Time Series methods[1]

☐ White noise process

The basic building block for all the process is a sequence $\{\varepsilon_t\}_{t=1}^{\infty}$ whose elements have mean zero and variance σ^2 and for which the ε's are independent of each other when $Y_t = \varepsilon_t$, Y_t follows a white noise process with zero and variance σ^2. This process often modifies real-world news, errors, etc. As the formation, $Y_t = \varepsilon_t, \{\varepsilon_t\}_{t=1}^{\infty} \sim (o, \sigma^2)$, which satisfies the following E$Y_t = 0$, Var$Y_t = \sigma^2 < \infty$ for all t and $\gamma_j \equiv cov(Y_t, Y_{t-j}) = 0$ where j is time interval for all j.

☐ Weekly stationarity

The time series Y_t is said to be weakly stationarity or covariance-stationary if E$Y_t = \mu < \infty$, Var $Y_t = \sigma^2 < \infty$ and Cov($Y_t, Y_{t+k}) = f(k) < \infty$. This process is a requirement for the following models.

☐ MA Model

The moving-average process (MA model) expresses the current value of a time series as a linear function of current and past random shocks with finite lag length. The notation MA(q) refers to the order q: $Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \cdots + \ \theta_q\varepsilon_{t-q}$ which satisfies the moments EY_t = E$Y_{t+1} = \mu$ and Var$Y_t = \left(1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2\right)\sigma^2$ that means Var MA(1) < Var MA(2).

☐ AR Model

The autoregressive process (AR model) has variables linearly represented in their prior values and stochastic terms (traditionally Gaussian distributions). This model can used to describe specific time-varying processes in nature, economics, behavior, etc. The notation AR(p) refers to the order p: $Y_t = \mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \ \phi_q Y_{t-q} + \varepsilon_t$. Expressing this equation in terms of Lag operator (L) $[LY_t = Y_{t-1},\ L^k Y_t = Y_{t-k}]$, $\phi_p(L)Y_t = \mu + \varepsilon_t$, where $\phi_p(x) = 1 - \phi_1 x - \phi_2 x^2 + \cdots + \ \phi_q x^p = 0$ is called as the characteristic equation. If all roots of the characteristic equation lie on or outside the unit circle, the time series satisfies stationarity, and the AR(p) can be expressed as MA (∞)

For example, assuming AR (1): $Y_t = \mu + \phi_1 Y_{t-1} + \varepsilon_t$ and the characteristic equation is $\phi_1(x) = 1 - \phi x = 0$. Th roots $|\phi| < 1$ means condition of stationarity is satisfied. Then

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[1] *The mathematical derivations presented in this section are based on the lecture notes of Professor Jouchi Nakajima (Hitotsubashi University) and Professor In-su Kim (Jeonbuk National University).*

rewrite as MA formation, $Y_t = \frac{\mu}{1-\phi} + \sum_{k=0}^{\infty} \phi^k \varepsilon_{t-k}$. The unconditional mean and variance are derived as follows: $E(Y_t) = \mu + \phi E(Y_{t-1})$, when the stationarity condition is satisfied, then $E(Y_t) = E(Y_{t-1})$. Therefore, the equation can be rewritten as: $E(Y_t) = \mu + \phi E(Y_t)$, solving for $E(Y_t)$, we obtain the **unconditional mean** $\boldsymbol{E(Y_t) = \frac{\mu}{1-\phi}}$. On the other hand, **unconditional variance** is derived as follows: $Var(Y_t) = Var\left(\frac{\mu}{1-\phi} + \sum_{k=0}^{\infty} \phi^k \varepsilon_{t-k}\right)$, since the variance of a constant is zero and the error terms ε_t are independently and identically distributed (i.i.d), we can simplify the expression to $\boldsymbol{\sum_{k=0}^{\infty} \phi^{2k} Var(\varepsilon_{t-k}) = \frac{\sigma_\varepsilon^2}{1-\phi^2}}$.

