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CUBIC SUBALGEBRAS AND FILTERS OF CI-ALGEBRAS

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Abstract. The notions of cubic subalgebras and cubic filters in CI-algebras are introduced, and related properties are investigated. Characterizations of cubic subalgebras are considered. Conditions for a cubic set to be a cubic filter are provided.

1. Introduction

As a generalization of a BCK-algebra, Kim and Kim [6] introduced the notion of a BE-algebra, and investigated several properties. The notion of CI-algebras is introduced by Meng [8] as a generalization of BE-algebras. Filter theory and properties in CI-algebras are studied by Kim [7], Meng [9] and Piekart et al. [10]. Fuzzy sets, which were introduced by Zadeh [11], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. Based on the (interval-valued) fuzzy sets, Jun et al. [3] introduced the notion of (internal, external) cubic sets, and investigated several properties. Jun et al. applied the notion of cubic sets to BCK/BCI-algebras (see [1, 2, 4, 5]).

In this paper, we discuss the notions of cubic subalgebras and cubic filters in CI-algebras. We investigated several related properties. We consider characterizations of cubic subalgebras. We provide conditions for a cubic set to be a cubic filter.

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2. Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

An algebra $(X; *, 1)$ of type $(2, 0)$ is called a *CI-algebra* if it satisfies the following properties:

(CI1) $x * x = 1$,
(CI2) $1 * x = x$,
(CI3) $x * (y * z) = y * (x * z)$, for all $x, y, z \in X$.

Let $(X; *_X, 1_X)$ and $(Y; *_Y, 1_Y)$ be two CI-algebras. A mapping $f : X \to Y$ is called a *homomorphism* form X to Y if for all $x, y \in X$, $f(x *_X y) = f(x) *_Y f(y)$.

Let $(X; *, 1)$ be a CI-algebra, A subset F of X is called a *filter* (see [8]) of X if

(F1) $1 \in F$;
(F2) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

Let I be a closed unit interval, i.e., $I = [0, 1]$. By an *interval number* we mean a closed subinterval $\overline{a} = [a^-, a^+]$ of I, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $D[0, 1]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) of two elements in $D[0, 1]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $D[0, 1]$. Consider two interval numbers $\overline{a}_1 := \left[a_1^-, a_1^+\right]$ and $\overline{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned} &\operatorname{rmin}\{\overline{a}_1, \overline{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right], \\ &\overline{a}_1 \succeq \overline{a}_2 \text{ if and only if } a_1^- \geq a_2^- \text{ and } a_1^+ \geq a_2^+, \end{aligned}$$

and similarly we may have $\overline{a}_1 \preceq \overline{a}_2$ and $\overline{a}_1 = \overline{a}_2$. To say $\overline{a}_1 \succ \overline{a}_2$ (resp. $\overline{a}_1 \prec \overline{a}_2$) we mean $\overline{a}_1 \succeq \overline{a}_2$ and $\overline{a}_1 \neq \overline{a}_2$ (resp. $\overline{a}_1 \preceq \overline{a}_2$ and $\overline{a}_1 \neq \overline{a}_2$). Let $\overline{a}_i \in D[0, 1]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \overline{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \overline{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

An *interval-valued fuzzy set* (briefly, *IVF set*) $\tilde{\mu}_A$ defined on a nonempty set X is given by

$$\tilde{\mu}_A := \left\{\left(x, \left[\mu_A^-(x), \mu_A^+(x)\right]\right) \mid x \in X\right\},$$

which is briefly denoted by $\tilde{\mu}_A = \left[\mu_A^-, \mu_A^+\right]$ where μ_A^- and μ_A^+ are two fuzzy sets in X such that $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in X$. For any IVF set $\tilde{\mu}_A$ on X and $x \in X$, $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ is called the degree of membership of an element x to $\tilde{\mu}_A$, in which $\mu_A^-(x)$ and $\mu_A^+(x)$ are refereed to as the lower and upper degrees, respectively, of membership of x to $\tilde{\mu}_A$.

