

Homework# 14

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P150 Problem2

证明. u_t times Lu , then integral it on Q_t , we can see:

$$\int_{Q_t} u_t L u dx dt = I_1(t) + I_2(t). \quad (1)$$

While

$$I_1(t) = - \int_{Q_t} \left(\sum_i b_i \frac{\partial u}{\partial x_i} + cu \right) u_t dx dt, \quad (2)$$

$$I_2(t) = \int_{Q_t} \left(\frac{\partial^2 u}{\partial t^2} - \sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) \right) u_t dx dt. \quad (3)$$

Then: set the vector function

$$F(x) = \left(\sum a_{1j} \frac{\partial u}{\partial x_j}, \dots, \sum a_{nj} \frac{\partial u}{\partial x_j} \right) \quad (4)$$

By (a_{ij}) elliptic, we can see:

$$\begin{aligned} & - \int_0^t \int_{\Omega} \nabla \cdot F dx dt \\ &= \int_0^t \int_{\Omega} \sum_{i,j} \frac{\partial^2 u}{\partial x_i \partial t} a_{ij} \frac{\partial u}{\partial x_j} dx dt - \int_0^t \int_{\partial\Omega} u_t F \cdot n dS dt. \\ &\leq \int_0^t \int_{\Omega} \sum_{i,j} \frac{\partial^2 u}{\partial x_i \partial t} a_{ij} \frac{\partial u}{\partial x_j} dx dt + \sigma \alpha \int_0^t \int_{\partial\Omega} u u_t dS dt. \end{aligned} \quad (5)$$

If we denote $A = (a_{ij})$, the second inequality is derived from:

$$\left(A \begin{bmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \end{bmatrix} \right) \cdot \mathbf{n} = \begin{bmatrix} \frac{\partial u}{\partial x_1} & \dots & \frac{\partial u}{\partial x_n} \end{bmatrix} A \mathbf{n} \geq \alpha \frac{\partial u}{\partial \mathbf{n}} = -\sigma \alpha \mathbf{n}. \quad (6)$$

Then by P145 Theorem 1.1, we can see that:

$$\begin{aligned} I_2(t) \leq & \frac{1}{2} \int_0^t \int_{\Omega} \frac{\partial}{\partial t} \left(\left(\frac{\partial u}{\partial t} \right)^2 + \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dx dt \\ & - \frac{1}{2} \int_0^t \int_{\Omega} \sum_{i,j} \frac{\partial a_{ij}}{\partial t} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx dt + \sigma \alpha \int_0^t \int_{\partial\Omega} uu_t dS dt \end{aligned} \quad (7)$$

On the other hand:

$$\int_0^t \int_{\partial\Omega} uu_t dS dt = \frac{1}{2} \left(\int_{\partial\Omega} u^2(\mathbf{x}, t) dS - \int_{\partial\Omega} u^2(\mathbf{x}, 0) dS \right) \quad (8)$$

Define the energy norm

$$E(t) = \int_{\Omega} (u^2 + u_t^2 + \sum u_{x_i}^2) dx + \int_{\partial\Omega} u^2 dS. \quad (9)$$

We can see:

$$\int_0^t \int_{\Omega} u_t L u dx dt = \frac{1}{2} \left[\int_{\Omega} \left(\frac{\partial u}{\partial t} \right)^2 + \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \int_{\partial\Omega} u^2 dS \right]_{t=0}^{t=t} + \tilde{I}_1(t). \quad (10)$$

Where:

$$|\tilde{I}_1(t)| \leq C \int_0^t E(\tau) d\tau. \quad (11)$$

Then, by the same tragedy with the proof of Theorem 1.1, we can derive the energy inequality as:

$$E(t) \leq C(E(0) + \int_{Q_t} f^2 dx dt). \quad (12)$$

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P150 Problem3

As $I(t) \in C^2(\mathbb{R})$, first we induce the following derivatives:

$$(e^{\lambda t} I(t))' = e^{\lambda t} (I'(t) + \lambda I(t)). \quad (13)$$

$$(e^{\lambda t} I(t))'' = e^{\lambda t} (\lambda^2 I(t) + 2\lambda I'(t) + I''(t)). \quad (14)$$

Then we can see:

$$(e^{\lambda t} I(t))'' + k(e^{\lambda t} I(t))' = e^{\lambda t} (I''(t) + (2\lambda + k)I'(t) + (\lambda^2 + \lambda k)I(t)). \quad (15)$$

If we set:

$$\begin{cases} 2\lambda + k = -C_1 \\ \lambda^2 + \lambda k = -C_2 \end{cases} \quad (16)$$

by the condition, we can derive that

$$(e^{\lambda t} I(t))'' + k(e^{\lambda t} I(t))' \leq e^{\lambda t} M. \quad (17)$$

Then set $h(t) = e^{kt}(e^{\lambda t} I(t))'$, we can derive that $h'(t) \leq e^{(\lambda+k)t} M$.

As $\lim_{t \rightarrow -\infty} h(t) = 0$, we can see that $h(t) \leq \frac{e^{(\lambda+k)t}}{\lambda+k} M$, so:

$$(e^{\lambda t} I(t))' \leq \frac{e^{\lambda t}}{\lambda+k} M \quad (18)$$

Then:

$$e^{\lambda t} I(t) \leq \frac{e^{\lambda t}}{\lambda(\lambda+k)} M \Rightarrow I(t) \leq \frac{M}{\lambda(\lambda+k)}. \quad (19)$$

By (16), $I(t) \leq -\frac{M}{C_2}$. If $C_2 \geq 0$, there is no such $I(t)$.

P155 Problem1

证明. First, consider the evaluation of $\|\partial_t u\|_{r-1}^2$, by (2.1), we can see:

$$\|\partial_t u(h)\|_{r-1}^2 \leq C_{r-1} \left(\|u_t(0, \cdot)\|_{r-1}^2 + \|u_{tt}(0, \cdot)\|_{r-2}^2 + \int_0^h \|\partial_t f(t, \cdot)\|_{r-2}^2 dt \right) \quad (20)$$

On the other hand, $Lu = u_{tt} - \tilde{L}u$, while \tilde{L} is an elliptic operator, so:

$$\|u_{tt}(0, \cdot)\|_{r-2}^2 \leq \|f + \tilde{L}u\|_{r-2}^2 \leq C\|f(0, \cdot)\|_{r-2}^2. \quad (21)$$

While C is a constant. So: by (20)

$$\|\partial_t u(h)\|_{r-1}^2 \leq C_{r-1} (\|\phi_1\|_{r-1}^2 + \int_0^h \|\partial_t f(t, \cdot)\|_{r-1}^2 dt + \|f(0, \cdot)\|_{r-2}^2). \quad (22)$$

If $j > 1$, we can do this evaluate in the same way. Then, choose $j \in [0, r]$ and sum them up, we can see:

$$\sum_{j=0}^r \|\partial_t^j u(h)\|_{r-j}^2 \leq C_r \left(\|\phi_0\|_r^2 + \|\phi\|_1^2 + \int_0^h \sum_{j=0}^{r-1} \|\partial_t^j f(t, \cdot)\|_{r-j-1}^2 dt + \sum_{j=0}^{r-2} \|\partial_t^j f(0, \cdot)\|_{r-j-2}^2 \right). \quad (23)$$

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