TMD phenomenology and nonperturbative structures

Ted Rogers
Jefferson Lab & Old Dominion University

Based on:

- J.O. Gonzalez, TCR, N. Sato, Phys.Rev.D 106 (2022) 3, 034002
- J.O. Gonzalez, T. Rainaldi, TCR (2023), Phys.Rev.D 107 (2023) 9, 094029
- F. Aslan, M. Boglione, J.O. Gonzalez, T. Rainaldi, TCR, A. Simonelli, 2401.14266 [hep-ph]

June 6, 2024, Transversity Workshop

"Structure"

https://www.bnl.gov/eic/

Precision 3D imaging of protons and nuclei

The Electron-Ion Collider will take three-dimensional precision snapshots of the internal structure of protons and atomic nuclei. As they pierce through the larger particles, the high-energy electrons will interact with the internal microcosm to reveal unprecedented details—zooming in beyond the simplistic structure of three valence quarks bound by a mysterious force. Recent experiments indicate that the gluons—which carry the strong force—multiply and appear to linger within particles accelerated close to the speed of light, and play a significant role in establishing key properties of protons and nuclear matter. By taking images at a range of energies, an EIC will reveal features of this "ocean" of gluons and the "sea" of quark-antiquark pairs that form when gluons interact—allowing scientists to map out the particles' distribution and movement within protons and nuclei, similar to the way medical imaging technologies construct 3D dynamic images of the brain. These studies may help reveal how the energy of the massless gluons is transformed through $E=mc^2$ to generate most of the mass of the visible universe.

Electron Ion Collider: The Next QCD Frontier : Understanding the glue that binds us all

- *3D-imaging.* The TMDs represent the intrinsic motion of partons inside the nucleon (confined motion!) and allow reconstruction of the nucleon structure in momentum space. Such information, when combined with the analogous information on the parton spatial distribution from GPDs, leads to a 3-dimensional imaging of the nucleon.

What I mean by a "hadron structure oriented" approach?

- TMD factorization is well understood at a formal level; theoretical properties of TMDs are well catalogued
- How does one construct phenomenological parametrizations that allow us to learn the most from using TMD factorization?

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^4\boldsymbol{q}_\mathrm{T}\,\mathrm{d}\Omega} = \mathcal{H}\sum\!\!\!\!\!\!\int \mathrm{d}^2\boldsymbol{k}_{a\mathrm{T}}\ \mathrm{d}^2\boldsymbol{k}_{b\mathrm{T}}\, f_{j/h_a}(x_a,\boldsymbol{k}_{a\mathrm{T}};\mu_Q,Q^2) f_{\bar{j}/h_b}(x_b,\boldsymbol{k}_{b\mathrm{T}};\mu_Q,Q^2)\delta^{(2)}(\boldsymbol{q}_\mathrm{T}-\boldsymbol{k}_{a\mathrm{T}}-\boldsymbol{k}_{b\mathrm{T}}) + \cdots$$

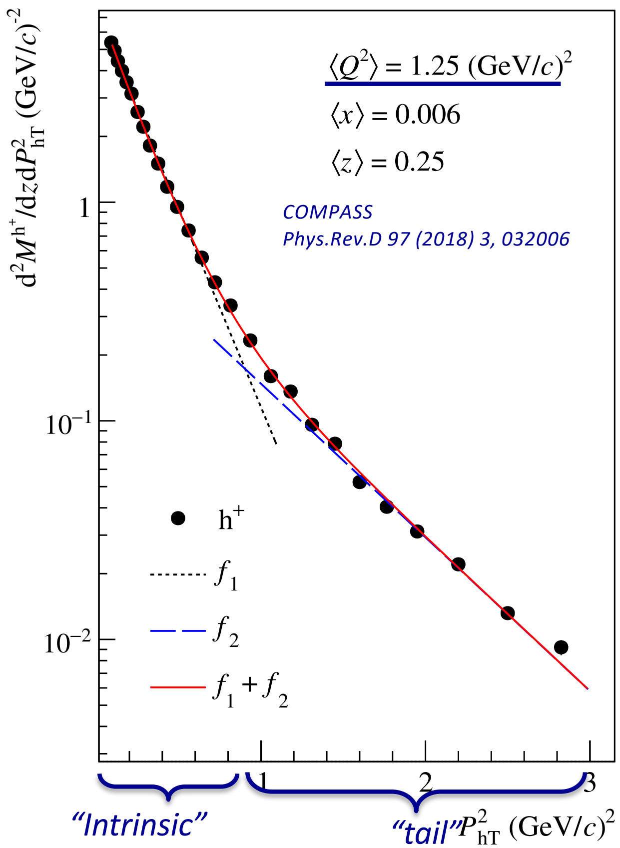
TMD factorization

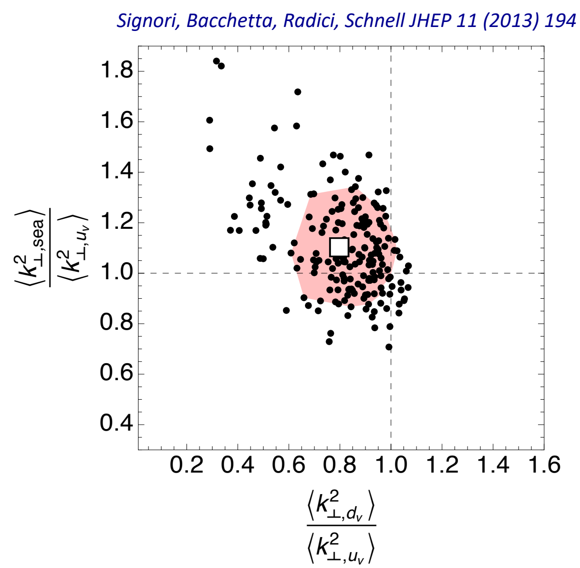
$$\underbrace{f(x,k_T;\mu_Q,Q)}_{\text{Abstract theory object}} \Longrightarrow \underbrace{f_{\mathrm{param}}(x,k_T;\mu_Q,Q)}_{\text{Function to be compared with / fitted to data}}$$

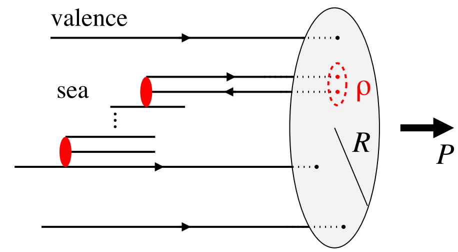
Abstract theory object

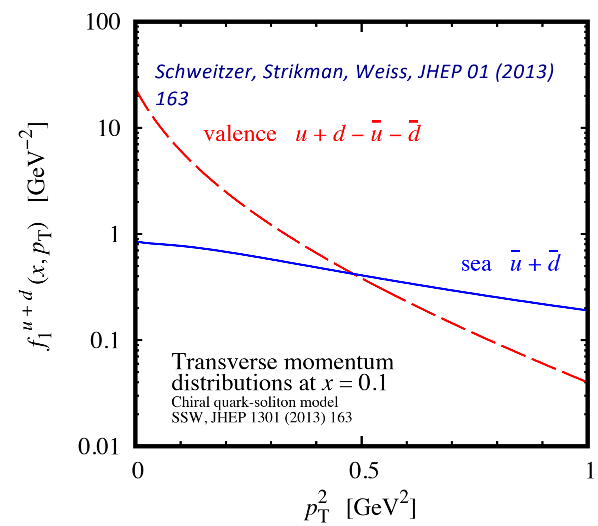
Function to be compared with / fitted to data

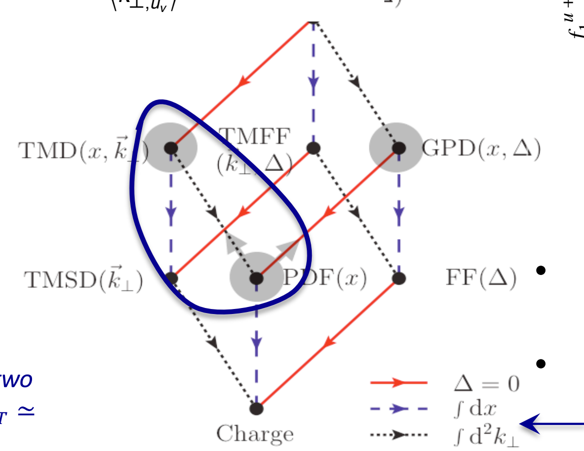
Nonperturbative structures in phenomenology











These integrals are often UV divergent!

- Models: χPT, lightcone wavefunctions, etc…
- Lattice

"… the two exponential functions in our parametrizations F_1 can be attributed to two completely different underlying physics mechanisms that overlap in the region $P_{hT} \simeq 1\ \mathrm{GeV}^2$."

Ingredients of TMD factorization

- Factorization $\frac{\mathrm{d}\sigma}{\mathrm{d}^4\boldsymbol{q}_\mathrm{T}\,\mathrm{d}\Omega} = \mathcal{H} \sum\!\!\!\!\!\!\int \mathrm{d}^2\boldsymbol{k}_{a\mathrm{T}}\ \mathrm{d}^2\boldsymbol{k}_{b\mathrm{T}}\, f_{j/h_a}(x_a, \boldsymbol{k}_{a\mathrm{T}}; \mu_Q, Q^2) f_{\bar{\jmath}/h_b}(x_b, \boldsymbol{k}_{b\mathrm{T}}; \mu_Q, Q^2) \delta^{(2)}(\boldsymbol{q}_\mathrm{T} - \boldsymbol{k}_{a\mathrm{T}} - \boldsymbol{k}_{b\mathrm{T}}) \quad + \cdots$

- Evolution $\frac{\partial \ln \tilde{f}_{j/p}(x, b_T; \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T; \mu)$ $\quad \frac{\mathrm{d} \ln \tilde{f}_{j/p}(x, b_T; \mu, \zeta)}{\mathrm{d} \ln \mu} = \gamma(\alpha_s(\mu); \zeta/\mu^2)$ $\quad \frac{\mathrm{d}\tilde{K}(b_T; \mu)}{\mathrm{d} \ln \mu} = -\gamma_K(\alpha_s(\mu))$

- Small b_T, large k_T (OPE) with collinear factorization

$$\tilde{f}_{j/p}(x, b_T; \mu, \zeta) = \int_x^1 \frac{\mathrm{d}\xi}{\xi} \tilde{C}_{j/k}(x/\xi, b_T; \zeta, \mu, \alpha_s(\mu)) f_{k/p}(\xi; \mu) + O\left(b_T \Lambda_\mathrm{QCD}\right) \qquad \tilde{K}(b_T; \mu) = \tilde{K}_\mathrm{pert}\left(b_T \mu, \alpha_s(\mu)\right) + O\left(b_T \Lambda_\mathrm{QCD}\right)$$

Or

$$f_{j/p}(x, k_T; \mu, \zeta) = \frac{1}{k_T^2} \left[\int_x^1 \frac{\mathrm{d}\xi}{\xi} C_{j/k}(x/\xi, k_T; \zeta, \mu, \alpha_s(\mu)) f_{k/p}(\xi; \mu) + O\left(\frac{\Lambda_\mathrm{QCD}}{k_T}\right) \right]$$

- In pheno applications, all are combined into a cross-section expression to be compared with data, with the help of perturbative calculations, fits, & nonperturbative models

Where is the intrinsic/nonperturbative hadron structure in pheno?