3. Cubic subalgebras

Definition 3.1 ([1, 3])**.** Let X be a nonempty set. A *cubic set* $\mathscr{A}$ in X is a structure

$$\mathscr{A} = \{\langle x, \tilde{\mu}_A(x), \lambda(x)\rangle : x \in X\}$$

which is briefly denoted by $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ where $\tilde{\mu}_A = \left[\mu_A^-, \mu_A^+\right]$ is an IVF set in X and λ is a fuzzy set in X.

Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic set in a set X, $r \in [0,1]$ and $[s,t] \in D[0,1]$. The set

$$\mathcal{C}(\mathscr{A}; [s,t], r) := \{x \in X \mid \tilde{\mu}_A(x) \succeq [s,t],\ \lambda(x) \leq r\}$$

is called the *cubic level set* of $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ (see [1]).

Denote by $\mathcal{C}(X)$ the family of cubic sets in a set X. In what follows, let X denote a CI-algebra unless otherwise specified.

Definition 3.2. A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle \in \mathcal{C}(X)$ is called a *cubic subalgebra* of X if it satisfies:

$$(\forall x, y \in X)\,(\tilde{\mu}_A(x * y) \succeq \text{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\})\,. \tag{3.1}$$

$$(\forall x, y \in X)\,(\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\})\,. \tag{3.2}$$

Example 3.3. Consider a CI-algebra $X = \{1, a, b, c\}$ in which the $*$-operation is given by Table 1.

Table 1. $*$-operation

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	c	c	c	1

We define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and λ by

$$\tilde{\mu}_A = \begin{pmatrix} 1 & a & b & c \\ [0.6, 0.9] & [0.4, 0.8] & [0.3, 0.7] & [0.1, 0.3] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} 1 & a & b & c \\ 0.2 & 0.2 & 0.6 & 0.7 \end{pmatrix},$$

respectively. Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subalgebra of X.

Proposition 3.4. *If $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subalgebra of X, then $\tilde{\mu}_A(1) \succeq \tilde{\mu}_A(x)$ and $\lambda(1) \leq \lambda(x)$ for all $x \in X$.*

Proof. It is straightforward. □

Theorem 3.5. *For a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle \in \mathcal{C}(X)$, the following are equivalent:*

(1) *$\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subalgebra of X.*
(2) *The nonempty cubic level set of $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a subalgebra of X.*

Proof. Assume that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subalgebra of X. Let $x, y \in \mathcal{C}(\mathscr{A}; [s,t], r)$ for all $r \in [0,1]$ and $[s,t] \in D[0,1]$. Then $\tilde{\mu}_A(x) \succeq [s,t]$, $\lambda(x) \leq r$, $\tilde{\mu}_A(y) \succeq [s,t]$ and $\lambda(y) \leq r$. It follows from (3.1) and (3.2) that

$$\tilde{\mu}_A(x * y) \succeq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \succeq [s,t]$$

and $\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\} \leq r$ so that $x * y \in \mathcal{C}(\mathscr{A}; [s,t], r)$. Therefore the nonempty cubic level set of $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a subalgebra of X.

Conversely, assume that $\mathcal{C}(\mathscr{A}; [s,t], r)$ is a subalgebra of X for all $r \in [0,1]$ and $[s,t] \in D[0,1]$ with $\mathcal{C}(\mathscr{A}; [s,t], r) \neq \emptyset$. Suppose that (3.1) is not true and (3.2) is valid. Then there exist $[s_0, t_0] \in D[0,1]$ and $a, b \in X$ such that

$$\tilde{\mu}_A(a * b) \prec [s_0, t_0] \preceq \mathrm{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$$

