\|_{r-1}^2 \mathrm{d}t\right). \tag{24}$$