CSS etc

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q^2\,\mathrm{d}y\,\mathrm{d}q_{\mathrm{T}}^2} = \frac{4\pi^2\alpha^2}{9Q^2 s}\sum_{j,j_A,j_B} e_j^2 \int \frac{\mathrm{d}^2\boldsymbol{b}_{\mathrm{T}}}{(2\pi)^2} e^{i\boldsymbol{q}_{\mathrm{T}}\cdot\boldsymbol{b}_{\mathrm{T}}}$$
$$\times \int_{x_A}^1 \frac{\mathrm{d}\xi_A}{\xi_A} f_{j_A/A}(\xi_A;\mu_{b_*})\ \tilde{C}^{\mathrm{CSS1,\ DY}}_{j/j_A}\left(\frac{x_A}{\xi_A}, b_*; \mu_{b_*}^2, \mu_{b_*}, C_2, a_s(\mu_{b_*})\right)$$
$$\times \int_{x_B}^1 \frac{\mathrm{d}\xi_B}{\xi_B} f_{j_B/B}(\xi_B;\mu_{b_*})\ \tilde{C}^{\mathrm{CSS1,\ DY}}_{\bar{\jmath}/j_B}\left(\frac{x_B}{\xi_B}, b_*; \mu_{b_*}^2, \mu_{b_*}, C_2, a_s(\mu_{b_*})\right)$$
$$\times \exp\left\{-\int_{\mu_{b_*}^2}^{\mu_Q^2}\frac{\mathrm{d}\mu'^2}{\mu'^2}\left[A_{\mathrm{CSS1}}(a_s(\mu');C_1)\ln\left(\frac{\mu_Q^2}{\mu'^2}\right) + B_{\mathrm{CSS1,\ DY}}(a_s(\mu');C_1,C_2)\right]\right\}$$
$$\times \exp\left[-g^{\mathrm{CSS1}}_{j/A}(x_A, b_{\mathrm{T}}; b_{\max}) - g^{\mathrm{CSS1}}_{\bar{\jmath}/B}(x_B, b_{\mathrm{T}}; b_{\max}) - g^{\mathrm{CSS1}}_{K}(b_{\mathrm{T}}; b_{\max})\ln(Q^2/Q_0^2)\right]$$
$$+ \text{suppressed corrections.}$$

Parton-model-like descriptions in early pheno

$$f(x,k_T;Q) = f(x;Q)\underbrace{\frac{e^{\frac{-k_T^2}{4B}}}{4\pi B}}_{??}$$

The b* method to sequester collinear factorization

Collins & Soper, Nucl. Phys. B197, 446 (1982)

Start with exact b_T-space TMD pdf: $\tilde{f}(x, b_T; \mu_{Q_0}, Q_0)$

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Start with exact b_T-space TMD pdf: $\tilde{f}(x, b_T; \mu_{Q_0}, Q_0)$

Define a regulator

$$\boldsymbol{b}_* = \boldsymbol{b}_*(b_\mathrm{T}) = \begin{cases} \boldsymbol{b}_\mathrm{T} & b_T \ll b_\mathrm{max} \\ \boldsymbol{b}_\mathrm{max} & b_T \gg b_\mathrm{max} \end{cases}$$

Classic example: $\boldsymbol{b}_* = \dfrac{\boldsymbol{b}_\mathrm{T}}{\sqrt{1 + b_\mathrm{T}^2/b_\mathrm{max}^2}}$

Multiply by one

$$\tilde{f}(x, b_T; \mu_{Q_0}, Q_0) = \tilde{f}(x, b^*(b_T); \mu_{Q_0}, Q_0) \left(\frac{\tilde{f}(x, b_T; \mu_{Q_0}, Q_0)}{\tilde{f}(x, b^*(b_T); \mu_{Q_0}, Q_0)} \right)$$

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Multiply by one

$$\tilde{f}(x, b_T; \mu_{Q_0}, Q_0) = \underbrace{\tilde{f}(x, b^*(b_T); \mu_{Q_0}, Q_0)}_{\text{Expand in OPE, implement evolution}} \underbrace{\left(\frac{\tilde{f}(x, b_T; \mu_{Q_0}, Q_0)}{\tilde{f}(x, b^*(b_T); \mu_{Q_0}, Q_0)} \right)}_{e^{-g(x, b_T)}}$$

"nonperturbative function"

No dependence on b_*

$$\delta \left[\tilde{f}(x, b^*(b_T); \mu_{Q_0}, Q_0) e^{-g(x, b_T)} \right] = 0$$

$$\frac{d}{db_\mathrm{max}} \left[\tilde{f}(x, b^*(b_T); \mu_{Q_0}, Q_0) e^{-g(x, b_T)} \right] = 0$$

$$-g(x, b_T) \approx -a b_T^2 \ldots$$

Ansatz…. Something else?

Our approach

- $$\frac{\mathrm{d}\sigma}{\mathrm{d}^4\boldsymbol{q}_\mathrm{T}\,\mathrm{d}\Omega} = \mathcal{H} \sum\!\!\!\!\!\!\!\int \mathrm{d}^2\boldsymbol{k}_{a\mathrm{T}}\;\mathrm{d}^2\boldsymbol{k}_{b\mathrm{T}}\, f_{j/h_a}(x_a, \boldsymbol{k}_{a\mathrm{T}}; \mu_Q, Q^2) f_{\bar{j}/h_b}(x_b, \boldsymbol{k}_{b\mathrm{T}}; \mu_Q, Q^2) \delta^{(2)}(\boldsymbol{q}_\mathrm{T} - \boldsymbol{k}_{a\mathrm{T}} - \boldsymbol{k}_{b\mathrm{T}})$$

Keep this form

Our approach

- $$\frac{\mathrm{d}\sigma}{\mathrm{d}^4\boldsymbol{q}_\mathrm{T}\,\mathrm{d}\Omega} = \mathcal{H}\sum\!\!\!\!\!\int \mathrm{d}^2\boldsymbol{k}_{a\mathrm{T}}\;\mathrm{d}^2\boldsymbol{k}_{b\mathrm{T}}\;f_{j/h_a}(x_a,\boldsymbol{k}_{a\mathrm{T}};\mu_Q,Q^2)f_{\bar{\jmath}/h_b}(x_b,\boldsymbol{k}_{b\mathrm{T}};\mu_Q,Q^2)\delta^{(2)}(\boldsymbol{q}_\mathrm{T}-\boldsymbol{k}_{a\mathrm{T}}-\boldsymbol{k}_{b\mathrm{T}})$$ *Keep this form*

- Start by optimizing the input scale treatment with fixed scale:

  Equivalent to imposing small b_T OPE

  – Smoothly/continuously interpolate between nonperturbative small-k_T model and a large-k_T perturbative tail *See also (Grewal, Kang, Qiu Phys.Rev.D 101 (2020))*

  – Impose $$\pi\int^{\mu_{Q_0}^2}\mathrm{d}k_T^2 f_{j/p}(x,k_T;\mu_{Q_0},Q_0^2) = f_{j/p}(x;\mu_{Q_0}) + \Delta_{j/p} + \text{power suppressed}$$

  $f_{j/p}(x;\mu_{Q_0})$: $\overline{\mathrm{MS}}$

  $\Delta_{j/p}$: $\sum_{j'}\mathcal{C}^{\Delta}_{j/j'}\otimes f_{j'/p}$

  See also: del Rio, Prokudin, Scimemi, Vladimirov (2024)

Our approach

- $$\frac{d\sigma}{d^4\boldsymbol{q}_T\, d\Omega} = \mathcal{H} \sum\!\!\!\!\!\!\int d^2\boldsymbol{k}_{aT}\, d^2\boldsymbol{k}_{bT}\, f_{j/h_a}(x_a, \boldsymbol{k}_{aT}; \mu_Q, Q^2) f_{\bar{j}/h_b}(x_b, \boldsymbol{k}_{bT}; \mu_Q, Q^2) \delta^{(2)}(\boldsymbol{q}_T - \boldsymbol{k}_{aT} - \boldsymbol{k}_{bT})$$ *Keep this form*

- Start by optimizing the input scale treatment with fixed scale:
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  See also: del Rio, Prokudin, Scimemi, Vladimirov (2024)

  Equivalent to imposing small b_T OPE

- Very small $b_T \ll \frac{1}{Q_0}$ description is not yet optimal
  - RG transform to $\bar{Q}_0 \sim \frac{1}{b_T}$ for $b_T \ll \frac{1}{Q_0}$, Q_0 otherwise
- Evolve to $Q \gg Q_0$ & refine fits

More details

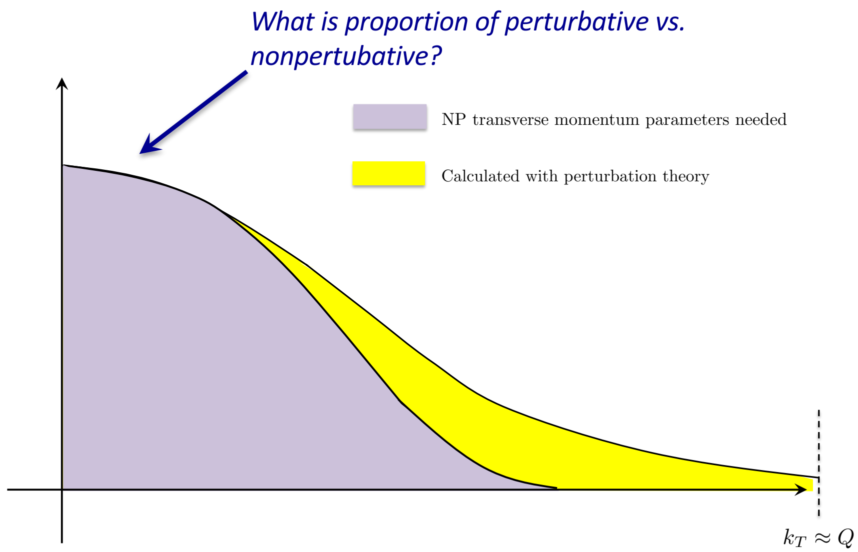
- Recipe to transform a NP TMD parametrization into an evolved parametrization at other scales:
  - *Sec. VI of Phys.Rev.D 106 (2022) 3, 034002*
- No b_{max} or b_* necessary
- No g-functions necessary
- Approach is equivalent to standard TMD factorization, CSS, etc, just with additional effective pheno constraints on the g-functions
- It is straightforward to translate between g-functions, b_* etc, and HSO
  - *Sec. IX of Phys.Rev.D 106 (2022) 3, 034002,*
  - *App. B of 2401.14266 [hep-ph]*

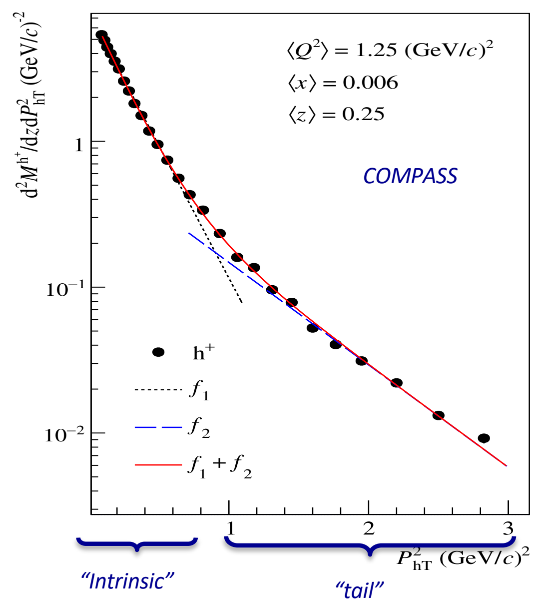
Conventional organization

- Highly sensitive to arbitrary choices near the perturbative/nonperturbative boundary

