Time Series methods[1]

☐ Stochastic Volatility (SV) Model

The SV model is used to describe the dynamics of assert returns where the volatility is not constant but follows its own latent stochastic process.

The SV Formula:

$$\text{Observation Equation: } R_t = E(R_t|\, I_{t-1}) + \varepsilon_t$$

$$\text{Error Term: } \varepsilon_t = \sigma_t z_t$$

$$\text{Stochastic Volatility Process: } \log \sigma_t^2 = \omega + \phi \log(\sigma_{t-1}^2) + \eta_t,\ \eta_t \sim N(0, \sigma_\eta^2)$$

In observation Equation, the return R_t is composed of its conditional expectation and an error term ε_t. The z_t of error term follows a standard normal distribution N (0, 1). The log-volatility follows an AR (1) process. It is assumed that $|\phi| < 1$ to ensure the process is stationary. For the first period, the initial log-variance is assumed to follow the stationary distribution of the AR (1) model as $\log \sigma_1^2 \sim N\left(\frac{\omega}{1-\phi}, \frac{\sigma_\eta^2}{1-\phi^2}\right)$.

In financial engineering, the Ornstein-Uhlenbeck (OU) process is often used. The SV model is essentially a discrete-time approximation of this continuous process. Unlike ARCH models, the volatility σ_t^2 is unknown at time $t-1$ because of random error η_t. Because σ_t^2 is unobserved, the likelihood function $L(\theta)$ requires integrating over all possible volatility states (a T-dimensional integral), making MLE computationally difficult:

$$\begin{aligned} L(\theta) = f(\{\varepsilon_t\}_{t=1}^T|\theta) &= \int_0^\infty \cdots \int_0^\infty f(\{\varepsilon_t\}_{t=1}^T, \{\sigma_t^2\}_{t=1}^T|\theta)\, d\sigma_1^2 \cdots d\sigma_T^2 \\ &= \int_0^\infty \cdots \int_0^\infty f(\{\varepsilon_t\}_{t=1}^T|\{\sigma_t^2\}_{t=1}^T)\, f(\{\sigma_t^2\}_{t=1}^T|\theta)\, d\sigma_1^2 \cdots d\sigma_T^2 \\ &= \int_0^\infty \cdots = \int_0^\infty \left[\prod_{t=1}^T f(\varepsilon_t|\sigma_t^2)\right] f(\sigma_1^2|\theta) \\ &\quad \cdot \left[\prod_{t=2}^T f(\sigma_t^2|\sigma_{t-1}^2, \theta)\right] d\sigma_1^2 \cdots d\sigma_T^2, \end{aligned}$$

[1] *The mathematical derivations presented in this section are based on the lecture notes of Professor Jouchi Nakajima (Hitotsubashi University) and Professor In-su Kim (Jeonbuk National University).*

$$where\ f(\varepsilon_t|\sigma_t^2) = \frac{1}{\sqrt{2\pi\sigma_t^2}}\exp(-\frac{\varepsilon_t^2}{2\sigma_t^2})$$

$$f(\sigma_t^2|\sigma_{t-1}^2,\theta) = \frac{1}{\sqrt{2\pi\sigma_\eta^2\sigma_t^2}}\exp(-\frac{\{\log(\sigma_t^2)-\omega-\phi\log(\sigma_{t-1}^2)\}^2}{2\sigma_\eta^2}),$$

$$f(\sigma_1^2|\theta) = \frac{1}{\sqrt{2\pi\sigma_\eta^2/\,(1-\phi^2)\sigma_1^2}}\exp(-\frac{\{\log(\sigma_1^2)-\omega/(1-\phi)\}^2}{2\sigma_\eta^2/(1-\phi^2)}).$$

To estimate the stochastic volatility model, the Kalman Filter is employed by converting the non-linear SV model into a Linear State Space representation. By squaring and taking the log of the errors, we get $\log(\varepsilon_t^2) = \log(\sigma_t^2) + \log(z_t^2)$. The distribution of $\log(z_t^2)$ is not normal (it is skewed), but for the filter, it is approximated as $N(-1.27, \frac{\pi^2}{2})$.

The State Space Formula:

Measurement Equation: $y_t = a + bx_t + u_t,\ u_t{\sim}N(0,\sigma_u^2$

Transition Equation: $x_t = \omega + \phi x_{t-1} + \eta_t,\ \eta_t \sim N(0,\sigma_\eta^2)$

If both u_t and η_t are normal, it is called a Linear Gaussian State Space model.

