Homework# 6

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P10 Problem7

Assume $\phi_\mu(x)$ is a foundamental sequence in $\mathscr{S}(\mathbb{R}^n)$, by definition, $\forall m \in \mathbb{N}$ and $\epsilon > 0$, $\exists N$ s.t. $\forall \mu, \nu > N$, we can see

$$\sup_{|\alpha|\leq m, x\in\mathbb{R}^n} (1+|x|^2)^{\frac{m}{2}} |\partial^\alpha[\phi_\mu - \phi_\nu](x)| < \epsilon. \tag{1}$$

(1) $\forall m$, $\{\|\phi_\nu\|_m\}$ is bounded, so $\exists M_m$ s.t.

$$\sup_{|\alpha|\leq m, x\in\mathbb{R}^n} (1+|x|^2)^{\frac{m}{2}} |\partial^\alpha \phi_\nu(x)| \leq M_m. \tag{2}$$

So:

$$|\partial^\alpha \phi_\nu(x)| \leq \frac{M_m}{(1+|x|^2)^{\frac{m}{2}}}. \tag{3}$$

Then, on any closed ball $|x| \leq R$, $\{\partial^\alpha \phi_\nu\}$ is a fundamental sequence. So on $|x| \leq R|$, there exists a uniform limit $\psi_\alpha(x)$ s.t.

$$|\psi_\alpha(x)| \leq \frac{M_m}{(1+|x|^2)^{\frac{m}{2}}}. \tag{4}$$

(2)$\psi_\alpha(x) = \partial^\alpha \psi_0(x)$. In fact we only need to show

$$\psi_{(1,0,\cdots,0)} = \partial_1 \psi_0(x). \tag{5}$$

$\forall R > 0$, when $|x| \leq R$, we can see:

$$\partial_1(\phi_\nu) \rightrightarrows \psi_{(1,0,\cdots,0)} \tag{6}$$

and

$$\phi_\nu \rightrightarrows \psi_0. \tag{7}$$

$\forall \epsilon > 0$, set $M_1' > 0$ and $N_0 \in \mathbb{N}$ s.t.

$$\int_{|x_1|>M_1'} \frac{\mathrm{d}x_1}{(1+|x_1|^2)^{\frac{1}{2}}} < \frac{\epsilon}{4M_1}, \tag{8}$$

and

$$|\psi_{(1,0,\cdots,0)(x)-\partial_{x_1}\phi_\nu(x)}| < \frac{\epsilon}{4M_1'} \tag{9}$$

So:

$$|\int_{-\infty}^{x_1} \psi_{(1,0,\cdots,0)}(x', x_2, \cdots, x_n)\mathrm{d}x' - \phi_\nu(x_1, \cdots, x_n)| < \epsilon. \tag{10}$$

So

$$\psi_{(1,0,\cdots,0)} = \partial_1 \psi_0(x). \tag{11}$$

(3)$\|\phi_\nu - \psi_0\| \to 0$. In fact, $\forall \epsilon > 0$, $\exists R > 0$ s.t. when $|x| > R$,

$$\sup_{|\alpha|\le m} |\partial^\alpha[\phi_\nu(x) - \psi_0(x)]| \le \frac{M_m}{(1+|x|^2)^{\frac{m}{2}}} < \epsilon. \tag{12}$$

Then choose N s.t. when $\nu > N$, on $|x| \le R$, we have

$$\sup_{|\alpha|\le m} |\partial^\alpha[\phi_\nu(x) - \psi_0(x)]| < \epsilon. \tag{13}$$

So

$$\partial^\alpha \phi_\nu(x) \rightrightarrows \partial^\alpha \psi_0(x). \tag{14}$$

Then we can see $\mathscr{S}(\mathbb{R}^n)$ is complete.