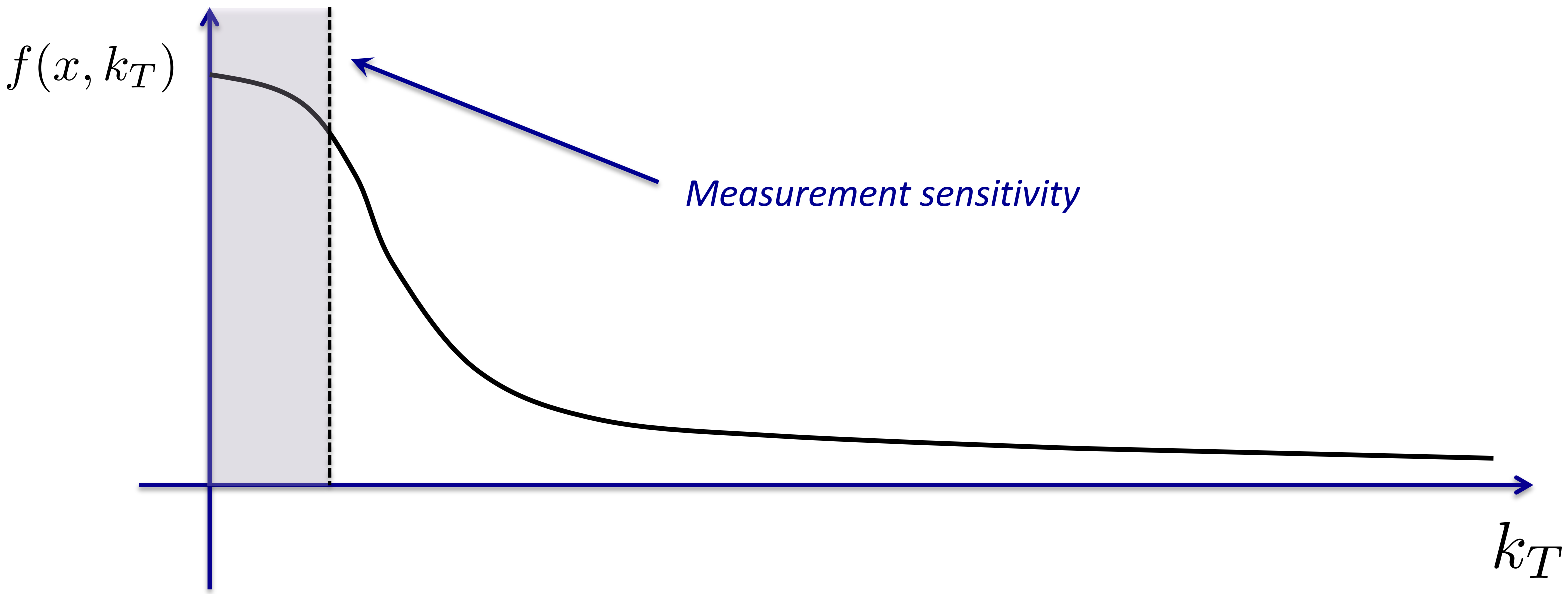
- Examples $\delta \left[\tilde{f}(x, b^*(b_T); \mu_{Q_0}, Q_0) e^{-g(b_T)} \right] \neq 0$ $\qquad \frac{d}{db_{\max}} \left[\tilde{f}(x, b^*(b_T); \mu_{Q_0}, Q_0) e^{-g(b_T)} \right] \neq 0$

$$\pi \int^{\mu_{Q_0}^2} \mathrm{d}k_T^2 f_{j/p}(x, k_T; \mu_{Q_0}, Q_0^2) = f_{j/p}(x; \mu_{Q_0}) + \Delta_{j/p} + O\left(\frac{1}{Q_0 b_{\max}}\right) \longleftarrow \quad b_{\max} \sim 1/Q_0$$





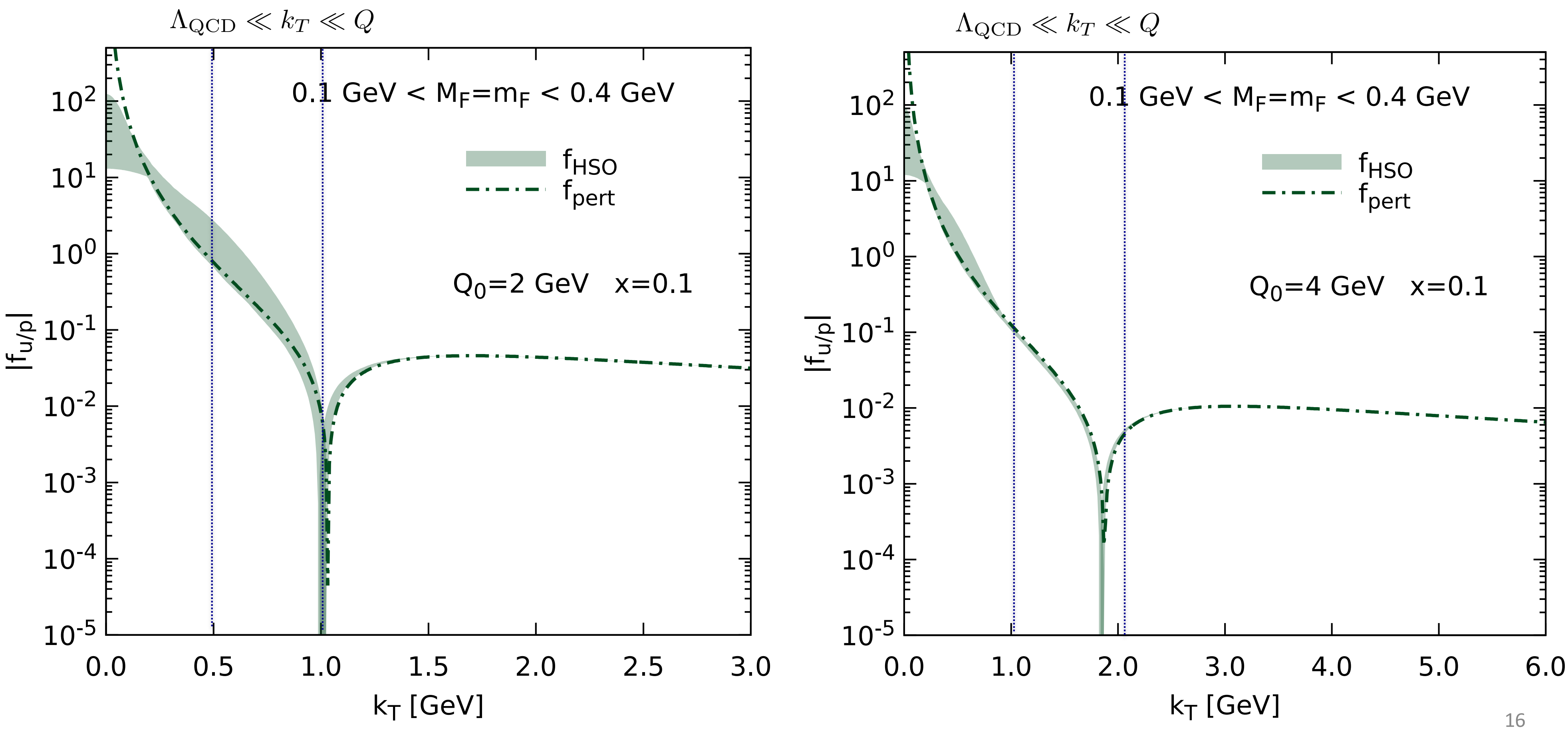
Why worry about "large" k_T at, e.g., Jlab?



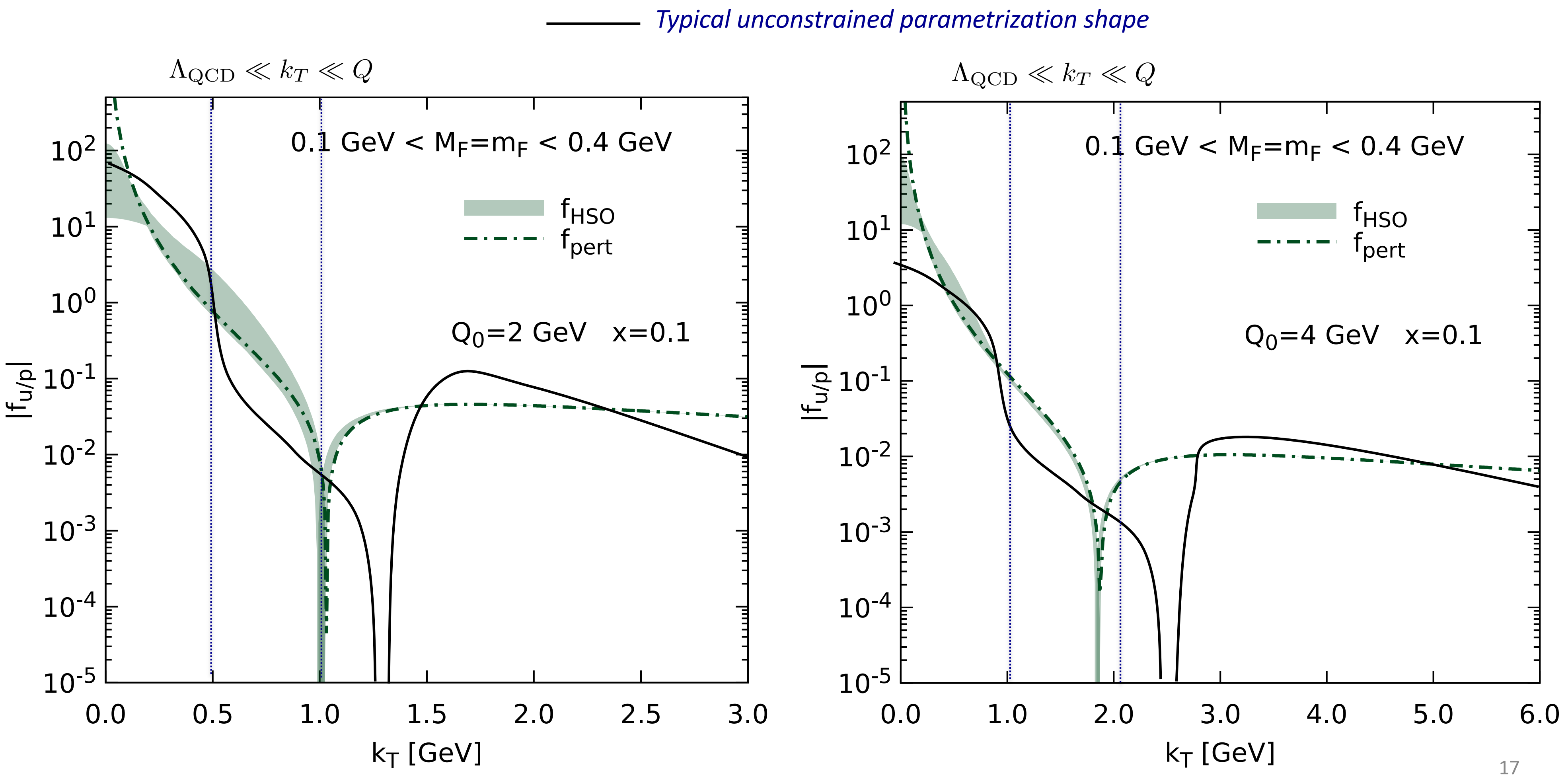
- TMD pdf exists for all k_T. Large/small k_T, consistency constrains small k_T
- How to compare low/moderate Q TMD pdf to large Q pdf?