and $\lambda(a * b) \leq \max\{\lambda(a), \lambda(b)\}$. It follows that $a, b \in \mathcal{C}(\mathscr{A}; [s_0, t_0], \max\{\lambda(a), \lambda(b)\})$ but $a * b \notin \mathcal{C}(\mathscr{A}; [s_0, t_0], \max\{\lambda(a), \lambda(b)\})$. This is a contradiction. If (3.1) is true and (3.2) is not valid, then $\tilde{\mu}_A(a * b) \succeq \mathrm{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$ and

$$\lambda(a * b) > r_0 \geq \max\{\lambda(a), \lambda(b)\}$$

for some $r_0 \in [0,1]$ and $a, b \in X$. Thus $a, b \in \mathcal{C}(\mathscr{A}; \mathrm{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, r_0)$ but $a * b \notin \mathcal{C}(\mathscr{A}; \mathrm{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, r_0)$, which is a contradiction. Assume that there exist $[s_0, t_0] \in D[0,1]$, $r_0 \in [0,1]$ and $a, b \in X$ such that

$$\tilde{\mu}_A(a * b) \prec [s_0, t_0] \preceq \mathrm{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}$$

and $\lambda(a * b) > r_0 \geq \max\{\lambda(a), \lambda(b)\}$. Then $a, b \in \mathcal{C}(\mathscr{A}; [s_0, t_0], r_0)$ but $a * b \notin \mathcal{C}(\mathscr{A}; [s_0, t_0], r_0)$. This is also a contradiction. Hence (3.1) and (3.2) are valid. Therefore $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subalgebra of X. □

Theorem 3.6. *If $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subalgebra of X, then the set*

$$S := \{x \in X \mid \tilde{\mu}_A(x) = \tilde{\mu}_A(1),\ \lambda(x) = \lambda(1)\}$$

is a subalgebra of X.

Proof. Let $x, y \in S$. Then $\tilde{\mu}_A(x) = \tilde{\mu}_A(1) = \tilde{\mu}_A(y)$ and $\lambda(x) = \lambda(1) = \lambda(y)$. It follows from (3.1) and (3.2) that

$$\tilde{\mu}_A(x * y) \succeq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = \tilde{\mu}_A(1)$$

and $\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\} = \lambda(1)$ so from Proposition 3.4 that $\tilde{\mu}_A(x * y) = \tilde{\mu}_A(1)$ and $\lambda(x * y) = \lambda(1)$. Hence $x * y \in S$, and so S is a subalgebra of X. □

Theorem 3.7. *For a subset S of X, let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle \in \mathcal{C}(X)$ be defined by*

$$\tilde{\mu}_A(x) = \begin{cases} [s, t] & \text{if } x \in S, \\ \overline{0} = [0, 0] & \text{otherwise} \end{cases}$$

and

$$\lambda(x) = \begin{cases} 0 & \text{if } x \in S \\ r & \text{otherwise} \end{cases}$$

where $r, s, t \in (0, 1]$ with $s < t$. Then

(1) *If S is a subalgebra of X, then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subalgebra of X and $\mathcal{C}(\mathscr{A}; [s, t], r) = S$.*
(2) *If $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subalgebra of X, then S is a subalgebra of X.*

Proof. (1) Assume that S is a subalgebra of X. Obviously $\mathcal{C}(\mathscr{A}; [s, t], r) = S$. Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$ and so

$$\tilde{\mu}_A(x * y) = [s, t] = \mathrm{rmin}\{[s, t], [s, t]\} = \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$$

and $\lambda(x * y) = 0 = \max\{0, 0\} = \max\{\lambda(x), \lambda(x)\}$. If $x, y \notin S$, then $\tilde{\mu}_A(x) = \overline{0} = [0, 0] = \tilde{\mu}_A(y)$ and $\lambda(x) = r = \lambda(y)$. Hence

$$\tilde{\mu}_A(x * y) \geq \overline{0} = [0, 0] = \mathrm{rmin}\{\overline{0}, \overline{0}\} = \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$$

