Homework# 6

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2022 年 4 月 4 日

P10 Problem6

证明. Assume $\alpha \in \mathbb{Z}^n$ is a multi-index and consider the operator ∂_α. Set $f, g \in S$, $S = C_c^\infty(\mathbb{R}^n)$ or $S = C^\infty(\mathbb{R}^n)$, we can see:

$$\partial_\alpha(k_1 f + k_2 g) = k_1 \partial_\alpha f + k_2 \partial_\alpha g. \tag{1}$$

So $P(x, \partial)$ is a linear operator. Then we should show that ∂_i is continuous.

For $f, g \in S$, by the definition of norm, we can see:

$$\|\partial_i f - \partial_i g\| = \sup_{x \in K, \alpha \in \mathbb{Z}^n} \|\partial_i \partial_\alpha (f-g)(x)\| \leq \sup_{x \in K, \alpha \in \mathbb{Z}^n} \|\partial_\alpha (f-g)(x)\| = \|f-g\|. \tag{2}$$

It means that ∂_i is Lip-1 continuous. So $P(x, \partial)$ is continuous. □

P27 Problem2

$$\begin{aligned} \mathrm{supp} u &= \{x : |x| < 1\} \\ \mathrm{supp} u_\epsilon &= \{x : |x| < 1 + \epsilon\} \end{aligned} \tag{3}$$

P27 Problem3

证明. By definition, $\forall \phi \in C_c^\infty(\mathbb{R}^n)$:

$$\langle f_m, \phi \rangle = \int_{\mathbb{R}^n} f_m \phi \mathrm{d}x. \tag{4}$$

$$\langle \delta, \phi \rangle = \phi(0) = \int_{\mathbb{R}^n} f_m \phi(0) \mathrm{d}x. \tag{5}$$

It means that:

$$|\langle f_m, \phi \rangle - \langle \delta, \phi \rangle| = |\int_{\mathbb{R}^n} f_m(x)(\phi(x) - \phi(0)) \mathrm{d}x|. \tag{6}$$

$\phi \in C_c^\infty(\mathbb{R}^n) \Rightarrow \forall \epsilon > 0, \exists \delta_0, \forall |x| < \delta_0, |\phi(x) - \phi(0)| \leq \epsilon$. So by (6), we can see:

$$\begin{aligned}(6) &\leq \int_{\mathbb{R}^n} f_m(x)|\phi(x) - \phi(0)|\mathrm{d}x \\ &= \int_{|x|\leq\delta_0} f_m(x)|\phi(x) - \phi(0)|\mathrm{d}x + \int_{|x|\geq\delta_0} f_m(x)|\phi(x) - \phi(0)|\mathrm{d}x \\ &< \epsilon + M \int_{|x|\geq\delta_0} f_m(x)\mathrm{d}x.\end{aligned} \tag{7}$$

As $f_m \rightrightarrows 0$, and $\int_{|x|\leq\delta_0} f_m(x)\mathrm{d}x \to 1$, we can see:$\int_{|x|\geq\delta_0} f_m(x)\mathrm{d}x \to 0$, which means:

$$\exists M, \forall m \geq M, \int_{|x|\geq\delta_0} f_m(x)\mathrm{d}x < \frac{\epsilon}{M}. \tag{8}$$

So $\exists M$, s.t. $\forall m > M$, $(6) < 2\epsilon$. It means that $\forall \phi \in C_c^\infty(\mathbb{R}^n)$, $\langle f_m, \phi\rangle \to \langle \delta, \phi\rangle$. So $f_m \to \delta$. □

P27 Problem5

$f_\epsilon(x) = \frac{2x}{x^2+\epsilon^2}$. We claim:

$$f_\epsilon(x) \to g(x) := P.V.(\frac{2}{x}). \tag{9}$$

Then, for $\phi \in C_c^\infty(\mathbb{R}^n)$, consider:

$$\begin{aligned}\langle f_\epsilon, \phi\rangle - \langle g, \phi\rangle &= \int_{\mathbb{R}} \frac{2x}{x^2+\epsilon^2}\phi(x)\mathrm{d}x - \lim_{\delta\to 0}\int_{|x|\geq\delta} \frac{2\phi(x)}{x}\mathrm{d}x \\ &= \lim_{\delta\to 0}\int_{|x|\leq\delta} \frac{2x}{x^2+\epsilon^2}\phi(x)\mathrm{d}x + \lim_{\delta\to 0}\int_{|x|\geq\delta} 2\phi(x)(\frac{x}{x^2+\epsilon^2} - \frac{1}{x})\mathrm{d}x \\ &= -2\lim_{\delta\to 0}\int_{|x|\geq\delta} \frac{\epsilon^2\phi(x)}{x(x^2+\epsilon^2)}\mathrm{d}x.\end{aligned} \tag{10}$$

Then, as $\phi \in C_c^\infty(\mathbb{R}^n)$, for $\epsilon \to 0$, we can see $(10) \to 0$. So $\lim_{\epsilon\to 0} f_\epsilon$ exists in $\mathcal{D}'(\mathbb{R}^n)$, and its limitation is exactly $P.V.(\frac{2}{x})$.

P27 Problem8

(1)

$$\begin{aligned}
&\left\langle x^k\delta^{(m)}(x),\varphi(x)\right\rangle \\
=&\left\langle \delta^{(m)}(x),x^k\varphi(x)\right\rangle \\
=&(-1)^m\left\langle \delta(x),(x^k\varphi(x))^(m)\right\rangle \\
=&(-1)^m\sum_{j=0}^{m}\binom{m}{j}(x^k)^{(j)}\varphi^{(m-j)}(x)|_{x=0} \\
=&(-1)^m\sum_{j=0}^{m}\binom{m}{j}\frac{x^{k-j}\varphi^{(m-j)}(x)(k-j)!j!}{k!}|_{x=0} \\
=&\begin{cases} 0 & (m<k) \\ \binom{m}{k}(-1)^m\varphi^{(m-k)} & (m\geq k). \end{cases}
\end{aligned} \tag{11}$$

(2)

$$\langle\delta(ax),\varphi(x)\rangle \overset{t=ax}{=} \frac{1}{a}\left\langle\delta(t),\varphi(\frac{t}{a})\right\rangle = \frac{\varphi(0)}{a}. \tag{12}$$

(3)Don't know.

P27 Problem9

$$\begin{aligned}
\left\langle\frac{\partial^2 H}{\partial x\partial y},\varphi\right\rangle &= \left\langle H,\frac{\partial^2\varphi}{\partial x\partial y}\right\rangle \\
&= \int_I \frac{\partial^2\varphi}{\partial x\partial y}\mathrm{d}x\mathrm{d}y \\
&= \varphi(0,0).
\end{aligned} \tag{13}$$

So $\frac{\partial^2 H}{\partial x\partial y}=\delta$.