Individual TMD pdfs?

Typical unconstrained parametrization shape



Individual TMD pdfs?



Conventional organization & its complications

(An example typical of conventional approach)

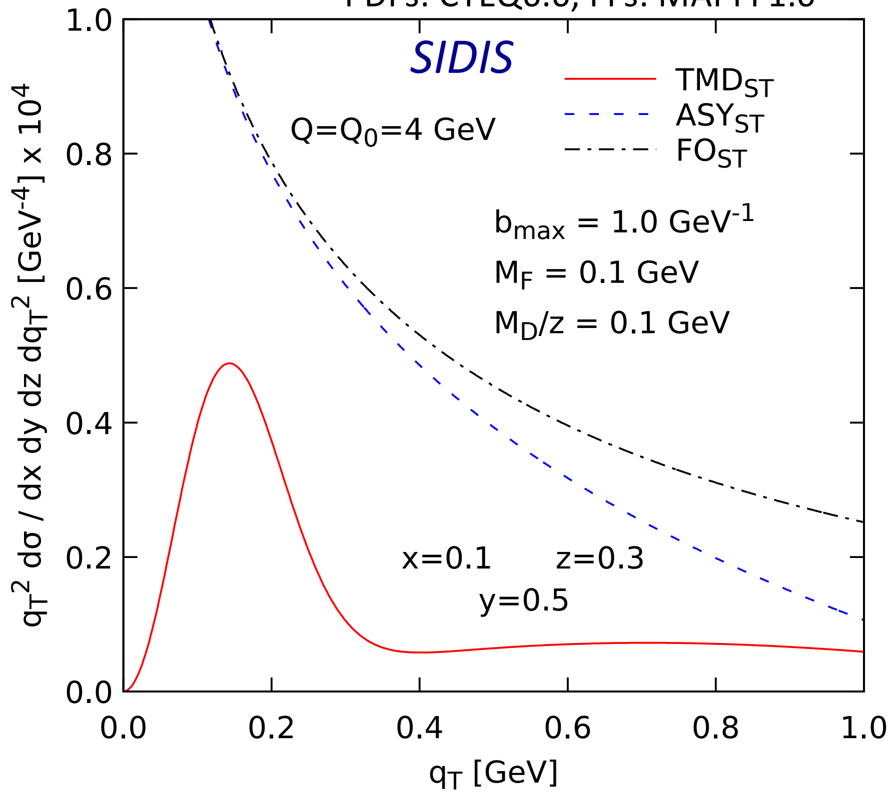
$$g_{j/p}(x, b_{\mathrm{T}}) = \frac{1}{4} M_F^2 \, b_{\mathrm{T}}^2$$

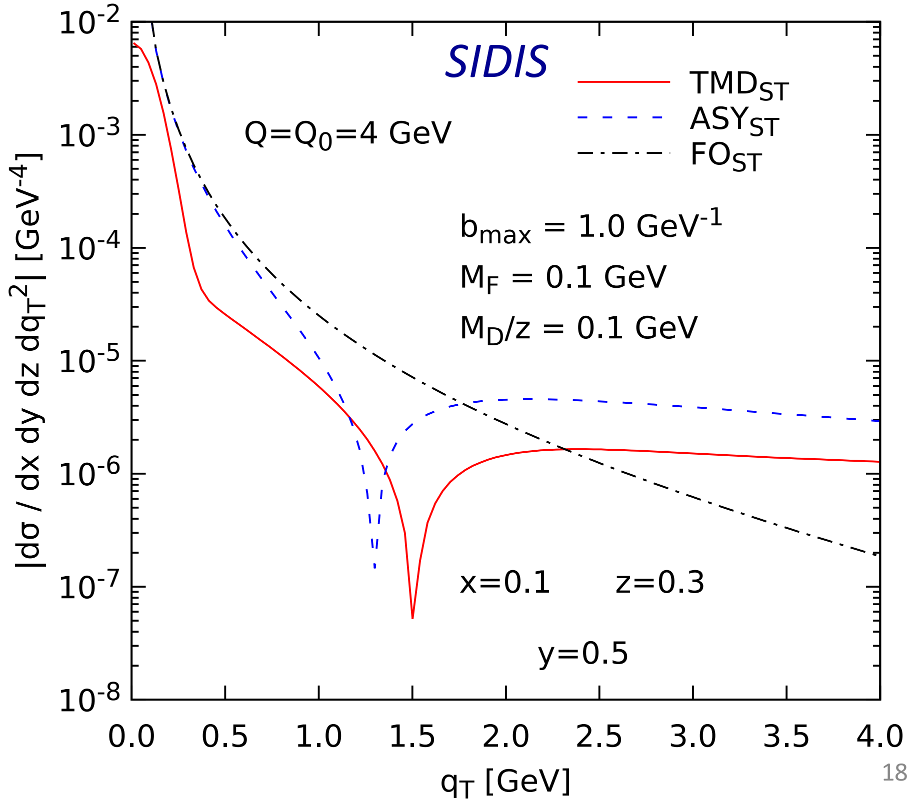
$$g_{h/j}(z, b_{\mathrm{T}}) = \frac{1}{4 z^2} M_D^2 \, b_{\mathrm{T}}^2$$

Red = TMD factorization

Blue = Large q_T approximation

PDFs: CTEQ6.6, FFs: MAPFF1.0





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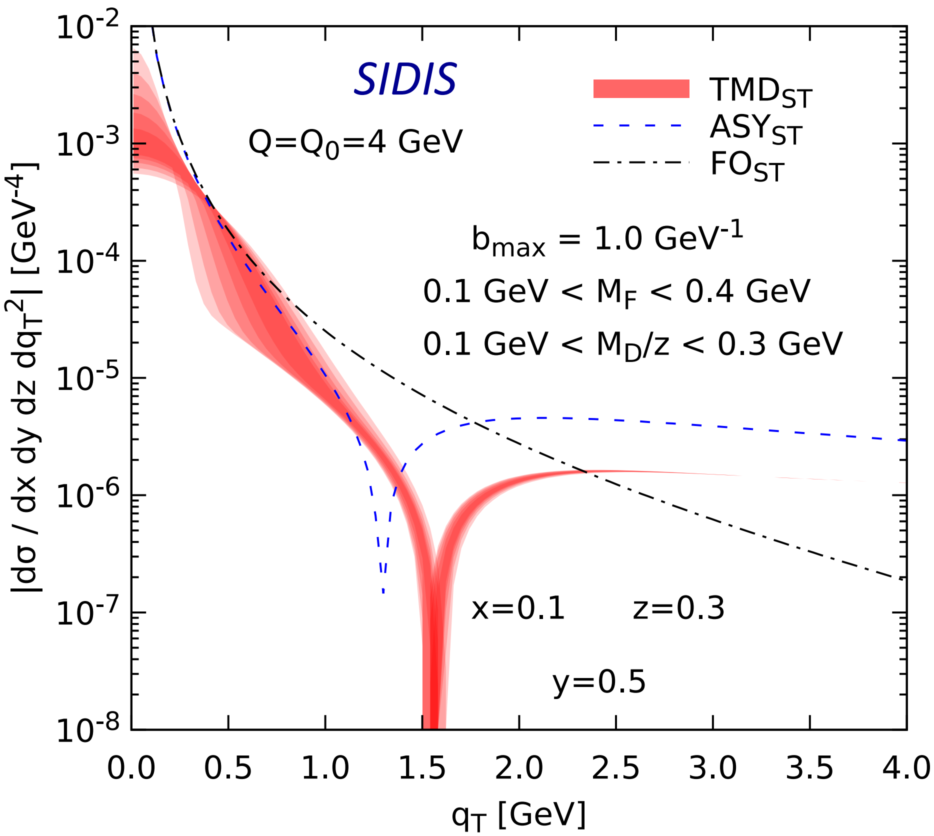
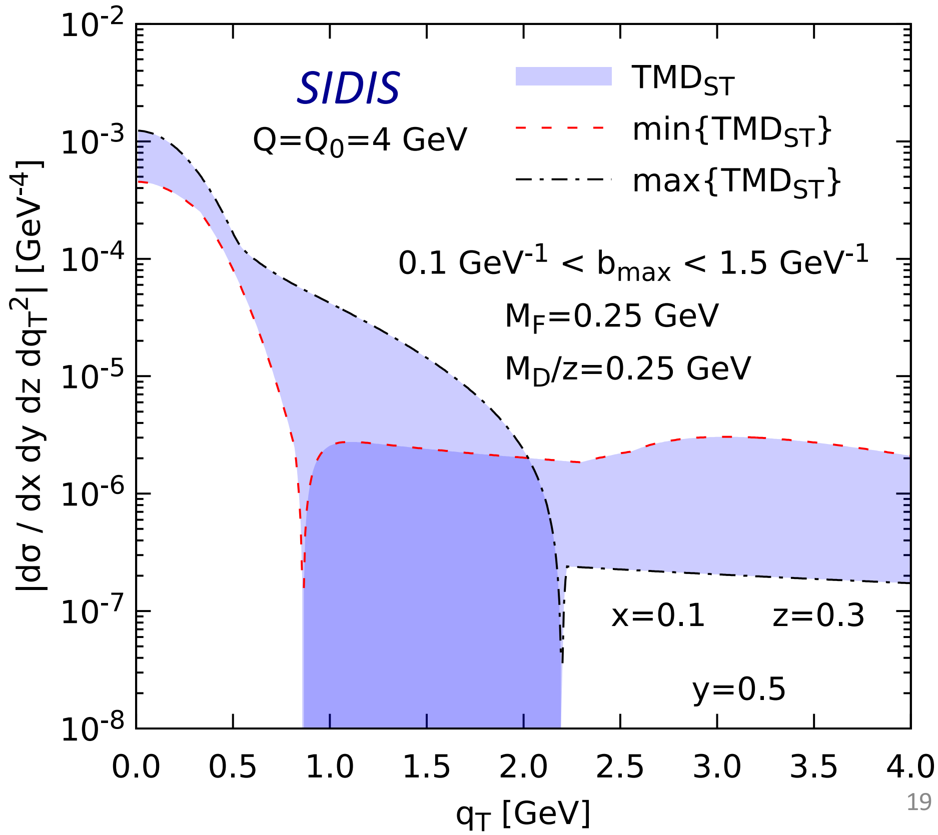
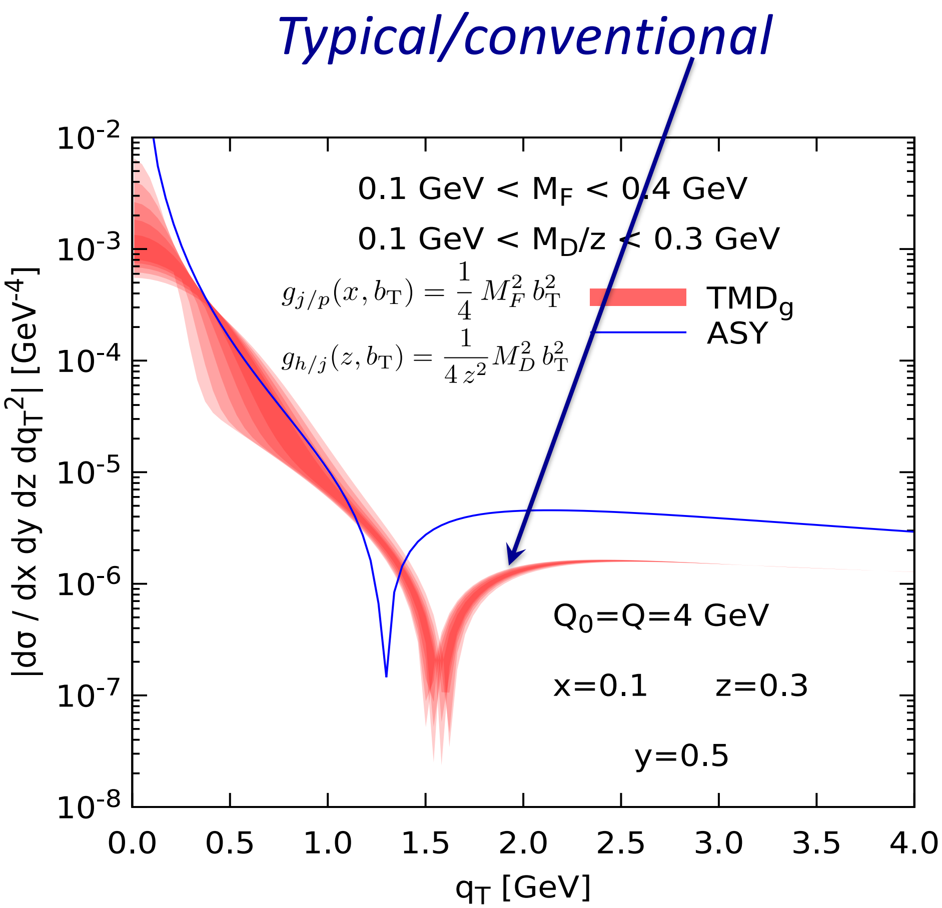
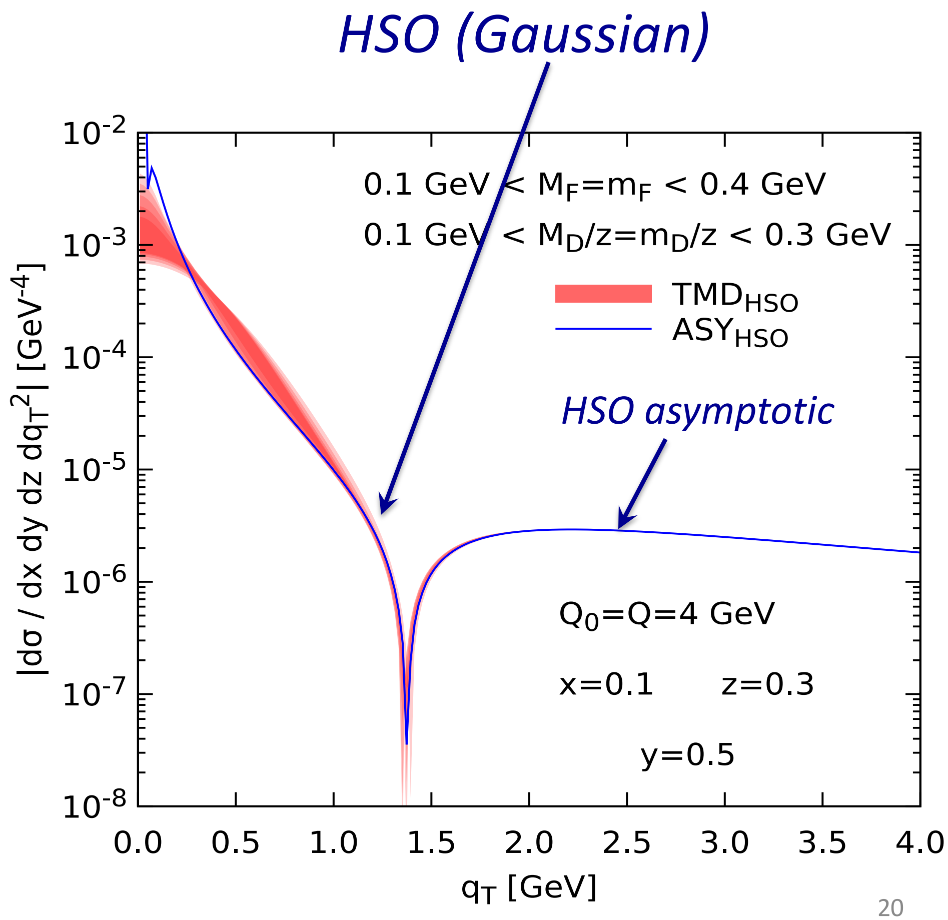