and $\lambda(x * y) \leq r = \max\{r, r\} = \max\{\lambda(x), \lambda(y)\}$. If $x \in S$ and $y \notin S$, then $\tilde{\mu}_A(x) = [s, t]$, $\tilde{\mu}_A(y) = \overline{0}$, $\lambda(x) = 0$ and $\lambda(y) = r$. It follows that

$$\tilde{\mu}_A(x * y) \geq \overline{0} = \mathrm{rmin}\{[s, t], \overline{0}\} = \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$$

and $\lambda(x * y) \leq r = \max\{0, r\} = \max\{\lambda(x), \lambda(y)\}$. Similarly for the case $x \notin S$ and $y \in S$, we have $\tilde{\mu}_A(x * y) \geq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\lambda(x*y) \leq \max\{\lambda(x), \lambda(y)\}$. Therefore $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subalgebra of X.

(2) Suppose that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subalgebra of X. Let $x, y \in S$. Then $\tilde{\mu}_A(x) = [s,t] = \tilde{\mu}_A(y)$ and $\lambda(x) = 0 = \lambda(y)$, and so

$$\tilde{\mu}_A(x * y) \geq \mathrm{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} = \mathrm{rmin}\{[s,t],[s,t]\} = [s,t]$$

and $\lambda(x*y) \leq \max\{\lambda(x), \lambda(y)\} = 0$. Thus $x * y \in S$, and therefore S is a subalgebra of X. □

Let X and Y be given sets. A mapping $f : X \to Y$ induces two mappings $\mathcal{C}_f : \mathcal{C}(X) \to \mathcal{C}(Y)$, $\mathscr{A} \mapsto \mathcal{C}_f(\mathscr{A})$, and $\mathcal{C}_f^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$, $\mathscr{B} \mapsto \mathcal{C}_f^{-1}(\mathscr{B})$, where $\mathcal{C}_f(\mathscr{A})$ is given by

$$\mathcal{C}_f(\tilde{\mu}_A)(y) = \begin{cases} \underset{y=f(x)}{\mathrm{rsup}}\ \tilde{\mu}_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ \overline{0} = [0,0] & \text{otherwise} \end{cases}$$

$$\mathcal{C}_f(\lambda)(y) = \begin{cases} \inf\limits_{y=f(x)} \lambda(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & \text{otherwise} \end{cases}$$

for all $y \in Y$; and $\mathcal{C}_f^{-1}(\mathscr{B})$ is defined by $\mathcal{C}_f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x))$ and $\mathcal{C}_f^{-1}(\kappa)(x) = \kappa(f(x))$ for all $x \in X$. Then the mapping $\mathcal{C}_f$ (resp. $\mathcal{C}_f^{-1}$) is called a *cubic transformation* (resp. *inverse cubic transformation*) induced by f. A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in X has the *cubic property* if for any subset $T$of X there exists $x_0 \in T$ such that $\tilde{\mu}_A(x_0) = \underset{x \in T}{\mathrm{rsup}}\, \tilde{\mu}_A(x)$ and $\lambda(x_0) = \inf\limits_{x \in T} \lambda(x)$.

Theorem 3.8. *For a homomorphism $f : X \to Y$ of CI-algebras, let $\mathcal{C}_f : \mathcal{C}(X) \to \mathcal{C}(Y)$ and $\mathcal{C}_f^{-1} : \mathcal{C}(Y) \to \mathcal{C}(X)$ be the cubic transformation and inverse cubic transformation, respectively, induced by f.*

1. *If $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle \in \mathcal{C}(X)$ is a cubic subalgebra of X which has the cubic property, then $\mathcal{C}_f(\mathscr{A})$ is a cubic subalgebra of Y.*
2. *If $\mathscr{B} = \langle \tilde{\mu}_B, \kappa \rangle \in \mathcal{C}(Y)$ is a cubic subalgebra of Y, then $\mathcal{C}_f^{-1}(\mathscr{B})$ is a cubic subalgebra of X.*