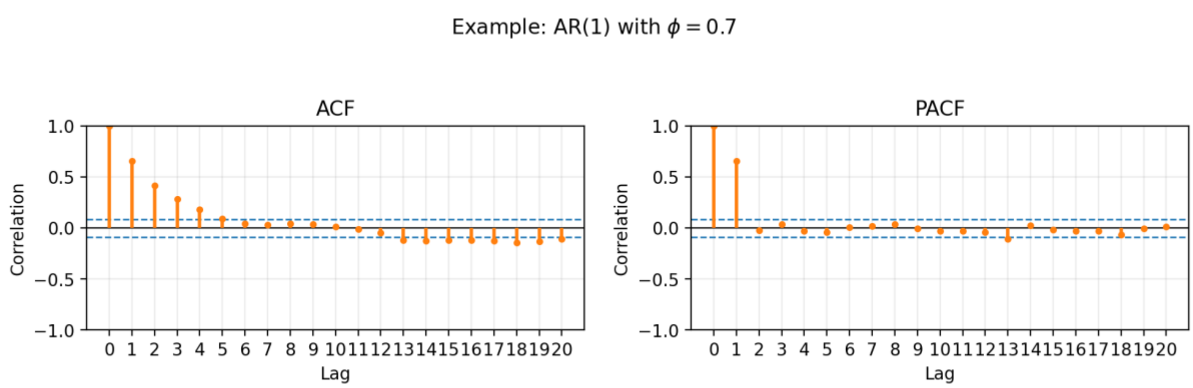
☐ ARIMA Model

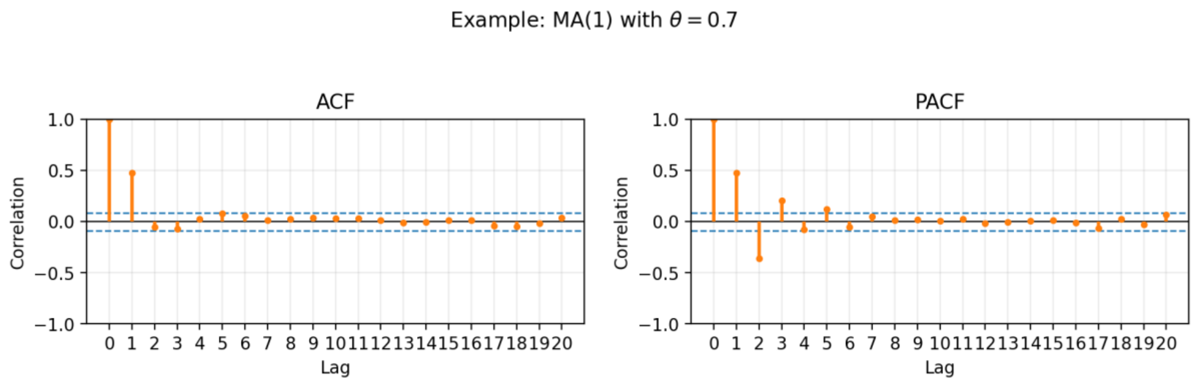
The ARIMA (Autoregressive Integrated Moving-Average) model is designed to capture various temporal structures in data. It is particularly useful for forecasting when the data exhibits a stable patten over time. The model is defined by three main parameters: **AR(p)** part uses the relationship between an observation and a certain number of lagged observations. **MA(q)** part captures the relationship between an observation and a residual error from aw moving-average process applied to lagged observations. Finally, **I(d)** involve differencing the raw observations to make the time series stationary.

Using the lag operator (L), where $LY_t = Y_{t-1}$, the ARIMA(p, d, q) model can be expressed as: $\phi_p(L)(1-L)^d Y_t = \mu + \theta_q(L)\varepsilon_t,\ \varepsilon_t \sim WN(0, \sigma^2)$. To checking parameters (p, q, i), first, we used **Augmented Dickey-Fuller (ADF) test** to identify stationarity. Then, determine initial p and q using **ACF (Autocorrelation Function)** and **PACF (Partial Autocorrelation Function)**.

Definition ADF and PACF:

ACF measures the correlation between a time series and tis own past values. Then, we can select the MA(q) order in ARIMA modeling. PACF are the direct correlation between the current value and a lagged value after removing the effects of shorter lags. It helps select the AR(q) order in ARIME modeling





Concept	Measures	Helps Identify	Pattern
ACF	Total correlation between X_t and X_{t-k}	MA(q) order	Sharp cutoff → MA; Slow decay → AR
PACF	Direct correlation between X_t and X_{t-k}	AR(p) order	Sharp cutoff → AR; Slow decay → MA

☐ ARCH Model

The ARCH model focus on Autoregressive Conditional Heteroskedasticity, proposed by Engle (1982). Its teats the variance of the current error term as a function of the size of previous time periods' error terms.