Compare standard/unconstrained with HSO ($O(\alpha_s)$)

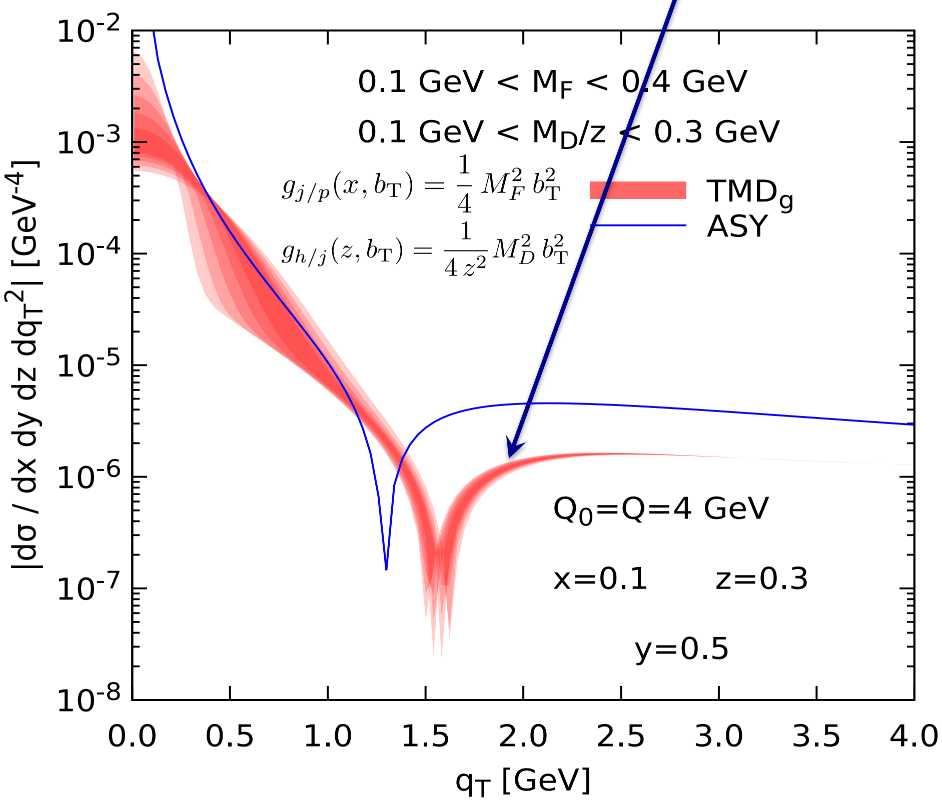


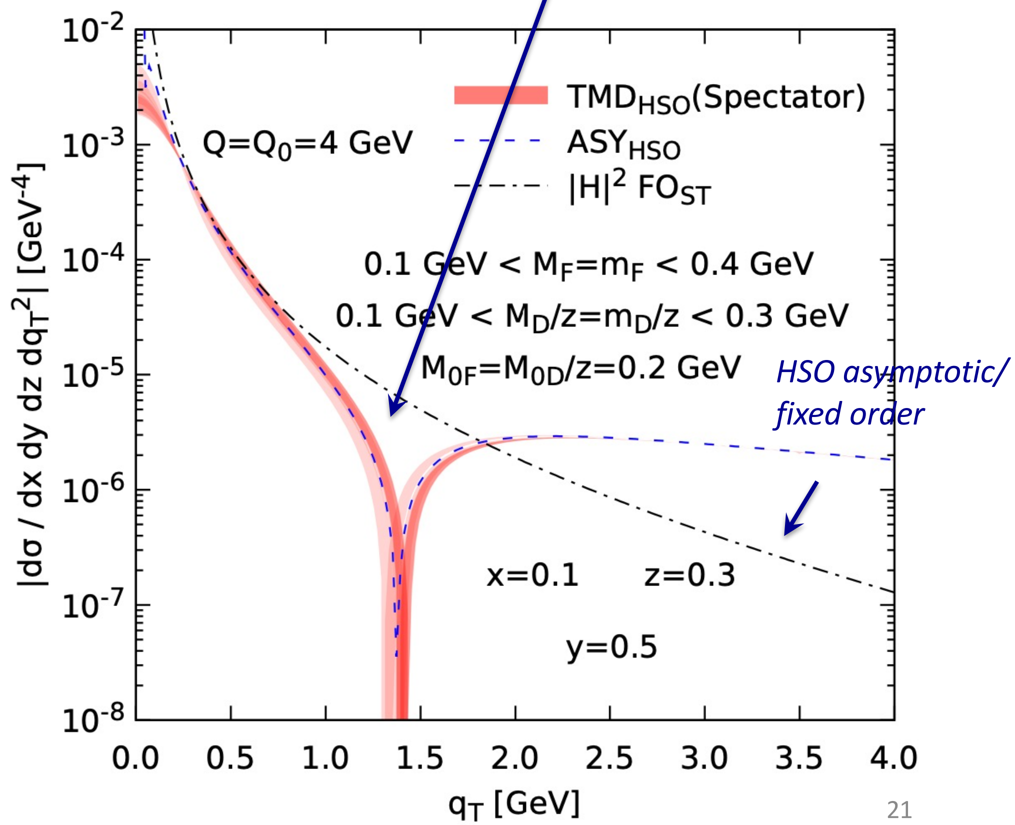


Compare standard/unconstrained with HSO ($O(\alpha_s)$)

Typical/conventional

HSO (Spectator model)

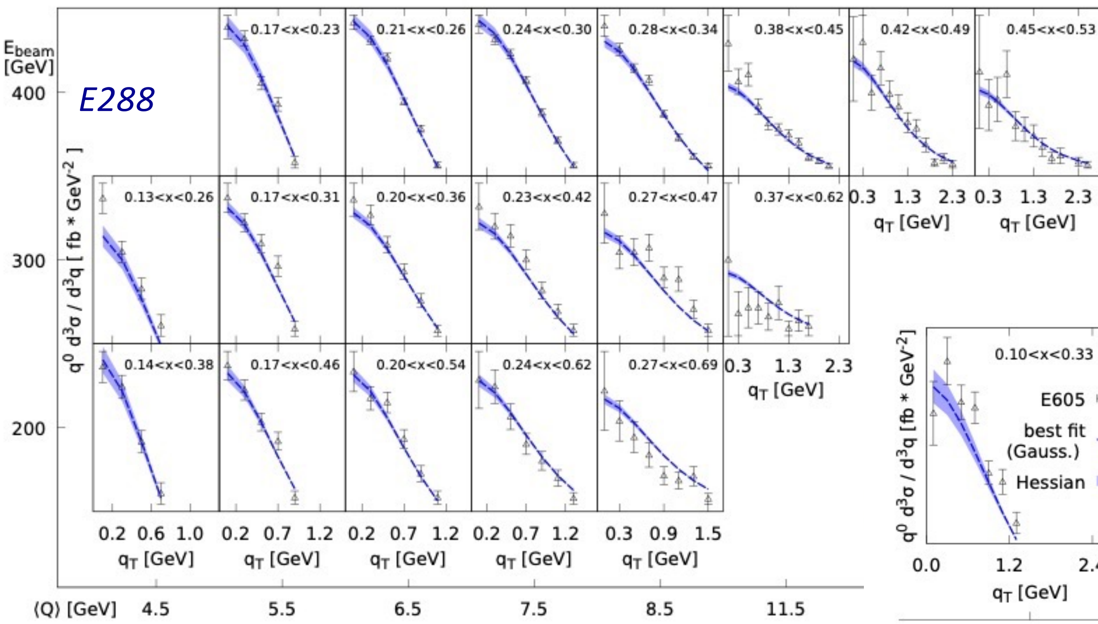




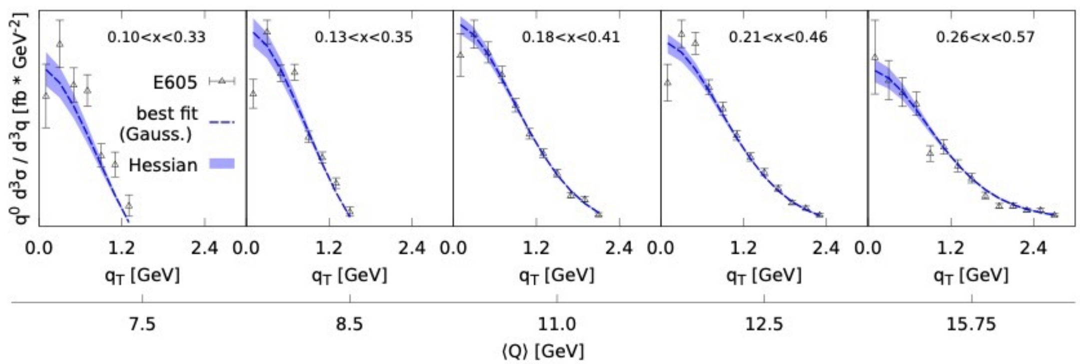
Examples *Proof of principle applied to Drell-Yan*

