Homework# 3

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P85 Problem1

证明. Set $w(t) := u(\phi(x,t),t)$ such that

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\partial u}{\partial t} + b \cdot Du. \tag{1}$$

it means that $\frac{\partial \phi}{\partial t} = b$, which means $\phi(x,t) = x_0 + bt$. After this modify, we can see:

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} + cw = 0 \\ \quad w_{x_0}(0) = u(x_0, 0) \end{cases} \tag{2}$$

this ODE suggests:

$$w(t) = e^{-ct} u(x_0, 0). \tag{3}$$

So:

$$u(x,t) = e^{-ct} u(x - bt, 0) = e^{-ct} g(x - bt). \tag{4}$$

□

P88 Problem18

证明.

$$\left.\begin{aligned} v_{tt} &= u_{ttt} \\ \Delta v &= \Delta u_t \end{aligned}\right\} \Rightarrow \frac{\partial (u_{tt} - \Delta u)}{\partial t} = 0 \Rightarrow v_{tt} - \Delta v = 0. \tag{5}$$

on $\mathbb{R}^n \times \{t = 0\}$, we can see $v = u_t = h$, and $v_t(x,0) = \lim_{t \to 0} \Delta u(x,t)$.

For $u \equiv 0$ when $t = 0$, we can see $\Delta u(x,0) = 0$. As Δu is continuous, we can see $v_t(x,0) = 0$. □

P88 Problem19

(a)

$$\begin{aligned} u_{xy} = 0 &\Rightarrow \frac{\partial u_x}{\partial y} = 0 \Rightarrow u_x = f(x) + C_1. \\ u_{xy} = 0 &\Rightarrow \frac{\partial u_y}{\partial x} = 0 \Rightarrow u_y = g(y) + C_2. \end{aligned} \tag{6}$$

It means that $u(x, y) = F(x) + G(y) + C$.

(b)

$$\begin{aligned} \frac{\partial u_\xi}{\partial \eta} &= \frac{\partial u_\xi}{\partial t}\frac{\partial t}{\partial \eta} + \frac{\partial u_\xi}{\partial x}\frac{\partial x}{\partial \eta} \\ &= u_{\xi x} - u_{\xi t} \\ &= \frac{\partial u_x}{\partial \xi} - \frac{\partial u_t}{\partial \xi} \\ &= (\frac{\partial u_x}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u_x}{\partial t}\frac{\partial t}{\partial \xi}) - (\frac{\partial u_t}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u_t}{\partial t}\frac{\partial t}{\partial \xi}) \\ &= u_{xx} + u_{xt} - u_{xt} - u_{tt} = u_{xx} - u_{tt}. \end{aligned} \tag{7}$$

(c)

$$\begin{aligned} u_{tt} - u_{xx} = 0 \Rightarrow u_{\xi\eta} = 0 \Rightarrow u(\xi, \eta) = f(\xi) + g(\eta) \\ \Rightarrow u(x, t) = F(x - t) + G(x + t). \end{aligned} \tag{8}$$

$$\begin{cases} u(x, 0) = F(x) + G(x) = g(x) \\ u_t(x, 0) = -F'(x) + G'(x) = h(x) \end{cases} \Rightarrow \begin{cases} G'(x) = \dfrac{h(x) + g'(x)}{2} \\ F'(x) = \dfrac{g'(x) - h(x)}{2} \end{cases} \tag{9}$$

It means that

$$u(x, t) = \frac{1}{2}(g(x + t) + g(x - t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\mathrm{d}y. \tag{10}$$

(d)

Right-moving: $g(\xi) \equiv 0$ when $\xi < x$.

Left-moving: $g(\xi) \equiv 0$ when $\xi > x$.

P89 Problem20

证明. If u satisfies the wave equation, it means:

$$\sum_{i=1}^{n} u_{x_i x_i} = \frac{1}{c^2} u_{tt}. \tag{11}$$

Set $r = |x|$, 11 means that

$$v_{rr} + \frac{n-1}{r} v = \frac{1}{c^2} v_{tt}. \tag{12}$$

Suppose $u = \alpha(r)\phi(t - \beta(r))$, according to 12, we can see:

$$\alpha''\phi - 2\alpha'\beta'\phi' - \alpha\beta''\phi' + \alpha(\beta')^2\phi'' + \frac{n-1}{r}(\alpha'\phi - \alpha\phi'\beta') = \frac{\alpha}{c^2}\phi'' \tag{13}$$

According to 13 and $\beta(0) = 0$, we can see:

$$\alpha(\beta')^2 = \frac{\alpha}{c} \Rightarrow \beta = \frac{r}{c}, \beta' = \frac{1}{c}, \beta'' = 0. \tag{14}$$

and:

$$\begin{aligned} \alpha'' + \frac{n-1}{r}\alpha' &= 0 \\ 2\alpha' + \frac{n-1}{r}\alpha &= 0 \end{aligned} \tag{15}$$

From the first equation in 15, we can see $\alpha' = \frac{C}{r^{n-1}}$. It means that if $n \neq 2$, $\alpha = C_1 r^{2-n} + C_2$, if $n = 2$, $\alpha = C_1 \log(r) + C_2$. If it satisfies the second equation in 15, it means:

$$2(2-n) + (n-1) = 0 \Rightarrow n = 3. \tag{16}$$

In special case ,when $n = 1$, the equation 15 is also correct. So $n = 1$ or $n = 3$.

When $n = 1$, $\alpha = C$, $\beta = \frac{r}{c}$. When $n = 3$, $\alpha = \frac{k}{r}$, $\beta = \frac{r}{c}$. □

P89 Problem21

(a)

$$\begin{aligned} E_{tt} &= \mathrm{curl}(B_t) \\ &= \mathrm{curl}(-\mathrm{curl} E) \\ &= \mathrm{curl}(\frac{\partial E_2}{\partial x_3} - \frac{\partial E_3}{\partial x_2}, \frac{\partial E_3}{\partial x_1} - \frac{\partial E_1}{\partial x_3}, \frac{\partial E_1}{\partial x_2} - \frac{\partial E_2}{\partial x_1}). \\ &= \Delta E. \end{aligned} \tag{17}$$

The final equation is derived by the condition $\nabla \cdot E = 0$. In the same way, we can see:$B_{tt} = \Delta B$.

(b)

For $w = \nabla \cdot u$, get divergence on the given equation, we can see:

$$\begin{aligned} &\nabla \cdot u_{tt} - \mu\nabla \cdot \Delta u - (\lambda + \mu)\nabla \cdot (\nabla(\nabla \cdot u)) = 0 \\ \Rightarrow &w_{tt} - (\lambda + \mu)w - \mu\nabla(\Delta u) = 0 \\ \Rightarrow &w_{tt} - (\lambda + 2\mu)\Delta w = 0. \end{aligned} \tag{18}$$

For $w = \nabla \times u$, in the same way, we can see $w_{tt} - \mu \Delta w = 0$.