Homework# 11

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P76 Problem2

证明. By definition:

$$p_m(x,\xi) = (i\xi_t)^2 - a^2((i\xi_x)^2 + (i\xi_y)^2) = -\xi_t^2 + a^2(\xi_x^2 + \xi_y^2). \tag{1}$$

The cone: $t^2 = a^2(x^2 + y^2)$, we choose a direction outside this cone, the direction vector $l = (r, \cos\theta, \sin\theta)$, while $r > a$. Choose vector ξ is not parallel to l, then check the function $p_m(x, \lambda l + \xi)$, we can see:

$$g(\lambda) = p_m(x, \lambda l+\xi) = (a^2-r^2)\lambda^2 + (2a\xi_y \sin\theta + 2a\xi_x \cos\theta - 2\xi_t r)\lambda + (a^2\xi_x^2 + a^2\xi_y^2 - \xi_t^2). \tag{2}$$

As the direction ξ lies outside the given cone, it's clear that:

$$a^2\xi_x^2 + a^2\xi_y^2 - \xi_t^2 > 0. \tag{3}$$

And $r > a$ means $a^2 - r^2 < 0$. Then we derive the discriminant of (2):

$$\Delta = (2a\xi_y \sin\theta + 2a\xi_x \cos\theta - 2\xi_t r)^2 - 4(a^2 - r^2)(a^2\xi_x^2 + a^2\xi_y^2 - \xi_t^2) > 0. \tag{4}$$

It means $p_m(x,\xi)$ always has two different roots. Then this equation is strictly hyperbolic. □

P77 Problem4

The characteristic surface $\phi(t,x,y) = 0$ satisfies two conditions:

$$\begin{cases} \phi_t^2 - a^2(\phi_x^2 + \phi_y^2) = 0 \\ \phi(0, 3\lambda, \lambda) = 0 \end{cases} \tag{5}$$

Solve this ODEs, we can see the characteristic surface is

$$x - 3y + \sqrt{10}at = 0, \tag{6}$$

or

$$x - 3y - \sqrt{10}at = 0. \tag{7}$$

P106 Problem1

Assume u is the foundamental solution of the operator

$$P(\mathrm{d}) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + c\frac{\mathrm{d}}{\mathrm{d}x}. \tag{8}$$

Then we can see $P(\mathrm{d})u = \delta$. Then:

$$\begin{aligned}
&\mathrm{d}(e^{cx}\frac{\mathrm{d}u}{\mathrm{d}x}) = e^{cx}\delta = \delta \\
\Rightarrow &e^{cx}\frac{\mathrm{d}u}{\mathrm{d}x} = H(x) \\
\Rightarrow &\frac{\mathrm{d}u}{\mathrm{d}x} = e^{-cx}H(x) \\
\Rightarrow &u(x) = \begin{cases} 0, x \leq 0 \\ \dfrac{1 - e^{-cx}}{c}, x > 0 \end{cases}
\end{aligned} \tag{9}$$

So $u(x)$ is the foundamental solution of the operator $P(\mathrm{d})$.

P106 Problem3

Assume the foundamental solution is $T(x, y)$, do Fourier transformation on $T(x, y)$ related to y, mark $\hat{T}(x) = F_y[T]$, we can see:

$$\begin{aligned}
&\frac{\mathrm{d}\hat{T}}{\mathrm{d}x} - \eta\hat{T} + c\hat{T} = \delta(x) \\
\Rightarrow &\hat{T} = e^{(\eta-c)x}H(x) + C(\eta)e^{(\eta-c)x}.
\end{aligned} \tag{10}$$

We should choose the form of $C(\eta)$ to make $\hat{T}$ an $\mathscr{S}'$-contribution, just set:

$$C(\eta) = \begin{cases} -1, \eta > c \\ 0, \eta < c \end{cases} \tag{11}$$

We can see:

$$\hat{T} = \begin{cases} -H(-x)e^{(\eta-c)x}, \eta > c \\ H(x)e^{(\eta-c)x}, \eta < c \end{cases} \tag{12}$$

Do inverse FT on (12), we can see when $x \neq 0$,

$$\begin{aligned} T(x,y) &= \frac{H(x)}{2\pi}\int_{-\infty}^{c} e^{(\eta-c)x}e^{iy\eta}\mathrm{d}\eta - \frac{H(-x)}{2\pi}\int_{c}^{+\infty} e^{(\eta-c)x}e^{iy\eta}\mathrm{d}\eta. \\ &= \frac{e^{icy}}{2\pi}\frac{1}{x+iy}. \end{aligned} \tag{13}$$

P106 Problem5

For the foundamental solution $E(x,t)$, we can see:

$$(I-\Delta)E = \delta. \tag{14}$$

Do Fourier transformation on (14), we can see:

$$\hat{E}(\xi) = \frac{1}{1+|\xi|^2}. \tag{15}$$

It means that:

$$E(x) = \int_{\mathbb{R}^n} \frac{e^{i\xi\cdot x}}{1+|\xi|^2}\mathrm{d}\xi. \tag{16}$$

$E(x)$ is C^∞ on $\mathbb{R}^n \setminus \{0\}$, so $I-\Delta$ is semi-elliptic.

P106 Problem6

Do Fourier transformation on $u(x,t)$ related to x, mark the result as $\hat{u}$, we can see:

$$\begin{cases} \dfrac{\mathrm{d}^2\hat{u}}{\mathrm{d}t^2} + \xi^2\hat{u} = 0, \\ \hat{u}(\xi,0) = 0, \\ \dfrac{\mathrm{d}\hat{u}}{\mathrm{d}t}(\xi,0) = 1. \end{cases} \tag{17}$$

It means that

$$\hat{u}(\xi,t) = \frac{\sin|\xi|t}{|\xi|}. \tag{18}$$

Then, we should get the inverse FT of (18). It's clear that:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\sin(|\xi|t)}{|\xi|}e^{i\xi x}\mathrm{d}\xi \\ &= \frac{1}{\pi}\int_0^\infty \frac{\sin(\xi t)\cos(\xi x)}{\xi}\mathrm{d}\xi \\ &= \frac{1}{4}(\mathrm{sgn}(t+x) + \mathrm{sgn}(t-x)) \end{aligned} \tag{19}$$

To derive the D'Alembert's formula, consider the BVP of wave function:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = \delta(t,x) \\ \qquad u|_{t=0} = 0, \\ \quad \dfrac{\partial u}{\partial t}|_{t=0} = \varphi(x). \end{cases} \tag{20}$$

Mark the foundamental solution (19) as $E(x,t)$, the solution of wave equation (20) is:

$$u(x,t) = E(x,t) * \varphi(x) = \frac{1}{2t}\int_{x-t}^{x+t} \varphi(\xi)\mathrm{d}\xi. \tag{21}$$

Then by DuHamel's Principle, D'Alembert formula is true.