Homework# 1

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P13 Problem3

证明. Proof by induction. First, consider the case $k = 1$. We can see:

$$LHS = x_1 + x_2 + \cdots + x_n.$$

set $\beta_i = (0, 0, \cdots, 1, \cdots, 0)$ where 1 comes in the i-th element, we can see:

$$RHS = \sum_{i=1}^{n} x^{\beta_i} = \sum_{i=1}^{n} x_i = LHS.$$

So the equation holds when $k = 1$. Assume the result holds for $k = k_0$, i.e.

$$(x_1 + \cdots + x_n)^{k_0} = \sum_{|\alpha|=k_0} \binom{|\alpha|}{\alpha} x^{\alpha}. \tag{1}$$

consider the case for $k_0 + 1$, i.e.

$$\begin{aligned}(x_1 + \cdots + x_n)^{k_0+1} &= \sum_{|\alpha|=k_0} \binom{|\alpha|}{\alpha} x^{\alpha} \\ &= \sum_{|\alpha|=k_0} \sum_{j=1}^{n} \binom{|\alpha|}{\alpha} x_j x^{\alpha} \\ &= \sum_{|\beta|=k_0+1} \sum_{j=1}^{n} \binom{|\beta_j|}{\beta_j} x^{\beta}.\end{aligned} \tag{2}$$

If $\beta = (x_1, \cdots, x_n)$, $\beta_j = (x_1, \cdots, x_j - 1, \cdots, x_n)(x_j \geq 1)$. It suffices to show that

$$\binom{|\beta|}{\beta} = \sum_{j=1}^{n} \binom{|\beta_j|}{\beta_j}. \tag{3}$$

We can see:

$$\begin{aligned} RHS &= k_0!\sum_{j=1}^{n}\frac{1}{\beta_1!\cdots(\beta_j-1)!\cdots\beta_n!} \\ &= k_0!\frac{\beta_1+\cdots+\beta_n}{\beta_1!\cdots\beta_n!} \\ &= LHS. \end{aligned} \tag{4}$$

So (3) is true. By induction, the multinomial theorem is true. □

P13 Problem4

证明. If $|\alpha| = 1$, by the formula of partial derivative, we can see:

$$\frac{\partial(uv)}{\partial x_i} = v\frac{\partial u}{\partial x_i} + u\frac{\partial v}{\partial x_i}.$$

So $D^\alpha(uv) = uD^\alpha v + vD^\alpha u$ when $|\alpha| = 1$.

Assume the result is true for $|\alpha| = k_0$, consider $|\beta| = k_0 + 1$, mark $\beta = (k_1, \cdots, k_n)$, WLOG, assume $k_1 \geq 1$, $\alpha = (k_1 - 1, \cdots, k_n)$. Then:

$$\begin{aligned} D^\beta(uv) &= D^\alpha(vu_1 + uv_1) \\ &= D^\alpha(u_1 v) + D^\alpha(v_1 u) \\ &= \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}D^\gamma u_1 D^{\alpha-\gamma}v + \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}D^\gamma u D^{\alpha-\gamma}v_1 \\ &= \sum_{\gamma\leq\alpha}\left[\binom{\alpha}{\gamma}D^{\gamma'}uD^{\alpha'-\gamma'}v + \binom{\alpha}{\gamma}D^\gamma u D^{\alpha-\gamma'}v\right]. \end{aligned} \tag{5}$$

where $\gamma' = (\gamma_1 + 1, \gamma_2, \cdots, \gamma_n)$, $\alpha' = (\alpha_1 + 1, \cdots, \alpha_n)$.

Finally, consider the coefficient in front of $D^{\gamma'}uD^{\alpha'-\gamma'}v$, it is:

$$\binom{\alpha}{\gamma} + \binom{\alpha}{\gamma'} = \binom{\alpha'}{\gamma'}.$$

By induction, the result is true. □

P13 Problem5

证明. Set $g(t) = f(tx)$. First, we should prove a lemma:

$$g^{(k)}(t) = \sum_{|\alpha|=k}\frac{k!}{\alpha!}x^\alpha D^\alpha f. \tag{6}$$

For $k = 1$, we can see: $g'(t) = \frac{\mathrm{d}f(tx_1,\cdots,tx_n)}{\mathrm{d}t} = \sum_{i=1}^{n} x_i f_{x_i}$, satisfies (6).

Assume (6) is true for $k = k_0$, consider $k = k_0 + 1$, we can see:

$$\begin{aligned} g^{(k+1)}(t) &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha (D^\alpha f)' \\ &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha \sum_{i=1}^{n} \frac{\partial^{\beta_i} f}{\partial x_1^{\alpha_1} \cdots \partial x_i^{\alpha_i+1} \cdots \partial x_n^{\alpha_n}} \\ &= \sum_{|\beta|=k+1} \frac{(k+1)!}{\beta!} x^\beta D^\beta f. \end{aligned} \tag{7}$$

In this equation, $\beta_i = (\alpha_1, \cdots, \alpha_i + 1, \cdots, \alpha_n)$.