F. Aslan, M. Boglione, J.O. Gonzalez, T. Rainaldi, TCR, A. Simonelli, 2401.14266 [hep-ph]

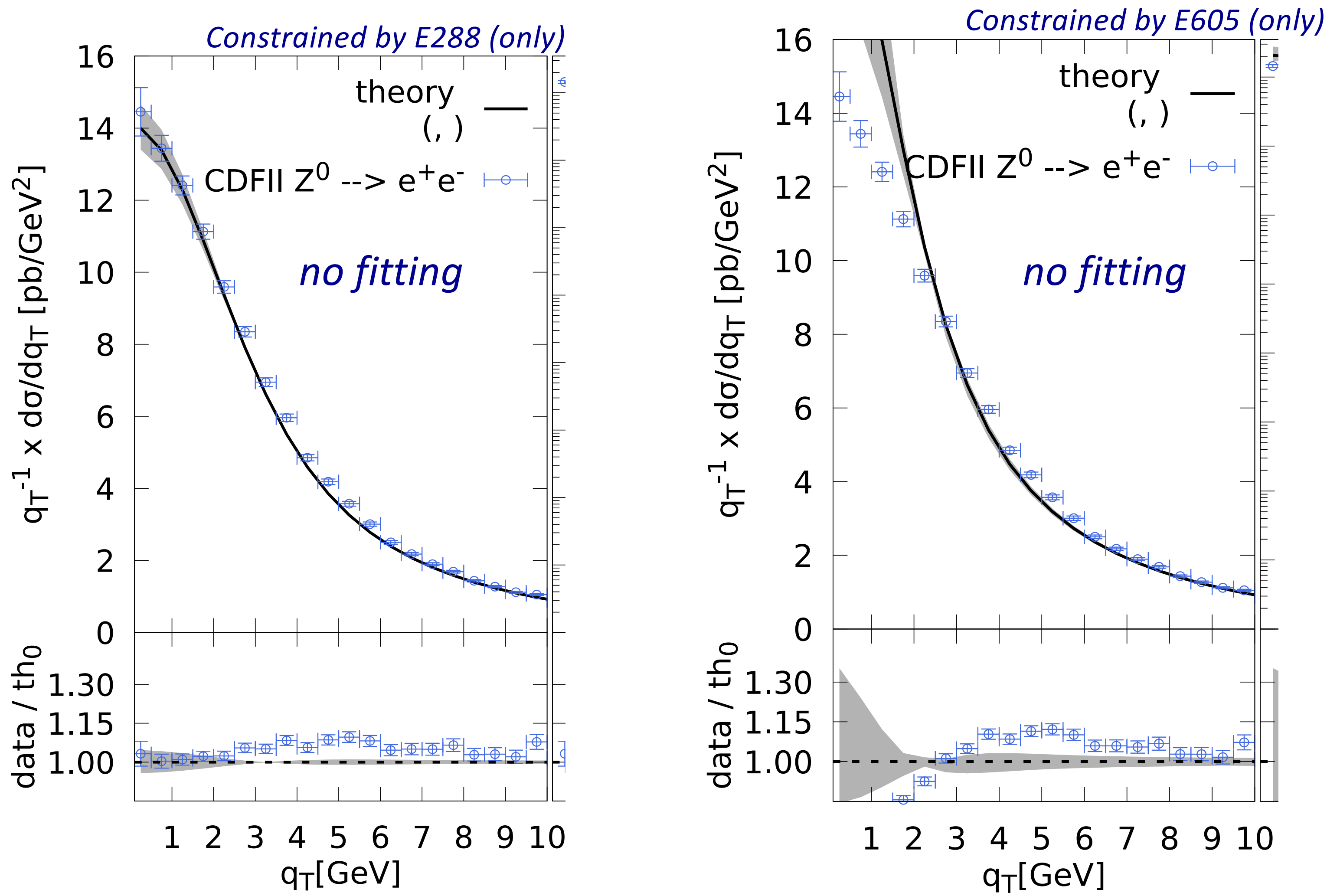
- Can we fit to a moderate Q Drell-Yan set *(high sensitivity to intrinsic TM),* and predict (or "postdict") very large Q *(less sensitivity to intrinsic TM)*?
- Do fit E288 (only) & fit E605 (only). Then evolve upward.
- In doing so, can we distinguish between nonperturbative descriptions?



Gaussian fits		
	E288 (130 pts.)	E605 (52 pts.)
$\chi^2_{\rm dof}$	1.04	1.68
M_0 (GeV)	0.0576	0.404
M_1 (GeV)	0.403	0.290
b_K	2.12	0.744
N(nuisance)	1.29	1.28



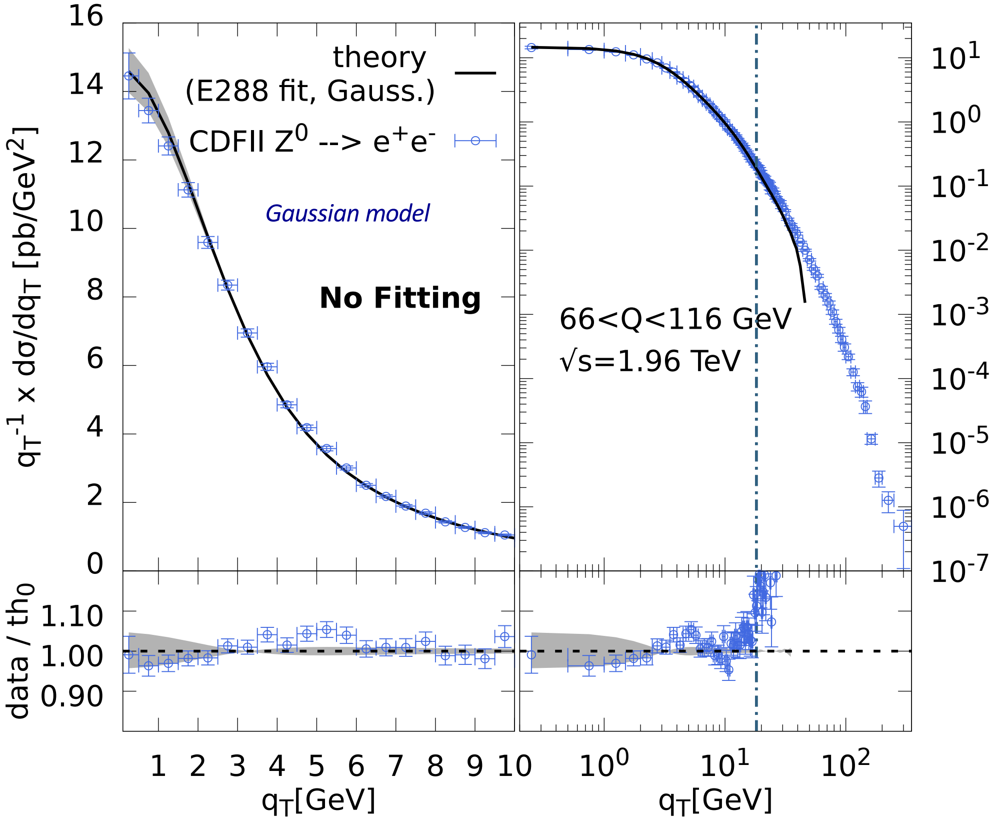
Examples

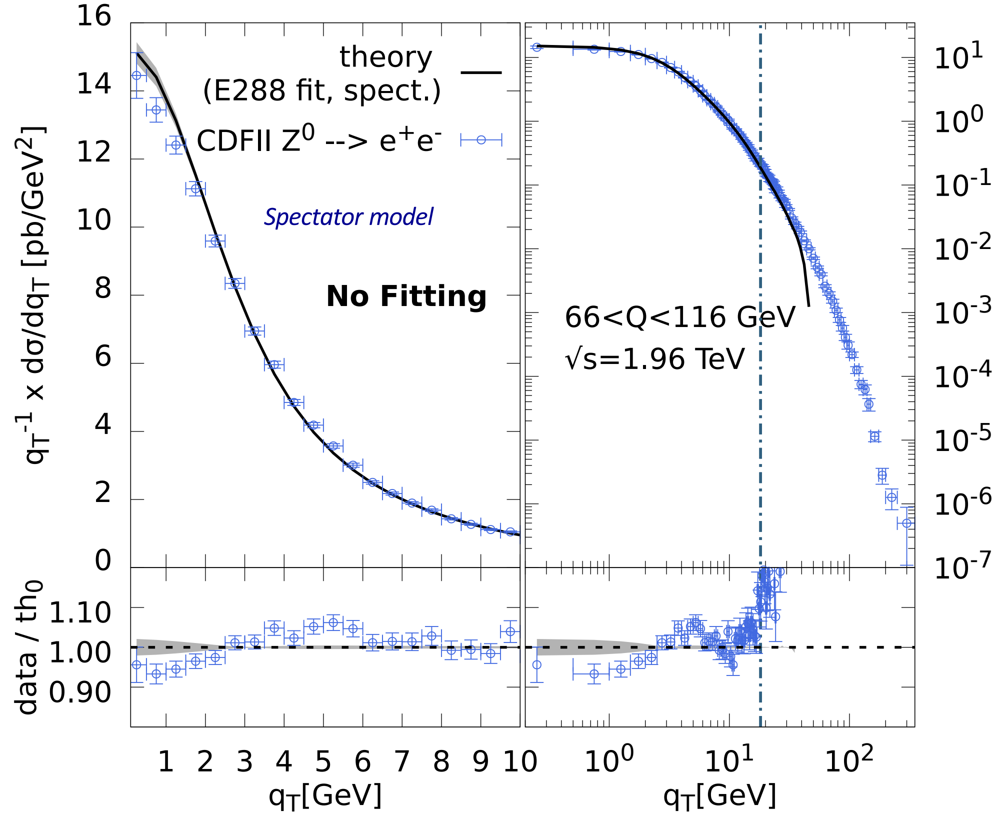


(Systematic correlated uncertainties not displayed)