To filtering and smoothing algorithms, the Kalman Filter recursively estimates the unobserved state x_t using two steps**: Prediction Step,** uses information up to $t-1$ to predict the state at t ($x_{t|t-1}\ and\ P_{t|t-1}$). **Updating Step,** adjusts prediction using the newly observed y_t to find the filtered state ($x_{t|t}\ and\ P_{t|t}$). Finally, **smoothing** is performed by moving backward from the end of the sample (T) to the beginning to refine the state estimates using the entire date set.

The State Space Algorithm:

Predicted State Mean: $x_{t|t-1} = \omega + \phi x_{t-1|t-1}$

Predicted State Variance: $P_{t|t-1} = \phi^2 P_{t-1|t-1} + \sigma_\eta^2$

Predicted Error: $v_t = y_t - a - bx_{t|t-1}$

Innovation Variance: $F_t = b^2 P_{t|t-1} + \sigma_u^2$

Updated State Mena: $x_{t|t} = x_{t|t-1} + \frac{bP_{t|t-1}}{F_t}v_t$

Updated State Variance: $P_{t|t} = P_{t|t-1} - \frac{b^2P_{t|t-1}^2}{F_t}$

First step predicts the current state and its variance using information available up to time $t-1$. Once the new observation y_t is available, the predictions are updated to find the filtered state.

The Kalman Filter is also used to calculate the Likelihood L for parameter estimation. The density of the observations is given by $f(y_t|\tilde{y}_{t-1}) = \frac{1}{\sqrt{2\pi F_t}}\exp(-\frac{v_t^2}{2F_t})$. The total likelihood is the product of these densities $from\ t = 1\ to\ T.$ The process typically starts at $t = 1$ using the stationary distribution of the AR (1) process: $x_{1|0} = \frac{\omega}{1-\phi}$ and $P_{1|0} = \frac{\sigma_\eta^2}{1-\phi^2}$.

Next, the smoothing start from the final period T and moves backward in time for $T - 1$ down to 1. The goal is to find the density $f(x_t|\tilde{y}_T)$. This is achieved by integrating the joint density of consecutive state: $f(x_t|\tilde{y}_T) = \int f(x_t, x_{t-1}|\tilde{y}_T)\, dx_{t+1}$. This process utilizes the results already calculated during the forward pass of the Kalman Filter:

$$\int f(x_t, x_{t-1}|\tilde{y}_T) = f(x_{t+1}|\tilde{y}_T) f(x_t|x_{t+1}, \tilde{y}_T)$$

$$= f(x_{t+1}|\tilde{y}_T) f(x_t|x_{t+1}, \tilde{y}_T)$$

$$f(x_{t+1}|\tilde{y}_T) \frac{f(x_{t+1}|x_t, \tilde{y}_T) f(x_t|\tilde{y}_T)}{f(x_{t+1}|\tilde{y}_T)}$$

$$f(x_t|\tilde{y}_T) = \int f(x_t, x_{t-1}|\tilde{y}_T)\, dx_{t+1}$$

$$= f(x_t|\tilde{y}_T) \frac{\int f(x_{t+1}|\tilde{y}_T)\, f(x_{t+1}|x_t)}{f(x_{t+1}|\tilde{y}_T)}\, dx_{t+1}$$

In a Linear Gaussian State Space model, the smoothed distribution remains normal, defined by its mean $x_{t|T}$ and variance $P_{t|T}$:

Smoothed State Mean: $x_{t|T} = x_{t|t}\, P_t^*\left(x_{t+1|T} - x_{t+1|t}\right)$

Smoothed State Variance: $P_{t|T} = P_{t|t} + (P_t^*)^2(P_{t+1|T} - P_{t+1|t})$

Smoothing Gain (P_t^*): $P_t^* = \frac{\phi P_{t|t}}{P_{t+1|t}}$

To perform smoothing, we must follow these two steps. **Forward Pass,** run the Kalman Filter to compute and store the filtered values $x_{t|t-1}$, $P_{t|t-1}$, and $P_{t|t}$ for $all\ t = 1, \dots, T$. Next, **Backward Pass,** start from the final filtered state $x_{T|T}$ and $P_{T|T}$, Then apply the smoothing equations recursively backward to $t = 1$.