So, by induction, we can see $g^{(k)}(t) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha D^\alpha f$.

By Taylor extension:

$$g(t) = \sum_{i=0}^{k} \frac{g^{(i)}(0)}{i!} t^i + \frac{g^{(k+1)}(\xi)}{(k+1)!} t^{k+1}.$$

So $f(x) = g(1) = \sum_{|\alpha|\le k} \frac{1}{\alpha!} D^\alpha f(0) x^\alpha + O(|x|^{k+1})$. □

P85 Problem4

证明. Define $u_\epsilon = u + \epsilon|x|^2$, we can see:

$$\Delta u_\epsilon = \Delta u + 2n\epsilon = 2n\epsilon \text{ on } U. \tag{8}$$

Assume u_ϵ gets its maximal in U, assume $u(x_0) = \max_{x\in U} u(x)$, for $u(x_0)$ is maximal, it's clear that $\nabla^2 u(x_0)$ is semi-negative definite, which means $\Delta u(x_0) \le 0$, contradict!

So:

$$u(x) \le u_\epsilon(x) \le \max_{\partial U} u_\epsilon(x) \le \max_{x\in\partial U} u + \epsilon C. \text{(C is a constant.)} \tag{9}$$

Set $\epsilon \to 0^+$, we can see $u(x) \le \max_{\partial U} u(x)\ \forall x \in U$. □

P85 Problem5

(a).Set

$$h(r) = ⨍_{\partial B(x,r)} v(y) \mathrm{d}S(y). \tag{10}$$

Then:

$$\begin{aligned} h'(r) &= \fint_{\partial B(0,1)} Dv(x+rz)\cdot z\mathrm{d}S(z) \\ &= \fint_{\partial B(x,r)} Dv(y)\cdot \frac{y-x}{r}\mathrm{d}S(y) \\ &= \fint_{\partial B(x,r)} Du\cdot n\mathrm{d}S(y) \\ &= \frac{r}{n}\fint_{B(x,r)} \Delta u\mathrm{d}y \geq 0. \end{aligned} \tag{11}$$

What's more:

$$\begin{aligned} \int_{B(x,r)} v\mathrm{d}y &= \int_0^r \int_{\partial B(x,\tau)} v\mathrm{d}S(y)\mathrm{d}\tau \\ &= \int_0^r n\alpha(n)t^{n-1}h(t)\mathrm{d}t \\ &\geq v(x)\int_0^r n\alpha(n)t^{n-1}\mathrm{d}t \\ &= v(x)\alpha(n)r^n. \end{aligned} \tag{12}$$

So $v(x) \leq \fint_{B(x,r)} v\mathrm{d}y$.

(b).Assume $v(x)$ gets its maximal in U, we can mark the set:

$$E := \{x \in U : v(x) = \max_{\bar{U}} v\}. \tag{13}$$

It's trivial that E is a closed set related to U. In the other hand, by (a), we can see: if $x_0 \in E$, $\exists\delta$ s.t. $B(x_0,\delta) \subset E$. So E is an open set.

So E is clopen. U is an open region, so $E = U$, in this condition v is a constant function, satisfy the result.

(c).

$$\frac{\partial^2 v}{\partial x_i^2} = \phi'' u(\frac{\partial u}{\partial x_i})^2 + \phi'(u)\frac{\partial^2 u}{\partial x_i^2}. \tag{14}$$

As ϕ is convex, we can see $\phi''(x) \geq 0$. So:

$$\Delta v \geq \phi'(u)\Delta u = 0. \tag{15}$$

Which means that v is subharmonic.

(d).

$$\frac{\partial^2 v}{\partial x_i^2} = 2\sum_{j=1}^n (u_{x_i x_j})^2 + 2\sum_{j=1}^n u_{x_j} u_{x_i x_i x_j} \tag{16}$$

As $\Delta u = 0 \Rightarrow \forall j, \Delta u_{x_j} = 0$. So:

$$\Delta v = 2\sum_{i,j=1}^n (u_{ij})^2 \geq 0 \Rightarrow -\Delta v \leq 0. \tag{17}$$

So v is subharmonic.

P86 Problem6

证明. Set $F := \max_{\bar{U}} |f|$, $G := \max_{\partial U} |g|$, $v := u + \frac{|x|^2}{2n}F$, $w := -u + \frac{|x|^2}{2n}F$. We can see v and w both subharmonic, which means:

$$u \leq G - C_1 F, u \geq C_2 F - G.$$

So $|u| \leq C(G + F)$. □