In an ARCH model, the return R_t consists of an expected return and a shock ε_t. The volatility of this shock, σ_t^2, is modeled as; $\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2$, where σ_t^2 is conditional variance at time t, ε_{t-1}^2 is past squared shock (the "ARCH" term), and as constraints, we require $\omega > 0 \; and \; \alpha \geq 0$ to ensure that the predicted variance is always positive. The limitation is that in the long-term persistence in volatility ARCH model often requires a very high number of lag (q), making the model complex and difficult to estimate.

☐ GARCH Model

To solve the complexity of ARCH, Bollerslev (1986) introduced GARC (Generalized ARCH). The specification GARCH (1, 1) model is the most widely used version because it captures volatility dynamics efficiently with very few parameters. This model variance as: $\sigma_t^2 = \omega + \beta_1 \sigma_{t-1}^2 + \alpha_1 \varepsilon_{t-1}^2$, where $\beta_1 \sigma_{t-1}^2$ means the GARCH term, which includes the model's own past variance, allowing the impact of shock to persist over time. And $\alpha_1 \varepsilon_{t-1}^2$ is the ARCH term, representing news/shocks from the previous period.

As properties of GARCH (1, 1), this model can be mathematically rewritten as an ARCH model with infinite lags; Assuming $\beta_1 < 1$, $\sigma_t^2 = \omega + \beta_1 L \sigma_t^2 + \alpha_1 \varepsilon_{t-1}^2$. Using the expansion, $(1 - \beta_1 L)\sigma_t^2 = \omega + \alpha_1 \varepsilon_{t-1}^2$, $\sigma_t^2 = \frac{\omega}{1-\beta_1 L} + \frac{\alpha_1 \varepsilon_{t-1}^2}{1-\beta_1 L}$ then, $\sigma_t^2 = \frac{\omega}{1-\beta_1 L} + \alpha_1 \sum_{t=1}^{\infty} \beta^{i-1} \varepsilon_{t-i}^2$, which means ARCH(∞). In the stationarity, for the volatility to be stable and eventually return to a long-run average, the condition $\alpha + \beta < 1$ must hold. Also, the sum $\alpha + \beta$ measures how slowly the volatility shock dies out. If the sum is close to 1, the "memory" of the shock lasts a long time. Finally, if the process is stationary, the volatility will converge to the constant value $\frac{\omega}{1-\alpha-\beta}$ as $t \to \infty$.

☐ GJP-GARCH Model

Developed by Glosten, Jagannathan, and Runkle (1993), this model introduces a "threshold" or indicator function to give extra weight to negative shocks.

The Formula.

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\varepsilon_{t-1}^2 + \gamma D_{t-1}^- \varepsilon_{t-1}^2,$$

$$D_{t-1}^- = \begin{cases} 0, & \varepsilon_{t-1} \geq 0 \\ 1, & \varepsilon_{t-1} < 0 \end{cases} \quad \omega > 0, \ \ \alpha, \beta, \gamma \ \geq 0,$$

Indicator variable ($\boldsymbol{D_{t-1}^-}$) is a dummy variable that equals 1 if the previous shock was negative and 0 otherwise. The parameter (γ) captures the asymmetry impact.

$$So, \qquad \sigma_t^2 = \begin{cases} \omega + \beta\sigma_{t-1}^2 + \alpha\varepsilon_{t-1}^2 & \varepsilon_{t-1} \geq 0 \\ \omega + \beta\sigma_{t-1}^2 + (\alpha+\gamma)\varepsilon_{t-1}^2 & \varepsilon_{t-1} < 0 \end{cases}$$

This model reflects "Asymmetric Impact", where Good news ($\varepsilon_{t-1} \geq 0$) and Bad news ($\varepsilon_{t-1} < 0$).

☐ EGARCH Model

The Exponential GARCH model, introduced by Nelson (1991), takes a different approach by modeling the logarithm of variance.

The Formula.

$$\log(\sigma_t^2) = \omega + \beta[\log(\sigma_{t-1}^2) - \omega] + \theta z_{t-1} + \gamma[|z_{t-1}| - E(|z_{t-1}|)]$$

Since its $\log(\sigma_t^2)$, the variance σ_t^2 is guaranteed to be positive regardless of the parameter values. This means we don't need to impose strict non-negativity constraints like $\alpha + \beta \geq 0$. The term θz_{t-1} allows the model to react differently based on the sign of the standardized shock (z_{t-1}). Also, because it is log-linear, the model can easily incorporate other variables that might take negative values.