Examples

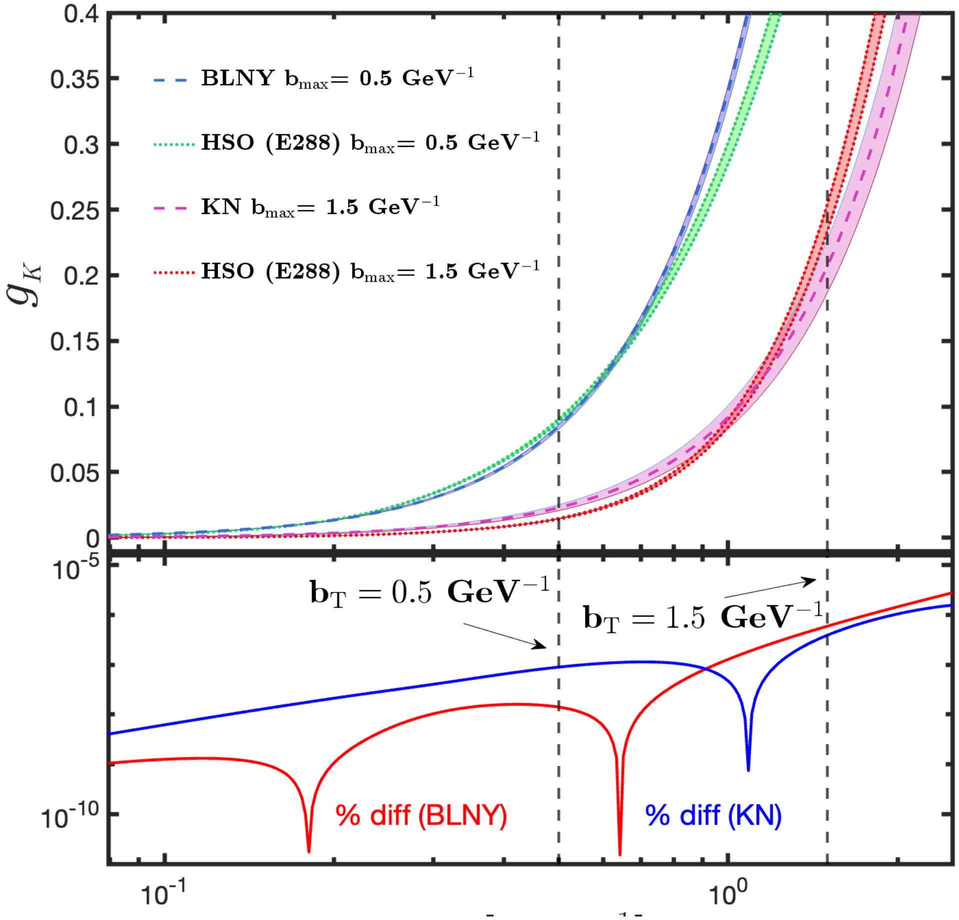
Constrained by E288 (only)





Note: Things get much more difficult when more data are included

Consistency checks

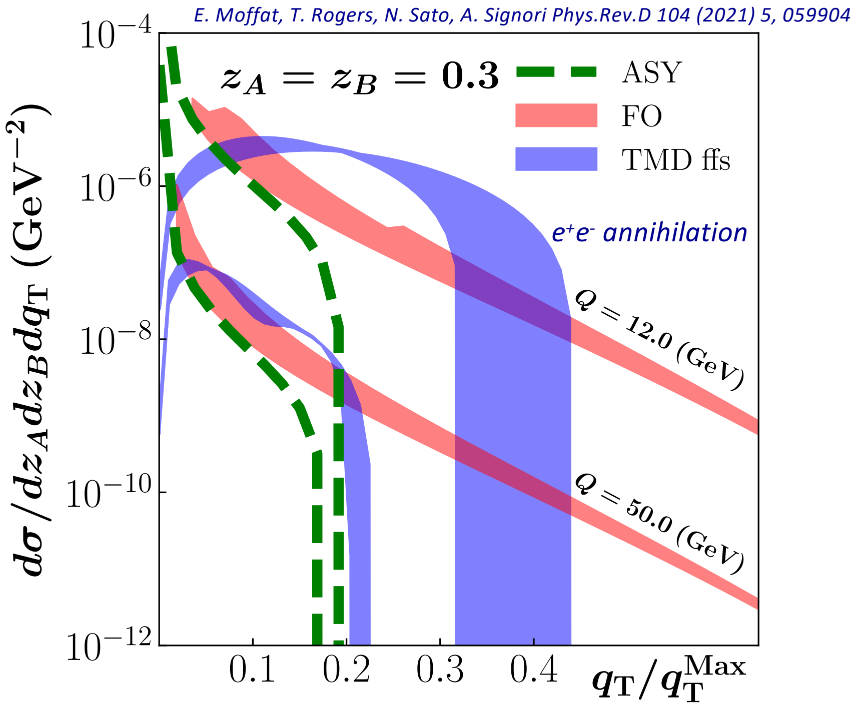


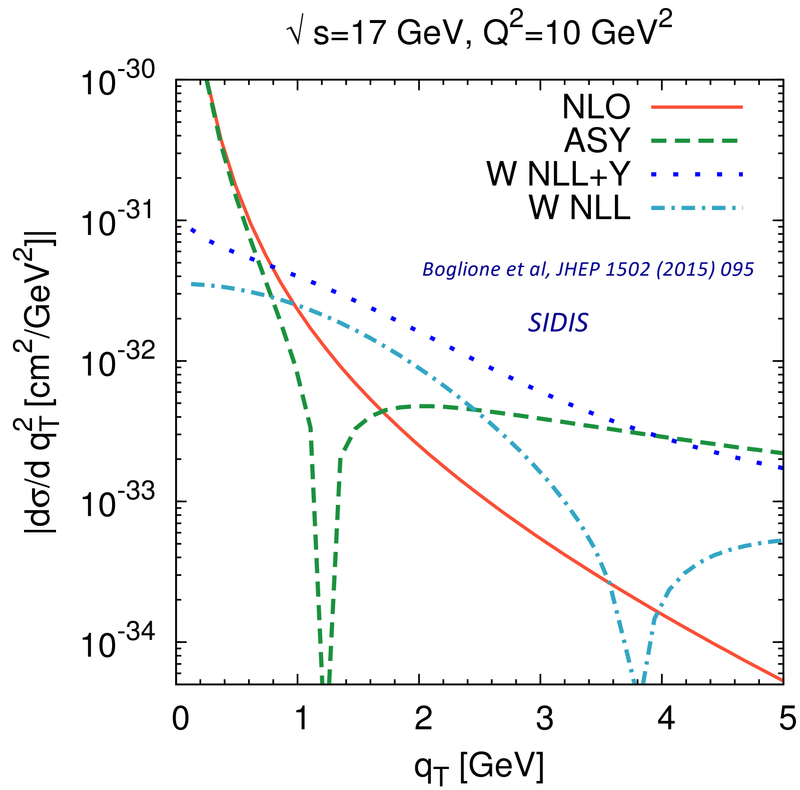
Summary

- Goals: Maximum use of input from theory and theory constraints
- Switching to a more hadron-structure-oriented approach to pheno with TMD factorization improves consistency in the large transverse behavior of TMD correlation functions
- Necessary for understanding the <u>shapes</u> of nonperturbative distributions, separating perturbative and nonperturbative parts, etc.
- Not a new formalism, just a way of organizing parametrizations; **HSO = "standard CSS"!**
- Next steps:
  - Order α_s^2
  - Spin dependent TMDs

See video:
https://youtu.be/7Wqx9yhBXuI

Practical difficulties





- TMD factorization at very large Q effectively resums powers of $\ln \frac{Q}{k_T}$ in the region $\Lambda_{\text{QCD}} \ll k_T \ll Q$. What to do at $Q \approx Q_0$ where no such region exists?

Separating large and small transverse coordinates

F. Aslan et al, Phys.Rev.D 107 (2023) 7, 074031

- Illustrate in a Yukawa theory where everything can be calculated exactly

- Use the OPE $\boldsymbol{b}_*(b_\mathrm{T}) = \dfrac{\boldsymbol{b}_\mathrm{T}}{\sqrt{1 + b_\mathrm{T}^2/b_\mathrm{max}^2}}$

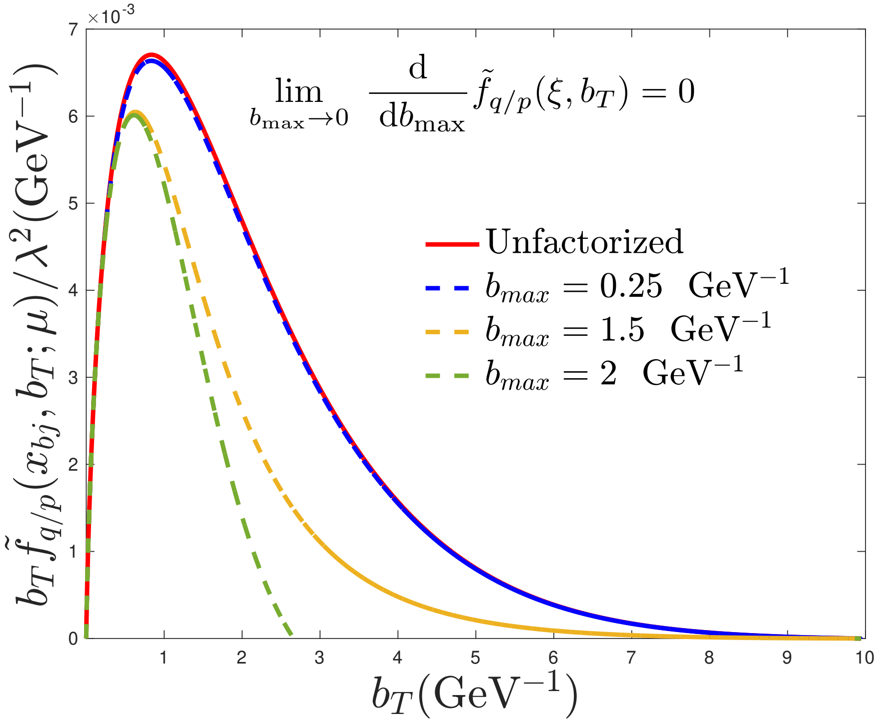
$$\tilde{f}_{q/p}(x_\mathrm{bj}, \boldsymbol{b}_\mathrm{T}; \mu) = \tilde{f}^\mathrm{OPE}_{q/p}(x_\mathrm{bj}, \boldsymbol{b}_*; \mu) \exp\{-g_{q/p}(x_\mathrm{bj}, \boldsymbol{b}_\mathrm{T})\} + \mathcal{O}(m^2 b_\mathrm{max}^2)$$

