

PROBLEM 1

Claim. The set of rational numbers (\mathbb{Q}) is countable.

Proof. Let $x \in \mathbb{Q}$. Then, by definition, we can write $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$. This fraction can be uniquely mapped to a tuple $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. This means that there exists an injective mapping $\mathbb{Q} \rightarrow \mathbb{Z} \times \mathbb{Z}$ that maps each fraction to a tuple. Thus, we have:

$$|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| \leq |\mathbb{N}|$$

By the countability of $|\mathbb{N}|$, we have shown that \mathbb{Q} is countable.

□

PROBLEM 2

Let E be an event, and let $1_E : \Omega \rightarrow \{0, 1\}$ be an indicator random variable such that:

$$1_E(\omega) = \begin{cases} 1 & \omega \in E \\ 0 & \omega \notin E \end{cases}$$

Claim. The expectation of E is equal to the probability that $1_E = 1$. In other words:

$$\mathbb{E}[1_E] = \mathbb{P}[E]$$

Proof.

$$\begin{aligned} \mathbb{E}[1_E] &= \mathbb{P}[1_E = 1] \cdot 1 + \mathbb{P}[1_E = 0] \cdot 0 \\ &= \mathbb{P}[1_E = 1] \\ &= \mathbb{P}[\{\omega \in \Omega : 1_E(\omega) = 1\}] \\ &= \mathbb{P}[E] \end{aligned}$$

□

PROBLEM 3

Claim. $e^{iy} = \cos y + i \sin y$ for $y \in \mathbb{R}$

Proof. We begin by noting that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$. Plugging in $x = yi$ for the exponent in e^x gives:

$$\begin{aligned} e^{yi} &= \sum_{n=0}^{\infty} \frac{(yi)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{y^{4n}}{(4n)!} + \sum_{n=0}^{\infty} i \cdot \frac{y^{4n+1}}{(4n+1)!} + \sum_{n=0}^{\infty} -1 \cdot \frac{y^{4n+2}}{(4n+2)!} + \sum_{n=0}^{\infty} -i \cdot \frac{y^{4n+3}}{(4n+3)!} \\ &= \sum_{n=0}^{\infty} \frac{y^{4n}}{(4n)!} - \frac{y^{4n+2}}{(4n+2)!} + i \left(\sum_{n=0}^{\infty} \frac{y^{4n+1}}{(4n+1)!} - \frac{y^{4n+3}}{(4n+3)!} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} \right) \\ &= \cos y + i \sin y \end{aligned}$$

This equality holds because the series representations of \sin and \cos are convergent. Thus, $e^{iy} = \cos y + i \sin y$ for $y \in \mathbb{R}$, as required. \square

PROBLEM 4

Claim. Two identical decks of n cards have a k -matching with probability:

$$\pi_k = \frac{1}{k!} \left(1 - \sum_{i=1}^{n-k} \frac{(-1)^i}{i!} \right)$$

Proof. First, we pick and order k cards to be matched. The probability of the selected card orders matching is $\frac{1}{k!}$.

Now, we consider the probability that the remaining $n - k$ cards *do not* match. Similar to the examples from class and the textbook, this is an instance of an indexed union of sets $A_i \in \mathcal{F}$ such that A_i is the set of permutations in which $f(i) = i$. But since we are looking for the probability of this not happening, we consider the probability of the complement, computed as 1 minus the probability of this union of events.

By the Inclusion-Exclusion principle, based on the notes from class, we have that

$$\mathbb{P} \left[\bigcup_{i=1}^{n-k} A_i \right] = \sum_{i=1}^{n-k} \frac{(-1)^i}{i!}$$

Putting everything together, we get the following, and we're done.

$$\pi_k = \frac{1}{k!} \left(1 - \sum_{i=1}^{n-k} \frac{(-1)^i}{i!} \right)$$

□