Homework# 9

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P56 Problem1

证明. Assume $u_\nu(x) \in C_c^\infty(\mathbb{R}_+^n)$, given $\mathbf{x} \in \mathbb{R}_+^n$, create a line segment $l(t) = t\mathbf{x}$, by Newton-Leibniz formula, we can see:

$$u_\nu(x) = \int_0^1 Du_\nu(\mathbf{x}t) \cdot \mathbf{x}\mathrm{d}t. \tag{1}$$

So, for a Cauchy sequence in $C_c^\infty(\mathbb{R}_+^n)$, we can see:

$$\begin{aligned} |u_\nu(x) - u_\mu(x)| &\leq \int_0^1 |(Du_\nu(\mathbf{x}t) - Du_\mu(\mathbf{x}t)) \cdot \mathbf{x}|\mathrm{d}t \\ &\leq C\|u_\nu - u_\mu\|_{H^{1,p}}. \end{aligned} \tag{2}$$

The final step is derived by the condition of compact support. It follows that $\{u_\nu(x)\}$ is uniformly convergent in $\mathbb{R}_+^n$, and its limitation $u(x)$ must satisfies that $\lim_{x\to 0} u(x) = 0$.

However, if we set

$$v(x) = e^{-|x|^2}, \tag{3}$$

we can see $v \in H^{m,p}(\mathbb{R}_+^n)\forall m \geq 1$, while $\lim_{x\to 0} v(x) = 1$, contradict!

So $C_c^\infty(\mathbb{R}_+^n)$ isn't dense in $H^{m,p}(\mathbb{R}_+^n)$. □

P56 Problem4

Claim: $H(x) \in W^{0,\infty}(\mathbb{R})$.

证明. First of all, $H(x)$ is a bounded function in $\mathbb{R}$, so $H(x) \in L^\infty(\mathbb{R})$. It means that $H(x) \in W^{0,\infty}(\mathbb{R})$.

$\int_0^\infty 1\mathrm{d}x$ diverge, it means that $H(x) \notin L^p \forall p < \infty$.

And $DH(x) = \delta(x)$, $\delta(x) \notin L^\infty(\mathbb{R})$, so $H(x) \notin W^{k,\infty}(\mathbb{R})\forall k \geq 1$. □

P56 Problem5

证明. By definition,

$$\|\delta - \alpha_\epsilon\|_{H^s(\mathbb{R}^n)} = \sup_{\|\phi\|_{H^{-s}(\mathbb{R}^n)=1}} \langle \alpha_\epsilon - \delta, \phi \rangle . \tag{4}$$

As $-s > \frac{n}{2}$, by Sobolev Embedding theorem(P62 Thm5.4), set $m = -s$, $\exists k < (-s)$ s.t. $\frac{k-1}{n} \leq \frac{1}{2} < \frac{k}{n}$, we can see: for $\phi \in H^{-s}(\mathbb{R}^n)$, $\phi \in C(\mathbb{R}^n)$ is always true.

Then:

$$\langle \alpha_\epsilon - \delta, \phi \rangle = \int_{|x|\leq\epsilon} \alpha_\epsilon(x)|\phi(x) - \phi(0)|\mathrm{d}x. \tag{5}$$

As ϕ is continuous, we can see: $\forall \epsilon_0 > 0$, $\exists \delta$, $\forall |x| \leq \delta$, $|\phi(x) - \phi(0)| \leq \epsilon_0$. It means when $\epsilon \leq \delta$, (5)$< \epsilon_0$. So we can see when $\epsilon \to 0$, (5) $\to 0$. It means that $\alpha_\epsilon(x) \to \delta(H^s(\mathbb{R}^n))$. □

P56 Problem7

Statement:$u \in H^{m,p}(\Omega) \Leftrightarrow u$ is a restriction of function in $H^{m,p}(\mathbb{R}^n)$.

证明. First, consider the case $m \in \mathbb{Z}^+$. We only need to show the necessity, which means that $\forall u \in H^{m,p}(\Omega)$, we can extend it to a function in $H^{m,p}(\mathbb{R}^n)$.

First, we should show that if $\forall$ open set Ω_1 such that $\Omega \subset \Omega_1$, we can extend u to $H^{m,p}(\Omega_1)$, then u can be extended to $H^{m,p}(\mathbb{R}^n)$. In fact, mark the function in $H^{m,p}(\Omega_1)$ as u_1, set $\eta \in C_c^\infty(\Omega_1)$ s.t. $\eta(x) = 1$ on Ω, then ηu_1 is just the extension of u on space $\mathbb{R}^n$. So we just need to show that u can be extended to Ω_1. By localization, we just need to extend a function $u \in H^{m,p}(\mathbb{R}^n_+)$ to $H^{m,p}(\mathbb{R}^n)$.