- $\pi \int^{\mu_{Q_0}^2} \mathrm{d}k_T^2 f_{j/p}(x, k_T; \mu_{Q_0}, Q_0^2) \approx f_{j/p}(x; \mu_{Q_0})$

violated by ≈ 40% in model

$m_\mathrm{quark} = 0.3$ GeV $\quad m_\mathrm{hadron} = 1.0$ GeV

$m_\mathrm{spectator} = 1.5$ GeV



An $O(\alpha_s)$ example

$$f_{\text{inpt},i/p}(x,\boldsymbol{k}_\text{T};\mu_{Q_0},Q_0^2) = \frac{1}{2\pi}\frac{1}{k_\text{T}^2+m_{f_{i,p}}^2}\left[A^f_{i/p}(x;\mu_{Q_0}) + B^f_{i/p}(x;\mu_{Q_0})\ln\frac{Q_0^2}{k_\text{T}^2+m_{f_{i,p}}^2}\right] + \frac{1}{2\pi}\frac{1}{k_\text{T}^2+m_{f_{g,p}}^2}A^{f,g}_{i/p}(x;\mu_{Q_0})$$
$$+ C^f_{i/p}\, f_{\text{core},i/p}(x,\boldsymbol{k}_\text{T};Q_0^2)$$

$$D_{\text{inpt},h/j}(z,z\boldsymbol{k}_\text{T};\mu_{Q_0},Q_0^2) = \frac{1}{2\pi z^2}\frac{1}{k_\text{T}^2+m_{D_{h,j}}^2}\left[A^D_{h/j}(z;\mu_{Q_0}) + B^D_{h/j}(z;\mu_{Q_0})\ln\frac{Q_0^2}{k_\text{T}^2+m_{D_{h,j}}^2}\right] + \frac{1}{2\pi z^2}\frac{1}{k_\text{T}^2+m_{D_{h,g}}^2}A^{D,g}_{h/j}(z;\mu_{Q_0})$$
$$+ C^D_{h/j}\, D_{\text{core},h/j}(z,z\boldsymbol{k}_\text{T};Q_0^2)$$

- C^f & C^D constrained by:

$$f^c_{i/p}(x;\mu_{Q_0}) \equiv 2\pi\int_0^{\mu_{Q_0}} \mathrm{d}k_\text{T}\, k_\text{T} f_{i/p}(x,\boldsymbol{k}_\text{T};\mu_{Q_0},Q_0^2) = f_{i/p}(x;\mu_{Q_0}) + \mathcal{H}_{i/i'}\otimes f_{i'/p} + \text{p.s.}$$
$$d^c_{h/j}(z;\mu_{Q_0}) \equiv 2\pi z^2\int_0^{\mu_{Q_0}} \mathrm{d}k_\text{T}\, k_\text{T} D_{h/j}(z,z\boldsymbol{k}_\text{T};\mu_{Q_0},Q_0^2) = d_{h/j}(z;\mu_{Q_0}) + \mathcal{H}_{j'/j}\otimes d_{h/j'} + \text{p.s.}$$

An $O(\alpha_s)$ example with $\overline{\mathrm{MS}}$ pdfs and ffs

- Parametrizing the very small transverse momentum

A. Gaussian model (very commonly used)

$$f^{\rm Gauss}_{{\rm core},i/p}(x,\boldsymbol{k}_{\rm T};Q_0^2) = \frac{e^{-k_{\rm T}^2/M_{\rm F}^2}}{\pi M_{\rm F}^2}\,, \qquad D^{\rm Gauss}_{{\rm core},h/j}(z,z\boldsymbol{k}_{\rm T};Q_0^2) = \frac{e^{-z^2\,k_{\rm T}^2/M_{\rm D}^2}}{\pi M_{\rm D}^2}$$

B. Spectator model

$$f^{\rm Spect}_{{\rm core},i/p}(x,\boldsymbol{k}_{\rm T};Q_0^2) = \frac{6M_{0\rm F}^6}{\pi\,(2M_{\rm F}^2+M_{0\rm F}^2)}\,\frac{M_{\rm F}^2+k_{\rm T}^2}{(M_{0\rm F}^2+k_{\rm T}^2)^{\,4}}\,, \quad D^{\rm Spect}_{{\rm core},h/j}(z,z\boldsymbol{k}_{\rm T};Q_0^2) = \frac{2M_{0\rm D}^4}{\pi\,(M_{\rm D}^2+M_{0\rm D}^2)}\,\frac{M_{\rm D}^2+k_{\rm T}^2z^2}{(M_{0\rm D}^2+k_{\rm T}^2z^2)^{\,3}}$$

Backup

$$A^{f}_{i/p}(x;\mu_{Q_0}) \equiv \sum_{ii'} \delta_{i'i} \frac{\alpha_s(\mu_{Q_0})}{\pi} \left\{ \left[(P_{i'i} \otimes f_{i'/p})(x;\mu_{Q_0}) \right] - \frac{3C_F}{2} f_{i'/p}(x;\mu_{Q_0}) \right\} ,$$

$$B^{f}_{i/p}(x;\mu_{Q_0}) \equiv \sum_{i'i} \delta_{i'i} \frac{\alpha_s(\mu_{Q_0}) C_F}{\pi} f_{i'/p}(x;\mu_{Q_0}) \,,$$

$$A^{f,g}_{i/p}(x;\mu_{Q_0}) \equiv \frac{\alpha_s(\mu_{Q_0})}{\pi} \left[(P_{ig} \otimes f_{g/p})(x;\mu_{Q_0}) \right] ,$$

$$C^{f}_{i/p} \equiv \frac{1}{N^{f}_{i/p}} \Bigg[f_{i/p}(x;\mu_{Q_0}) - A^{f}_{i/p}(x;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{f_{i,p}}}\right) - B^{f}_{i/p}(x;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{f_{i,p}}}\right) \ln\left(\frac{Q_0^2}{\mu_{Q_0} m_{f_{i,p}}}\right) ,$$

$$- A^{f,g}_{i/p}(x;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{f_{g,p}}}\right) + \frac{\alpha_s(\mu_{Q_0})}{2\pi} \left\{ \sum_{ii'} \delta_{i'i} [\mathcal{C}^{i/i'}_{\Delta} \otimes f_{i'/p}](x;\mu_{Q_0}) + [\mathcal{C}^{i/g}_{\Delta} \otimes f_{g/p}](x;\mu_{Q_0}) \right\} \Bigg] .$$

$$P_{ig}(x) = T_F \left[x^2 + (1-x)^2 \right] ,$$

$$\mathcal{C}^{i/i}_{\Delta}(x) = C_F(1-x) - C_F \frac{\pi^2}{12} \delta(1-x) \,,$$

$$\mathcal{C}^{g/p}_{\Delta}(x) = 2T_F x(1-x) \,,$$

$$N^{f}_{i/p} \equiv 2\pi \int_0^{\infty} dk_{\mathrm{T}} k_{\mathrm{T}} \, f_{\mathrm{core},i/p}(x, \boldsymbol{k}_{\mathrm{T}}; Q_0^2)$$

Backup

$$A^D_{h/j}(z;\mu_{Q_0}) \equiv \sum_{jj'} \delta_{j'j} \frac{\alpha_s(\mu_{Q_0})}{\pi} \left\{ \left[(P_{jj'} \otimes d_{h/j'})(z;\mu_{Q_0}) \right] - \frac{3C_F}{2} d_{h/j'}(z;\mu_{Q_0}) \right\} ,$$

$$B^D_{h/j}(z;\mu_{Q_0}) \equiv \sum_{jj'} \delta_{j'j} \frac{\alpha_s(\mu_{Q_0}) C_F}{\pi} d_{h/j'}(z;\mu_{Q_0}) ,$$

$$A^{D,g}_{h/j}(z;\mu_{Q_0}) \equiv \frac{\alpha_s(\mu_{Q_0})}{\pi} \left[(P_{gj} \otimes d_{h/g})(z;\mu_{Q_0}) \right] ,$$

$$C^D_{h/j} \equiv \frac{1}{N^D_{h/j}} \left[d_{h/j}(z;\mu_{Q_0}) - A^D_{h/j}(z;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{D_{h,j}}}\right) - B^D_{h/j}(z;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{D_{h,j}}}\right) \ln\left(\frac{Q_0^2}{\mu_{Q_0} m_{D_{h,j}}}\right) , \right.$$

$$\left. - A^{D,g}_{h/j}(z;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{D_{h,g}}}\right) + \frac{\alpha_s(\mu_{Q_0})}{2\pi} \left\{ \sum_{jj'} \delta_{j'j} [\mathcal{C}^{j'/j}_{\Delta} \otimes d_{h/j'}](z;\mu_{Q_0}) + [\mathcal{C}^{g/j}_{\Delta} \otimes d_{h/g}](z;\mu_{Q_0}) \right\} \right] .$$

$$P_{qq}(z) = P_{\bar{q}\bar{q}}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right] ,$$

$$P_{gq}(z) = C_F \frac{1+(1-z)^2}{z} ,$$

$$\mathcal{C}^{q/q}_{\Delta}(z) = 2P_{qq}(z) \ln z + C_F(1-z) - C_F \frac{\pi^2}{12} \delta(1-z) ,$$

$$\mathcal{C}^{g/q}_{\Delta}(z) = 2P_{gq}(z) \ln z + C_F z ,$$

$$N^D_{h/j} \equiv 2\pi z^2 \int_0^\infty dk_T k_T \, D_{\mathrm{core},h/j}(z, z\boldsymbol{k}_T; Q_0^2) .$$

Backup

$$
\begin{aligned}
f^c_{\mathrm{inpt},i/p}(x;\mu_{Q_0}) =& 2\pi \int_0^{\mu_{Q_0}} \mathrm{d}k_\mathrm{T}\, k_\mathrm{T} f_{\mathrm{inpt},i/p}(x,\boldsymbol{k}_\mathrm{T};\mu_{Q_0},Q_0^2) = \\
& C^f_{i/p}\, f^c_{\mathrm{core},i/p}(x;\mu_{Q_0}) + \frac{1}{2} A^{f,g}_{i/p}(x;\mu_{Q_0}) \ln\left(1+\frac{\mu^2_{Q_0}}{m^2_{f_{g,p}}}\right) \\
& + \frac{1}{2} A^f_{i/p}(x;\mu_{Q_0}) \ln\left(1+\frac{\mu^2_{Q_0}}{m^2_{f_{i,p}}}\right) + \frac{1}{4} B^f_{i/p}(x;\mu_{Q_0}) \left[\ln^2\left(\frac{m^2_{f_{i,p}}}{Q_0^2}\right) - \ln^2\left(\frac{\mu^2_{Q_0}+m^2_{f_{i,p}}}{Q_0^2}\right)\right] \\
& = f_{i/p}(x;\mu_{Q_0}) + O\left(\alpha_s(\mu_0), \frac{m^2}{Q_0^2}\right) ,
\end{aligned}
$$

$$
\begin{aligned}
d^c_{\mathrm{inpt},h/j}(z;\mu_{Q_0}) =& 2\pi z^2 \int_0^{\mu_{Q_0}} \mathrm{d}k_\mathrm{T}\, k_\mathrm{T} D_{\mathrm{inpt},h/j}(z,z\boldsymbol{k}_\mathrm{T};\mu_{Q_0},Q_0^2) = \\
& C^D_{h/j}\, d^c_{\mathrm{core},h/j}(z;\mu_{Q_0}) + \frac{1}{2} A^{D,g}_{h/j}(z;\mu_{Q_0}) \ln\left(1+\frac{\mu^2_{Q_0}}{m^2_{D_{h,g}}}\right) \\
& + \frac{1}{2} A^D_{h/j}(z;\mu_{Q_0}) \ln\left(1+\frac{\mu^2_{Q_0}}{m^2_{D_{h,j}}}\right) + \frac{1}{4} B^D_{h/j}(z;\mu_{Q_0}) \left[\ln^2\left(\frac{m^2_{D_{h,j}}}{Q_0^2}\right) - \ln^2\left(\frac{\mu^2_{Q_0}+m^2_{D_{h,j}}}{Q_0^2}\right)\right] \\
& = d_{h/j}(z;\mu_{Q_0}) + O\left(\alpha_s(\mu_0), \frac{m^2}{Q_0^2}\right) ,
\end{aligned}
$$