Proof. (1) Given $f(x), f(y) \in f(X)$, let $x_0 \in f^{-1}(f(x))$ and $y_0 \in f^{-1}(f(y))$ be such that

$$\tilde{\mu}_A(x_0) = \underset{a \in f^{-1}(f(x))}{\mathrm{rsup}} \tilde{\mu}_A(a),\ \lambda(x_0) = \inf_{a \in f^{-1}(f(x))} \lambda(a),$$

and

$$\tilde{\mu}_A(y_0) = \underset{b \in f^{-1}(f(y))}{\mathrm{rsup}} \tilde{\mu}_A(b),\ \lambda(y_0) = \inf_{b \in f^{-1}(f(y))} \lambda(b),$$

respectively. Then

$$\begin{aligned}\mathcal{C}_f(\tilde{\mu}_A)(f(x)*f(y)) &= \underset{z\in f^{-1}(f(x)*f(y))}{\text{rsup}} \tilde{\mu}_A(z)\\ &\succeq \tilde{\mu}_A(x_0*y_0) \succeq \text{rmin}\{\tilde{\mu}_A(x_0), \tilde{\mu}_A(y_0)\}\\ &= \text{rmin}\Big\{\underset{a\in f^{-1}(f(x))}{\text{rsup}} \tilde{\mu}_A(a), \underset{b\in f^{-1}(f(y))}{\text{rsup}} \tilde{\mu}_A(b)\Big\}\\ &= \text{rmin}\{\mathcal{C}_f(\tilde{\mu}_A)(f(x)), \mathcal{C}_f(\tilde{\mu}_A)(f(y))\},\end{aligned}$$

$$\begin{aligned}\mathcal{C}_f(\lambda)(f(x)*f(y)) &= \inf_{z\in f^{-1}(f(x)*f(y))} \lambda(z)\\ &\leq \lambda(x_0*y_0) \leq \max\{\lambda(x_0), \lambda(y_0)\}\\ &= \max\Big\{\inf_{a\in f^{-1}(f(x))} \lambda(a), \inf_{b\in f^{-1}(f(y))} \lambda(b)\Big\}\\ &= \max\{\mathcal{C}_f(f(x)), \mathcal{C}_f(f(y))\}.\end{aligned}$$

Therefore $\mathcal{C}_f(\mathscr{A})$ is a cubic subalgebra of Y.

(2) For any $x, y \in X$, we have

$$\begin{aligned}\mathcal{C}_f^{-1}(\tilde{\mu}_B)(x*y) &= \tilde{\mu}_B(f(x*y)) = \tilde{\mu}_B(f(x)*f(y))\\ &\succeq \text{rmin}\{\tilde{\mu}_B(f(x)), \tilde{\mu}_B(f(y))\}\\ &= \text{rmin}\{\mathcal{C}_f^{-1}(\tilde{\mu}_B)(x), \mathcal{C}_f^{-1}(\tilde{\mu}_B)(y)\},\end{aligned}$$

$$\begin{aligned}\mathcal{C}_f^{-1}(\kappa)(x*y) &= \kappa(f(x*y)) = \kappa(f(x)*f(y))\\ &\leq \max\{\kappa(f(x)), \kappa(f(y))\}\\ &= \max\{\mathcal{C}_f^{-1}(\kappa)(x), \mathcal{C}_f^{-1}(\kappa)(y)\}.\end{aligned}$$

Hence $\mathcal{C}_f^{-1}(\mathscr{B})$ is a cubic subalgebra of X. □

4. Cubic filters

Definition 4.1. A cubic set $\mathscr{A} = \langle\tilde{\mu}_A, \lambda\rangle \in \mathcal{C}(X)$ is called a *cubic filter* of X if it satisfies: for all $x, y \in X$,

$$(\forall x, y \in X)\,(\tilde{\mu}_A(1) \succeq \tilde{\mu}_A(x),\ \lambda(1) \leq \lambda(x))\,. \tag{4.1}$$

$$(\forall x, y \in X)\,(\tilde{\mu}_A(y) \succeq \text{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(x*y)\})\,. \tag{4.2}$$

$$(\forall x, y \in X)\,(\lambda(y) \leq \max\{\lambda(x),\ \lambda(x*y)\})\,. \tag{4.3}$$

Example 4.2. Consider a CI-algebra $X = \{1, a, b, c\}$ in which the $*$-operation is given by Table 2.