P10 Problem8

证明. It's clear that

$$\begin{aligned} \|x^\alpha \partial^p (u_\epsilon(x) - u(x))\| &= \|x^\alpha \int_{\mathbb{R}^n} (u^p(x-y) - u^p(x))\alpha_\epsilon(y)\mathrm{d}y\| \\ &\le \int_{\mathbb{R}^n} x^\alpha |u^p(x-y) - u^p(x)|\alpha_\epsilon(y)\mathrm{d}y \end{aligned} \tag{15}$$

Then it suffices to show that

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^n} x^\alpha |u^p(x-y) - u^p(x)|\alpha_\epsilon(y)\mathrm{d}y = 0. \tag{16}$$

As $u \in \mathscr{S}(\mathbb{R}^n)$,(16) means

$$\forall \epsilon > 0, \exists K, \forall |x| \ge K, \int_{\mathbb{R}^n} x^\alpha |u^p(x-y) - u^p(x)|\alpha_\epsilon(y)\mathrm{d}y < \epsilon. \tag{17}$$

Then consider the compact set $\{x : |x| \le K\}$. As $u_\epsilon \to u$ on $C_c^\infty(\mathbb{R}^n)$, we can see:

$$\forall \epsilon > 0, \exists \delta_0, \forall \delta < \delta_0, |x| \le K, \int_{\mathbb{R}^n} x^\alpha |u^p(x-y) - u^p(x)|\alpha_\delta(y)\mathrm{d}y < \epsilon \tag{18}$$

By the above two equations, we can see

$$u_\epsilon \to u(\mathscr{S}(\mathbb{R}^n)). \tag{19}$$

□

P27 Problem14

证明. For the derivative of contribution $\mathcal{F}$ is continuous, we can see $\exists\ f \in C(\mathbb{R}^n)$ such that:

$$\langle \partial_k \mathcal{F}, \phi \rangle = -\langle \mathcal{F}, \partial_k \phi \rangle = \langle f, \phi \rangle \tag{20}$$

$\forall \phi \in C_c^\infty(\mathbb{R}^n)$. By the continuous of function f, there exists primitive functions of f such that $\partial_k(F) = f$. Then it suffices to show that

$$\langle \mathcal{F}, \phi \rangle = \langle F, \phi \rangle . \tag{21}$$

Set $\phi = \partial_k \psi$, we can see:

$$\begin{aligned} \langle \mathcal{F}, \phi \rangle &= -\langle f, \psi \rangle \\ &= -\int_{\mathbb{R}^n} f \psi \mathrm{d}x \\ &= -\int_{\mathbb{R}^n} \partial_k F \psi \mathrm{d}x \\ &= \int_{\mathbb{R}^n} F \partial_k \psi \mathrm{d}x \\ &= \langle F, \phi \rangle . \end{aligned} \tag{22}$$

So $\mathcal{F} = F$. Then $\partial_k F = f$, so this contribution is continuous and derivable.

□

P27 Problem15

证明. First of all, set $\varphi \in C_c^\infty(\mathbb{R})$, consider a function

$$h(y) = \langle g_x, \varphi(x+y) \rangle . \tag{23}$$

Set $\zeta(x) = 1$ when $x \geq a$, $\zeta(x) = 0$ when $x \leq a-1$, and ζ is a C^∞ function, so

$$h(y) = \langle g_x, \zeta(x)\varphi(x+y) \rangle . \tag{24}$$

As $\varphi(x)$ has compact support, we can see when y is big enough, $\zeta(x)\varphi(x+y) \equiv 0$, which means $h(y) \equiv 0$ when y is big enough.

Then:

$$\langle f_y, h(y) \rangle = \langle \zeta(y) f_y, h(y) \rangle = \langle f_y, \zeta(y) h(y) \rangle . \tag{25}$$

As $h(y)$ has support $(-\infty, c)$, $c < +\infty$, and $\zeta(y) = 0$ when $y < a-1$, it means: $\zeta(y)h(y) = 0$ when $y < a-1$ or $y > c$. So $\zeta(y)h(y)$ has compact support, which means $\zeta(y)h(y) \in C_c^{+\infty}(\mathbb{R})$, $f * g \in \mathcal{D}'(\mathbb{R})$, the convolution exists.

