Homework# 10

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Evans P308 Problem7

By the given condition and **Gauss-Green Formula**, we can see:

$$\begin{aligned}\int_{\partial U}|u|^p\mathrm{d}S &\leq \int_{\partial U}|u|^p\alpha\cdot\mu\mathrm{d}S\\ &=\int_U|u|^p(\nabla\cdot\alpha)\mathrm{d}x+\int_U\alpha\cdot(\nabla|u|^p)\mathrm{d}x\\ &\leq C\int_U(|u|^p+|\nabla|u|^p|)\mathrm{d}x.\end{aligned} \tag{1}$$

Since

$$\nabla|u|^p=p|u|^{p-1}(\mathrm{sgn}u)\nabla u, \tag{2}$$

we have for $p=1$,

$$\int_{\partial U}|u|\mathrm{d}S\leq C\int_U(|u|+|\nabla u|)\mathrm{d}x. \tag{3}$$

If $p>1$, by (2), we can see:

$$\int_U|\nabla|u|^p|\mathrm{d}x\leq C\int_U p|u|^{p-1}|\nabla u|\mathrm{d}x. \tag{4}$$

Then by **Young's Inequality**, we can see:

$$|\nabla u||u|^{p-1}\leq\frac{|\nabla u|^p}{p}+\frac{|u|^{q(p-1)}}{q}\leq\frac{|\nabla u|^p}{p}+\frac{|u|^p}{q}. \tag{5}$$

Combine (1), (2) and (5), we can see:

$$\int_{\partial U}|u|^p\mathrm{d}S\leq\mathrm{Const}\int_U(|\nabla U|^p+|u|^p)\mathrm{d}x. \tag{6}$$

P309 Problem8

证明. If T is a bounded and linear operator, by the definition, we have:

$$\|Tu\| \leq C\|u\| \forall u \in L^p(\Omega). \tag{7}$$

It means that:

$$\int_{\partial\Omega} |u|^p \mathrm{d}x \leq C \int_{\Omega} |u|^p \mathrm{d}x, \forall u \in L^p(\Omega). \tag{8}$$

In fact, $\forall \epsilon > 0$, there exists $u \in C^\infty(\Omega)$ such that $u|_{\partial\Omega} \equiv 1$, $u(x) = 0$ when $\mathrm{dist}(x, \partial\Omega) > \epsilon$. When $\epsilon \to 0$, the left side of (8) equivalent the length of $\partial\Omega$, and the right side of (8) converges to 0, contradict!

So T isn't bounded in general. □

P309 Problem10

(a)

证明. The first step is an integration by parts:

$$\begin{aligned}
\int_U |Du|^p \mathrm{d}x &= \int_U \nabla u \cdot \nabla u |Du|^{p-2} \mathrm{d}x \\
&= -\int_U u \nabla \cdot (\nabla u |Du|^{p-2}) \mathrm{d}x \\
&= -\int_U u(\Delta u |Du|^{p-2} + (p-2)(\nabla u^T D^2 u \nabla u)|Du|^{p-4}) \\
&\leq C \int_U u|Du|^{p-2}|D^2 u| \mathrm{d}x
\end{aligned} \tag{9}$$

Then, as $p \neq 2$, by Holder Inequality, we can see:

$$\int_U u|Du|^{p-2}|D^2u|\mathrm{d}x \leq (\int_U |u|^{\frac{p}{2}}|D^2u|^{\frac{p}{2}}\mathrm{d}x)^{\frac{2}{p}}(\int_U |Du|^p \mathrm{d}x)^{\frac{p-2}{p}}. \tag{10}$$

By (9), (10) and Cauchy inequality, we can see:

$$\int_U |Du|^p \mathrm{d}x \leq C(\int_U |u|^p \mathrm{d}x)^{\frac{1}{2}}(\int_U |D^2u|^p \mathrm{d}x)^{\frac{1}{2}}. \tag{11}$$

□

(b)

证明. by (a), we can see:

$$\int_U |Du|^{2p} \mathrm{d}x \leq C \int_U u|Du|^{2p-2}|D^2u|\mathrm{d}x. \tag{12}$$

Then by Holder inequality:

$$
\begin{aligned}
(12) &\leq C\|u\|_{L^\infty}\int_U |Du|^{2p-2}|D^2u|\mathrm{d}x \\
&\leq C\|u\|_{L^\infty}(\int_U |Du|^{2p}\mathrm{d}x)^{\frac{p-1}{p}}(\int_U |D^2u|^p\mathrm{d}x)^{\frac{1}{p}}
\end{aligned}
\tag{13}
$$

By (12) and (13),

$$
\|Du\|_{L^{2p}} \leq C\|u\|_{L^\infty}^{\frac{1}{2}}\|D^2u\|_{L^p}^{\frac{1}{2}}. \tag{14}
$$

□

P309 Problem14

证明. Mark w_n as the area of n-dimension ball, we can see:

$$
\begin{aligned}
\int_U |u|^n\mathrm{d}x &= \omega_n\int_0^1 |\log\log(1+\frac{1}{r})|^n r^{n-1}\mathrm{d}r \\
&= \omega_n\int_1^\infty \frac{1}{t^n}\frac{|\log\log(1+t)|^n}{t}\mathrm{d}t
\end{aligned}
\tag{15}
$$

Since $\frac{|\log\log(1+t)|^n}{t} \to 0$ when $t\to\infty$, $\exists T$ such that $\frac{|\log\log(1+t)|^n}{t} < 1$ is true $\forall t > T$. It means that:

$$
(15) \leq \omega_n\int_1^T \frac{1}{t^n}\frac{|\log\log(1+t)|^n}{t}\mathrm{d}t + \int_1^\infty \frac{1}{t^n}\mathrm{d}t < +\infty. \tag{16}
$$

Then it's time to consider the case of derivatives. In fact:

$$
\frac{\partial u}{\partial x_i} = \frac{1}{\log(1+\frac{1}{|x|})}\cdot\frac{1}{1+\frac{1}{|x|}}\cdot\frac{-x_i}{|x|^3}. \tag{17}
$$

By (17):

$$
\begin{aligned}
\int_U |u_{x_i}|^n\mathrm{d}x &\leq \omega_n\int_0^1 (\frac{1}{\log(1+\frac{1}{r})}\frac{1}{1+\frac{1}{r}}\frac{1}{r^2})^n r^{n-1}\mathrm{d}r \\
&= \omega_n\int_1^{+\infty} \frac{1}{|\log(1+t)|^n}\frac{1}{(1+t)^n}t^{n-1}\mathrm{d}t \\
&\leq \frac{\omega_n}{n}\int_{\log 2}^{+\infty}\frac{1}{s^n}\frac{1}{e^{sn}}e^{sn}\mathrm{d}s \\
&< \infty.
\end{aligned}
\tag{18}
$$

It means $u \in W^{1,n}(U)$. □

P311 Problem21