Table 2. $*$-operation

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	c
c	c	c	c	1

We define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and λ by

$$\tilde{\mu}_A = \begin{pmatrix} 1 & a & b & c \\ [0.5, 0.8] & [0.4, 0.7] & [0.4, 0.7] & [0.1, 0.3] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} 1 & a & b & c \\ 0.2 & 0.2 & 0.2 & 0.6 \end{pmatrix},$$

respectively. Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic filter of X.

Proposition 4.3. *Every cubic filter $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ of X satisfies: for all $a, b, x, y, z \in X$,*

(1) $x * y = 1 \;\Rightarrow\; \tilde{\mu}_A(y) \succeq \tilde{\mu}_A(x),\ \lambda(y) \leq \lambda(x)$,

(2) $a * (b * x) = 1 \;\Rightarrow \begin{pmatrix} \tilde{\mu}_A(x) \succeq \mathrm{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\} \\ \lambda(x) \leq \max\{\lambda(a), \lambda(b)\} \end{pmatrix}$.

(3) $\begin{cases} \tilde{\mu}_A(x * z) \succeq \mathrm{rmin}\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}, \\ \lambda(x * z) \leq \max\{\lambda(x * (y * z)), \lambda(y)\} \end{cases}$

(4) $\tilde{\mu}_A(x) \preceq \tilde{\mu}_A((x * y) * y),\ \lambda(x) \geq \lambda((x * y) * y)$,

(5) $\begin{cases} \tilde{\mu}_A((a * (b * x)) * x) \succeq \mathrm{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}, \\ \lambda((a * (b * x)) * x) \leq \max\{\lambda(a), \lambda(b)\}. \end{cases}$

Proof. (1) Assume that $x * y = 1$ for all $x, y \in X$. Then

$$\tilde{\mu}_A(x) = \mathrm{rmin}\{\tilde{\mu}_A(1), \tilde{\mu}_A(x)\} = \mathrm{rmin}\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(x)\} \preceq \tilde{\mu}_A(y)$$

and

$$\lambda(x) = \max\{\lambda(1), \lambda(x)\} = \max\{\lambda(x * y), \lambda(x)\} \geq \lambda(y).$$

(2) Let $a, b, x \in X$ be such that $a * (b * x) = 1$. Then

$$\begin{aligned} \tilde{\mu}_A(x) &\succeq \mathrm{rmin}\{\tilde{\mu}_A(b * x), \tilde{\mu}_A(b)\} \\ &\succeq \mathrm{rmin}\{\mathrm{rmin}\{\tilde{\mu}_A(a * (b * x)), \tilde{\mu}_A(a)\}, \tilde{\mu}_A(b)\} \\ &= \mathrm{rmin}\{\mathrm{rmin}\{\tilde{\mu}_A(1), \tilde{\mu}_A(a)\}, \tilde{\mu}_A(b)\} \\ &= \mathrm{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\} \end{aligned}$$

and

$$\begin{aligned}\lambda(x) &\leq \max\{\lambda(b*x), \lambda(b)\}\\ &\leq \max\{\max\{\lambda(a*(b*x)), \lambda(a)\}, \lambda(b)\}\\ &= \max\{\max\{\lambda(1), \lambda(a)\}, \lambda(b)\}\\ &= \max\{\lambda(a), \lambda(b)\}.\end{aligned}$$

(3) Using (4.2), (4.3) and (CI3), we have

$$\tilde{\mu}_A(x*z) \succeq \text{rmin}\{\tilde{\mu}_A(y*(x*z)), \tilde{\mu}_A(y)\} = \text{rmin}\{\tilde{\mu}_A(x*(y*z)), \tilde{\mu}_A(y)\}$$

and

$$\lambda(x*z) \leq \max\{\lambda(y*(x*z)), \lambda(y)\} = \max\{\lambda(x*(y*z)), \lambda(y)\}$$

for all $x, y, z \in X$.