Set sequence $\{u^{(\nu)}\}$ such that $u^{(\nu)} \in C^\infty(\bar{\mathbb{R}}^n_+)$, and $u^{(\nu)} \to u$ on $H^{m,p}(\mathbb{R}^n_+)$, set $v^{(\nu)}$ as:

$$v^{(\nu)}(x', x_n) = \begin{cases} u^{(\nu)}(x', x_n), x_n \geq 0 \\ \displaystyle\sum_{j=1}^m C_j u^{(\nu)}(x', -jx_n), x_n < 0 \end{cases} \tag{6}$$

while C_j is defined as

$$\sum_{j=1}^m (-j)^k C_j = 1, \forall 0 \leq k \leq m-1. \tag{7}$$

Then we can see that $\{v^{(\nu)}\}$ is a foundamental sequence in $H^{m,p}(\mathbb{R}^n)$, which means $\{v^{(\nu)}\}$ converges to $v \in H^{m,p}(\mathbb{R}^n)$, and the norm of v in $H^{m,p}(\mathbb{R}^n)$ can be controled by $\|u\|$. So when $m > 0$, the extension is available.

If $m = 0$, we can see $H^0(\Omega) = L^p(\Omega)$, just set zero-extension for u to the outside of Ω, then get a function in $H^{0,p}(\mathbb{R}^n)$.

For $m < 0$, assume $\frac{1}{p} + \frac{1}{q} = 1, m_1 = -m$, then $u \in H^{m,p}(\Omega)$ is a linear continuous functional on $H_0^{m_1,q}$. Then set the distribution $\tilde{u}$ as:

$$\langle \tilde{u}, \phi \rangle = \sup_{E_\phi} \langle u, \phi - E_\phi \rangle . \tag{8}$$

while E_ϕ is an extension for ϕ to $H^{m,q}(\mathbb{R}^n)$, which satisfies

$$\|E_\phi\|_{H^{m_1}(\mathbb{R}^n)} \leq C_0 \|\phi\|_{H^{m_1}(\mathbb{R}^n \setminus \bar{\Omega})}. \tag{9}$$

It's clear that $\tilde{u}$ is linear on ϕ, we just need to show $\tilde{u}$ is continuous. In fact:

$$\begin{aligned} |\langle \tilde{u}, \phi \rangle| &= \sup_{E_\phi} |\langle u, \phi - E_\phi \rangle| \\ &\leq C \sup_{E_\phi} \|\phi - E_\phi\|_{H^{m_1,q}(\Omega)} \\ &\leq C(\|\phi\|_{H^{m_1,q}(\Omega)} + C_0 \|\phi\|_{H^{m_1,q}(\mathbb{R}^n \setminus \bar{\Omega})}) \\ &\leq C' \|\phi\|_{H^{m_1,q}(\mathbb{R}^n)}. \end{aligned} \tag{10}$$

So $\tilde{u} \in H^{m,p}(\mathbb{R}^n)$ and $\langle \tilde{u}, \phi \rangle = \langle u, \phi \rangle$. So on Ω, we can see $\tilde{u} = u$, Q.E.D. □

P70 Problem1

证明. First, consider the case when u has a compact support set. In this case, $\forall \epsilon > 0$, $u \in H^{m,p-\epsilon}(\mathbb{R}^n)$. Then by theorem 5.1, we can see $u \in H^{m-k,q(\epsilon)}(\mathbb{R}^n)$, while

$$q(\epsilon) = (\frac{1}{p - \epsilon} - \frac{k}{n})^{-1}. \tag{11}$$

When ϵ is small enough, $q(\epsilon)$ can be arbitrary big. So $u \in L^q(\mathbb{R}^n)$ is true for all q.

For u not have compact support, by the above analysis, we can see $u \in H^{m-k,q}_{loc}(\mathbb{R}^n)$. Set the unit decomposition $a(x), b(x), \sigma(x), K_i$ the same as theorem 5.2, we can see:$u = \sum_i b_i u_i$. Then it's time to estimate the $H^{m-k,q}$ norm of u.

By the definition:

$$
\begin{aligned}
\|u\|^q_{H^{m-k,q}(\mathbb{R}^n)} &= \sum_i \int_{K_i} \sum_{|\alpha|\le m-k} |D^\alpha u|^q \mathrm{d}x \\
&\le C \sum_i \sum_{|\alpha|\le m-k} \|D^\alpha (b_i u)\|^q_{L^q(\Omega_i)} \\
&\le C \sum_i \|b_i u\|^q_{H^{m,p}(\Omega_i)} \\
&\le C \left(\sum_i \|b_i u\|^p_{H^{m,p}(\Omega_i)} \right)^{\frac{q}{p}} \\
&\le CN \left(\int_{\mathbb{R}^n} \left(\sum_{|\alpha|\le m} |\partial^\alpha u|^p \right) \mathrm{d}x \right)^{\frac{q}{p}} \\
&\le CN \|u\|^q_{H^{m,p}(\mathbb{R}^n)}.
\end{aligned}
\tag{12}
$$

So the result is true for $m \ge k > 1$. $\square$