If $\phi(x) \equiv 0$ when $x \leq a^*$(a^* is a constant), it means $h(y)$ itself has a compact support K, just choose a^* s.t. $K \cap [a, +\infty) = \emptyset$, we can see $\langle f * g, \phi \rangle = 0$. So $\mathrm{supp}(f * g) \subset [a^*, +\infty)$. □

P28 Problem16

In this problem, $\phi \in C_c^\infty(\mathbb{R}^n)$. (1)

$$\begin{aligned}
\left\langle tH(t) * e^t H(t), \phi \right\rangle &= \langle tH(t), \langle e^y H(y), \phi(t+y) \rangle \rangle \\
&= \left\langle tH(t), \left\langle e^{w-t} H(w-t), \phi(w) \right\rangle \right\rangle \\
&= \left\langle tH(t), \int_t^{+\infty} e^{w-t} \phi(w) \mathrm{d}w \right\rangle \\
&= \int_0^\infty t \int_t^\infty e^{w-t} \phi(w) \mathrm{d}w \mathrm{d}t \\
&= \int_0^{+\infty} \phi(w) \int_0^w t e^{w-t} \mathrm{d}t \mathrm{d}w \\
&= \int_0^{+\infty} (e^w - w - 1) \phi(w) \mathrm{d}w \\
&= \left\langle (e^t - t - 1) H(t), \phi \right\rangle
\end{aligned} \tag{26}$$

(2)

$$\begin{aligned}
\langle H(t)\sin(t) * H(t)\cos(t), \phi \rangle &= \langle H(t)\sin(t), \langle H(y)\cos y, \phi(t+y) \rangle \rangle \\
&= \langle H(t)\sin(t), \langle H(w-t)\cos(w-t), \phi(w) \rangle \rangle \\
&= \left\langle H(t)\sin(t), \int_t^{+\infty} \cos(w-t)\phi(w)\mathrm{d}w \right\rangle \\
&= \int_0^{+\infty} \sin(t) \int_t^{+\infty} \cos(w-t)\phi(w)\mathrm{d}w \\
&= \int_0^{+\infty} \phi(w) \int_0^w \sin(x)\cos(w-x)\mathrm{d}x\mathrm{d}w \\
&= \int_0^{+\infty} \frac{w \sin w}{2} \phi(w) \mathrm{d}w \\
&= \left\langle \frac{tH(t)}{2} \sin t, \phi \right\rangle.
\end{aligned} \tag{27}$$

(3)

$$
\begin{aligned}
\langle (f(t)H(t)) * H(t), \phi \rangle &= \langle f(t)H(t), \langle H(y), \phi(t+y) \rangle \rangle \\
&= \langle f(t)H(t), \langle H(w-t), \phi(w) \rangle \rangle \\
&= \left\langle f(t)H(t), \int_x^{+\infty} \phi(w)\mathrm{d}w \right\rangle \\
&= \int_0^{+\infty} f(t)\mathrm{d}t \int_t^{\infty} \phi(w)\mathrm{d}w \\
&= \int_0^{+\infty} \phi(w) \int_0^{w} f(\tau)\mathrm{d}\tau\mathrm{d}w \\
&= \left\langle H(t) \int_0^{t} f(\tau)\mathrm{d}\tau, \phi \right\rangle .
\end{aligned} \tag{28}
$$

P28 Problem18

By definition,$T * \phi = \langle T_y, \phi(x-y) \rangle$.

If $\phi_n \to \phi$ in $C_c^{\infty}(\mathbb{R}^n)$, it means $\exists$ compact set K such that $\phi_n \rightrightarrows \phi$ on K. Then:

$$
\begin{aligned}
\|T * \phi_n - T * \phi\| &= \| \langle T_y, \phi_n(x-y) - \phi(x-y) \rangle \| \\
&\le \langle T_y, |\phi_n(x-y) - \phi(x-y)| \rangle \\
&\to 0.
\end{aligned} \tag{29}
$$

The final step is derived from the linear property of functional T_y and $\phi_n \rightrightarrows \phi$. So it means that the given map is continuous on ϕ.

By the definiton of the convergence of contribution, if $T_n \to T$, it means that $\forall \phi \in C_c^{\infty}(\mathbb{R}^n)$, $\langle T_n - T, \phi \rangle = 0$. It means that the given map is continuous on T.