(4) If we take $y = (x*y)*y$ in (4.2) and (4.3), then

$$\begin{aligned}\tilde{\mu}_A((x*y)*y) &\succeq \text{rmin}\{\tilde{\mu}_A(x*((x*y)*y)), \tilde{\mu}_A(x)\}\\ &= \text{rmin}\{\tilde{\mu}_A((x*y)*(x*y)), \tilde{\mu}_A(x)\}\\ &= \text{rmin}\{\tilde{\mu}_A(1), \tilde{\mu}_A(x)\} = \tilde{\mu}_A(x)\end{aligned}$$

and

$$\begin{aligned}\lambda((x*y)*y) &\leq \max\{\lambda(x*((x*y)*y)), \lambda(x)\}\\ &= \max\{\lambda((x*y)*(x*y)), \lambda(x)\}\\ &= \max\{\lambda(1), \lambda(x)\} = \lambda(x)\end{aligned}$$

by using (CI3), (CI1) and (4.1).

(5) Using (3) and (4), we get

$$\begin{aligned}\tilde{\mu}_A((a*(b*x))*x) &\succeq \text{rmin}\{\tilde{\mu}_A((a*(b*x))*(b*x)), \tilde{\mu}_A(b)\}\\ &\succeq \text{rmin}\{\tilde{\mu}_A(a), \tilde{\mu}_A(b)\}\end{aligned}$$

and

$$\begin{aligned}\lambda((a*(b*x))*x) &\leq \max\{\lambda((a*(b*x))*(b*x)), \lambda(b)\}\\ &\leq \max\{\lambda(a), \lambda(b)\}\end{aligned}$$

for all $a, b, x \in X$. □

As a generalization of Proposition 4.3(2), we have the following result.

Proposition 4.4. *If a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle \in \mathcal{C}(X)$ is a cubic filter of X, then*

$$\prod_{i=1}^{n} a_i * x = 1 \;\Rightarrow\; \begin{cases} \tilde{\mu}_A(x) \succeq \text{rmin}\{\tilde{\mu}_A(a_i) \mid i = 1, 2, \cdots, n\}\\ \lambda(x) \leq \max\{\lambda(a_i) \mid i = 1, 2, \cdots, n\}\end{cases} \tag{4.4}$$

for all $x, a_1, \cdots, a_n \in X$, *where*

$$\prod_{i=1}^{n} a_i * x = a_n * (a_{n-1} * (\cdots (a_1 * x) \cdots)).$$

Proof. The proof is by induction on n. Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic filter of X. By (1) and (2) of Proposition 4.3, we know that the condition (4.4) is valid for $n = 1, 2$. Assume that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ satisfies the condition (4.4) for $n = k$, that is,

$$\prod_{i=1}^{k} a_i * x = 1 \;\Rightarrow\; \begin{cases} \tilde{\mu}_A(x) \succeq \operatorname{rmin}\{\tilde{\mu}_A(a_i) \mid i = 1, 2, \cdots, k\} \\ \lambda(x) \leq \max\{\lambda(a_i) \mid i = 1, 2, \cdots, k\} \end{cases}$$

for all $x, a_1, \cdots, a_k \in X$. Suppose that $\prod_{i=1}^{k+1} a_i * x = 1$ for all $x, a_1, \cdots, a_k, a_{k+1} \in X$. Then

$$\tilde{\mu}_A(a_1 * x) \succeq \operatorname{rmin}\{\tilde{\mu}_A(a_i) \mid i = 2, 3, \cdots, k+1\}$$

and

$$\lambda(a_1 * x) \leq \max\{\tilde{\mu}_A(a_i) \mid i = 2, 3, \cdots, k+1\}.$$

Since $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic filter of X, it follows from (4.2) and (4.3) that

$$\begin{aligned} \tilde{\mu}_A(x) &\succeq \operatorname{rmin}\{\tilde{\mu}_A(a_1 * x), \tilde{\mu}_A(a_1)\} \\ &\succeq \operatorname{rmin}\{\operatorname{rmin}\{\tilde{\mu}_A(a_i) \mid i = 2, 3, \cdots, k+1\}, \tilde{\mu}_A(a_1)\} \\ &= \operatorname{rmin}\{\tilde{\mu}_A(a_i) \mid i = 1, 2, \cdots, k+1\} \end{aligned}$$

and

$$\begin{aligned} \lambda(x) &\leq \max\{\lambda(a_1 * x), \lambda(a_1)\} \\ &\leq \max\{\max\{\lambda(a_i) \mid i = 2, 3, \cdots, k+1\}, \lambda(a_1)\} \\ &= \max\{\lambda(a_i) \mid i = 1, 2, \cdots, k+1\}. \end{aligned}$$

This completes the proof. □

We now provide conditions for a cubic set to be a cubic filter.

Theorem 4.5. *If a cubic set* $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle \in \mathcal{C}(X)$ *satisfies two conditions* (4.1) *and Proposition 4.3(2), then* $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ *is a cubic filter of* X.

Proof. Since $x * ((x * y) * y) = 1$ for all $x, y \in X$, it follows from Proposition 4.3(2) that

$$\tilde{\mu}_A(y) \succeq \operatorname{rmin}\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(x)\}$$

and

$$\lambda(y) \leq \max\{\max(x * y), \max(x)\}$$

for all $x, y \in X$. Therefore $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic filter of X. □

Theorem 4.6. *If a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle \in \mathcal{C}(X)$ satisfies two conditions* (4.1) *and Proposition 4.3(3), then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic filter of X.*

Proof. If we take $x = 1$ in Proposition 4.3(3) and use (CI2), then

$$\tilde{\mu}_A(z) = \tilde{\mu}_A(1*z) \succeq \text{rmin}\{\tilde{\mu}_A(1*(y*z)), \tilde{\mu}_A(y)\} = \text{rmin}\{\tilde{\mu}_A(y*z), \tilde{\mu}_A(y)\}$$

and

$$\lambda(z) = \lambda(1 * z) \leq \max\{\lambda(1 * (y * z)), \lambda(y)\} = \max\{\lambda(y * z), \lambda(y)\}$$

for all $y, z \in X$. Hence $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic filter of X. □

Theorem 4.7. *If a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle \in \mathcal{C}(X)$ satisfies Proposition 4.3(5) and*

$$(\forall x, y \in X)\, (\tilde{\mu}_A(y * x) \succeq \tilde{\mu}_A(x),\ \lambda(y * x) \leq \lambda(x))\,, \tag{4.5}$$

then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic filter of X.

Proof. Using (CI1), (CI2) and Proposition 4.3(5), we have

$$\tilde{\mu}_A(y) = \tilde{\mu}_A(1*y) = \tilde{\mu}_A(((x*y)*(x*y))*y) \succeq \text{rmin}\{\tilde{\mu}_A((x*y), \tilde{\mu}_A((x)\}$$

and

$$\lambda(y) = \lambda(1 * y) = \lambda(((x * y) * (x * y)) * y) \leq \text{rmin}\{\lambda((x * y), \lambda((x)\}$$

for all $x, y \in X$. If we take $y = x$ in (4.5), then $\tilde{\mu}_A(1) = \tilde{\mu}_A(x*x) \succeq \tilde{\mu}_A(x)$ and $\lambda(1) = \lambda(x * x) \leq \lambda(x)$ for all $x \in X$. Consequently, $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic filter of X. □

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